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The asymptotics of the area-preserving mean curvature and the Mullins–Sekerka flow in two dimensions

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Abstract

We provide the first general result for the asymptotics of the area preserving mean curvature flow in two dimensions showing that flat flow solutions, starting from any bounded set of finite perimeter, converge with exponential rate to a finite union of equally sized disjoint disks. A similar result is established also for the periodic two-phase Mullins–Sekerka flow.

1 Introduction

In this paper we address the long-time behaviour of two physically relevant area preserving nonlocal geometric flows in the plane: the area-preserving mean curvature and the Mullins–Sekerka flow.

We start by recalling that a smooth flow of sets $(E_t)_{t\in[0,T)} \subset \mathbb{R}^2$, for some $T > 0$, is a solution to the area preserving mean curvature flow if it satisfies

$$V_t = -\kappa_{E_t} + \overline{\kappa}_{E_t} \quad \text{on } \partial E_t, \tag{1.1}$$

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where V_t denotes the normal velocity, κ_{E_t} the curvature and $\overline{\kappa}_{E_t} := \fint_{\partial E_t} \kappa_{E_t}\, d\mathcal{H}^1$ the integral average of the curvature of the evolving boundary ∂E_t. Such a geometric flow has been proposed in the physical literature as a model for coarsening phenomena. For example, one can consider systems that, after a first relaxation time, can be described by two subdomains of nearly pure phases far from equilibrium, evolving in a way to decrease the total interfacial length between the phases while keeping their area constant (for the physical background see [8, 40, 48, 49]). An important feature of the flow is that it can be regarded as a gradient flow of the perimeter with respect to a suitable (formal) L^2-type Riemannian structure.

The second geometric evolution we consider, the two-phase Mullins–Sekerka flow in the flat torus $\mathbb{T}^2$, is governed by the law

$$\begin{cases} V_t = [\partial_{\nu_t} u_t] & \text{on } \partial E_t, \\ -\Delta u_t = 0 & \text{in } \mathbb{T}^2 \backslash \partial E_t, \\ u_t = \kappa_{E_t} & \text{on } \partial E_t, \end{cases} \tag{1.2}$$

where ν_t denotes the external normal to ∂E_t, $[\partial_{\nu_t} u_t]$ denotes the jump of the normal derivative of u_t at ∂E_t, i.e., $[\partial_{\nu_t} u_t] := \partial_{\nu_t} u_t^+ - \partial_{\nu_t} u_t^-$, with u_t^+ and u_t^- denoting the restrictions of u_t to $\mathbb{T}^2 \backslash E_t$ and E_t respectively, and κ_{E_t} is as before the curvature of the evolving boundary. Let us notice that the choice of the flat torus $\mathbb{T}^2$ instead of a bounded domain Ω is made to avoid in the first place boundary effects. The Mullins–Sekerka flow is a nonlocal geometric flow arising from physics. It can be seen as a quasistatic variant of the Stefan problem (see [33]) and it was originally proposed as an isotropic model for solidification and liquefaction phenomena when the specific heat is negligible, see [42]. Moreover, it arises as a singular limit of the Cahn–Hilliard equation, see [3, 44]. Common features with (1.1) are the area preserving character and the gradient flow structure (this time with respect to a suitable H^{-1}-Riemannian structure).

It is well-known that, in general, smooth solutions of (1.1) may develop singularities in finite time, such as disappearance and coalescence of components, pinch-offs and curvature blow-up, even in two dimensions (see for instance [18, 36, 37]). The same can be expected for the flow (1.2). The possible singular behaviour of (1.1) and (1.2) is even wilder than that of the unconstrained mean curvature flow, due to their nonlocal character and the subsequent lack of a comparison principle. Thus, for a well defined global-in-time evolution one has to introduce suitable notion of weak solution which is capable of handling singularities, changes in topology and, possibly, rough initial data. This is a well-established feature of curvature flows, and for several geometric motions, different definitions of weak solutions have been introduced in the literature.

Due to the lack of a comparison principle and based on the underlying gradient flow structure, a natural choice for (1.1) and (1.2) is the minimizing movement approach proposed for the mean curvature flow independently by Almgren, Taylor and Wang [4] and by Luckhaus and Sturzenhecker [34], and adapted to the volume constrained case in [41]. Note that Luckhaus and Sturzenhecker [34] introduce a similar variational scheme for (1.2) as well, see also [7, 45] where the same scheme is further analyzed. Recently, the first author and Niinikoski [29] proved the consistency of the flat flow solutions for the volume preserving mean curvature flow with the classical solutions (see also [32] for a weak–strong uniqueness result). We recall that the minimizing movement method is based on implicit time-discretization and recursive minimization of suitable incremental problems. The limiting time-continuous evolutions constructed in this way are usually referred to as *flat flows*. We refer to Sects. 3 and 4 for the precise definition of flat flow solution of (1.1) and (1.2), respectively.

Once a global-in-time weak solution has been constructed, it is a natural problem to investigate its asymptotics. The focus of the paper is the long-time behaviour of flat flows in two dimensions. Previous results on the long-time convergence of volume preserving flows are mostly confined to the case of smooth solutions starting from specific classes of initial regular sets, see for instance [19, 27, 43] for the volume preserving mean curvature flow and [1, 10, 20, 25] for the Mullins–Sekerka flow. For less general initial data, the long time behaviour of the volume preserving mean curvature flow starting from convex and star-shaped sets (see [6, 30]) has been characterized only up to (possibly diverging in the case of [6]) translations. Finally, concerning flat flow solutions the most general result is due to [28] where the asymptotic convergence to finitely many disjoint balls is proven in two and three dimensions for arbitrary bounded initial sets of finite perimeter, but only up to (possibly diverging) translations and without a convergence rate.

In our main result we are able to rule out translations and we provide in two dimensions the first full convergence result for the asymptotics of the area preserving mean curvature and the Mullins–Sekerka flow. We show that every flat flow solutions of (1.1) starting from *any* set of finite perimeter asymptotically converge, with *exponential rate*, to a finite disjoint union of (possibly tangent) equally sized discs. Under the additional assumption that the perimeter of the initial set is smaller than 2, we establish a similar result also for (1.2). Note that such an additional condition is assumed for simplicity to rule out lamellae as possible limiting sets (see Sect. 4 for further details). We refer to the next section for the precise statements.

Let us finally mention that the analysis of this paper extends in two dimensions the results proven in [39] (see also [14] for related results in the flat torus) for the discrete minimizing movements of the volume preserving mean curvature flow to the time-continuous limiting evolutions.

1.1 Statement of the main results

In the previous work [39] three of the authors prove that in all dimensions the discrete approximate volume preserving mean curvature flow converges exponentially fast to a disjoint union of balls with equal size. This is the optimal convergence result but it leaves open the question of the convergence of the limiting flat flow. On the other hand, in [28] the first author and Niinikoski prove that the limiting flat flow converges in low dimensions $\mathbb{R}^2$ and $\mathbb{R}^3$ to a disjoint union of balls, up to possible translations of the components. Again this result does not prove the full convergence nor does it provide any rate of convergence. In both papers it was observed that a key technical issue is to prove a quantitative version of the Alexandrov theorem, which in the classical form states that the only compact smooth hypersurfaces with constant mean curvature are union of spheres. In this paper we develop this idea further and observe that we may prove a geometric inequality, very much related to the quantitative Alexandrov theorem, which implies the full convergence of the flow and also gives the exponential rate of convergence.

There has been a lot of recent research on generalizations and quantifications of the Alexandrov theorem. We refer to [11] for an overview of this challenging problem, and mention the works [15–17] on the characterization of critical sets of the isoperimetric problem and [12, 13, 31] on quantification of the Alexandrov theorem.

We state our quantitative version in a form that is suitable for the study of Eq. (1.1). We denote the length of the boundary or more generally the perimeter of a set E by $P(E)$ and by $|E|$ its area. We also denote by $P_d = 2\sqrt{\pi m d}$ the perimeter of the disjoint union of d equally sized disks with total area m. Our first result reads as follows.

Theorem 1.1 *Let $m, M > 0$ and let $E \subset \mathbb{R}^2$ be a bounded open set of class C^2, with $|E| = m$ and $P(E) \leq M$. Then there exists a constant $C(m, M) > 0$ such that*

$$\min_{d\in\mathbb{N}} |P(E) - P_d| \leq C\|\kappa_E - \overline{\kappa}_E\|^2_{L^2(\partial E)}.$$

Moreover, if $\delta_0 > 0$ and $d \in \mathbb{N}$, are such that $P_d \leq P(E) \leq P_{d+1} - \delta_0$, then it holds

$$P(E) - P_d \leq C_0\|\kappa_E - \overline{\kappa}_E\|^2_{L^2(\partial E)}, \tag{1.3}$$

with $C_0 = C_0(m, M, \delta_0)$.

The novelty of the above result is that on the right-hand-side we have quadratic dependence on the curvature which is optimal (see Remark 2.2). One may compare this result to the quantification of the Willmore energy [46] or to the optimal quantitative isoperimetric inequality [24], which both have similar scaling. The inequalities in Theorem 1.1 are geometric and do not measure how close the set E is to the union of disks. In the planar case the closeness of E to the union of disks is proven in [23] (see also Proposition 2.1 in Section 2). The above result is proven in the planar case but it could be true also in higher dimensions.

As we already mentioned, the motivation for the geometric inequality in Theorem 1.1 is the proof of the asymptotic convergence of the area-preserving mean curvature flow equation (1.1).

Theorem 1.2 *Let $\{E(t)\}_{t\geq 0}$ be an area-preserving flat flow for (1.1) (see Definition 3.1) starting from a bounded set of finite perimeter $E(0) \subset \mathbb{R}^2$. Then, there exist $d \in \mathbb{N}$ disjoint open disks in the plane $D_r(x_1), \dots, D_r(x_d)$, with $\pi r^2 d = |E(0)|$, and there exists a constant $C > 1$ such that, setting $E_\infty = \bigcup_{i=1}^d D_r(x_i)$, it holds*

$$\sup_{x\in E(t)\Delta E_\infty} dist(x, \partial E_\infty) + |P(E(t)) - P(E_\infty)| \leq Ce^{-\frac{t}{C}} \tag{1.4}$$

for all $t \geq 0$.

The above theorem gives the full characterization and quantitative speed of the convergence of the asymptotics of Eq. (1.1). We expect the result to be sharp, in the sense that the flow may, indeed, converge to a union of tangent disks. In [23, Theorem 1.4] the authors consider the case when the initial set is a union of two ellipses and show that Eq. (1.1) is well defined and smooth for all times and converges to two tangent disks. In particular, we may not improve the Hausdroff convergence in Theorem 1.2 to C^1-convergence of the sets. The exponential convergence rate is optimal but we note that the flow may in fact converge to the limiting disks also in finite time. This is the case when we consider as an initial set a union of two disks D_1, D_2, which are far apart and D_2 is much smaller than D_1. Then along the flow the larger disk grows and the smaller one shrinks until it vanishes completely and the flow reaches its equilibrium state in finite time. The same phenomenon occurs when D_2 is only slightly smaller than D_1 but the time to reach the equilibrium state tends to infinity when the size of D_2 gets closer to the size of D_1. This shows that we cannot bound the constant C by a universal constant, but it may depend on the initial set in a rather complicated way.

We note that our method can be also used to study asymptotic behavior of other geometric flows, and to emphasize this we also address the asymptotics of the two-phase Mullins–Sekerka flow (1.2). To avoid boundary effects we consider periodic conditions and set the problem in the flat torus $\mathbb{T}^2$ and, as a further simplification, we consider initial configurations with perimeter smaller than that of the single lammella (alternatively, we can think that the size of the torus is big enough compared to the perimeter of the initial set).

The main result is the following. We denote the perimeter of a set E in the flat torus by $P_{\mathbb{T}^2}(E)$.

Theorem 1.3 *Let $\{E(t)\}_{t\geq 0}$ be a flat flow solution to the Mullins–Sekerka flow (1.2) in the flat torus $\mathbb{T}^2$ starting from a set of finite perimeter $E(0) \subset \mathbb{T}^2$, with $P_{\mathbb{T}^2}(E) < 2$. Then, there exist $d \in \mathbb{N}$ disjoint open disks $D_r(x_1), \ldots, D_r(x_d)$, with $\pi r^2 d = |E(0)|$, and there exists a constant $C > 1$ such that it holds*

$$|E(t)\Delta E_\infty| + |P(E(t)) - P(E_\infty)| \leq Ce^{-\frac{t}{C}}$$

for all $t \geq 0$, where E_∞ either coincides with $\bigcup_{i=1}^d D_r(x_i)$ or with its complement in $\mathbb{T}^2$.

The proof of Theorem 1.3 is similar to that of the previous theorem. We use Theorem 1.1 and a result by Schätzle [47] to obtain a functional inequality (see Corollary 4.3), which is in the spirit of the quantitative Alexandrov theorem, stated now in terms of the potential u_t.

We remark that one could also consider the one-phase model for the Mullins–Sekerka as in [9] in the whole $\mathbb{R}^2$ and expect the above convergence to hold also in this case. We also expect the convergence of the sets in Theorem 1.3 to hold with respect to Hausdorff distance but we do not prove it here.

1.2 Structure of the paper

Section 2 is purely geometric and in Proposition 2.1 we prove our quantitative version of the Alexandrov theorem which then implies Theorem 1.1 as a corollary. In Sect. 3, we first introduce the incremental minimization problem for the minimizing movements scheme, and recall some basic results related to its minimizers. Then we recall the construction of the flat flow and give the proof of Theorem 1.2 at the end of the section. In Sect. 4, we introduce the incremental minimization problem and the flat flow for the Mullins–Sekerka equation. We then state and prove in Proposition 4.2 a crucial functional inequality which is related to Proposition 2.1. The section concludes with the proof of Theorem 1.3.

2 A sharp quantitative Alexandrov theorem in two-dimensions

Let us first recall that for measurable sets $E \subset \mathbb{R}^2$, the perimeter is defined by

$$P(E) := \sup\left\{\int_E \operatorname{div} X\, dx : X \in C^1_c(\mathbb{R}^2, \mathbb{R}^2),\ \|X\|_{L^\infty} \le 1\right\}.$$

If $P(E) < \infty$ we say that E is a set of finite perimeter. We also recall that if E is regular enough, say a domain with Lipschitz boundary, then $P(E) = \mathcal{H}^1(\partial E)$. For the general properties of sets of finite perimeter we refer to the monographs [5, 35].

In the following we fix the prescribed area $m > 0$ of a set E and a constant $M > 0$ representing an upper bound for the perimeter of E. For $d \in \mathbb{N}$ we denote by P_d the perimeter of any union of d disjoint disks with equal areas m/d, i.e.,

$$P_d := 2\sqrt{\pi m d}.$$

For a set of $E \subset \mathbb{R}^2$ of class C^2 we denote by κ_E its curvature (with the sign defined so that κ_E is positive for convex sets) and we set

$$\overline{\kappa}_E := \fint_{\partial E} \kappa_E \, d\mathcal{H}^1 = \frac{1}{\mathcal{H}^1(\partial E)} \int_{\partial E} \kappa_E \, d\mathcal{H}^1.$$

In [23] it is proven that if $E \subset \mathbb{R}^2$ is a set of class C^2 with area $|E| = m$ and $\|\kappa_E - \overline{\kappa}_E\|^2_{L^1(\partial E)} \le \varepsilon_0$, for ε_0 small enough, then E is C^1-diffeomorphic to a disjoint union of disks $D_1, \dots, D_d$ and it holds

$$|P(E) - P_d| \le C\|\kappa_E - \overline{\kappa}_E\|_{L^1(\partial E)}.$$

Our first result improves the above inequality by showing that a similar estimate holds with quadratic right-hand side, which is the optimal scaling of the quantitative Alexandrov theorem. We also consider L^2-norms as this is more natural in our variational framework. We state this in the following proposition.

Proposition 2.1 *Let $m, M > 0$. There exist $\varepsilon_0 = \varepsilon_0(m, M) \in (0, 1)$ and $C_0 = C_0(m, M) > 1$ with the following property: Let $E \subset \mathbb{R}^2$ be a bounded open set of class C^2, with $|E| = m$ and $P(E) \le M$, such that $\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \le \varepsilon_0$. Then E*

is diffeomorphic to a union of d disjoint disks $D_1, \ldots, D_d$, with equal areas m/d and dist$(D_i, D_j) > 0$ for $i \neq j$, and

$$|P(E) - P_d| \leq C_0 \|\kappa_E - \overline{\kappa}_E\|^2_{L^2(\partial E)}. \tag{2.1}$$

Moreover, d is bounded from above by a constant depending only on m, M.

Finally, for ε_0 sufficiently small, the boundary of every connected component of the set E can be parametrized as a normal graph over one of the discs D_i with $C^{1,\frac{1}{2}}$ norm of the parametrization vanishing as $\varepsilon_0 \to 0$.

Proof Let E be as in the statement and let $E_1, \ldots, E_d$ be the collection of its connected components. For each component E_i we denote by Γ_i the outer component of ∂E_i and by $\hat{E}_i$ the bounded region enclosed by Γ_i, i.e., the set obtained by filling the "holes" of E_i.

We split the proof into several steps. Notice that in what follows $m_0 \in (0, 1)$ and $M_0 > 1$ will denote "universal" constants, i.e, constants depending only on m, M, which may change from line to line.

Step 1. We claim that

$$|\hat{E}_{\bar{k}}| \geq m_0 \quad \text{for some } \bar{k} \in \{1, \ldots, d\}. \tag{2.2}$$

Indeed, by translating the components if necessary we may assume that $\operatorname{dist}(\hat{E}_i, \hat{E}_j) > \sqrt{2}$. Setting $Q := (0, 1) \times (0, 1)$, we may use [39, Lemma 2.1] to infer that there exist $z \in \mathbb{Z}^2$ such that

$$|E \cap (z + Q)| \geq c \min\left\{\frac{m^2}{M^2}, 1\right\},$$

with $c > 0$ a universal constant. Since $z + Q$ can only intersect one component $\hat{E}_i$, the claim follows.

Step 2. We claim that

$$|\overline{\kappa}_E| \leq M_0. \tag{2.3}$$

To this aim, note that by the Isoperimetric Inequality and by (2.2), we have

$$\mathcal{H}^1(\partial \hat{E}_{\bar{k}}) \geq 2\sqrt{\pi m_0}. \tag{2.4}$$

Now,

$$\int_{\partial \hat{E}_{\bar{k}}} \left|\kappa_E - \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})}\right|^2 d\mathcal{H}^1 \leq \int_{\partial E} |\kappa_E - \overline{\kappa}_E|^2 \, d\mathcal{H}^1 \leq \varepsilon_0^2,$$

where we used the simply connectedness of $\hat{E}_{\bar{k}}$ and Gauss–Bonnet Theorem to get $\overline{\kappa}_{\hat{E}_{\bar{k}}} = 2\pi/\mathcal{H}^1(\partial \hat{E}_{\bar{k}})$. Note that here and repeatedly in the sequel we use that

$\min_a \int |f-a|^2 d\mathcal{H}^1 = \int |f-\bar{f}|^2 d\mathcal{H}^1$, with $\bar{f}$ the average of f. In turn,

$$\frac{1}{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})}\Big|2\pi - \mathcal{H}^1(\partial \hat{E}_{\bar{k}})\overline{\kappa}_E\Big|^2 = \int_{\partial \hat{E}_{\bar{k}}} \Big|\frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})} - \overline{\kappa}_E\Big|^2 d\mathcal{H}^1$$
$$\leq 2\int_{\partial \hat{E}_{\bar{k}}} \Big|\kappa_E - \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})}\Big|^2 d\mathcal{H}^1$$
$$+ 2\int_{\partial E} \Big|\kappa_E - \overline{\kappa}_E\Big|^2 d\mathcal{H}^1 \leq 4\varepsilon_0^2 \leq 4.$$

Hence $\Big|2\pi - \mathcal{H}^1(\partial \hat{E}_{\bar{k}})\overline{\kappa}_E\Big| \leq 2\sqrt{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})}$, so that, using also (2.4),

$$2\sqrt{\pi m_0}|\overline{\kappa}_E| \leq 2\pi + |2\pi - \mathcal{H}^1(\partial \hat{E}_{\bar{k}})\overline{\kappa}_E| \leq 2\pi + 2\sqrt{\mathcal{H}^1(\partial \hat{E}_{\bar{k}})} \leq 2\pi(1+\sqrt{M}),$$

and the claim follows.
Step 3. We claim that

$$\mathcal{H}^1(\Gamma) \geq m_0 \qquad \text{for any component } \Gamma \text{ of } \partial E. \tag{2.5}$$

Indeed, using again Gauss–Bonnet Theorem and Jensen inequality,

$$M|\overline{\kappa}_E|^2 + 1 \geq \mathcal{H}^1(\Gamma)|\overline{\kappa}_E|^2 + \varepsilon_0^2 \geq \mathcal{H}^1(\Gamma)|\overline{\kappa}_E|^2 + \int_\Gamma |\kappa_E - \overline{\kappa}_E|^2\, d\mathcal{H}^1$$
$$\geq \frac{1}{2}\int_\Gamma |\kappa_E|^2\, d\mathcal{H}^1 \geq \frac{1}{2\mathcal{H}^1(\Gamma)}\Big(\int_\Gamma \kappa_E\, d\mathcal{H}^1\Big)^2 = \frac{1}{2\mathcal{H}^1(\Gamma)}4\pi^2,$$

and the claim follows taking into account (2.3).
Step 4. We claim that if ε_0 is sufficiently small, then E has $d \leq M_0$ connected components which are simply connected.

We argue by contradiction. Suppose there exists a connected component E_i which is not simply connected. Then there exists a component $\Gamma \subset \partial E$ contained in $\hat{E}_i$ such that $\int_\Gamma \kappa_E\, d\mathcal{H}^1 = -2\pi$. We observe that then it holds

$$⨍_\Gamma \kappa_E d\mathcal{H}^1 = -\frac{2\pi}{\mathcal{H}^1(\Gamma)} \qquad \text{and} \qquad ⨍_{\partial \hat{E}_i} \kappa_E d\mathcal{H}^1 = \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_i)}$$

and therefore

$$\int_\Gamma \Big|\kappa_E + \frac{2\pi}{\mathcal{H}^1(\Gamma)}\Big|^2 d\mathcal{H}^1 + \int_{\partial \hat{E}_i} \Big|\kappa_E - \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_i)}\Big|^2 d\mathcal{H}^1 \leq \int_{\partial E} |\kappa_E - \overline{\kappa}_E|^2 d\mathcal{H}^1 \leq \varepsilon_0^2.$$

We then infer that by (2.5)

$$\frac{16\pi^2}{M^2} \leq \Big|\frac{2\pi}{\mathcal{H}^1(\Gamma)} + \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_i)}\Big|^2 \leq 2\Big|\overline{\kappa}_E + \frac{2\pi}{\mathcal{H}^1(\Gamma)}\Big|^2 + 2\Big|\overline{\kappa}_E - \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_i)}\Big|^2$$

$$\leq 2 \fint_{\Gamma} \left| \kappa_E + \frac{2\pi}{\mathcal{H}^1(\Gamma)} \right|^2 d\mathcal{H}^1 + 2 \fint_{\partial \hat{E}_i} \left| \kappa_E - \frac{2\pi}{\mathcal{H}^1(\partial \hat{E}_i)} \right|^2 d\mathcal{H}^1 \leq \frac{2\varepsilon_0^2}{m_0}.$$

Therefore, for ε_0 sufficiently small we reach a contradiction.

Every component of E is thus simply connected and by (2.5) their perimeter is bounded from below. Therefore the number d of the components is bounded from above $d \leq M_0$. Note that in particular $\overline{\kappa}_E = \frac{2\pi d}{\mathcal{H}^1(\partial E)}$.

Step 5. Let us show that if ε_0 is sufficiently small, then each connected component E_i is a nearly spherical set, parametrized over a disks $D_{r_i}(x_i)$ with $|D_{r_i}(x_i)| = |E_i|$ and the $C^{1,\frac{1}{2}}$ norm of the parametrization is infinitesimal with $\varepsilon_0 \to 0$.

We adapt the argument of [23, Lemma 3.2]. Let us fix a component E_i and denote its perimeter by l_i, i.e. $\mathcal{H}^1(\partial E_i) = l_i$. By Gauss–Bonnet it holds $\overline{\kappa}_{E_i} = \frac{2\pi}{l_i}$. Since the boundary ∂E_i is connected we may parametrize it by a unit speed curve $\gamma : [0, l_i] \to \mathbb{R}^2$, $\gamma(s) = (x(s), y(s))$ with counterclockwise orientation. Define $\theta(s) := \int_0^s \kappa_{E_i}(\gamma(\tau))\, d\tau$ so that $\theta(0) = 0$ and $\theta(l_i) = 2\pi$. Then, for every $0 \leq s_1 < s_2 \leq l_i$, it holds by Hölder's inequality

$$\begin{aligned} |\theta(s_2) - s_2\overline{\kappa}_E - (\theta(s_1) - s_1\overline{\kappa}_E)| &\leq \int_{s_1}^{s_2} |\kappa_E - \overline{\kappa}_E| \\ &\leq \|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} |s_2 - s_1|^{\frac{1}{2}} \\ &\leq \varepsilon_0 |s_2 - s_1|^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

In particular, applying (2.6) to $s_1 = 0$ and $s_2 = s \in [0, l_i]$ generic (recall that $M_0 > 1$ denote "universal" constants depending only on m, M, which may change from line to line)

$$\left|\theta(s) - s\overline{\kappa}_E\right| \leq M_0 \|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \leq M_0 \varepsilon_0, \tag{2.7}$$

and for $s_2 = l_i$ it yields

$$|2\pi - l_i\overline{\kappa}_E| \leq M_0 \|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \leq M_0 \varepsilon_0. \tag{2.8}$$

By possibly rotating the set E_i we have

$$x'(s) = -\sin\theta(s) \qquad \text{and} \qquad y'(s) = \cos\theta(s) \quad \text{for all } s \in (0, l_i).$$

We obtain by (2.7) and (2.8) that

$$\begin{aligned} \left| x'(s) + \sin\left(\frac{2\pi s}{l_i}\right) \right| + \left| y'(s) - \cos\left(\frac{2\pi s}{l_i}\right) \right| &\leq M_0 \|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \\ &\leq M_0 \varepsilon_0 \end{aligned} \tag{2.9}$$

for all $s \in [0, l_i]$. Integrating (2.9) we deduce that there are numbers a and b such that

$$\left|x(s) - a - \frac{l_i}{2\pi}\cos\left(\frac{2\pi s}{l_i}\right)\right| + \left|y(s) - b - \frac{l_i}{2\pi}\sin\left(\frac{2\pi s}{l_i}\right)\right| \\ \le M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \le M_0\varepsilon_0 \tag{2.10}$$

for all $s \in [0, l_i]$. We set $x_i = (a, b)$ and note that from (2.10) we infer that

$$D_{\frac{l_i}{2\pi} - M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}}(x_i) \subset E_i \subset D_{\frac{l_i}{2\pi} + M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}}(x_i). \tag{2.11}$$

In particular, if r_i is chosen in such a way that $|E_i| = \pi r_i^2 = |D_{r_i}(x_i)|$, then (2.11) yields

$$\frac{l_i}{2\pi} - M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \le r_i \le \frac{l_i}{2\pi} + M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}, \tag{2.12}$$

and

$$\left|x(s) - a - r_i\cos\left(\frac{2\pi s}{l_i}\right)\right| + \left|y(s) - b - r_i\sin\left(\frac{2\pi s}{l_i}\right)\right| \\ \le M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \le M_0\varepsilon_0 \qquad \forall\, s \in [0, l_i]. \tag{2.13}$$

By (2.9) and (2.13) the boundary of the component E_i is parametrized by a small perturbation of the boundary of the disc $\partial D_{r_i}(x_i)$ given by $c : [0, l_i] \to \mathbb{R}^2$ with $c(s) = r_i(\cos(\frac{2\pi s}{l_i}), \sin(\frac{2\pi s}{l_i}))$:

$$\gamma(s) = c(s) + \sigma(s) \\ \|\sigma\|_{L^\infty} + \|\sigma'\|_{L^\infty} \le M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)} \le M_0\varepsilon_0. \tag{2.14}$$

Now it is a simple consequence of (2.6), (2.9) and (2.14) to verify that ∂E_i is as nearly spherical sets over $D_{r_i}(x_i)$, with $|D_{r_i}(x_i)| = |E_i|$, by functions $f_i \in C^{1,1/2}(\partial D_{r_i}(x_i))$ with

$$\|f_i\|_{C^{1,1/2}} \le \omega(\|\kappa_E - \overline{\kappa}_E\|_{L^2}),$$

for suitable increasing modulus of continuity ω, with $\omega(0^+) = 0$.

Step 6. Quantitative Alexandrov Theorem.

We use the quantitative Alexandrov theorem proven in [39] to infer that, if f_i is the parametrization of the component E_i, then

$$\|f_i\|^2_{H^1(\partial D_{r_i}(x_i))} \le C\|\kappa_{E_i} - \overline{\kappa}_{E_i}\|^2_{L^2(\partial E_i)}.$$

Recall that r_i is such that $|E_i| = |D_{r_i}(x_i)|$. By the area formula, see e.g. [39, (1.3)], and a simple linearization we infer that

$$0 \le P(E_i) - P(D_{r_i}(x_i)) \le C\|f_i\|^2_{H^1(\partial D_{r_i}(x_i))}.$$

Summing over the connected components yields

$$\begin{aligned}\|\kappa_E - \overline{\kappa}_E\|^2_{L^2(\partial E)} &\ge \sum_{i=1}^d \|\kappa_{E_i} - \overline{\kappa}_{E_i}\|^2_{L^2(\partial E_i)} \ge c\sum_{i=1}^d \|f_i\|^2_{H^1(\partial D_{r_i}(x_i))} \\ &\ge c\,\Big|\sum_{i=1}^d P(E_i) - P(D_{r_i}(x_i))\Big|.\end{aligned} \tag{2.15}$$

Step 7. Conclusion.

Let $r > 0$ be such that the disk $D_r(x_i)$ has area $|D_r(x_i)| = m/d$, where $m = |E|$. In other words $\sum_{i=1}^d P(D_r(x_i)) = 2\pi\, rd = P_d$. Recall that the disks $D_{r_i}(x_i)$ are defined such that $|D_{r_i}(x_i)| = |E_i|$ for every component E_i and thus

$$\sum_i^d r_i^2 = dr^2. \tag{2.16}$$

Recall also that by the previous estimates it holds $m_0 \le r_i, r, d \le M_0$. By (2.11) and (2.12) we infer that

$$D_{r_i - M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}}(x_i) \subset E_i \subset D_{r_i + M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}}(x_i). \tag{2.17}$$

and therefore

$$|r - r_i| \le M_0\|\kappa_E - \overline{\kappa}_E\|_{L^2(\partial E)}. \tag{2.18}$$

Thus, by simple algebra, by (2.16) and by (2.12), if $d > 1$ we deduce

$$\begin{aligned}\Big|P_d - \sum_{i=1}^d P(D_{r_i}(x_i))\Big| &= 2\pi\Big|d\,r - \sum_{i=1}^d r_i\Big| = 2\pi\left|\sqrt{d}\left(\sum_{i=1}^d r_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^d r_i\right| \\ &\le M_0\left(d\sum_{i=1}^d r_i^2 - \left(\sum_{i=1}^d r_i\right)^2\right) \\ &= M_0 \sum_{1\le i<j\le d} (r_i - r_j)^2 \\ &\le C\sum_{i=1}^d (r_i - r)^2 \overset{(2.18)}{\le} M_0\|\kappa_E - \overline{\kappa}_E\|^2_{L^2(\partial E)}.\end{aligned}$$

Hence, the inequality (2.1) then follows by combining the above estimate with (2.15). Finally, by the very same argument of step 5 and by (2.18) we deduce that the connected components E_i can be parametrized as nearly spherical sets over the discs $D_r(x_i)$. □

Proposition 2.1 immediately implies the sharp geometric inequality in the plane stated in Theorem 1.1.

Proof of Theorem 1.1 Let $\varepsilon_0 > 0$ be from Proposition 2.1. If $\|\kappa_E - \bar{\kappa}_E\|_{L^2(\partial E)} \leq \varepsilon_0$ then the inequality holds by Proposition 2.1. If $\|\kappa_E - \bar{\kappa}_E\|_{L^2(\partial E)} \geq \varepsilon_0$, then the inequality holds trivially as

$$|P(E) - P_d| \leq 3M \leq \frac{3M}{\varepsilon_0^2}\|\kappa_E - \bar{\kappa}_E\|^2_{L^2(\partial E)}.$$

The inequality (1.3) follows similarly. □

Remark 2.2 The exponent 2 in Theorem 1.1 is optimal. Indeed, let E_f be a nearly spherical set with same area and barycenter of the disc, parametrized (on the unitary circle) by a smooth function $f : \partial D_1 \to \mathbb{R}$ with C^1 norm small enough; then, by Fuglede inequality [22]

$$P(E_f) - P(D_1) \geq c\|f\|^2_{H^1(\partial D_1)}.$$

If in addition f is in the second eigenspace of the Laplace–Beltrami operator on the circle, as pointed out in [39, Remark 1.5] we have

$$\|f\|^2_{H^1(\partial D_1)} \geq c\|H_{E_f} - \bar{H}_{E_f}\|^2_{L^1(\partial D_1)}.$$

Combining the above inequalities, the optimality of the exponent follows.

3 The asymptotics of the area preserving curvature flow in the plane

Let us first introduce the setting for the construction of the flat flows. We use the notation from [39] and refer to [39, 41] for a more detailed introduction. We denote the signed distance function by d_E and define it as

$$d_E(x) = \operatorname{dist}(x, E) - \operatorname{dist}(x, \mathbb{R}^2 \setminus E).$$

Then clearly $|d_E(x)| = \operatorname{dist}(x, \partial E)$.

We fix the volume $m > 0$ and the time step $h > 0$, and given a bounded set E we consider the minimization problem

$$\min\left\{P(F) + \frac{1}{h}\int_F d_E\,dx + \frac{1}{\sqrt{h}}\big||F| - m\big|\right\} \tag{3.1}$$

and note that the minimizer exists but might not be unique. We define the dissipation of a set F with respect to a set E as

$$\mathcal{D}(F, E) := \int_{F \Delta E} \operatorname{dist}(x, \partial E)\, dx \tag{3.2}$$

and observe that we may write the minimization problem (3.1) as

$$\min \left\{ P(F) + \frac{1}{h}\mathcal{D}(F, E) + \frac{1}{\sqrt{h}}\big||F| - m\big| \right\}.$$

Let us then recall the construction of the flat flow for the volume preserving mean curvature flow (1.1) from [41]. Let $E(0) \subset \mathbb{R}^2$ be a bounded set of finite perimeter which coincides with its Lebesgue representative. We fix a minimizer of (3.1), with $E = E(0)$, denote it by $E_1^{(h)}$ and consider its Lebesgue representative. We construct the discrete-in-time evolution $\{E_k^{(h)}\}_{k\in\mathbb{N}}$ by recursion such that assuming that $E_k^{(h)}$ is defined we set $E_{k+1}^{(h)}$ to be a minimizer of (3.1) with $E = E_k^{(h)}$. By [41, Lemma 3.1] it holds for all $k = 0, 1, \ldots$

$$P(E_{k+1}^{(h)}) + \frac{1}{\sqrt{h}}\big||E_{k+1}^{(h)}| - m\big| + \frac{1}{h}\mathcal{D}\left(E_{k+1}^{(h)}, E_k^{(h)}\right) \le P(E_k^{(h)}) + \frac{1}{\sqrt{h}}\big||E_k^{(h)}| - m\big|. \tag{3.3}$$

Also the set $E_{k+1}^{(h)}$ is $C^{2,\alpha}$-regular and satisfies the Euler–Lagrange equation (see [41, Lemma 3.7])

$$\frac{d_{E_k^{(h)}}}{h} = -\kappa_{E_{k+1}^{(h)}} + \lambda_{k+1}^{(h)} \qquad \text{on } \partial E_{k+1}^{(h)}, \tag{3.4}$$

in the classical sense, where $\lambda_{k+1}^{(h)}$ is the Lagrange multiplier due to the volume penalization. Finally we define the approximate flat flow $\{E^{(h)}(t)\}_{t\ge 0}$ by setting

$$E^{(h)}(t) = E_k^{(h)} \qquad \text{for } t \in [kh, (k+1)h).$$

Definition 3.1 A flat flow solution of (1.1) is any family of sets $\{E(t)\}_{t\ge0}$ which is a cluster point of $\{E^{(h)}(t)\}_{t\ge0}$, i.e.,

$$E^{(h_n)}(t) \to E(t) \quad \text{as } h_n \to 0 \quad \text{in } L^1 \quad \text{for almost every } t > 0.$$

By [41, Theorem 2.2] there exists a flat flow starting from $E(0)$ such that $P(E(t)) \le P(E(0))$ and $|E(t)| = m$ for every $t \ge 0$.

We are interested in the long time behavior of the flow. To this aim we need two technical lemmas. The first lemma is algebraic.

Lemma 3.2 *Let $K \in \mathbb{N}$ and $\{a_k\}_{k\in\{1,\dots,K\}}$ be a sequence of non-negative numbers and let $\mathcal{I} \subset \{1,\dots,K\}$. Assume that there exists $c > 1$ such that*

$$\sum_{k=i}^{K} a_k \le c a_i$$

for every $i \in \{1,\dots,K\}\setminus\mathcal{I}$. Then,

$$\sum_{k=i+1}^{K} a_k \le \Big(1 - \frac{1}{c}\Big)^{i-|\mathcal{I}|} S$$

for every $i \in \{1,\dots,K\}$, where $S := \sum_{k=1}^{K} a_k$ and $|\mathcal{I}|$ denotes the cardinality of $\mathcal{I}$.

Proof Set $F(i) := \sum_{k=i}^{K} a_k$ and note that by assumption $F(i) \le c(F(i) - F(i+1))$ for every $i \in \{1,\dots,K\}\setminus\mathcal{I}$. Hence, we have

$$F(i+1) \le \begin{cases} \Big(1-\frac{1}{c}\Big)F(i) & \text{if } i \notin \mathcal{I}, \\ F(i) & \text{if } i \in \mathcal{I}. \end{cases}$$

By iterating the previous estimate (note that at least $K - |\mathcal{I}|$ times the first instance must hold), we conclude. □

The second lemma is in the spirit of Ekeland variational principle.

Lemma 3.3 *Let $d \in \mathbb{N}$ and $D_r(x_1),\dots,D_r(x_d)$ be disjoint disks and denote $F = \bigcup_{i=1}^{d} D_r(x_i)$. Then there is a constant C, which depends only on d and r, such that for every set of finite perimeter $E \subset \mathbb{R}^2$ it holds*

$$P(F) \le P(E) + C|E\Delta F|^{\frac{1}{3}}.$$

Proof Let us fix a set E and let $\rho \le r/10$ be a positive number whose choice will be clear later. We begin by constructing a set F_ρ of class $C^{1,1}$, which contains the union of disks $F \subset F_\rho$, satisfies interior and exterior ball condition with radius ρ and

$$P(F) \le P(F_\rho) + C\sqrt{\rho}, \quad \text{and} \quad |F\Delta F_\rho| \le C\rho^{\frac{3}{2}}. \tag{3.5}$$

Let $x_1,\dots,x_d$ be the centerpoints of the disks. If it holds $|x_i - x_j| > 2r + \rho$ for every $i \ne j$ we simply choose $F_\rho = F$. If $|x_i - x_j| \le 2r + \rho$ for some $i \ne j$ we connect the disks $D_r(x_i)$ and $D_r(x_j)$ with a thin neck around the midpoint $(x_i + x_j)/2$ as follows. We first enlarge the disks by ρ and consider the union $\tilde{F}_\rho^{ij} := D_{r+\rho}(x_i) \cup D_{r+\rho}(x_j)$, which overlap around the midpoint $(x_i + x_j)/2$. We then decrease the union back by ρ and define

$$F_\rho^{i,j} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathbb{R}^2\setminus\tilde{F}_\rho^{ij}) > \rho\}.$$

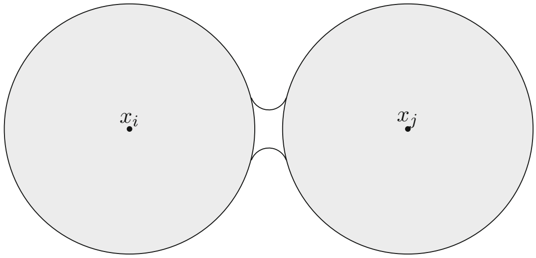

Fig. 1 If two disks are close to each other, we connect them with a neck given by two arcs

Since $|x_i - x_j| \leq 2r + \rho$, the set $F_\rho^{i,j}$ is connected and contains the disks $D_r(x_i)$ and $D_r(x_j)$. The part of the boundary of $F_\rho^{i,j}$, which is not contained in $\bar{D}_r(x_i) \cup \bar{D}_r(x_j)$, consists of two arcs, see Fig. 1. In particular, the set $F_\rho^{i,j}$ satisfies interior and exterior ball condition with radius ρ. We repeat the same construction for all disks $D_r(x_i)$ and $D_r(x_j)$ which are close to each other in the sense that $|x_i - x_j| \leq 2r + \rho$, and obtain F_ρ which satisfies (3.5).

The rest of the proof follows from standard calibration argument (see e.g. [2, Proof of Theorem 4.3]) and we only give the sketch of the argument. We construct a vector field $X \in C^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$X(x) = \nabla d_{F_\rho}(x)\zeta(x)$$

where $0 \leq \zeta \leq 1$ is a smooth cut-off function such that $\zeta(x) = 1$ for $|d_{F_\rho}(x)| \leq \rho/4$, $\zeta(x) = 0$ for $|d_{F_\rho}(x)| \geq \rho/2$ and $|\nabla\zeta| \leq C/\rho$. In particular, it holds $|X| \leq 1$ in $\mathbb{R}^2$ and $X = \nu_{F_\rho}$ on ∂F_ρ. Moreover, since F_ρ satisfies interior and exterior ball condition with radius ρ it holds $|\Delta d_{F_\rho}(x)| \leq C/\rho$ for $|d_{F_\rho}(x)| \leq \rho/2$. Therefore by the divergence theorem

$$P(F_\rho) - P(E) \leq \int_{F_\rho \Delta E} |\mathrm{div}(X)|\, dx \leq \frac{C}{\rho}|F_\rho \Delta E|.$$

We combine the above inequality with (3.5) and deduce

$$P(F) \leq P(E) + \frac{C}{\rho}|E\Delta F| + C\sqrt{\rho}.$$

Choosing $\rho = \min\{|E\Delta F|^{\frac{2}{3}}, r/10\}$ yields the claim. □

We may now give the proof of the convergence of the area-preserving mean curvature flow.

Proof of Theorem 1.2 Let $\{E(t)\}_{t\geq 0}$ be an area-preserving flat flow and let $\{E^{(h_n)}(t)\}_{t\geq 0}$ be an approximate flow converging to $E(t)$. Set

$$f_n(t) = P(E^{(h_n)}(t)) + \frac{1}{\sqrt{h_n}}\big||E^{(h_n)}(t)| - m\big|.$$

By (3.3) the f_n's are monotone non-increasing functions which are bounded by $P(E(0))$. Therefore, by Helly's selection theorem, up to passing to a further subsequence (not relabeled), the functions f_n's converge pointwise to some non-increasing function $f_\infty : [0, +\infty) \to \mathbb{R}$. Set $F_\infty = \lim_{t\to+\infty} f_\infty(t)$. In what follows we also set

$$v_t^{(h_n)} = \frac{d_{E_k^{(h_n)}}}{h_n}, \qquad \text{where } k = \left\lfloor \frac{t}{h_n} \right\rfloor - 1$$

the approximate velocity of the approximate flow at time t. Moreover, C will denote a positive constant, which may change from line to line and might depend on the flat flow itself (but not on h_n nor on the discrete step of the minimizing movements).

We divide the proof in two cases.

Case 1: There exists $d \in \mathbb{N}\setminus\{0\}$ such that either $P_d < F_\infty < P_{d+1}$ or $F_\infty = P_d$ and $f_\infty(t) > P_d$ for every $t \in [0, +\infty)$.

In this case, there exists $\bar{t} > 0$ such that, for every $T > \bar{t}$ there exist $\bar{n} \in \mathbb{N}\setminus\{0\}$ such that

$$P_d \le f_n(t) < P_{d+1} \qquad \text{and} \qquad P_{d+1} - f_n(t) \ge \frac{P_{d+1} - F_\infty}{2} =: \delta_0 \tag{3.6}$$

for every $n \ge \bar{n}$ and $t \in [\bar{t}, T]$. Set $\mathcal{I}^{(h_n)} = \left\{ i \in \left\{ \lfloor \frac{\bar{t}}{h_n} \rfloor, \ldots, \lfloor \frac{T}{h_n} \rfloor \right\} : |E_i^{(h_n)}| \ne m \right\}$. By [41, Cor. 3.10] there exists a constant $C_T > 0$ such that

$$|\mathcal{I}^{(h_n)}| \le C_T \tag{3.7}$$

for n sufficiently large. For every $i \notin \mathcal{I}^{(h_n)}$ we have by iterating (3.3) and using (3.6)

$$\frac{1}{h_n} \sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \le P(E_i^{(h_n)}) - f_n(T) \le P(E_i^{(h_n)}) - P_d.$$

Then by (1.3) and by the Euler–Lagrange equation (3.4)

$$\begin{aligned}
\frac{1}{h_n} \sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) &\le P(E_i^{(h_n)}) - P_d \\
&\le C_0 \|\kappa_{E_i^{(h_n)}} - \overline{\kappa}_{E_i^{(h_n)}}\|^2_{L^2(\partial E_i^{(h_n)})} \\
&\le C_0 \|\kappa_{E_i^{(h_n)}} - \lambda_i^{(h_n)}\|^2_{L^2(\partial E_i^{(h_n)})} \\
&= \frac{C_0}{h_n^2} \int_{\partial E_i^{(h_n)}} d^2_{E_{i-1}^{(h_n)}} \, d\mathcal{H}^1.
\end{aligned} \tag{3.8}$$

In [41] it is proven (formula after (3.25)) that

$$\int_{\partial E_i^{(h_n)}} d^2_{E_{i-1}^{(h_n)}}\, d\mathcal{H}^1 \le C\mathcal{D}\left(E_i^{(h_n)}, E_{i-1}^{(h_n)}\right). \tag{3.9}$$

Therefore from (3.8) we conclude

$$\sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathcal{D}\left(E_k^{(h_n)}, E_{k-1}^{(h_n)}\right) \le \frac{C_0'}{h_n}\mathcal{D}\left(E_i^{(h_n)}, E_{i-1}^{(h_n)}\right).$$

Setting $a_k^{(h_n)} = h_n^{-1}\mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)})$ we have that for every $i \in \left\{\lfloor \frac{\bar{t}}{h_n} \rfloor, \dots, \lfloor \frac{T}{h_n} \rfloor\right\} \setminus \mathcal{I}^{(h_n)}$ it holds

$$\begin{aligned}\sum_{k=i}^{\lfloor \frac{T}{h_n} \rfloor} a_k^{(h_n)} &\le \frac{C_0' + h_n}{h_n} a_i^{(h_n)} \\ &\le \frac{2C_0'}{h_n} a_i^{(h_n)}.\end{aligned}$$

Moreover it holds by (3.3) $\sum_{k=1}^{\infty} a_k \le P(E(0)) \le M$. By Lemma 3.2 we infer that

$$\sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} a_k^{(h_n)} \le M\left(1 - \frac{h_n}{2C_0'}\right)^{i - C_T - \frac{\bar{t}}{h_n}} \qquad \text{for all } i = \lfloor \frac{\bar{t}}{h_n} \rfloor, \dots, \lfloor \frac{T}{h_n} \rfloor.$$

In other words for every $t \in [\bar{t}, T]$ we have

$$\begin{aligned}\sum_{k=\lfloor \frac{t}{h_n} \rfloor + 1}^{\lfloor \frac{T}{h_n} \rfloor} h_n^{-1}\mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) &\le M\left(1 - \frac{h_n}{2C_0'}\right)^{\lfloor \frac{t}{h_n} \rfloor - C_T - \frac{\bar{t}}{h_n}} \\ &\le Ce^{-\frac{t}{2C_0'}} \end{aligned} \tag{3.10}$$

for $h_n \le h_0(T)$.

By [41, Proposition 3.4], it holds

$$|E_i^{(h_n)} \Delta E_{i-1}^{(h_n)}| \le C\ell P(E_i^{(h_n)}) + \frac{C}{\ell}\int_{E_i^{(h_n)} \Delta E_{i-1}^{(h_n)}} |d_{E_{i-1}^{(h_n)}}|\, dx$$

for all $\ell \leq \frac{1}{C}\sqrt{h_n}$. Therefore, by the inequality above and by (3.10) we infer that for every $\bar{t} \leq t < s \leq T$ we have

$$
\begin{aligned}
|E^{(h_n)}(t)\Delta E^{(h_n)}(s)| &= \sum_{i=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{s}{h_n}\rfloor} |E_i^{(h_n)}\Delta E_{i-1}^{(h_n)}| \\
&\leq C \sum_{i=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{s}{h_n}\rfloor} \left(\ell P(E_i^{(h_n)}) + \frac{1}{\ell}\int_{E_i^{(h_n)}\Delta E_{i-1}^{(h_n)}} |d_{E_{i-1}^{(h_n)}}|\,dx \right) \\
&\leq C P(E(0))\ell \frac{s-t}{h_n} + \frac{C}{\ell} \sum_{i=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{s}{h_n}\rfloor} \mathcal{D}\left(E_i^{(h_n)}, E_{i-1}^{(h_n)}\right) \\
&\leq CM\ell \frac{s-t}{h_n} M + \frac{C h_n}{\ell} e^{-\frac{t}{2C_0'}},
\end{aligned}
$$

for all $\ell \leq \frac{1}{C}\sqrt{h_n}$ and $h_n \leq h_0$. In particular, choosing $\ell = \frac{h_n}{e^{\alpha t}}$ with $\alpha = \frac{1}{4C_0'}$ and $s \leq t+1$, we have

$$
|E^{(h_n)}(t)\Delta E^{(h_n)}(s)| \leq CMe^{-\frac{t}{4C_0'}}.
$$

Passing to the limit as $h_n \to 0$, we get

$$
|E(t)\Delta E(s)| \leq CMe^{-\frac{t}{4C_0'}} \qquad \text{for all } \bar{t} \leq t \leq s \leq t+1. \tag{3.11}
$$

Hence, we deduce that $E(t)$ converges exponentially fast to a set of finite perimeter E_∞ in L^1 and $|E_\infty| = m$.

We now show that the limiting set E_∞ is the union of disjoint open disks with the same radius. Denote by S_∞ the countable set of discontinuity points of f_∞ and note that for any $t \in (0,+\infty)\setminus S_\infty$ and any sequence $t_n \to t$ we have $f_n(t_n) \to f_\infty(t)$.

Fix $t \geq \bar{t}$, $0 < \alpha < \frac{1}{2C_0'}$, and an open set $A(t)$ such that $S_\infty \cap [t, t+e^{-\alpha t}] \subset A(t) \subset [t, t+e^{-\alpha t}]$ and $|A(t)| \leq e^{-\alpha' t}$, with $\alpha' > \alpha$. By (3.9) and (3.10) we have

$$
\begin{aligned}
&\int_{[t,t+e^{-\alpha t}]\setminus A(t)} \left(\int_{\partial E^{(h_n)}(s)} (v_s^{(h_n)})^2 d\mathcal{H}^1 \right) ds \leq \frac{1}{h_n} \sum_{i=\lfloor \frac{t}{h_n}\rfloor}^{\lfloor \frac{t+e^{-\alpha t}}{h_n}\rfloor} \int_{\partial E_i^{(h_n)}} d^2_{E_{i-1}^{(h_n)}}\, d\mathcal{H}^1 \\
&\leq C \sum_{i=\lfloor \frac{t}{h_n}\rfloor}^{\lfloor \frac{t+e^{-\alpha t}}{h_n}\rfloor} \frac{1}{h_n}\mathcal{D}\left(E_i^{(h_n)}, E_{i-1}^{(h_n)}\right) \leq Ce^{-\frac{t}{2C_0'}},
\end{aligned} \tag{3.12}
$$

for n sufficiently large. By possibly increasing $\bar{t}$ we have $|[t, t+e^{-\alpha t}]\backslash A(t)| > \frac{1}{2}e^{-\alpha t}$ for $t \geq \bar{t}$. Moreover by (3.7) it holds

$$|\{s \in [t, t+e^{-\alpha t}] : |E^{(h_n)}(s)| \neq m\}| \to 0, \quad \text{as } n \to \infty.$$

Then by (3.12) and by the mean value theorem there exists $s_n \in [t, t+e^{-\alpha t}]\backslash A(t)$ such that

$$\|\kappa_{E^{(h_n)}(s_n)} - \overline{\kappa}_{E^{(h_n)}(s_n)}\|^2_{L^2(\partial E^{(h_n)}(s_n))} \leq \int_{\partial E^{(h_n)}(s_n)} (v^{(h_n)}_{s_n})^2 d\mathcal{H}^1 \leq Ce^{-\left(\frac{1}{2C_0'}-\alpha\right)t}, \tag{3.13}$$

$|E^{(h_n)}(s_n)| = m$, and thus, in particular, $f_n(s_n) = P(E^{(h_n)}(s_n))$. From Proposition 2.1 and (3.13), we infer that, for $t \geq \tilde{t}$, where $\tilde{t}$ is sufficiently large, $E^{(h_n)}(s_n)$ is diffeomorphic to a union of d disjoint disks and

$$|P(E^{(h_n)}(s_n)) - P_d| \leq Ce^{-\left(\frac{1}{2C_0'}-\alpha\right)t}. \tag{3.14}$$

In particular, passing to the limit in $h_n \to 0$ (up to a further not relabelled subsequence, if needed), there exists $s_t \in [t, t+e^{-\alpha t}]\backslash A(t)$ such that $s_n \to s_t$ and thus $E^{(h_n)}(s_n) \to E(s_t)$ in L^1 and $P(E^{(h_n)}(s_n)) = f_n(s_n) \to f_\infty(s_t)$. In fact, by the uniform $C^{1,\frac{1}{2}}$-bounds provided by (3.13) and Proposition 2.1 we deduce that $P(E^{(h_n)}(s_n)) \to P(E(s_t))$ and thus $f_\infty(s_t) = P(E(s_t))$, and that $E(s_t)$ is the union of d nearly spherical sets parametrized over d disjoint open disks $D_r(x_i(t))$, $i = 1, \ldots d$ of volume m/d, with $C^{1,\frac{1}{2}}$-norm of the parametrizations (exponentially) small. In particular, setting $F(t) := \cup_{i=1}^d D_r(x_i(t))$, we have that $\sup_{x \in E(s_t)\Delta F(t)} \operatorname{dist}(x, \partial F(t))$ decays exponentially to zero as $t \to +\infty$, E_∞ is a union of d disjoint open disks of volume m/d, and $F(t) \to E_\infty$ in the Hausdorff sense exponentially fast.

Summarizing, and recalling also the first inequality in (3.6) and (3.14), we have shown that for every t sufficiently large, there exists $s_t \in [t, t+e^{-\alpha t}]$ such that $E(s_t)$ is the union of d disjoint nearly spherical sets parametrized over the disjoint open disks of E_∞ and

$$\begin{aligned} P_d \leq f_\infty(s_t) = P(E(s_t)) &\leq P_d + Ce^{-(1/C-\alpha)t}, \\ \sup_{x \in E(s_t)\Delta E_\infty} \operatorname{dist}(x, \partial E_\infty) &\leq Ce^{-\frac{t}{C}}, \end{aligned} \tag{3.15}$$

for a suitable constant $C > 1$.

From the first inequality in (3.15) and by the monotonicity of f_∞ we obtain for all s sufficiently large that by choosing t such that $s = t + e^{-\alpha t}$ it holds

$$P(E(s)) \leq f_\infty(s) \leq f_\infty(s_t) \leq P_d + Ce^{-(1/C-\alpha)(s-e^{-\alpha t})} \leq P_d + Ce^{-\frac{(1/C-\alpha)s}{2}}.$$

On the other hand, by Lemma 3.3 and (3.11) we obtain

$$P_d \leq P(E(t)) + C|E(t)\Delta E_\infty|^{\frac{1}{3}} \leq P(E(t)) + C'e^{-\frac{t}{12C_0'}}.$$

Hence, we have the exponential convergence of the perimeters in (1.4).

The first part of the inequality in (1.4) follows from the second inequality in (3.15) and from [28, Lemma 4.3].

Case 2: There exist $d \in \mathbb{N}\setminus\{0\}$ and $\bar{t} > 0$ such that $F_\infty = P_d = f_\infty(t)$ for every $t \geq \bar{t}$.

In this case, using the monotonicity of the functions f_n's, we deduce that for every $T > \bar{t}$ the functions f_n converge uniformly to $f_\infty \equiv F_\infty$ in $[\bar{t}, T]$. In particular, using that

$$\frac{1}{h_n}\mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \leq f_n((k-1)h_n) - f_n(kh_n),$$

we deduce that for every $t \in [\bar{t} + h_n, T]$ we have

$$\begin{aligned}\sum_{k=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{T}{h_n}\rfloor} \frac{1}{h_n}\mathcal{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \;&\leq\; f_n\Big(\lfloor \frac{t}{h_n}\rfloor h_n\Big) - f_n\Big(\lfloor \frac{T}{h_n}\rfloor h_n\Big)\\ &=: b_n \to F_\infty - F_\infty = 0\end{aligned}$$

as $h_n \to 0$. Arguing as above, for every $\bar{t} + h_n \leq t < s \leq T$, we get

$$|E^{(h_n)}(t)\Delta E^{(h_n)}(s)| \leq C\ell\frac{s-t}{h_n}P(E(0)) + \frac{C}{\ell}\sum_{i=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{s}{h_n}\rfloor} \mathcal{D}(E_i^{(h_n)}, E_{i-1}^{(h_n)}),$$

for all $\ell \leq \frac{1}{C}\sqrt{h_n}$ and, choosing $\ell = \sqrt{b_n}h_n$, we conclude that

$$|E^{(h_n)}(t)\Delta E^{(h_n)}(s)| \leq C\sqrt{b_n}(s-t)P(E(0)) + C\sqrt{b_n} \to 0,$$

that is $E(t) = E(s)$ for every $\bar{t} < t < s < T$.

The final part of the proof consists in showing that the limiting set E_∞ is the union of disjoint open disks with the same radius. We have

$$\begin{aligned}\int_t^T \int_{\partial E^{(h_n)}(t)} (v_t^{(h_n)})^2 d\mathcal{H}^1 &= \frac{1}{h_n}\sum_{i=\lfloor \frac{t}{h_n}\rfloor}^{\lfloor \frac{T}{h_n}\rfloor} \int_{\partial E_i^{(h_n)}} d^2_{E_{i-1}^{(h_n)}}\\ &= \sum_{i=\lfloor \frac{t}{h_n}\rfloor}^{\lfloor \frac{T}{h_n}\rfloor} \frac{1}{h_n}\mathcal{D}(E_i^{(h_n)}, E_{i-1}^{(h_n)}) = o(1).\end{aligned}$$

By the mean value theorem, for every T sufficiently large there exists $t_n \in [T, T+1]$ such that

$$\|\kappa_{E^{(h_n)}(t_n)} - \overline{\kappa}_{E^{(h_n)}(t_n)}\|^2_{L^2(\partial E^{(h_n)}(t_n))} \le \int_{\partial E^{(h_n)}(t_n)} (v_{t_n}^{(h_n)})^2 d\mathcal{H}^1 = o(1).$$

As before, by Proposition 2.1 the sets $E^{(h_n)}(t_n)$ are nearly spherical and converge to the union of d disjoint open balls. From here the conclusion follows. ◻

4 The asymptotics of the 2D Mullins–Sekerka flow

Let us first construct a flat flow solution for the Mullins–Sekerka flow in the 2-dimensional flat torus. The construction in the case of bounded domain is due to Luckhaus and Sturzenhecker [34] and the same construction can be applied to the periodic setting with obvious changes. We denote the perimeter of a set E in the flat torus $\mathbb{T}^2$ by $P_{\mathbb{T}^2}(E)$ and recall that it is defined as

$$P_{\mathbb{T}^2}(E) := \sup\left\{\int_E \operatorname{div} X\, dx : X \in C^1(\mathbb{T}^2, \mathbb{R}^2),\ \|X\|_{L^\infty} \le 1\right\}.$$

Here $X \in C^1(\mathbb{T}^2, \mathbb{R}^2)$ means that the $\mathbb{Z}^2$-periodic extension of X to $\mathbb{R}^2$ is continuously differentiable. For a given set of finite perimeter $E \subset \mathbb{T}^2$, with $|E| = m$, we consider the minimization problem

$$\min\left\{P_{\mathbb{T}^2}(F) + \frac{h}{2}\int_{\mathbb{T}^2} |\nabla U_{F,E}|^2\, dx : \quad \text{with } |F| = |E| = m\right\}, \tag{4.1}$$

where the function $U_{F,E} \in H^1(\mathbb{T}^2)$ is the solution of

$$-\Delta U_{F,E} = \frac{1}{h}(\chi_F - \chi_E) \tag{4.2}$$

with zero average. As proven in [34, 45] there exists a minimizer for (4.1), but it might not be unique. Concerning the regularity of the minimizers we may argue as in [2, Theorem 2.8.] (see also [39, Proposition 2.2]) to deduce that the minimizing set F is $C^{3,\alpha}$-regular. Let us briefly sketch the argument. First, we may replace the volume constraint in (4.1) by volume penalization as in [2, 21] and conclude that the minimizer is a Λ-minimizer of the perimeter. This implies that the minimizer is $C^{1,\alpha}$-regular and satisfies the associated Euler–Lagrange equation

$$U_{F,E} = -\kappa_F + \lambda \qquad \text{on } \partial F$$

in a weak sense (see, for instance, [34, 45]), where λ is the Lagrange multiplier. Since $U_{F,E}$ is the solution of (4.2), by standard elliptic regularity it holds $U_{F,E} \in C^{1,\alpha}(\mathbb{T}^2)$.

Then by the Euler–Lagrange equation we deduce that F is in fact $C^{3,\alpha}$-regular and the Euler–Lagrange equation holds in the classical sense.

Let us denote

$$\mathfrak{D}(F, E) := \int_{\mathbb{T}^2} |\nabla U_{F,E}|^2 \, dx \tag{4.3}$$

where $U_{F,E}$ is defined in (4.2). We define the H^{-1}-norm of a function f on the torus $\mathbb{T}^2$ with $\int_{\mathbb{T}^2} f = 0$ by duality as

$$\|f\|_{H^{-1}(\mathbb{T}^2)} := \sup \left\{ \int_{\mathbb{T}^2} \varphi \, f \, dx : \|\nabla \varphi\|_{L^2(\mathbb{T}^2)} \leq 1 \right\}.$$

Then, integrating (4.2) by parts yields

$$\|\chi_F - \chi_E\|^2_{H^{-1}(\mathbb{T}^2)} \leq h^2 \, \|\nabla U_{F,E}\|^2_{L^2(\mathbb{T}^2)} = h^2 \, \mathfrak{D}(F, E). \tag{4.4}$$

We fix the time step $h > 0$ and our initial set $E(0) \subset \mathbb{T}^2$ and let $E_1^{(h)}$ be a minimizer of (4.1) with $E(0) = E$. We construct the discrete-in-time evolution $(E_k^{(h)})_{k \in \mathbb{N}}$ as before by induction such that, assuming that $E_k^{(h)}$ is defined, we set $E_{k+1}^{(h)}$ to be a minimizer of (4.1) with $E = E_k^{(h)}$ and denote the associated potential for short by $U_{k+1}^{(h)}$, which is the solution of

$$-\Delta U_{k+1}^{(h)} = \frac{1}{h} \left(\chi_{E_{k+1}^{(h)}} - \chi_{E_k^{(h)}} \right) \tag{4.5}$$

with zero average. The Euler–Lagrange equation now reads as

$$U_{k+1}^{(h)} = -\kappa_{E_{k+1}^{(h)}} + \lambda_{k+1}^{(h)} \qquad \text{on } \partial E_{k+1}^{(h)}. \tag{4.6}$$

By a direct energy comparison (formula (3.6) in [45]) we obtain

$$P_{\mathbb{T}^2}(E_{k+1}^{(h)}) + \frac{h}{2} \mathfrak{D}(E_{k+1}^{(h)}, E_k^{(h)}) \leq P_{\mathbb{T}^2}(E_k^{(h)}), \tag{4.7}$$

where $\mathfrak{D}(E_{k+1}^{(h)}, E_k^{(h)})$ is defined in (4.3).

As before we define the approximate flat flow $\{E^{(h)}(t)\}_{t \geq 0}$ by setting

$$E^{(h)}(t) = E_k^{(h)} \qquad \text{for } t \in [kh, (k+1)h)$$

and we call a *flat flow solution* of (1.2) any cluster point $\{E(t)\}_{t \geq 0}$ of $\{E^{(h)}(t)\}_{t \geq 0}$, as $h \to 0$; i.e.,

$$E^{(h_n)}(t) \to E(t) \quad \text{in } L^1 \text{ for almost every } t > 0 \text{ and for some } h_n \to 0.$$

Arguing exactly as in [45, Proposition 3.1] we may conclude that there exists a flat flow starting from $E(0)$ such that $P_{\mathbb{T}^2}(E(t)) \leq P_{\mathbb{T}^2}(E(0))$, $|E(t)| = |E(0)|$ for every $t \geq 0$ and $\{E(t)\}_{t \geq 0}$ satisfies the equation (1.2) in a weak sense.

To proceed, we need the analogue of Proposition 2.1 for the Mullins–Sekerka flow. To this aim we first prove the following lemma, which is similar to [47, Lemma 2.1].

Lemma 4.1 *Let $E \subset \mathbb{T}^2$ be a set of class C^3, with $|E| \leq \frac{1}{2}$ and $P_{\mathbb{T}^2}(E) < 2$, and let $u_E \in C^1(\mathbb{T}^2)$ be a function with zero average such that $\|\nabla u_E\|_{L^2(\mathbb{T}^2)} \leq M$ and*

$$\kappa_E = -u_E + \lambda \quad \text{on } \partial E \quad \text{for some } \lambda \in \mathbb{R}. \tag{4.8}$$

Then it holds

$$\sup_{x \in \mathbb{T}^2, \rho > 0} \frac{\mathcal{H}^1(\partial E \cap D_\rho(x))}{\rho} \leq K, \tag{4.9}$$

where the constant $K > 0$ depends only on $|E|$ and M.

Proof We note that by (4.8) for every $X \in C^1(\mathbb{T}^2; \mathbb{R}^2)$ it holds

$$\int_{\partial E} \operatorname{div}_\tau X \, d\mathcal{H}^1 = \int_E \operatorname{div}\big((-u_E + \lambda) X\big) \, dx. \tag{4.10}$$

Therefore the statement follows from [47, Lemma 2.1] once we bound the Lagrange multiplier $\lambda \in \mathbb{R}$. To this aim, and for future purpose, we show that there is $\delta > 0$ such that every component E_i of E is contained in a cube $Q_{1-\delta}(x_i) := (1-\delta)^2 + \{x_i\}$ for some x_i.

Let us first show that every component Γ_i of the boundary ∂E divides the torus $\mathbb{T}^2$ in two components and thus it is the boundary of a set. Indeed, if this is not the case then necessarily $\mathcal{H}^1(\Gamma_i) \geq 1$. Since Γ_i is not a boundary of a set then ∂E must have another component, say Γ_j, such that $\mathcal{H}^1(\Gamma_j) \geq 1$. But this implies $P_{\mathbb{T}^2}(E) = \mathcal{H}^1(\partial E) \geq 2$, which contradicts the assumption $P_{\mathbb{T}^2}(E) < 2$.

Let us next show that Γ_i is contained in a cube $Q_{1-\delta}(x_i)$ for some x_i. Let $\pi_1 : \mathbb{T}^2 \to \mathbb{T}$ be the projection onto the x_1-axis i.e., $\pi_1(x_1, x_2) = x_1$. Then we deduce from $\mathcal{H}^1(\Gamma_i) < 2$ and from the fact that Γ_i is the boundary of a set that $\mathcal{H}^1(\pi_1(\Gamma_i)) < 1$. Similarly it holds $\mathcal{H}^1(\pi_2(\Gamma_i)) < 1$, where π_2 is the projection onto the x_2-axis. This implies that $\Gamma_i \subset Q_{1-\delta}(x_i)$ for some $\delta > 0$ and x_i. Let us from now on denote the set enclosed by Γ_i which is inside the cube $Q_{1-\delta}(x_i)$ by F_i.

Let $\Gamma_1, \ldots, \Gamma_n$ be the components of the boundary ∂E which enclose the sets $F_1, \ldots, F_n$. Let us show that

$$E \subset \bigcup_{i=1}^n F_i. \tag{4.11}$$

Since $F_i \subset Q_{1-\delta}(x_i)$ we have by the Isoperimetric Inequality $2\sqrt{\pi|F_i|} \le \mathcal{H}^1(\Gamma_i)$. Therefore by the assumption on the perimeter, $P_{\mathbb{T}^2}(E) < 2$, we have

$$4\pi\Big|\bigcup_{i=1}^n F_i\Big| \le 4\pi\sum_{i=1}^n |F_i| \le \sum_{i=1}^n \mathcal{H}^1(\Gamma_i)^2 \le \Big(\sum_{i=1}^n \mathcal{H}^1(\Gamma_i)\Big)^2 \le P_{\mathbb{T}^2}(E)^2 < 4.$$

Therefore $\big|\bigcup_{i=1}^n F_i\big| < \frac{1}{\pi} < \frac{1}{3}$. Since, $|E| \le \frac{1}{2}$ then necessarily $E \subset \bigcup_{i=1}^n F_i$.

We conclude from (4.11) that a component E_j of E is contained in F_i for some i. Therefore since $F_i \subset Q_{1-\delta}(x_i)$, then also $E_j \subset Q_{1-\delta}(x_i)$.

We may finally bound the Lagrange multiplier in (4.10) by a standard argument. Indeed, let E_j be a component of E. Since $E_j \subset Q_{1-\delta}(x_j)$ we may define $X \in C_0^1\big(Q_{1-\delta/3}(x_j)\big)$ such that $X(x) = x$ in E_j and $X(x) = 0$ in $E\setminus E_j$. We apply (4.10) with this choice of X and have

$$\begin{aligned} P_{\mathbb{T}^2}(E_j) = \int_{\partial E_j} \operatorname{div}_\tau x \, d\mathcal{H}^1 &= \int_{E_j} \operatorname{div}\big((-u_E + \lambda)x\big)\, dx \\ &= -\int_{E_j} \operatorname{div}(u_E\, x)\, dx + 2\lambda|E_j|. \end{aligned}$$

We have $\big|\int_{E_j} \operatorname{div}(u_E\, x)\, dx\big| \le C\|u_E\|_{H^1(E_j)}$. By repeating the argument for every component we obtain by the Poincaré inequality

$$\begin{aligned} |\lambda||E| &\le P_{\mathbb{T}^2}(E) + C\|u_E\|_{H^1(E)} \le P_{\mathbb{T}^2}(E) + C\|u_E\|_{H^1(\mathbb{T}^2)} \\ &\le P_{\mathbb{T}^2}(E) + C\|\nabla u_E\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

This yields the required bound on the Lagrange multiplier. □

We also recall the result by Meyers–Ziemer [38, Theorem 4.7] which implies that if E satisfies (4.9) then for every $\varphi \in C^1(\mathbb{T}^2)$ it holds

$$\Big|\int_{\partial E} \varphi \, d\mathcal{H}^1\Big| \le C\|\varphi\|_{W^{1,1}(\mathbb{T}^2)}, \tag{4.12}$$

with C depending on K (and thus on $|E|$ and M).

We are now ready to state and prove the analogue of Proposition 2.1, which is suited for the Mullins–Sekerka flow.

Proposition 4.2 *Let $E \subset \mathbb{T}^2$ be a set of class C^3, with $|E| = m \le \frac{1}{2}$ and $P_{\mathbb{T}^2}(E) < 2$, and let $u_E \in C^1(\mathbb{T}^2)$ be a function with zero average such that*

$$\kappa_E = -u_E + \lambda \quad on\ \partial E$$

for some $\lambda \in \mathbb{R}$. Then, there exist $\varepsilon_0 = \varepsilon_0(m) \in (0,1)$ and $C_0 = C_0(m) > 1$ such that if

$$\|\nabla u_E\|_{L^2(\mathbb{T}^2)} \le \varepsilon_0$$

then E is diffeomorphic to a union of d disjoint disks D_1, ..., D_d with equal areas m/d and $dist(D_i, D_j) > 0$ for $i \neq j$. Moreover,

$$|P_{\mathbb{T}^2}(E) - P_d| \leq C_0 \|\nabla u_E\|^2_{L^2(\mathbb{T}^2)}$$

and for ε_0 sufficiently small the boundary of every connected component of the set E can be parametrized as a normal graph over one of the disc D_i with $C^{1,\frac{1}{2}}$ norm of the parametrization vanishing as $\varepsilon_0 \to 0$.

We note that we need the assumption $P_{\mathbb{T}^2}(E) < 2$ to exclude the case when E is a strip or a union of strips.

Proof We recall that the argument in the proof of Lemma 4.1 implies that every component E_i of E is contained in a cube $Q_{1-\delta}(x_i)$ for some x_i. By Lemma 4.1, we can apply (4.12) with $\varphi = u_E^2$ and obtain

$$\int_{\partial E} u_E^2 \, d\mathcal{H}^1 \leq C \|u_E^2\|_{W^{1,1}(\mathbb{T}^2)} \leq C \|u_E\|^2_{H^1(\mathbb{T}^2)} \leq C \|\nabla u_E\|^2_{L^2(\mathbb{T}^2)},$$

where the last inequality follows from Poincaré inequality. Since u_E satisfies (4.8) we deduce by the assumption $\|\nabla u_E\|_{L^2(\mathbb{T}^2)} \leq \varepsilon_0$ that

$$\int_{\partial E} |\kappa_E - \overline{\kappa}_E|^2 \, d\mathcal{H}^1 \leq \int_{\partial E} |\kappa_E - \lambda|^2 \, d\mathcal{H}^1 = \int_{\partial E} u_E^2 \, d\mathcal{H}^1 \leq C \|\nabla u_E\|^2_{L^2(\mathbb{T}^2)} \leq C\varepsilon_0^2.$$

Hence, the claim follows from Proposition 2.1. □

Proposition 4.2 immediately implies the following corollary.

Corollary 4.3 *Let $E \subset \mathbb{T}^2$ be a set of class C^3, with $|E| = m \leq \frac{1}{2}$ and $P_{\mathbb{T}^2}(E) < 2$ and let $u_E \in C^1(\mathbb{T}^2)$ be a function with zero average such that $\kappa_E = -u_E + \lambda$ on ∂E for some $\lambda \in \mathbb{R}$. If $\delta_0 > 0$ and $d \in \mathbb{N}$, are such that $P_d \leq P(E) \leq P_{d+1} - \delta_0$, then it holds*

$$P(E) - P_d \leq C_0 \|\nabla u_E\|^2_{L^2(\mathbb{T}^2)}$$

for $C_0 = C_0(m, \delta_0)$.

We also need the following lemma which is essentially a restatement of [34, Lemma 3.1]. The proof can also be found in [9, Lemma 2], but we recall it for the reader's convenience.

Lemma 4.4 *Let $\varphi \in BV(\mathbb{T}^2)$. There is a constants $C > 1$ and $\rho_0 > 0$ such for all $\rho \leq \rho_0$ it holds*

$$\|\varphi\|_{L^1(\mathbb{T}^2)} \leq C\rho \, \|\varphi\|_{BV(\mathbb{T}^2)} + C\rho^{-1} \|\varphi\|_{H^{-1}(\mathbb{T}^2)}.$$

Proof Let us fix $\rho > 0$ and let $\eta_\rho(x) = \rho^{-2}\eta(\frac{x}{\rho})$ be the standard mollifier. Then we write

$$\|\varphi\|_{L^1(\mathbb{T}^2)} \leq \int_{\mathbb{T}^2} |\varphi - \varphi * \eta_\rho|\, dx + \int_{\mathbb{T}^2} |\varphi * \eta_\rho|\, dx.$$

Let us first bound the second term on the RHS. Since $\|\eta_\rho\|_{H^1(\mathbb{T}^2)} \leq C/\rho$ we obtain by the definition of the H^{-1}-norm

$$\begin{aligned}\int_{\mathbb{T}^2} |\varphi * \eta_\rho|\, dx &= \int_{\mathbb{T}^2} \Big| \int_{\mathbb{T}^2} \varphi(y)\eta_\rho(y-x)\, dy \Big| dx \\ &\leq \|\varphi\|_{H^{-1}(\mathbb{T}^2)} \|\eta_\rho\|_{H^1(\mathbb{T}^2)} \leq C\rho^{-1}\, \|\varphi\|_{H^{-1}(\mathbb{T}^2)}.\end{aligned}$$

We bound the first term by change of variables

$$\begin{aligned}\int_{\mathbb{T}^2} |\varphi - \varphi * \eta_\rho|\, dx &= \int_{\mathbb{T}^2} \Big| \int_{\mathbb{T}^2} \big(\varphi(x) - \varphi(x+\rho y)\big)\eta(y)\, dy \Big|\, dx \\ &= \int_{\mathbb{T}^2} \Big| \int_{\mathbb{T}^2} \int_0^\rho -\frac{\partial}{\partial \tau}\varphi(x+\tau y)\eta(y)\, d\tau dy \Big|\, dx \\ &\leq C\rho\, \|\varphi\|_{BV(\mathbb{T}^2)}.\end{aligned}$$

□

We are ready to prove the convergence of the Mullins–Sekerka flow in the flat torus $\mathbb{T}^2$.

Proof of Theorem 1.3 The proof is similar to the proof of Theorem 1.2 but we highlight the main differences. Let $\{E(t)\}_{t\geq 0}$ be a flat flow for the Mullins–Sekerka flow and let $\{E^{(h_n)}(t)\}_{t\geq 0}$ be an approximate flow converging to $E(t)$. Since $\{\mathbb{T}^2 \setminus E(t)\}_{t\geq 0}$ is a flat flow starting from $\mathbb{T}^2 \setminus E(0)$, by replacing $E(0)$ with its complement in $\mathbb{T}^2$ if needed, we may assume without loss of generality that $|E(0)| \leq \frac{1}{2}$. We will show that in this case the limiting set is a finite union of disjoint open discs with equal radii.

Arguing as before we deduce that by (4.7) the functions

$$f_n(t) = P_{\mathbb{T}^2}(E^{(h_n)}(t))$$

are monotone non-increasing with $f_n(t) < 2$ and (possibly up to a further unrelabelled subsequence) converge pointwise to a non-increasing function $f_\infty : [0, +\infty) \to \mathbb{R}$. Set $F_\infty = \lim_{t\to+\infty} f_\infty(t)$. Again we divide the proof in two cases.

Case 1: There exists $d \in \mathbb{N}\setminus\{0\}$ such that either $P_d < F_\infty < P_{d+1}$, or $F_\infty = P_d$ and $f_\infty(t) > P_d$ for every $t \in [0, +\infty)$. In this case, there exists $\bar{t} \geq 1$ such that, for every $T > \bar{t}$ there exist $\bar{n} \in \mathbb{N}\setminus\{0\}$ such that

$$P_d \leq f_n(t) < P_{d+1} \qquad \text{and} \qquad P_{d+1} - f_n(t) \geq \frac{P_{d+1} - F_\infty}{2} =: \delta_0 \tag{4.13}$$

for every $n \geq \bar{n}$ and $t \in [\bar{t}, T]$. By summing (4.7) and using (4.13) we obtain for every $i \in \left\{\lfloor \frac{\bar{t}}{h_n} \rfloor, \dots, \lfloor \frac{T}{h_n} \rfloor\right\}$ that

$$\frac{h_n}{2} \sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \leq P_{\mathbb{T}^2}(E_i^{(h_n)}) - P_{\mathbb{T}^2}\big(E_{\lfloor \frac{T}{h_n} \rfloor}^{(h_n)}\big) \leq P_{\mathbb{T}^2}(E_i^{(h_n)}) - P_d, \tag{4.14}$$

where $\mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)})$ is defined in (4.3). Then by (4.14) and by Corollary 4.3 it holds

$$\begin{aligned}\frac{h_n}{2} \sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) &\leq P_{\mathbb{T}^2}(E_i^{(h_n)}) - P_d \leq C_0 \|\nabla U_i^{(h)}\|_{L^2(\mathbb{T}^2)}^2 \\ &= C_0 \mathfrak{D}(E_i^{(h_n)}, E_{i-1}^{(h_n)}).\end{aligned}$$

Therefore we conclude

$$\sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} \mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \leq \frac{2C_0}{h_n} \mathfrak{D}(E_i^{(h_n)}, E_{i-1}^{(h_n)}).$$

Setting $a_k^{(h_n)} = h_n\, \mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)})$ we have that for every $i \in \left\{\lfloor \frac{\bar{t}}{h_n} \rfloor, \dots, \lfloor \frac{T}{h_n} \rfloor\right\}$ it holds

$$\sum_{k=i}^{\lfloor \frac{T}{h_n} \rfloor} a_k^{(h_n)} \leq \frac{2C_0 + h_n}{h_n} a_i^{(h_n)} \leq \frac{3C_0}{h_n} a_i^{(h_n)}$$

and by applying (4.14) with $i = \lfloor \frac{\bar{t}}{h_n} \rfloor$ yields

$$\sum_{k=\lfloor \frac{\bar{t}}{h_n} \rfloor + 1}^{\lfloor \frac{T}{h_n} \rfloor} a_k^{(h_n)} \leq P_{\mathbb{T}^2}\left(E_{\lfloor \frac{\bar{t}}{h_n} \rfloor}^{(h_n)}\right) \leq P_{\mathbb{T}^2}(E(0)) < 2.$$

Therefore Lemma 3.2, with $\mathcal{I}$ being the empty set this time, implies

$$\sum_{k=i+1}^{\lfloor \frac{T}{h_n} \rfloor} a_k^{(h_n)} \leq 2\left(1 - \frac{h_n}{3C_0}\right)^{i - \frac{\bar{t}}{h_n}} \qquad \text{for all } i = \lfloor \frac{\bar{t}}{h_n} \rfloor, \dots, \lfloor \frac{T}{h_n} \rfloor.$$

In other words for every $t \in [\bar t, T]$ we have

$$\sum_{k=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{T}{h_n}\rfloor} h_n\,\mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)}) \le 2\left(1-\frac{h_n}{3C_0}\right)^{\lfloor \frac{t}{h_n}\rfloor - \frac{\bar t}{h_n}} \le Ce^{-\frac{t}{3C_0}}$$

for $h_n \le h_0(T)$. Then, by (4.4) and by the above inequality we have that for $\bar t \le t < s \le T$ with $s \le t+1$ it holds

$$\begin{aligned}
\|\chi_{E^{(h_n)}(s)} - \chi_{E^{(h_n)}(t)}\|_{H^{-1}(\mathbb{T}^2)} &\le \sum_{k=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{s}{h_n}\rfloor} \|\chi_{E_k^{(h_n)}} - \chi_{E_{k-1}^{(h_n)}}\|_{H^{-1}(\mathbb{T}^2)} \\
&\le \frac{\sqrt{s-t}}{\sqrt{h_n}}\Big(\sum_{k=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{T}{h_n}\rfloor} \|\chi_{E_k^{(h_n)}} - \chi_{E_{k-1}^{(h_n)}}\|^2_{H^{-1}(\mathbb{T}^2)}\Big)^{\frac12} \\
&\le \frac{1}{\sqrt{h_n}}\Big(\sum_{k=\lfloor \frac{t}{h_n}\rfloor+1}^{\lfloor \frac{T}{h_n}\rfloor} h_n^2\,\mathfrak{D}(E_k^{(h_n)}, E_{k-1}^{(h_n)})\Big)^{\frac12} \\
&\le C\,e^{-\frac{t}{6C_0}},
\end{aligned} \tag{4.15}$$

when $h_n \le h_0(T)$.

Recall for all $t > 0$ it holds $\|\chi_{E^{(h_n)}(t)}\|_{\mathrm{BV}(\mathbb{T}^2)} \le \|\chi_{E(0)}\|_{\mathrm{BV}(\mathbb{T}^2)} \le 3$. We use Lemma 4.4 and (4.15) to deduce

$$\begin{aligned}
&\|\chi_{E^{(h_n)}(s)} - \chi_{E^{(h_n)}(t)}\|_{L^1(\mathbb{T}^2)} \\
&\quad\le C\varepsilon\,\|\chi_{E^{(h_n)}(s)} - \chi_{E^{(h_n)}(t)}\|_{\mathrm{BV}(\mathbb{T}^2)} + C\varepsilon^{-1}\|\chi_{E^{(h_n)}(s)} - \chi_{E^{(h_n)}(t)}\|_{H^{-1}(\mathbb{T}^2)} \\
&\quad\le C\varepsilon + C\varepsilon^{-1}e^{-\frac{t}{6C_0}}.
\end{aligned}$$

Choosing $\varepsilon = e^{-\frac{t}{12C_0}}$ yields

$$\|\chi_{E^{(h_n)}(s)} - \chi_{E^{(h_n)}(t)}\|_{L^1(\mathbb{T}^2)} \le Ce^{-\frac{t}{12C_0}}.$$

Letting $h_n \to 0$ we obtain for the limit flow

$$|E(s)\Delta E(t)| \le C\,e^{-\frac{t}{12C_0}}.$$

From here we conclude that $E(t)$ converges to a set of finite perimeter E_∞ exponentially fast.

We may characterize the limit set E_∞ as a disjoint union of open disks $D_r(x_1), \dots, D_r(x_d)$ thanks to Proposition 4.2 by arguing as in the proof of Theorem 1.2. Similarly, we obtain the convergence of the perimeters. We leave the details for the reader.

Also the argument for the Case 2, when there exist $d \in \mathbb{N}\setminus\{0\}$ and $\bar{t} > 0$ such that $F_\infty = P_d = f_\infty(t)$ for every $t \geq \bar{t}$, follows by the same argument as in the proof of Theorem 1.2. □

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