Mathematische Annalen (2024) 388:1969–1999
https://doi.org/10.1007/s00208-023-02564-3

Mathematische Annalen

Divisibility of polynomials and degeneracy of integral points

Erwan Rousseau[1] · Amos Turchet[2] · Julie Tzu-Yueh Wang[3]

Received: 23 June 2021 / Revised: 18 July 2022 / Accepted: 7 January 2023 /
Published online: 30 January 2023

Abstract

We prove several statements about arithmetic hyperbolicity of certain blow-up varieties. As a corollary we obtain multiple examples of simply connected quasi-projective varieties that are pseudo-arithmetically hyperbolic. This generalizes results of Corvaja and Zannier obtained in dimension 2 to arbitrary dimension. The key input is an application of the Ru–Vojta's strategy. We also obtain the analogue results for function fields and Nevanlinna theory with the goal to apply them in a future paper in the context of Campana's conjectures.

Mathematics Subject Classification 11J87 · 11J97 · 14G05 · 32A22

1 Introduction

The goal of this project is to generalize the results of our previous paper [26] to higher dimensions. In [26] we dealt with two competing conjectures that aim to characterize algebraic varieties defined over a number field k that have a potentially dense set of k-rational points. On one hand Campana conjectured that the class of these varieties is the class of *special varieties*, introduced in [6], while the Weak Specialness Conjecture

✉ Amos Turchet
amos.turchet@uniroma3.it

Erwan Rousseau
erwan.rousseau@univ-brest.fr

Julie Tzu-Yueh Wang
jwang@math.sinica.edu.tw

1 UMR 6205, Laboratoire de Mathematiques de Bretagne Atlantique, Univ Brest, CNRS, 6 avenue Le Gorgeu, 29238 Brest Cedex 3, France

2 Dipartimento di Matematica e Fisica, Università degli studi Roma 3, L.go S. L. Murialdo 1, 00146 Rome, Italy

3 Institute of Mathematics, Academia Sinica, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan

(see [14, Conjecture 1.2]) predicts that these should be the weakly special varieties, i.e. varieties that do not admit any étale cover that dominates a variety of general type. In [26, Theorem 4.2] we constructed examples of quasi-projective threefolds that are not special but weakly-special (see also [3, 7] for other constructions), and proved in [26, Theorem 6.5] that such examples possess properties that contradict function field and analytic analogues of the Weak-Specialness conjecture.

In order to generalize these results in higher dimensions we need two ingredients: the first one, that is the focus of the present paper, is the construction of simply connected varieties X where we have a good control on the distribution of integral points and entire curves. The second one, which will be addressed in a forthcoming paper, is the construction of weakly-special varieties Z fibered over X and the study of the orbifold hyperbolicity of the base X.

In [26] we used as "arithmetic input" a construction of Corvaja and Zannier in [10] of a simply connected quasi-projective surface whose integral points are not Zariski dense. The key observation in [10] was that the study of the distribution of integral points in such surfaces is connected to divisibility problems of polynomials evaluated at S-integers. In fact many classical problems in Diophantine Geometry, such as (certain cases of) Siegel's finiteness theorem or the S-unit equation, can be rephrased via divisibility of polynomials. In this paper we use this observation to obtain several new results that extend [10] to higher dimensions.

The first result is a generalization of [10, Theorem 4] to an arbitrary number of variables.

Theorem 1 *Let $n \geq 2$. Let k be a number field, let S be a finite set of places including the Archimedean ones, and let $\mathcal{O}_S$ be the ring of S-integers. Let $F_1, \ldots, F_r, G \in \mathcal{O}_S[t_0, \ldots, t_n]$ be absolutely irreducible homogeneous polynomials with $F_1, \ldots, F_r$ of the same degree. Suppose that the hypersurfaces defined by $F_1, \ldots, F_r$ and G are in general position, i.e. any intersection of $n+1$ hypersurfaces is empty, and $\deg(F_i) \geq \deg(G)$ for every i. Then there exists a closed subset $Z \subset \mathbb{P}^n$, independent of k and S, such that there are only finitely many points $(x_0, \ldots, x_n) \in \mathbb{P}^n(\mathcal{O}_S)\setminus Z$ such that one of the following holds:*

(i) *$r \geq 2n+1$ and $F_i(x_0, \ldots, x_n) \mid G(x_0, \ldots, x_n)$ in the ring $\mathcal{O}_S$, for $i = 1, \ldots, r$; or*
(ii) *$r \geq n+2$ and $\prod_{i=1}^r F_i(x_0, \ldots, x_n) \mid G(x_0, \ldots, x_n)$ in the ring $\mathcal{O}_S$.*

In [10, Theorem 4] the original Theorem was obtained in the case $n = 2$. Moreover, in Theorem 1, we obtain a stronger conclusion, namely the existence of an exceptional set Z independent of the field of definition. The above Theorem yields the following Corollary that generalizes the classical S-unit equation (that is the case $g = 1$).

Corollary 1 (Compare with [10, Corollary 1]) *Let $g \in \mathcal{O}_S[t_1, \ldots, t_n]$ be a polynomial of degree ≤ 1 such that $g(0, \ldots, 0) \neq 0$, $g(1, 0, \ldots, 0) \neq 0, \ldots, g(0, \ldots, 0, 1) \neq 0$. The n-tuples $(x_1, \ldots, x_n) \in \mathcal{O}_S^n$ such that $\left(\left(1 - \sum_{i=1}^n x_i\right) \prod_{i=1}^n x_i\right) \mid g(x_1, \ldots, x_n)$ are not Zariski-dense in $\mathbb{A}^n$.*

Proof Apply Theorem 1 (ii) to the linear forms $t_0,\ldots,t_n$, and $t_0 - \sum_{i=1}^n t_i$.

As we will see, both results follow from a more general statement, Theorem 9 in Sect. 4.

We mentioned above that divisibility results as the ones of Theorem 1 and Corollary 1, are related to degeneracy of integral points on varieties. The first statement in this direction is the following theorem that studies certain blow up of $\mathbb{P}^n$ along intersections of hypersurfaces.

Theorem 2 *Let $n \geq 2$, $r \geq 2n+1$ and $D_0, D_1, \ldots, D_r$ be hypersurfaces in general position in $\mathbb{P}^n$ defined over k. Let $\pi : X \to \mathbb{P}^n$ be the blowup of the union of subschemes $D_i \cap D_0$, $1 \leq i \leq r$, and let $\widetilde{D}_i$ be the strict transform of D_i. Let $D = \widetilde{D}_1 + \cdots + \widetilde{D}_r$. Then $X \backslash D$ is arithmetically pseudo-hyperbolic.*

This is the key result needed for the future applications to the study of weakly special varieties. In fact we can use Theorem 2 to construct simply connected varieties whose integral points are not Zariski dense, thus generalizing Corvaja and Zannier's construction in arbitrary dimension.

Proposition 1 (Compare with [10, Theorem 3]) *In the setting of Theorem 2, suppose that the divisor $D_0 + D_1 + \cdots + D_r$ has simple normal crossing singularities. Then the variety $X \backslash D$ appearing in Theorem 2 is simply connected.*

Proof If $n = 2$ this was done in [26, Example 4.4]. If $n \geq 3$, consider a loop around $\widetilde{D}_i$. Now observe that, if E is the exceptional divisor over $D_i \cap D_0$, the generic fiber of the restriction $\pi : E \backslash D \to D_i \cap D_0$ is isomorphic to $\mathbb{C}$. Thus the loop becomes homotopically trivial in $X \backslash D$.

Proposition 1 will be used in a subsequent paper to discuss analogues of a question of Hassett and Tschinkel in [15, Problem 3.7] for function fields and entire curves.

Along the same lines we generalize to arbitrary dimensions [10, Corollary 2].

Theorem 3 *Let $n \geq 2$ and let $H_1, \ldots, H_{2n}$ be $2n$ hyperplanes in general position on $\mathbb{P}^n$ defined over k. Choose $n+1$ points P_i, $1 \leq i \leq n+1$ such that $P_i \in H_i$, $1 \leq i \leq n+1$, and $P_i \notin H_j$ if $i \neq j$ for $1 \leq j \leq 2n$. Let $\pi : X \to \mathbb{P}^n$ be the blowup of the $n+1$ points P_i, $1 \leq i \leq n+1$, and let $D \subset X$ be the strict transform of $H_1 + \cdots + H_{2n}$. Then $X \backslash D$ is arithmetically pseudo-hyperbolic.*

Finally we obtain a generalization of [10, Proposition 1, Theorem 7] as follows.

Theorem 4 *Let $n \geq 2$ and $q \geq 3n$ be two integers; for every index $i \in \mathbb{Z}/q\mathbb{Z}$, let H_i be a hyperplane in $\mathbb{P}^n$ defined over k. Suppose that the H_i's are in general position. For each index $i \in \mathbb{Z}/q\mathbb{Z}$ let P_i be the intersection point $\cap_{j=0}^{n-1} H_{i+j}$. Let $\pi : X \to \mathbb{P}^n$ be the blow-up of the points $P_1, \ldots, P_q$, let $\widetilde{H}_i \subset X$ be the strict transform of H_i, and let $D = \widetilde{H}_1 + \cdots + \widetilde{H}_q$. Then $X \backslash D$ is arithmetically pseudo-hyperbolic.*

Theorem 5 *Let $n \geq 2$, $q \geq 3n$ be an integer; for every index $i \in \mathbb{Z}/q\mathbb{Z}$, let F_i, $1 \leq i \leq q$ be linear form in $k[t_0, \ldots, t_n]$ in general position. Then there exists a Zariski closed subset Z of $\mathbb{P}^n$ such that the set of points $[x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$*

satisfying, for each $i \in \mathbb{Z}/q\mathbb{Z}$, the equality of ideals

$$F_i(x_0, \dots, x_n) \cdot (x_0, \dots, x_n) = \prod_{j=i-n+1}^{i} (F_j(x_0, \dots, x_n), \dots, F_{j+n-1}(x_0, \dots, x_n))$$

is contained in Z.

We also mention that most of these results can be rephrased as hyperbolicity of complements of divisors in certain varieties that are higher dimensional analogues of Del Pezzo surfaces. For example Theorem 2 applies to open subsets of the blow up of $\mathbb{P}^3$ in $r \geq 7$ lines. Interestingly enough the condition $r \geq 7$ characterizes precisely the blow-ups that are *not* weakly Fano (and hence not Mori dream spaces).

`Ideas of the proof` The main technical tool to obtain the proof of the previous results, as in our previous paper [26], is to apply (a generalization of) the main theorem of Ru–Vojta (see Theorem 7). In fact in [26] we have already proven that the Ru–Vojta method can be used to recover the main theorem of [9], that was used in [10] to obtain the degeneracy results that we are generalizing in this paper. However, in this situation, the computations of the constant β, which is the crucial part of the proof, is less direct and make use of several ingredients, among them an adaptation of Autissier's ideas of [1].

Moreover, by carefully tracing the exceptional set, and following a strategy already discussed by Levin in [20], we can obtain a stronger result, namely pseudo-arithmetic hyperbolicity instead of degeneracy of integral points. In particular, this shows that in our statements, the closed subset outside of which the integral points are finite, does not depend on the field of definition (as expected in the stronger versions of the conjectures of Lang and Vojta); we refer to [26, Section 3] for more details and discussions. In fact our results are indeed instances of the Lang-Vojta conjecture for integral points.

The paper is organized as follows: in Sect. 2 we collect some preliminary definitions and properties of local heights and we link divisibility problems with integral points. In Sect. 3 we state the Main Theorem of Ru–Vojta with better control of the exceptional set. In Sect. 4 we prove Theorems 1 and Theorem 2. In Sect. 5 we compute β in several cases and we prove Theorem 3. In Sect. 6 we prove Theorem 4 and Theorem 5. In Sect. 7 we collect the analogue results for holomorphic maps, while in Sect. 8 we present the results over function fields, together with the proof of the key Ru–Vojta statement.

`Divisibility and integral points` As observed in [10], there is a connection between distribution of integral points on certain rational quasi-projective varieties, and divisibility problems. In fact, Corvaja and Zannier show the following: given points $P_1, \dots, P_n \in \mathbb{P}^2$ that are the intersection of two curves $\mathcal{C}_1, \mathcal{C}_2$, let $\pi : X \to \mathbb{P}^2$ be the blow up along $P_1, \dots, P_n$. Then, for a point $Q \in \mathbb{P}^2$, $Q \neq P_i$, one can relate the condition that $\pi^{-1}(Q)$ is integral on X with respect to the strict transform of $\mathcal{C}_1$, with a divisibility condition for the polynomials defining C_1 and C_2 (locally).

We formalize this in Lemma 1, where we generalize to arbitrary dimensions [10, Lemma 1].

In fact, divisibility conditions are connected to the celebrated Vojta's conjectures (as in [31, Conjecture 3.4.3]) in many ways: Silverman in [29] observed that GCD results for S-units in number fields, as in the seminal paper [5], are related to Vojta's conjecture on certain blow-ups. Since then, a number of articles have been devoted to exploit this connection. We cite for example [2, 8, 11–13, 22, 23, 25, 30, 35–37].

2 Heights and integral points

We collect here standard facts and definitions on local and global Weil heights and integral points. We refer to [18, Chapter 10], [16, B.8], [22, Section 2.3] or [28, Section 2] for more details about this section. We have decided to avoid the use of integral models to discuss integral points since it is more natural in the arithmetic context of the Ru–Vojta method.

Let k be a number field and M_k be the set of places of k, normalized so that it satisfies the product formula

$$\prod_{v \in M_k} |x|_v = 1, \qquad \text{for } x \in k^\times.$$

For a point $[x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, the standard logarithmic height is defined by

$$h([x_0 : \cdots : x_n]) = \sum_{v \in M_k} \log \max\{|x_0|_v, \ldots, |x_n|_v\},$$

and it is independent of the choice of coordinates $x_0, \ldots, x_n$ by the product formula.

A M_k-*constant* is a family $\{\gamma_v\}_{v \in M_k}$ of real numbers, with all but finitely many equal to zero. Equivalently it is a real-valued function $\gamma : M_k \to \mathbb{R}$ which is zero almost everywhere. Given two families $\{\lambda_{1v}\}$ and $\{\lambda_{2v}\}$, we say $\lambda_{1v} \le \lambda_{2v}$ holds up to an M_k-constant if there exists an M_k-constant $\{\gamma_v\}$ such that $\lambda_{2v} - \lambda_{1v} \ge \gamma_v$ for all $v \in M_k$. We say $\lambda_{1v} = \lambda_{2v}$ up to an M_k-constant if $\lambda_{1v} \le \lambda_{2v}$ and $\lambda_{2v} \le \lambda_{1v}$ up to M_k-constants.

Let V be a projective variety defined over a number field k. The classical theory of heights associates to every Cartier divisor D on V a *height function* $h_D : V(k) \to \mathbb{R}$ and a *local Weil function* (or *local height function*) $\lambda_{D,v} : V(k) \setminus \operatorname{Supp}(D) \to \mathbb{R}$ for each $v \in M_k$, such that

$$\sum_{v \in M_k} \lambda_{D,v}(P) = h_D(P) + O(1)$$

for all $P \in V(k) \setminus \operatorname{Supp}(D)$.

We also recall some basic properties of local Weil functions associated to closed subschemes from [28, Section 2]. Given a closed subscheme Y on a projective variety V defined over k, we can associate to each place $v \in M_k$ a function

$$\lambda_{Y,v} : V \setminus \operatorname{Supp}(Y) \to \mathbb{R}.$$

Intuitively, for each $P \in V$ and $v \in M_k$, we think of

$$\lambda_{Y,v}(P) = -\log(v\text{-adic distance from } P \text{ to } Y).$$

To describe $\lambda_{Y,v}$ more precisely, we use (see for example [28, Lemma 2.5.2]) that for a closed subscheme Y of V, there exist effective divisors $D_1, \ldots, D_r$ such that $Y = \cap D_i$. Then, the function $\lambda_{Y,v}$ can be described as follows:

Definition-Theorem 6 ([31, Lemma 2.5.2], [28, Theorem 2.1 (d)(h)]) *Let k be a number field, and M_k be the set of places of k. Let V be a projective variety over k and let $Y = \cap D_i \subset V$ be a closed subscheme of V. We define the (local) Weil function for Y with respect to $v \in M_k$ as*

$$\lambda_{Y,v} = \min_i \{\lambda_{D_i,v}\}, \tag{2.1}$$

This is independent of the choices of the D_i's up to an M_k-constant, and satisfies

$$\lambda_{Y_1,v}(P) \leq \lambda_{Y_2,v}(P)$$

up to an M_k-constant, whenever $Y_1 \subseteq Y_2$. Moreover, if $\pi : \widetilde{V} \to V$ is the blowup of V along Y with exceptional divisor E, $\lambda_{Y,v}(\pi(P)) = \lambda_{E,v}(P)$ up to an M_k-constant, as functions on $\widetilde{V}(k) \backslash E$.

The height function for a closed subscheme Y of V is defined, for $P \in V(k) \backslash Y$, by

$$h_Y(P) := \sum_{v \in M_k} \lambda_{Y,v}(P).$$

We also define two related functions for a closed subscheme Y of V, depending on a finite set of places S of k: the *proximity function* $m_{Y,S}$ and the *counting function* $N_{Y,S}$, for $P \in V(k) \backslash Y$, as

$$m_{Y,S}(P) := \sum_{v \in S} \lambda_{Y,v}(P) \quad \text{and} \quad N_{Y,S}(P) := \sum_{v \in M_k \backslash S} \lambda_{Y,v}(P) = h_Y(P) - m_{Y,S}(P).$$

We can now define the notion of (D, S)-integral points following Vojta.

Definition 1 ([33, Definition 13.1]) Let k be a number field and M_k be the set of places on k. Let $S \subset M_k$ be a finite subset containing all Archimedean places. Let X be a projective variety over k, and let D be an effective divisor on X. A set $R \subseteq X(k) \backslash \operatorname{Supp} D$ is a (D, S)-*integral* set of points if there is a Weil function $\{\lambda_{D,v}\}$ for D and an M_k-constant $\{\gamma_v\}$ such that for all $v \notin S$, $\lambda_{D,v}(P) \leq \gamma_v$ for all $P \in R$.

By the uniqueness (up to an M_k-constant) of Weil functions for a Cartier divisor D (see [33, Theorem 9.8 (d)]), one can use a fixed Weil function λ_D in Definition 1 (after adjusting $\{\gamma_v\}$).

Finally we recall the definition of arithmetic hyperbolicity.

Definition 2 Let X and D be as above. We say that $X\backslash D$ is *arithmetically pseudo-hyperbolic* if there exists a proper closed subset $Z \subset X$ such that for any number field $k' \supset k$, every finite set of places S of k' containing the Archimedean places, and every set R of (k'-rational) (D, S)-integral points on X, the set $R\backslash Z$ is finite. We say that $X\backslash D$ is *arithmetically hyperbolic* if it is pseudo-arithmetically hyperbolic with $Z = \emptyset$.

The main tool for relating questions of divisibility between values of polynomials to integrability for points on varieties is established in the following lemma. We state it in terms of local heights since it is more convenient and it admits an explicit analogue using local equations as in [10].

Lemma 1 (Compare to [10, Lemma 1]) *Let X be a projective variety over a number field k, and let $S \subset M_k$ be a finite subset containing all Archimedean places. Let D be an effective Cartier divisor of X and W be a closed subscheme of X such that the codimension of $D \cap W$ is at least 2. Let $\pi : \widetilde{X} \to X$ be the blowup along some closed subscheme of X containing $D \cap W$ such that $\pi^* D = \widetilde{D} + \pi^{-1}(D \cap W)$, where $\widetilde{D}$ is the strict transform of D. Let R be a set of points in $\widetilde{X}(k)$. Then the following are equivalent.*

(i) $\lambda_{\widetilde{D},v}(P) = 0$ *up to a M_k-constant for $P \in R$ and $v \notin S$,*
(ii) $\lambda_{D,v}(\pi(P)) \le \lambda_{W,v}(\pi(P))$ *up to a M_k-constant for $P \in R$ and $v \notin S$.*

Proof Let $Y = D \cap W$. The functorial property of Weil functions implies that

$$\lambda_{D,v}(\pi(P)) = \lambda_{\pi^* D,v}(P) = \lambda_{\widetilde{D},v}(P) + \lambda_{Y,v}(\pi(P)) \tag{2.2}$$

up to a M_k-constant. On the other hand, it follows from (2.1) that

$$\lambda_{Y,v}(\pi(P)) = \min\{\lambda_{D,v}(\pi(P)), \lambda_{W,v}(\pi(P))\} \tag{2.3}$$

up to a M_k-constant for any $v \in M_k$. Then the equivalence of (i) and (ii) can be easily deduced from (2.2) and the (2.3).

3 Ru–Vojta theorem and some basic propositions

We first recall the following definitions and geometric properties from [27].

Definition 3 Let $\mathcal{L}$ be a big line sheaf and let D be a nonzero effective Cartier divisor on a projective variety X. We define

$$\beta_{\mathcal{L},D} := \lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathcal{L}^N(-mD))}{N \cdot h^0(X, \mathcal{L}^N)}.$$

If A is a big (Cartier) divisor we let $\beta_{A,D} := \beta_{\mathcal{O}(A),D}$.

The constant β is the crucial ingredient in Ru–Vojta's main Theorem. Before stating it we recall the following definition.

Definition 4 Let $D_1, \ldots, D_q$ be effective Cartier divisors on a variety X of dimension n.

(i) We say that $D_1, \ldots, D_q$ are *in general position* if for any $I \subset \{1, \ldots, q\}$, we have

$$\dim(\cap_{i \in I} \operatorname{Supp} D_i) \leq n - \#I \qquad \text{with } \dim \emptyset = -\infty.$$

(ii) We say that $D_1, \ldots, D_q$ *intersect properly* if for any $I \subset \{1, \ldots, q\}$, $x \in \cap_{i \in I} \operatorname{Supp} D_i$, and local equations ϕ_i for D_i in x, the sequence $(\phi_i)_{i \in I}$ is a regular sequence in the local ring $\mathcal{O}_{X,x}$.

Remark 1 If $D_1, \ldots, D_q$ intersect properly, then they are in general position. By [24, Theorem 17.4], the converse holds if X is Cohen–Macaulay.

The following is the main arithmetic Theorem of Ru and Vojta.

Theorem 7 [27, General Theorem (Arithmetic Part)] *Let k be a number field and let M_k be the set of places of k. Let $S \subset M_k$ be a finite subset containing the Archimedean places. Let X be a projective variety defined over k. Let $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let $\mathcal{L}$ be a big line sheaf on X. Then for any $\epsilon > 0$, there exists a proper Zariski-closed subset $Z \subset X$, independent of k and S, such that*

$$\sum_{i=1}^{q} \beta_{\mathcal{L},D_i} m_{D_i,S}(x) \leq (1+\epsilon) h_{\mathcal{L}}(x)$$

holds for all but finitely many x in $X(k) \backslash Z$.

We stress that the result is in fact stronger than the original statement, since the exceptional set Z does not depend on k and S. This can be obtained by carefully tracing the exceptional sets in the proof with the following version, due to Vojta in [32], of Schmidt's subspace theorem, which gives a better control on the exceptional sets.

Theorem 8 *Let k be a number field and M_k be the set of places on k. Let $S \subset M_k$ be a finite subset containing the Archimedean places. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ defined over k with the corresponding Weil functions $\lambda_{H_1}, \ldots, \lambda_{H_q}$. Then there exist a finite union of hyperplanes Z, depending only on $H_1, \ldots, H_q$ (and not on k or S), such that for any $\epsilon > 0$,*

$$\sum_{v \in S} \max_I \sum_{i \in I} \lambda_{H_i,v}(P) \leq (n+1+\epsilon) h(P)$$

holds for all but finitely many points P in $\mathbb{P}^n(k) \backslash Z$, where the maximum is taken over subsets $\{1, \ldots, q\}$ such that the linear forms defining H_i for $i \in I$ are linearly independent.

We end this section with a useful lemma about local height functions.

Lemma 2 [35, Lemma 5.2] *Let $D_1, \ldots, D_q$ be effective divisors of a projective variety V of dimension n, defined over k, in general position. Then*

$$\sum_{i=1}^{q} \lambda_{D_i,v}(P) = \max_I \sum_{j\in I} \lambda_{D_j,v}(P),$$

up to a M_k- constant, where $v \in M_k$, I runs over all index subsets of $\{1, \ldots, q\}$ with n elements for all $x \in V(k)$.

4 Proof of Theorem 1 and Theorem 2

In this section we will prove Theorem 1 and Theorem 2. These will be obtained as a consequence of the following more general statement. From now on, we denote by k a number field, and by S a finite set of places of k.

Theorem 9 *Let V be a Cohen–Macaulay projective variety of dimension n defined over k. Let $D_0, D_1, \ldots, D_r$, $r \geq n+1$, be effective Cartier divisors of V defined over k in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ for $0 \leq i \leq r$. Then there exists a proper Zariski closed subset Z of V, independent of k and S, such that for any M_k constant $\{\gamma_v\}$, there are only finitely many $P \in V(k)\backslash Z$ such that the following holds.*

(i) $r \geq 2n+1$ *and* $\frac{1}{d_i}\lambda_{D_i,v}(P) \leq \frac{1}{d_0}\lambda_{D_0,v}(P) + \gamma_v$ *for* $v \notin S$ *and* $1 \leq i \leq r$; *or*
(ii) $r \geq n+2$ *and* $\sum_{i=1}^{r} \frac{1}{d_i}\lambda_{D_i,v}(P) \leq \frac{1}{d_0}\lambda_{D_0,v}(P) + \gamma_v$ *for* $v \notin S$.

Here, $\equiv$ denotes numerical equivalence of divisors, and $\lambda_{D_i,v}$ is a Weil function of D_i at v.

The following theorem can be deduced from Theorem 9 using Lemma 1.

Theorem 10 *Let V be a Cohen–Macaulay projective variety of dimension n defined over k. Let $D_0, D_1, \ldots, D_r$, $r \geq 2n+1$, be effective Cartier divisors of V defined over k in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ for all i. Let $\pi : \widetilde{V} \to V$ be the blowup along the union of subschemes $D_i \cap D_0$, $1 \leq i \leq r$, and let $\widetilde{D}_i$ be the strict transform of D_i. If $D = \widetilde{D}_1 + \cdots + \widetilde{D}_r$, then $\widetilde{V}\backslash D$ is arithmetically pseudo-hyperbolic.*

Then, Theorem 2 is a direct consequence of Theorem 10. We now show that Theorem 9 implies Theorem 1.

Proof of Theorem 1 Let $D_i := [F_i = 0]$ for $1 \leq i \leq r$, and $D_0 = [G = 0]$.

Recall the following standard local Weil function for D_i

$$\lambda_{D_i,v}(P) := -\log \frac{|F_i(x_0, \ldots, x_n)|_v}{\max\{|x_0|_v^{d_i}, \ldots, |x_n|_v^{d_i}\}},$$

where $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)\backslash D_i$, $F_0 = G$ and $d_i = \deg F_i$, $0 \leq i \leq r$. Since the coefficients of F_i and G are in $\mathcal{O}_S$, for integral points $P = (x_0, \ldots, x_n) \in \mathcal{O}_S^{n+1}$, the condition that $F_i(x_0,\ldots,x_n)$ divides $G(x_0,\ldots,x_n)$ in the ring $\mathcal{O}_S$ implies that $|G(x_0,\ldots,x_n)|_v \le |F_i(x_0,\ldots,x_n)|_v \le 1$ for $v \notin S$. Then $|G(x_0,\ldots,x_n)^{d_i}|_v \le |F_i(x_0,\ldots,x_n)^{d_0}|_v$ for $v \notin S$ as $d_i \ge d_0$, and hence for $v \notin S$,

$$\frac{1}{d_i}\lambda_{D_i,v}(P) - \frac{1}{d_0}\lambda_{D_0,v}(P) = -\frac{1}{d_0 d_i}\log\left|\frac{F_i(x_0,\ldots,x_n)^{d_0}}{G(x_0,\ldots,x_n)^{d_i}}\right|_v \le 0$$

Therefore, Theorem 1 (i) is a consequence of Theorem 9 (i). The proof for (ii) is the same.

4.1 Basic properties and one technical lemma

We will recall some basic results and one technical lemma from [35]. We start with [35, Proposition 2.4], which is an immediate consequence of [16, Theorem B.3.2.(f)].

Proposition 2 *Let X be a projective variety defined over k, and A be an ample Cartier divisor on X defined over k. Let D be a Cartier divisor D defined over k with $D \equiv A$. Let $\epsilon > 0$. Then there exists a constant c_ϵ such that for all $P \in X(k)$*

$$(1-\epsilon)h_A(P) - c_\epsilon \le h_D(P) \le (1+\epsilon)h_A(P) + c_\epsilon.$$

The following theorem is a reformulation of [21, Theorem 3.2] by applying Proposition 2.

Theorem 11 *Let X be a projective variety of dimension n defined over k. Let $D_1,\ldots,D_q$ be effective Cartier divisors on X, defined over k, in general position. Suppose that there exists an ample Cartier divisor A on X and positive integer d_i such that $D_i \equiv d_i A$ for all i and all $v \in S$. Let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$, independent of S and k, such that for all but finitely many points $P \in X(k)\backslash Z$,*

$$\sum_{i=1}^{q}\frac{1}{d_i}N_{D_i,S}(P) > (q-n-1-\epsilon)h_A(P).$$

The following proposition follows from [17, Proposition 5.5].

Proposition 3 *Let X be a Cohen–Macaulay scheme over k and $Y \subset X$ be a locally complete intersection subscheme. Let $\pi : \widetilde{X} \mapsto X$ be the blowup of X along Y. Then $\widetilde{X}$ is a Cohen–Macaulay scheme. Moreover, if Z is an irreducible subscheme of Y,*

$$\dim \pi^{-1}(Z) = \dim Z + \operatorname{codim} Y - 1.$$

Finally, we recall the following technical lemma which is a modified version of [35, Lemma 4.7].

Lemma 3 *Let V be a projective variety of dimension n, and let Y be a closed subscheme of V of codimension at least 2. Let $\pi : \widetilde{V} \to V$ be the blowup along Y, and let*

$E = \pi^{-1}(Y)$ be the exceptional divisor. Let A be and ample Cartier divisor on V and let D be an effective Cartier divisor such that $D \equiv A$. Then, for all sufficiently large ℓ, the sheaf $\mathcal{L} = \mathcal{O}(\ell(n+1)\pi^*A - E)$ is ample and

$$\beta_{\mathcal{L},\pi^*D}^{-1} \leq \frac{1}{\ell}\left(1 + O\left(\frac{1}{\ell^2}\right)\right) \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

4.2 Proof of Theorem 9

We begin with the following proposition on general position for pullbacks.

Proposition 4 *Let V be a Cohen–Macaulay projective variety, and let $D_0, D_1, \ldots, D_r$ be ample effective Cartier divisors of V in general position. Let $Y = \cup_{i=1}^r (D_i \cap D_0)$ and $\pi : \widetilde{V} \to V$ be the blowup along the subscheme Y. Then, the following holds:*

(i) *$\pi^* D_i = \widetilde{D}_i + \pi^{-1}(D_i \cap D_0)$ for each $1 \leq i \leq r$, where $\widetilde{D}_i$ is the strict transform of D_i.*
(ii) *$\pi^* D_1, \ldots, \pi^* D_r$ are in general position.*

Proof Since $D_0, \ldots, D_r$ are in general position, for every $i \neq j$ the intersection $D_0 \cap D_i \cap D_j$ has codimension at least 3. Hence, the support of $\pi^{-1}(D_0 \cap D_j)$ is not a subset of the support of $\pi^{-1}(D_i)$, which implies (i).

To show (ii), we first note that if $r \geq n$, then the intersection of any $n+1$ of $\pi^* D_i$, $0 \leq i \leq r$, is empty since $D_0, D_1, \ldots, D_r$ are in general position. Next, let $I \subset \{1, \ldots, r\}$ with $\#I \leq n$. We claim that $\dim(\cap_{i \in I} \operatorname{Supp} \pi^* D_i) \leq n - \#I$. Let W be an irreducible component of $\cap_{i \in I} \operatorname{Supp} \pi^* D_i$. If $\pi(W) \subset Y$, then $\pi(W)$ is a subset of $(\cap_{i \in I} D_i) \cap D_0$ and hence $\dim \pi(W) < n - \#I$. Then $\dim W \leq n - \#I$ by Proposition 3. It remains to consider when $\pi(W)$ is not a subset of Y, which implies that $W \backslash \pi^{-1}(Y)$ is not empty and is contained in $\cap_{i \in I} \operatorname{Supp} \widetilde{D}_i \backslash \pi^{-1}(Y)$. Since $(\cap_{i \in I} \widetilde{D}_i) \backslash \pi^{-1}(Y)$ and $(\cap_{i \in I} D_i) \backslash Y$ are isomorphic, this shows that $\dim W \leq n - \#I$.

We can now prove Theorem 9.

Proof of Theorem 9 Let c be the least common multiple of $d_0, d_1, \ldots, d_r$. Let $A_0 = cA$, $D_i' := \frac{c}{d_i} D_i \equiv A_0$, for $0 \leq i \leq r$. For $P \in V(k)$ satisfying (i) and $v \notin S$, we have

$$\lambda_{D_i',v}(P) = \frac{c}{d_i}\lambda_{D_i,v}(P) \leq \frac{c}{d_0}\lambda_{D_0,v}(P) = \lambda_{D_0',v}(P),$$

up to a M_k constant. Similarly, if $P \in V(k)$ satisfies (ii), then

$$\sum_{i=1}^r \lambda_{D_i',v}(P) = \sum_{i=1}^r \frac{c}{d_i}\lambda_{D_i,v}(P) \leq \frac{c}{d_0}\lambda_{D_0,v}(P) = \lambda_{D_0',v}(P)$$

up to a M_k constant. Therefore, by replacing D_i by D_i', $0 \leq i \leq r$, and A by A_0, we may assume that $D_i \equiv A$ for each $0 \leq i \leq r$ and replace (i) by

$$\lambda_{D_i,v}(P) \leq \lambda_{D_0,v}(P) + \gamma_v, \quad \text{for } 1 \leq i \leq r; \tag{4.1}$$

and, when $v \notin S$, replace (ii) by

$$\sum_{i=1}^{r} \lambda_{D_i,v}(P) \leq \lambda_{D_0,v}(P) + \gamma_v. \tag{4.2}$$

Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^{r} Y_i$. Since D_0 is in general position with each D_i, $1 \leq i \leq r$, D_0 and each D_i intersect properly by Remark 1 (at page 6). Hence, Y is a local complete intersection. Let $\pi : \widetilde{V} \to V$ be the blowup along Y, and $E = \pi^{-1}(Y)$ be the exceptional divisors. Since Y is a local complete intersection, by Proposition 3, $\widetilde{V}$ is Cohen–Macaulay and hence by Proposition 4, $\pi^* D_1, \ldots, \pi^* D_r$ intersect properly. Let ℓ be a fixed sufficiently large integer such that the line sheaf $\mathcal{L} = \mathcal{O}(\ell(n+1)\pi^* A - E)$ is ample and Lemma 3 holds true, i.e.

$$\beta_{\mathcal{L},\pi^* D_i}^{-1} \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

Let $\epsilon' = \ell^{-5/2}$. Theorem 7 applied with $\epsilon = \epsilon'$, $X = \widetilde{V}$, $\mathcal{L}$, the divisors $\pi^* D_i$ (for $1 \leq i \leq r$) and $q = r$, gives a proper Zariski closed subset $\widetilde{Z} \subset \widetilde{V}$, independent of k and S such that

$$\begin{aligned}\sum_{i=1}^{r} m_{\pi^* D_i,S}(x) &\leq \left(\frac{1}{\ell}(1 + \frac{1}{\ell\sqrt{\ell}}) + \epsilon'\right) h_{\ell(n+1)\pi^* A - E}(x) \\ &\leq \left(1 + \frac{2}{\ell\sqrt{\ell}}\right)(n+1)h_{\pi^* A}(x) - \frac{1}{\ell} h_E(x)\end{aligned}$$

holds for all x outside the proper Zariski-closed subset $\widetilde{Z}$ of $\widetilde{V}(k)$. By the functorial properties of the local height functions, $h_D = m_{D,S} + N_{D,S}$, and $h_E = h_Y \circ \pi$, we have

$$\left(r - n - 1 - \frac{2(n+1)}{\ell\sqrt{\ell}}\right) \cdot h_A(\pi(x)) + \frac{1}{\ell} h_Y(\pi(x)) \leq \sum_{i=1}^{r} N_{D_i,S}(\pi(x)) \tag{4.3}$$

holds for all $\pi(x)$ outside the proper Zariski-closed subset $Z = \pi(\widetilde{Z})$ of $V(k)$. Furthermore, it follows from Lemma 2 and Proposition 2 with $\epsilon = \frac{1}{\ell^2}$ that for all $P \in V(k)$,

$$\sum_{i=1}^{r} N_{D_i,S}(P) \leq n N_{D_0,S}(P) + O(1) \leq \left(n + \frac{1}{\ell^2}\right) h_A(P) + O(1). \tag{4.4}$$

On the other hand, for all $P = \pi(x) \in V(k)$ such that $r \geq 2n+1$ and (4.1) holds, i.e. $\lambda_{D_i,v}(P) \leq \lambda_{D_0,v}(P) + \gamma_v$ for each $1 \leq i \leq r$, we have

$$h_Y(P) \geq N_{Y,S}(P) = \sum_{i=1}^{r} \sum_{v \notin S} \min\{\lambda_{D_i,v}(P), \lambda_{D_0,v}(P)\}$$

$$= \sum_{i=1}^{r} N_{D_i,S}(P) - \sum_{i=1}^{r} \sum_{v \notin S} \max\{0, \gamma_v\}. \tag{4.5}$$

If we apply Theorem 11 with $\epsilon = \frac{1}{\ell^2}$, then there exists a proper Zariski-closed subset Z' of $V(k)$, independent of S and k, such that, for all $P \in V(k)\backslash Z'$,

$$\sum_{i=1}^{r} N_{D_i,S}(P) \geq \left(r - n - 1 - \frac{1}{\ell^2}\right) h_A(P). \tag{4.6}$$

Combining Eqs. (4.5) and (4.6), together with the fact that $r \geq 2n+1$, we get that, for all $P \in V(k)\backslash Z'$,

$$h_Y(P) \geq \left(1 - \frac{1}{\ell^2}\right) h_A(P). \tag{4.7}$$

We now use (4.4) to get an upper bound for the right hand side of (4.3) and use (4.7) for the left hand side. Then we have that

$$\left(r - 2n - 1 + \frac{1}{\ell} - \frac{2(n+1)}{\ell\sqrt{\ell}} - \frac{2}{\ell^2}\right) \cdot h_A(\pi(x)) \leq O(1) \tag{4.8}$$

holds for all but finitely many $P \in V(k)$ outside $Z \cup Z'$. Since A is ample, $r \geq 2n+1$, and $\frac{1}{\ell} - \frac{2(n+1)}{\ell\sqrt{\ell}} - \frac{2}{\ell^2} > 0$, there are only finitely many $P \in V(k)$ such that (4.8) holds. This shows (i).

We are left considering when $r \geq n+2$ and (4.2) holds. In this case, we have similarly to (4.4)

$$\sum_{i=1}^{r} N_{D_i,S}(P) \leq N_{D_0,S}(P) + O(1) \leq \left(1 + \frac{1}{\ell^2}\right) h_A(P) + O(1)$$

for all $P \in V(k)$. Together with (4.3), (4.5) and (4.6), we have that

$$\left(r - n - 2 + \frac{1}{\ell} - \frac{2(n+1)}{\ell\sqrt{\ell}} - \frac{2}{\ell^2}\right) \cdot h_A(\pi(x)) \leq O(1)$$

holds for all but finitely many $P \in V(k)$ outside a proper Zariski-closed $Z \cup Z'$. Since A is ample, $r \geq n+2$, and $\frac{1}{\ell} - \frac{2(n+1)}{\ell\sqrt{\ell}} - \frac{2}{\ell^2} > 0$, this implies (ii).

Proof of Theorem 10 Since the property of being arithmetically pseudo-hyperbolic is independent of the multiplicity of the divisors, we may assume that there exists a positive constant d such that $D_i \equiv dA$ for $0 \leq i \leq r$.

Let $Y_i = D_i \cap D_0$, $Y = \cup_{i=1}^{r} Y_i$, and $\pi : \widetilde{V} \to V$ be the blowup along Y. By Proposition 4, $\pi^* D_i = \widetilde{D}_i + \pi^{-1}(D_i \cap D_0)$. Let R be a set of (D, S)-integral points, where $D = \widetilde{D}_1 + \cdots + \widetilde{D}_r$. Then there exists M_k-constant $\{\gamma_v\}$ such that for all $v \notin S$, $\lambda_{D,v}(P) \leq \gamma_v$ for all $P \in R$. By Lemma 1, we have for each $1 \leq i \leq r$

$$\lambda_{D_i,v}(\pi(P)) \leq \lambda_{D_0,v}(\pi(P)) \quad \text{up to a} \quad M_k\text{-constant for} \quad P \in R \text{ and } v \notin S. \tag{4.9}$$

Since $r \geq 2n+1$, Theorem 9 (i) implies that there exists a proper Zariski closed subset Z of V, independent of k and S, such that there are only finitely many $\pi(P) \in V(k)\backslash Z$, i.e. $P \notin \pi^{-1}(Z)$, satisfying (4.9). Since the choice of Z is independent of the M_k-constant, it implies that $\tilde{V}\backslash D$ is arithmetically pseudo-hyperbolic.

5 Proof of Theorem 3

In this section we prove Theorem 3. The main technical result is a computation of the constant β. To this end we generalize some construction of Autissier removing some hypotheses.

5.1 Background results and computing β

We start by recalling some basic properties on global sections of line bundles, and refer to [20, Section 7.3] for further references and proofs.

Lemma 4 *Suppose D is a nef divisor on a nonsingular projective variety X. Let $n = \dim X$. Then $h^0(X, \mathcal{O}(ND)) = (D^n/n!)N^n + O(N^{n-1})$. In particular, $D^n > 0$ if and only if D is big.*

We will also make use of two basic exact sequences (see [20, Lemma 7.7]).

Lemma 5 *Let D be an effective divisor on a projective variety X with inclusion map $i : D \to X$. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have exact sequences*

$$0 \to \mathcal{L} \otimes \mathcal{O}(-D) \to \mathcal{L} \to i_*(i^*\mathcal{L}) \to 0,$$
$$0 \to H^0(X, \mathcal{L} \otimes \mathcal{O}(-D)) \to H^0(X, \mathcal{L}) \to H^0(D, i^*\mathcal{L}).$$

Lemma 6 [20, Lemma 7.9] *Let X be a nonsingular projective variety of dimension n. Let D and E be any divisor on X, and let F be a nef divisor on X. Then we have*

$$h^0(X, \mathcal{O}(ND + E - mF)) \leq h^0(X, \mathcal{O}(ND)) + O(N^{n-1}) \quad \text{for all } m, N \geq 0,$$

where the implied constant is independent of m and N.

We will use the following lemma and its corollary, which are modification of [1, Lemma 4.2, Corollary 4.3] where we weaken the original hypothesis on B.

Lemma 7 *Let X be a nonsingular projective variety of dimension $n \geq 2$. Let B be a nonsingular subvariety of X of codimension 1 that is also a nef Cartier divisor. Let A be a nef Cartier divisor on X such that $A - B$ is also nef. Let $\delta > 0$ be a positive real number. Then, for any positive integers N and m with $1 \leq m \leq \delta N$, we have*

$$h^0(X, \mathcal{O}(NA - mB)) \geq \frac{A^n}{n!}N^n - \frac{A^{n-1}.B}{(n-1)!}N^{n-1}m$$

$$+\frac{(n-1)A^{n-2}.B^2}{n!}N^{n-2}\min\{m^2,N^2\}-O(N^{n-1}), \tag{5.1}$$

where the implicit constant depends on δ.

Proof We will follow the proof of [1, Lemma 4.2] with necessary modification. We first note when $m \le N$, (5.1) follows from the proof in [1, Lemma 4.2], since this part of proof only need the assumption that A, B and $A-B$ are nef.

For the case that $m > N$, we let $N \le j \le m$. Let $i : B \to X$ be the inclusion map. From Lemma 5, we have an exact sequence

$$0 \to H^0(X, \mathcal{O}(NA-(j+1)B)) \to H^0(X, \mathcal{O}(NA-jB)) \to H^0(B, i^*\mathcal{O}(NA-jB)).$$

Therefore, we have

$$h^0(X, \mathcal{O}(NA-(j+1)B)) \ge h^0(X, \mathcal{O}(NA-jB)) - h^0(B, i^*\mathcal{O}(NA-jB)).$$

Hence,

$$h^0(X, \mathcal{O}(NA-mB)) \ge h^0(X, \mathcal{O}(NA-NB)) - \sum_{j=N}^{m-1} h^0(B, i^*\mathcal{O}(NA-jB)). \tag{5.2}$$

Since B is a nef divisor on X, $i^*\mathcal{O}(B)$ is nef. Applying Lemma 6 to B, which is a non-singular subvariety of X, and the divisors corresponding to $i^*\mathcal{O}(A)$ and $i^*\mathcal{O}(B)$, we have

$$h^0(B, i^*\mathcal{O}(NA-jB)) \le h^0(B, i^*\mathcal{O}(NA)) + O(N^{n-2}) = \frac{A^{n-1}B}{(n-1)!}N^{n-1} + O(N^{n-2}). \tag{5.3}$$

Then, from (5.2), (5.3), Lemma 4, and the estimate of $h^0(X, \mathcal{O}(NA-NB))$ in the first case, it follows that

$$\begin{aligned} h^0(X, \mathcal{O}(NA-mB)) &\ge h^0(X, \mathcal{O}(NA-NB)) - (m-N)\frac{A^{n-1}B}{(n-1)!}N^{n-1} - O(N^{n-2}) \\ &\ge \frac{A^n}{n!}N^n - \frac{A^{n-1}.B}{(n-1)!}N^{n-1}m + \frac{(n-1)A^{n-2}.B^2}{n!}N^n - O(N^{n-1}). \end{aligned}$$

This shows (5.1) for the case that $m > N$.

We use Lemma 7 to obtain a lower bound on the β constant in terms of intersection numbers.

Corollary 2 *Let X be a nonsingular projective variety of dimension $n \ge 2$. Let B be a nonsingular subvariety of X of codimension 1 that is also a nef Cartier divisor on X.*

Let A be a big and nef Cartier divisor on X such that $A-B$ is nef $A^{n-1}.B>0$ and $A^{n-2}.B^2\geq 0$. Then

$$\beta_{A,B}\geq \frac{A^n}{2nA^{n-1}.B}+\frac{(n-1)A^{n-2}.B^2}{A^n}g\left(\frac{A^n}{nA^{n-1}.B}\right), \tag{5.4}$$

where $g:\mathbb{R}^+\to\mathbb{R}^+$ is the function given by $g(x)=x^3/3$ if $x\leq 1$ and $g(x)=x-2/3$ for $x\geq 1$.

Proof Let

$$b=\frac{A^n}{nA^{n-1}.B}\qquad \text{and}\qquad a=(n-1)A^{n-2}.B^2.$$

We first consider the case when $A^{n-2}.B^2>0$. For N sufficiently large and such that bN is an integer, Lemma 7 implies that

$$\begin{aligned}
&\sum_{m=1}^{\infty}h^0(NA-mB)\\
&\quad\geq \sum_{m=1}^{bN}\left(\frac{A^n}{n!}N^n-\frac{A^{n-1}.B}{(n-1)!}N^{n-1}m+\frac{a}{n!}N^{n-2}\min\{m^2,N^2\}\right)+O(N^n)\\
&\quad\geq \left(\frac{A^n}{n!}b-\frac{A^{n-1}.B}{(n-1)!}\cdot\frac{b^2}{2}+\frac{a}{n!}g(b)\right)N^{n+1}+O(N^n)\\
&\quad=\left(\frac{b}{2}+\frac{a}{A^n}g(b)\right)A^n\frac{N^{n+1}}{n!}+O(N^n).
\end{aligned}$$

Then (5.4) follows. When $A^{n-2}.B^2=0$, the same computation gives $\beta_{A,B}\geq\dfrac{b}{2}$, which implies that (5.4) holds also in this case.

5.2 Proof of Theorem 3

We apply Corollary 2 to the setting of Theorem 3.

Lemma 8 *Let $H_1,\ldots,H_{2n}$ be $2n$ hyperplanes in general position on $\mathbb{P}^n$ and choose $n+1$ points P_i such that $P_i\in H_i$, $1\leq i\leq n+1$, and $P_i\notin H_j$ if $i\neq j$ for $1\leq j\leq 2n$. Let $\pi:X\to\mathbb{P}^n$ be the blowup of the $n+1$ points P_i, and let $\widetilde{H}_i$ be the strict transform of H_i. Finally, let ℓ be a sufficiently large integer and let $A=\sum_{i=1}^{n+1}\ell\widetilde{H}_i+\widetilde{H}_{n+2}$. Then A is big and nef and*

$$\beta_{A,\widetilde{H}_1}=\cdots=\beta_{A,\widetilde{H}_{n+1}}\geq\frac{(n+1)\ell}{2n}, \tag{5.5}$$

$$\beta_{A,\widetilde{H}_{n+2}}=\cdots=\beta_{A,\widetilde{H}_{2n}}>\frac{(n+1)\ell}{2n}-\frac{\ell}{2n(n+1)^{n-2}}. \tag{5.6}$$

Proof Let $\pi : X \to \mathbb{P}^n$ be the blowup of the points P_i, as in the hypotheses. Let $E_i = \pi^{-1}(P_i)$, be the exceptional divisors. Then

$$\pi^* H_i = \begin{cases} \widetilde{H}_i + E_i, & \text{for } 1 \le i \le n+1 \\ \widetilde{H}_i, & \text{for } n+2 \le i \le 2n. \end{cases}$$

Moreover,

$$A := \sum_{i=1}^{n+1} \ell \widetilde{H}_i + \widetilde{H}_{n+2} \sim (\ell(n+1)+1)\pi^* H - \ell E, \tag{5.7}$$

where $E = E_1 + \cdots + E_{n+1}$ and H is a (generic) hyperplane in $\mathbb{P}^n$. Then

$$A^n = ((\ell(n+1)+1)\pi^* H - \ell E)^n = \big((n+1)^n - (n+1)\big)\ell^n + O(\ell^{n-1}). \tag{5.8}$$

We now show that $\widetilde{H}_i$ is nef for $1 \le i \le 2n$. Let C be an irreducible curve on X. Then

$$\widetilde{H}_i.C = \left\{ \begin{matrix} \pi^* H.C - E_i.C, & \text{for } 1 \le i \le n+1 \\ \pi^* H.C, & \text{for } n+2 \le i \le 2n. \end{matrix} \right\}.$$

If $C \subset E$, then C is contained in exactly one of the E_i. Let us assume that $C \subset E_j$ for some $1 \le j \le n+1$. Then $E_j.C < 0$, $E_i.C = 0$ if $i \ne j$ and $\pi^* H.C = H.\pi_* C = 0$. Hence, $\widetilde{H}_j.C > 0$ and $\widetilde{H}_i.C = 0$ for $1 \le i \ne j \le 2n$. Otherwise, $\pi_*(C)$ is a curve in $\mathbb{P}^n$. Then for $1 \le i \le n+1$,

$$\begin{aligned} \widetilde{H}_i.C = \pi^* H.C - E_i.C &= H.\pi_* C - \text{multi}_{P_i} \pi_*(C) \\ &= \deg \pi_* C - \text{multi}_{P_i} \pi_*(C) \ge 0, \end{aligned}$$

since $\text{multi}_{P_i} \pi_*(C) \le \deg \pi_* C$; and for $n+1 \le i \le 2n$,

$$\widetilde{H}_i.C = \pi^* H.C = H.\pi_* C = \deg \pi_* C > 0.$$

Therefore, $\widetilde{H}_i$ is nef for each $1 \le i \le 2n$ and hence A is also nef. Since A is nef and $A^n > 0$ by (5.8) as $n \ge 2$, Lemma 4 implies that A is big.

Next, we estimate the following intersection numbers using (5.7) and (5.8), and we obtain

$$\begin{aligned} A^{n-1}.\widetilde{H}_i &= ((n+1)^{n-1} - 1)\ell^{n-1} + O(\ell^{n-2}) \\ A^{n-2}.\widetilde{H}_i^2 &= ((n+1)^{n-2} - 1)\ell^{n-2} + O(\ell^{n-3}), \end{aligned} \tag{5.9}$$

for $1 \le i \le n+1$, and

$$\begin{aligned} A^{n-1}.\widetilde{H}_i &= (n+1)^{n-1}\ell^{n-1} + O(\ell^{n-2}) \\ A^{n-2}.\widetilde{H}_i^2 &= (n+1)^{n-2}\ell^{n-2} + O(\ell^{n-3}) \end{aligned} \tag{5.10}$$

for $n+2 \le i \le 2n$. It follows that $A^{n-1}.\widetilde{H}_i > 0$ for $n+2 \le i \le 2n$, and $A^{n-2}.\widetilde{H}_i^2 \ge 0$ for $1 \le i \le n+1$. Then our assertions (5.5) and (5.6) can be easily obtained from Corollary 2 by (5.8), (5.9) and (5.10) and by noting that for $i \le n+2$, $A - \widetilde{H}_i$ is still nef.

We can now prove Theorem 3.

Proof of Theorem 3 Let $\pi : X \to \mathbb{P}^n$ be the blowup of the $n+1$ points P_i, $1 \le i \le n+1$, such that $P_i \in H_i$, and $P_i \notin H_j$ if $j \ne i$. Let $E_i = \pi^{-1}(P_i)$, $1 \le i \le n+1$, be the exceptional divisors. We note that X is smooth and the strict transforms $\widetilde{H}_1, \dots, \widetilde{H}_{2n}$ are in general position.

Let ℓ be a sufficiently large integer to be determined later. Let $A = \sum_{i=1}^{n+1} \ell \widetilde{H}_i + \widetilde{H}_{n+2} \sim (\ell(n+1)+1)\pi^* H - \ell E$, where $E = E_1 + \cdots + E_{n+1}$. By Lemma 8, A is big and nef and there exist constants β and $\widetilde{\beta}$ such that

$$\begin{aligned} \beta \cdot \ell &= \beta_{A,\widetilde{H}_1} = \cdots = \beta_{A,\widetilde{H}_{n+1}}, \\ \widetilde{\beta} \cdot \ell &= \beta_{A,\widetilde{H}_{n+2}} = \cdots = \beta_{A,\widetilde{H}_{2n}}, \end{aligned}$$

and

$$\delta := (n-1)\widetilde{\beta} + 2\beta - 2 > \frac{(n-1)}{2n}\left(n - 1 - \frac{1}{(n+2)^{n-2}}\right) \ge 0 \tag{5.11}$$

since $n \ge 2$. Then we let

$$\epsilon := \frac{\delta}{4(n+3)} > 0.$$

Applying Theorem 7 to ϵ, X, $\mathcal{L} = \mathcal{O}(A)$ and $\widetilde{H}_i$, $1 \le i \le 2n$, there exists a proper Zariski closed subset $Z \subset X$, independent of k and S, such that

$$\beta \sum_{i=1}^{n+1} m_{\widetilde{H}_i,S}(x) + \widetilde{\beta} \sum_{i=n+2}^{2n} m_{\widetilde{H}_i,S}(x) \le (1+\epsilon)\left(n+1+\frac{1}{\ell}\right) h_{\pi^* H}(x) - (1+\epsilon) h_E(x) \tag{5.12}$$

holds for all x in $X(k) \backslash Z$. For the set R of (D, S)-integral points, with $D = \widetilde{H}_1 + \cdots + \widetilde{H}_{2n}$, let $c_R := \sum_{v \notin S} \max\{0, \gamma_v\}$, where $\{\gamma_v\}$ is the M_k constant from Definition 1. Then for $x \in R$,

$$\sum_{i=1}^{2n} N_{\pi^* H_i,S}(x) - N_{E,S}(x) = \sum_{i=1}^{2n} N_{\widetilde{H}_i,S}(x) \le c_R,$$

and hence

$$N_{E,S}(x) \ge \sum_{i=1}^{2n} N_{H_i,S}(\pi(x)) - O(1). \tag{5.13}$$

Moreover, since $x \in R$,

$$\sum_{i=1}^{n+1} m_{\widetilde{H}_i,S}(x) \geq \sum_{i=1}^{n+1} h_{\widetilde{H}_i}(x) - O(1) = (n+1)h_{\pi^*H}(x) - h_E(x) - O(1), \tag{5.14}$$

and

$$\sum_{i=n+2}^{2n} m_{\widetilde{H}_i,S}(x) \geq \sum_{i=n+2}^{2n} h_{\widetilde{H}_i}(x) - O(1) = (n-1)h_{\pi^*H}(x) - O(1). \tag{5.15}$$

Using (5.14) and (5.15), and assuming $\ell > \frac{1}{\epsilon}$, if $\beta < 1$ we can rewrite (5.12) as

$$(1-\beta)h_E(x) \leq \big(n+1-(n+1)\beta-(n-1)\widetilde{\beta}+(n+3)\epsilon\big)h_{\pi^*H}(x) + O(1). \tag{5.16}$$

If $\beta \geq 1$ we have

$$(\beta-1)h_E(x) \geq \big((n+1)(\beta-1)+(n-1)\widetilde{\beta}-(n+3)\epsilon\big)h_{\pi^*H}(x) - O(1).$$

Since $h_E(x) \leq (n+1)h_{\pi^*H}(x)$, the latter case immediately implies that $h(\pi(x)) = h_{\pi^*H}(x) \leq O(1)$, which can only be satisfied for finitely many $\pi(x) \in \mathbb{P}^n(k)$. Therefore we will assume that $\beta < 1$.

By Lemma 2 and the fact that $m_{H_i,S}(\pi(x)) + N_{H_i,S}(\pi(x)) = h(\pi(x)) + O(1)$, we can derive from Theorem 8 that there exists a finite union of hyperplanes W, independent of k and S, such that for any $\epsilon' > 0$

$$\sum_{i=1}^{2n} N_{H_i,S}(\pi(x)) \geq (n-1-\epsilon')h(\pi(x)) - O(1) \tag{5.17}$$

for all but finitely many $\pi(x)$ in $\mathbb{P}^n(k)\backslash W$.

Since $h_E(x) \geq N_E(x)$, we can deduce from (5.13) and (5.17) that for all but finitely many $\pi(x)$ in $\mathbb{P}^n(k)\backslash W$

$$h_E(x) \geq (n-1-\epsilon')h(\pi(x)) - O(1). \tag{5.18}$$

Then we derive from (5.11), (5.16), and (5.18) that

$$(\delta-(1-\beta)\epsilon'-(n+3)\epsilon)h(\pi(x)) \leq O(1)$$

for all but finitely $x \in R$ outside $Z \cup \pi^*(W) \cup E$. Let $\epsilon' \leq \delta/4(1-\beta)$. Then, by definition of ϵ,

$$\frac{\delta}{2}h(\pi(x)) \leq O(1),$$

which can only be satisfied for finitely many $\pi(x) \in \mathbb{P}^n(k)$. Therefore, there are only finitely many $x \in R$ outside $Z \cup \mathrm{Supp}\pi^*(W) \cup \mathrm{Supp}E$.

6 Proof of Theorem 4 and Theorem 5

We fix the notation we will use throughout this section. Let $n \geq 2$, $q \geq 3n$ be integers. For every index $i \in \mathbb{Z}/q\mathbb{Z}$, let H_i be a hyperplane in $\mathbb{P}^n$ defined over k. Suppose that $H_1, \ldots, H_q$ are in general position. For each index $i \in \mathbb{Z}/q\mathbb{Z}$, let P_i be the intersection point $\cap_{j=0}^{n-1} H_{i+j}$. Let $\pi : X \to \mathbb{P}^n$ be the blow-up over the points $P_1, \ldots, P_q$ and $E_i = \pi^{-1}(P_i)$, $1 \leq i \leq q$, be the exceptional divisors. Let $\widetilde{H}_i \subset X$ be the corresponding strict transform of H_i and let $D = \widetilde{H}_1 + \cdots + \widetilde{H}_q$. Since P_i is the intersection point $\cap_{j=0}^{n-1} H_{i+j}$, we have

$$\pi^* H_i = \widetilde{H}_i + \sum_{j=i-n+1}^{i} E_j, \tag{6.1}$$

and

$$D = \sum_{i=1}^{q} \widetilde{H}_i \sim q\pi^* H - n \sum_{i=1}^{q} E_i. \tag{6.2}$$

6.1 Key lemmas

We collect here the key lemmas for computing the constant β.

Lemma 9 *Let $n \geq 2$, $q \geq 3n$ and let D and $\widetilde{H}_i$ be as defined above. Then, for every $1 \leq i \leq q$ and $0 \leq m \leq n$, the divisor $D - m\widetilde{H}_i$ is nef.*

Proof Recall that $\pi : X \to \mathbb{P}^n$ is the blowup of the points P_i, as described above.

It is clear that it suffices to show $D - m\widetilde{H}_q$ is nef if $q \geq 3n$ and $0 \leq m \leq n$ by rearranging the index. By (6.1) and (6.2), we have

$$D - m\widetilde{H}_q \sim (q-m)\pi^* H - n \sum_{i=1}^{q-n} E_i - (n-m) \sum_{i=q-n+1}^{q} E_i. \tag{6.3}$$

Let C be an irreducible curve on X. If $\pi_*(C)$ is not a curve in $\mathbb{P}^n$, then $\pi_*(C) = P_i$ for some i. Hence, $\pi^* H.C = H.\pi_* C = 0$, $E_j.C = 0$ for $1 \leq j \neq i \leq q$ and $E_i.C > 0$. Therefore, (6.3) gives $(D - m\widetilde{H}_q).C \geq 0$ if $0 \leq m \leq n$.

If $\pi_*(C)$ is a curve in $\mathbb{P}^n$ and $\pi_*(C)$ is not in any of the H_i, from (6.3) we have

$$
\begin{aligned}
&(D - m\widetilde{H}_q).C\\
&\quad = (q-m)H.\pi_*(C) - n\sum_{i=1}^{q-n}\mathrm{multi}_{P_i}\pi_*(C) - (n-m)\sum_{i=q-n+1}^{q}\mathrm{multi}_{P_i}\pi_*(C).
\end{aligned}
\tag{6.4}
$$

It suffices to find $q-m$ hyperplanes passing through $P_1, \dots, P_q$ with described multiplicity as the equation above. We note that each H_i contains exactly n points, $P_{i-n+1}, \dots, P_i$, among the P_j's; and each point P_i is contained in exactly n hyperplanes, $H_i, \dots, H_{n+i-1}$, among the H_j's. We first consider the points P_i's contained in $H_n, \dots, H_{q-n}$ and denote these points with multiplicities as a formal sum below.

$$
n\sum_{i=n}^{q-2n+1} P_i + \sum_{i=1}^{n-1} i(P_i + P_{q-n+1-i}) = n\sum_{i=1}^{q-n} P_i - \sum_{i=1}^{n-1}(n-i)(P_i + P_{q-n+1-i})
\tag{6.5}
$$

Recall that $q \geq 3n$. The last sum in the right hand side of (6.5) contains $n(n-1)$ points counting multiplicity and the multiplicities of these points range from one to $n-1$. Therefore, we may choose $n-1$ hyperplanes $L_1, \dots, L_{n-1}$ containing these $n(n-1)$ points (counting multiplicity). Then together with (6.5), we have

$$
(q-n)H.\pi_*(C) = \sum_{i=n}^{q-n} H_i.\pi_*(C) + \sum_{i=1}^{n-1} L_i.\pi_*(C) \geq n\sum_{i=1}^{q-n}\mathrm{multi}_{P_i}\pi_*(C). \tag{6.6}
$$

Finally, since $P_{q-n+1}, \dots, P_q \in H_q$, we have

$$
(n-m)H.\pi_*(C) = (n-m)H_q.\pi_*(C) \geq (n-m)\sum_{i=q-n+1}^{q}\mathrm{multi}_{P_i}\pi_*(C). \tag{6.7}
$$

Then $(D - m\widetilde{H}_q).C \geq 0$ if $q \geq 3n$ and $0 \leq m \leq n$ by (6.4), (6.6) and (6.7).

Finally, we consider the case where $\pi_*(C)$ is contained in some H_b, where $1 \leq b \leq q$.

Suppose that $\pi_*(C) \subset \cap_{t=0}^{a} H_{b-t}$ and $\pi_*(C) \not\subset H_{b-a-1}$. Clearly, $0 \leq a \leq n-2$ since the H_i are in general position and $\pi_*(C)$ is a curve. Then $\pi_*(C) \cap \{P_1, \dots, P_q\} \subseteq \{P_{b-n+1}, \dots, P_{b-a}\}$, which is contained in $H_{b-a-1} \cup H_{b-a+j}$, for $a+1 \leq j \leq n-1$. Since $\pi_*(C) \subset \cap_{t=0}^{a} H_{b-t}$, it cannot be contained in every H_{b-a+j}, $a+1 \leq j \leq n-1$. Suppose that $\pi_*(C)$ is not contained in H_{j_0}, for some $b-a+1 \leq j_0 \leq b-a+n-1$. Then we have

$$
2H.\pi_*(C) = (H_{b-a-1} + H_{j_0}).\pi_*(C) \geq \sum_{i=1}^{q}\mathrm{multi}_{P_i}\pi_*(C).
$$

Then, by (6.4) since $m \leq n$ and $q \geq 3n$, we have $(D - m\widetilde{H}_q).C \geq (q - 2n - m)H.\pi_*(C) \geq 0$.

Lemma 10 *Let $n \geq 2$ and $q \geq 3n$. Let D and $\widetilde{H}_i$, be as above. Then D is big and*

$$\beta_{D,\widetilde{H}_1} = \cdots = \beta_{D,\widetilde{H}_q} > 1.$$

Proof Since D is nef by Lemma 9, we have

$$h^0(X, \mathcal{O}(ND)) = \frac{D^n}{n!} \cdot N^n + O(N^{n-1})$$

by Lemma 4. It follows from (6.2) that

$$D^n = \left(q\pi^* H - n\sum_{i=1}^{q} E_i\right)^n = q^n - n^n q. \tag{6.8}$$

Therefore, D is big if $q^{n-1} > n^n$, which is satisfied when $n \geq 2$ and $q \geq 3n$.

By the Hirzebruch–Riemann–Roch theorem, adapting the arguments in [1, Lemma 4.2], we obtain

$$\chi(X; ND - m\widetilde{H}_i) = \frac{1}{n!}(ND - m\widetilde{H}_i)^n + O(N^{n-1}),$$

where $\chi(X; \cdot)$ is the Euler characteristic.

Since D and $D - b\widetilde{H}_i$ are nef for $0 \leq b \leq n$, $h^i(X, \mathcal{O}(ND - m\widetilde{H}_i) = O(N^{n-i})$ if $m \leq nN$ (see e.g. [19, Theorem 1.4.40]). Therefore,

$$h^0(X, \mathcal{O}(ND - m\widetilde{H}_i)) = \frac{(ND - m\widetilde{H}_i)^n}{n!} \cdot N^n + O(N^{n-1}), \quad \text{for } m \leq nN. \tag{6.9}$$

By (6.1) and (6.2), we can compute

$$D^k.\widetilde{H}_i^{n-k} = q^k - n^{k+1}, \quad \text{for } k \leq n-1.$$

Then

$$\begin{aligned}(ND - m\widetilde{H}_i)^n &= (q^n - n^n q)N^n + \sum_{k=0}^{n-1}\binom{n}{k}(q^k - n^{k+1})(-1)^{n-k}N^k m^{n-k} \\ &= (qN - m)^n - n(nN - m)^n - n^n(q - n)N^n. \end{aligned} \tag{6.10}$$

In particular, for $m = nN$

$$(ND - n\widetilde{H}_i)^n = (q - n)\big((q - n)^{n-1} - n^n\big)N^n \geq 0$$

since $q \geq 3n$ and $n \geq 2$. Since (6.10) is a decreasing function in m, the right hand side of (6.10) is nonnegative for $m \leq nN$. By (6.9) and (6.10), we have

$$\begin{aligned} & n! \sum_{m=0}^{nN} h^0(X, \mathcal{O}(ND - m\widetilde{H}_i)) \\ &= \sum_{m=0}^{nN} (qN-m)^n - n(nN-m)^n - n^n(q-n)N^n + O(N^n) \\ &= \big(q^{n+1} - (q-n)^{n+1} - n^{n+2} - n^{n+1}(n+1)(q-n)\big)\frac{N^{n+1}}{n+1} + O(N^n). \end{aligned}$$

Together with (6.8), it yields

$$\beta_{D,\widetilde{H}_i} \geq \beta := \frac{q^{n+1} - (q-n)^{n+1} - n^{n+2} - n^{n+1}(n+1)(q-n)}{(n+1)(q^n - n^n q)}.$$

We now show that $\beta > 1$. Let

$$\begin{aligned} f(q) :&= (\beta - 1)(n+1)(q^n - n^n q) \\ &= q^{n+1} - (q-n)^{n+1} - n^{n+2} - n^{n+1}(n+1)(q-n) - (n+1)(q^n - n^n q) \\ &= q^{n+1} - (q-n)^{n+1} - (n+1)q^n - (n^2-1)n^n(q-n) + n^{n+1}. \end{aligned} \tag{6.11}$$

We will need to show that $f(q) > 0$ if $q \geq 3n$.

$$\begin{aligned} f'(q) &= (n+1)(q^n - (q-n)^n) - (n+1)nq^{n-1} - n^{n+2} + n^n \\ &= n(n+1)(q^{n-1} + q^{n-2}(q-n) + \cdots + (q-n)^{n-1}) - (n+1)nq^{n-1} - n^{n+2} + n^n \\ &= n(n+1)(q^{n-2}(q-n) + \cdots + (q-n)^{n-1}) - n^{n+2} + n^n \\ &> (n^3 - n)(q-n)^{n-1} - n^{n+2} \geq 0 \quad \text{if} \quad q \geq 3n \quad \text{and} \quad n \geq 2. \end{aligned}$$

Therefore, it suffices to show that $f(3n) > 0$. By (6.11),

$$f(3n) = n^n\big((2n-1)\cdot 3^n - n\cdot 2^{n+1} - (2n^2-3)n\big).$$

It is easy to check that $f(3n) > 0$ for $n = 2$. We now assume that $n \geq 3$. Then

$$\begin{aligned} & (2n-1)\cdot 3^n - n\cdot 2^{n+1} - (2n^2-3)n \\ &\geq n\cdot 3^n - n\cdot 2^{n+1} + (n-1)3^n - (3n^2-3)n \\ &\geq 9n\cdot 3^{n-2} - 8n\cdot 2^{n-2} + (n-1)(3^n - 3n(n+1)) > 0. \end{aligned}$$

This show that $f(3n) > 0$ for $n \geq 3$ as well.

6.2 Proof of Theorem 4 and Theorem 5

Proof of Theorem 4 We note that since X is smooth we need to verify that $\widetilde{H}_1, \dots, \widetilde{H}_q$ are in general position in order to apply Theorem 7. Let $W = \widetilde{H}_1 \cap \widetilde{H}_2 \cap \dots \cap \widetilde{H}_i$ (after reindexing) $1 \le i \le n$. Following the proof of Proposition 4, it suffices to consider when $\pi(W) \subset \{P_1, \dots, P_q\}$. Since W is irreducible, it implies that $\pi(W)$ is some $P_j \in H_1 \cap H_2 \cap \dots \cap H_i$. Since $H_1, \dots, H_q$ are hyperplanes in general position, they intersect transversally. Thus, the codimension of W is i.

Since $n \ge 2$ and $q \ge 3n$, it follows from Lemma 10 that D is big and $\beta := \beta_{D,\widetilde{H}_1} = \dots = \beta_{D,\widetilde{H}_q} > 1$. Theorem 7 with $\epsilon = \frac{1}{2}(\beta - 1)$ implies that there exists a proper Zariski closed set $Z \subset X$ independent of k and S such that

$$\beta \cdot \sum_{i=1}^{q} m_{\widetilde{H}_i,S}(x) \le (1+\epsilon)h_D(x)$$

for all but finitely many $x \in X(k)\backslash Z$. Let R be a set of (D, S)-integral points. Then

$$\sum_{i=1}^{q} m_{\widetilde{H}_i,S}(x) = h_D(x) + O(1),$$

where the constant depends only on R. Hence,

$$\frac{1}{2}(\beta - 1)h_D(x) = (\beta - 1 - \epsilon)h_D(x) \le O(1) \tag{6.12}$$

for all but finitely many $x \in R$ outside Z. Since D is big, for a given ample divisor A, there exists a positive real constant c and a proper Zariski closed set Z' of X, depending only on A and D such that $h_A(x) \le c h_D(x) + O(1)$ for all $x \in X(\bar{k})$ outside of Z' (see [33, Proposition 10.11]). Therefore, (6.12) implies that there are only finitely many $x \in R$ outside $Z \cup Z'$.

Proof of Theorem 5 We denote by $\mathbf{x}$ the point $[x_0 : \dots : x_n] \in \mathbb{P}^n(k)$. Up to enlarging the set S, we can suppose that $x_0, \dots, x_n$ are S-integers and that the ring $\mathcal{O}_S$ is a unique factorization domain. Let $H_i = [F_i = 0]$ for $1 \le i \le q$. For each index $j \in \mathbb{Z}/q\mathbb{Z}$, let P_j be the intersection point $\cap_{\ell=0}^{n-1} H_{j+\ell}$. By Definition 6, the identity of ideals in $\mathcal{O}_S$

$$F_i(x_0, \dots, x_n) \cdot (x_0, \dots, x_n) = \prod_{j=i-n+1}^{i} (F_j(x_0, \dots, x_n), \dots, F_{j+n-1}(x_0, \dots, x_n)) \tag{6.13}$$

implies that for $v \notin S$

$$\lambda_{H_i,v}(\mathbf{x}) = \sum_{j=i-n+1}^{i} \lambda_{P_j,v}(\mathbf{x}),$$

up to a M_k constant. On the other hand, let $\pi : X \to \mathbb{P}^n$ be the blow-up over the points $P_1, \ldots, P_q$ and let $\widetilde{H}_i \subset X$ be the corresponding strict transform of H_i. It follows from (6.1) that

$$\lambda_{H_i,v}(\pi(Q)) = \lambda_{\widetilde{H}_i,v}(Q) + \sum_{j=i-n+1}^{i} \lambda_{P_i,v}(\pi(Q))$$

up to a M_k constant for $Q \in X$, $v \notin S$. If $Q \notin \cup_{i=1}^{q} \operatorname{Supp}(E_i)$, then $Q = \pi^{-1}(\mathbf{x})$ for some $\mathbf{x} \neq P_i$, $1 \leq i \leq q$. Therefore, for $\mathbf{x} := [x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \backslash \{P_1, \ldots, P_q\}$ satisfying (6.13) we have

$$\lambda_{\widetilde{H}_i,v}(\pi^{-1}(\mathbf{x})) = 0 \quad \text{up to a} \quad M_k \quad \text{constant for} \quad v \notin S. \tag{6.14}$$

By Theorem 4, there exists a Zariski closed subset W of X such that the set of points $\pi^{-1}(\mathbf{x})$ satisfying (6.14) are contained in W. Therefore, the points $\mathbf{x} \in \mathbb{P}^n(k) \backslash \{P_1, \ldots, P_q\}$ satisfying the identity (6.13) are contained in the Zariski closure of $\pi(W)$.

7 Degeneracy of holomorphic maps

In this section, we give the analytic versions of the arithmetic statements obtained in the previous sections. This imply several results on Brody hyperbolicity.

Theorem 12 *Let $n \geq 2$, $F_1, \ldots, F_r, G \in \mathbb{C}[X_1, \ldots, X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, \ldots, h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds*

(i) *$r \geq 2n + 1$ and $\frac{G(h_1,\ldots,h_n)}{F(h_1,\ldots,h_n)}$ is holomorphic, for $i = 1, \ldots, r$; or*

(ii) *$r \geq n + 2$ and $\frac{G(h_1,\ldots,h_n)}{\prod_{i=1}^{r} F_i(h_1,\ldots,h_n)}$ is holomorphic.*

Then $h_1, \ldots, h_n$ are algebraically dependent.

This can be seen as a generalization of Borel's Theorem [4] stating that nowhere vanishing entire functions $h_1, \ldots, h_{n+1}$ satisfying the identity $h_1 + \cdots + h_{n+1} = 1$ are dependent. Indeed, we have the following corollary.

Corollary 3 *Let $h_1, \ldots, h_n$ be holomorphic functions on $\mathbb{C}$ such that $\frac{1}{(h_1 \ldots h_n).(1-\sum_{i=1}^{n} h_i)}$ is holomorphic. Then $h_1, \ldots, h_n$ are algebraically dependent.*

We recall the following definition.

Definition 5 We say that a complex variety X is Brody pseudo-hyperbolic if there exists a proper closed subset $Z \subset X$ such that any (non-constant) entire curve $f : \mathbb{C} \to X$ is contained in Z i.e. $f(\mathbb{C}) \subset Z$.

Then we can rephrase in the analytic setting the main theorems of this paper.

Theorem 13 *Let $n \geq 2$, $r \geq 2n+1$ and $D_0, D_1, \ldots, D_r$ be hypersurfaces in general position on $\mathbb{P}^n(\mathbb{C})$. Let $\pi : X \to \mathbb{P}^n$ be the blowup long the union of subschemes $D_i \cap D_0$, $1 \leq i \leq r$, and let $\widetilde{D}_i$ be the strict transform of D_i. Let $D = \widetilde{D}_1 + \cdots + \widetilde{D}_r$. Then $X \backslash D$ is Brody pseudo-hyperbolic.*

Theorem 14 *Let $n \geq 2$ and $H_1, \ldots, H_{2n}$ be $2n$ hyperplanes in general position on $\mathbb{P}^n(\mathbb{C})$. Choose $n+1$ points P_i, $1 \leq i \leq n+1$ such that $P_i \in H_i$, $1 \leq i \leq n+1$, and $P_i \notin H_j$ if $i \neq j$ for $1 \leq j \leq 2n$. Let $\pi : X \to \mathbb{P}^n$ be the blowup of the $n+1$ points P_i, $1 \leq i \leq n+1$, and let $D \subset X$ be the strict transform of $H_1 + \cdots + H_{2n}$. Then $X \backslash D$ is Brody pseudo-hyperbolic.*

Theorem 15 *Let $n \geq 2$, $q \geq 3n$ be an integer; for every index $i \in \mathbb{Z}/q\mathbb{Z}$, let H_i be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Suppose that H_i's are in general position. Let for each index $i \in \mathbb{Z}/q\mathbb{Z}$, P_i be the intersection point $\cap_{j=0}^{n-1} H_{i+j}$. Let $\pi : X \to \mathbb{P}^n$ be the blow-up over the points $P_1, \ldots, P_q$ and let $\widetilde{H}_i \subset X$ be the corresponding strict transform of H_i and let $D = \widetilde{H}_1 + \cdots + \widetilde{H}_q$. Then $X \backslash D$ is Brody pseudo-hyperbolic.*

The proofs of the above statements are the same as the arithmetic ones replacing Theorem 7 by its analytic analogue. Its generalization is obtained using Vojta's version of Schmidt's subspace theorem [32], which gives a better control on the exceptional sets.

Theorem 16 *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ with the corresponding Weil functions $\lambda_{H_1}, \ldots, \lambda_{H_q}$. Then there exists a finite union of hyperplanes Z, depending only on $H_1, \ldots, H_q$, such that for any $\epsilon > 0$, and any (non-constant) entire curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset Z$*

$$\int_0^{2\pi} \max_I \sum_{i \in I} \lambda_{H_i}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq_{\text{exc}} (n+1+\epsilon) T_f(r)$$

holds, where $\leq_{\text{exc}}$ means that the inequality holds for all $r \in \mathbb{R}^+$ except a set of finite Lebesgue measure., where the maximum is taken over subsets $\{1, \ldots, q\}$ such that the linear forms defining H_i for $i \in I$ are linearly independent.

By carefully tracing the exceptional sets with Theorem 16, the general analytic Theorem of Ru and Vojta can be stated as follows.

Theorem 17 [27, General Theorem (Analytic Part)] *Let X be a complex projective variety of dimension n and let $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let $\mathcal{L}$ be a big line bundle. Let $f : \mathbb{C} \to X$ be a Zariski dense entire curve. Then, for every $\varepsilon > 0$, there exists a proper Zariski-closed subset $Z \subset X$, such that for any (non-constant) entire curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset Z$,*

$$\sum_{j=1}^{q} \beta_{\mathcal{L}, D_j} m_f(r, D_j) \leq_{\text{exc}} (1+\varepsilon) T_{\mathcal{L}, f}(r)$$

holds, where $\leq_{\text{exc}}$ means that the inequality holds for all $r \in \mathbb{R}^+$ except a set of finite Lebesgue measure.

8 Function fields

In this section we give the analogue statements over function fields of the theorems obtained in the previous sections. For this section we let κ be an algebraically closed field of characteristic zero. Let $\mathcal{C}$ be a non-singular projective curve defined over κ and let $K = \kappa(\mathcal{C})$ denote its function field. We refer to [26, Section 7.2] for the basic definitions of heights and proximity functions in the function field setting. We recall the definition of algebraic hyperbolicity.

Definition 6 Let (X, D) be a pair of a non-singular projective variety X defined over κ and a normal crossing divisor D on X. We say that (X, D) is *algebraically hyperbolic* if there exists an ample line bundle $\mathcal{L}$ on X and a positive constant α such that, for every non-singular projective curve $\mathcal{C}$ and every morphism $\varphi : \mathcal{C} \to X$ the following holds:

$$\deg \varphi^*\mathcal{L} \le \alpha \cdot \left(2g(\mathcal{C}) - 2 + N_\varphi^{[1]}(D)\right), \tag{8.1}$$

where $N_\varphi^{[1]}(D)$ is the cardinality of the support of $\varphi^*(D)$.

We say that (X, D) is *pseudo algebraically hyperbolic* if there exists a proper closed subvariety Z of X such that (8.1) holds for every morphism $\varphi : \mathcal{C} \to X$ such that $\varphi(\mathcal{C})$ is not contained in Z.

We can now rephrase Theorems 2, 3 and 5.

Theorem 18 *Let $n \ge 2$, $r \ge 2n+1$ and $D_0, D_1, \ldots, D_r$ be hypersurfaces in general position on $\mathbb{P}^n$ defined over κ. Let $\pi : X \to \mathbb{P}^n$ be the blowup long the union of subschemes $D_i \cap D_0$, $1 \le i \le r$, and let $\widetilde{D}_i$ be the strict transform of D_i. Let $D = \widetilde{D}_1 + \cdots + \widetilde{D}_r$. Then $X\backslash D$ is algebraically pseudo-hyperbolic.*

Theorem 19 *Let $n \ge 2$ and $H_1, \ldots, H_{2n}$ be $2n$ hyperplanes in general position on $\mathbb{P}^n$ defined over κ. Choose $n+1$ points P_i, $1 \le i \le n+1$ such that $P_i \in H_i$, $1 \le i \le n+1$, and $P_i \notin H_j$ if $i \ne j$ for $1 \le j \le 2n$. Let $\pi : X \to \mathbb{P}^n$ be the blowup of the $n+1$ points P_i, $1 \le i \le n+1$, and let $D \subset X$ be the strict transform of $H_1 + \cdots + H_{2n}$. Then $X\backslash D$ is algebraically pseudo-hyperbolic.*

Theorem 20 *Let $n \ge 2$, $q \ge 3n$ be an integer; for every index $i \in \mathbb{Z}/q\mathbb{Z}$, let H_i be a hyperplane in $\mathbb{P}^n$ defined over k. Suppose that H_i's are in general position. Let for each index $i \in \mathbb{Z}/q\mathbb{Z}$, P_i be the intersection point $\cap_{j=0}^{n-1} H_{i+j}$. Let $\pi : X \to \mathbb{P}^n$ be the blow-up over the points $P_1, \ldots, P_q$ and let $\widetilde{H}_i \subset X$ be the corresponding strict transform of H_i and let $D = \widetilde{H}_1 + \cdots + \widetilde{H}_q$. Then $X\backslash D$ is algebraically pseudo-hyperbolic.*

We remark that, even if we stated the results in the so-called *split case*, our proofs carry over almost verbatim to the non-split case as well.

As in the analytic setting, the proofs of the above statements follow the same lines of the proof of our arithmetic results and the same strategy as in our previous paper [26] with two modifications. On one hand we can use the results in Sect. 7 instead of [20, Theorem 8.3 B] for the case in which $2g(\mathcal{C}) - 2 + N_\varphi^{[1]}(D) \le 0$. On the other hand we replace the use of Theorem 7 with the following analogue that uses a version of the Schmidt subspace theorem over function fields obtained in [34, Theorem 1]. In particular this gives a better control on the exceptional set.

Theorem 21 *Let $X \subset \mathbb{P}^m$ be a projective variety over κ of dimension n, let $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X, and let $\mathcal{L}$ be a big line sheaf. Then for any $\epsilon > 0$, there exist constants c_1 and c_2, independent of the curve $\mathcal{C}$ and the set S, and a finite collection of hypersurfaces $\mathcal{Z}$ (over κ) in $\mathbb{P}^m$ of degree at most c_2 such that for any map $x = [x_0 : \cdots : x_m] : \mathcal{C} \to X$, where $x_i \in K$, outside the augmented base locus of $\mathcal{L}$ we have either*

$$\sum_{i=1}^{q} \beta_{\mathcal{L},D_i} m_{D_i,S}(x) \leq (1+\epsilon) h_{\mathcal{L}}(x) + c_1 \max\{1, 2g(\mathcal{C}) - 2 + |S|\},$$

or the image of x is contained in $\mathcal{Z}$.

Proof The proof is similar to the first part of the proof of [26, Theorem 7.6]. We will follow its argument and notation and only indicate the modification. Let $\epsilon > 0$ be given. Since $\mathcal{L}$ is a big line sheaf, there is a constant c such that $\sum_{i=1}^{q} h_{D_i}(x) \leq c h_{\mathcal{L}}(x)$ for all $x \in X(K)$ outside the augmented base locus B of $\mathcal{L}$. By the properties of the local heights, together with the fact that $m_{D_i,S} \leq h_{D_i} + O(1)$, we can choose $\beta_i \in \mathbb{Q}$ for all i such that

$$\sum_{i=1}^{q} (\beta_{\mathcal{L},D_i} - \beta_i) m_{D_i,S}(x) \leq \frac{\epsilon}{2} h_{\mathcal{L}}(x)$$

for all $x \in X \backslash B(K)$. Therefore, we can assume that $\beta_{\mathcal{L},D_i} = \beta_i \in \mathbb{Q}$ for all i and also that $\beta_i \neq 0$ for each i. From now on we will assume that the point $x \in X(K)$ does *not* lie on B.

Choose positive integers N and b such that

$$\left(1 + \frac{n}{b}\right) \max_{1 \leq i \leq q} \frac{\beta_i N h^0(X, \mathcal{L}^N)}{\sum_{m \geq 1} h^0(X, \mathcal{L}^N(-mD_i))} < 1 + \epsilon . \tag{8.2}$$

Then, using [26, Theorem 7.5] with the same notation, we obtain

$$\begin{aligned} &\frac{b}{b+n} \left(\min_{1 \leq i \leq q} \sum_{m \geq 1} \frac{h^0(\mathcal{L}^N(-mD_i))}{\beta_i} \right) \sum_{i=1}^{q} \beta_i \lambda_{D_i,\mathfrak{p}}(x) \\ &\quad \leq \max_{1 \leq i \leq T_1} \lambda_{\mathcal{B}_i,\mathfrak{p}}(x) + O(1) = \max_{1 \leq i \leq T_1} \sum_{j \in J_i} \lambda_{s_j,\mathfrak{p}}(x) + O(1). \end{aligned} \tag{8.3}$$

Let $M = h^0(X, \mathcal{L}^N)$, let the set $\{\phi_1, \ldots, \phi_M\}$ be a basis of the vector space $H^0(X, \mathcal{L}^N)$, and let

$$\Phi = [\phi_1, \ldots, \phi_M] : X \dashrightarrow \mathbb{P}^{M-1}(\kappa)$$

be the corresponding rational map. By [34, Theorem 1], there exists a constant c_1' and a finite collection of linear subspaces $\mathcal{R}$ over κ such that, whenever $\Phi \circ x$ is not in $\mathcal{R}$, we have the following

$$\sum_{\mathfrak{p}\in S}\max_{J}\sum_{j\in J}\lambda_{s_j,\mathfrak{p}}(x)\le M\,h_{\mathcal{L}^N}(x)+c_1'(2g-2+|S|), \tag{8.4}$$

here the maximum is taken over all subsets J of $\{1,\dots,T_2\}$ for which the sections s_j, $j\in J$, are linearly independent (with the same notation as in the proof of [26, Theorem 7.1]). We first consider when $\phi_1,\dots,\phi_M$ are linearly independent over κ. Combining (8.3) and (8.4) gives

$$\sum_{i=1}^{q}\beta_i m_{D_i,S}(x)\le\left(1+\frac{n}{b}\right)\max_{1\le i\le q}\frac{\beta_i}{\sum_{m\ge 1}h^0(\mathcal{L}^N(-mD_i))}\,M\,h_{\mathcal{L}^N}(x)+c_1'(2g-2+|S|)+O(1).$$

Using (8.2) and the fact that $h_{\mathcal{L}^N}(x)=Nh_{\mathcal{L}}(x)$, we have

$$\sum_{i=1}^{q}\beta_i m_{D_i,S}(x)\le(1+\epsilon)\,h_{\mathcal{L}}(x)+c_1'(2g-2+|S|)+O(1),$$

which implies the first case of the Theorem.

To conclude we note that, if $\Phi\circ x$ is in one of the linear subspace of $\mathcal{R}$ over κ in $\mathbb{P}^{M-1}$, then $a_1\phi_1(x)+\dots+a_M\phi_M(x)=0$, where $H=\{a_1z_1+\dots+a_Mz_M=0\}$ is one of the hyperplanes (over κ) in $\mathbb{P}^{M-1}$ coming from $\mathcal{R}$.

On the other hand, since $\phi_1,\dots,\phi_M$ is a basis of $H^0(X,\mathcal{L}^N)$, it follows that $\Phi(X)$ is not contained in H, hence $x(\mathcal{C})$ is contained in is the hypersurface coming from $a_1\phi_1+\dots+a_M\phi_M=0$ in $\mathbb{P}^m$ (as $X\subset\mathbb{P}^m$) whose degree is bounded independently of $\mathcal{C}$ and x as wanted. Moreover, since $\mathcal{R}$ is a finite collection of linear subspaces over κ in $\mathbb{P}^{M-1}$, there are only finitely many H and hence the number of hypersurfaces obtained above is also finite.

Acknowledgements We thank Pietro Corvaja and Umberto Zannier for several discussions. We thank Pascal Autissier and Min Ru for several comments on a preliminary version of our paper. We thank the anonymous referee for their comments and suggestions that greatly improved the paper.

Funding Open access funding provided by Universitá degli Studi Roma Tre within the CRUI-CARE Agreement. ER was supported by Institut Universitaire de France and the ANR project "FOLIAGE", ANR-16-CE40-0008. AT was partially supported by PRIN "Advances in Moduli Theory and Birational Classification" and is a member of GNSAGA-INdAM. JW was supported in part by Taiwan's MoST grants 108-2115-M-001-001-MY2 and 110-2115-M-001-009-MY3.

Data availability Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Autissier, P.: Géométries, points entiers et courbes entières. Ann. Sci. Éc. Norm. Supér. (4) **42**(2), 221–239 (2009)
2. Barroero, F., Capuano, L., Turchet, A.: Geometric divisibility sequences on abelian and semiabelian varieties. https://arxiv.org/abs/2205.05562 (2022) **(preprint)**
3. Bogomolov, F., Tschinkel, Y.: Special elliptic fibrations. In: The Fano Conference, pp. 223–234. Univ. Torino, Turin (2004)
4. Borel, E.: Sur les zéros des fonctions entières. Acta Math. **20**(1), 357–396 (1897)
5. Bugeaud, Y., Corvaja, P., Zannier, U.: An upper bound for the G.C.D. of $a^n - 1$ and $b^n - 1$. Math. Z. **243**(1), 79–84 (2003)
6. Campana, F.: Orbifolds, special varieties and classification theory. Ann. Inst. Fourier (Grenoble) **54**(3), 499–630 (2004)
7. Campana, F., Păun, M.: Variétés faiblement spéciales à courbes entières dégénérées. Compos. Math. **143**(1), 95–111 (2007)
8. Capuano, L., Turchet, A.: Lang–Vojta conjecture over function fields for surfaces dominating $\mathbb{G}_m^2$. Eur. J. Math. **8**(2), 573–610 (2022)
9. Corvaja, P., Zannier, U.: On integral points on surfaces. Ann. Math. (2) **160**(2), 705–726 (2004)
10. Corvaja, P., Zannier, U.: Integral points, divisibility between values of polynomials and entire curves on surfaces. Adv. Math. **225**(2), 1095–1118 (2010)
11. Corvaja, P., Zannier, U.: Algebraic hyperbolicity of ramified covers of $\mathbb{G}_m^2$ (and integral points on affine subsets of $\mathbb{P}_2$). J. Differ. Geom. **93**(3), 355–377 (2013)
12. Guo, J., Sun, C.-L., Wang, J.T.-Y.: Some cases of Vojta's conjectures related to algebraic tori over function fields. https://arxiv.org/abs/2106.15881 (2021) **(preprint)**
13. Guo, J., Wang, J.T.-Y.: Asymptotic GCD and divisible sequences for entire functions. Trans. Am. Math. Soc. **371**(9), 6241–6256 (2019)
14. Harris, J., Tschinkel, Y.: Rational points on quartics. Duke Math. J. **104**(3), 477–500 (2000)
15. Hassett, B., Tschinkel, Y.: Density of integral points on algebraic varieties. In: Rational Points on Algebraic Varieties, Progr. Math., vol. 199, pp. 169–197. Birkhäuser, Basel (2001)
16. Hindry, M., Silverman, J.H.: Diophantine Geometry. Graduate Texts in Mathematics, vol. 201. Springer, New York (2000)
17. Kovács, S.J.: Rational Singularities (2020)
18. Lang, S.: Fundamentals of Diophantine Geometry. Springer, New York (1983)
19. Lazarsfeld, R.: Positivity in Algebraic Geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 48. Springer, Berlin (2004)
20. Levin, A.: Generalizations of Siegel's and Picard's theorems. Ann. Math. (2) **170**(2), 609–655 (2009)
21. Levin, A.: On the Schmidt subspace theorem for algebraic points. Duke Math. J. **163**(15), 2841–2885 (2014)
22. Levin, A.: Greatest common divisors and Vojta's conjecture for blowups of algebraic tori. Invent. Math. **215**(2), 493–533 (2019)
23. Levin, A., Wang, J.T.-Y.: Greatest common divisors of analytic functions and Nevanlinna theory on algebraic tori. J. Reine Angew. Math. **767**, 77–107 (2020)
24. Matsumura, H.: Commutative Ring Theory, Cambridge Studies in Advanced Mathematics, vol. 8. Cambridge University Press, Cambridge (1986). Translated from the Japanese by M. Reid
25. Pasten, H., Wang, J.T.-Y.: GCD bounds for analytic functions. Int. Math. Res. Not. IMRN **2017**(1), 47–95 (2017)
26. Rousseau, E., Turchet, A., Wang, J.T.-Y.: Nonspecial varieties and generalised Lang–Vojta conjectures. Forum Math. Sigma **9**, 1–29 (2021)

27. Ru, M., Vojta, P.: A birational nevanlinna constant and its consequences. Am. J. Math. **142**(3), 957–991 (2020)
28. Silverman, J.H.: Arithmetic distance functions and height functions in Diophantine geometry. Math. Ann. **279**(2), 193–216 (1987)
29. Silverman, J.H.: Generalized greatest common divisors, divisibility sequences, and Vojta's conjecture for blowups. Monatsh. Math. **145**(4), 333–350 (2005)
30. Turchet, A.: Fibered threefolds and Lang–Vojta's conjecture over function fields. Trans. Am. Math. Soc. **369**(12), 8537–8558 (2017)
31. Vojta, P.: Diophantine Approximations and Value Distribution Theory. Lecture Notes in Mathematics, vol. 1239. Springer, Berlin (1987)
32. Vojta, P.: A refinement of Schmidt's subspace theorem. Am. J. Math. **111**(3), 489–518 (1989)
33. Vojta, P.: Diophantine approximation and Nevanlinna theory. In: Arithmetic Geometry, Lecture Notes in Math., vol. 2009, pp. 111–224. Springer, Berlin (2011)
34. Wang, J.T.-Y.: An effective Schmidt's subspace theorem over function fields. Math. Z. **246**(4), 811–844 (2004)
35. Wang, J.T.-Y., Yasufuku, Y.: Greatest common divisors of integral points of numerically equivalent divisors. Algebra Number Theory **15**(1), 287–305 (2021)
36. Yasufuku, Y.: Vojta's conjecture on blowups of $\mathbb{P}^n$, greatest common divisors, and the *abc* conjecture. Monatsh. Math. **163**(2), 237–247 (2011)
37. Yasufuku, Y.: Integral points and Vojta's conjecture on rational surfaces. Trans. Am. Math. Soc. **364**(2), 767–784 (2012)