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Relation between Hénon maps with biholomorphic escaping sets

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Abstract

Let H and F be two Hénon maps with biholomorphically equivalent escaping sets, then there exist affine automorphisms A_1 and A_2 in $\mathbb{C}^2$ such that

$$F = A_1 \circ H \circ A_2$$

in $\mathbb{C}^2$.

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1 Introduction

In the complex plane the simplest examples of holomorphic dynamical systems with non-trivial dynamical behaviour are the polynomial maps of degree greater than or equal to 2. The linear polynomials, in other words, the automorphisms of the complex plane, generate trivial dynamics. In contrast, the class of polynomial automorphisms of $\mathbb{C}^2$ is large and possesses rich dynamical features. A dynamical classification of these maps was given by Friedland–Milnor [7]. They showed that any polynomial automorphism of $\mathbb{C}^2$ is conjugate to one of the following maps:

- an *affine map*;
- an *elementary map*, i.e., the maps of the form $(x, y) \mapsto (ax + b, sy + p(x))$ with $as \neq 0$, where p is a polynomial in single variable of degree strictly greater than one;
- a finite composition of Hénon maps, where Hénon maps are the maps of the form

$$(x, y) \mapsto (y, p(y) - \delta x) \tag{1.1}$$

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with $\delta \neq 0$ and p a polynomial in single variable of degree $d \geq 2$.

The degree of a single Hénon map H of the form (1.1) is defined to be the degree of the polynomial p. The degree of composition of Hénon maps $H_n \circ \cdots \circ H_1$ is defined to be $d_n \cdots d_1$ where $d_i = \deg(H_i)$, for $1 \leq i \leq n$.

Hénon maps are generalization of classical real quadratic Hénon maps introduced by astronomer Michel Hénon. The relevance of these maps in complex dynamics became apparent in the above-mentioned classification theorem of Friedland–Milnor. Moreover it turned out that the composition of Hénon maps are the only dynamically non-trivial polynomial automorphisms in $\mathbb{C}^2$, which naturally drew attention of many foremost researchers towards these maps. The pioneering work on Hénon maps was done by Bedford–Smillie [1–3], Fornæss–Sibony [8] and Hubbard–Oberste–Vorth [9, 10].

As in the case of polynomials in the complex plane, the orbit of any point in $\mathbb{C}^2$ under the iterations of a Hénon map H (or more generally, finite composition of Hénon maps) either diverges to infinity or always remains bounded. The collection of points $I_H^+ \subseteq \mathbb{C}^2$ which escape to infinity are called the *escaping set* of H and the collection of points $K_H^+ \subseteq \mathbb{C}^2$ whose orbits remain bounded are called the *non-escaping set* of H. The union of escaping set I_H^+ and the interior of non-escaping set K_H^+ is the largest set of normality of the sequence of maps $\{H^n\}_{n\geq 1}$, that is, $I_H^+ \cup \operatorname{int} K_H^+$ is the Fatou set of H. The escaping sets and the interiors of non-escaping sets of Hénon maps can be thought of as analogues of the unbounded components and the bounded components of Fatou sets of polynomials in the complex plane. However, the non-escaping sets of Hénon maps are not bounded in $\mathbb{C}^2$. The common boundary set J_H^+ of the escaping set I_H^+ and the non-escaping set K_H^+ is the Julia set of H.

The present article addresses a *rigidity* property of Hénon maps of the form (1.1). *To what extent do the escaping sets of Hénon maps determine the Hénon maps?* In other words, if the escaping sets of a pair of Hénon maps H and F of degree d are biholomorphically equivalent, then are these two Hénon maps closely related?

This question is first studied in a recent work of Bonnot–Radu–Tanase [4], where they prove that H and F coincide, for $d = 2$. Further, they produce examples to show that for $d \geq 3$, H and F might not be even conjugate to each other. In this article we establish a precise relation between H and F of any degree $d \geq 2$ with biholomorphic escaping sets (Theorem 1.2).

The rigidity question raised here is conceived based on an explicit description of analytic structure of the escaping sets given by Hubbard–Oberste–Vorth in [9]. A convenient description of the escaping set I_H^+ of a given Hénon map H is given in terms of logarithmic rate of escape function, the so-called Green's function G_H^+ of the Hénon map H. One can show that I_H^+ is precisely where the Green's function is strictly positive. Further, $G_H^+ : I_H^+ \to \mathbb{R}_+$ is a pluri-harmonic submersion and the level sets of G_H^+ are three dimensional manifolds, which are naturally foliated by copies of $\mathbb{C}$. The Green's function G_H^+ is inextricably related to the Böttcher function ϕ_H^+ of H, which is one of the key ingredients in describing the analytic structure of I_H^+. The Böttcher functions of Hénon maps can be considered as analogues of Böttcher functions of polynomials in $\mathbb{C}$ near infinity and they are defined in appropriate neighbourhoods of a point at infinity $[0 : 1 : 0]$ in $\mathbb{P}^2$. To be a bit more precise, for $R > 0$ sufficiently large, ϕ_H^+ is defined on the open set $V_R^+ = \big\{(x, y) \in \mathbb{C}^2 : |y| > \max\{|x|, R\}\big\} \subseteq I_H^+$

by means of approaching the point $[0:1:0]$ by the n-fold iteration H^n, then returning back by appropriate d^n-th root of the mapping $y \mapsto y^{d^n}$ and finally taking the limit as $n \to \infty$. Consequently, the range of ϕ_H^+ lies inside $\mathbb{C}\backslash\bar{\mathbb{D}}$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$ and $G_H^+ \equiv \log|\phi_H^+|$ in V_R^+. Although the Böttcher function ϕ_H^+ does not extend analytically to I_H^+, it extends along curves in I_H^+ starting in V_R^+ and defines a multi-valued analytic map in I_H^+. Let $\tilde{I}_H^+$ be the covering of I_H^+ obtained as the Riemann domain of ϕ_H^+ and let ϕ_H^+ lifts as a single-valued holomorphic function $\tilde{\phi}_H^+ : \tilde{I}_H^+ \to \mathbb{C}\backslash\bar{\mathbb{D}}$. Further, $\tilde{I}_H^+$ is biholomorphically equivalent to the domain $\{(z,\zeta) : z \in \mathbb{C}, |\zeta| > 1\}$. By construction $\zeta = \tilde{\phi}_H^+$ and thus in this new coordinate the level sets of $\tilde{\phi}_H^+$ simply straightens out. The Hénon map H lifts as a map $\tilde{H} : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ and one can write down $\tilde{H}$ explicitly (see 2.6).

The following result relies on two main ingredients, also used by Bonnot–Radu–Tanase in [4]: The above-mentioned explicit description of the covering $\tilde{I}_H^+$ of I_H^+ and a method given by Bousch in [5].

Theorem 1.1 *Let $H(x,y) = (y, p_H(y) - \delta_H x)$ and $F(x,y) = (y, p_F(y) - \delta_F x)$ be a pair of Hénon maps, where $p_H(y) = y^d + \sum_{i=0}^{d-2} a_i^H y^i$ and $p_F(y) = y^d + \sum_{i=0}^{d-2} a_i^F y^i$. Let I_H^+ and I_F^+ be escaping sets of H and F, respectively and let I_H^+ and I_F^+ be biholomorphically equivalent. Then $\beta p_H(y) = \alpha p_F(\alpha y)$, for some $\alpha, \beta \in \mathbb{C}$ with $\alpha^{d+1} = \beta$ and $\beta^{d-1} = 1$. Further, we have $\delta_H = \gamma\delta_F$, with $\gamma^{d-1} = 1$. Therefore,*

$$F \equiv L \circ B \circ H \circ B, \tag{1.2}$$

where $B(x,y) = (\gamma\alpha\beta^{-1}x, \alpha^{-1}y)$ and $L(x,y) = (\gamma^{-1}\beta x, \beta y)$, for all $(x,y) \in \mathbb{C}^2$.

Now note that p_H and p_F in Theorem 1.1 are monic and centered (next to highest coefficients vanish), whereas for an arbitrary Hénon map the associated polynomial in one variable is not necessarily monic and centered. However, it is not hard to see that up to conjugation any arbitrary Hénon map is of the form as in Theorem 1.1. Here goes a brief justification. Up to conjugation by an affine automorphism of $\mathbb{C}$, any polynomial in one variable is a monic and centered polynomial of the same degree. In particular, there exists an affine automorphism σ_H of $\mathbb{C}$ such that $\sigma_H^{-1} \circ p_H \circ \sigma_H = \hat{p}_H$ in $\mathbb{C}$, where $\hat{p}_H$ is monic and centered. Thus if we consider the affine automorphism $A_H(x,y) = (\sigma_H(x), \sigma_H(y))$, for $(x,y) \in \mathbb{C}^2$, then $A_H^{-1} \circ H \circ A_H = \hat{H}$, where $\hat{H}(x,y) = (y, \hat{p}_H(y) - \delta_H x)$, for all $(x,y) \in \mathbb{C}^2$ with $\deg(\hat{p}_H) = \deg(p_H)$. Clearly, $A_H(I_H^+) = I_{\hat{H}}^+$. Similarly, there exist an affine map A_F and a Hénon map $\hat{F}$ such that $A_F^{-1} \circ F \circ A_F = \hat{F}$ and $A_F(I_F^+) = I_{\hat{F}}^+$, where $\hat{F}(x,y) = (y, \hat{p}_F(y) - \delta_F x)$, for all $(x,y) \in \mathbb{C}^2$ with $\hat{p}_F$ monic and centered. Therefore, once we prove Theorem 1.1, the following result is obtained immediately.

Theorem 1.2 *Let H and F be two Hénon maps with biholomorphically equivalent escaping sets, then there exist affine automorphisms A_1 and A_2 in $\mathbb{C}^2$ such that*

$$F = A_1 \circ H \circ A_2$$

in $\mathbb{C}^2$.

2 Preliminaries

Let

$$H(x, y) = (y, p(y) - \delta x) \tag{2.1}$$

be a Hénon map, where p is a monic and centered polynomial in single variable of degree $d \geq 2$ and $\delta \neq 0$. In this section, we see a few fundamental definitions and a couple of known results on Hénon maps pertaining to the theme of the present article.
Filtration: For $R > 0$, let

$$\begin{aligned} V_R^+ &= \{(x, y) \in \mathbb{C}^2 : |x| < |y|, |y| > R\}, \\ V_R^- &= \{(x, y) \in \mathbb{C}^2 : |y| < |x|, |x| > R\}, \\ V_R &= \{(x, y) \in \mathbb{C}^2 : |x|, |y| \leq R\}. \end{aligned}$$

This is called a filtration. For a given Hénon map H, there exists $R > 0$ sufficiently large such that

$$H(V_R^+) \subset V_R^+,\ H(V_R^+ \cup V_R) \subset V_R^+ \cup V_R$$

and

$$H^{-1}(V_R^-) \subset V_R^-,\ H^{-1}(V_R^- \cup V_R) \subset V_R^- \cup V_R.$$

Escaping and non-escaping sets: The set

$$I_H^+ = \Big\{(x, y) \in \mathbb{C}^2 : \|H^n(x, y)\| \to \infty \text{ as } n \to \infty\Big\}$$

is called the *escaping set* of H and the set

$$K_H^+ = \Big\{(x, y) \in \mathbb{C}^2 : \text{ the sequence } \big\{H^n(x, y)\big\}_{n \geq 1} \text{ is bounded}\Big\}$$

is called the *non-escaping set* of H. One can prove that $K_H^+ \subset V_R \cup V_R^-$ and

$$I_H^+ = \mathbb{C}^2 \setminus K_H^+ = \bigcup_{n=0}^{\infty} H^{-n}(V_R^+). \tag{2.2}$$

Any Hénon map H can be extended meromorphically to $\mathbb{P}^2$ with an isolated indeterminacy point $[0 : 1 : 0]$. In fact, one can prove that the points in I_H^+ under iteration of H converges to the point $[0 : 1 : 0]$ uniformly (on compacts).
Green's function: The *Green's function* of H is defined to be

$$G_H^+(x, y) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \left\| H^n(x, y) \right\|,$$

for all $(x, y) \in \mathbb{C}^2$, where $\log^+(t) = \max\{\log t, 0\}$. It turns out that G_H^+ is non-negative everywhere in $\mathbb{C}^2$, plurisubharmonic in $\mathbb{C}^2$, pluriharmonic on $\mathbb{C}^2 \setminus K_H^+$ and vanishes precisely on K_H^+. Further,

$$G_H^+ \circ H = dG_H^+$$

in $\mathbb{C}^2$. The Green's function G_H^+ has logarithmic growth near infinity, i.e., there exist $R > 0$ and $L > 0$ such that

$$\log^+|y| - L \leq G_H^+(x, y) \leq \log^+|y| + L,$$

for $(x, y) \in \overline{V_R^+ \cup V_R}$.
Böttcher function: Let $H^n(x, y) = ((H^n)_1(x, y), (H^n)_2(x, y))$, for $(x, y) \in \mathbb{C}^2$. Note that $y_n = (H^n)_2(x, y)$ is a polynomial in x and y of degree d^n. The function

$$\phi_H^+(x, y) := \lim_{n \to \infty} y_n^{\frac{1}{d^n}} = y.\frac{y_1{}^{\frac{1}{d}}}{y}.\cdots.\frac{y_{n+1}^{\frac{1}{d^{n+1}}}}{y_n^{\frac{1}{d^n}}}\cdots,$$

for $(x, y) \in V_R^+$, defines an analytic function from V_R^+ to $\mathbb{C} \setminus \bar{\mathbb{D}}$ and is called the *Böttcher function* of the Hénon map H. Further,

$$\phi_H^+ \circ H(x, y) = \left(\phi_H^+(x, y)\right)^d,$$

for all $(x, y) \in V_R^+$ and

$$\phi_H^+(x, y) \sim y \text{ as } \|(x, y)\| \to \infty$$

in V_R^+. Comparing the definitions of Green's function and Böttcher functions it follows instantly that

$$G_H^+ = \log|\phi_H^+|$$

in V_R^+.

For a detailed treatment of the above discussion, the inquisitive readers can see [1–3, 9, 11].
Now we present one of the technical ingredients required for the proof of Theorem 1.1. A series of change of coordinates we see here is a part of the standard theory of Hénon maps. However, most of the results presented here are paraphrasing of a couple of lemmas appearing in the beginning of the appendix of [4]. However the genesis of these results goes back to [6, 9].

For $M > 0$ sufficiently large, let

$$U_R^+ = \left\{(x, y) \in V_R^+ : |\phi_H^+(x, y)| > M \max\{R, |x|\}\right\}.$$

One can easily check that $U_R^+ \subseteq V_R^+$ and $H(U_R^+) \subseteq U_R^+$.

Lemma 2.1 [11, Lemma 7.3.7],[6, Prop. 2.2] *There exists a holomorphic function ψ_H on U_R^+ such that*

- $\psi_H \circ H(x, y) = (\delta/d)\psi_H(x, y) + Q\left(\phi_H^+(x, y)\right)$, *for all* $(x, y) \in U_R^+$, *where* Q *is a monic polynomial of degree* $d + 1$;
- *the map* $\Phi_H = (\psi_H, \phi_H^+) : U_R^+ \to \mathbb{C}^2$ *is an injective holomorphic map.*

The map $(x, y) \mapsto (x, \zeta) = (x, \phi_H^+(x, y))$ maps U_R^+ biholomorphically to $\left\{(x, y) \in \mathbb{C}^2 : |y| > M \max\{R, |x|\}\right\}$. Let $(x, \zeta) \mapsto (x, y(x, \zeta))$ be the inverse map. It turns out that

$$\psi_H(x, y) = \zeta \int_0^x \frac{\partial y}{\partial \zeta}(u, \zeta)du,$$

for $(x, y) \in U_R^+$ and $Q(\zeta)$ is the polynomial part of the power series expansion of

$$\zeta^d \int_0^{y(0,\zeta)} \frac{\partial y}{\partial \zeta}(u, \zeta^d)du \tag{2.3}$$

(see proof of [11, Lemma 7.3.7] for details).

Next we consider the following change of coordinate near $p = [0 : 1 : 0]$:

$$T : (x, y(x, \zeta)) = (x, y) \mapsto \left(\frac{x}{y}, \frac{1}{y}\right) = (t, w).$$

Note that $T^2 = \text{Id}$ and H takes the following form in (t, w)-coordinate:

$$\begin{aligned}(t, w) \mapsto (x, y) \mapsto (y, p(y) - \delta x) \mapsto &\left(\frac{y}{p(y) - \delta x}, \frac{1}{p(y) - \delta x}\right)\\ &= \left(\frac{w^{d-1}}{w^d p(1/w) - \delta t w^{d-1}}, \frac{w^d}{w^d p(1/w) - \delta t w^{d-1}}\right).\end{aligned}$$

Lemma 2.2 *With the above notations, near* $p = [0 : 1 : 0]$ *we have*

$$1/\zeta = w(1 + w\alpha(t, w)), \tag{2.4}$$

where $\alpha(t, w)$ *is a power series in* t, w.

Proof Note that

$$\frac{1}{\zeta} = \frac{1}{\phi_H^+(x, y)} = \lim_{n\to\infty} \left[\frac{1}{(H^n(x, y))_2}\right]^{\frac{1}{d^n}} = \lim_{n\to\infty} \left[(T H^n T^{-1}(t, w))_2\right]^{\frac{1}{d^n}}.$$

Equivalently,

$$\frac{1}{\zeta} = w.\left[\left(\frac{w^d}{w^d p(\frac{1}{w}) - \delta t w^{d-1}}\right)/w^d\right]^{\frac{1}{d}}.\left[\left(\frac{w_1^d}{w_1^d p(\frac{1}{w_1}) - \delta t_1 w_1^{d-1}}\right)/w_1^d\right]^{\frac{1}{d^2}}\cdots$$

$$= w.\left[\frac{1}{w^d p(\frac{1}{w}) - \delta t w^{d-1}}\right]^{\frac{1}{d}}.\left[\frac{1}{w_1^d p(\frac{1}{w_1}) - \delta t_1 w_1^{d-1}}\right]^{\frac{1}{d^2}}\cdots = wX(t,w)$$

(here the d^n-th roots, for $n \geq 1$, are taken to be the principal branches of roots). One can check that the above series converges. Further, since $\left|\phi_H^+(x,y)/y\right|$ is bounded in V_R^+, it follows that $X(t,w)$ is also bounded in its domain of definition, which is a subset of $\{|t| < 1\} \times \{0 < |w| < 1/R\}$. Thus $X(t,w)$ has a power series expansion. Also note that $X(t,0) = 1$, for all t. So $X(t,w) = 1 + w\alpha(t,w)$, where $\alpha(t,w)$ is a power series in t and w. Therefore, finally we obtain (2.4). □

Next we consider the following change of variables: $(t,w) \mapsto (r,s) = (x/\zeta, 1/\zeta)$, which is a biholomorphism and is tangent to the identity (see [4, Lemma 4.4]). Therefore,

$$t = r + T_2(r,s) \text{ and } w = s + S_2(r,s),$$

where T_2 and S_2 both are power series with monomials of degree at least 2.

Lemma 2.3 *With the above notations, we have*

$$y = \zeta\left(1 + C/\zeta + U\left(x/\zeta, 1/\zeta\right)\right),$$

where $C \in \mathbb{C}$ and U is a power series in two variables with monomials of degree at least 2.

Proof By Lemma 2.2, we have $1/\zeta = w(1+w\alpha(t,w))$. Thus $y/\zeta = (1+w\alpha(t,w))$. Therefore, by replacing $t = r + T_2(r,s)$ and $w = s + S_2(r,s)$, we get

$$\begin{aligned} y &= \zeta\left(1 + (s + S_2(r,s))(\alpha(r + T_2(r,s), s + S_2(r,s)))\right) \\ &= \zeta\left(1 + s\beta(r,s) + S_2(r,s)\beta(r,s)\right), \end{aligned}$$

where $\beta(r,s) = \alpha(r + T_2(r,s), s + S_2(r,s))$. Now since $(r,s) = (x/\zeta, 1/\zeta)$, we have

$$y = \zeta\left(1 + C/\zeta + U\left(x/\zeta, 1/\zeta\right)\right),$$

where $C \in \mathbb{C}$ is the constant term of the power series expansion of β and U is a power series in two variables without constant term. □

Therefore,

$$y(x,\zeta) = \left(\zeta + C + \frac{A^0_{-1}}{\zeta} + \cdots\right) + x\left(\frac{A^1_{-1}}{\zeta^{m_1}} + \cdots\right) \\ + x^2\left(\frac{A^2_{-1}}{\zeta^{m_2}} + \cdots\right) + \cdots + x^n\left(\frac{A^n_{-1}}{\zeta^{m_n}} + \cdots\right) + \cdots,$$

where $m_1 = 1, m_n = n-1$, for $n \geq 2$ and $A^0_{-1}, A^1_{-1}, A^2_{-1}$ and so on are in $\mathbb{C}$.

The following lemma is one of the main takeaways from this section.

Lemma 2.4 *With the above notations, we have*

$$y(0,\zeta) = \zeta + \frac{D_1}{\zeta} + \frac{D_2}{\zeta^2} \cdots,$$

and

$$\zeta(0,y) = y + \frac{L_1}{y} + \frac{L_2}{y^2} \cdots,$$

where D_i's and L_i's are constants. Further, $Q(\zeta)$ is given by the polynomial part of $\zeta^d y(0,\zeta)$ and therefore, Q is centered and monic.

Proof Since p is monic and centered in (2.1), it follows from [9, Prop. 5.2] that

$$\zeta(0,y) = y + \frac{L_1}{y} + \frac{L_2}{y^2} \cdots.$$

Now since $y = y(0,\zeta(0,y))$,

$$y = y(0,\zeta(0,y)) = \zeta(0,y) + D_0 + \frac{D_1}{\zeta(0,y)} + \cdots = \left(y + \frac{L_1}{y} + \frac{L_2}{y^2} + \cdots\right) + D_0 \\ + D_1\left(y + \frac{L_1}{y} + \frac{L_2}{y^2} + \cdots\right)^{-1} + \cdots.$$

Thus $D_0 = 0$ and consequently, we have

$$y(0,\zeta) = \zeta + \frac{D_1}{\zeta} + \frac{D_2}{\zeta^2} + \cdots.$$

The last assertion follows immediately from (2.3). □

As we shall see in Section 3 and in Section 4, the particular forms of the power series representations of $y(0,\zeta)$ and $\zeta(0,y)$ as obtained in Lemma 2.4 and the knowledge of analytic structure of the escaping sets of Hénon maps are the key players in unravelling the relation between any pair of Hénon maps with biholomorphic escaping sets.

Here we give a brief description of analytic structure of Hénon maps. For a detailed account of analytic structure of escaping sets of Hénon maps the inquisitive readers can look at [6, 9, 11]. For any Hénon map H, it turns out that the Riemann surface of the Böttcher function ϕ_H^+ is a covering space of the escaping set I_H^+ and it is isomorphic to $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$. For a Hénon map H of degree d, the fundamental group of I_H^+ is

$$\mathbb{Z}[1/d] = \left\{k/d^n : k, n \in \mathbb{Z}\right\}$$

and the covering $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ of I_H^+ arises corresponding to the subgroup $\mathbb{Z} \subseteq \mathbb{Z}[1/d]$. Therefore, I_H^+ is a quotient of $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ by some discrete subgroup of the automorphisms of $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ isomorphic to $\mathbb{Z}[1/d]/\mathbb{Z}$. For each element $\left[k/d^n\right] \in \mathbb{Z}[1/d]/\mathbb{Z}$, there exists a unique deck transformation γ_{k/d^n} from $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ to $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ such that

$$\gamma_{k/d^n} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} z + \frac{d}{\delta} \sum_{l=0}^{n-1} \left(\frac{d}{\delta}\right)^l \left(Q(\zeta^{d^l}) - Q\left(\left(e^{\frac{2k\pi i}{d^n}} \zeta \right)^{d^l} \right) \right) \\ e^{\frac{2k\pi i}{d^n}} \zeta \end{bmatrix}, \tag{2.5}$$

for $n \geq 0$ and $k \geq 1$. Further, H lifts as a holomorphic map

$$\tilde{H}(z, \zeta) = \left((\delta/d)z + Q(\zeta), \zeta^d\right) \tag{2.6}$$

from $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ to $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$.

3 A brief idea of the proof of the main theorem

In this section we sketch the main idea of the proof of Theorem 1.1.

Let $H(x, y) = (y, p_H(y) - \delta_H x)$ and $F(x, y) = (y, p_F(y) - \delta_F x)$ be a pair of Hénon maps of degree d. Thus the fundamental groups of both I_H^+ and I_F^+ are $\mathbb{Z}[1/d]$. Let a be a biholomorphism between I_H^+ and I_F^+. Any biholomorphism between I_H^+ and I_F^+, which induces identity as an isomorphism between the fundamental groups of I_H^+ and I_F^+, can be lifted as a biholomorphism from $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ to $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$. Now since any group isomorphism of $\mathbb{Z}[1/d]$ is of the form $x \mapsto \pm d^s x$, for some $s \in \mathbb{Z}$, the map a up to pre-composition with some n-fold iterates of F (or F^{-1}), i.e., the map $F^{\pm n} \circ a$, for some $n \in \mathbb{N}$, induces identity map between the fundamental groups of the escaping sets I_H^+ and I_F^+. Thus, it is harmless to assume that a lifts as a biholomorphism A from $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ to $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$. Therefore, $\pi \circ A = a \circ \pi$ and consequently, the fiber of any point $p \in I_H^+$ of the natural projection map $\pi_H : \mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}}) \to I_H^+$ maps into the fiber of the point $a(p) \in I_F^+$ of the projection map $\pi_F : \mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}}) \to I_F^+$ by the biholomorphism A. Fiber of any point of the projection maps π_H and π_F are captured by the group of deck transformation of $\mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$, which is isomorphic to $\mathbb{Z}[1/d]/\mathbb{Z}$. Thus, it turns out that if $(z, \zeta) \in \mathbb{C} \times (\mathbb{C} \backslash \bar{\mathbb{D}})$ is in the fiber of any point $p \in I_H^+$ (or I_F^+), then any other point in the same fiber will be of the form $\gamma_{k/d^n}(z, \zeta)$, for $k \geq 1$ and

$n \geq 0$, where γ_{k/d^n} is the deck transformation corresponding to $[k/d^n] \in \mathbb{Z}[1/d]/\mathbb{Z}$. An explicit description of this fibers can be written down (see (2.5)). Now adapting an idea of Bousch ([5]), one can narrow down the possible forms of A by comparing the fibers of p and $a(p)$. In fact, in our case, A has a very simple form $(u, v) \mapsto (u+\gamma, \alpha v)$, with $|\alpha| = 1$ and $\gamma \in \mathbb{C}$. Thus we obtain an explicit expression for the image of the foliation $\tilde{\phi}_H^+ = \zeta_0$ under the map A. These foliations come down to the corresponding escaping sets and induce rigidity on escaping sets. As a consequence, one expects a close relation between H and F, which is validated in our main theorem.

The above-mentioned idea was employed by Bonnot–Radu–Tanase in [4] in establishing the relation between H and F with biholomorphic escaping sets when $\deg(H) = \deg(F) = 2$. They attempts to extract the relation between the coefficients of p_H and p_F by directly comparing the fibers of p and $a(p)$. Although their approach works for lower degrees (for $d = 2, 3$), since precise computations can be carried out in these cases but as degree increases, to extract the relation between the coefficients of p_H and p_F seems very difficult by performing such direct calculations.

We take a different approach towards this problem. For two Hénon maps H and F with biholomorphic escaping sets, we first establish the relation between coefficients of Q_H and Q_F (see Section 3). The explicit relation between Q_H and Q_F is used to obtain a neat relation between the polynomials p_H and p_F, namely, $\beta p_H(y) = \alpha p_F(\alpha y)$, for all $y \in \mathbb{C}$ with $\alpha^{d+1} = \beta$ and $\beta^{d-1} = 1$ (see Section 4). To establish the relation between H and F, we are yet to investigate the relation between the Jacobians δ_H and δ_F. It turns out that $\delta_H = \gamma\delta_F$, with $\gamma^{d-1} = 1$, thanks to [4]. It is noteworthy that all our calculations in Section 3 and in Section 4 are performed under the assumption that $\delta_H = \delta_F$. In Section 5, we outline how to handle the case when δ_H and δ_F are different.

4 Relation between Q_H and Q_F

Let H and F be two Hénon maps as in Theorem 1.1 with bihholomorphic escaping sets I_H^+ and I_F^+, respectively. As indicated in the Introduction, for now we assume $\delta_H = \delta_F = \delta$. Later (in Section 6), we handle the case when $\delta_H \neq \delta_F$.

Lifting biholomorphisms between escaping sets: Recall from Section 2 that the fundamental group of escaping set of any Hénon map of degree d is $\mathbb{Z}[1/d]$ and $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ is the covering of the escaping set corresponding to the subgroup $\mathbb{Z} \subseteq \mathbb{Z}[1/d]$. Let $\pi_H : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to I_H^+$ and $\pi_F : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to I_F^+$ be the covering maps. Now note that a biholomorphism a from I_H^+ to I_F^+ can be lifted as an automorphism A of $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ if and only if the induced group isomorphism

$$\pi_1(a) : \mathbb{Z}[1/d] \to \mathbb{Z}[1/d]$$

is $\pm$Id (identity maps). It is easy to see that any group isomorphism of $\mathbb{Z}[1/d]$ is of the form $x \mapsto \pm d^s x$, for some $s \in \mathbb{Z}$. Thus, there exists $n \in \mathbb{N}$ such that $F^{\pm n} \circ a$ induces identity on $\mathbb{Z}[1/d]$ and thus lifts as an automorphism of $\mathbb{C} \times \left(\mathbb{C}\backslash\bar{\mathbb{D}}\right)$. Therefore, without loss of generality, we assume that a lifts as an automorphism A of $\mathbb{C} \times \left(\mathbb{C}\backslash\bar{\mathbb{D}}\right)$.

$$\begin{array}{ccc} \mathbb{C} \times \left(\mathbb{C}\backslash\bar{\mathbb{D}}\right) & \xrightarrow{A} & \mathbb{C} \times \left(\mathbb{C}\backslash\bar{\mathbb{D}}\right) \\ \pi_H \downarrow & & \downarrow \pi_F \\ I_H^+ & \xrightarrow{a} & I_F^+ \end{array}$$

It follows from (2.5) that for H there exists a polynomial Q_H of degree $d+1$ such that for each element $\left[k/d^n\right] \in \mathbb{Z}[1/d]/\mathbb{Z}$, with $n \geq 0$ and $k \geq 1$, there exists a unique deck transformation γ_{k/d^n} from $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ to $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ of the form

$$\gamma_{k/d^n} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} z + \frac{d}{\delta} \sum_{l=0}^{n-1} \left(\frac{d}{\delta}\right)^l \left(Q_H(\zeta^{d^l}) - Q_H\left(\left(e^{\frac{2k\pi i}{d^n}} \zeta\right)^{d^l}\right)\right) \\ e^{\frac{2k\pi i}{d^n}} \zeta \end{bmatrix}. \tag{4.1}$$

The same holds for the Hénon map F. Thus if $\tilde{p} = (z, \zeta) \in \mathbb{C} \times \left(\mathbb{C}\backslash\bar{\mathbb{D}}\right)$ lies in the $\pi_H^{-1}(p)$ for some $p \in I_H^+$, then

$$\pi_H^{-1}(p) = \{\gamma_{k/d^n}(z, \zeta) : n \geq 0, k \geq 1\}.$$

Form of lifts: First note that any automorphism A of $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ is of the form:

$$A(z, \zeta) = (A_1(z, \zeta), A_2(z, \zeta)) = (\beta(\zeta)z + \gamma(\zeta), \alpha\zeta), \tag{4.2}$$

where $|\alpha| = 1$ and β, γ are holomorphic maps from $\mathbb{C}\backslash\bar{\mathbb{D}}$ to $\mathbb{C}^*$ and $\mathbb{C}$, respectively (see [5, Section 3]).

Note that since $a \circ \pi_H = \pi_F \circ A$, if $(z, \zeta), (z', \zeta') \in \pi_H^{-1}(p)$, for some $p \in I_H^+$, then $A(z, \zeta), A(z', \zeta') \in \pi_F^{-1}(a(p))$. Now let (z, ζ) and (z', ζ') be in the same fiber of π_H, then using (4.1), we have

$$(\zeta'/\zeta)^{d^n} = 1$$

and

$$z' = z + \frac{d}{\delta} \sum_{l=0}^{n-1} \left(\frac{d}{\delta}\right)^l \left(Q_H(\zeta^{d^l}) - Q_H\left(\left(e^{\frac{2k\pi i}{d^n}} \zeta\right)^{d^l}\right)\right),$$

for some $n \in \mathbb{N}$. Therefore the difference between the first coordinates of A, i.e.,

$$\begin{aligned} A_1(z', \zeta') - A_1(z, \zeta) = &\left(\beta(\zeta') - \beta(\zeta)\right) u \\ &+ \beta(\zeta') \frac{d}{\delta} \sum_{l=0}^{n-1} \left(\frac{d}{\delta}\right)^l \left(Q_H(\zeta^{d^l}) - Q_H\left(\left(e^{\frac{2k\pi i}{d^n}} \zeta\right)^{d^l}\right)\right) \\ &+ \left(\gamma(\zeta') - \gamma(\zeta)\right). \end{aligned} \tag{4.3}$$

Now since $A(z,\zeta)$ and $A(z',\zeta')$ are in the same fiber of π_F, the difference $A_1(z,\zeta) - A_1(z',\zeta')$ is a function of $\alpha\zeta$ and $\alpha\zeta'$. Thus it follows from (4.3) that $\beta(\zeta) = \beta(\zeta')$, i.e.,

$$\beta(\zeta) = \beta\left(\zeta . e^{\frac{2\pi i k}{d^n}}\right),$$

for all $k \geq 1$ and for all $n \geq 0$. Therefore, $\beta(\zeta) \equiv \beta$ in $\mathbb{C}$. Thus it follows form (4.3) that

$$A_1(z',\zeta') - A_1(z,\zeta) = \Delta_H(\zeta,\zeta') + \gamma(\zeta') - \gamma(\zeta), \tag{4.4}$$

where

$$\Delta_H(\zeta,\zeta') = \beta\frac{d}{\delta}\sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left(Q_H(\zeta^{d^l}) - Q_H\left(\left(e^{2k\pi i/d^n}\zeta\right)^{d^l}\right)\right).$$

On the other hand since $A(z,\zeta)$ and $A(z',\zeta')$ are in the same fiber of π_F,

$$\begin{aligned} A_1(z',\zeta') - A_1(z,\zeta) &= \Delta_F(\alpha\zeta,\alpha\zeta') \\ &= \frac{d}{\delta}\sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left(Q_F(\alpha^{d^l}\zeta^{d^l}) - Q_F\left(\left(e^{\frac{2k\pi i}{d^n}}\alpha\zeta\right)^{d^l}\right)\right). \end{aligned} \tag{4.5}$$

Since γ is a holomorphic function on $\mathbb{C}\setminus\bar{\mathbb{D}}$, comparing (4.4) and (4.5), it follows that for a fixed $\zeta \in \mathbb{C}\setminus\bar{\mathbb{D}}$, the modulus of the difference between $\Delta_H(\zeta,\zeta')$ and $\Delta_F(\alpha\zeta,\alpha\zeta')$ is uniformly bounded for all $v' = e^{2k\pi i/d^n}\zeta$ with $n \geq 0$ and $k \geq 1$. Note that for $\zeta' = e^{2k\pi i/d^n}\zeta$, the difference $\Delta_H(\zeta,\zeta') - \Delta_F(\alpha\zeta,\alpha\zeta')$ is a polynomial of degree $(d+1)d^{n-1}$ and it can be written as

$$\begin{aligned} &\beta\sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left(Q_H\left(\zeta^{d^l}\right) - Q_H\left(\left(e^{\frac{2k\pi i}{d^n}}\zeta\right)^{d^l}\right)\right) \\ &\quad - \sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left(Q_F\left(\alpha^{d^l}\zeta^{d^l}\right) - Q_F\left(\left(e^{\frac{2k\pi i}{d^n}}\alpha\zeta\right)^{d^l}\right)\right) \\ &= \sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left[\beta Q_H\left(\zeta^{d^l}\right) - Q_F\left(\alpha^{d^l}\zeta^{d^l}\right)\right] \\ &\quad - \sum_{l=0}^{n-1}\left(\frac{d}{\delta}\right)^l \left[\beta Q_H\left(\left(e^{\frac{2k\pi i}{d^n}}\zeta\right)^{d^l}\right) - Q_F\left(\left(e^{\frac{2k\pi i}{d^n}}\alpha\zeta\right)^{d^l}\right)\right] \\ &= \frac{d^{n-1}}{\delta^{n-1}}\left[\beta\zeta^{d^{n-1}(d+1)} - (\alpha\zeta)^{d^{n-1}(d+1)} - \beta\left(e^{\frac{2\pi i}{d^n}}\zeta\right)^{d^{n-1}(d+1)} + \left(\alpha\zeta e^{\frac{2\pi i}{d^n}}\right)^{d^{n-1}(d+1)}\right] \\ &\quad + R(\zeta) = \left(\frac{d}{\delta}\right)^{n-1}\left(\beta - \alpha^{d^{n-1}(d+1)}\right)\left(1 - e^{\frac{2\pi i}{d}}\right)\zeta^{d^{n-1}(d+1)}\left[1 + \tilde{R}(\zeta)\right], \end{aligned}$$

where $R(\zeta) = O\left(\zeta^{d^n}\right)$ and

$$\tilde{R}(\zeta) = R(\zeta)(d/\delta)^{-n+1}\left(\beta - \alpha^{d^{n-1}(d+1)}\right)^{-1}\left(1 - e^{\frac{2\pi i}{d}}\right)^{-1}\zeta^{-d^{n-1}(d+1)}.$$

We claim that $\alpha^{d^{n-1}(d+1)} \to \beta$, as $n \to \infty$. If not, then there exists a subsequence $\{n_l\}_{l\geq 1}$ such that

$$\left|\alpha^{d^{n_l-1}(d+1)} - \beta\right| > c > 0, \tag{4.6}$$

for all $l \geq 1$. Thus, if $|d/\delta| \geq 1$, then (note that R is a polynomial of degree at most d^n)

$$|\tilde{R}(\zeta)| \leq \frac{4nK_1}{|\zeta|^{d^{n-1}}},$$

for some $K_1 > 1$. On the other hand, if $|d/\delta| < 1$, then

$$|\tilde{R}(\zeta)| \leq \frac{4nK_2}{|\zeta|^{d^{n-1}}}\left(\frac{\delta}{d}\right)^{n-1},$$

for some $K_2 > 1$. Therefore, since the modulus of $\Delta_H\left(\zeta, e^{\frac{2\pi i}{d^n}}\zeta\right) - \Delta_F\left(\alpha\zeta, \alpha e^{\frac{2\pi i}{d^n}}\zeta\right)$ is uniformly bounded, for all $n \geq 1$, we get a contradiction if (4.6) holds. Thus

$$\alpha^{d^n(d+1)} \to \beta, \tag{4.7}$$

as $n \to \infty$. Also,

$$\alpha^{d^{n+1}(d+1)} \to \beta \tag{4.8}$$

as $n \to \infty$ and thus dividing (4.8) by (4.7) and taking the limit, we get

$$\alpha^{(d+1)(d-1)d^n} \to 1, \tag{4.9}$$

as $n \to \infty$. Therefore, we get

$$\beta^{d-1} = 1.$$

On the other hand, it follows from (4.7) that $\left(\alpha^{d+1}\right)^{d^n} \to \beta$ and also note that β is a repelling fixed point for the map $z \mapsto z^d$. Therefore, (4.7) holds if and only if

$$\alpha^{d+1} = \beta.$$

Thus $\left(\alpha^{d^n(d+1)}-\beta\right)=0$, for all $n\geq 1$, which in turn gives $\gamma(\zeta)=\gamma\left(e^{\frac{2k\pi i}{d^n}}\zeta\right)$, for all $k\geq 1$ and $n\geq 0$. Thus $\gamma\equiv\gamma_0$, for some $\gamma_0\in\mathbb{C}$.
Relation between Q_H and Q_F: Note that it follows from Lemma 2.4 that next to the highest degree coefficients of Q_H and Q_F vanish. Let

$$Q_H(\zeta)=\zeta^{d+1}+A^H_{d-1}\zeta^{d-1}+A^H_{d-2}\zeta^{d-2}+\cdots+A^H_1\zeta+A^H_0 \tag{4.10}$$

and

$$Q_F(\zeta)=\zeta^{d+1}+A^F_{d-1}\zeta^{d-1}+A^F_{d-2}\zeta^{d-2}+\cdots+A^F_1\zeta+A^F_0. \tag{4.11}$$

Since γ is identically constant in the complex plane, taking $k,n=1$, it follows from (4.4) and (4.5) that

$$\begin{aligned}\beta\frac{d}{\delta}\left[Q_H(\zeta)-Q_H\left(e^{\frac{2\pi i}{d}}\zeta\right)\right]&=\Delta_H(\zeta,\zeta')=\Delta_F(\alpha\zeta,\alpha\zeta')\\&=\frac{d}{\delta}\left[Q_F(\alpha\zeta)-Q_F\left(e^{\frac{2\pi i}{d}}\alpha\zeta\right)\right].\end{aligned} \tag{4.12}$$

Comparing coefficients of both sides of (4.12), we get

$$\beta A^H_{d-k}=\alpha^{d-k}A^F_{d-k},$$

for $1\leq k\leq(d-1)$ and since $\beta=\alpha^{d+1}$, equivalently we get

$$A^H_{d-k}=\alpha^{-(k+1)}A^F_{d-k}, \tag{4.13}$$

for $1\leq k\leq(d-1)$. Note that the relation between the constant terms of Q_H and Q_F, i.e., the relation between A^H_0 and A^F_0 cannot be extracted from (4.12). However, as we are going to see in the next section that the explicit relation between A^H_{d-i} and A^F_{d-i}, for $1\leq i\leq(d-1)$, obtained in (4.13) is sufficient to track down the relation between p_H and p_F.

5 Relation between p_H and p_F

It follows from Lemma 2.4 that there exist

$$y_H(\zeta)\equiv y_H(0,\zeta)=\zeta+\frac{D^H_1}{\zeta}+\frac{D^H_2}{\zeta^2}+\cdots+\frac{D^H_{d-1}}{\zeta^{d-1}}+O\left(\frac{1}{\zeta^d}\right) \tag{5.1}$$

and

$$y_F(\zeta)\equiv y_F(0,\zeta)=\zeta+\frac{D^F_1}{\zeta}+\frac{D^F_2}{\zeta^2}+\cdots+\frac{D^F_{d-1}}{\zeta^{d-1}}+O\left(\frac{1}{\zeta^d}\right), \tag{5.2}$$

with $D_i^H, D_i^F \in \mathbb{C}$ for $i \geq 1$, such that $Q_H(\zeta)$ and $Q_F(\zeta)$ are polynomial parts of $\zeta^d y_H(0, \zeta)$ and $\zeta^d y_F(0, \zeta)$, respectively. Therefore, the precise relation obtained in (4.13) determines a similar relation between the first few corresponding coefficients of the power series expansion of $y_H(\zeta)$ and $y_F(\zeta)$, namely

$$D_k^H = \alpha^{-(k+1)} D_k^F, \tag{5.3}$$

for $1 \leq k \leq (d-1)$.

Since we assume that the Jacobian determinants of H and F are the same, that is, $\delta_H = \delta_F = \delta$, understanding the relation between the polynomials p_H and p_F is sufficient to capture the relation between H and F. We show in this section that the relation between the coefficients of p_H and p_F can be extracted via the Böttcher coordinates of H and F, namely via the functions ζ_H and ζ_F, respectively. Let

$$\zeta_H(y) \equiv \zeta_H(0, y) = y + \frac{L_1^H}{y} + \frac{L_2^H}{y^2} + \cdots + \frac{L_{d-1}^H}{y^{d-1}} + O\left(\frac{1}{y^d}\right) \tag{5.4}$$

and

$$\zeta_F(y) \equiv \zeta_F(0, y) = y + \frac{L_1^F}{y} + \frac{L_2^F}{y^2} + \cdots + \frac{L_{d-1}^F}{y^{d-1}} + O\left(\frac{1}{y^d}\right). \tag{5.5}$$

Recall from Section 2 that $\zeta_H \circ y_H(0, \zeta) = \zeta$ and $\zeta_F \circ y_F(0, \zeta) = \zeta$, for all $\zeta \in \mathbb{C}$ with $|\zeta| > R$, where $R > 0$ is sufficiently large. Thus implementing (5.1), (5.2), (5.4) and (5.5) together, we get

$$\begin{aligned}
&\left[\frac{D_1^H}{\zeta} + \frac{D_2^H}{\zeta^2} + \cdots + \frac{D_{d-1}^H}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right] \\
&\quad + \frac{L_1^H}{\zeta}\left[1 + \frac{D_1^H}{\zeta^2} + \frac{D_2^H}{\zeta^3} + \cdots + \frac{D_{d-1}^H}{\zeta^{d-1}}\right. \\
&\quad \left. + O\left(\frac{1}{\zeta^d}\right)\right]^{-1} + \frac{L_2^H}{\zeta^2}\left[1 + \frac{D_1^H}{\zeta^2} + \frac{D_2^H}{\zeta^3}\right. \\
&\quad \left. + \cdots + \frac{D_{d-1}^H}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right]^{-2} + \cdots = 0
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&\left[\frac{D_1^F}{\zeta} + \frac{D_2^F}{\zeta^2} + \cdots + \frac{D_d^F}{\zeta^d} + O\left(\frac{1}{\zeta^d}\right)\right] \\
&\quad + \frac{L_1^F}{\zeta}\left[1 + \frac{D_1^F}{\zeta^2} + \frac{D_2^F}{\zeta^3} + \cdots + \frac{D_{d-1}^F}{\zeta^{d-1}}\right.
\end{aligned}$$

$$+O\left(\frac{1}{\zeta^d}\right)\Bigg]^{-1}+\frac{L_2^F}{\zeta^2}\Bigg[1+\frac{D_1^F}{\zeta^2}+\frac{D_2^F}{\zeta^3}$$
$$+\cdots+\frac{D_{d-1}^F}{\zeta^{d-1}}+O\left(\frac{1}{\zeta^d}\right)\Bigg]^{-2}+\cdots=0. \tag{5.7}$$

Note that expanding (5.6) and (5.7), one can express L_k^H and L_k^F, for any $k \geq 1$, in terms of $D_1^H, D_2^H, \ldots$ and $D_1^F, D_2^F, \ldots$, respectively. In fact, using the relation between D_k^H and D_k^F, for $1 \leq k \leq (d-1)$, obtained in (5.3), one can establish an explicit relation between L_k^H and L_k^F for $1 \leq k \leq (d-1)$.
Claim $\mathcal{L}$: For $1 \leq k \leq d-1$, $L_k^H = \alpha^{-(k+1)} L_k^F$.
Note that implementing claim $\mathcal{L}$ to (5.4) and (5.5), we obtain

$$\alpha\zeta_H(y) - \zeta_F(\alpha y) = O\left(1/y^d\right). \tag{5.8}$$

As we shall see shortly that the above relation between ζ_H and ζ_F plays the key role in establishing the relation between p_H and p_F. For now assuming claim $\mathcal{L}$ is true and thus assuming (5.8) holds, let us first determine the relation between p_H and p_F. We see a proof of the claim $\mathcal{L}$ in the end of the present section.
Relation between p_H and p_F: Recall from Section 2 that for $|y| > R$, with R large enough, one can write

$$\zeta_H(y) \equiv \zeta_H(0, y) = y.\left(\frac{p_H(y)}{y^d}\right)^{\frac{1}{d}}\left(\frac{p_H(y_{1,H})}{y_{1,H}^d}\right)^{\frac{1}{d^2}}\cdots \tag{5.9}$$

and

$$\zeta_F(y) \equiv \zeta_F(0, y) = y.\left(\frac{p_F(y)}{y^d}\right)^{\frac{1}{d}}\left(\frac{p_F(y_{1,F})}{y_{1,F}^d}\right)^{\frac{1}{d^2}}\cdots, \tag{5.10}$$

where $y_{1,H} = (H(x, y))_2 = p_H(y) - \delta x$ and $y_{1,F} = (F(x, y))_2 = p_F(y) - \delta x$. Thus $\zeta_H(y) - O(1/y^d)$ and $\zeta_F(y) - O(1/y^d)$ are determined by

$$y.\left(\frac{p_H(y)}{y^d}\right)^{\frac{1}{d}} \text{ and } y.\left(\frac{p_F(y)}{y^d}\right)^{\frac{1}{d}},$$

respectively. The power series expansion of the holomorphic function $(1+z)^{1/d}$ (principal branch) is

$$1 + (1/d)z + ((1/d-1)/2d)\,z^2 + \cdots,$$

for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Now note that

$$
\begin{aligned}
y.\left(\frac{p_H(y)}{y^d}\right)^{\frac{1}{d}} &= y.\left(\frac{y^d + a_{d-2}^H y^{d-2} + \cdots + a_1^H y + a_0^H}{y^d}\right)^{\frac{1}{d}} \\
&= y\left[1 + \frac{1}{d}\left(\frac{a_{d-2}^H}{y^2} + \cdots + \frac{a_0^H}{y^d}\right)\right. \\
&\quad \left. + \frac{1}{2d}\left(\frac{1}{d} - 1\right)\left(\frac{a_{d-2}^H}{y^2} + \cdots + \frac{a_0^H}{y^d}\right)^2 + \cdots\right]
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
\alpha y.\left(\frac{p_F(\alpha y)}{(\alpha y)^d}\right)^{\frac{1}{d}} &= \alpha y.\left(\frac{(\alpha y)^d + a_{d-2}^F (\alpha y)^{d-2} + \cdots + a_1^F (\alpha y) + a_0^F}{(\alpha y)^d}\right)^{\frac{1}{d}} \\
&= \alpha y\left[1 + \frac{1}{d}\left(\frac{a_{d-2}^F}{\alpha^2 y^2} + \cdots + \frac{a_0^F}{\alpha^d y^d}\right)\right. \\
&\quad \left. + \frac{1}{2d}\left(\frac{1}{d} - 1\right)\left(\frac{a_{d-2}^F}{\alpha^2 y^2} + \cdots + \frac{a_0^F}{\alpha^d y^d}\right)^2 + \cdots\right],
\end{aligned}
\tag{5.12}
$$

for all $y \in \mathbb{C}$ with $|y|$ large enough. Since (5.8) holds, comparing (5.11) and (5.12), we get

$$
a_{d-2}^F = \alpha^2 a_{d-2}^H.
$$

Let us assume that

$$
a_{d-k}^F = \alpha^k a_{d-k}^H,
$$

up to some k, where $2 \le k \le (d-1)$. Now it follows from (5.11) that the coefficient of $1/y^k$ in the expansion of $\alpha \zeta_H(y)$ is

$$
c_{k+1}^H = \alpha\left[\left(a_{d-(k+1)}^H / d\right) + G\left(a_{d-2}^H, \cdots, a_{d-k}^H\right)\right],
\tag{5.13}
$$

where G is defined as follows. The function G is the polynomial in $(k-1)$ complex variables determined by the coefficient of $1/y^{k+1}$ of the power series expansion of

$$
\begin{aligned}
&\left(\frac{p_H(y)}{y^d}\right)^{\frac{1}{d}} - \frac{1}{d}\left(\frac{a_{d-2}^H}{y^2} + \cdots + \frac{a_0^H}{y^d}\right) - 1 \\
&= \frac{1}{2d}\left(\frac{1}{d} - 1\right)\left(\frac{a_{d-2}^H}{y^2} + \cdots + \frac{a_0^H}{y^d}\right)^2 + \cdots
\end{aligned}
\tag{5.14}
$$

(see 5.11). In other words, if the coefficient of $1/y^{k+1}$ in (5.14) is $\sum_{i=1}^{n} g_i \left(\Pi_{j=2}^{k}(a_{d-j}^{H})^{i(j)}\right)$ with $g_i \in \mathbb{C}$, then

$$G(x_1, x_2, \ldots, x_{k-1}) = \sum_{i=1}^{n} g_i \left(\Pi_{j=2}^{k} x_j{}^{i(j)}\right).$$

Similarly, it follows from (5.12) that the coefficient of $1/y^k$ in the expansion of $\zeta_F(\alpha y)$ is

$$c_{k+1}^{F} = \frac{a_{d-(k+1)}^{F}}{\alpha^k d} + \alpha G\left(\frac{a_{d-2}^{F}}{\alpha^2}, \cdots, \frac{a_{d-k}^{F}}{\alpha^k}\right) = \frac{a_{d-(k+1)}^{F}}{\alpha^k d} + \alpha G(a_{d-2}^{H}, \cdots, a_{d-k}^{H}). \tag{5.15}$$

Since $c_{k+1}^{H} = c_{k+1}^{F}$, comparing (5.13) and (5.15), we have

$$a_{d-(k+1)}^{F} = \alpha^{k+1} a_{d-(k+1)}^{H}.$$

Therefore, the following relation between the corresponding coefficients of p_H and p_F holds:

$$a_{d-k}^{F} = \alpha^k a_{d-k}^{H}, \tag{5.16}$$

for $2 \leq k \leq d$.

Now since $\beta = \alpha^{d+1}$ and (5.16) holds, we have

$$\begin{aligned}
p_F(y) &= y^d + a_{d-2}^{F} y^{d-2} + \cdots + a_1^{F} y + a_0^{F} \\
&= y^d + \alpha^2 a_{d-2}^{H} y^{d-2} + \alpha^3 a_{d-3}^{H} y^{d-3} + \cdots + \alpha^{d-1} a_1^{H} y + \alpha^d a_0^{H} \\
&= y^d + \beta\alpha^{-(d-1)} a_{d-2}^{H} y^{d-2} + \beta\alpha^{-(d-2)} a_{d-3}^{H} y^{d-3} + \cdots + \beta\alpha^{-2} a_1^{H} y + \beta\alpha^{-1} a_0^{H} \\
&= \beta\alpha^{-1}\left((\alpha^{-1}y)^d + a_{d-2}^{H}(\alpha^{-1}y)^{d-2} + a_{d-3}^{H}(\alpha^{-1}y)^{d-3} + \cdots + a_1^{H}(\alpha^{-1}y) + a_0^{H}\right) \\
&= (\beta\alpha^{-1}) p_H(\alpha^{-1}y).
\end{aligned}$$

In other words, we obtain

$$\alpha p_F(\alpha y) = \beta p_H(y), \tag{5.17}$$

for all $y \in \mathbb{C}$, with $\alpha^{d+1} = \beta$ and $\beta^{d-1} = 1$.

Before giving a formal proof of the claim $\mathcal{L}$, let us establish the relation between L_k^H and L_k^F with bare hands when the common degree d of the Hénon maps H and F is small. To start with, first note that the coefficients of $1/\zeta^k$ vanish in both (5.6) and (5.7), for all $k \geq 1$.

• Let $d = 2$. Thus by (5.3), we have $D_1^H = \alpha^{-2} D_1^F$. The coefficients of $1/\zeta$ in (5.6) and (5.7) are $D_1^H + L_1^H$ and $D_1^F + L_1^F$, respectively and both vanish. Thus

$$L_1^H = -D_1^H = -\alpha^{-2} D_1^F = \alpha^{-2} L_1^F.$$

• Let $d = 3$. By (5.3), we have $D_i^H = \alpha^{-(i+1)} D_i^F$, for $i = 1, 2$. Since the coefficients of $1/\zeta$ vanish in (5.6) and (5.7), as before we can show that $L_1^H = \alpha^{-2} L_1^F$. Using the fact that coefficients of $1/\zeta^2$, which are

$$D_2^H + L_1^H.0 + L_2^H \text{ and } D_2^F + L_1^F.0 + L_2^F,$$

vanish, we have

$$L_2^H = -D_2^H = -\alpha^{-3} D_2^F = \alpha^{-3} L_2^F.$$

• Let $d = 4$. By (5.3), we have $D_i^H = \alpha^{-(i+1)} D_i^F$, for $1 \le i \le 3$. Now since the coefficients $1/\zeta^i$ vanish in (5.6) and (5.7), as before we can show that $L_i^H = \alpha^{-(i+1)} L_i^F$, for $1 \le i \le 2$. Now the coefficients of $1/\zeta^3$ in (5.6) and (5.7) are

$$D_3^H + L_1^H.\left(C_1 D_1^H\right) + L_2^H.0 + L_3^H \text{ and } D_3^F + L_1^F.\left(C_1 D_1^F\right) + L_2^F.0 + L_3^F,$$

respectively for some $C_1 \in \mathbb{C}$ and they vanish. Therefore,

$$L_3^H = -D_3^H - L_1^H.\left(C_1 D_1^H\right),$$

which implies

$$L_3^H = -\alpha^{-4} D_3^F - \alpha^{-4} L_1^F.\left(C_1 D_1^F\right) = \alpha^{-4} L_3^F.$$

• Let $d = 5$. Thus we have $D_i^H = \alpha^{-(i+1)} D_i^F$, for $1 \le i \le 4$. Using the same arguments as before we can show $L_i^H = \alpha^{-(i+1)} L_i^F$, for $1 \le i \le 3$. Now the coefficients of $1/\zeta^4$ in (5.6) and (5.7) are

$$D_4^H + L_1^H.\left(C_2 D_2^H\right) + L_2^H.\left(C_3 D_1^H\right) + L_3^H.0 + L_4^H$$

and

$$D_4^F + L_1^F.\left(C_2 D_2^F\right) + L_2^F.\left(C_3 D_1^F\right) + L_3^F.0 + L_4^F,$$

for some $C_2, C_3 \in \mathbb{C}$ and since they vanish, a simple calculation as above gives that

$$L_4^H = \alpha^{-5} L_4^F.$$

• Let $d = 6$. Thus we have $D_i^H = \alpha^{-(i+1)} D_i^F$, for $1 \le i \le 5$. As before we can show $L_i^H = \alpha^{-(i+1)} L_i^F$, for $1 \le i \le 4$. The coefficients of $1/\zeta^5$ in (5.6) and (5.7) are M_5^X. Here for $X = H, F$,

$$M_5^X = D_5^X + L_1^X . \left(C_4 D_3^X + C_5 (D_1^X)^2\right) + L_2^X . \left(C_6 D_2^X\right) + L_3^X .(C_7 D_1^X) + L_4^X .0 + L_5^X,$$

where C_4, C_5 and so on are from $\mathbb{C}$. Since M_5^H and M_5^F vanish, calculations as above give us

$$L_5^H = \alpha^{-6} L_5^F.$$

• Let $d = 7$. Thus $D_i^H = \alpha^{-(i+1)} D_i^F$, for $1 \le i \le 6$. Hence we can show $L_i^H = \alpha^{-(i+1)} L_i^F$, for $1 \le i \le 5$ as before. The coefficients of $1/\zeta^6$ in (5.6) and (5.7) are M_6^X. Here for $X = H, F$,

$$\begin{aligned} M_6^X = D_6^X &+ L_1^X . \left(C_8 D_4^X + C_9 D_1^X D_2^X\right) \\ &+ L_2^X . \left(C_{10} D_3^X + C_{11} (D_1^X)^2\right) + L_3^X .(C_{12} D_2^X) \\ &+ L_4^X .(C_{13} D_1^X) + L_5^X .0 + L_6^X, \end{aligned}$$

where C_8, C_9 and so on are from $\mathbb{C}$. Since M_6^H and M_6^F vanish, we get

$$L_6^H = \alpha^{-7} L_6^F.$$

• Let $d = 8$. So $D_i^H = \alpha^{-(i+1)} D_i^F$, for $1 \le i \le 7$. Hence we can show $L_i^H = \alpha^{-(i+1)} L_i^F$, for $1 \le i \le 6$ just as before. Now the the coefficients of $1/\zeta^7$ in (5.6) and (5.7) are M_7^X. Here for $X = H, F$,

$$\begin{aligned} M_7^X = D_7^X &+ L_1^X . \left(C_{14} D_5^X + C_{15} D_1^X D_3^X + C_{16} (D_2^X)^2 + C_{17} (D_1^X)^3\right) \\ &+ L_2^X . \left(C_{18} D_4^X + C_{19} D_1^X D_2^X\right) \\ &+ L_3^X . \left(C_{20} D_3^X + C_{21} (D_1^X)^2\right) \\ &+ L_4^X .(C_{22} D_2^X) + L_5^X .(C_{23} D_1^X) + L_6^X .0 + L_7^X, \end{aligned}$$

where C_{14}, C_{15} and so on are from $\mathbb{C}$. A simple calculation as above gives us

$$L_7^H = \alpha^{-8} L_7^F.$$

As promised earlier, now we see a proof of the claim $\mathcal{L}$.

Proof of Claim $\mathcal{L}$

First we make a few observations, which follow immediately by chasing the expression of the coefficients of $1/\zeta^s$ in (5.6) and (5.7), for $s \geq 1$ and then using the fact that all of them vanish.
(O1) For any $s \geq 1$, the coefficients of $1/\zeta^s$ in (5.6) and (5.7) vanish. Further, they are of the form $D_s^H + R_s^H + L_s^H$ and $D_s^F + R_s^F + L_s^F$, respectively, where R_s^H and R_s^F are linear combinations of products of powers of D_i^H's, L_j^H's and D_i^F's, L_j^F's, respectively, with $1 \leq i, j < s$.
(O2) For $s \geq 1$, one can write

$$R_s^H = \sum_{i=1}^{s-1} L_i^H R_{i,s}^H \text{ and } R_s^F = \sum_{i=1}^{s-1} L_i^F R_{i,s}^F.$$

Let $2 \leq i \leq s-1$. It follows from (5.6) that $R_{i,s}^H$ is the coefficient of $1/\zeta^{s-i}$ of the power series expansion of

$$\left(1 + \frac{D_1^H}{\zeta^2} + \frac{D_2^H}{\zeta^3} + \cdots + \frac{D_{d-1}^H}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right)^{-i},$$

and similarly, $R_{i-1,s-1}^H$ is the coefficient of $1/\zeta^{s-i}$ of the power series expansion of

$$\left(1 + \frac{D_1^H}{\zeta^2} + \frac{D_2^H}{\zeta^3} + \cdots + \frac{D_{d-1}^H}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right)^{-(i-1)}.$$

Therefore, for $2 \leq i \leq s-1$, if

$$R_{i,s}^H = C(i,s,1)D^H(i,s,1) + C(i,s,2)D^H(i,s,2) + \cdots + C(i,s,i_s)D^H(i,s,i_s) \tag{5.18}$$

and

$$R_{i,s}^F = C(i,s,1)D^F(i,s,1) + C(i,s,2)D^F(i,s,2) + \cdots + C(i,s,i_s)D^F(i,s,i_s), \tag{5.19}$$

where for $1 \leq j \leq i_s$, $D^H(i,s,j)$ and $D^F(i,s,j)$ are products of powers of D_l^H's and D_l^F's $(1 \leq l < s)$, respectively, with $C(i,s,j) \in \mathbb{C}$, then

$$\begin{aligned} R_{i-1,s-1}^H = \tilde{C}(i,s,1)D^H(i,s,1) + \tilde{C}(i,s,2)D^H(i,s,2) \\ + \cdots + \tilde{C}(i,s,i_s)D^H(i,s,i_s) \end{aligned} \tag{5.20}$$

and

$$R^F_{i-1,s-1} = \tilde{C}(i,s,1)D^F(i,s,1) + \tilde{C}(i,s,2)D^F(i,s,2) + \cdots + \tilde{C}(i,s,i_s)D^F(i,s,i_s), \tag{5.21}$$

with $\tilde{C}(i,s,j) \in \mathbb{C}$, for $1 \le j \le i_s$.
(O3) Let $s \ge 1$ and $i = 1$. Then

$$R^H_{1,s} = C(1,s,1)D^H(i,s,1) + C(1,s,2)D^H(i,s,2) + \cdots + C(1,s,1_s)D^H(1,s,1_s), \tag{5.22}$$

where for $1 \le j \le 1_s$,

$$D^H(1,s,j) = D^H_{s_{j_1}} D^H_{s_{j_2}} \cdots D^H_{s_{j_{m(s_j)}}},$$

with $s_{j_1}, s_{j_2}, \ldots, s_{j_{m(s_j)}} \in \{1,2,\ldots,s-1\}$ ($s_{j_1}, s_{j_2}, \ldots, s_{j_{m(s_j)}}$ are not possibly all distinct), i.e., $D^H_{s_{j_1}}, D^H_{s_{j_2}}, \ldots, D^H_{s_{j_{m(s_j)}}} \in \left\{D^H_1, \ldots, D^H_{s-1}\right\}$. Now $R^H_{1,s}$ is the coefficient of $1/\zeta^{s-1}$ of the power series expansion of

$$\left(1 + \frac{D^H_1}{\zeta^2} + \frac{D^H_2}{\zeta^3} + \cdots + \frac{D^H_{d-1}}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right)^{-1}.$$

Similarly, $R^H_{1,s+1}$ is the coefficient of $1/\zeta^s$ of the power series expansion of

$$\left(1 + \frac{D^H_1}{\zeta^2} + \frac{D^H_2}{\zeta^3} + \cdots + \frac{D^H_{d-1}}{\zeta^{d-1}} + O\left(\frac{1}{\zeta^d}\right)\right)^{-1}.$$

Therefore, if

$$R^H_{1,s} = \sum_{j=1}^{1_s} C(1,s,j) D^H_{s_{j_1}} D^H_{s_{j_2}} \cdots D^H_{s_{j_{m(s_j)}}}, \tag{5.23}$$

then

$$R^H_{1,s+1} = \sum_{j=1}^{1_s} \Big[C^{(1)}(1,s,j) D^H_{s_{j_1}+1} D_{s_{j_2}} \cdots D^H_{s_{j_{m(s_j)}}} + C^{(2)}(1,s,j) D^H_{s_{j_1}} D^H_{s_{j_2}+1} \cdots D^H_{s_{j_{m(s_j)}}} + \cdots + C^{(m(s_j))}(1,s,j) D^H_{s_{j_1}}.D_{s_{j_2}} \cdots D^H_{s_{j_{m(s_j)}}+1} \Big] + C_s (D^H_1)^{\frac{s}{2}}, \tag{5.24}$$

with $C_s, C^{(k)}(1,s,j) \in \mathbb{C}$, for $1 \le j \le 1_s$ and $1 \le k \le m(s_j)$. Further, $C_s = 0$, if s is not divisible by 2 and $C_s \ne 0$, if s is divisible by 2. The same relation holds between $R^F_{1,s}$ and $R^F_{1,s+1}$.

Let $d \geq 4$. Note that we have already established claim $\mathcal{L}$ for $d = 2, 3$ just before starting the present proof. Let $1 \leq k \leq d-2$. Then we prove that if for $1 \leq s \leq k$,

(•) $L_s^H = \alpha^{-(s+1)} L_s^F$, and

(••) for each fixed pair (i, s) satisfying $2 \leq i \leq s-1$, $D^H(i, s, j) = \alpha^{-(s-i)} D^F(i, s, j)$, for $1 \leq j \leq i_s$ and $R_{1,s}^H = \alpha^{-(s-1)} R_{1,s}^F$ (which in turn gives $R_{i,s}^H = \alpha^{-(s-i)} R_{i,s}^F$, for $1 \leq i \leq s-1$),

then $L_{k+1}^H = \alpha^{-(k+2)} L_{k+1}^F$.

Now observe that performing the same calculations, which we did immediately before starting this proof, (•) and (••) hold for a first few k's with $1 \leq k \leq d-2$. The coefficients of $1/\zeta^{k+1}$ in (5.6) and (5.7) are

$$D_{k+1}^H + L_1^H R_{1,k+1}^H + L_2^H R_{2,k+1}^H + \cdots + L_k^H R_{k,k+1}^H + L_{k+1}^H, \tag{5.25}$$

and

$$D_{k+1}^F + L_1^F R_{1,k+1}^F + L_2^F R_{2,k+1}^F + \cdots + L_k^F R_{k,k+1}^F + L_{k+1}^F, \tag{5.26}$$

respectively (note that both vanish). Let $2 \leq i \leq k$. Then by (5.18), it follows that

$$\begin{aligned} R_{i,k+1}^H = C(i, k+1, 1) D^H(i, k+1, 1) + \cdots \\ + C(i, k+1, i_{k+1}) D^H(i, k+1, i_{k+1}), \end{aligned} \tag{5.27}$$

and thus by (5.20)

$$\begin{aligned} R_{i-1,k}^H = \tilde{C}(i, k+1, 1) D^H(i, k+1, 1) \\ + \cdots + \tilde{C}(i, k+1, i_{k+1}) D^H(i, k+1, i_{k+1}). \end{aligned} \tag{5.28}$$

Therefore, clearly for $1 \leq j \leq i_{k+1}$, $D^H(i, k+1, j) = D^H(i-1, k, r_j)$, for some $1 \leq r_j \leq (i-1)_k$. Similar conclusion holds for $D^F(i, k+1, j)$, i.e., for $1 \leq j \leq i_{k+1}$, $D^F(i, k+1, j) = D^F(i-1, k, r_j)$. Therefore, using (••) for $s = k$, we have

$$\begin{aligned} D^H(i, k+1, j) = D^H(i-1, k, r_j) = \alpha^{-(k-i+1)} D^F(i-1, k, r_j) \\ = \alpha^{-(k-i+1)} D^F(i, k+1, j), \end{aligned}$$

for $2 \leq i \leq k$ and for $1 \leq j \leq i_{k+1}$.

Therefore,

$$R_{i,k+1}^H = \alpha^{-(k+1-i)} R_{i,k+1}^F, \tag{5.29}$$

for $2 \leq i \leq k$. By (••), we have $R_{1,k}^H = \alpha^{-(k-1)} R_{1,k}^F$. Thus comparing (5.23) and (5.24) along with using the fact that $D_k^H = \alpha^{-(k+1)} D_k^F$, for $1 \leq k \leq d-1$ (see (5.3)), we obtain

$$R_{1,k+1}^H = \alpha^{-k} R_{1,k+1}^F. \tag{5.30}$$

Now as mentioned before (5.25) and (5.26) both are equal to zero. Therefore using (5.29), (5.30) and (5.3), we get

$$\begin{aligned} L_{k+1}^H &= -\left(D_{k+1}^H + L_1^H R_{1,k+1}^H + L_2^H R_{2,k+1}^H + \cdots + L_k^H R_{k,k+1}^H\right) \\ &= -\alpha^{-(k+2)}\left(D_{k+1}^F + L_1^F R_{1,k+1}^F + L_2^F R_{2,k+1}^F + \cdots + L_k^F R_{k,k+1}^F\right) \\ &= \alpha^{-(k+2)} L_{k+1}^F. \end{aligned}$$

Thus the claim follows.

6 Hénon maps with different Jacobians

Let $H(x, y) = (y, p_H(y) - \delta_H x)$ and $F(x, y) = (y, p_F(y) - \delta_F x)$ be such that $\delta_H \neq \delta_F$. As before, we assume that $\deg H = \deg F = d \geq 2$. It follows from [4, Thm. 2.7] that if I_H^+ and I_F^+ are biholomorphic, then

$$\delta_H^{d-1} = \delta_F^{d-1} = \delta. \tag{6.1}$$

Now instead of working with H and F, we work with $\tilde{H} = H^{d-1}$ and $\tilde{F} = F^{d-1}$. Note that since (6.1) holds the Jacobians of $\tilde{H}$ and $\tilde{F}$ are the same. Further, $I_{H^{d-1}}^+ = I_{F^{d-1}}^+$.

If $\tilde{H} : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ is the lift of $H : I_H^+ \to I_H^+$, then clearly $\tilde{H}^{d-1} : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ is the lift of $H^{d-1} : I_H^+ \to I_H^+$. It follows from (2.6) that

$$\tilde{H}(z, \zeta) = \left((\delta_H/d)z + Q_H(\zeta), \zeta^d\right).$$

A simple calculation gives

$$\tilde{H}^{d-1}(z, \zeta) = \left((\delta_H/d)^{d-1}z + \tilde{Q}_H(\zeta), \zeta^{d^{d-1}}\right),$$

where

$$\tilde{Q}_H(\zeta) = (\delta_H/d)^{d-2}Q_H(\zeta) + (\delta_H/d)^{d-3}Q_H(\zeta^d) + \cdots + Q_H\left(\zeta^{d^{d-2}}\right).$$

Note that $\deg \tilde{H} = \deg \tilde{F} = \tilde{d} = d^{d-1}$ and $\det J_H = \det J_H = \delta$, where J_H and J_F are the Jacobian matrices of H and F, respectively. Also it is easy to see that $\mathbb{Z}[1/d^n] = \mathbb{Z}[1/d]$ for all $n \geq 1$, and in particular, $\mathbb{Z}[1/\tilde{d}] = \mathbb{Z}[1/d^{d-1}] = \mathbb{Z}[1/d]$. As in (4.1), one can show that for each element $\left[k/\tilde{d}^n\right] \in \mathbb{Z}[1/\tilde{d}]/\mathbb{Z}$, there exists a unique deck transformation $\gamma_{k/\tilde{d}^n}$ of $\mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ such that

$$\gamma_{k/\tilde{d}^n}\begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} z + \frac{\tilde{d}}{\delta}\sum_{l=0}^{n-1}\left(\frac{\tilde{d}}{\delta}\right)^l \left(\tilde{Q}_H(\zeta^{\tilde{d}^l}) - \tilde{Q}_H\left(\left(e^{\frac{2k\pi i}{\tilde{d}^n}}\zeta)\right)^{\tilde{d}^l}\right)\right) \\ e^{\frac{2k\pi i}{\tilde{d}^n}}\zeta \end{bmatrix}, \quad (6.2)$$

for each $n \geq 0, k \geq 1$ and for all $(z, \zeta) \in \mathbb{C}\times(\mathbb{C}\backslash\bar{\mathbb{D}})$. Thus, if $\tilde{p} = (z, \zeta) \in \mathbb{C}\times\left(\mathbb{C}\backslash\bar{\mathbb{D}}\right)$ is a point in the fiber of $p \in I_H^+ = I_{\tilde{H}}^+$, then the other points in the fiber are precisely $\gamma_{k/\tilde{d}^n}(z, \zeta)$ with $n \geq 0$ and $k \geq 1$.

Note that the main hindrance to run a similar set of calculations as in Section 3 for any arbitrary pair of Hénon maps H and F of the same degree is that the Jacobians of H and F are possibly different. But if we work with H^{d-1} and F^{d-1}, there is no such issues. Since (6.2) holds, a moment's thought assures that exactly similar calculations as in Section 3 run smoothly for the maps H^{d-1} and F^{d-1}. Therefore, if $a : I_{\tilde{H}}^+ \to I_{\tilde{H}}^+$ is a biholomorphism and if $A : \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}}) \to \mathbb{C} \times (\mathbb{C}\backslash\bar{\mathbb{D}})$ is a lift, i.e, A is of the form $(z, \zeta) \mapsto (\beta(z)\zeta + \gamma(\zeta), \alpha\zeta)$, then as in Section 3, we can show

- $\beta(\zeta) \equiv \beta$ in $\mathbb{C}$;
- $\left(\alpha^{d^{d-2}}\right)^{(d+1)d^{n-1}} \to \beta$ as $n \to \infty$ and consequently, $\alpha^{d^{d-2}(d+1)} = \beta$ with $\beta^{d-1} = 1$.

Thus as in (4.12) we obtain

$$\beta\frac{\tilde{d}}{\delta}\left[\tilde{Q}_H(\zeta) - \tilde{Q}_H\left(e^{\frac{2\pi i}{\tilde{d}}}\zeta\right)\right] = \frac{\tilde{d}}{\delta}\left[\tilde{Q}_F(\alpha\zeta) - \tilde{Q}_F\left(e^{\frac{2\pi i}{\tilde{d}}}\alpha\zeta\right)\right], \quad (6.3)$$

for all $\zeta \in \mathbb{C}\backslash\bar{\mathbb{D}}$, with $\alpha^{d^{d-2}(d+1)} = \beta$ and $\beta^{d-1} = 1$. Expanding (6.3), we get

$$\begin{aligned} &\beta\frac{\delta_H^{d-2}}{d^{d-2}}\left[Q_H(\zeta) - Q_H\left(e^{\frac{2\pi i}{d}}\zeta\right)\right] + \beta\frac{\delta_H^{d-3}}{d^{d-3}}\left[Q_H(\zeta^d) - Q_H\left(\left(e^{\frac{2\pi i}{d}}\zeta\right)^d\right)\right] + \\ &\qquad \cdots + \beta\left[Q_H\left(\zeta^{d^{d-2}}\right) - Q_H\left(\left(e^{\frac{2\pi i}{d}}\zeta\right)^{d^{d-2}}\right)\right] \\ &= \frac{\delta_F^{d-2}}{d^{d-2}}\left[Q_F(\alpha\zeta) - Q_F\left(e^{\frac{2\pi i}{d}}\alpha\zeta\right)\right] + \frac{\delta_F^{d-3}}{d^{d-3}}\left[Q_F\left((\alpha\zeta)^d\right) - Q_F\left(\left(e^{\frac{2\pi i}{d}}\alpha\zeta\right)^d\right)\right] + \\ &\qquad \cdots + \left[Q_F\left((\alpha\zeta)^{d^{d-2}}\right) - Q_F\left(\left(e^{\frac{2\pi i}{d}}\alpha\zeta\right)^{d^{d-2}}\right)\right]. \end{aligned} \quad (6.4)$$

While comparing the coefficients of the polynomials appearing in the right hand side and in the left hand side of (6.4), note that if $\zeta^{d^r l} = \zeta^{d^s m}$ for $0 \leq r, s \leq d-2$ and $1 \leq l, m \leq d+1$ with $r \neq s$ (without loss of generality $r > s$, say), then $m = d^{r-s}l$. Since $d \geq 2$ and $0 \leq r, s \leq d-2$ and $1 \leq l, m \leq d+1$, clearly $r = s+1$ and $l = 1$.

Thus since next to the highest coefficients of both Q_H and Q_F vanish (Q_H and Q_F are of the form (4.10) and (4.11), respectively), we obtain

$$\beta\left[Q_H\left(\zeta^{d^{d-2}}\right) - Q_H\left(\left(e^{\frac{2\pi i}{\tilde{d}}}\zeta\right)^{d^{d-2}}\right)\right]$$
$$= Q_F\left((\alpha\zeta)^{d^{d-2}}\right) - Q_F\left(\left(e^{\frac{2\pi i}{\tilde{d}}}\alpha\zeta\right)^{d^{d-2}}\right). \tag{6.5}$$

Comparing both sides of (6.5), we get

$$\beta A^H_{d-k} = \tilde{\alpha}^{d-k} A^F_{d-k},$$

for $1 \leq k \leq (d-1)$, where $\tilde{\alpha} = \alpha^{d^{d-2}}$. Equivalently, we get

$$A^H_{d-k} = \tilde{\alpha}^{-(k+1)} A^F_{d-k} \tag{6.6}$$

for $1 \leq k \leq (d-1)$, with $\tilde{\alpha}^{d+1} = 1$. Note that (6.6) is an analogue of (4.13) and therefore, using the same set of arguments as in Section 4, we get

$$\beta p_H(y) = \tilde{\alpha} p_F(\tilde{\alpha} y), \tag{6.7}$$

for all $y \in \mathbb{C}$, with $\tilde{\alpha}^{d+1} = \beta$ and $\beta^{d-1} = 1$.

7 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1: It follows from (5.17), (6.7) and (6.1) that

$$\alpha p_F(\alpha y) = \beta p_H(y),$$

with $\alpha^{d+1} = 1$ and

$$\delta_F = \gamma \delta_H,$$

with $\alpha^{d^2-1} = 1$ and $\beta^{d-1} = \gamma^{d-1} = 1$. Thus if

$$A(x, y) = (\alpha x, \beta\alpha^{-1} y) \text{ and } B(x, y) = (\gamma\alpha\beta^{-1} x, \alpha^{-1} y), \tag{7.1}$$

then a simple calculation gives

$$F \equiv A \circ H \circ B$$

in $\mathbb{C}^2$. Now note that if we write $A = (A_1, A_2)$ and $B = (B_1, B_2)$, then

$$(A_1(x, y))^{d-1} = (B_1(x, y))^{d-1} \text{ and } (A_2(x, y))^{d-1} = (B_2(x, y))^{d-1}.$$

Thus,

$$A_1(x, y) = \mu_1 B_1(x, y) \text{ and } A_2(x, y) = \mu_2 B_2(x, y), \tag{7.2}$$

for some $\mu_1, \mu_2 \in \mathbb{C}$, with $\mu_1^{d-1} = \mu_2^{d-1} = 1$. Now comparing (7.1) and (7.2), we obtain $\mu_1 = \gamma^{-1}\beta$ and $\mu_2 = \beta$. Therefore,

$$A(x, y) = L_\mu \circ B(x, y),$$

for all $(x, y) \in \mathbb{C}^2$, where $L_\mu(x, y) = (\mu_1 x, \mu_2 y)$. Thus,

$$F \equiv L_\mu \circ B \circ H \circ B.$$

Proof of Theorem 1.2: Implementing the idea discussed just after stating the Theorem 1.1 in the Introduction, proof of Theorem 1.2 follows immediately once we prove Theorem 1.1.

It would be interesting to investigate the converse of Theorem 1.2.

Question: If H and F are two Hénon maps such that

$$F = A_1 \circ H \circ A_2$$

in $\mathbb{C}^2$, where A_1 and A_2 are affine automorphisms in $\mathbb{C}^2$, then how are the escaping sets of H and F related?

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