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On the shape of Meissner solutions to the 2-dimensional Ginzburg–Landau system

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Abstract

This paper concerns the asymptotic behavior of the stable solution $(f_\lambda, \mathbf{Q}_\lambda)$ of the full Meissner state equation for a two-dimensional superconductor with penetration depth λ and Ginzburg–Landau parameter κ, and subjected to an applied magnetic field $\mathcal{H}^e$. It is known that the solution is stable if the minimum value of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ is larger than $1/3$, and the solution loses its stability when the minimum value reached $1/3$. It has been conjectured that the location of the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ has connection with the location of vortex nucleation of the superconductor. In this paper, we prove that if the penetration depth λ is small, the solution $(f_\lambda, \mathbf{Q}_\lambda)$ exhibits boundary layer behavior, and $(1 - f_\lambda, \mathbf{Q}_\lambda)$ exponentially decays in the normal direction away from the boundary. Moreover, the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ locate near the set $S(\mathcal{H}^e)$, which is determined by the applied magnetic field $\mathcal{H}^e$ and the geometry of the domain. In the special case where the applied magnetic field $\mathcal{H}^e$ is constant, the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ locate near the maximum points of the curvature of the domain boundary.

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Contents

1 Introduction

1.1 The equation

This paper concerns the asymptotic behavior, as $\lambda \to 0$, of the solutions of the following equation:

$$\begin{cases} -\frac{\lambda^2}{\kappa^2}\Delta f = (1-|f|^2-|\mathbf{Q}|^2)f & \text{in } \Omega, \\ \lambda^2\mathrm{curl}^2\mathbf{Q} + |f|^2\mathbf{Q} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{n}\cdot\nabla f = 0, \quad \lambda\mathrm{curl}\mathbf{Q} = \mathcal{H}^e & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, and $\mathcal{H}^e$ is a given function. f and $\mathbf{Q}$ are unknown, where f is a scalar function and $\mathbf{Q} = (Q_1, Q_2)$ is a vector field. For a vector field $\mathbf{Q}$ in two dimensions,

$$\mathrm{curl}\mathbf{Q} = \partial_2 Q_1 - \partial_1 Q_2, \quad \mathrm{curl}^2\mathbf{Q} = (\partial_2(\mathrm{curl}\mathbf{Q}), -\partial_1(\mathrm{curl}\mathbf{Q})).$$

Equation (1.1) is called the *Meissner equation*, as it describes the Meissner states of a type II superconductor occupying a cylinder of infinite hight with its axis along the x_3-axis and a cross section Ω in the x_1x_2-plane, and subjected to an axial applied magnetic field $\mathbf{H} = (0, 0, \mathcal{H}^e)$. κ and λ are positive constants, among them, λ is the penetration depth of the superconductor (generally $0 < \lambda \ll 1$), and κ is the Ginzburg-Landau parameter given by the ratio of the penetration depth and the coherence length.

1.2 Motivation from phase transformation of Meissner states

In the classical theory of superconductivity, the electromagnetic behavior of a superconductor is described by a global minimizer of the Ginzburg-Landau energy functional. A superconductor of type II is subjected to an increasing magnetic field will undergo phase transitions, and there exist three critical values for the strength of the applied field, denoted by H_{C_1}, H_{C_2} and H_{C_3} respectively, with $H_{C_1} < H_{C_2} < H_{C_3}$. If the applied field is below H_{C_1}, it will be excluded from the bulk of the superconductor and the sample is in a superconducting state, which is also called a *Meissner state*. This phenomenon is the well-known Meissner effect. If its strength of the applied magnetic field is raised to above H_{C_1} but still below H_{C_2}, the applied field will penetrate the sample through some vortices, and the sample is in a mixed state so that both superconducting and normal regions coexist. If the applied field increases to exceed H_{C_2}, but remains below H_{C_3}, the superconductor will be in a surface superconducting state. In this state superconductivity persists only within some thin sheathes near the surface of the sample. If the applied magnetic field is raised above H_{C_3}, superconductivity will be totally destroyed and the entire sample will be in a normal state.

These phenomena have been extensively studied by many mathematicians, see for instance [28–31] for the mathematical theory of the mixed states when the applied magnetic field is between H_{C_1} and H_{C_2}, and see [12, 13, 16, 20, 23, 24] and references therein for the analysis of surface superconductivity when the applied field is between H_{C_2} and H_{C_3}.

Physicists have discovered that, superconductivity can be described by a critical point of the Ginzburg-Landau functional, which is not necessary to be a global minimizer. For type II superconductors, the Meissner state is metastable and persists up to the so-called superheating field H_{sh} which is higher than H_{C_1}, see [18, 21, 32]. As the applied field increases further and reaches H_{sh}, it begins to penetrate the sample and vortices start to nucleate. See [6, 7] and the references therein for the mathematical discussions on the critical field H_{sh} and nucleation of vortices.

We believe that one more critical field is needed in order to understand the phase transitions of the Meissner states. This critical field, denoted by H_S, lies in between H_{C_1} and H_{sh}, and it is a critical value of the strength of the applied magnetic field for a Meissner state to lose local stability. That is, if the applied field is below H_S, the Meissner states are locally stable; while if the applied field reaches H_S, some Meissner states will be locally instable. For comparison, the first critical field H_{C_1} is the critical value of the strength of the increasing applied magnetic field at which some Meissner solutions start to lose global stability.

To explain this critical field H_S, let us recall that in the Ginzburg-Landau theory [15], superconducting behaviors of a sample are described by a critical point $(\psi, \mathbf{A})$ of the Ginzburg-Landau functional. Let us consider a type II superconductor occupying a cylinder in $\mathbb{R}^3$ with its axis along the x_3-axis, subjected to an axial applied magnetic field $(0, 0, \mathcal{H}^e)$, where $\mathcal{H}^e(x_1, x_2) > 0$ is a smooth function. For simplicity, we may also call the function $\mathcal{H}^e$ the *applied field*. Then the Ginzburg-Landau energy is reduced to the two-dimensional functional of the following form

$$\mathcal{E}[\Psi, \mathbf{A}] = \int_\Omega \left\{ \left| \left(\frac{\lambda}{\kappa}\nabla - i\mathbf{A}\right)\Psi \right|^2 + \frac{1}{2}(1-|\Psi|^2)^2 \right\} dx + \int_{\mathbb{R}^2} |\lambda \mathrm{curl}\mathbf{A} - \mathcal{H}^e|^2 dx, \tag{1.2}$$

where Ω is the cross section of the cylinder, Ψ is a complex-valued function called *order parameter* with $|\Psi|^2$ representing the density of superconducting electron pairs, $\mathbf{A}$ is the magnetic potential and curl$\mathbf{A}$ is the induced magnetic field. The Euler-Lagrange equation of the functional $\mathcal{E}$ is called the Ginzburg-Landau equation:

$$\begin{cases} -(\frac{\lambda}{\kappa}\nabla - i\mathbf{A})^2\Psi = (1-|\Psi|^2)\Psi & \text{in } \Omega, \\ \lambda^2 \mathrm{curl}^2\mathbf{A} + \Psi^2\mathbf{A} = \frac{i\lambda}{2\kappa}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) & \text{in } \Omega, \\ \lambda\mathrm{curl}^2\mathbf{A} = \mathrm{curl}\mathcal{H}^e & \text{in } \Omega^c, \\ \mathbf{n}\cdot\left(\frac{\lambda}{\kappa}\nabla - i\mathbf{A}\right)\Psi = 0, \quad [\mathbf{n}\times\mathbf{A}] = \mathbf{0}, \quad [\mathrm{curl}\mathbf{A}] = 0 & \text{on } \partial\Omega, \\ \lambda\mathrm{curl}\mathbf{A} - \mathcal{H}^e \to 0 & \text{as } |x|\to\infty, \end{cases} \tag{1.3}$$

where $[\cdot]$ represents the jump in the enclosed quantity across $\partial\Omega$, and

$$\mathrm{curl}\mathcal{H}^e = (\partial_2(\mathcal{H}^e), -\partial_1(\mathcal{H}^e)).$$

For convenience we call a cylindrical superconductor that can be described by the equation (1.3) as a *two-dimensional superconductor*, and call a superconductor occupying a bounded domain in $\mathbb{R}^3$ and can be described by the Ginzburg-Landau equation on the three-dimensional domain as a *three-dimensional superconductor*.

A Meissner state is represented by a solution $(\Psi, \mathbf{A})$ of (1.3) such that the order parameter Ψ does not have zero points over $\bar{\Omega}$, and such a solution is called *Meissner solution*. If a solution $(\psi, \mathbf{A})$ is such that Ψ has zero points, then the zero points are called *vortices* and $(\psi, \mathbf{A})$ is called a *vortex solution*. Existence of Meissner solutions and vortex solutions of (1.3) have been extensively studied, and very rich results have been established, see for instance [19, 29, 31] and the references therein.[1]

If the applied field is below H_{C_1}, then the global minimizers of the Ginzburg-Landau energy have no zero points, hence they are Meissner states. In other words, those Meissner solutions are globally stable with respect to the Ginzburg-landau energy [28–30]. If the applied field increases to exceed H_{C_1} but is still below H_S, the solutions are no longer global minimizers, but they are still locally stable with respect to some energy functional which may be called *Meissner energy* and will be defined later. If the applied field increases further to exceed H_S but is still below H_{sh}, some Meissner solutions continuous to exist but become instable with respect to the Meissner energy. When the applied field reaches H_{sh}, then some Meissner solutions will change to vortex solutions, namely the order parameters will have zeroes. So the phase transitions of Meissner states with the applied magnetic filed increasing along H_{C_1}, H_S and H_{sh} have different nature, comparing with the phase transitions of the global minimizers

[1] See also [31, Chapter 11] and [9] for the corresponding results of type I superconductors.

with the applied field increasing along H_{C_1}, H_{C_2} and H_{C_3}. Therefore it will be useful to study the whole process how a stable Meissner state loses its local stability and then produces vortices and changes into a mixed state, and find the location where the vortices begin to nucleate.

To study these problems, we start with the Meissner equation derived by Chapman [6, 7]. Let $(\Psi, \mathbf{A})$ be a Meissner solution and suppose that Ψ can be written as

$$\Psi = f e^{i\chi},$$

where f is a positive function and χ is a smooth real function. Then we let

$$\mathbf{A} = \mathbf{Q} + \frac{\lambda}{\kappa}\nabla\chi.$$

Plugging this $(\Psi, \mathbf{A})$ into (1.3), we see that $(f, \mathbf{Q})$ satisfies the following equation:

$$\begin{cases} -\frac{\lambda^2}{\kappa^2}\Delta f = (1-|f|^2-|\mathbf{Q}|^2)f & \text{in } \Omega, \\ \lambda^2\mathrm{curl}^2\mathbf{Q} + |f|^2\mathbf{Q} = \mathbf{0} & \text{in } \Omega, \\ \lambda\mathrm{curl}^2\mathbf{Q} = \mathrm{curl}\mathcal{H}^e & \text{in } \Omega^c, \\ \mathbf{n}\cdot\nabla f = 0, \quad [\mathbf{n}\times\mathbf{Q}] = \mathbf{0}, \quad [\mathrm{curl}\mathbf{Q}] = 0 & \text{on } \partial\Omega, \\ \lambda\mathrm{curl}\mathbf{Q} \to \mathcal{H}^e & \text{as } |x|\to\infty. \end{cases} \tag{1.4}$$

In the two dimensional case, we can write the third and last equalities in (1.4) as follows:

$$\begin{cases} \partial_2(\lambda\mathrm{curl}\mathbf{Q} - \mathcal{H}^e) = 0, \quad \partial_1(\lambda\,\mathrm{curl}\mathbf{Q} - \mathcal{H}^e) = 0 \quad \text{in } \Omega^c, \\ \lambda\,\mathrm{curl}\mathbf{Q} - \mathcal{H}^e \to 0 \quad \text{as } |x|\to\infty. \end{cases}$$

This gives that

$$\lambda\mathrm{curl}\mathbf{Q} = \mathcal{H}^e \quad \text{in } \Omega^c.$$

Therefore, (1.4) is reduced to (1.1) if the condition $[\mathbf{n}\times\mathbf{Q}] = \mathbf{0}$ is ignored.[2]

On the other hand, if $(f, \mathbf{Q})$ is a solution of (1.4) with $0 < f(x) \le 1$, then for any smooth real-valued function χ,

$$(\Psi, \mathbf{A}) = (f e^{i\chi}, \mathbf{Q} + \frac{\lambda}{\kappa}\nabla\chi)$$

is a solution of the Ginzburg-Landau equation (1.3).

Equation (1.1) can be further simplified by taking large κ limit. From the first equality in (1.1) one formally gets $(1-|f|^2-|\mathbf{Q}|^2)f = 0$. For a Meissner state, one

[2] See also [5, 25, 26] for the derivation of (1.4) and (1.1).

expects that $f > 0$, which implies that $|f|^2 = 1 - |\mathbf{Q}|^2$. Plugging this into the second equation in (1.1), we get the limiting equation for $\mathbf{Q}$:

$$\begin{cases} -\lambda^2 \text{curl}^2 \mathbf{Q} = (1 - |\mathbf{Q}|^2)\mathbf{Q} & \text{in } \Omega, \\ \lambda \text{curl} \mathbf{Q} = \mathcal{H}^e & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

For our convenience we may say (1.5) is the special case of (1.1) with $\kappa = \infty$.

Equation (1.5) with $\mathcal{H}^e$ equalling to a positive constant has been studied in [4, 6, 7, 27]. Chapman [6] showed that the solution $\mathbf{Q}$ of (1.5) is stable with respect to the energy associated with (1.5) if $\max_{x\in\bar{\Omega}} |\mathbf{Q}(x)| < 1/\sqrt{3}$, and as the applied field $\mathcal{H}^e$ increases, the solution begins to loss such stability when the maximum value of $|\mathbf{Q}(x)|$ reaches $1/\sqrt{3}$. Berestycki, Bonnet and Chapman [4] showed that the maximum points of $|\mathbf{Q}(x)|$ locate on the domain boundary. Chapman [7] used the asymptotic analysis to derive that the maximum points of $|\mathbf{Q}(x)|$ locate on the most negative points of the boundary curvature, which has been rigorously proved for small λ by Pan and Kwek [27].

The analysis in [6, 7] suggests that the loss of certain stability of Meissner states will lead to generation of vortices, and Chapman conjectured that the location of the maximum points of $|\mathbf{Q}(x)|$ is the location where the first vortices will appear. This conjecture motivates our study on the change of stability of the Meissner solutions of (1.1).

For reader's convenience, we now state the definition of stability of a solution $(f, \mathbf{Q})$ of (1.1). We define the *Meissner energy functional* associated with the equation (1.1) by

$$\mathcal{E}_\Omega[f, \mathbf{Q}] = \int_\Omega \left\{ \frac{\lambda^2}{\kappa^2} |\nabla f|^2 + |f|^2|\mathbf{Q}|^2 + \frac{1}{2}(1 - |f|^2)^2 \right\} dx + \int_\Omega |\lambda \text{curl} \mathbf{Q} - \mathcal{H}^e|^2 dx.$$

Then the second order differential of the functional $\mathcal{E}_\Omega$ is given by the following:

$$\langle \mathcal{E}_\Omega''[f, \mathbf{Q}], [g, \mathbf{B}] \rangle = 2 \int_\Omega \left\{ \frac{\lambda^2}{\kappa^2} |\nabla g|^2 + |f\mathbf{B} + 2g\mathbf{Q}|^2 + 3g^2(|f|^2 - |\mathbf{Q}|^2 - \frac{1}{3}) \right\} dx + 2 \int_\Omega |\lambda \text{curl} \mathbf{B}|^2 dx.$$

Set

$$\mathcal{W}(\Omega) = [H^1(\Omega) \cap L^\infty(\Omega)] \times [H^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)].$$

Definition 1 Let $(f, \mathbf{Q})$ be a solution of (1.1) and assume $(f, \mathbf{Q}) \in \mathcal{W}(\Omega)$.

(a) We say $(f, \mathbf{Q})$ is a *Meissner solution* of (1.1) if $f(x) > 0$ over $\bar{\Omega}$.

(b) We say $(f, \mathbf{Q})$ is *stable* (with respect to the Meissner equation (1.1)) if $\mathcal{E}''_\Omega$ is non-negative on $\mathcal{W}(\Omega)$, namely if

$$\langle \mathcal{E}''_\Omega[f, \mathbf{Q}], [g, \mathbf{B}]\rangle \geq 0 \quad \text{for all } (g, \mathbf{B}) \in \mathcal{W}(\Omega).$$

Existence and uniqueness of a stable Meissner solution of (1.1) have been discussed in [5].[3] If $(f, \mathbf{Q}) \in \mathcal{W}(\Omega)$ and if

$$|f(x)|^2 - |\mathbf{Q}(x)|^2 > \frac{1}{3}, \quad 0 < f(x) \leq 1 \quad \text{for all } x \in \Omega,$$

then $(f, \mathbf{Q})$ is stable, and it is the case if κ is sufficiently large. The solution loses its stability when the minimum value of $|f(x)|^2 - |\mathbf{Q}(x)|^2$ reaches $1/3$.

Although the physical meaning of the critical fields H_S and the superheating field H_{sh} are clear, mathematically we need a careful definition of these fields. Since one can describe a Meissner state by using either the Ginzburg-Landau model (1.3), or the full Meissner model (1.4), or the reduced Meissner system (1.1), there are many options to define these critical fields. As in this paper we use the reduced system (1.1) to describe the Meissner states, we shall give a definition of stability based on (1.1).

Let us consider the applied magnetic field of the form

$$\mathcal{H}^e = \sigma \mathcal{H},$$

where $\mathcal{H}$ is a continuous and positive-valued function defined over $\bar{\Omega}$, and $\sigma > 0$. Then we define the critical fields H_S and H_{sh} as follows.

Definition 2

$$\begin{aligned} H_S(\mathcal{H}) &= \sup\{H > 0 : \text{ all Meissner solutions of (1.1) with } \mathcal{H}^e = \sigma\mathcal{H} \\ &\qquad \text{are stable if } 0 \leq \sigma < H\}, \\ H_{sh}(\mathcal{H}) &= \inf\{H > 0 : \text{ Equation (1.1) with } \mathcal{H}^e = \sigma\mathcal{H} \\ &\qquad \text{has no Meissner solutions if } \sigma > H\}. \end{aligned} \tag{1.6}$$

Then we let

$$H_S = H_S(1), \qquad H_{sh} = H_{sh}(1).$$

The above discussions suggest the following problems:

Problem (A). Find the value of the critical field H_S. Examine how a stable Meissner solution $(f, \mathbf{Q})$ of (1.1) starts to lose its stability as the strength of the applied magnetic field $\mathcal{H}^e$ increases and reaches this critical value. In particular, find the location of the minimum points of $|f(x)|^2 - |\mathbf{Q}(x)|^2$ (with minimum value $1/3$).

[3] Uniqueness of the stable Meissner solution of the system in a three dimensional domain can be directly derived from Lemma 3.1 in [26].

Problem (B). Find the value of the critical value H_{sh}. Examine how an instable Meissner solution $(f, \mathbf{Q})$ of (1.1) starts to nucleate vortices and find the location of the first vortices.

Problem (C). Verify that if κ is large then

$$H_{C_1} < H_S < H_{\text{sh}}.$$

In this paper we investigate Problem (A).

1.3 Main results

At moment we do not know the precise value of H_S, so we start with Meissner solutions in a *weak* magnetic field, that is, $\max_{\bar{\Omega}} |\mathcal{H}^e(x)|$ is sufficiently small. Under a weak magnetic field, a Meissner solution $(f, \mathbf{Q})$ of (1.1) is stable, hence

$$d_{f,\mathbf{Q}} > \frac{1}{3},$$

here we denote

$$d_{f,\mathbf{Q}} := \inf_{x \in \bar{\Omega}} \left\{ |f(x)|^2 - |\mathbf{Q}(x)|^2 \right\}. \tag{1.7}$$

We let $\mathcal{H}^e$ increase and look for a Meissner solution $(f_0, \mathbf{Q}_0)$ which first loses its stability, hence $d_{f_0,\mathbf{Q}_0}$ first achieves the value $\frac{1}{3}$, and find the position of the minimum points of $|f_0(x)|^2 - |\mathbf{Q}_0(x)|^2$. Due to some technical reason, instead of analyze the solution $(f_0, \mathbf{Q}_0)$ with $d_{f_0,\mathbf{Q}_0} = \frac{1}{3}$, we consider first an approximation problem as follows. We fix $\kappa > 0$ and take a small number $\delta > 0$. Let $(f, \mathbf{Q})$ be a solution of (1.1) satisfying the following inequality

$$|f(x)|^2 - |\mathbf{Q}(x)|^2 \geq \frac{1}{3} + \delta^2, \quad 0 < f(x) \leq 1, \quad x \in \bar{\Omega}. \tag{1.8}$$

We show that the minimum points of $|f(x)|^2 - |\mathbf{Q}(x)|^2$ locate near the domain boundary, and $(1 - f(x), \mathbf{Q}(x))$ decays exponentially in the normal direction away from the boundary if the penetration depth λ is small. Denote

$$\mathrm{d}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|.$$

Let h^* be the number defined in Definition 2.3 in Sect. 2.

Theorem 1.1 (Decay estimate) *Let Ω be a bounded domain in $\mathbb{R}^2$ with a C^3 boundary $\partial\Omega$, and let $\mathcal{H}^e$ be a C^3 function on $\bar{\Omega}$ satisfying $\|\mathcal{H}^e\|_{C^0(\partial\Omega)} < h^*$. There exists a positive constant λ_0 such that, if $\lambda \in (0, \lambda_0)$ and if $(f_\lambda, \mathbf{Q}_\lambda)$ is a solution of system* (1.1) *satisfying* (1.8)*, then for any $0 < \alpha < \min\{\sqrt{2}\kappa, 2\}$ and any $0 < \beta < 1$, we* *have*

$$|1 - f_\lambda(x)| \le C_1 e^{-\alpha \mathrm{d}(x,\partial\Omega)/\lambda}, \qquad |\mathbf{Q}_\lambda(x)| \le C_2 e^{-\beta \mathrm{d}(x,\partial\Omega)/\lambda}, \quad x \in \bar{\Omega},$$

where the constants C_1 and C_2 depend only on Ω, $\mathcal{H}^e$, κ, δ, α and β.

Remark 1 (a) Theorem 1.1 says that if a superconductor is in a stable Meissner state and is subjected to a weak magnetic field, then in the interior of the sample we have $(f_\lambda, \mathbf{Q}_\lambda) \sim (1, \mathbf{0})$, which shows that the induced magnetic field vanishes away from a thin layer around the surface of the sample, hence the applied magnetic field does not penetrate the bulk and will not destroy the superconductivity in the interior, and the material is almost in a perfectly superconducting state except a boundary sheath. This is the mathematical description of the Meissner effect.

(b) Intuitively, the decay behavior of $|1 - f_\lambda(x)|$ and $|\mathbf{Q}_\lambda(x)|$ can be explained in the following way. If the boundary conditions in (1.1) were ignored, formally we can derive from the equations that, $f_\lambda(x) \sim 1$ and $|\mathbf{Q}_\lambda(x)| \sim 0$ in the interior of the domain as $\lambda \to 0$. Then the linearization of the second equality of (1.1) around $(f, \mathbf{Q}) = (1, \mathbf{0})$ gives the London equation

$$\lambda^2 \mathrm{curl}^2 \mathbf{H} + \mathbf{H} = \mathbf{0}, \quad \mathrm{div} \mathbf{H} = 0 \quad \text{in } \Omega.$$

By the Agmon's estimate [2] we can show that the non-zero solutions of the above equation are exponentially decay

$$|\mathbf{H}(x)| \le C e^{-d(x,\partial\Omega)/\lambda}, \quad x \in \Omega,$$

from which we can derive the decay behavior of $|\mathbf{Q}_\lambda(x)|$. The linearization of the first equality of (1.1) around $(f, \mathbf{Q}) = (1, \mathbf{0})$ gives

$$-\frac{\lambda^2}{\kappa^2} \Delta w + 2w = |\mathbf{q}|^2 \quad \text{in } \Omega,$$

where $\mathbf{q}$ is a variation of $\mathbf{Q}$. Using the Agmon's estimate again we can show that

$$|w(x)| \le C e^{-\min\{\sqrt{2}\kappa, 2\} d(x,\partial\Omega)/\lambda}, \quad x \in \Omega,$$

from which we can obtain the decay behavior of the function $1 - f_\lambda(x)$.

To determine precise location of the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$, we need carefully analyze the behavior of the solution $(f_\lambda, \mathbf{Q}_\lambda)$ in a thin layer around the domain boundary. We shall derive an asymptotic expansion of $(f_\lambda, \mathbf{Q}_\lambda)$ around any given point $X_0 \in \partial\Omega$ for small λ:

$$\left\| f_\lambda(x) - \hat{f}_0(\psi^{-1}(x)/\lambda) - \lambda \hat{f}_1(\psi^{-1}(x)/\lambda) \right\|_{C^0\left(\overline{\mathcal{U}_{0,\lambda} \cap \Omega}\right)} \le O(\lambda^2) \qquad (1.9)$$

and

$$\left\| \mathbf{Q}_\lambda(x) - \hat{\mathbf{Q}}_0(\psi^{-1}(x)/\lambda) - \lambda\hat{\mathbf{Q}}_1(\psi^{-1}(x)/\lambda) \right\|_{C^0\left(\overline{\mathcal{U}_{0,\lambda} \bigcap \Omega}\right)} \le O(\lambda^2), \quad (1.10)$$

where

- $\mathcal{U}_{0,\lambda}$ is an open neighbourhood of the point X_0 with diameter λ;
- $x = \psi(y)$ is a diffeomorphism straightening a boundary portion of $\partial\Omega$ around X_0;
- the scalar function $\hat{f}_0(\cdot)$ and the vector field $\hat{\mathbf{Q}}_0(\cdot)$ are determined by the strength of the magnetic field (see (5.18) in section 5);
- the scalar function $\hat{f}_1(\cdot)$ and the vector field $\hat{\mathbf{Q}}_1(\cdot)$ are defined by equations involving the strength of the magnetic field and the curvature k of the domain boundary (see (5.19) in section 5).

Moreover we shall show that $\hat{f}_1(\cdot)$ and the first component of $\hat{\mathbf{Q}}_1(\cdot)$ are monotonic with respect to the curvature k of $\partial\Omega$, see for the more precise description in Theorem 5.2. This monotonicity property together with the estimates (1.9) and (1.10) will lead to the determination of the location of the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ as described in the following Theorem 1.2. To state the result of Theorem 1.2 we need the concept of *sub-convergence*.

Definition 3 Let $\{P_\lambda\}$ be a family of points indexed by the parameter λ. We say that the points $\{P_\lambda\}$ sub-converge to the set S as λ tends to zero, if for any sequence $\lambda_n \to 0$ there exists a subsequence $\{\lambda_{n_j}\}$ and a point $P \in S$ which depends on the subsequence, such that $\lim_{j\to\infty} P_{\lambda_{n_j}} = P$.

For the given function $\mathcal{H}^e$ we set

$$\partial\Omega(\mathcal{H}^e) = \left\{x \in \partial\Omega \ : \ \mathcal{H}^e(x) = \|\mathcal{H}^e\|_{C^0(\partial\Omega)}\right\}, \quad (1.11)$$

and

$$S(\mathcal{H}^e) = \left\{x \in \partial\Omega(\mathcal{H}^e) \ : \ k(x) = \max_{y\in\partial\Omega(\mathcal{H}^e)} k(y)\right\}, \quad (1.12)$$

where $k(x)$ is the curvature function of $\partial\Omega$.

Theorem 1.2 *Assume Ω is a bounded domain in $\mathbb{R}^2$ with a C^3 boundary $\partial\Omega$, and let $\mathcal{H}^e$ be a C^3 function on $\bar{\Omega}$ satisfying $\|\mathcal{H}^e\|_{C^0(\partial\Omega)} < h^*$. Suppose $(f_\lambda, \mathbf{Q}_\lambda)$ is the solution of system (1.1) satisfying (1.8). Then, as λ tends to zero, the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ sub-converge to the set $S(\mathcal{H}^e)$ defined by (1.12).*

If $\mathcal{H}^e = h$ is a positive constant, then

$$\partial\Omega(h) = \partial\Omega, \qquad S(h) = \{x \in \partial\Omega : \ k(x) = \max_{y\in\partial\Omega} k(y)\},$$

that is, $S(h)$ is the set of the maximum points of the curvature function of $\partial\Omega$.

Corollary 1.3 *Assume Ω is a bounded domain in $\mathbb{R}^2$ with a C^3 boundary $\partial\Omega$, and $\mathcal{H}^e = h$ is a positive constant. Suppose $(f_\lambda, \mathbf{Q}_\lambda)$ is the solution of system (1.1) satisfying (1.8). Then, as λ tends to zero, the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ sub-converge to the set of the maximum points of the curvature of the domain boundary.*

Remark 2 It is interesting to compare the result on (1.1) in Corollary 1.3 with those on (1.5) in [7, 27].[4] In the limiting process as $\kappa \to \infty$, the solution $(f_\lambda, \mathbf{Q}_\lambda)$ of (1.1) corresponds to the solution $\mathbf{Q}$ of (1.5) by the relation $|f|^2 = 1 - |\mathbf{Q}|^2$, hence the minimum points of $|f_\lambda|^2 - |\mathbf{Q}_\lambda|^2$ correspond to the maximum points of $|\mathbf{Q}(x)|$. However, if $\mathcal{H}^e = h$ is a positive constant, as λ tends to zero, the minimum points of $|f_\lambda|^2 - |\mathbf{Q}_\lambda|^2$ sub-converge to the *maximum points of the curvature* (see Corollary 1.3), while the maximum points of $|\mathbf{Q}|$ sub-converge to the *minimum points of the curvature* (see [7, 27]). This difference reflects the multi-scale nature of (1.1). In fact the behavior of the Meissner states depends on two parameters, the Ginzburg–Landau parameter κ and the penetration depth λ, among other physical parameters. Then:

- If we fix κ and send λ to zero as in this paper, then we have the situation of Corollary 1.3. The minimum points of $|f|^2 - |\mathbf{Q}|^2$ sub-converge to the maximum points of the boundary curvature.
- If we first send κ to infinity (and we get (1.5)) and then send λ to zero, then we have the situation of [7, 27]. In this case $|f|^2 - |\mathbf{Q}|^2 \sim 1 - 2|\mathbf{Q}|^2$, and the minimum points of $|f|^2 - |\mathbf{Q}|^2$ correspond to the maximum points of $|\mathbf{Q}|^2$, which sub-converge to the minimum points of the boundary curvature.

We expect that if we let $\kappa \to \infty$ and $\lambda \to 0$, the minimum points of $|f|^2 - |\mathbf{Q}|^2$ will sub-converge to points on the boundary, and the location of the limiting positions depends on the relative scale of κ and λ. We will study the multiple-scales phenomena of the Meissner solutions in the later future.

In order to establish the uniform convergence estimates (1.9) and (1.10), we need a C^0 estimate of the solution to a semilinear Maxwell system (or called semilinear curl-curl system) for the vector field $\mathbf{Q}_\lambda$, which is a degenerately elliptic system without comparison principle and maximum principle, hence the C^0 estimate does not follow from the standard theory of elliptic systems. Our strategy to prove (1.9) and (1.10) is as follows:

— We first prove the global H^1 estimate for the remainder terms in (1.9) and (1.10) by the method of matched asymptotic expansions;

— Then we deduce an H^2 estimate of the remainder terms near the domain boundary by the difference quotient technique, which yields the C^0 regularity of the remainder terms by the Sobolev imbedding theorem.

Let us mention that the method of the proof of (1.9) and (1.10) in this paper is different from that used by Pan and Kwek in [27], where the estimates for the solutions were proved by applying the maximum principle to a divergence-type elliptic equation for the scalar function $H_\lambda = \lambda\,\mathrm{curl}\mathbf{Q}_\lambda$.

[4] See also [3] for the three dimensional system.

1.3.1 Organization of this paper

The formal expansion for $(f_\lambda, \mathbf{Q}_\lambda)$ with respect to λ is derived in section 2. Then we establish the uniform estimation for the asymptotic expansion of the solution $(f_\lambda, \mathbf{Q}_\lambda)$ in section 3. In section 4, we prove the exponential decay estimate (Theorem 1.1) of $1 - f_\lambda$ and $|\mathbf{Q}_\lambda|$. Finally in section 5, by applying (1.9) and (1.10) we give the proof of Theorem 1.2. Further remarks will be given in section 6. The proofs of the theorems involve lengthy computations and technical details, which will be given in appendices. Among them, in appendix A we prove the uniqueness of the solution to a limiting system in the half space (see (2.11)), which is associated with the leading order term of the expansions of $(f_\lambda, \mathbf{Q}_\lambda)$; in appendix B we prove the exponential decay estimate for the solutions to some ODEs; in appendix C and appendix D we give the details of the calculations for the formal expansion for $(f_\lambda, \mathbf{Q}_\lambda)$.

Throughout the paper, the bold typeface is used to indicate vector quantities; normal typeface will be used for scalars and the components of vectors. We shall use the letter C to denote a positive constant which is independent of λ, but the numerical value may be vary line to line.

2 Formal asymptotic solution to system (1.1)

As stated in the introduction, we shall find the location of the minimum points of $|f_\lambda|^2 - |\mathbf{Q}_\lambda|^2$ for small λ, and we need first prove the uniform convergence of the approximation solutions as λ tends to zero. The proof is based on the method of matched asymptotic expansions of the solution $(f_\lambda, \mathbf{Q}_\lambda)$ in term of λ. The construction of the inner expansions in a thin tubular neighborhood of the domain boundary of scale λ requires detailed analysis on the behavior of the solutions near the domain boundary, which will be carried out in this section.

To start with, let us first introduce a new local coordinate system near a boundary point $X_0 \in \partial\Omega$. Let $\mathcal{U}$ denote a neighborhood of X_0. The portion of the boundary $\partial\Omega$ located inside $\mathcal{U}$ can be represented as $u = u(s)$ with $u(0) = X_0$, where s is the arc length variable of $\partial\Omega$. Then $\tau(s) = u'(s)$ is the unit tangent vector. Let $\mathbf{n}(s) = (n_1, n_2)$ be the unit outer normal at $x \in \partial\Omega$. We introduce new variables y_1 and y_2, with $y_1 = s$, such that for any $x \in \bar{\Omega} \cap \mathcal{U}$ we have a diffeomorphism map ψ given by

$$x = \psi(y_1, y_2) = u(y_1) - y_2\mathbf{n}(y_1). \tag{2.1}$$

Let

$$g(y_1, y_2) = |\det D\psi| = 1 - k(y_1)y_2, \tag{2.2}$$

where $k(y_1)$ is the curvature of $\partial\Omega$ at the point $x = \psi(y_1, 0) \in \partial\Omega$. Then we have a new orthogonal coordinate framework $\{\mathbf{E}_1, \mathbf{E}_2\}$ as follows:

$$\mathbf{E}_1(y) = \frac{\partial_1\psi}{|\partial_1\psi|} = \tau(y_1), \qquad \mathbf{E}_2(y) = -\mathbf{n}(y_1).$$

Now we introduce the following notations. For any function $f(x)$ defined on $\mathcal{U}$ we define a function of y and write it by $\hat{f}(y)$, such that

$$\hat{f}(y) := f(\psi(y)).$$

For a vector field $\mathbf{Q}(x)$ depending on the variable x, we define a vector field $\hat{\mathbf{Q}}(y)$ with variable y by

$$\hat{\mathbf{Q}}(y) := \mathbf{Q}(\psi(y)).$$

We shall call $\hat{f}(y)$ and $\hat{\mathbf{Q}}(y)$ the *representations of* $f(x)$ *and* $\mathbf{Q}(x)$ *in the coordinates* y respectively.

Using the framework $\{\mathbf{E}_1, \mathbf{E}_2\}$ we can write $\hat{\mathbf{Q}}(y)$ as

$$\hat{\mathbf{Q}}(y) = \hat{Q}_1(y)\mathbf{E}_1 + \hat{Q}_2(y)\mathbf{E}_2,$$

where $\hat{Q}_1(y)$ and $\hat{Q}_2(y)$ are scalar functions. Then $\text{curl}\mathbf{Q}(x)$ and $\text{curl}^2\mathbf{Q}(x)$ can be represented by

$$\text{curl}\mathbf{Q}(x) = \frac{1}{g}\left[\partial_1\hat{Q}_2 - \partial_2(g\hat{Q}_1)\right]$$

and

$$\text{curl}^2\mathbf{Q}(x) = \mathcal{M}_1(y)\mathbf{E}_1(y) + \mathcal{M}_2(y)\mathbf{E}_2(y),$$

where

$$\begin{aligned}\mathcal{M}_1(y) &\equiv \partial_2\left(\frac{1}{g}\left[\partial_1\hat{Q}_2 - \partial_2(g\hat{Q}_1)\right]\right),\\ \mathcal{M}_2(y) &\equiv -\frac{1}{g}\partial_1\left(\frac{1}{g}\left[\partial_1\hat{Q}_2 - \partial_2(g\hat{Q}_1)\right]\right).\end{aligned} \tag{2.3}$$

In the above, ∂_j denotes $\frac{\partial}{\partial y_j}$ for $j = 1, 2$. Also, we have

$$\Delta_x f = \Delta_y \hat{f},$$

where Δ_y is defined by

$$\Delta_y\hat{f} = \frac{1}{g}\left(\partial_1\left(\frac{1}{g}\partial_1\hat{f}\right) + \partial_2\left(g\partial_2\hat{f}\right)\right). \tag{2.4}$$

For simplicity, we introduce the operators

$$\mathscr{C}\mathrm{url}_y\,\hat{\mathbf{Q}} = \frac{1}{g}\left[\partial_1 \hat{Q}_2 - \partial_2(g\hat{Q}_1)\right], \quad \mathscr{C}\mathrm{url}_y^2\,\hat{\mathbf{Q}} = (\mathcal{M}_1(y), \mathcal{M}_2(y)). \tag{2.5}$$

Let $(f_\lambda(x), \mathbf{Q}_\lambda(x))$ be a solution of (1.1), and let $\hat{f}_\lambda(y)$ and $\hat{\mathbf{Q}}_\lambda(y)$ be the representations of $f_\lambda(x)$ and $\mathbf{Q}_\lambda(x)$ in the coordinates y respectively. We introduce re-scaled variables

$$y = \lambda z.$$

In the neighborhood of X_0, we then define the rescaled vector fields (which will be called the z-coordinates):

$$\tilde{f}_\lambda(z) = \hat{f}_\lambda(\lambda z) = \hat{f}_\lambda(y) \quad \text{and} \quad \tilde{\mathbf{Q}}_\lambda(z) = \hat{\mathbf{Q}}_\lambda(\lambda z) = \hat{\mathbf{Q}}_\lambda(y). \tag{2.6}$$

In the following, for convenience of notation, we may drop the subscript λ and denote $\tilde{f}_\lambda(z)$ by $\tilde{f}(z)$, and $\tilde{\mathbf{Q}}_\lambda(z)$ by $\tilde{\mathbf{Q}}(z)$. Then system (1.1) can be rewritten by

$$\begin{cases} -\frac{1}{\kappa^2}\Delta_z \tilde{f} = (1 - |\tilde{f}|^2 - |\tilde{\mathbf{Q}}|^2)\tilde{f} & \text{in } \tilde{\mathbf{Q}}_z, \\ \mathscr{C}\mathrm{url}_z^2\,\tilde{\mathbf{Q}} + |\tilde{f}|^2\tilde{\mathbf{Q}} = \mathbf{0} & \text{in } \tilde{\mathbf{Q}}_z, \\ \frac{\partial \tilde{f}}{\partial \mathbf{n}} = 0, \quad \mathscr{C}\mathrm{url}_z\,\tilde{\mathbf{Q}} = \tilde{\mathcal{H}}^e & \text{on } \tilde{T}_z, \end{cases} \tag{2.7}$$

where the operators $\mathscr{C}\mathrm{url}_z$ and Δ_z are defined by

$$\mathscr{C}\mathrm{url}_z := \lambda\,\mathscr{C}\mathrm{url}_y, \quad \Delta_z := \lambda^2\Delta_y, \quad y = \lambda z, \tag{2.8}$$

and $\tilde{\Omega}_z$ and $\tilde{T}_z$ represent the images of the domain $\Omega \cap \mathcal{U}$ and of the boundary $\partial\Omega \cap \mathcal{U}$ under the $z-$coordinate system respectively.

Now we begin to derive the formal asymptotic solution in the (y_1, z_2) coordinates, where $z_2 = y_2/\lambda$. Let us assume that the inner expansion of the solution in the neighborhood of X_0 has the form

$$\begin{aligned} \hat{f}_\lambda(y) &= \hat{f}_0(y_1, z_2) + \lambda \hat{f}_1(y_1, z_2) + \lambda^2 \hat{f}_2(y_1, z_2) + O(\lambda^3), \\ \hat{\mathbf{Q}}_\lambda(y) &= \hat{\mathbf{Q}}_0(y_1, z_2) + \lambda\hat{\mathbf{Q}}_1(y_1, z_2) + \lambda^2\hat{\mathbf{Q}}_2(y_1, z_2) + O(\lambda^3). \end{aligned} \tag{2.9}$$

We emphasize that $\hat{f}_\lambda(y_1, z_2)$ and $\hat{\mathbf{Q}}_\lambda(y_1, z_2)$ have multi-scales with y_1 in the scale $O(1)$ and z_2 in the scale $O(\frac{1}{\lambda})$ for small λ.

2.1 The leading order term

We first derive the leading order term $(\hat{f}_0(y_1, z_2)), \hat{\mathbf{Q}}_0(y_1, z_2))$. We shall prove a uniform $C^{2,\alpha}$ estimate for $(\tilde{f}_\lambda(z), \tilde{\mathbf{Q}}_\lambda(z))$ on any bounded z-domain, which yields estimates of $(\hat{f}, \hat{\mathbf{Q}})$ inside any boundary layer.

Lemma 2.1 *Assume Ω is a bounded domain in $\mathbb{R}^2$ with a $C^{2,\alpha}$ boundary, $0 < \alpha < 1$ and $\mathcal{H}^e(x)$ is a $C^{2,\alpha}$ function on $\bar{\Omega}$. Let $(f_\lambda, \mathbf{Q}_\lambda)$ be a solution of (1.1) satisfying (1.8), and $(\tilde{f}_\lambda, \tilde{\mathbf{Q}}_\lambda)$ be the rescaled pair. Then for small λ, we have*

$$\|\tilde{f}_\lambda\|_{C^{2,\alpha}(\tilde{\Omega}_z \cap B_R^+(0))} + \|\tilde{\mathbf{Q}}_\lambda\|_{C^{2,\alpha}(\tilde{\Omega}_z \cap B_R^+(0))} \leq C,$$

where C depends only on Ω, $\mathcal{H}^e$, κ, δ and α, but is independent of R and λ.

Proof The proof is quite similar to that of Lemma 9.2 in [26], we here omit it. □

Next we show that

$$(\tilde{f}_\lambda, \tilde{\mathbf{Q}}_\lambda) \quad \text{converges in } C^2_{loc}(\mathbb{R}^2_+) \text{ as } \lambda \to 0. \tag{2.10}$$

Proof of (2.10) From Lemma 2.1 and by Arzela-Ascoli's theorem (see the compactness result [14, Lemma 6.36]), for any sequence $\lambda_n \to 0$, there exists a subsequence $\{\lambda_{n_j}\}$ such that, as $j \to \infty$, $(\tilde{f}_{\lambda_{n_j}}, \tilde{\mathbf{Q}}_{\lambda_{n_j}})$ converges in $C^2_{\text{loc}}(\mathbb{R}^2_+)$ to the solution $(\bar{f}_0(z_1, z_2)), \bar{\mathbf{Q}}_0(z_1, z_2))$ of the following system

$$\begin{cases} -\dfrac{1}{\kappa^2}\Delta \bar{f}_0 = (1 - |\bar{f}_0|^2 - |\bar{\mathbf{Q}}_0|^2)\bar{f}_0 & \text{in } \mathbb{R}^2_+, \\ \text{curl}^2\bar{\mathbf{Q}}_0 + |\bar{f}_0|^2\bar{\mathbf{Q}}_0 = \mathbf{0} & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial \bar{f}_0}{\partial \mathbf{n}} = 0, \quad \text{curl}\bar{\mathbf{Q}}_0 = \mathcal{H}^e(X_0) & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.11}$$

Moreover, because $(f_\lambda, \mathbf{Q}_\lambda)$ satisfies the condition (1.8), so $(f, \mathbf{Q}) = (\bar{f}_0, \bar{\mathbf{Q}}_0)$ satisfies the following

$$|f(z)|^2 - |\mathbf{Q}(z)|^2 \geq \frac{1}{3} + \delta^2 \quad \text{and} \quad 0 < f(z) \leq 1, \quad \forall z \in \mathbb{R}^2_+. \tag{2.12}$$

From Lemma A.1, the solution of (2.11) satisfying (2.12) is unique. Hence $(\bar{f}_0, \bar{\mathbf{Q}}_0)$ is the unique solution of (2.11) satisfying (2.12). It follows that the whole sequence $(\tilde{f}_{\lambda_n}, \tilde{\mathbf{Q}}_{\lambda_n})$ actually converges to $(\bar{f}_0(z_1, z_2), \bar{\mathbf{Q}}_0(z_1, z_2))$. Therefore $(\tilde{f}_\lambda, \tilde{\mathbf{Q}}_\lambda)$ converges to $(\bar{f}_0(z_1, z_2), \bar{\mathbf{Q}}_0(z_1, z_2))$ in $C^2_{\text{loc}}(\mathbb{R}^2_+)$ as $\lambda \to 0$. Hence (2.10) is proved. □

In the following we show that if $\mathcal{H}^e(X_0)$ is small, then the unique solution of (2.11) satisfying (2.12) has the form

$$\bar{f}_0(z_1, z_2) = f_0(z_2), \quad \bar{\mathbf{Q}}_0(z_1, z_2) = (Q^1_0(z_2), 0). \tag{2.13}$$

To prove this conclusion, we only need to show that, if $\mathcal{H}^e(X_0)$ is small, (2.11) has a solution of this form and it satisfies (2.12). Then the uniqueness result of Lemma A.1 implies that this solution is the only solution of (2.11) satisfying (2.12).

Plugging (2.13) into (2.11) we see that $(f_0(z_2), Q_0^1(z_2))$ satisfies the following ODEs:

$$\begin{cases} -\frac{1}{\kappa^2} f_0'' = (1 - |f_0|^2 - |Q_0^1|^2) f_0 & \text{in } \mathbb{R}_+, \\ -(Q_0^1)'' + |f_0|^2 Q_0^1 = 0 & \text{in } \mathbb{R}_+, \\ f_0'(0) = 0, \qquad (Q_0^1)'(0) = -h_0, \\ f_0(\infty) = 1, \qquad (Q_0^1)(\infty) = 0, \end{cases} \tag{2.14}$$

where $f_0' = \frac{df_0}{dz_2}$, and $h_0 = \mathcal{H}^e(X_0) > 0$. We look for the solution of (2.14) satisfying (2.12).

Proposition 2.2 *If* (2.14) *has a solution* $(f_0, Q_0^1) \in C^3(\mathbb{R}_+) \times C^3(\mathbb{R}_+)$, *then it is the unique solution of* (2.14) *satisfying* (2.12), *and for any* $0 < \alpha_1 < \min\{2, \sqrt{2}\kappa\}$ *and any* $0 < \beta_1 < 1$ *we have*

$$f_0'(z_2) > 0, \quad |1 - f_0(z_2)| \le C e^{-\alpha_1 z_2},$$
$$(Q_0^1)'(z_2) < 0, \quad 0 < Q_0^1(z_2) \le C e^{-\beta_1 z_2}$$

for all $z_2 > 0$, *where* $C = C(\mathcal{H}^e, \kappa, \alpha_1, \beta_1)$.

Proof *Step 1*. Assume (2.14) has a solution (f_0, Q_0^1) satisfying (2.12). Then $(f_0, Q_0^1) \in C^3(\mathbb{R}_+) \times C^3(\mathbb{R}_+)$. By the maximum principle, it is easy to see that

$$Q_0^1(z_2) > 0, \quad (Q_0^1)'(z_2) < 0 \quad \text{for all } 0 < z_2 < \infty. \tag{2.15}$$

Since $f_0(\infty) = 1$, by the comparison principle (or see Proposition B.2 in appendix B), we easily obtain that: for any $0 < \beta_1 < 1$, there exists a constant $C > 0$ depending on β_1 and $\mathcal{H}^e$ such that

$$|Q_0^1(z_2)| \le C e^{-\beta_1 z_2} \quad \text{for all } z_2 > 0.$$

Next we show that $f_0'(z_2) \ge 0$ for all $z_2 > 0$. Suppose not, then there exist two numbers c_2 and c_3 with $0 < c_2 < c_3$ such that

$$f_0''(c_2) \le 0, \quad f_0''(c_3) \ge 0, \quad f_0(c_2) > f_0(c_3). \tag{2.16}$$

From the first equation in (2.14), we have

$$(1 - |f_0|^2 - |Q_0^1|^2)|_{z_2 = c_2} \ge 0, \qquad (1 - |f_0|^2 - |Q_0^1|^2)|_{z_2 = c_3} \le 0. \tag{2.17}$$

From (2.15) we have $Q_0^1(c_2) > Q_0^1(c_3)$, and by (2.16) we have $f_0(c_2) > f_0(c_3)$. Then

$$(1-|f_0|^2-|Q_0^1|^2)|_{z_2=c_2} < (1-|f_0|^2-|Q_0^1|^2)|_{z_2=c_3}.$$

This is a contradiction with (2.17).

Now we show that

$$f_0'(z_2) > 0 \quad \text{for all } z_2 > 0.$$

Otherwise, suppose there exists $c_4 \in (0, \infty)$ such that $f_0'(c_4) = 0$, then $f_0'''(c_4) \geq 0$. This is a contradiction with

$$f_0'''(c_4) = -(1-|f_0|^2-|Q_0^1|^2)f_0' + (2f_0f_0' + 2Q_0^1(Q_0^1)')f_0|_{z_2=c_4} < 0.$$

Therefore the strict inequality holds.

Let $w(z_2) = 1 - f_0(z_2)$. Then w satisfies

$$\begin{cases} -\frac{1}{\kappa^2}w'' + w(2+|Q_0^1|^2 - 3w + w^2) = |Q_0^1|^2 & \text{in } \mathbb{R}_+, \\ w'(0) = 0 \text{ and } w(\infty) = 0. \end{cases}$$

Note that $2 + |Q_0^1|^2 - 3w + w^2 \to 2$ as $z_2 \to +\infty$. Then from Proposition B.2 in appendix B, for any $0 < \alpha_1 < \min\{2, \sqrt{2}\kappa\}$ there exists a constant $C > 0$ such that

$$w(z_2) \leq C(\mathcal{H}^e, \kappa, \alpha_1, \beta_1)e^{-\alpha_1 z_2} \quad \text{for all } z_2 > 0.$$

Step 2. We show that (2.14) has at most one solution satisfying (2.12). Define the space

$$\mathscr{V} = \left\{(u, v) : u', \quad 1-u, \quad v', \quad v \in L^2(\mathbb{R}_+), \quad u'(0) = 0, \quad v'(0) = 0\right\},$$

which is a reflexive Banach space equipped with the norm

$$\|(u, v)\| = \|1-u\|_{L^2(\mathbb{R}_+)} + \|u'\|_{L^2(\mathbb{R}_+)} + \|v\|_{L^2(\mathbb{R}_+)} + \|v'\|_{L^2(\mathbb{R}_+)}.$$

Set

$$\mathscr{U} = \left\{(f_0, P_0^1) \in \mathscr{V} : \ 0 \leq f_0 \leq 1, \ \ |f_0|^2 - (P_0^1 + h_0e^{-z_2})^2 \geq \frac{1}{3} + \frac{1}{2}\delta^2\right\},$$

and define a functional $\mathcal{E}$ in $\mathscr{U}$ by

$$\mathcal{E}[f_0, P_0^1] = \int_0^\infty \Big\{\frac{1}{\kappa^2}|f_0'|^2 + |f_0|^2(P_0^1 + h_0e^{-z_2})^2 + \frac{1}{2}(1-|f_0|^2)^2 + |(P_0^1)'|^2 - 2h_0e^{-z_2}P_0^1\Big\}dz_2.$$

It is easy to see that $\mathscr{U}$ is a closed and convex subset of $\mathscr{V}$, and $\mathcal{E}$ is strictly convex, coercive and weakly lower semi-continuous on $\mathscr{U}$ with respect to the norm inherited from $\mathscr{V}$. Therefore, $\mathcal{E}$ has a unique minimizer $(f_0, P_0^1) \in \mathscr{U}$.

Let (f_0, Q_0^1) be a solution of (2.14) satisfying (2.12). From step 1, we see that $f_0'(z_2) > 0$ for all $z_2 > 0$. It follows that $0 < f_0(z_2) < 1$ for all $z_2 \geq 0$. Let $P_0^1 = Q_0^1 - h_0 e^{-z_2}$. Then from (2.12), we see that (f_0, P_0^1) lies in the interior of $\mathscr{U}$, and it is a critical point of the strictly convex functional $\mathcal{E}$. Hence (f_0, P_0^1) is the unique minimizer of $\mathcal{E}$ in $\mathscr{U}$. This shows that if (2.14) has a solution satisfying (2.12) then it is unique. □

Another proof of uniqueness of the solution of (2.14) satisfying (2.12) will be given in Lemma A.1 in Appendix A.

Definition 2.3 We define

$$h^* = \sup\Big\{h : \text{(2.14) has a solution } (f_0, Q_0^1) \text{ satisfying } |f_0|^2 - |Q_0^1|^2 > \frac{1}{3} \\ \text{and } 0 < f_0 \leq 1 \text{ for all } h_0 \in (0, h)\Big\}.$$

Proposition 2.4 *We have*

$$\frac{\sqrt{2}}{3} \leq h^* \leq \frac{\sqrt{6}}{3}. \tag{2.18}$$

The proof of Proposition 2.4 will be given in section 5 after Theorem 5.1.

From Propositions 2.2 and 2.4, for any $0 < h_0 < h^*$, (2.14) has a unique solution (f_0, Q_0^1) satisfying (2.12) for some positive constant δ. Then we define $\hat{f}_0(0, z_2)$ and $\hat{\mathbf{Q}}_0(0, z_2)$ by letting

$$\hat{f}_0(0, z_2) = f_0(z_2), \quad \hat{\mathbf{Q}}_0(0, z_2) = (Q_0^1(z_2), 0).$$

Moreover, for each $y_1 \neq 0$, we can define $\hat{f}_0(y_1, z_2)$ and $\hat{\mathbf{Q}}_0(y_1, z_2)$ by using the equations (5.18) in section 5. We will see later that $(\hat{f}_0(y_1, z_2), \hat{\mathbf{Q}}_0(y_1, z_2))$ gives the leading order term of the asymptotic expansions at X_0, which provides the information how the minimum points of $f^2 - |\mathbf{Q}|^2$ depend on the intensity of the applied magnetic filed.

Based on Proposition 2.2, we have the exponential decay in the z_2−direction for $1 - \hat{f}_0(y_1, z_2)$ and $\hat{\mathbf{Q}}_0(y_1, z_2)$ which will be used later.

Proposition 2.5 *Let $\mathcal{H}^e$ be a C^3 function on $\bar{\Omega}$ satisfying $\|\mathcal{H}^e\|_{C^0(\partial\Omega)} < h^*$. Then for any $0 < \alpha_1 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_1 < 1$, we have*

$$|1 - \hat{f}_0(y_1, z_2)| + \sum_{\substack{0 \leq i \leq 3,\, 0 \leq j \leq 2 \\ i^2 + j^2 \neq 0}} |\partial^{i+j}_{y_1^i z_2^j} \hat{f}_0(y_1, z_2)| \leq C e^{-\alpha_1 z_2},$$

$$\sum_{0 \leq i \leq 3,\, 0 \leq j \leq 2} |\partial^{i+j}_{y_1^i z_2^j} \hat{\mathbf{Q}}_0(y_1, z_2)| \leq C e^{-\beta_1 z_2},$$

where the constants C depend only on $\mathcal{H}^e$, κ, α_1 and β_1.

The proof will be given in appendix B.

2.2 The first order term

Next we derive the first order term $(\hat{f}_1(y_1, z_2), \hat{\mathbf{Q}}_1(y_1, z_2))$ of the expansions, which will be useful to determine how the geometry of the domain influences the distribution of the minimum points of $f^2 - |\mathbf{Q}|^2$.

We first consider the values of this term for $y_1 = 0$. Set, for $z_2 \geq 0$,

$$f_1(z_2) := \hat{f}_1(0, z_2), \qquad \mathbf{Q}_1(z_2) \equiv (Q_1^1(z_2), Q_1^2(z_2)) := \hat{\mathbf{Q}}_1(0, z_2).$$

For convenience, we write $\partial_{y_1} \hat{\mathbf{Q}}_0(0, z_2)$ as follows:

$$\partial_{y_1} \hat{\mathbf{Q}}_0(0, z_2) = (q(z_2), 0). \tag{2.19}$$

Substituting (2.9) into system (1.1) under the $z-$coordinates, equating the coefficients of λ, and then considering the problem at $(0, z_2)$, we obtain a system for $(f_1(z_2), (Q_1^1(z_2), Q_1^2(z_2)))$ in the variable $z_2 \in \mathbb{R}_+$:

$$\begin{cases} -\frac{1}{\kappa^2} f_1'' + (3|f_0|^2 + |Q_0^1|^2 - 1) f_1 = -2 f_0 Q_0^1 Q_1^1 - \frac{k_0}{\kappa^2} f_0' & \text{in } \mathbb{R}_+, \\ -(Q_1^1)'' + |f_0|^2 Q_1^1 = -2 f_0 Q_0^1 f_1 - k_0 \partial_2 Q_0^1 & \text{in } \mathbb{R}_+, \\ q' + |f_0|^2 Q_1^2 = 0 & \text{in } \mathbb{R}_+, \\ f_1'(0) = 0, \quad (Q_1^1)'(0) = k_0 Q_0^1(0) & \text{on } z_2 = 0, \end{cases} \tag{2.20}$$

where $k_0 = k(X_0)$ is the value of the curvature of $\partial\Omega$ at the point X_0, and κ is the Ginzburg-Landau parameter. The detailed derivation of (2.20) will be given in appendix C.

From the third equation of (2.20), we immediately obtain that

$$Q_1^2(z_2) = -q'(z_2)|f_0|^{-2}(z_2). \tag{2.21}$$

From Proposition 2.5 we see that

$$|Q_1^2(z_2)| \leq C(\mathcal{H}^e, \kappa, \alpha_1, \beta_1) e^{-\beta_1 z_2} \quad \text{for all } z_2 > 0,$$

where $0 < \beta_1 < 1$.

Applying Proposition B.2 in appendix B to (2.20), we get the following

Proposition 2.6 *There exists a solution $(f_1(z_2), (Q_1^1(z_2), Q_1^2(z_2)))$ to system (2.20) such that, for any $0 < \alpha_2 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_2 < 1$, we have*

$$|f_1(z_2)| \leq C e^{-\alpha_2 z_2}, \qquad |Q_1^1(z_2)| \leq C e^{-\beta_2 z_2} \qquad \textit{for all } z_2 > 0,$$

where the constants C depend only on Ω, $\mathcal{H}^e$, κ, α_2 and β_2.

Proof Using the estimate in Proposition 2.5, we have

$$|f_0^{'}| \le Ce^{-\alpha_1 z_2}, \quad |\partial_2 Q_0^1| \le Ce^{-\beta_1 z_2} \quad \text{for all } z_2 > 0,$$

where $C = C(\kappa, \alpha_1, \beta_1, \mathcal{H}^e)$. Then by noting that

$$\begin{pmatrix} (3|f_0|^2 + |Q_0^1|^2 - 1) & 2f_0 Q_0^1 \\ 2f_0 Q_0^1 & |f_0|^2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as } z_2 \to +\infty,$$

and from Proposition B.2 in appendix B, we have the solution $(f_1(z_2), (Q_1^1(z_2), Q_1^2(z_2)))$ to system (2.20), and $|f_1(z_2)| \le C(\kappa, \mathcal{H}^e)$ for all $z_2 > 0$. Now applying Proposition B.2 in appendix B again, from the second equation in (2.20), for any $0 < \beta_2 < 1$, we have $|Q_1^1(z_2)| \le C(\kappa, \beta_2, \mathcal{H}^e)e^{-\beta_2 z_2}$ for all $z_2 > 0$, where we have taken $\beta_1 = (\beta_2 + 1)/2$. At last, using the estimate on $Q_1^1(z_2)$, and by the first equation in (2.20) we can obtain the estimate for f_1. □

Similarly, for each $y_1 \ne 0$, we can also define $\hat{f}_1(y_1, z_2)$ and $\hat{\mathbf{Q}}_1(y_1, z_2)$ (see the equations (5.19) in section 5), and we also have

Proposition 2.7 *Let $\mathcal{H}^e$ be a C^3 function on $\bar{\Omega}$ satisfying $\|\mathcal{H}^e\|_{C^0(\partial\Omega)} < h^*$. Then for any $0 < \alpha_2 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_2 < 1$, we have*

$$\sum_{0\le i\le 3,\, 0\le j\le 2} |\partial^{i+j}_{y_1^i z_2^j} \hat{f}_1(y_1, z_2)| \le Ce^{-\alpha_2 z_2},$$
$$\sum_{0\le i\le 3,\, 0\le j\le 2} |\partial^{i+j}_{y_1^i z_2^j} \hat{\mathbf{Q}}_1(y_1, z_2)| \le Ce^{-\beta_2 z_2}$$

for all $z_2 > 0$, where the constants C depend only on Ω, $\mathcal{H}^e$, κ, α_2 and β_2.

The proof is similar to that of Proposition 2.5, we here omit it.

2.3 The second order term

Next we look for the second order term $(\hat{f}_2(y_1, z_2), \hat{\mathbf{Q}}_2(y_1, z_2))$ in the expansion at X_0, which will be needed to derive the uniform estimation for the approximation solution.

We first derive the values of this term at $y_1 = 0$. Let, for $z_2 \ge 0$,

$$f_2(z_2) = \hat{f}_2(0, z_2), \qquad \mathbf{Q}_2(z_2) = \hat{\mathbf{Q}}_2(0, z_2) = (Q_2^1(z_2), Q_2^2(z_2)).$$

Substituting (2.9) into (1.1) under the $z-$coordinates, equating the coefficients of λ^2, and then considering this problem at $(0, z_2)$, we obtain the equations of $(f_2(z_2), (Q_2^1(z_2), Q_2^2(z_2)))$ for $z_2 \in \mathbb{R}_+$:

$$\begin{cases} -\frac{1}{\kappa^2} f_2'' + (3|f_0|^2 + |Q_0^1|^2 - 1) f_2 = -2 f_0 Q_0^1 Q_2^1 - r_1 & \text{in } \mathbb{R}_+, \\ -(Q_2^1)'' + |f_0|^2 Q_2^1 = -2 f_0 Q_0^1 f_2 - r_2 & \text{in } \mathbb{R}_+, \\ |f_0|^2 Q_2^2 + (\partial_{z_2 y_1} \hat{Q}_1^1\Big|_{y_1=0} - k_0 q - k_0' Q_0^1 + k_0 z_2 q') + 2 f_0 f_1 Q_1^2 = 0 & \text{in } \mathbb{R}_+, \\ f_2'(0) = 0, \quad (Q_2^1)'(0) = \partial_{y_1} \hat{Q}_1^2\Big|_{y_1=0} + k_0 Q_1^1 & \text{on } z_2 = 0, \end{cases} \tag{2.22}$$

where $k_0 = k(0)$ is the curvature of $\partial\Omega$ at the point X_0, $k_0' = \frac{\partial k}{\partial s}(0)$, $q = q(z_2)$ is the function defined in (2.19), and

$$\begin{aligned} r_1(z_2) = & -\frac{1}{\kappa^2}(\partial_{y_1 y_1} \hat{f}_0\Big|_{y_1=0} - k_0 (f_1)' - k^2(0) z_2 (f_0)') \\ & + f_0(2 Q_0^1 Q_2^1 + |f_1|^2 + |Q_1^1|^2 + |Q_1^2|^2) + f_1(2 f_0 f_1 + 2 Q_0^1 Q_1^1), \\ r_2(z_2) = & \ \partial_{z_2 y_1} \hat{Q}_1^2\Big|_{y_1=0} + k_0 (Q_1^1)' + k_0 Q_0^1 + k_0^2 z_2 (Q_0^1)' + |f_1|^2 Q_0^1 + 2 f_0 f_1 Q_1^1. \end{aligned}$$

The detailed calculations will be given in appendix D. It is easy to see that

$$Q_2^2(z_2) = -|f_0|^{-2}(\partial_{z_2 y_1} \hat{Q}_1^1\Big|_{y_1=0} - k_0 q - k_0' Q_0^1 + k_0 z_2 q' + 2 f_0 f_1 Q_1^2). \tag{2.23}$$

From Proposition 2.5 and Proposition 2.7, we have

$$|Q_2^2(z_2)| \leq C e^{-\beta_2 z_2} \quad \text{for all } z_2 > 0,$$

where $0 < \beta_2 < 1$ and $C = C(\kappa, \Omega, \beta_2, \mathcal{H}^e)$.

Proposition 2.8 *There exists a solution $(f_2(z_2), (Q_2^1(z_2), Q_2^2(z_2)))$ to system (2.22) such that, for any $0 < \alpha_3 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_3 < 1$, there exist constants C such that*

$$|f_2(z_2)| \leq C e^{-\alpha_3 z_2}, \qquad |Q_2^1(z_2)| \leq C e^{-\beta_3 z_2} \qquad \textit{for all } z_2 > 0,$$

where $C = C(\mathcal{H}^e, \kappa, \alpha_3, \beta_3)$.

Proof Using the estimate in Proposition 2.5 and in Proposition 2.7, it is easy to see that, for any $0 < \alpha_2 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_2 < 1$, there exist constants C such that

$$|r_1(z_2)| \leq C e^{-\alpha_2 z_2}, \quad |r_2| \leq C e^{-\beta_2 z_2} \quad \text{for all } z_2 > 0,$$

where we have taken $\alpha_1 = (\alpha_2 + 1)/2$, $\beta_1 = (\beta_2 + 1)/2$, and we have used the same letter C to denote constants depending on κ, α_2, Ω, β_2 and $\mathcal{H}^e$. Note that the matrix

$$\begin{pmatrix} 3|f_0|^2 - |Q_0^1|^2 - 1 & 2f_0 Q_0^1 \\ 2f_0 Q_0^1 & |f_0|^2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as } z_2 \to +\infty.$$

Then by taking $\alpha_2 = (\alpha_3 + 1)/2$, $\beta_2 = (\beta_3 + 1)/2$, the conclusion of this proposition can be obtained by Proposition B.2 in appendix B. □

For each $y_1 \neq 0$ we can also define $\hat{f}_2(y_1, z_2)$ and $\hat{\mathbf{Q}}_2(y_1, z_2)$, and we have

Proposition 2.9 *Let $\mathcal{H}^e$ be a C^3 function on $\bar{\Omega}$ satisfying $\|\mathcal{H}^e\|_{C^0(\partial\Omega)} < h^*$. Then for any $0 < \alpha_3 < \min\{2, \sqrt{2}\kappa\}$ and any $0 < \beta_3 < 1$, we have*

$$\sum_{0 \le i \le 3,\, 0 \le j \le 2} |\partial^{i+j}_{y_1^i z_2^j} \hat{f}_2(y_1, z_2)| \le C e^{-\alpha_3 z_2},$$
$$\sum_{0 \le i \le 3,\, 0 \le j \le 2} |\partial^{i+j}_{y_1^i z_2^j} \hat{\mathbf{Q}}_2(y_1, z_2)| \le C e^{-\beta_3 z_2}$$

for all $z_2 > 0$, where the constants C depend only on Ω, $\mathcal{H}^e$, κ, α_3 and β_3.

The proof is similar to that of Proposition 2.5, and we omit it.

3 Uniform estimation for the approximation solution

In this section we shall construct an approximation solution to system (1.1), then we shall apply the method of matched asymptotic expansions (for the detail see [17]) to derive estimates of this solution with respect to the parameter λ, from which we can derive that the approximation solution we constructed is a global one.

To construct the approximation solution we need an inner asymptotic expansion valid inside the boundary layer, and an outer asymptotic expansion valid outside the boundary layer.

The outer expansion is $(1, 0)$. In fact, we write the outer expansion in the form

$$U_f(x, \lambda) = 1 + \sum_{k=1}^{\infty} \lambda^k h_k^f(x), \qquad \lambda \to 0,$$
$$\mathbf{U_Q}(x, \lambda) = \sum_{k=1}^{\infty} \lambda^k \mathbf{h}_k^{\mathbf{Q}}(x), \qquad \lambda \to 0.$$

The right sides of these equalities should be understood as formal expansions in the powers of λ. Substituting these expressions of U_f and $\mathbf{U_Q}$ into system (1.1) and equating the coefficients of the powers λ^k for each $k \geq 1$, we find that

$$h_k^f(x) = 0, \qquad \mathbf{h}_k^{\mathbf{Q}}(x) = 0, \quad x \in \Omega, \quad k \geq 1.$$

The inner expansion of the form (2.9) can be construct as in section 2. In fact, for each $y_1 \neq 0$, we can find the leading order term $(\hat{f}_0(y_1, z_2), \hat{\mathbf{Q}}_0(y_1, z_2))$, the first order term $(\hat{f}_1(y_1, z_2), \hat{\mathbf{Q}}_1(y_1, z_2))$, and the second order terms $(\hat{f}_2(y_1, z_2), \hat{\mathbf{Q}}_2(y_1, z_2))$ by the processes similar to that for the solutions of (2.11), of system (2.20) and of system (2.22). For more details see (5.18) and (5.19) below.

To construct the global approximation solution $(f_{ap}, \mathbf{Q}_{ap}(x))$, we fix a neighborhood $\mathcal{N}_0$ of the boundary $\partial\Omega$ such that, for each point $X_0 \in \partial\Omega$, there is a ball $B_\epsilon(X_0)$ and a $C^{2,\alpha}$ diffeomorphism that straightens the portion of $\partial\Omega$ that lies in $\mathcal{N}_0 \bigcap B_\epsilon(X_0)$. Set

$$d_0 := \operatorname{dist}(\partial\Omega, \Omega\backslash\mathcal{N}_0), \quad \sigma_n := \left\{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \le d_0/n\right\}. \tag{3.1}$$

Then we define a smooth function $\chi(x)$ by

$$\chi(x) = \begin{cases} 1, & x \in \sigma_4; \\ \text{smooth}, & x \in \sigma_2\backslash\sigma_4; \\ 0, & x \in \Omega\backslash\sigma_2. \end{cases}$$

Note that $\chi(x)$ is independent of λ.

Then we define the approximation solution by

$$\begin{aligned} f_{ap}(x) &= \chi(x)\big(\hat{f}_0(y_1, z_2) + \lambda\hat{f}_1(y_1, z_2) + \lambda^2\hat{f}_2(y_1, z_2)\big) + 1 - \chi(x), \\ \mathbf{Q}_{ap}(x) &= \chi(x)\big(\hat{\mathbf{Q}}_0(y_1, z_2) + \lambda\hat{\mathbf{Q}}_1(y_1, z_2) + \lambda^2\hat{\mathbf{Q}}_2(y_1, z_2)\big), \end{aligned} \tag{3.2}$$

where $z_2 = y_2/\lambda$, $x = \psi(y_1, y_2)$ and ψ is defined by (2.1). Since $\chi(x) = 0$ outside of a neighborhood of X_0, we can extend the approximation solution by zero outside of the support of χ, such that the approximation solution $(f_{ap}, \mathbf{Q}_{ap}(x))$ is defined everywhere in $\bar{\Omega}$.

Now we define an operator $\mathscr{L}_\lambda$ as follows. For a scalar function f and a vector field $\mathbf{Q}$,

$$\mathscr{L}_\lambda(f, \mathbf{Q}) := \left(-\frac{\lambda^2}{\kappa^2}\Delta f - (1 - f^2 - |\mathbf{Q}|^2)f, \quad \lambda^2\operatorname{curl}^2\mathbf{Q} + f^2\mathbf{Q}\right). \tag{3.3}$$

Lemma 3.1 *Let*

$$\mathbf{b}(x, \lambda) = (b_1(x, \lambda), \mathbf{b}_2(x, \lambda)) := \mathscr{L}_\lambda(f_{ap}(x), \mathbf{Q}_{ap}(x)). \tag{3.4}$$

Then there exists a constant λ_0 such that for any $\lambda \in (0, \lambda_0)$ we have

$$\|b_1\|_{C^0(\bar{\Omega})} + \|\mathbf{b}_2\|_{C^0(\bar{\Omega})} + \|\lambda\nabla\mathbf{b}_2\|_{C^0(\bar{\Omega})} + \|\lambda^2\nabla \operatorname{div}\mathbf{b}_2\|_{C^0(\bar{\Omega})} \le C\left(\Omega, \kappa, \mathcal{H}^e\right)\lambda^3. \tag{3.5}$$

The proof of Lemma 3.1 will be given in appendix D.

Now we introduce the remainder terms R_f and $\mathbf{R_Q}$ by letting

$$R_f = f - f_{ap}, \qquad \mathbf{R_Q} = \mathbf{Q} - \mathbf{Q}_{ap}. \tag{3.6}$$

Then $(R_f, \mathbf{R_Q})$ satisfies the equations

$$\begin{cases} -\frac{\lambda^2}{\kappa^2}\Delta R_f = (1 - |f|^2 - ff_{ap} - |f_{ap}|^2 - |\mathbf{Q}|^2)R_f \\ \quad +f_{ap}(\mathbf{Q} + \mathbf{Q}_{ap}) \cdot \mathbf{R_Q} + b_1 & \text{in } \Omega, \\ \lambda^2 \text{curl}^2 \mathbf{R_Q} + |f|^2 \mathbf{R_Q} + (f + f_{ap})R_f \mathbf{Q}_{ap} = \mathbf{b}_2 & \text{in } \Omega, \end{cases} \tag{3.7}$$

and boundary conditions

$$\begin{cases} \dfrac{\partial R_f}{\partial \mathbf{n}} = 0, \qquad \lambda\, \text{curl} \mathbf{R_Q} = \mathcal{B}_3, \\ \mathbf{n} \cdot \mathbf{R_Q} = |f_{ap}|^{-2}\left[\lambda |f|^{-2}(|f|^2 - |f_{ap}|^2)\nabla_{\tan}\mathcal{H}^e + \mathcal{B}_4\right] \quad \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where b_1 and $\mathbf{b}_2$ are defined by (3.4), $\mathcal{B}_3$ and $\mathcal{B}_4$ are given by (E.2) and (E.4) in Appendix E respectively. The derivation of (3.8) is lengthy and will be given in Appendix E.

In the following, we derive the H^1, H^2 and C^0 estimates of $(R_f, \mathbf{R_Q})$ in terms of b_1, $\mathbf{b}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$. We need the following space

$$\mathcal{H}(\Omega, \text{curl}) = \{\mathbf{u} \in L^2(\Omega, \mathbb{R}^2) \;:\; \text{curl}\mathbf{u} \in L^2(\Omega)\}.$$

Lemma 3.2 (H^1 estimate) *Let* $(f, \mathbf{Q})$ *be the solution of* (1.1) *satisfying* (1.8)*, and let* $(R_f, \mathbf{R_Q})$ *be defined by* (3.6) *with* $R_f \in H^1(\Omega)$ *and* $\mathbf{R_Q} \in \mathcal{H}(\Omega, curl)$*. Then there exists a constant* $\lambda_0 > 0$ *such that, for any* $0 < \lambda < \lambda_0$ *we have*

$$\begin{aligned} &\|R_f\|_{L^2(\Omega)} + \|\lambda\nabla R_f\|_{L^2(\Omega)} + \|\mathbf{R_Q}\|_{L^2(\Omega)} + \|\lambda\nabla\mathbf{R_Q}\|_{L^2(\Omega)} \\ &\quad \le C\left(\|b_1\|_{L^2(\Omega)} + \|\mathbf{b}_2\|_{L^2(\Omega)} + \|\lambda\, \text{div}\, \mathbf{b}_2\|_{L^2(\Omega)} + \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)} + \|\mathcal{B}_4\|_{H^{1/2}(\partial\Omega)}\right), \end{aligned} \tag{3.9}$$

where the constant C depends only on Ω, $\mathcal{H}^e$, κ *and* δ*, but not on* λ.

Proof *Step 1*. Note that $(R_f, \mathbf{R_Q})$ can be viewed as a weak solution of (3.7) in the sense of

$$\begin{aligned} &\int_\Omega \Big\{\frac{\lambda^2}{\kappa^2}\nabla R_f \cdot \nabla B + \Big((|f|^2 + ff_{ap} + |f_{ap}|^2 + |\mathbf{Q}|^2 - 1)R_f \\ &\quad - f_{ap}(\mathbf{Q} + \mathbf{Q}_{ap}) \cdot \mathbf{R_Q}\Big)B\Big\}dx = \int_\Omega b_1 B dx, \end{aligned} \tag{3.10}$$

and

$$\int_\Omega \lambda^2 \operatorname{curl} \mathbf{R_Q} \cdot \operatorname{curl} \mathbf{D} + \Big(|f|^2 \mathbf{R_Q} + (f + f_{\mathrm{a}p}) R_f \mathbf{Q}_{\mathrm{a}p}\Big) \cdot \mathbf{D} dx \\ = \int_\Omega \mathbf{b}_2 \cdot \mathbf{D} dx + \int_{\partial\Omega} \lambda^2 \operatorname{curl} \mathbf{R_Q} \cdot (\mathbf{n} \times \mathbf{D}) dS \tag{3.11}$$

for all $B \in H^1(\Omega)$ and $\mathbf{D} \in \mathcal{H}(\Omega, \operatorname{curl})$. Taking $B = R_f$ and $\mathbf{D} = \mathbf{R_Q}$ in (3.10) and (3.11) respectively, and then adding the two equalities together we get

$$\int_\Omega \Big(\frac{\lambda^2}{\kappa^2} |\nabla R_f|^2 + \lambda^2 |\operatorname{curl} \mathbf{R_Q}|^2 + (|f|^2 + f f_{\mathrm{a}p} + |f_{\mathrm{a}p}|^2 + |\mathbf{Q}|^2 - 1)|R_f|^2 \\ + |f|^2 |\mathbf{R_Q}|^2 + R_f (f \mathbf{Q}_{\mathrm{a}p} - f_{\mathrm{a}p} \mathbf{Q}) \cdot \mathbf{R_Q}\Big) dx \\ = \int_\Omega (b_1 R_f + \mathbf{b}_2 \cdot \mathbf{R_Q}) dx + \int_{\partial\Omega} (\lambda \mathbf{n} \times \mathbf{R_Q}) \mathcal{B}_3 dS. \tag{3.12}$$

Using (3.6) we can derive

$$R_f (f \mathbf{Q}_{\mathrm{a}p} - f_{\mathrm{a}p} \mathbf{Q}) \cdot \mathbf{R_Q} = |R_f|^2 \mathbf{Q}_{\mathrm{a}p} \cdot \mathbf{R_Q} - f_{\mathrm{a}p} R_f |\mathbf{R_Q}|^2,$$

and

$$|f|^2 |\mathbf{R_Q}|^2 - f_{\mathrm{a}p} R_f |\mathbf{R_Q}|^2 = \Big(|f|^2 - f_{\mathrm{a}p} f + |f_{\mathrm{a}p}|^2\Big) |\mathbf{R_Q}|^2.$$

Then

$$|f|^2 |\mathbf{R_Q}|^2 + R_f (f \mathbf{Q}_{\mathrm{a}p} - f_{\mathrm{a}p} \mathbf{Q}) \cdot \mathbf{R_Q} \\ = |R_f|^2 \mathbf{Q}_{\mathrm{a}p} \cdot \mathbf{R_Q} + \Big[(f - \frac{1}{2} f_{\mathrm{a}p})^2 + \frac{3}{4} |f_{\mathrm{a}p}|^2\Big] |\mathbf{R_Q}|^2.$$

Step 2. We claim that, for any given $\epsilon > 0$, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ we have

$$|R_f|^2 |\mathbf{Q}_{\mathrm{a}p}(x)| < \epsilon \quad \text{for all } x \in \Omega.$$

Indeed, from Proposition 2.5, Proposition 2.7 and Proposition 2.9, there exists $0 < \beta_3 < 1$ such that

$$|\hat{\mathbf{Q}}_0| + |\hat{\mathbf{Q}}_1| + |\hat{\mathbf{Q}}_2| \le C(\Omega, \mathcal{H}^e, \beta_3) e^{-\beta_3 z_2},$$

which shows that

$$|\mathbf{Q}_{\mathrm{a}p}(x)| \le M_0 e^{-\beta_3 \frac{\operatorname{dist}(x, \partial\Omega)}{\lambda}}$$

for some M_0 depending on Ω, $\mathcal{H}^e$ and β_3. Now we choose R_0 sufficiently large such that

$$M_0 e^{-\beta_3 R_0} < \epsilon.$$

This implies that

$$|R_f|^2 |\mathbf{Q}_{ap}(x)| \le |f - f_{ap}|^2 |\mathbf{Q}_{ap}(x)| \le |\mathbf{Q}_{ap}(x)| < \epsilon \quad \text{if } \mathrm{dist}(x, \partial\Omega) > \lambda R_0.$$

On the other hand, from lemma 2.1 and the uniqueness of the solution to (2.11) satisfying (2.12), we conclude that,

$$\|\tilde{f}_\lambda - \hat{f}_0(\lambda z_1, z_2)\|_{C^0(\tilde{\Omega}_z \cap B^+_{R_0}(0))} \to 0,$$

as $\lambda \to 0$. Here we keep the notation used in section 2. Therefore, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ and any $x_0 \in \partial\Omega$ we have

$$|f - f_{ap}|^2 = |R_f(x)|^2 < \frac{\epsilon}{M_0} \quad \text{if } x \in \Omega \cap B_{\lambda R_0}(x_0),$$

where we have used the boundedness of $\hat{f}_1$ and $\hat{f}_2$. Then

$$|R_f|^2 |\mathbf{Q}_{ap}(x)| < \frac{\epsilon}{M_0} M_0 < \epsilon \quad \text{if } \mathrm{dist}(x, \partial\Omega) \le \lambda R_0.$$

Now the claim is proved.

Step 3. By the trace theorem on $\mathcal{H}(\Omega, \mathrm{curl})$, we have

$$\begin{aligned}\left|\int_{\partial\Omega} (\lambda \mathbf{n} \times \mathbf{R_Q}) \mathcal{B}_3 dS\right| &\le \lambda \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)} \|\mathbf{n} \times \mathbf{R_Q}\|_{H^{-1/2}(\partial\Omega)} \\ &\le \lambda \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)} (\|\mathrm{curl}\mathbf{R_Q}\|_{L^2(\partial\Omega)} + \|\mathbf{R_Q}\|_{L^2(\partial\Omega)}).\end{aligned}$$

Note that, there exists $\lambda_2 > 0$ such that for any $\lambda \in (0, \lambda_2)$ we have

$$|f|^2 > \frac{1}{3} + \frac{\delta^2}{2}, \qquad |f_{ap}|^2 > \frac{1}{3} + \frac{\delta^2}{2}.$$

Then taking ϵ in the claim sufficiently small, and using the Cauchy's inequality, we obtain from (3.12) that

$$\begin{aligned}&\|R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|_{L^2(\Omega)} + \|\mathbf{R_Q}\|_{L^2(\Omega)} + \|\lambda \mathrm{curl}\mathbf{R_Q}\|_{L^2(\Omega)} \\ &\quad \le C \left(\|b_1\|_{L^2(\Omega)} + \|\mathbf{b}_2\|_{L^2(\Omega)} + \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)}\right),\end{aligned} \tag{3.13}$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$.

Denote by

$$\hat{f}_{\mathrm{a}p} = \hat{f}_0(y_1, z_2) + \lambda \hat{f}_1(y_1, z_2) + \lambda^2 \hat{f}_2(y_1, z_2).$$

From Proposition 2.5, Proposition 2.7 and Proposition 2.9, it follows that

$$|\hat{f}_{\mathrm{a}p}| + |\nabla_{y_1} \hat{f}_{\mathrm{a}p}| + |\lambda \nabla_{y_2} \hat{f}_{\mathrm{a}p}| \le C(\Omega, \mathcal{H}^e, \kappa). \tag{3.14}$$

Then by (3.2) we have

$$|\lambda \nabla f_{\mathrm{a}p}(x)| \le C(\Omega, \mathcal{H}^e, \kappa) \quad \text{for all } x \in \Omega.$$

From the second equation of (3.7), we have

$$\operatorname{div}\left(|f|^2 \mathbf{R}_{\mathbf{Q}} + (f + f_{\mathrm{a}p}) R_f \mathbf{Q}_{\mathrm{a}p} - \mathbf{b}_2\right) = 0. \tag{3.15}$$

From this and (3.13), and using the fact $\lambda \nabla f = \lambda \nabla \mathbf{R}_f + \lambda \nabla f_{\mathrm{a}p}$, we find

$$\|\lambda \operatorname{div} \mathbf{R}_{\mathbf{Q}}\|_{L^2(\Omega)} \le C\left(\|b_1\|_{L^2(\Omega)} + \|\mathbf{b}_2\|_{L^2(\Omega)} + \|\lambda \operatorname{div} \mathbf{b}_2\|_{L^2(\Omega)} + \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)}\right),$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$.

We now consider the estimate for $\mathbf{n} \cdot \mathbf{R}_{\mathbf{Q}}$. From (E.6) in appendix E, we have

$$\begin{aligned}
&\|\mathbf{n} \cdot \mathbf{R}_{\mathbf{Q}}\|_{H^{1/2}(\partial\Omega)} \\
&\quad \le \|f_{\mathrm{a}p}^{-2}\|_{C^1(\partial\Omega)} \left(\|\mathcal{B}_4\|_{H^{1/2}(\partial\Omega)} + C(\Omega, \mathcal{H}^e)\|\lambda |f|^{-2}(|f|^2 - |f_{\mathrm{a}p}|^2)\|_{H^{1/2}(\partial\Omega)}\right) \\
&\quad \le C\left(\|\mathcal{B}_4\|_{H^{1/2}(\partial\Omega)} + \|R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|_{L^2(\Omega)}\right),
\end{aligned}$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$. In the last inequality we have used the trace theorem on $H^1(\Omega)$, and the inequalities:

$$\begin{aligned}
&\|f_{\mathrm{a}p}\|_{C^1(\partial\Omega)} \le C(\Omega, \mathcal{H}^e, \kappa, \delta) \quad \text{since (3.14)}, \\
&\frac{1}{3} < |f|^2 \le 1, \qquad \frac{1}{3} < |f_{\mathrm{a}p}|^2 \le 1, \quad |\lambda \nabla f_{\mathrm{a}p}| \le C(\Omega, \mathcal{H}^e, \kappa, \delta).
\end{aligned}$$

We apply the following div-curl-gradient inequality (see [11, P.212, Corollary 1])

$$\begin{aligned}
\|\nabla \mathbf{R}_{\mathbf{Q}}\|_{L^2(\Omega)} \le C(\Omega)\big(&\|\mathbf{R}_{\mathbf{Q}}\|_{L^2(\Omega)} + \|\operatorname{div} \mathbf{R}_{\mathbf{Q}}\|_{L^2(\Omega)} \\
&+ \|\operatorname{curl} \mathbf{R}_{\mathbf{Q}}\|_{L^2(\Omega)} + \|\mathbf{n} \cdot \mathbf{R}_{\mathbf{Q}}\|_{H^{1/2}(\partial\Omega)}\big).
\end{aligned} \tag{3.16}$$

Then using (3.13) and the estimate on $\operatorname{div} \mathbf{R}_{\mathbf{Q}}$ and $\nu \cdot \mathbf{R}_{\mathbf{Q}}$ obtained above, we get (3.9).
□

Therefore, by applying the estimate of $\mathbf{b}$ in Ω (see Lemma 3.1), the estimate of $\mathcal{B}_3$ (see (E.3)) and the estimate of $\mathcal{B}_4$ on $\partial\Omega$ (see (E.5)) in appendix E, for small λ we have

$$\|R_f\|_{L^2(\Omega)} + \|\lambda\nabla R_f\|_{L^2(\Omega)} + \|\mathbf{R_Q}\|_{L^2(\Omega)} + \|\lambda\nabla\mathbf{R_Q}\|_{L^2(\Omega)} \le C\lambda^3, \tag{3.17}$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$.

Next, we establish the H^2 estimate for $(R_f, \mathbf{R_Q})$.

Lemma 3.3 *Let* $(f, \mathbf{Q})$ *be the solution of* (1.1) *satisfying* (1.8)*, and let* $(R_f, \mathbf{R_Q})$ *be the solution of* (3.7) *with* $R_f \in H^1(\Omega)$ *and* $\mathbf{R_Q} \in \mathcal{H}(\Omega, curl)$*. Then there exists a constant* $\lambda_0 > 0$ *such that, for any* $0 < \lambda < \lambda_0$ *we have*

$$\|\lambda^2\nabla^2 R_f\|_{L^2(\Omega)} + \|\lambda^2\nabla^2\mathbf{R_Q}\|_{L^2(\Omega)} \le C\lambda^3, \tag{3.18}$$

where the constant C *depends only on* Ω*,* $\mathcal{H}^e$*,* κ *and* δ*, but not on* λ*.*

Proof *Step 1*. By the usual difference quotient method [5], from the first equation of (3.7) and the boundary condition for R_f in (3.8) we immediately obtain that

$$\|\lambda^2\nabla^2 R_f\|_{L^2(\Omega)} \le C\left(\|R_f\|_{L^2(\Omega)} + \|\mathbf{R_Q}\|_{L^2(\Omega)} + \|b_1\|_{L^2(\Omega)}\right),$$

where the constant C depends on Ω. From (3.5) and (3.17), we have

$$\|\lambda^2\nabla^2 R_f\|_{L^2(\Omega)} \le C(\Omega, \mathcal{H}^e, \kappa, \delta)\lambda^3. \tag{3.19}$$

Step 2. Let $H = \lambda\text{curl}\mathbf{R_Q}$. From the second equation of (3.7), we can deduce that H satisfies

$$\begin{cases} -\lambda^2\Delta H + f^2 H = F & \text{in } \Omega, \\ H = \mathcal{B}_3 & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} &\mathbf{R_Q} = (R_{\mathbf{Q}}^1, R_{\mathbf{Q}}^2) \\ &F = \lambda\text{curl}(\mathbf{b}_2 - (R_f + 2f_{ap})R_f\mathbf{Q}_{ap}) - \lambda(\partial_1(f^2)R_{\mathbf{Q}}^2 - \partial_2(f^2)R_{\mathbf{Q}}^1). \end{aligned}$$

Then by the Cauchy's inequality, and using $f \ge 1/3$ we have

$$\|\lambda\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \le C(\Omega)\left(\|F\|_{L^2(\Omega)} + \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)}\right),$$

[5] We refer to [14, Theorem 8.8] for the interior H^2 estimates, [14, Theorem 8.12] for the boundary H^2 estimates.

where we have used the inequality

$$\lambda^2 \left| \int_{\partial\Omega} \frac{\partial H}{\partial \mathbf{n}} H dS \right| \leq \lambda^2 \|H\|_{H^{1/2}(\partial\Omega)} \left\| \frac{\partial H}{\partial \mathbf{n}} \right\|_{H^{-1/2}(\partial\Omega)}$$

$$\leq C(\Omega) \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)} \left(\left\| \lambda^2 \nabla H \right\|_{L^2(\Omega)} + \left\| \lambda^2 \Delta H \right\|_{L^2(\Omega)} \right).$$

From the expressions of f_{ap} and $\mathbf{Q}_{ap}$ (see (3.2)), then by Proposition 2.5, Proposition 2.7 and Proposition 2.9 we have

$$|f_{ap}(x)| + |\mathbf{Q}_{ap}(x)| + |\lambda \nabla f_{ap}(x)| + |\lambda \nabla \mathbf{Q}_{ap}(x)| \leq C(\Omega, \mathcal{H}^e, \kappa, \delta) \quad \text{for all } x \in \Omega. \tag{3.20}$$

This gives that

$$\|F\|_{L^2(\Omega)} \leq C(\Omega, \mathcal{H}^e, \kappa, \delta) \left(\|\lambda \nabla \mathbf{b}_2\|_{L^2(\Omega)} + \|R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|_{L^2(\Omega)} \right).$$

Therefore, we have

$$\|\lambda \nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \leq C \left(\|\lambda \nabla \mathbf{b}_2\|_{L^2(\Omega)} + \|R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|_{L^2(\Omega)} + \|\mathcal{B}_3\|_{H^{1/2}(\partial\Omega)} \right),$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$. From (3.5), (3.7) and the estimate on $\mathcal{B}_3$ (see (E.3)), we can conclude that

$$\|\nabla(\text{curl} \mathbf{R_Q})\|_{L^2(\Omega)} \leq C(\Omega, \mathcal{H}^e, \kappa, \delta)\lambda. \tag{3.21}$$

By (3.15), we have

$$\text{div}\, \mathbf{R_Q} = \left[\text{div}(\mathbf{b}_2 - (f + f_{ap}) R_f \mathbf{Q}_{ap}) - 2 f \nabla f \cdot \mathbf{R_Q} \right] f^{-2}.$$

Then using Hölder's inequality and $f = R_f + f_{ap}$, we have

$$\begin{aligned} &\|\lambda^2 \nabla \,\text{div}\, \mathbf{R_Q}\|_{L^2(\Omega)} \\ &\quad \leq C\big(\|R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|_{L^2(\Omega)} + \|\lambda \nabla R_f\|^2_{L^4(\Omega)} + \|\lambda^2 \nabla^2 R_f\|_{L^2(\Omega)} \\ &\quad\quad + \|\lambda \,\text{div}\, \mathbf{b}_2\|_{L^2(\Omega)} + \|\lambda^2 \nabla \,\text{div}\, \mathbf{b}_2\|_{L^2(\Omega)} \big) + 6\lambda^2 \|\nabla R_f\|_{L^4(\Omega)} \|\nabla \mathbf{R_Q}\|_{L^4(\Omega)}, \end{aligned}$$

where we have used the estimates in (3.5), (3.20) and $1/3 < f \leq 1$, $C = C(\Omega, \mathcal{H}^e, \kappa, \delta)$. Note that

$$\|\nabla R_f\|_{L^4(\Omega)} \leq C(\Omega) \left(\|\nabla^2 R_f\|_{L^2(\Omega)} + \|\nabla R_f\|_{L^2(\Omega)} \right) \leq C(\Omega, \mathcal{H}^e, \kappa, \delta)\lambda \tag{3.22}$$

by the Sobolev imbedding theorem. Then by (3.5) and (3.17), it follows that

$$\|\nabla(\operatorname{div}\mathbf{R_Q})\|_{L^2(\Omega)} \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda(1+\|\nabla\mathbf{R_Q}\|_{L^4(\Omega)}). \tag{3.23}$$

We now give the estimate of $\nu\cdot\mathbf{R_Q}$. From (3.8), it follows that

$$\begin{aligned}\lambda\|\nu\cdot\mathbf{R_Q}\|_{H^{3/2}(\partial\Omega)} &\le C(\Omega,\mathcal{H}^e,\kappa,\delta)\left(\|\mathcal{B}_4\|_{H^{3/2}(\partial\Omega)}+\|\lambda^2|f|^{-2}(|f|^2-|f_{ap}|^2)\|_{H^{3/2}(\partial\Omega)}\right)\\ &\le C(\Omega,\mathcal{H}^e,\kappa,\delta)\left(\|\mathcal{B}_4\|_{H^{3/2}(\partial\Omega)}+\|\lambda^2|f|^{-2}(|f|^2-|f_{ap}|^2)\|_{H^2(\Omega)}\right),\end{aligned}$$

where we have used the boundedness of $\|f_{ap}\|_{C^2(\partial\Omega)}$ and $\|\nabla_{\tan}\mathcal{H}^e\|_{C^2(\partial\Omega)}$. Using (3.22) and (3.20), we have

$$\|\lambda^2|f|^{-2}(|f|^2-|f_{ap}|^2)\|_{H^2(\Omega)} \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda^3.$$

Then by the estimate on $\mathcal{B}_4$ (see (E.5)), we now obtain that

$$\|\nu\cdot\mathbf{R_Q}\|_{H^{3/2}(\partial\Omega)} \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda^2. \tag{3.24}$$

By applying the div-curl-gradient inequality (see [3, section 2])

$$\|\mathbf{u}\|_{H^2(\Omega)} \le C(\Omega)\left(\|\mathbf{u}\|_{H^1(\Omega)}+\|\operatorname{div}\mathbf{u}\|_{H^1(\Omega)}+\|\operatorname{curl}\mathbf{u}\|_{H^1(\Omega)}+\|\nu\cdot\mathbf{u}\|_{H^{3/2}(\partial\Omega)}\right),$$

we at last obtain that

$$\|\nabla^2\mathbf{R_Q}\|_{L^2(\Omega)} \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda(1+\|\nabla\mathbf{R_Q}\|_{L^4(\Omega)}),$$

where we have used (3.21), (3.23), (3.17) and (3.24). By the Sobolev imbedding theorem, then choosing λ sufficiently small, we have

$$\|\nabla^2\mathbf{R_Q}\|_{L^2(\Omega)} \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda(1+\|\nabla\mathbf{R_Q}\|_{L^2(\Omega)}) \le C(\Omega,\mathcal{H}^e,\kappa,\delta)\lambda.$$

We end our proof. □

We use the notations introduced in section 2. Let $X_0\in\partial\Omega$ be fixed and $\mathcal{U}$ be a neighborhood of X_0. We assume that ψ defined by (2.1) is a diffeomorphism from $B_R^+(0)$ onto $\mathcal{U}\bigcap\Omega$. Here $B_R^+(0)$ denotes an open half ball with the center at the origin and the radius R. Let $\tilde{R}_f$ and $\tilde{\mathbf{R}}_\mathbf{Q}$ be the representations of R_f and $\mathbf{R_Q}$ under the z-coordinate system respectively. Then we have the estimate

$$\|\tilde{R}_f\|_{H^2(B_1^+(0))}+\|\tilde{\mathbf{R}}_\mathbf{Q}\|_{H^2(B_1^+(0))} \le C\lambda^2,$$

where the constant C depends on $\mathcal{H}^e$, Ω, κ and δ, but not on λ.

Applying the Sobolev imbedding theorem ([1, Lemma 5.17]), we can derive the C^0 estimate for $\tilde{R}_f$ and $\tilde{\mathbf{R}}_\mathbf{Q}$ on a half ball $\overline{B_1^+}$.

Theorem 3.4 *Let* $(f, \mathbf{Q})$ *be the solution of system* (1.1) *satisfying* (1.8)*, and let* $(\tilde{R}_f, \tilde{\mathbf{R}}_{\mathbf{Q}})$ *be the solution of system* (3.7) *under the* z*-coordinate system. Then there exists a constant* $\lambda_0 > 0$ *such that, for any* $0 < \lambda < \lambda_0$ *we have*

$$\|\tilde{R}_f\|_{C^0(\overline{B_1^+})} + \|\tilde{\mathbf{R}}_{\mathbf{Q}}\|_{C^0(\overline{B_1^+})} \leq C\lambda^2,$$

where the constant C *depends only on* Ω*,* $\mathcal{H}^e$*,* κ *and* δ*, but not on* λ*.*

Proof of (1.9) *and* (1.10) The inequalities (1.9) and (1.10) follow from Theorem 3.4 immediately. □

4 Decay estimate for Meissner solutions

In this section we prove Theorem 1.1. We shall follow the notations in section 3. We also introduce the new variable $t = \frac{x}{\lambda}$, and set

$$\Omega_\lambda = \left\{ t \in \Omega_\lambda \ : \ t = \frac{x}{\lambda}, \quad x \in \Omega \right\}.$$

Let $(f, \mathbf{Q})$ be the solution of (1.1), and let $(f_{\mathrm{a}p}, \mathbf{Q}_{\mathrm{a}p})$ be the approximation solution constructed in section 3 in the x-coordinates. Let $\check{f}$, $\check{R}_f$, $\check{f}_{\mathrm{a}p}$, $\check{\mathbf{Q}}$, $\check{\mathbf{R}}_{\mathbf{Q}}$, $\check{\mathbf{Q}}_{\mathrm{a}p}$, $\check{\mathcal{B}}_3$ and $\check{\mathbf{b}}$ be the representations of f, R_f, $f_{\mathrm{a}p}$, $\mathbf{Q}$, $\mathbf{R}_{\mathbf{Q}}$, $\mathbf{Q}_{\mathrm{a}p}$, $\mathcal{B}_3$ and $\mathbf{b}$ in the t-coordinate system respectively. Then $(\check{R}_f, \check{\mathbf{R}}_{\mathbf{Q}})$ satisfies

$$\begin{cases} \frac{1}{\kappa^2}\Delta \check{R}_f = (|\check{f}|^2 + \check{f}\check{f}_{\mathrm{a}p} + |\check{f}_{\mathrm{a}p}|^2 + |\check{\mathbf{Q}}|^2 - 1)\check{R}_f - \check{f}_{\mathrm{a}p}(\check{\mathbf{Q}} + \check{\mathbf{Q}}_{\mathrm{a}p}) \cdot \check{\mathbf{R}}_{\mathbf{Q}} - \check{b}_1 & \text{in } \Omega_\lambda, \\ \mathrm{curl}^2 \check{\mathbf{R}}_{\mathbf{Q}} + |\check{f}|^2 \check{\mathbf{R}}_{\mathbf{Q}} + (\check{f} + \check{f}_{\mathrm{a}p})\check{R}_f \check{\mathbf{Q}}_{\mathrm{a}p} = \check{\mathbf{b}}_2 & \text{in } \Omega_\lambda, \\ \frac{\partial \check{R}_f}{\partial \mathbf{n}} = 0, \quad \mathrm{curl}\check{\mathbf{R}}_{\mathbf{Q}} = \check{\mathcal{B}}_3 & \text{on } \partial\Omega_\lambda. \end{cases} \tag{4.1}$$

Lemma 4.1 (Schauder estimate) *Let* $(f, \mathbf{Q})$ *be the solution of* (1.1) *satisfying* (1.8)*. Then there exists a constant* C *depending on* Ω*,* $\|\mathcal{H}^e\|_{C^3(\partial\Omega)}$*,* κ *and* δ*, but independent of* λ*, such that*

$$\left\| \check{f} \right\|_{C^3(\Omega_\lambda)} + \left\| \check{\mathbf{Q}} \right\|_{C^3(\Omega_\lambda)} \leq C. \tag{4.2}$$

Proof The proof is similar to that of Lemma 9.2 in [26], and we give only the outline of the proof here. For any number $m > 0$, let $B_m(x_0)$ denote a ball with radius m and center $x_0 \in \Omega_\lambda$, and

$$\mathcal{O}_m := B_m(x_0) \cap \Omega_\lambda.$$

Step 1. The scaled function $\check{f}$ satisfies the following equation in Ω_λ:

$$-\frac{1}{\kappa^2}\Delta \check{f} = (1 - |\check{f}|^2 - |\check{\mathbf{Q}}|^2)\check{f} \quad \text{in } \Omega_\lambda, \qquad \frac{\partial \check{f}}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega_\lambda. \tag{4.3}$$

From (1.8) we see that

$$\frac{1}{3} \le \check{f}(t) \le 1, \quad |\check{\mathbf{Q}}(t)|^2 \le \frac{2}{3}, \quad t \in \Omega_\lambda.$$

By the L^p estimate of elliptic equations we see that $\check{f} \in W^{2,p}(\mathcal{O}_1)$ for any $1 < p < \infty$, and hence $\check{f} \in C^{1,\alpha}(\mathcal{O}_1)$ for any $0 < \alpha < 1$.

Step 2. $\check{\mathbf{Q}}$ satisfies the following equation on Ω_λ:

$$\begin{cases} \mathrm{curl}^2\check{\mathbf{Q}} + |\check{f}|^2\check{\mathbf{Q}} = 0 & \text{in } \Omega_\lambda, \\ \mathrm{curl}\check{\mathbf{Q}} = \check{\mathcal{H}}^e, \quad \nu \cdot \check{\mathbf{Q}} = -\check{f}^{-2}\nabla_{\tan}\check{\mathcal{H}}^e & \text{on } \partial\Omega_\lambda. \end{cases} \tag{4.4}$$

From (4.4) we can derive the integral estimate of $\mathrm{curl}\check{\mathbf{Q}}$. From the first equality we see that

$$\mathrm{div}\,\check{\mathbf{Q}} = 2\check{f}^{-1}\nabla\check{f} \cdot \check{\mathbf{Q}} \in L^2(\mathcal{O}_1). \tag{4.5}$$

Then we use the cut-off argument and use the div-curl-gradient inequality for vector fields vanishing on $\partial\mathcal{O}_{1/2}$ to get an estimate on the norm $\|\check{\mathbf{Q}}\|_{H^1(\mathcal{O}_{1/2})}$. It follows that

$$\mathrm{div}\,\check{\mathbf{Q}} = 2\check{f}^{-1}\nabla\check{f} \cdot \check{\mathbf{Q}} \in H^1(\mathcal{O}_{1/2}).$$

We further use the difference quotient method to derive an estimate for $\|\check{\mathbf{Q}}\|_{H^2(\mathcal{O}_{1/3})}$. By this and the Sobolev imbedding theorem we find that $\check{\mathbf{Q}} \in W^{1,p}(\mathcal{O}_{1/3})$ for any $1 < p < \infty$, hence

$$\check{\mathbf{Q}} \in C^\alpha(\overline{\mathcal{O}}_{1/3}, \mathbb{R}^2) \quad \text{for any } 0 < \alpha < 1. \tag{4.6}$$

From this and (4.5) we see that

$$\mathrm{div}\check{\mathbf{Q}} \in C^\alpha(\overline{\mathcal{O}}_{1/3}) \quad \text{for any } 0 < \alpha < 1. \tag{4.7}$$

Step 3. Now we denote

$$\check{\mathbf{Q}}(t) = (\check{Q}^1(t), \check{Q}^2(t)), \qquad \check{H}(t) = \mathrm{curl}\check{\mathbf{Q}}(t).$$

$\check{H}$ is a solution of the following Dirichlet problem

$$\begin{cases} \Delta\check{H} + \partial_2(f^2\check{Q}^1) - \partial_1(f^2\check{Q}^2) = 0 & \text{in } \Omega_\lambda, \\ \check{H} = \check{\mathcal{H}}^e & \text{on } \partial\Omega_\lambda. \end{cases} \tag{4.8}$$

Applying the interior L^p estimates of elliptic equations and using the result obtained in step 2, we see that $\check{H} \in W^{2,p}(\mathcal{O}_{1/4})$ for any $1 < p < \infty$. This and the Sobolev imbedding theorem imply that $\text{curl}\check{\mathbf{Q}} = \check{H} \in C^{1,\alpha}(\mathcal{O}_{1/4})$. From this, (4.6) and (4.7), and applying the div-curl-gradient inequality

$$\|\mathbf{u}\|_{C^{k+1,\alpha}(\bar{D})} \le C(D,k,\alpha)\{\|\mathbf{u}\|_{C^{k,\alpha}(\bar{D})} + \|\text{div}\mathbf{u}\|_{C^{k,\alpha}(\bar{D})} + \|\text{curl}\mathbf{u}\|_{C^{k,\alpha}(\bar{D})} + \|\nu\cdot\mathbf{u}\|_{C^{k+1,\alpha}(\partial D)}\}, \tag{4.9}$$

with $k = 0$ to $\zeta\check{\mathbf{Q}}$, where ζ is a suitable cut-off function, we obtain $\check{\mathbf{Q}} \in C^{1,\alpha}(\mathcal{O}_{1/5})$.

Step 4. Using equation (4.3) again we can show that $\check{f} \in C^{3,\alpha}(\mathcal{O}_{1/6})$. From this and (4.7) we get $\text{div}\,\check{\mathbf{Q}} \in C^{1,\alpha}(\mathcal{O}_{1/6})$. Applying Schauder estimates to (4.8) we get $\text{curl}\check{\mathbf{Q}} = \check{H} \in C^{2,\alpha}(\mathcal{O}_{1/7})$. Then using (4.9) with $k = 1$ we get $\check{\mathbf{Q}} \in C^{2,\alpha}(\mathcal{O}_{1/8})$. From this and (4.5) we see that $\text{div}\,\check{\mathbf{Q}} \in C^{2,\alpha}(\mathcal{O}_{1/8})$. So using (4.9) with $k = 2$ we find $\check{\mathbf{Q}} \in C^{3,\alpha}(\mathcal{O}_{1/9})$. □

Combining (3.5) and Lemma 4.1 we have

$$\|\check{\mathbf{b}}\|_{C^2(\Omega_\lambda)} \le C\lambda^3,$$

where C is independent of λ. Then by the scaling argument and using Lemma 3.2 and Lemma 4.1, we find

$$\|\check{R}_f\|_{H^1(\Omega_\lambda)} + \|\check{\mathbf{R}}_{\mathbf{Q}}\|_{H^1(\Omega_\lambda)} \le C\lambda^2, \tag{4.10}$$

where C depends on Ω, κ, δ and $\mathcal{H}^e$, but is independent of λ.

We now establish the interior C^α estimate for $(\check{R}_f, \check{\mathbf{R}}_{\mathbf{Q}})$. Denote

$$d(t) = \text{dist}(t, \partial\Omega_\lambda), \qquad \omega_n := \{t \in \Omega_\lambda : d(t) \ge n\}. \tag{4.11}$$

Lemma 4.2 *Let $(f, \mathbf{Q})$ be the solution of (1.1) satisfying (1.8). Then there exists a constant C depending on Ω, $\mathcal{H}^e$, κ and δ, but not on λ, such that*

$$\|\check{R}_f\|_{C^0(\omega_1)} + \|\check{\mathbf{R}}_{\mathbf{Q}}\|_{C^0(\omega_1)} \le C\lambda^2, \tag{4.12}$$

where ω_1 is defined in (4.11) for $n = 1$.

Proof Using (4.10) and applying Sobolev imbedding theorem (see [1, Chapter 6]), we can show that, for any $1 < p < \infty$ and any ball $B_1(x_0) \subset \Omega_\lambda$ we have

$$\|\check{\mathbf{R}}_{\mathbf{Q}}\|_{L^p(B_1(x_0))} + \|\check{R}_f\|_{L^p(B_1(x_0))} \le C(p)\left(\|\check{R}_f\|_{H^1(\Omega_\lambda)} + \|\check{\mathbf{R}}_{\mathbf{Q}}\|_{H^1(\Omega_\lambda)}\right) \le C\lambda^2.$$

where the constant C in the right side depends on Ω, $\mathcal{H}^e$, κ, δ and p, but is independent of λ. Then we apply the interior $W^{1,p}$ elliptic estimates to (4.1) (see Theorem 2.2 in [8, Chapter 10]) and find that

$$\|\nabla\check{R}_f\|_{L^p(B_{\frac{1}{2}}(x_0))} + \|\nabla\check{\mathbf{R}}_{\mathbf{Q}}\|_{L^p(B_{\frac{1}{2}}(x_0))} \le C(p)\left(\|\check{\mathbf{R}}_{\mathbf{Q}}\|_{L^p(B_1(x_0))} + \|\check{R}_f\|_{L^p(B_1(x_0))}\right)$$

$$\leq C\lambda^2.$$

Taking $p > 2$ in this inequality and applying the Sobolev imbedding theorem again, we obtain (4.12). □

Proof of Theorem 1.1 The proof is based on the Agmon's estimate [2].

First, from the expressions of f_{ap} and $\mathbf{Q}_{ap}$ given in (3.2), and using Lemma 4.2, we see that, for any positive constants $\beta_4 < 1$ and $\alpha_4 < \min\{2, \sqrt{2}\kappa\}$, there exists N_0 depending on α_4 and β_4, such that for any x satisfying $\mathrm{d}(x, \partial\Omega) > N_0\lambda$ we have

$$f(x) = f_{ap}(x) + R_f(x) > \beta_4, \quad \kappa^2(f^2 + f + |\mathbf{Q}|^2) > \alpha_4^2. \tag{4.13}$$

Step 1. We prove the exponential decay of $\mathbf{Q}$.

Multiplying the second equation of (1.1) by $\eta_0^2\mathbf{Q}$ with $\eta_0 \in H_0^1(\Omega)$, and integrating over Ω, we obtain

$$\int_\Omega \left(\lambda^2|\mathrm{curl}(\eta_0\mathbf{Q})|^2 + |\eta_0 f\mathbf{Q}|^2\right) dx = \lambda^2 \int_\Omega |\nabla\eta_0 \times \mathbf{Q}|^2 dx. \tag{4.14}$$

Take

$$\mathrm{d}(x) = \mathrm{d}(x, \partial\Omega), \quad \eta_0(x) = \zeta_0(x)e^{\beta_4\,\mathrm{d}(x)/\lambda},$$

where $0 < \beta_4 < 1$, and $\zeta_0 \in C_0^\infty(\Omega, [0, 1])$ is a cutoff function satisfying

$$\zeta_0(x) = \begin{cases} 1, & \text{if } \mathrm{d}(x) > (N_0 + 1)\lambda, \\ 0, & \text{if } \mathrm{d}(x) < N_0\lambda, \end{cases}$$

and $|\nabla\zeta_0(x)| \leq 2/\lambda$ for all x. Plugging this η_0 into (4.14), and using (4.13) and the estimate $|\mathbf{Q}| \leq 1$, we derive

$$\int_\Omega e^{2\beta_4\,\mathrm{d}(x)/\lambda}|\mathbf{Q}|^2 dx \leq C, \tag{4.15}$$

where the constant C depends on Ω, β_4, κ, δ and $\mathcal{H}^e$, but not on λ.

Next, we let

$$\mathbf{A}(x) = e^{\beta_4\,\mathrm{d}(x)/\lambda}\mathbf{Q}(x).$$

Then from (4.14), (4.2) and (4.15), we have

$$\int_\Omega |\lambda\mathrm{curl}\mathbf{A}|^2 dx \leq C, \tag{4.16}$$

where $C = C(\Omega, \beta_4, \kappa, \delta, \mathcal{H}^e)$. From (1.1), we have $\operatorname{div}(f^2\mathbf{Q}) = 0$ in Ω. Then we obtain that

$$\lambda f \operatorname{div} \mathbf{A} + 2\lambda \nabla f \cdot \mathbf{A} - f \mathbf{c}_1(x) \cdot \mathbf{A} = 0 \quad \text{in } \Omega, \tag{4.17}$$

where

$$\mathbf{c}_1(x) = \beta_4 \nabla \mathrm{d}(x). \tag{4.18}$$

Using $|\lambda \nabla f| \le C$ (see (4.2)) and (4.15), we have

$$\int_\Omega |\lambda \operatorname{div} \mathbf{A}|^2 dx \le C(\Omega, \mathcal{H}^e, \kappa, \delta, \beta_4). \tag{4.19}$$

Note that $\mathbf{n} \cdot \mathbf{A} = \mathbf{n} \cdot \mathbf{Q} = -\lambda f^{-2} \nabla_{\tan} \mathcal{H}^e$ on $\partial\Omega$. Then there exists a constant C depending on Ω, $\mathcal{H}^e$, κ and δ, such that

$$\begin{aligned} \|\mathbf{n} \cdot \mathbf{A}\|_{H^{1/2}(\partial\Omega)} &\le \|\mathcal{H}^e\|_{C^2(\partial\Omega)} \|\lambda f^{-2}\|_{H^{1/2}(\partial\Omega)} \\ &\le C(\Omega) \|\mathcal{H}^e\|_{C^2(\partial\Omega)} \|\lambda f^{-2}\|_{H^1(\Omega)} \le C, \end{aligned}$$

where in this inequality we have used the trace theorem on $H^1(\Omega)$ and $\|\lambda \nabla f\|_{L^2(\Omega)} \le C$. Applying (4.15), (4.16), (4.19) and then by the div-curl-gradient inequality (3.16), we have

$$\int_\Omega |\lambda \nabla \mathbf{A}|^2 dx \le C, \tag{4.20}$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta, \beta_4)$.

From (1.1), we can derive that $\mathbf{A}$ is a weak solution of the following system:

$$\lambda^2 \operatorname{curlcurl} \mathbf{A} - \lambda \operatorname{curl} E - \mathbf{F}(x) = 0 \quad \text{in } \Omega, \tag{4.21}$$

where $\mathbf{A} = (A_1, A_2)$,

$$\begin{aligned} &E(x) = \beta_4(\partial_1 \mathrm{d}(x) A_2 - \partial_2 \mathrm{d}(x) A_1), \quad \mathbf{c}_2(x) = \beta_4 \operatorname{curl} \mathrm{d}(x) = \beta_4(\partial_2 \mathrm{d}(x), -\partial_1 \mathrm{d}(x)), \\ &\operatorname{curl} E = (\partial_2 E, -\partial_1 E), \quad \mathbf{F}(x) = (\lambda \operatorname{curl} \mathbf{A}) \mathbf{c}_2(x) - E(x) \mathbf{c}_2(x) - f^2 \mathbf{A}. \end{aligned}$$

Denote by

$$\mathbf{G}(x) = f^{-1} \left(2\lambda \nabla f \cdot \mathbf{A} - f \mathbf{c}_1(x) \cdot \mathbf{A} \right),$$

and let

$$\check{\mathbf{A}}(t) = \mathbf{A}(\lambda t), \quad \check{\mathbf{F}}(t) = \mathbf{F}(\lambda t), \quad \check{E}(t) = E(\lambda t), \quad \check{G}(t) = G(\lambda t).$$

From (4.21) and (4.17), for any $\mathbf{\Phi} \in H_0^1(B_2(t_0))$ with $B_2(t_0) \subset \Omega_t$ being a disc of the center t_0 and radius 2, we have

$$\int_{B_2(t_0)} \nabla\check{\mathbf{A}} \cdot \nabla\mathbf{\Phi} dt = \int_{B_2(t_0)} \mathrm{curl}\check{\mathbf{A}} \cdot \mathrm{curl}\mathbf{\Phi} dt + \int_{B_2(t_0)} \mathrm{div}\,\check{\mathbf{A}} \cdot \mathrm{div}\,\mathbf{\Phi} dt$$
$$= \int_{B_2(t_0)} \check{E}\mathrm{curl}\mathbf{\Phi} dt + \int_{B_2(t_0)} \check{\mathbf{F}} \cdot \mathbf{\Phi} dt + \int_{B_2(t_0)} \check{G}\,\mathrm{div}\,\mathbf{\Phi} dt.$$

Since $\mathrm{div}\,\check{\mathbf{F}} = 0$ in $B_2(t_0)$, we can find $\check{H} \in L^p(B_2(t_0))$ (see Lemma 3 in [11, Chapter IX]) such that $\mathrm{curl}\check{H} = \check{\mathbf{F}}$ and $\|\check{H}\|_{L^p(B_2(t_0))} \le C(p)\|\check{\mathbf{F}}\|_{L^2(B_2(t_0))}$ for any $2 \le p < \infty$. Now we can apply the interior $W^{1,p}$ elliptic estimates to $\check{\mathbf{A}}$ (see Theorem 2.2 in [8, Chapter 10]) and find that

$$\|\nabla\check{\mathbf{A}}\|_{L^p(B_1(t_0))}$$
$$\le C(p)\left(\|\check{E}\|_{L^p(B_2(t_0))} + \|\check{G}\|_{L^p(B_2(t_0))} + \|\check{H}\|_{L^p(B_2(t_0))} + \|\check{\mathbf{A}}\|_{H^1(B_2(t_0))}\right).$$

Taking $p = 3$ in the last inequality and applying the Sobolev imbedding theorem, we obtain

$$\|\nabla\check{\mathbf{A}}\|_{L^3(B_1(t_0))} \le C\|\check{\mathbf{A}}\|_{H^1(B_2(t_0))} \le C\lambda^{-1},$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta, \beta_4)$.

Since $W^{1,3}$ is continuously embedded into C^0, and then using the arbitrariness of the ball $B_1(t_0) \subset \Omega_\lambda$, we have

$$\|\check{\mathbf{A}}\|_{C^0(\omega_2)} \le C\lambda^{-1}, \tag{4.22}$$

where $C = C(\Omega, \beta_4, \kappa, \delta, \mathcal{H}^e)$, and ω_2 is defined by (4.11). Since $\partial\Omega \in C^3$, then there exists a positive constant μ depending on Ω such that the distance function $\mathrm{d}(x) \in C^3(\Gamma_\mu)$ (see [14, Lemma 14.16]), where

$$\Gamma_\mu = \{x \in \bar{\Omega} : \mathrm{d}(x) < \mu\}. \tag{4.23}$$

Using the equations (4.17) and (4.21), then by the Schauder's estimate [14, Theorem 6.2] on $\Gamma_{\mu,\lambda}$ we have

$$\|\check{\mathbf{A}}\|_{C^1(\omega_3 \bigcap \Gamma_{\frac{\mu}{2},\lambda})} \le C\lambda^{-1},$$

where $C = C(\Omega, \beta_4, \kappa, \delta, \mathcal{H}^e)$, and $\Gamma_{\mu,\lambda} = \left\{t = x/\lambda \in \Omega_\lambda : x \in \Gamma_\mu\right\}$. Then

$$|\lambda \mathrm{curl}\mathbf{Q}(x)| \le C(\Omega, \beta_4, \kappa, \delta, \mathcal{H}^e)\lambda^{-1}e^{-\beta_4\,\mathrm{d}(x)/\lambda} \quad \text{for } x \in \partial\Gamma_{\frac{\mu}{2}} \setminus \partial\Omega.$$

For any $0 < \beta < \beta_4$, there exists $\lambda_1 > 0$ such that, for any $\lambda \in (0, \lambda_1)$ we have

$$|\lambda \mathrm{curl}\mathbf{Q}(x)| \leq C_1 e^{-\beta\, \mathrm{d}(x)/\lambda} \quad \text{for } x \in \partial\Gamma_{\frac{\mu}{2}} \backslash \partial\Omega \tag{4.24}$$

and by (4.22)

$$|\mathbf{Q}(x)| \leq C_2 e^{-\beta\, \mathrm{d}(x)/\lambda} \quad \text{for } x \in \Omega \backslash \Gamma_{\frac{\mu}{4}}, \tag{4.25}$$

where the constants C_1 and C_2 depending on $\Omega, \beta, \kappa, \delta$ and $\mathcal{H}^e$ can be taken the same number. Let

$$H = \lambda \mathrm{curl}\mathbf{Q}, \quad B(x) = e^{\beta\, \mathrm{d}(x)/\lambda} H.$$

Then from the second equation of (1.1), we see that $B(x)$ satisfies

$$\lambda^2 \,\mathrm{div}(f^{-2}\nabla B) - 2\lambda\beta f^{-2}\nabla d \cdot \nabla B + (f^{-2}\beta^2 - \lambda\beta f^{-2}\Delta d - 1)B = 0 \quad \text{for } x \in \Gamma_{\frac{\mu}{2}}.$$

There exists positive constants ε (depending on $\Omega, \beta, \kappa, \delta, \mathcal{H}^e$) and λ_2 (depending on $\varepsilon, \Omega, \beta, \kappa, \delta, \mathcal{H}^e$) such that for any $\lambda \in (0, \lambda_2)$ we have

$$f^{-2}\beta^2 - \lambda\beta f^{-2}\Delta d - 1 < -\varepsilon \quad \text{for } x \in \Gamma_{\frac{\mu}{2}} \backslash \Gamma_{\lambda N_0},$$

where we have used the first inequality in (4.13). By the maximum principle [14, Theorem 3.7], we have

$$\|B\|_{C^0(\Gamma_{\frac{\mu}{2}} \backslash \Gamma_{\lambda N_0})} \leq \|B\|_{C^0(\partial(\Gamma_{\frac{\mu}{2}} \backslash \Gamma_{\lambda N_0}))} \leq \max(e^{\beta N_0}, C_1),$$

where C_1 is given in (4.24). By the Schauder's estimate [14, Theorem 6.2] again, we have

$$\|\lambda \nabla B\|_{C^0(\Gamma_{\frac{\mu}{4}} \backslash \Gamma_{\lambda(N_0+1)})} \leq C(\Omega, \mathcal{H}^e)\|B\|_{C^0(\Gamma_{\frac{\mu}{2}} \backslash \Gamma_{\lambda N_0})}.$$

Then we have

$$|\lambda^2 \mathrm{curl}^2\mathbf{Q}(x)| \leq 2|\lambda \nabla \mathbf{H}(x)| \leq C e^{-\beta\, \mathrm{d}(x)/\lambda} \quad \text{for } x \in \Gamma_{\frac{\mu}{4}} \backslash \Gamma_{\lambda(N_0+1)},$$

where $C = C(\Omega, \beta, \kappa, \delta, \mathcal{H}^e)$. Using the second equation in (1.1), we have

$$|\mathbf{Q}(x)| = |\lambda^2 f^{-2} \mathrm{curlcurl}\mathbf{Q}(x)| \leq C e^{-\beta\, \mathrm{d}(x)/\lambda} \quad \text{for } x \in \Gamma_{\frac{\mu}{4}},$$

where we have used $|\mathbf{Q}(x)| \leq 1$ for $x \in \Gamma_{\lambda(N_0+1)}$, $C = C(\Omega, \beta, \kappa, \delta, \mathcal{H}^e)$. Combining this inequality with the estimate in (4.25), we obtain the exponential decay estimate for $\mathbf{Q}$.

Step 2. We prove the exponential decay of f.

Let $g = 1 - f$. Multiplying the first equation of (1.1) by $\eta_1^2 g$ with $\eta_1 \in H_0^1(\Omega)$, and integrating over Ω, we obtain

$$\begin{aligned}\int_\Omega \left(\frac{\lambda^2}{\kappa^2}|\nabla(\eta_1 g)|^2 + (2 - 3g + |g|^2 + |\mathbf{Q}|^2)(\eta_1 g)^2\right) dx \\ = \int_\Omega |\eta_1|^2 |\mathbf{Q}|^2 g dx + \frac{\lambda^2}{\kappa^2}\int_\Omega |g\nabla\eta_1|^2 dx.\end{aligned} \tag{4.26}$$

Take

$$\eta_1(x) = \zeta_1(x) e^{\alpha_4 \mathrm{d}(x)/\lambda},$$

where $0 < \alpha_4 < \min\{\sqrt{2}\kappa, 2\beta\}$, and $\zeta_1 \in C_0^\infty(\Omega, [0, 1])$ is a cutoff function satisfying

$$\zeta_1(x) = \begin{cases} 1, & \text{if } \mathrm{d}(x) > (N_0 + 1)\lambda, \\ 0, & \text{if } \mathrm{d}(x) < N_0\lambda, \end{cases}$$

and $|\nabla\zeta_1(x)| \le 2/\lambda$ for all x. Plugging this η_1 into (4.26), using (4.13) and the fact $|g| < 1$, we get

$$\int_\Omega e^{2\alpha_4 \mathrm{d}(x)/\lambda}|g|^2 dx \le C,$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta, \alpha_4, \beta)$, but C does not depend on λ.

Now we set

$$h(x) = e^{\alpha_4 \mathrm{d}(x)/\lambda} g(x).$$

Using (4.26) again, we have

$$\|h\|_{L^2(\Omega)} + \|\lambda\nabla h\|_{L^2(\Omega)} \le C(\Omega, \mathcal{H}^e, \kappa, \delta, \alpha_4, \beta).$$

Then h is a weak solution of

$$\frac{\lambda^2}{\kappa^2}\Delta h - \frac{\lambda}{\kappa^2}\operatorname{div}(h\mathbf{v}) = r(x) \quad \text{in } \Omega,$$

where

$$\begin{aligned}&\mathbf{v}(x) = \alpha_4 \nabla \mathrm{d}(x), \\ &r(x) = \frac{\mathbf{v}}{\kappa^2}\cdot(\lambda\nabla h - h\mathbf{v}) + (2 - 3g + |g|^2 + |\mathbf{Q}|^2)h - |\mathbf{Q}|^2 e^{\alpha_4 \mathrm{d}(x)/\lambda}.\end{aligned}$$

Let

$$\check{h}(t) = h(\lambda t), \quad \check{\mathbf{d}}(t) = \check{\mathbf{d}}(\lambda t), \quad \check{r}(t) = \check{r}(\lambda t)$$

Then for any $\varphi \in H_0^1(B_2)$ with $B_2(t_0) \subset \Omega_t$ being a ball with the center t_0 and radius 2, we have

$$\int_{B_2(t_0)} \frac{1}{\kappa^2} \nabla \check{h} \cdot \nabla \varphi dt = \int_{B_2(t_0)} \frac{1}{\kappa^2} \check{h} \check{\mathbf{d}} \cdot \nabla \varphi dt - \int_{B_2(t_0)} \check{r} \cdot \varphi dt$$

We look for $\phi \in H_0^1(B_2(t_0)) \bigcap H^2(\Omega)$ such that

$$\int_{B_2(t_0)} \check{r} \cdot \varphi dt = \int_{B_2(t_0)} \nabla \phi \cdot \nabla \varphi dt \quad \text{for any } \varphi \in H_0^1(B_2)$$

and ϕ satisfies $\|\nabla \phi\|_{L^p(B_2(t_0))} \leq C(p)\|\check{r}\|_{L^2(B_2(t_0))}$ with $2 < p < \infty$. Now we can apply the interior $W^{1,p}$ elliptic estimates to $\check{h}$ (see Theorem 2.2 in [8, Chapter 10]) and find that

$$\|\nabla \check{h}\|_{L^p(B_1(t_0))} \leq C(p,\kappa) \left(\|\check{h}\|_{L^p(B_2(t_0))} + \|\nabla \phi\|_{L^p(B_2(t_0))} + \|\check{h}\|_{H^1(B_2(t_0))} \right).$$

Taking $p = 3$ in the last inequality and applying the Sobolev imbedding theorem, we obtain

$$\|\nabla \check{h}\|_{L^3(B_1(t_0))} \leq C(\|\check{h}\|_{H^1(B_2(t_0))} + 1) \leq C\lambda^{-1}.$$

where $C = C(\Omega, \mathcal{H}^e, \kappa, \delta, \alpha_4, \beta)$. By the Sobolev imbedding theorem again, we have $\check{h} \in C^0(\omega_2)$, where ω_2 is defined by (4.11). Then

$$|g(x)| \leq C(\Omega, \mathcal{H}^e, \kappa, \delta, \alpha_4, \beta)\lambda^{-1} e^{-\alpha_4 \,\mathrm{d}(x)/\lambda} \quad \text{for } x \in \bar{\Omega}.$$

For any $0 < \alpha < \alpha_4$, there exists $\lambda_3 > 0$ such that, for any $\lambda \in (0, \lambda_3)$ we have

$$|g(x)| \leq C(\Omega, \mathcal{H}^e, \kappa, \delta, \alpha, \beta) e^{-\alpha \,\mathrm{d}(x)/\lambda} \quad \text{for } x \in \Omega \backslash \Gamma_{\frac{\mu}{2}}, \tag{4.27}$$

where $\Gamma_{\frac{\mu}{2}}$ is defined in (4.23). Let

$$u(x) = g(x) e^{\alpha \,\mathrm{d}(x)/\lambda}.$$

Then $u(x)$ satisfies

$$\frac{\lambda^2}{\kappa^2} \Delta u - \frac{2\alpha\lambda}{\kappa^2} \nabla \mathrm{d} \cdot \nabla u = s(x) \quad \text{in } \Omega,$$

where

$$s(x)=\left(-\frac{\alpha^2}{\kappa^2}+\frac{\lambda\alpha}{\kappa^2}\Delta \mathrm{d}+2-3g+|g|^2+|\mathbf{Q}|^2\right)u-|\mathbf{Q}|^2e^{\alpha\,\mathrm{d}(x)/\lambda}.$$

Since $0<\alpha<\min\{\sqrt{2}\kappa, 2\}$, there exist $\varepsilon_1>0$ (depending on $\Omega, \mathcal{H}^e, \kappa, \alpha, \delta$), $\lambda_4>0$ (depending on $\varepsilon_1, \Omega, \mathcal{H}^e, \kappa, \alpha, \delta$) and N_1 (depending on $\Omega, \mathcal{H}^e, \kappa, \alpha, \delta$) such that, for any $\lambda\in(0,\lambda_4)$ and any $x\in\Gamma_{\frac{\mu}{2}}\backslash\Gamma_{\lambda N_1}$ we have

$$-\frac{\alpha^2}{\kappa^2}+\frac{\lambda\alpha}{\kappa^2}\Delta \mathrm{d}+2-3g+|g|^2+|\mathbf{Q}|^2>\varepsilon_1>0.$$

Using the maximum principle [14, Corollary 3.2], we then can deduce that

$$\|u\|_{C^0(\Gamma_{\frac{\mu}{2}}\backslash\Gamma_{\lambda N_1})}\le\|u\|_{C^0(\partial(\Gamma_{\frac{\mu}{2}}\backslash\Gamma_{\lambda N_1}))}+\varepsilon_1^{-1}\sup_{x\in\Gamma_{\frac{\mu}{2}}\backslash\Gamma_{\lambda N_1}}|\mathbf{Q}|^2e^{\alpha\,\mathrm{d}(x)/\lambda}\le C,$$

where $C=C(\Omega,\beta,\alpha,\kappa,\delta,\mathcal{H}^e)$. Combining this inequality with the estimate in (4.27), and then using the boundedness of $|g(x)|\le 1$ for $x\in\Gamma_{\lambda N_1}$, we obtain the exponential decay estimate for $1-f$. □

5 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We shall first establish two results for some ordinary differential systems. The first result is Theorem 5.1, which will be used to show how the minimum points of $|f|^2-|\mathbf{Q}|^2$ depend on the applied field $\mathcal{H}^e$, and the second is Theorem 5.2 which is needed to prove how the minimum points of $|f|^2-|\mathbf{Q}|^2$ depend on the curvature of $\partial\Omega$.

We first establish the mixed monotonicity on h_0, of the solution of (2.14) satisfying (2.12). Existence of such solution has been proved in Proposition 2.2. For simplicity of notation, we take positive constants $h_1>h_2$, and denote the solution (f_0, Q_0^1) of (2.14) with $h_0=h_i$ by (f_i, g_i), $i=1,2$. So (f_i, g_i) is the solution of the following problem:

$$\begin{cases}-\frac{1}{\kappa^2}(f_i)''=(1-|f_i|^2-|g_i|^2)f_i & \text{in } \mathbb{R}_+,\\ -(g_i)''+|f_i|^2g_i=0 & \text{in } \mathbb{R}_+,\\ (f_i)'(0)=0,\quad f_i(\infty)=1, & \\ (g_i)'(0)=-h_i,\quad g_i(\infty)=0. & \end{cases}\tag{5.1}$$

Theorem 5.1 *Let (f_i, g_i) be the solution of (5.1) satisfying*

$$\inf_{\mathbb{R}_+}(|f_i|^2-|g_i|^2)\ge\frac{1}{3}+\delta^2$$

for $i = 1, 2$. *If* $h_1 > h_2 > 0$, *then*

$$f_1(z_2) < f_2(z_2), \quad g_1(z_2) > g_2(z_2) \quad \textit{for all } z_2 \geq 0.$$

Proof We prove the monotonicity property by an iterative method. As the process is very technical, we describe the main idea of the proof first.

We first let $f_i^{(0)} = 1$, and solve the second equation of (5.1) with $f_i = f_i^{(0)}$, together with the boundary conditions given on the last line of (5.1), and get $g_i^{(0)}$.

Next we solve the first equation of (5.1) with $g_i = g_i^{(0)}$, together with the boundary conditions given on the third line, and obtain the solution $f_i^{(1)}$.

Then we solve the second equation of (5.1) again but with $f_i = f_i^{(1)}$, together with the boundary conditions on the last line, and obtain the solution $g_i^{(1)}$.

Then we solve the first equation of (5.1) again but with $g_i = g_i^{(1)}$, together with the boundary conditions on the third line, and obtain the solution $f_i^{(2)}$.

We iterate this process and obtain two sequences

$$\{(f_i^{(k)}, g_i^{(k)})\}_{k=1}^{\infty}, \quad i = 1, 2.$$

We claim that these two sequences have the following mixed monotonicity property:

(i) For each $i = 1, 2$,

$$1 \geq f_i^{(k)} > f_i^{(k+1)} > \frac{\sqrt{3}}{3}, \qquad 0 < g_i^{(k)} < g_i^{(k+1)} < \frac{\sqrt{6}}{3}, \quad \text{for all } k = 1, 2, \cdots.$$

(ii)

$$f_1^{(k)} < f_2^{(k)}, \qquad g_1^{(k)} > g_2^{(k)}, \quad \text{for all } k = 1, 2, \cdots.$$

The monotonicity properties (i) and (ii) will be proved later, see step 2 and step 3 in the detailed proof.

From the monotonicity property (i), and by the elliptic estimates, the sequence $(f_i^{(k)}, g_i^{(k)})$ converges in $C^{2,\alpha}_{loc}(\mathbb{R}_+)$ to a solution (f_i, g_i) of (5.1).

From the monotonicity property (ii), we can show that $f_1(z_2) \leq f_2(z_2)$ and $g_1(z_2) \geq g_2(z_2)$ for all $z_2 \geq 0$. Then by the maximum principle we can show that the strict inequalities hold for $z_2 > 0$.

Now we give the detailed proof of the theorem.

Step 1. Set $f^{(0)} = 1$. Let $g_1^{(0)}$ and $g_2^{(0)}$ be the solutions of the following equations

$$\begin{cases} -(g_1^{(0)})'' + |f^{(0)}|^2 g_1^{(0)} = 0 & \text{in } \mathbb{R}_+, \\ (g_1^{(0)})'(0) = -h_1, \quad (g_1^{(0)})(\infty) = 0, \end{cases}$$

and

$$\begin{cases} -(g_2^{(0)})'' + |f^{(0)}|^2 g_2^{(0)} = 0 & \text{in } \mathbb{R}_+, \\ (g_2^{(0)})'(0) = -h_2, \quad (g_2^{(0)})(\infty) = 0, \end{cases}$$

respectively. It is easy to see that

$$g_1^{(0)}(z_2) = h_1 e^{-z_2} > g_2^{(0)}(z_2) = h_2 e^{-z_2} \quad \text{for all } z_2 \geq 0.$$

Step 2. Let $g_{h_1} \in L^2(\mathbb{R}_+)$ and $g_{h_2} \in L^2(\mathbb{R}_+)$ be two given smooth functions and assume that $\frac{\sqrt{6}}{3} > g_{h_1}(z_2) > g_{h_2}(z_2) > 0$ for all $z_2 \geq 0$. Let $p_1(z_2)$ and $p_2(z_2)$ be the solutions of the following two problems

$$\begin{cases} -\frac{1}{\kappa^2} p_1'' = (1 - |p_1|^2 - |g_{h_1}|^2) p_1 & \text{in } \mathbb{R}_+, \\ p_1(\infty) = 1, \quad p_1'(0) = 0, \end{cases}$$

and

$$\begin{cases} -\frac{1}{\kappa^2} p_2'' = (1 - |p_2|^2 - |g_{h_2}|^2) p_2 & \text{in } \mathbb{R}_+, \\ p_2(\infty) = 1, \quad p_2'(0) = 0, \end{cases}$$

respectively. The existence and the uniqueness of the solution p_1 follows from the minimization problem of the functional

$$\min_{f \in \mathscr{W}} \int_0^\infty \Big\{ \frac{1}{\kappa^2} |f'(z_2)|^2 + |f(z_2)|^2 |g_{h_1}(z_2)|^2 + \frac{1}{2}(1 - |f(z_2)|^2)^2 \Big\} dz_2,$$

where

$$\mathscr{W} = \Big\{ u : u', \;\; 1 - u \in L^2(\mathbb{R}_+), \;\; u \in L^\infty(\mathbb{R}_+) \Big\}.$$

It is easy to see that $p_1 \geq 0$. By the standard elliptic estimates, the solution p_1 is smooth, and hence $p_1 > 0$. Similarly, we have $p_2 > 0$.

Claim 1 $p_1(z_2) \leq p_2(z_2)$ for all $z_2 \geq 0$.

Suppose otherwise Claim 1 were false. Then there exist $z_2^0 \in [0, \infty)$ and $z_2^1 \in (z_2^0, \infty]$ such that

$$p_1(z_2^0) \geq p_2(z_2^0), \quad p_1(z_2^1) = p_2(z_2^1), \quad p_1(z_2) > p_2(z_2) \quad \text{for all } z_2 \in (z_2^0, z_2^1). \tag{5.2}$$

This gives that

$$p_2'(z_2^0) \leq p_1'(z_2^0), \quad p_2'(z_2^1) \geq p_1'(z_2^1). \tag{5.3}$$

Indeed, if $z_2^0 = 0$, then $p_2'(z_2^0) = p_1'(z_2^0) = 0$; if $0 < z_2^0 < \infty$, then $p_2(z_2^0) = p_1(z_2^0)$, and $p_2'(z_2^0) \le p_1'(z_2^0)$ because of the last inequality in (5.2). So the first inequality in (5.3) is true. If $z_2^1 = \infty$, then $p_2'(z_2^1) = p_1'(z_2^1) = 0$; if $0 < z_2^1 < \infty$, then $p_2(z_2^1) = p_1(z_2^1)$, and $p_2'(z_2^1) \ge p_1'(z_2^1)$ because of the last inequality in (5.2). Therefore the second inequality in (5.3) is true.

From the equations for p_1 and p_2 we have

$$-\frac{1}{\kappa^2}(p_1 p_2' - p_2 p_1')' = p_1 p_2 \left(|p_1|^2 - |p_2|^2 - |g_{h_2}|^2 + |g_{h_1}|^2\right).$$

Integrating the above equality from z_2^0 to z_2^1, we then find that the left side of the resulted equality is

$$\frac{1}{\kappa^2}(p_1 p_2' - p_2 p_1')(z_2^0) - \frac{1}{\kappa^2}(p_1 p_2' - p_2 p_1')(z_2^1) \le 0.$$

However the right side of the resulted equality is

$$\int_{z_2^0}^{z_2^1} p_1 p_2 \left(|p_1|^2 - |p_2|^2 - |g_{h_2}|^2 + |g_{h_1}|^2\right) dz_2 > 0,$$

so we get a contradiction. Therefore Claim 1 is true.

Claim 2 $p_1(z_2) < p_2(z_2)$ for all $z_2 \ge 0$.

To prove this, let $w(z_2) = p_1(z_2) - p_2(z_2)$. From the equations of p_1 and p_2 we have

$$-\frac{1}{\kappa^2}w'' = w(1 - |p_1|^2 - |p_2|^2 - p_1 p_2 - |g_{h_2}|^2) + (|g_{h_2}|^2 - |g_{h_1}|^2)p_1. \quad (5.4)$$

Suppose there exists $z_2^3 \in [0, \infty)$ such that $w(z_2^3) = 0$. Then we have

$$w'(z_2^3) = 0; \qquad w''(z_2^3) \le 0 \ \text{ if } z_2^3 > 0, \qquad \lim_{z \to 0^+} w''(z) \le 0 \ \text{ if } z_2^3 = 0. \quad (5.5)$$

In fact, $w(z_2)$ is non-positive for any $z_2 \ge 0$, and $w(z_2^3) = 0$, so z_2^3 is a maximum point of w. If $z_2^3 > 0$, then we have obviously $w'(z_2^3) = 0$ and $w''(z_2^3) \le 0$. If $z_2^3 = 0$, then $w'(0) = p_1'(0) - p_2'(0) = 0$, which together with the fact that z_2^3 is a maximum point implies the last inequality in (5.5).

Note that

$$(|g_{h_2}|^2(z_2) - |g_{h_1}|^2(z_2))p_1(z_2) < 0 \quad \text{for all } z_2 > 0.$$

Then by (5.4) and by noting that $w(z_2^3) = 0$, we have that

$$w''(z_2^3) > 0 \ \text{ if } z_2^3 > 0, \qquad \lim_{z \to 0^+} w''(z) > 0 \ \text{ if } z_2^3 = 0.$$

This is a contradiction to (5.5). Therefore Claim 2 is true.

Step 3. Let $f_{h_1}(z_2)$ and $f_{h_2}(z_2)$ be given functions and assume that $0 < f_{h_1}(z_2) \le f_{h_2}(z_2) < 1$ for all $z_2 \ge 0$. Let q_1 and q_2 be the solutions of the following problems

$$\begin{cases} -q_1'' + f_{h_1}^2 q_1 = 0 & \text{in } \mathbb{R}_+, \\ q_1'(0) = -h_1, \quad q_1(\infty) = 0, \end{cases}$$

and

$$\begin{cases} -q_2'' + f_{h_2}^2 q_2 = 0 & \text{in } \mathbb{R}_+, \\ q_2'(0) = -h_2, \quad q_2(\infty) = 0, \end{cases}$$

respectively. We show that

$$q_1(z_2) > q_2(z_2) \quad \text{for all } z_2 \ge 0. \tag{5.6}$$

To prove (5.6), we introduce a function q_3 which is a solution of the following equation:

$$\begin{cases} -q_3'' + f_{h_2}^2 q_3 = 0 & \text{in } \mathbb{R}_+, \\ q_3'(0) = -h_1, \quad q_3(\infty) = 0. \end{cases}$$

By the maximum principle, it is easy to see that, for any $z_2 > 0$,

$$q_i'(z_2) < 0, \quad q_i(z_2) > 0 \quad \text{for } i = 1, 2, 3.$$

Claim 3 $q_1(z_2) \ge q_3(z_2)$ for all $z_2 \ge 0$, and $q_1(z_2) > q_3(z_2)$ for all $z_2 \ge 0$ if $0 < f_{h_1}(z_2) < f_{h_2}(z_2) < 1$.

Suppose Claim 3 were false. Then there exists $z_2^4 \in [0, \infty)$ and $z_2^5 \in (z_2^4, \infty]$ such that

$$q_1(z_2) < q_3(z_2) \quad \text{for } z_2 \in (z_2^4, z_2^5) \tag{5.7}$$

and

$$\begin{aligned} &q_1(z_2^4) \le q_3(z_2^4), \quad q_1'(z_2^4) \le q_3'(z_2^4) < 0, \\ &q_1(z_2^5) = q_3(z_2^5), \quad q_3'(z_2^5) \le q_1'(z_2^5) < 0. \end{aligned} \tag{5.8}$$

From the equations for q_1 and q_3 we find

$$(q_1 q_3' - q_3 q_1')' = q_1 q_3 \left(|f_{h_2}|^2 - |f_{h_1}|^2 \right). \tag{5.9}$$

Integrating (5.9) from z_2^4 to z_2^5 we get

$$0 \le \int_{z_2^4}^{z_2^5} q_1 q_3 \left(|f_{h_2}|^2 - |f_{h_1}|^2\right) dz_2$$
$$= (q_1 q_3' - q_3 q_1')(z_2^5) - (q_1 q_3' - q_3 q_1')(z_2^4) \le 0. \tag{5.10}$$

Then, since $f_{h_2}(z_2) \ge f_{h_1}(z_2) > 0$ for all $z_2 \ge 0$, $g_1(z_2)$ and $g_3(z_2)$ are positive functions, we must have

$$f_{h_2}(z_2) = f_{h_1}(z_2) \quad \text{for all } z_2 \in (z_2^4, z_2^5).$$

Moreover, we have $q_1(z_2^4) = q_3(z_2^4)$ and $q_1'(z_2^4) = q_3'(z_2^4)$ by (5.10), which gives that $q_1(z_2) = q_3(z_2)$ for $z_2 \in (z_2^4, z_2^5)$ by the uniqueness of the solutions for the initial value problem. This is a contradiction to (5.7). By a similar proof, using (5.10) again we have $q_1(z_2) > q_3(z_2)$ for all $z_2 \ge 0$ if $0 < f_{h_1}(z_2) < f_{h_2}(z_2) < 1$. Therefore Claim 3 is true.

Claim 4 $q_3(z_2) > q_2(z_2)$ for all $z_2 \ge 0$.

Suppose Claim 4 were false. Then there exists $z_2^6 \in [0, \infty)$ and $z_2^7 \in (z_2^6, \infty]$ such that

$$q_3(z_2^6) \le q_2(z_2^6), \quad q_3'(z_2^6) \le q_2'(z_2^6) < 0,$$
$$q_3(z_2^7) = q_2(z_2^7), \quad q_2'(z_2^7) \le q_3'(z_2^7) < 0.$$

From the equations for q_2 and q_3 we have

$$(q_2 q_3' - q_3 q_2')' = 0 \quad \text{for all } z_2 > 0.$$

Integrating this equality from z_2^6 to z_2^7 we get

$$(q_2 q_3' - q_3 q_2')(z_2^7) = (q_2 q_3' - q_3 q_2')(z_2^6). \tag{5.11}$$

For the left side term in (5.11) we have

$$q_2(z_2^7) q_3'(z_2^7) - q_3(z_2^7) q_2'(z_2^7) = q_2(z_2^7)(q_3'(z_2^7) - q_2'(z_2^7)) \ge 0.$$

If $z_2^6 = 0$, then the right side term in (5.11) is

$$q_2(0) q_3'(0) - q_3(0) q_2'(0) = -q_2(0) h_1 + q_3(0) h_2 \le q_2(0)(h_2 - h_1) < 0,$$

which is a contradiction.

If $z_2^6 > 0$, then from (5.11) we have

$$0 \le q_2(z_2^6) q_3'(z_2^6) - q_3(z_2^6) q_2'(z_2^6)$$
$$= [q_2(z_2^6) - q_3(z_2^6)] q_3'(z_2^6) + q_3(z_2^6)[q_3'(z_2^6) - q_3'(z_2^6)].$$

Each term in the right side of the above equality is non-positive. So we must have $q_2(z_2^6) = q_3(z_2^6)$ and $q_2'(z_2^6) = q_3'(z_2^6)$. Then we apply the existence and uniqueness theorem for the initial value problems of ordinary differential equations on the interval $[0, z_2^6]$, and find that $q_2(z_2) = q_3(z_2)$ for all $z_2 \in [0, z_2^6]$. In particular

$$-h_2 = q_2'(0) = q_3'(0) = -h_1,$$

which is a contradiction to the assumption that $h_1 > h_2$. Now Claim 4 is proved.

Combining Claims 3 and 4 we conclude that

$$q_1(z_2) \geq q_3(z_2) > q_2(z_2) \quad \text{for all } z_2 > 0.$$

Now (5.6) has been proved.

Step 4. Let $i = 1, 2$ and $k \geq 0$. Assume $f_i^{(k)}$ and $f_i^{(k+1)}$ are two given functions satisfying

$$0 < f_i^{(k+1)}(z_2) < f_i^{(k)}(z_2) < 1 \quad \text{for all } z_2 \geq 0,$$

and h_i is a given constant. Let $g_i^{(k)}$ and $g_i^{(k+1)}$ be the solutions of the following two problems

$$\begin{cases} -(g_i^{(k)})'' + |f_i^{(k)}|^2 g_i^{(k)} = 0 & \text{in } \mathbb{R}_+, \\ (g_i^{(k)})'(0) = -h_i, \quad g_i^{(k)}(\infty) = 0, \end{cases} \tag{5.12}$$

and

$$\begin{cases} -(g_i^{(k+1)})'' + |f_i^{(k+1)}|^2 g_i^{(k+1)} = 0 & \text{in } \mathbb{R}_+, \\ (g_i^{(k+1)})'(0) = -h_i, \quad g_i^{(k+1)}(\infty) = 0, \end{cases}$$

respectively. From Claim 3 in step 3 we see that

$$g_i^{(k)}(z_2) < g_i^{(k+1)}(z_2) \quad \text{for all } z_2 \geq 0.$$

Let $i = 1, 2$ and $k \geq 0$. Assume $g_i^{(k)}$ and $g_i^{(k+1)}$ are two given functions satisfying

$$0 < g_i^{(k)}(z_2) < g_i^{(k+1)}(z_2) \quad \text{for all } z_2 \geq 0.$$

Let $f_i^{(k+1)}$ and $f_i^{(k+2)}$ be the solution of the following problems

$$\begin{cases} -\frac{1}{\kappa^2}(f_i^{(k+1)})'' = (1 - |f_i^{(k+1)}|^2 - |g_i^{(k)}|^2) f_i^{(k+1)} & \text{in } \mathbb{R}_+, \\ (f_i^{(k+1)})'(0) = 0, \quad (f_i^{(k+1)})(\infty) = 1, \end{cases} \tag{5.13}$$

and

$$\begin{cases} -\frac{1}{\kappa^2}(f_i^{(k+2)})'' = (1 - |f_i^{(k+1)}|^2 - |g_i^{(k+1)}|^2) f_i^{(k+2)} & \text{in } \mathbb{R}_+, \\ (f_i^{(k+2)})'(0) = 0, \quad (f_i^{(k+2)})(\infty) = 1, \end{cases}$$

respectively. From step 2 we see that

$$0 < f_i^{(k+2)}(z_2) < f_i^{(k+1)}(z_2) \le 1 \quad \text{for all } z_2 \ge 0.$$

Step 5. Let $i = 1, 2$. Given constants h_1 and h_2, let (f_i, g_i) be the unique solution of (5.1) satisfying

$$0 < f_i(z_2) < 1, \quad g_i(z_2) > 0 \quad \text{for all } z_2 \ge 0.$$

We construct two sequences $\{f_i^{(k)}\}_{k=0}^{\infty}$ and $\{g_i^{(k)}\}_{k=0}^{\infty}$ as follows. First, we let $f_i^{(0)}(z_2) = 1$ for $i = 1, 2$. Then we obtain the sequences by induction as follows: if we know $f_i^{(k)}(z_2)$, then we solve the equation of (5.12) to obtain $g_i^{(k)}(z_2)$; if we know $g_i^{(k)}(z_2)$, then we solve the equation of (5.13) to obtain $f_i^{(k+1)}(z_2)$.

From Claim 3 in step 3, we have $g_i^{(0)}(z_2) < g_i(z_2)$ for all $z_2 \ge 0$, since $f_i(z_2) < f_i^{(0)}(z_2)$. Next using $g_i^{(0)}(z_2) < g_i(z_2)$, we have $f_i(z_2) < f_i^{(1)}(z_2)$ for all $z_2 \ge 0$ by Claim 2 in step 2. Then using $f_i(z_2) < f_i^{(1)}(z_2)$, we have $g_i^{(1)}(z_2) < g_i(z_2)$ for all $z_2 \ge 0$. By induction and from step 4, we finally obtain a sequence $(f_i^{(k)}, g_i^{(k)})$ satisfying, for all $z_2 \ge 0$,

$$1 = f_i^{(0)}(z_2) > f_i^{(1)}(z_2) > f_i^{(2)}(z_2) > \cdots > f_i^{(k)}(z_2) > f_i^{(k+1)}(z_2) > \cdots > f_i(z_2)$$

and

$$0 < g_i^{(0)}(z_2) < g_i^{(1)}(z_2) < g_i^{(2)}(z_2) < \cdots < g_i^{(k)}(z_2) < g_i^{(k+1)}(z_2) < \cdots < g_i(z_2).$$

Then, noting that $(f_i^{(k)}(z_2), g_i^{(k)}(z_2))$ satisfies the equations for f_i^k and g_i^k, using which we can derive the following estimate

$$\|f_i^{(k)}\|_{C^3(\mathbb{R}_+)} + \|g_i^{(k)}\|_{C^3(\mathbb{R}_+)} \le C_i,$$

where the norm $\|f\|_{C^3(\mathbb{R}_+)}$ is defined by

$$\|f\|_{C^3(\mathbb{R}_+)} = \sum_{j=0}^{3} \sup_{z_2 \in \mathbb{R}_+} |f^{(j)}(z_2)|,$$

and the constant C_i depends only on f_i and g_i, and hence depends only on h_i for each i. So we can apply the Arzela's theorem and using the uniqueness of the solution to

(5.1), to derive

$$\lim_{k\to\infty} f_i^{(k)}(z_2) = f_i(z_2), \quad \lim_{k\to\infty} g_i^{(k)}(z_2) = g_i(z_2) \quad \text{for each } z_2 \geq 0, i = 1, 2.$$

From step 2 and step 3, we have

$$f_1^{(k)}(z_2) < f_2^{(k)}(z_2), \quad g_1^{(k)}(z_2) > g_2^{(k)}(z_2), \qquad \text{for } z_2 \geq 0, \quad k = 1, 2, \cdots.$$

Therefore, $f_1(z_2) \leq f_2(z_2)$ and $g_1(z_2) \geq g_2(z_2)$ for all $z_2 \geq 0$. From step 3, we actually have $g_1(z_2) > g_2(z_2)$ for all $z_2 \geq 0$. Then applying the result of step 2, we finally obtain that $f_1(z_2) < f_2(z_2)$ and $g_1(z_2) > g_2(z_2)$ for all $z_2 \geq 0$. □

Proof of Proposition 2.4 *Step 1*. We prove that (2.14) has a solution (f_0, Q_0^1) satisfying (2.12) when $h_0 = \frac{\sqrt{2}}{3}$. We construct sequences $\{f_i\}$ and $\{g_i\}$ which solve the following problems

$$\begin{cases} -(g_i)'' + (f_i)^2 g_i = 0 & \text{in } \mathbb{R}_+, \\ (g_i)'(0) = -h_0, \quad g_i(\infty) = 0, \end{cases} \tag{5.14}$$

and

$$\begin{cases} -\frac{1}{\kappa^2}(f_{i+1})'' = (1 - |f_{i+1}|^2 - |g_i|^2) f_{i+1} & \text{in } \mathbb{R}_+, \\ (f_{i+1})'(0) = 0, \quad (f_{i+1})(\infty) = 1. \end{cases} \tag{5.15}$$

In this step we always let $h_0 = \frac{\sqrt{2}}{3}$.

Let $f_0 = \frac{\sqrt{6}}{3}$. Solving the equation (5.14) for $i = 0$, we get that $g_0 = \frac{\sqrt{3}}{3} e^{-\sqrt{6} z_2/3}$. Then we look for f_1 which solves (5.15) for $i = 0$ and $g_0 = \frac{\sqrt{3}}{3} e^{-\sqrt{6} z_2/3}$. In fact, f_1 can be obtained by minimization:

$$\min_{\substack{\nabla f \in L^2(\mathbb{R}_+), \\ 1-f \in L^2(\mathbb{R}_+), 0 \leq f \leq 1}} \int_0^\infty \left\{ \frac{1}{\kappa^2} |f'(z_2)|^2 + |f(z_2)|^2 |g_0(z_2)|^2 + \frac{1}{2}(1 - |f(z_2)|^2)^2 \right\} dz_2.$$

Using the fact that: $t^2|g_0|^2 + \frac{1}{2}(1-t^2)^2$ is monotonically decreasing with respect to t if $g_0^2 \leq \frac{1}{3}$ and $0 < t \leq \frac{\sqrt{6}}{3}$. Then we have

$$|f(z_2)|^2 |g_0(z_2)|^2 + \frac{1}{2}(1 - |f(z_2)|^2)^2 \geq \left(\frac{\sqrt{6}}{3}\right)^2 |g_0(z_2)|^2 + \frac{1}{2}\left(1 - \left(\frac{\sqrt{6}}{3}\right)^2\right)^2$$

if $f(z_2) \leq \frac{\sqrt{6}}{3}$. Therefore, the solution f_1 satisfies $f_1(z_2) \geq \frac{\sqrt{6}}{3}$ for all $z_2 \geq 0$. Actually, by the maximum principle, we have $f_1(z_2) > \frac{\sqrt{6}}{3}$ for all $z_2 \geq 0$.

Repeating this process we can solve equation (5.14) and equation (5.15) in turn to find the sequences $\{f_i\}$ and $\{g_i\}$. In particular, from step 3 in Theorem 5.1 (see Claim 3) we have $g_0(z_2) > g_1(z_2)$ for all $z_2 \geq 0$, since $f_0(z_2) < f_1(z_2)$; then from step 2 in Theorem 5.1 we have $f_1(z_2) < f_2(z_2)$ for all $z_2 \geq 0$. Repeating using step 2 and step 3 in Theorem 5.1, we obtain that, for all $z_2 \geq 0$,

$$\begin{aligned} &\frac{\sqrt{6}}{3} = f_0(z_2) < f_1(z_2) < f_2(z_2) < f_3(z_2) < \cdots < 1, \\ &\frac{\sqrt{3}}{3} \geq g_0(z_2) > g_1(z_2) > g_2(z_2) > g_3(z_2) > \cdots > 0. \end{aligned} \tag{5.16}$$

From the proof of step 5 in Theorem 5.1, the limit

$$(\lim_{i\to\infty} f_i(z_2), \lim_{i\to\infty} g_i(z_2)) := (f^{\sqrt{2}/3}(z_2), g^{\sqrt{2}/3}(z_2))$$

is the solution of (2.14) satisfying (2.12) when $h_0 = \frac{\sqrt{2}}{3}$.

Step 2. We prove that (2.14) has a solution (f_0, Q_0^1) satisfying (2.12) when $0 < h_0 < \frac{\sqrt{2}}{3}$. Let $(f^{\sqrt{2}/3}(z_2), g^{\sqrt{2}/3}(z_2))$ be the solution obtained in step 1 when $h_0 = \sqrt{2}/3$. For any given $0 < h_1 < \sqrt{2}/3$, similar to the construction of the sequences (f_i, g_i) in (5.16), we first let $f_0^{h_1}(z_2) = f^{\sqrt{2}/3}(z_2)$, and solve the equation (5.14) with $h_0 = h_1$ and $f_0(z_2) = f_0^{h_1}(z_2)$, then we can obtain the solution $g_0^{h_1}(z_2) < g^{\sqrt{2}/3}(z_2)$ for all $z_2 \geq 0$ by step 3 in Theorem 5.1; next we solve the equation (5.15) with $g_0(z_2) = g_0^h(z_2)$ to obtain the solution $f_1^{h_1}(z_2)$, and $f_1^{h_1}(z_2) > f_0^{h_1}(z_2)$ for all $z_2 \geq 0$ by step 2 in Theorem 5.1. Repeating solving (5.14) and (5.15), we obtain a sequence $(f_i^{h_1}, g_i^{h_1})$ satisfying

$$\begin{aligned} &f^{\sqrt{2}/3}(z_2) = f_0^{h_1}(z_2) < f_1^{h_1}(z_2) < f_2^{h_1}(z_2) < f_3^{h_1}(z_2) < \cdots < 1, \\ &g^{\sqrt{2}/3}(z_2) > g_0^{h_1}(z_2) > g_1(z_2)^{h_1}(z_2) > g_2^{h_1}(z_2) > g_3^{h_1}(z_2) > \cdots > 0 \end{aligned}$$

for all $z_2 \geq 0$. Using the proof of step 5 in Theorem 5.1 again, the limit

$$(\lim_{i\to\infty} f_i^{h_1}(z_2), \lim_{i\to\infty} g_i^{h_1}(z_2))$$

is the solution of (2.14) satisfying (2.12) when $h_0 = h_1 < \frac{\sqrt{2}}{3}$.

Step 3. We prove that (2.14) has no solutions satisfying (2.12) when $h_0 = \frac{\sqrt{6}}{3}$. Solving the equation (5.14) when $h_0 = \frac{\sqrt{6}}{3}$ and $f_0(z_2) = 1$, we obtain that $g_0(z_2) = \frac{\sqrt{6}}{3}e^{-z_2}$. If there exists a solution $(f^*(z_2), g^*(z_2))$ to equation (2.14) satisfying (2.12) when $h_0 = \frac{\sqrt{6}}{3}$, then from step 2 and step 3 in Theorem 5.1, it follows that $0 <$

$f^*(z_2) \le 1, \quad g^*(z_2) \ge g_0(z_2) > 0$. Therefore,

$$\min_{z_2\in\mathbb{R}_+}\left((f^*(z_2))^2-(g^*(z_2))^2\right) \le \min_{z_2\in\mathbb{R}_+}\left(1-g_0^2(z_2)\right)=\frac{1}{3}.$$

This is a contradiction with (2.12). This shows that there does not exist solutions of (2.14) satisfying (2.12) when $h_0=\frac{\sqrt{6}}{3}$. We now have the bound of h^*. □

Next we establish the mixed monotonicity of the solution of (2.20) on the parameter k_0 in the equations. For this purpose, we take two real constants $k_1 < k_2$, and compare the solutions $(f_{1,i}, Q^1_{1,i})$, $i = 1, 2$, of (2.20) with k_0 equal to k_i. For the convenience of our discussion we write the equations for $(f_{1,i}, Q^1_{1,i})$ as follows:

$$\begin{cases} \frac{1}{\kappa^2}(-f''_{1,i}+k_i f'_0)=(1-3|f_0|^2-|Q^1_0|^2)f_{1,i}-2f_0Q^1_0Q^1_{1,i} & \text{in } \mathbb{R}_+,\\ (-(Q^1_{1,i})''+k_i(Q^1_0)')+|f_0|^2Q^1_{1,i}+2f_0Q^1_0f_{1,i}=0 & \text{in } \mathbb{R}_+,\\ f'_{1,i}(0)=0, \quad (Q^1_{1,i})'(0)=k_iQ^1_0(0) & \text{on } z_2=0. \end{cases} \tag{5.17}$$

In (5.17) the functions f_0 and Q^1_0 are the solutions to equations (2.14).

Theorem 5.2 *Let $i = 1, 2$, and let $(f_{1,i}, Q^1_{1,i}) \in H^1(\mathbb{R}_+) \times H^1(\mathbb{R}_+)$ be the solution of* (5.17). *If $k_1 < k_2$, then we have*

$$f_{1,1}(z_2) > f_{1,2}(z_2), \quad Q^1_{1,1}(z_2) \le Q^1_{1,2}(z_2) \quad \textit{for all } z_2 \ge 0.$$

Proof Note that (5.17) is a linear equation of $(f_{1,i}, Q^1_{1,i})$, and when f_0 and Q^1_0 are fixed, the equation is linear in k_i. Hence in order to prove the conclusion, it suffices to prove that if $k_1 > 0$, then

$$f_{1,1}(z_2)<0, \quad Q^1_{1,1}(z_2)>0 \quad \text{for all } z_2 \ge 0.$$

Note that the solution $(f_{1,1}, Q^1_{1,1})$ is the unique minimizer of the following minimization problem

$$\min_{(f_{1,1},Q^1_{1,1})\in H^1(\mathbb{R}_+)\times H^1(\mathbb{R}_+)} \int_{\mathbb{R}_+} J[f_{1,1}, Q^1_{1,1}]dz_2,$$

where

$$\begin{aligned} J[f_{1,1}, Q^1_{1,1}] &= \frac{1}{\kappa^2}|f'_{1,1}|^2+|(Q^1_{1,1})'|^2+(3|f_0|^2+|Q^1_0|^2-1)|f_{1,1}|^2+4f_0Q^1_0f_{1,1}Q^1_{1,1}\\ &\quad +|f_0|^2|Q^1_{1,1}|^2+2k_1(Q^1_0)'Q^1_{1,1}+\frac{2}{\kappa^2}k_1f'_0f_{1,1}. \end{aligned}$$

From Proposition 2.2, we have

$$0<f_0(z_2)<1, \quad f'_0(z_2)>0, \quad Q^1_0(z_2)>0, \quad (Q^1_0)'(z_2)<0 \quad \text{for all } z_2>0.$$

It follows that

$$J[-|f_{1,1}|, |Q_{1,1}^1|] \leq J[f_{1,1}, Q_{1,1}^1].$$

This shows that the unique solution $(f_{1,1}, Q_{1,1}^1)$ of (5.17) satisfies

$$f_{1,1}(z_2) \leq 0, \quad Q_{1,1}^1(z_2) \geq 0 \quad \text{for all } z_2 \geq 0.$$

Suppose there exists a point $z_2^0 \geq 0$ such that $f_{1,1}(z_2^0) = 0$. Then z_2^0 is a maximum point of $f_{1,1}$. If $z_2^0 > 0$, then we obviously have $f_{1,1}''(z_2^0) \leq 0$. However, this is a contradiction, because from this and by the first equation of (5.17) we have

$$0 < -\frac{1}{\kappa^2}(f_{1,1}'' - k_1 f_0')(z_2^0) = \left[(1 - 3f_0^2 - (Q_0^1)^2)f_{1,1} - 2f_0 Q_0^1 Q_{1,1}^1\right](z_2^0) \leq 0.$$

If $z_2^0 = 0$, since $f_{1,1}'(0) = 0$ and

$$2f_0 Q_0^1 Q_{1,1}^1 + \frac{1}{\kappa^2}(k_1 f_0') > 0 \quad \text{if } z_2 > 0,$$

then there exists $\sigma > 0$ such that for $z_2 \in (0, \sigma)$ we have

$$\frac{1}{\kappa^2}(f_{1,1}'') = (3|f_0|^2 + |Q_0^1|^2 - 1)f_{1,1} + 2f_0 Q_0^1 Q_{1,1}^1 + \frac{1}{\kappa^2}(k_1 f_0') \geq 0 \ (\not\equiv 0).$$

Therefore, $f_{1,1}(z_2) \geq 0 \ (\not\equiv 0)$ for $z_2 \in (0, \sigma)$. This is a contradiction with $f_{1,1}(z_2) \leq 0$. Thus we have $f_{1,1}(z_2) < 0$ for any $z_2 \geq 0$ We finish the proof of this theorem. □

Proof of Theorem 1.2 From Theorem 1.1 we know that $1 - f_\lambda$ and $\mathbf{Q}_\lambda$ decay exponentially in the normal direction away from the boundary $\partial\Omega$. Therefore, in order to prove Theorem 1.2 we only need to analyze the asymptotic expansion of $(f_\lambda, \mathbf{Q}_\lambda)$ near the boundary $\partial\Omega$.

Let $\mathcal{N}_0$ be a neighbourhood of a point on $\partial\Omega$ in the x-coordinates, and let $(\hat{f}, \hat{\mathbf{Q}})$ be the representations of $(f_\lambda, \mathbf{Q}_\lambda)$ under the y-coordinates (see section 2). Then, in the coordinates (y_1, z_2) with $z_2 = y_2/\lambda$, $\hat{f}_\lambda$ and $\hat{\mathbf{Q}}_\lambda$ have the following representations:

$$\hat{f}_\lambda(y) = \hat{f}_0(y_1, z_2) + \lambda \hat{f}_1(y_1, z_2) + R_f(y_1, z_2, \lambda),$$
$$\hat{\mathbf{Q}}_\lambda(y) = \hat{\mathbf{Q}}_0(y_1, z_2) + \lambda \hat{\mathbf{Q}}_1(y_1, z_2) + \mathbf{R}_{\mathbf{Q}}(y_1, z_2, \lambda).$$

The leading order terms $\hat{f}_0(y_1, z_2)$ and $\hat{\mathbf{Q}}_0(y_1, z_2) = (\hat{Q}_0^1(y_1, z_2), 0)$ satisfy, for each fixed y_1, the following problem in the variable z_2:

$$\begin{cases} -\frac{1}{\kappa^2}\frac{\partial^2}{\partial z_2^2}\hat{f}_0 = (1 - |\hat{f}_0|^2 - |\hat{Q}_0^1|^2)\hat{f}_0 & \text{in } \mathbb{R}_+, \\ -\frac{\partial^2}{\partial z_2^2}\hat{Q}_0^1 + |\hat{f}_0|^2\hat{Q}_0^1 = 0 & \text{in } \mathbb{R}_+, \\ \partial_{z_2}\hat{f}_0(y_1, 0) = 0, \quad \partial_{z_2}\hat{Q}_0^1(y_1, 0) = -\hat{\mathcal{H}}^e(y_1), \\ \hat{f}_0(y_1, \infty) = 1, \quad \hat{Q}_0^1(y_1, \infty) = 0, \end{cases} \tag{5.18}$$

where $\hat{\mathcal{H}}^e(y_1)$ is the value of $\mathcal{H}^e$ at the point $x = \psi(y_1, 0) \in \partial\Omega$ and $\psi(\cdot, \cdot)$ is defined by (2.1).

The first order terms $\hat{f}_1(y_1, z_2)$ and $\hat{\mathbf{Q}}_1(y_1, z_2) = (\hat{Q}_1^1(y_1, z_2), \hat{Q}_1^2(y_1, z_2))$ satisfy, for any fixed y_1, the following problem in the variable z_2:

$$\begin{cases} \frac{1}{\kappa^2}(-\frac{\partial^2}{\partial z_2^2}\hat{f}_1 + k(y_1)\partial_{z_2}\hat{f}_0) = (1 - 3|\hat{f}_0|^2 - |\hat{Q}_0^1|^2)\hat{f}_1 - 2\hat{f}_0\hat{Q}_0^1\hat{Q}_1^1 & \text{in } \mathbb{R}_+, \\ (-\frac{\partial^2}{\partial z_2^2}\hat{Q}_1^1 + k(y_1)\partial_{z_2}\hat{Q}_0^1) + |\hat{f}_0|^2\hat{Q}_1^1 + 2\hat{f}_0\hat{Q}_0^1\hat{f}_1 = 0 & \text{in } \mathbb{R}_+, \\ \partial_{z_2}\hat{f}_1(y_1, 0) = 0, \quad \partial_{z_2}\hat{Q}_1^1(y_1, 0) = 0, \\ \hat{f}_1(y_1, \infty) = 1, \quad \hat{Q}_1^1(y_1, \infty) = 0, \end{cases} \tag{5.19}$$

where $k(y_1)$ is the curvature of $\partial\Omega$ at the point $x = \psi(y_1, 0)$.

The error terms R_f and $\mathbf{R_Q}$ defined in (3.6) satisfy the following inequality

$$|R_f(y_1, z_2, \lambda)| + |\mathbf{R_Q}(y_1, z_2, \lambda)| \le C\lambda^2, \quad \text{for any } x = \psi(y_1, \lambda z_2) \in \mathcal{N}_0,$$

where the constant C depends only on $\Omega, \mathcal{H}^e$, κ and δ, but not on λ and x, see Theorem 3.4.

Using the fact that $\hat{Q}_0^2(y_1, z_2) \equiv 0$, we have

$$\begin{aligned} &|\hat{f}_\lambda(y_1, z_2)|^2 - |\hat{\mathbf{Q}}_\lambda(y_1, z_2)|^2 \\ &\quad = (|\hat{f}_0(y_1, z_2)|^2 - |\hat{Q}_0^1(y_1, z_2)|^2) \\ &\qquad + \lambda(2\hat{f}_0(y_1, z_2)\hat{f}_1(y_1, z_2) - 2\hat{Q}_0^1(y_1, z_2)\hat{Q}_1^1(y_1, z_2)) + O(\lambda^2). \end{aligned} \tag{5.20}$$

We first check the leading order term $|\hat{f}_0(y_1, z_2)|^2 - |\hat{Q}_0^1(y_1, z_2)|^2$ in the right side of (5.20). From Proposition 2.2 we know that, for any fixed y_1 we have

$$\begin{aligned} &\hat{f}_0(y_1, z_2) > 0, \quad \hat{Q}_0^1(y_1, z_2) > 0, \\ &\partial_{z_2}\hat{f}_0(y_1, z_2) > 0, \quad (\partial_{z_2}\hat{Q}_0^1)(y_1, z_2) < 0 \quad \text{for all } z_2 > 0. \end{aligned}$$

Hence $|\hat{f}_0(y_1, z_0)|^2 - |\hat{Q}_0^1(y_1, z_2)|^2$ has a strict minimum at some point $(y_1, 0)$, which implies that $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ has a strict minimum on the domain boundary $\partial\Omega$.

Next we examine the location of the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ on boundary $\partial\Omega$. It follows from Theorem 5.1 that the function $\hat{f}_0(y_1, 0)$ is strictly decreasing with respect to the value of $\hat{\mathcal{H}}^e(y_1)$, and $\hat{Q}_0^1(y_1, 0)$ is strictly increasing with respect to the value of $\hat{\mathcal{H}}^e(y_1)$. Therefore, the minimum points of $|\hat{f}_0(y_1, z_2)|^2 - |\hat{Q}_0^1(y_1, z_2)|^2$ are located at the maximum points of $\hat{\mathcal{H}}^e(y_1)$. Since the function $2\hat{f}_0(y_1, z_2)\hat{f}_1(y_1, z_2) - 2\hat{Q}_0^1(y_1, z_2)\hat{Q}_1^1(y_1, z_2)$ is uniformly bounded, from (5.20) we see that the minimum points of $|\hat{f}_\lambda(y)|^2 - |\hat{\mathbf{Q}}_\lambda(y)|^2$ approach the set $\partial\Omega(\mathcal{H}^e)$ defined by (1.11) for small λ.

Note that the set $\partial\Omega(\mathcal{H}^e)$ may be large. To get more precise information about the location in $\partial\Omega(\mathcal{H}^e)$ of the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$, we need to check the values of the first order term

$$W(y_1) \equiv 2\hat{f}_0(y_1, 0)\hat{f}_1(y_1, 0) - 2\hat{Q}_0^1(y_1, 0)\hat{Q}_1^1(y_1, 0)$$

among all $y_1 \in \mathcal{C}$, where

$$\mathcal{C} = \{y_1 : \ x = \psi(y_1, 0) \in \partial\Omega(\mathcal{H}^e)\}.$$

Note that both functions $\hat{f}_1(y_1, 0)$ and $\hat{Q}_1^1(y_1, 0)$ depend on the curvature $k(y_1)$ of $\partial\Omega$, see (5.19). From Theorem 5.2, we see that $\hat{f}_1(y_1, 0)$ is strictly decreasing with respect to $k(y_1)$, and $\hat{Q}_1^1(y_1, 0)$ is increasing with respect to $k(y_1)$. Therefore, $W(y_1)$ is strictly decreasing with respect to $k(y_1)$, and hence the minimum points of $W(y_1)$ are located at the maximum points of $k(y_1)$ on $\mathcal{C}$.

Note that

$$\psi(\mathcal{C}) = S(\mathcal{H}^e),$$

where $S(\mathcal{H}^e)$ is the set defined in (1.12). Then the minimum points of $|f_\lambda(x)|^2 - |\mathbf{Q}_\lambda(x)|^2$ must sub-converge to the set $S(\mathcal{H}^e)$ as $\lambda \to 0$ (see Definition 3 in section 1). Now Theorem 1.2 is proved. □

6 Further remarks

6.1 Chapman's conjecture on vortex nucleation

Consider an applied magnetic field $\mathcal{H}^e = \sigma\mathcal{H}$, where $\mathcal{H}$ is a continuous and positive-valued function on $\bar{\Omega}$ and $\sigma > 0$. Let $(f^\sigma, \mathbf{Q}^\sigma)$ be a Meissner solution of (1.1). Theorem 1.2 suggests that, if the penetration depth λ is sufficiently small, then, as the applied magnetic field increases to a critical value $\sigma_0 = H_S(\mathcal{H})$, the minimum value $d_{f^\sigma, \mathbf{Q}^\sigma}$ will approach the value $1/3$ from above, and the minimum points of $|f^\sigma|^2 - |\mathbf{Q}^\sigma|^2$ will sub-converge to the maximum points of $\mathcal{H}$ over the set $\partial\Omega(\mathcal{H})$. In particular, if $\mathcal{H}^e = h$ is a positive constant and increases to H_S, the minimum points of $|f^\sigma|^2 - |\mathbf{Q}^\sigma|^2$ will sub-converge to the maximum points of the curvature of domain boundary. Therefore it is natural to generalize Chapman's conjecture in [7] to the case where the Ginzburg-Landau parameter is finite and the applied field is non-constant.

We first note that, under the assumption that the minimum of $|f^\sigma|$ is continuous with respect to the parameter σ, and by the definition of $\sigma_0 = H_{sh}(\mathcal{H})$, f^{σ_0} has zero points which are called the *vortices*, while for all $0 < \sigma < \sigma_0$, f^σ has no zero points, Then we say that the first vortices nucleate when $\sigma = \sigma_0$, and we look for the location of these vortices.

Conjecture 6.1 *As σ increases to $H_{\mathrm{sh}}(\mathcal{H})$, the first vortices will nucleate at points in the set $S(\mathcal{H})$ which is defined in (1.12) with $\mathcal{H}^e$ replaced by $\mathcal{H}$.*

6.2 Meissner states of three dimensional superconductors

The Meissner states of a three dimensional superconductor can be described by the three-dimensional version of equation (1.4) and approximately by the three-dimensional version of (1.1), and the limiting system obtained by letting κ tend to infinity is the three-dimensional version of (1.5).

The stable solutions $\mathbf{Q}$ of (1.5) in three dimensions have been studied by several authors, see [3, 22, 33] and the references therein. Monneau [22] proved that the maximum points of $|\mathbf{Q}(x)|$ occur on the boundary. Bates and Pan [3] proved that, as λ tends zero, the maximum points of $|\mathbf{Q}(x)|$ sub-converge to the maximum points of the module of the tangential component of the applied magnetic field. In the special case when the applied magnetic field is given by $\mathcal{H}^e = \sigma\mathbf{h}$ where $\mathbf{h}$ is a constant unit vector, the maximum points of $|\mathbf{Q}(x)|$ sub-converge to the subset of the boundary $\partial\Omega$ where $\mathbf{h}$ is tangential to $\partial\Omega$. Xiang [33] further obtained the geometric characterization of the limiting position of the maximum points of $|Q(x)|$. The Meissner states of anisotropic superconductors have been studied by Pan.

For the three-dimensional version of the system (1.4), existence, regularity and uniqueness of the stable solutions and the asymptotic behavior as κ tends to infinity have been studied in [26].

6.3 Comparison of Meissner effects and surface superconductivity

It would be interesting to compare the boundary layer behaviors of the solutions $(f, \mathbf{Q})$ of (1.1) which describe the Meissner effect of a superconductor in a weak magnetic field, with the boundary layer behaviors of the solutions $(\Psi, \mathbf{A})$ of the Ginzburg-Landau system (1.3) which describe the surface superconductivity of a type II superconductor subjected to an applied magnetic field lying in between the second critical field H_{C_2} and the third critical field H_{C_3}. In particular, for the cylindrical superconductors in an applied magnetic field $\mathcal{H}^e = \sigma$ we have the following conclusions:

— For the solutions $(\Psi, \mathbf{A})$ of (1.3), as κ tends to infinity while λ is fixed, if $\mathcal{H}^e$ is strong and lies below but very close to the critical field H_{C_3}, the maximum points of $|\Psi(x)|$ sub-converge to the maximum points of the curvature of the domain boundary. $|\Psi|$ exponentially decays in the normal direction away from the boundary, and it also exponentially decays on $\partial\Omega$ along the tangential direction away from the maximum points of the curvature of $\partial\Omega$. See for instance [10, 12, 13, 16, 23] and the references therein, from which we will see that the analysis of the concentration behavior of Ψ is more challenging due to the non-uniqueness of the solutions of (1.3).

— For the solutions $(f_\lambda, \mathbf{Q}_\lambda)$ of (1.1), as λ tends to zero while κ is fixed, if $\mathcal{H}^e$ is weak and below the critical field H_S, the minimum points of $f_\lambda^2(x) - |\mathbf{Q}_\lambda(x)|^2$ sub-converge to the maximum points of the curvature of the domain boundary. Moreover, $(1 - f_\lambda(x), \mathbf{Q}_\lambda(x))$ exponentially decays in the normal direction away from the boundary, see Theorems 1.1 and 1.2 in this paper. However, for any applied magnetic field $\mathcal{H}^e(x)$, $(1 - f_\lambda(x), \mathbf{Q}_\lambda(x))$ does not decay on $\partial\Omega$ along the tangential direction away from the set $S(\mathcal{H}^e)$. In fact, in the coordinates (y_1, z_2) with $z_2 = y_2/\lambda$, the Meissner solution $(\hat{f}_\lambda, \hat{\mathbf{Q}}_\lambda)$ has the following expansions:

$$\hat{f}_\lambda(y) = \hat{f}_0(y_1, z_2) + \lambda \hat{f}_1(y_1, z_2) + O(\lambda^2),$$
$$\hat{\mathbf{Q}}_\lambda(y) = \hat{\mathbf{Q}}_0(y_1, z_2) + \lambda \hat{\mathbf{Q}}_1(y_1, z_2) + O(\lambda^2).$$

For each fixed $y_1 \neq 0$, the leading order terms $\hat{f}_0(y_1, \cdot)$ and $\hat{\mathbf{Q}}_0(y_1, \cdot) = (\hat{Q}_0^1(y_1, \cdot), 0)$ is a solution of equation (5.18) and satisfies the condition (2.12), hence is uniquely determined by $\hat{\mathcal{H}}^e(y_1)$, and the profile of the solution $(\hat{f}_0(y_1, \cdot), \hat{\mathbf{Q}}_0(y_1, \cdot))$ is similar to that of $(\hat{f}_0(0, \cdot), \hat{\mathbf{Q}}_0(0, \cdot))$. Thus the solution $(\hat{f}_\lambda, \hat{\mathbf{Q}}_\lambda)$ does not decay along the tangential direction.

— For the solutions $\mathbf{Q}$ of (1.5), as λ tends to zero (while $\kappa = \infty$), if $\mathcal{H}^e = \sigma$ is weak and below the critical field H_S, the maximum points of $|\mathbf{Q}(x)|$ sub-converge to the minimum points of the curvature of the domain boundary, see [3, 27, 33]. Moreover, $\mathbf{Q}_\lambda(x)$ exponentially decays in the normal direction away from the boundary, but does not decay on $\partial\Omega$ along the tangential direction away from the minimum points of the curvature.

6.4 Meissner states in various setting

Remark 6.2 Let us emphasize that the stability of a Meissner solution stated in Definition 1 is with respect to the Meissner equation (1.1). $(f, \mathbf{Q})$ is a stable Meissner solution of (1.1) does not mean that it is also stable with respect to the full Ginzburg-Landau system on Ω.

Proof Recall that, if we restrict ourself in Ω, the Ginzburg-Landau functional on Ω has the following form

$$\mathcal{E}[\Psi, \mathbf{A}] = \int_\Omega \left\{ \left| \left(\frac{\lambda}{\kappa}\nabla - i\mathbf{A}\right)\Psi \right|^2 + \frac{1}{2}(1 - |\Psi|^2)^2 + |\lambda \mathrm{curl} \mathbf{A} - \mathcal{H}^e|^2 \right\} dx.$$

The Euler-Lagrange equation of this functional is the Ginzburg-Landau system on Ω.

Let $(f, \mathbf{Q})$ be a solution of (1.1). For any smooth pair $(g, \mathbf{B})$ we have

$$\mathcal{E}''\langle[f,\mathbf{Q}],[g,\mathbf{B}]\rangle = \int_\Omega \Big(\mathrm{Re}\Big[(\frac{\lambda}{\kappa}\nabla f - i\mathbf{Q}f)(i\mathbf{B}\bar{g})\Big] + \Big|\frac{\lambda}{\kappa}\nabla g - i\mathbf{B}f - i\mathbf{Q}g\Big|^2 + \frac{1}{2}(g\bar{f}+f\bar{g})^2 - (1-|f|^2)|g|^2 + \lambda^2|\mathrm{curl}\mathbf{B}|^2\Big)dx.$$

Take $g = 2if$ and $\mathbf{B} = \lambda\kappa^{-1}f^{-1}\nabla f$. Then we have

$$\begin{aligned}
&\mathrm{Re}\left[\left(\frac{\lambda}{\kappa}\nabla f - i\mathbf{Q}f\right)(i\mathbf{B}\bar{g})\right] = 2\frac{\lambda^2}{\kappa^2}|\nabla f|^2,\\
&\mathrm{curl}\,\mathbf{B} = \frac{\lambda}{\kappa}\mathrm{curl}(f^{-1}\nabla f) = \frac{\lambda}{\kappa}[\partial_1(f^{-1}\partial_2 f) - \partial_2(f^{-1}\partial_1 f)] = 0,\\
&\left|\frac{\lambda}{\kappa}\nabla g - i\mathbf{B}f - i\mathbf{Q}g\right|^2 = \frac{\lambda^2}{\kappa^2}|\nabla f|^2 + 4|\mathbf{Q}|^2|f|^2,\\
&\left(g\bar{f}+f\bar{g}\right)^2 = 0.
\end{aligned}$$

Therefore, we have

$$\mathcal{E}''\langle[f,\mathbf{Q}],[2if,\frac{\lambda}{\kappa}f^{-1}\nabla f]\rangle = \int_\Omega\left(3\frac{\lambda^2}{\kappa^2}|\nabla f|^2 + 4(|\mathbf{Q}|^2+|f|^2-1)|f|^2\right)dx.$$

Using system (1.1), we obtain that

$$\int_\Omega(|\mathbf{Q}|^2+f^2-1)f^2dx = -\int_\Omega\frac{\lambda^2}{\kappa^2}|\nabla f|^2dx,$$

which implies that

$$\mathcal{E}''\langle[f,\mathbf{Q}],[2if,\frac{\lambda}{\kappa}f^{-1}\nabla f]\rangle = -\frac{\lambda^2}{\kappa^2}\int_\Omega|\nabla f|^2dx < 0.$$

This shows that $(f, \mathbf{Q})$ is an unstable solution with respect to the full Ginzburg-Landau system in Ω. □

It has been proved in [28–30] that $H_{C_1} \sim C\frac{\log\kappa}{\kappa}$, and if the applied magnetic field is below H_{C_1}, then the global minimizers of the Ginzburg-Landau functional on Ω have no vortices hence they are Meissner solutions, and they are stable with respect to the full Ginzburg-Landau system in Ω. On the other hand, the study in [5–7] imply that $H_S \sim C$ for large κ. Proposition 2.4 and Remark 6.2 above show that, if the applied magnetic field $\mathcal{H}^e$ is such that system (1.1) has a solution satisfying (1.8) for any small λ and

$$H_0 \le \mathcal{H}^e < h^*,$$

where H_0 is any positive number, hence $H_{C_1} \ll \mathcal{H}^e < H_S$, the Meissner solutions $(f, \mathbf{Q})$ are stable with respect to the equation (1.1), but not with respect to the Ginzburg-Landau system in Ω.

It is interesting that for the applied magnetic field $\mathcal{H}^e$ much larger than H_{C_1}, more precisely $H_{C_1} < \mathcal{H}^e < C\kappa^{\alpha-1}$ with $0 < \alpha < \frac{1}{4}$, stable Meissner solutions of (1.3) can still be obtained for large Ginzburg-Landau parameter $\kappa \gg 1$ with λ fixed, see [29, Theorem 1] and [31, Theorem 11.1]. In this paper we consider the situation with fixed κ and with small λ, and the solutions we found exhibit boundary layer. It will be interesting to know if the Meissner solutions obtained in [29, 31] have boundary layer behavior when the applied magnetic field $\mathcal{H}^e \gg H_{C_1}$, $\kappa \gg 1$ and λ is small.

An interesting problem related to the critical fields H_S and H_{sh} for Meissner states is the supercooling field H_{sc} for vortex solutions, and the hysteretic behavior of the superconductors, which have been investigated by F.H. Lin and Q. Du in [19].

We would like to mention that the Meissner states of type I superconductors have also been investigated, and surprising phenomena have been explored in [9].

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Data Availability All data included in this study are available upon request by contact with the corresponding author.

Declarations

Conflict of interest The authors declared that they have no conflicts of interest to this work.

Appendix A: Uniqueness of the solution to system (2.11)

Lemma A.1 *If* (2.11) *has a solution* $(f_0, \mathbf{Q}_0) \in C^2(\mathbb{R}^2_+, \mathbb{R}^3)$ *satisfying* (2.12), *then it is unique.*

Proof The uniqueness has been proved in Proposition 2.2, where we used the fact that the functional $\mathcal{E}$ is strictly convex. Here we give a direct proof. The idea of the proof goes back to Lemma 4.2 in [26] where the case of the bounded domains was treated.

Let $(f_1, \mathbf{Q}_1)$ and $(f_2, \mathbf{Q}_2)$ be two solutions, both satisfying

$$|f_0|^2 - |\mathbf{Q}_0|^2 > \frac{1}{3} + \delta^2.$$

Let $h \in H^1(\mathbb{R}^2)$ and $\mathbf{B} \in H^1(\mathrm{curl}, \mathbb{R}^2)$, both with compact support. We have

$$\int_{\mathbb{R}^2_+} \Big\{ \frac{1}{\kappa^2} \nabla(f_1 - f_2) \cdot \nabla h - \Big[(1 - |f_1|^2 - |\mathbf{Q}_1|^2) f_1 - (1 - |f_2|^2 - |\mathbf{Q}_2|^2) f_2 \Big] h$$
$$+ (|f_1|^2 \mathbf{Q}_1 - |f_2|^2 \mathbf{Q}_2) \cdot \mathbf{B} + \mathrm{curl}(\mathbf{Q}_1 - \mathbf{Q}_2) \cdot \mathrm{curl}\mathbf{B} \Big\} dz = 0.$$

Take $h = \eta^2(f_1 - f_2)$ and $\mathbf{B} = \eta^2(\mathbf{Q}_1 - \mathbf{Q}_2)$, where η is a smooth function with compact support in $\mathbb{R}^2$. Then we have

$$\begin{aligned}&\int_{\mathbb{R}^2_+} \{\frac{1}{\kappa^2}|\nabla(\eta(f_1 - f_2))|^2 + |\mathrm{curl}(\eta(\mathbf{Q}_1 - \mathbf{Q}_2))|^2 dx \\ &+ \int_{\mathbb{R}^2_+}\int_0^1 \{|f_t(\mathbf{Q}_1 - \mathbf{Q}_2) + 2(f_1 - f_2)\mathbf{Q}_t|^2 + (3f_t^2 - 3|\mathbf{Q}_t|^2 - 1)|f_1 - f_2|^2\}\eta^2 dtdz \\ &= \int_{\mathbb{R}^2_+} \{\frac{1}{\kappa^2}|(f_1 - f_2)\nabla\eta|^2 + |(\mathbf{Q}_1 - \mathbf{Q}_2)\times\nabla\eta|^2 dx,\end{aligned} \tag{A.1}$$

where $f_t = f_1 + t(f_1 - f_2)$ and $\mathbf{Q}_t = \mathbf{Q}_1 + t(\mathbf{Q}_1 - \mathbf{Q}_2)$. Note that

$$\begin{aligned}&|f_t(\mathbf{Q}_1 - \mathbf{Q}_2) + 2(f_1 - f_2)\mathbf{Q}_t|^2 + (3f_t^2 - 3|\mathbf{Q}_t|^2 - 1)|f_1 - f_2|^2 \geq \frac{\delta^2}{9}|\mathbf{Q}_1 - \mathbf{Q}_2|^2, \\ &3f_t^2 - 3|\mathbf{Q}_t|^2 - 1 \geq \delta^2.\end{aligned}$$

Then

$$\begin{aligned}&|f_t(\mathbf{Q}_1 - \mathbf{Q}_2) + 2(f_1 - f_2)\mathbf{Q}_t|^2 + (3f_t^2 - 3|\mathbf{Q}_t|^2 - 1)|f_1 - f_2|^2 \\ &\quad\geq \frac{\delta^2}{18}|\mathbf{Q}_1 - \mathbf{Q}_2|^2 + \frac{\delta^2}{2}|f_1 - f_2|^2.\end{aligned}$$

Taking $\eta = e^{-\sigma r}\xi(r)$, where $\xi(r)$ is a smooth cut-off function such that $\xi(r) = 1$ for $r < R$, $\xi(r) = 0$ for $r > R + 1$, and $\xi'(r) \leq 2$. Then we have

$$\begin{aligned}&\int_{B_R^+}\left(\frac{\delta^2}{18}|\mathbf{Q}_1 - \mathbf{Q}_2|^2 + \frac{\delta^2}{2}|f_1 - f_2|^2\right)e^{-2\sigma r}dz \\ &\leq \frac{\sigma^2}{\kappa^2}\int_{B_{R+1}^+}|f_1 - f_2|^2 e^{-2\sigma r}dz + \sigma^2\int_{B_{R+1}^+}|\mathbf{Q}_1 - \mathbf{Q}_2|^2 e^{-2\sigma r}dz \\ &\quad + 4e^{-2\sigma R}\int_{B_{R+1}^+\setminus B_R^+}|\mathbf{Q}_1 - \mathbf{Q}_2|^2 dz + \frac{4}{\kappa^2}e^{-2\sigma R}\int_{B_{R+1}^+\setminus B_R^+}|f_1 - f_2|^2 dz.\end{aligned}$$

Letting $R \to \infty$ first and then letting $\sigma \to 0$ in the above inequality , we obtain that $f_1 = f_2$ and $\mathbf{Q}_1 = \mathbf{Q}_2$. □

Appendix B: Exponential decay for some ODEs

Consider the following system

$$\begin{cases} u'' = a_{11}(z_2)u + a_{12}(z_2)v + b_1(z_2) & \text{in } \mathbb{R}_+, \\ v'' = a_{21}(z_2)u + a_{22}(z_2)v + b_2(z_2) & \text{in } \mathbb{R}_+, \\ u'(0) = u_0, \quad v'(0) = v_0, \\ u(\infty) = 0, \quad v(\infty) = 0. \end{cases} \tag{B.1}$$

Definition B.1 We say that the coefficient matrix $A(z_2) = (a_{ij}(z_2))_{2\times 2}$ is elliptic if there exist positive constants λ and M such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(z_2)\xi_i\xi_j \le M|\xi|^2. \tag{B.2}$$

for all $\xi \in \mathbb{R}^2$ and almost every $z_2 \in \mathbb{R}_+$.

Proposition B.2 *Assume that the matrix $A(z_2) = (a_{ij}(z_2))_{2\times 2}$ is elliptic, and suppose there exist positive constants α_5, β_5, M_1 and M_2 such that*

$$|b_1(z_2)| \le M_1 e^{-\alpha_5 z_2}, \quad |b_2(z_2)| \le M_2 e^{-\beta_5 z_2}, \quad z_2 \ge 0. \tag{B.3}$$

Then system (B.1) has a unique solution $(u, v) \in C^2(\mathbb{R}_+) \cap H^1(\mathbb{R}_+)$. Moreover, for any real number μ satisfying $0 < \mu < \min\{\sqrt{\lambda}, \alpha_5, \beta_5\}$ we have

$$|u(z_2)| \le Ce^{-\mu z_2}, \quad |v(z_2)| \le Ce^{-\mu z_2}, \quad z_2 \ge 0, \tag{B.4}$$

where the constant C depends on the constants in (B.2) and (B.3).

Proof Replacing u by $u - u_0 e^{-\lambda z_2}$ and v by $v - v_0 e^{-\lambda z_2}$, we see that there is no loss of generality in assuming $u_0 = v_0 = 0$. Let us fix a constant μ with $0 < \mu < \min\{\sqrt{\lambda}, \alpha_5, \beta_5\}$, and take a function $\eta \in C^2(\mathbb{R}_+)$ satisfying

$$\eta(z_2) = 1 \quad \text{for } z_2 \in [0, 1], \quad e^{-\mu x}\eta(z_2) < 2 \quad \text{and} \quad |\eta'(z_2)| \le \mu\eta(z_2) \quad \text{for all } z_2 \ge 0. \tag{B.5}$$

Define a space

$$\mathscr{Y} = \left\{(u, v) : \ (\eta u) \in H^1(\mathbb{R}_+), \ (\eta v) \in H^1(\mathbb{R}_+), \ u'(0) = 0, \ v'(0) = 0\right\}.$$

Equipped with the norm

$$\|(u, v)\|_{\mathscr{Y}} = \left(\|\eta u\|^2_{H^1(\mathbb{R}_+)} + \|\eta v\|^2_{H^1(\mathbb{R}_+)}\right)^{1/2}$$

and the inner product

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \int_{\mathbb{R}_+} \{\eta^2(u_1u_2 + v_1v_2) + (\eta u_1)'(\eta u_2)' + (\eta v_1)'(\eta v_2)'\} dz_2,$$

$\mathscr{Y}$ is a Hilbert space.

Define a bilinear form $\mathcal{B}[(\cdot,\cdot),(\cdot,\cdot)]$ on $\mathscr{Y}$ by

$$\begin{aligned}
&\mathcal{B}[(u, v), (u^*, v^*)] \\
&= \int_{\mathbb{R}^+} \Big\{ (\eta u)'(\eta u^*)' + (\eta v)'(\eta v^*)' - \frac{\eta'^2}{\eta^2}(\eta u)(\eta u^*) - \frac{\eta'^2}{\eta^2}(\eta v)(\eta v^*) \\
&\quad - \frac{\eta'}{\eta}[(\eta u^*)'(\eta u) - (\eta u^*)(\eta u)'] - \frac{\eta'}{\eta}[(\eta v^*)'(\eta v) - (\eta v^*)(\eta v)'] \\
&\quad + (a_{11}(z_2)\eta u + a_{12}(z_2)\eta v)(\eta u^*) + (a_{21}(z_2)\eta u + a_{22}(z_2)\eta v)(\eta v^*) \Big\} dz_2.
\end{aligned}$$

Using the condition (B.2) on the coefficient matrix A and the assumption (B.5) on the function η, and by the Cauchy's inequality, there exists a constant K depending only on the constants in (B.2) and μ, such that for all (u, v) and (u^*, v^*) in $\mathscr{Y}$ we have

$$\begin{aligned}
\mathcal{B}[(u, v), (u^*, v^*)] &\le K \|(u, v)\|_{\mathscr{Y}} \|(u^*, v^*)\|_{\mathscr{Y}}, \\
\mathcal{B}[(u, v), (u, v)] &\ge \min\{1, \lambda - \mu^2\} \|(u, v)\|^2_{\mathscr{Y}}.
\end{aligned}$$

Therefore, $\mathcal{B}$ is bounded and coercive on $\mathscr{Y}$. Then the existence and uniqueness of the solution to (B.1) in $\mathscr{Y}$ follows from the Lax-Milgram lemma.

Set $\check{u} = \eta u$ and $\check{v} = \eta v$. Then $(\check{u}, \check{v})$ satisfies

$$\begin{cases}
\check{u}'' = \frac{2\eta'}{\eta}\check{u}' + \frac{\eta''}{\eta}\check{u} - \frac{2\eta'^2}{\eta^2}\check{u} + a_{11}(z_2)\check{u} + a_{12}(z_2)\check{v} + \eta b_1(z_2) & \text{in } \mathbb{R}_+, \\
\check{v}'' = \frac{2\eta'}{\eta}\check{v}' + \frac{\eta''}{\eta}\check{v} - \frac{2\eta'^2}{\eta^2}\check{v} + a_{21}(z_2)\check{u} + a_{22}(z_2)\check{v} + \eta b_2(z_2) & \text{in } \mathbb{R}_+, \\
\check{u}'(0) = 0, \quad \check{v}'(0) = 0, \\
\check{u}(\infty) = 0, \quad \check{v}(\infty) = 0.
\end{cases} \tag{B.6}$$

Also, we have

$$\begin{aligned}
&\min\{1, \lambda - \mu^2\} \left(\|\check{u}\|^2_{H^1(\mathbb{R}_+)} + \|\check{v}\|^2_{H^1(\mathbb{R}_+)} \right) \le \mathcal{B}[(u, v), (u, v)] \\
&= \int_{\mathbb{R}^+} \left(\eta b_1(z_2)\check{u} + \eta b_2(z_2)\check{v} \right) dz_2.
\end{aligned}$$

Then by the Cauchy's inequality we get

$$\|\check{u}\|_{H^1(\mathbb{R}_+)} + \|\check{v}\|_{H^1(\mathbb{R}_+)} \le C.$$

Since $H^1(\mathbb{R}_+)$ is continuously embedded into $C^0(\mathbb{R}_+)$, then we have

$$\|\check{u}\|_{C^0(\mathbb{R}_+)} + \|\check{v}\|_{C^0(\mathbb{R}_+)} \leq C.$$

This proves (B.4). □

Proof of Proposition 2.5 From Proposition 2.2, we obtain the decay estimate for $|\hat{f}_0(y_1, z_2)|$ and $|\hat{\mathbf{Q}}_0(y_1, z_2)|$ at $y_1 = 0$. Next we derive the estimates for $\partial_{y_1} \hat{f}_0(y_1, z_2)$ and $\partial_{y_1} \hat{\mathbf{Q}}_0(y_1, z_2)$ at $y_1 = 0$. Recall that

$$p(z_2) := \partial_{y_1} \hat{f}_0(0, z_2), \quad (q(z_2), 0) := \partial_{y_1} \hat{\mathbf{Q}}_0(0, z_2).$$

Then from the equation (5.18) in section 5, we see that $(p(z_2), q(z_2))$ satisfies

$$\begin{cases} \frac{1}{\kappa^2} p''(z_2) = (3|f_0|^2 + |Q_0^1|^2 - 1)p + 2Q_0^1 f_0 q & \text{in } \mathbb{R}_+, \\ q''(z_2) = 2f_0 Q_0^1 p + f_0^2 q & \text{in } \mathbb{R}_+, \\ p'(0) = 0, \quad q'(0) = -\hat{\mathcal{H}}^e_{y_1}(0), \\ p(\infty) = 0, \quad q(\infty) = 0. \end{cases} \tag{B.7}$$

Let $\lambda(z_2)$ be the minimum eigenvalue of the matrix

$$\begin{pmatrix} 3|f_0|^2 + |Q_0^1|^2 - 1 & 2f_0 Q_0^1 \\ 2f_0 Q_0^1 & |f_0|^2 \end{pmatrix}.$$

Then $\lambda(z_2) \to 1$ as $z_2 \to \infty$. Now we can apply Proposition B.2 to conclude that, for any real number β_1 satisfying $0 < \beta_1 < 1$ we have

$$|p(z_2)| + |q(z_2)| \leq C(\kappa, \beta_1, \Omega, \mathcal{H}^e) e^{-\beta_1 z_2}.$$

Applying Proposition B.2 again for the first equation in (B.7) and noting that $|Q_0^1| \leq Ce^{-\beta_1 z_2}$, for any real number α_1 satisfying $0 < \alpha_1 < \min\{2, \sqrt{2}\kappa\}$, we have

$$|p(z_2)| \leq C(\kappa, \alpha_1, \beta_1, \Omega, \mathcal{H}^e) e^{-\alpha_1 z_2}.$$

We derive the higher derivative estimates of $\hat{f}_0(y_1, z_2)$ and $\hat{\mathbf{Q}}_0(y_1, z_2)$ at $y_1 = 0$. From the equation (3.5), we see that

$$u(z_2) := \partial^i_{y_1^i} \hat{f}_0(0, z_2), \quad v(z_2) := \partial^i_{y_1^i} \hat{Q}_0^1(0, z_2) \quad \text{for } i = 2, 3$$

satisfy

$$\begin{cases} \frac{1}{\kappa^2}u''(z_2) = (3|f_0|^2 + |Q_0^1|^2 - 1)u + 2Q_0^1 f_0 v + F_i(z_2) & \text{in } \mathbb{R}_+, \\ v''(z_2) = 2f_0 Q_0^1 u + f_0^2 v + G_i(z_2) & \text{in } \mathbb{R}_+, \\ u'(0) = 0, \quad v'(0) = -\hat{\mathcal{H}}^e_{y_1^i}(0), \\ u(\infty) = 0, \quad v(\infty) = 0, \end{cases} \tag{B.8}$$

where

$$|F_i(z_2)| \le C(\kappa, \alpha_1, \beta_1, \Omega, \mathcal{H}^e)e^{-\min\{3\alpha_1, 2\beta_1\}z_2},$$
$$|G_i(z_2)| \le C(\kappa, \alpha_1, \beta_1, \Omega, \mathcal{H}^e)e^{-(\alpha_1+\beta_1)z_2}$$

for $i = 2, 3$. As the proof of the estimates of $p(z_2)$ and $q(z_2)$, by applying Proposition B.2 we can obtain the decay estimates of $\partial^i_{y_1^i}\hat{f}_0(0, z_2)$ and $\partial^i_{y_1^i}\hat{Q}_0^1(0, z_2)$ for $i = 2, 3$.

Applying the above argument to the equations of $\partial^i_{y_1^i}\hat{f}_0(0, z_2)$ and $\partial^i_{y_1^i}\hat{Q}_0^1(0, z_2)$ respectively, we immediately obtain the decay estimates of $|\partial^{i+2}_{y_1^i z_2^2}\hat{f}_0(y_1, z_2)|$ and $|\partial^{i+2}_{y_1^i z_2^2}\hat{\mathbf{Q}}_0(y_1, z_2)|$ for $i = 0, \cdots, 3$.

Integrating from z_2 to ∞ on both sides of the equations of $\partial^i_{y_1^i}\hat{f}_0(0, z_2)$ and $\partial^i_{y_1^i}\hat{Q}_0^1(0, z_2)$ respectively, we can obtain the decay estimates of $|\partial^{i+1}_{y_1^i z_2}\hat{f}_0(y_1, z_2)|$ and $|\partial^{i+1}_{y_1^i z_2}\hat{\mathbf{Q}}_0(y_1, z_2)|$ for $i = 0, \ldots, 3$.

Now we have proved Proposition 2.5 for $y_1 = 0$. Replacing $\hat{\mathcal{H}}^e(0)$ by $\hat{\mathcal{H}}^e(y_1)$ in (B.7) and in (B.8), then noting that $\hat{\mathcal{H}}^e \in C^3(\partial\Omega)$, we see that Proposition 2.5 also holds for $y_1 \neq 0$. Now we have completed the proof. □

Appendix C: Derivation of system (2.20)

To derive equation (2.20) we need the local coordinate expansions introduced in [24, section 3]. Here we keep the notations in section 2. We use $\mathfrak{R}_i(|y_1^3|)$, $i = 1, 2, \cdots$, to denote a function of y_1 and z_2 which is of order $(|y_1^3|)$ uniformly for z_2, and use $\mathfrak{R}_i(\lambda^k)$, $k > 0$, $i = 1, 2, \cdots$, to denote a function of y_1 and z_2 which is of order (λ^k).

For the function g defined in (2.2) we have, for $\lambda > 0$ small,

$$\begin{aligned} g(z) &= 1 - \lambda k(0)z_2 - \lambda^2 k'(0)z_1 z_2 + O(\lambda^3), \\ \frac{1}{g(z)} &= 1 + \lambda k(0)z_2 + \lambda^2\left(k^2(0)z_2^2 + k'(0)z_1 z_2\right) + O(\lambda^3), \end{aligned} \tag{C.1}$$

where $k'(0) = \frac{\mathrm{d}k}{\mathrm{d}s}(0) = \frac{\mathrm{d}k}{\mathrm{d}y_1}(0)$. For any fixed $z_2 \ge 0$, we have the formal asymptotic expansions for $\hat{f}_0(y_1, z_2)$ and $\hat{\mathbf{Q}}_0(y_1, z_2)$ with respect to the variable y_1 at the point

$(0, z_2)$:

$$\begin{aligned}\hat{f}_0(y_1, z_2) &= f_0 + y_1 \partial_{y_1} \hat{f}_0(0, z_2) + \frac{1}{2} y_1^2 \partial_{y_1^2} \hat{f}_0(0, z_2) + \mathfrak{R}_1(|y_1^3|), \\ \hat{\mathbf{Q}}_0(y_1, z_2) &= \mathbf{Q}_0 + y_1 \partial_{y_1} \hat{\mathbf{Q}}_0(0, z_2) + \frac{1}{2} y_1^2 \partial_{y_1^2} \hat{\mathbf{Q}}_0(0, z_2) + \mathfrak{R}_2(|y_1^3|),\end{aligned} \tag{C.2}$$

where

$$(f_0, \mathbf{Q}_0) = (\hat{f}_0(0, z_2), \hat{\mathbf{Q}}_0(0, z_2)) = (\hat{f}_0(0, z_2), (\hat{Q}_0^1(0, z_2), \hat{Q}_0^2(0, z_2)))$$

is the solution of (2.14), $(\hat{f}_0(y_1, z_2), \hat{\mathbf{Q}}_0(y_1, z_2))$ is the solution of (5.18).

Write

$$p(z_2) := \partial_{y_1} \hat{f}_0(0, z_2), \quad (q(z_2), 0) := \partial_{y_1} \hat{\mathbf{Q}}_0(0, z_2). \tag{C.3}$$

Then we take the expansions for $\tilde{f}$ and $\tilde{\mathbf{Q}}$ in (2.6) with respect to λ, and have

$$\tilde{f} = f_0 + \lambda(p z_1 + f_1) + \mathfrak{R}_3(\lambda^2), \quad \tilde{\mathbf{Q}} = \mathbf{Q}_0 + \lambda((q z_1, 0) + \mathbf{Q}_1) + \mathfrak{R}_4(\lambda^2), \tag{C.4}$$

where $f_1 = \hat{f}_1(0, z_2)$ and $\mathbf{Q}_1 = \hat{\mathbf{Q}}_1(0, z_2) = (Q_1^1, Q_1^2)$ are to be determined.

Firstly, we have

$$\begin{aligned}\lambda \mathrm{curl} \mathbf{Q}(x) &= \frac{1}{g} \left[\partial_{z_1} \tilde{Q}_2 - \partial_{z_2} (g \tilde{Q}_1) \right] \\ &= -(Q_0^1)' + \lambda (k(0) Q_0^1 - (Q_1^1)' - z_1 q'(z_2)) + \mathfrak{R}_5(\lambda^2).\end{aligned} \tag{C.5}$$

Then,

$$\begin{aligned}&\frac{1}{g} \left(\partial_{z_1} \left(\frac{1}{g} \partial_{z_1} \tilde{f} \right) + \partial_{z_2} \left(g \partial_{z_2} \tilde{f} \right) \right) \\ &\quad = (f_0)'' + \lambda \left((f_1)'' - k(0)(f_0)' + p''(z_2) z_1 \right) + \mathfrak{R}_6(\lambda^2)\end{aligned} \tag{C.6}$$

and

$$\begin{aligned}(1 - |\tilde{f}|^2 - |\tilde{\mathbf{Q}}|^2) \tilde{f} &= (1 - |f_0|^2 - |\mathbf{Q}_0|^2) f_0 + \lambda \big((1 - |f_0|^2 - |\mathbf{Q}_0|^2)(p z_1 + f_1) \\ &\quad - 2 f_0 (f_0 (p z_1 + f_1) + \mathbf{Q}_0 \cdot ((q z_1, 0) + \mathbf{Q}_1)) \big) + \mathfrak{R}_7(\lambda^2),\end{aligned} \tag{C.7}$$

where $p = p(z_2)$ and $q = q(z_2)$ are defined in (C.2). Using (C.5), for $\mathcal{M}_1(\lambda z)$ and $\mathcal{M}_2(\lambda z)$ defined by (2.3) we have

$$\begin{aligned}\mathcal{M}_1(\lambda z) &= -(Q_0^1)'' - \lambda \big[(Q_1^1)'' + q''(z_2) z_1 - k(0)(Q_0^1)' \big] + \mathfrak{R}_8(\lambda^2), \\ \mathcal{M}_2(\lambda z) &= \lambda q'(z_2) + \mathfrak{R}_9(\lambda^2).\end{aligned} \tag{C.8}$$

Also, we have

$$|\tilde{f}|^2\tilde{\mathbf{Q}} = |f_0|^2\mathbf{Q}_0 + \lambda\left[2f_0(pz_1 + f_1)\mathbf{Q}_0 + |f_0|^2((qz_1, 0) + \mathbf{Q}_1)\right] + \mathfrak{R}_{10}(\lambda^2). \tag{C.9}$$

We now consider the equations at the point $(0, z_2)$. We have

$$\frac{1}{g}\left(\partial_{z_1}\left(\frac{1}{g}\partial_{z_1}\tilde{f}\right) + \partial_{z_2}\left(g\partial_{z_2}\tilde{f}\right)\right) = (f_0)'' + \lambda\left((f_1)'' - k(0)\partial_2 f_0\right) + \mathfrak{R}_{11}(\lambda^2) \tag{C.10}$$

and

$$\begin{aligned}(1 - |\tilde{f}|^2 - |\tilde{\mathbf{Q}}|^2)\tilde{f} &= (1 - |f_0|^2 - |Q_0^1|^2)f_0 \\ &\quad + \lambda\big((1 - |f_0|^2 - |Q_0^1|^2)f_1 - 2f_0(f_0 f_1 + Q_0^1 Q_1^1)\big) + \mathfrak{R}_{12}(\lambda^2).\end{aligned} \tag{C.11}$$

For $\mathcal{M}_1(\lambda z)$ we have

$$\mathcal{M}_1(\lambda z) = -(Q_0^1)'' + \lambda\big[-(Q_1^1)'' + k(0)(Q_0^1)'\big] + \mathfrak{R}_{13}(\lambda^2). \tag{C.12}$$

Also, we have

$$|\tilde{f}|^2\tilde{\mathbf{Q}} = |f_0|^2\mathbf{Q}_0 + \lambda\left[2f_0 f_1\mathbf{Q}_0 + |f_0|^2\mathbf{Q}_1\right] + \mathfrak{R}_{14}(\lambda^2). \tag{C.13}$$

Comparing with the coefficients of λ, we obtain the equations (2.20) for the first order terms.

Appendix D: Derivation of system (2.22) and proof of (3.5)

We follow the notations used in section 2 and in appendix C. Let $\mathfrak{R}_i(\lambda^2)$ be the terms appear in appendix C, and it has been proved in section 3 that these terms have the order $O(\lambda^2)$ uniformly for y_1 and z_2. In this section we shall expand these terms in the form

$$\mathfrak{R}_i(\lambda^2) = \lambda^2 R_i + \mathfrak{R}_i(\lambda^3) \qquad \text{for } i = 3, \cdots, 14,$$

where R_i denotes a functions of y_1 and z_2 which is independent of λ, and $\mathfrak{R}_i(\lambda^3)$ denotes a function of y_1 and z_2 which is of the order $O(\lambda^3)$.

From the inner expansion (2.9) and the expansion (C.2), we have the expansions for the function $\mathfrak{R}_3$ and the vector field $\mathfrak{R}_4$ in (C.4):

$$\mathfrak{R}_3(\lambda^2) = \lambda^2\left(\frac{1}{2}z_1^2\partial_{y_1^2}\hat{f}_0(0, z_2) + z_1\partial_{y_1}\hat{f}_1(0, z_2) + f_2\right) + \mathfrak{R}_{16}(\lambda^3), \tag{D.1}$$

and

$$\mathfrak{R}_4(\lambda^2) = \lambda^2\Big(\frac{1}{2}z_1^2\partial_{y_1^2}\hat{Q}_0^1(0,z_2) + z_1\partial_{y_1}\hat{Q}_1^1(0,z_2) + Q_2^1(0,z_2), \\ z_1\partial_{y_1}\hat{Q}_1^2(0,z_2) + Q_2^2(0,z_2)\Big) + \mathfrak{R}_{17}(\lambda^3), \tag{D.2}$$

where

$$(\hat{f}_0(0,z_2), \hat{\mathbf{Q}}_0(0,z_2)) = (f_0, (Q_0^1, 0))$$

is the solution of (2.14),

$$(\hat{f}_1(0,z_2), \hat{\mathbf{Q}}_1(0,z_2)) = (f_1, (Q_1^1, Q_1^2))$$

is the solution of (2.20), and

$$(\hat{f}_2(0,z_2), \hat{\mathbf{Q}}_2(0,z_2)) = (f_2, (Q_2^1, Q_2^2))$$

is to be determined now.

From (C.5), we have

$$\begin{aligned}\mathfrak{R}_5(\lambda^2) =& \lambda^2\Big(\partial_{y_1}\hat{Q}_1^2 + k'(0)z_1Q_0^1 + k'(0)z_1z_2(Q_0^1)' + k(0)\left(z_1\partial_{y_1}\hat{Q}_0^1\big|_{y_1=0} + Q_1^1\right) \\ &- (Q_2^1)' + k(0)z_2\left[z_1\partial_{y_1z_2}\hat{Q}_0^1\big|_{y_1=0} + (Q_1^1)'\right] - \frac{1}{2}z_1^2\partial_{y_1^2z_2}\hat{Q}_0^1\big|_{y_1=0} \\ &- z_1\partial_{y_1z_2}\hat{Q}_1^1\big|_{y_1=0} + k(0)z_2\left[k(0)Q_0^1 - (Q_1^1)' - z_1\partial_{y_1z_2}\hat{Q}_0^1\big|_{y_1=0}\right] \\ &- (Q_0^1)'\left[k^2(0)z_2^2 + k'(0)z_1z_2\right]\Big) + \mathfrak{R}_{18}(\lambda^3).\end{aligned} \tag{D.3}$$

Then we have the expansions for $\mathfrak{R}_6$ in (C.6) and for $\mathfrak{R}_7$ in (C.7):

$$\begin{aligned}\mathfrak{R}_6(\lambda^2) =& \lambda^2\Big(\partial_{y_1^2}\hat{f}_0\big|_{y_1=0} + \frac{1}{2}z_1^2\partial_{z_2^2y_1^2}\hat{f}_0\big|_{y_1=0} + z_1\partial_{z_2^2y_1}\hat{f}_1\big|_{y_1=0} - k(0)\left[p'z_1 + (f_1)'\right] \\ &+ (f_2)'' - k(0)z_2\left[p''z_1 + (f_1)''\right] - (f_0)'k'(0)z_1 - (f_0)''k'(0)z_1z_2 \\ &+ k(0)z_2\left[(f_1)'' + p''z_1 - k(0)(f_0)' - k(0)z_2(f_0)''\right] \\ &+ (f_0)''\left[k^2(0)z_2^2 + k'(0)z_1z_2\right]\Big) + \mathfrak{R}_{19}(\lambda^3),\end{aligned}$$

and

$$\mathfrak{R}_7(\lambda^2) = \lambda^2\Big(\Big[\frac{1}{2}z_1^2\partial_{y_1^2}\hat{f}_0\big|_{y_1=0} + z_1\partial_{y_1}\hat{f}_1\big|_{y_1=0} + f_2\Big](1 - f_0^2 - |\mathbf{Q}_0|^2) + (pz_1 + f_1)[-2f_0(pz_1 + f_1) - 2Q_0^1(qz_1 + Q_1^1) - 2Q_0^2Q_1^2] + f_0\Big[-(pz_1 + f_1)^2 - 2f_0\left(\frac{1}{2}z_1^2\partial_{y_1^2}\hat{f}_0\big|_{y_1=0} + z_1\partial_{y_1}\hat{f}_1\big|_{y_1=0} + f_2\right) - (qz_1 + Q_1^1)^2 - 2Q_0^1\left(\frac{1}{2}z_1^2\partial_{y_1^2}\hat{Q}_0^1\big|_{y_1=0} + z_1\partial_{y_1}\hat{Q}_1^1\big|_{y_1=0} + Q_2^1\right) -(Q_1^2)^2 - 2Q_0^2(z_1\partial_{y_1}\hat{Q}_1^2\big|_{y_1=0} + Q_2^2)\Big]\Big) + \mathfrak{R}_{20}(\lambda^3).$$

From (D.1), (D.2) and (C.8), we see that

$$\begin{aligned}\mathfrak{R}_8(\lambda^2) =&\lambda^2\Big(\partial_{z_2y_1}\hat{Q}_1^2\big|_{y_1=0} - \frac{1}{2}z_1^2\partial_{z_2^2y_1^2}\hat{Q}_2^1\big|_{y_1=0} - z_1\partial_{z_2^2y_1}\hat{Q}_1^1\big|_{y_1=0} + k'(0)z_1z_2(Q_0^1)'' \\ &- (Q_2^1)'' + 2k(0)\left[q'z_1 + (Q_1^1)'\right] + k(0)z_2\left[q''z_1 + (Q_1^1)''\right] + k'(0)z_1(Q_0^1)' \\ &- k(0)\left[q'z_1 + (Q_1^1)' - k(0)Q_0^1\right] - (Q_0^1)''\left[k^2(0)z_2^2 + k'(0)z_1z_2\right] \\ &- k(0)z_2\left[q''z_1 + (Q_1^1)'' - k(0)(Q_0^1)' - k(0)z_2(Q_0^1)''\right]\Big) + \mathfrak{R}_{21}(\lambda^3)\end{aligned} \tag{D.4}$$

and

$$\begin{aligned}\mathfrak{R}_9(\lambda^2) =&\lambda^2\Big(z_1\partial_{z_2y_1^2}\hat{Q}_0^1\big|_{y_1=0} + \partial_{z_2y_1}\hat{Q}_1^1\big|_{y_1=0} - k(0)q - k'(0)Q_0^1 \\ &+ k(0)z_2q'\Big) + \mathfrak{R}_{22}(\lambda^3).\end{aligned} \tag{D.5}$$

From (D.1), (D.2) and (C.9), we have

$$\begin{aligned}\mathfrak{R}_{10}(\lambda^2) =&\lambda^2\Big(\Big[(pz_1 + f_1)^2 + 2f_0\left(\frac{1}{2}z_1^2\partial_{y_1^2}\hat{f}_0\big|_{y_1=0} + z_1\partial_{y_1}\hat{f}_1\big|_{y_1=0} + f_2\right)\Big]\mathbf{Q}_0 \\ &+ f_0^2\left(\frac{1}{2}z_1^2\partial_{y_1y_1}\hat{Q}_0^1\big|_{y_1=0} + z_1\partial_{y_1}\hat{Q}_1^1\big|_{y_1=0} + Q_2^1, z_1\partial_{y_1}\hat{Q}_1^2\big|_{y_1=0} + Q_2^2\right) \\ &+ 2f_0(pz_1 + f_1)\left[(qz_1, 0) + \mathbf{Q}_1\right]\Big) + \mathfrak{R}_{23}(\lambda^3).\end{aligned} \tag{D.6}$$

We now consider the equations at the point $(0, z_2)$. From the expression of $\mathfrak{R}_6(\lambda^3)$, it follows that,

$$\mathfrak{R}_{11}(\lambda^2) = \lambda^2\left(\partial_{y_1^2}\hat{f}_0\big|_{y_1=0} + f_2'' - k(0)(f_1)' - k^2(0)z_2(f_0)'\right) + \mathfrak{R}_{24}(\lambda^3). \tag{D.7}$$

From the expression of $\mathfrak{R}_7(\lambda^3)$, we have

$$\begin{aligned}\mathfrak{R}_{12}(\lambda^2) =&\lambda^2\Big(f_2(1-3f_0^2-|\mathbf{Q}_0|^2)+f_0\left[-f_1^2-(Q_1^1)^2-(Q_1^2)^2\right]\\&-2f_0Q_0^1Q_2^1+f_1\left[-2f_0f_1-2Q_0^1Q_1^1\right]\Big)+\mathfrak{R}_{25}(\lambda^3).\end{aligned} \tag{D.8}$$

From the expressions of $\mathfrak{R}_8(\lambda^3)$ and $\mathfrak{R}_9(\lambda^3)$, we have

$$\begin{aligned}\mathfrak{R}_{13}(\lambda^2) =&\lambda^2\Big(\partial_{z_2y_1}\hat{Q}_1^2\big|_{y_1=0}-(Q_2^1)''+k(0)(Q_1^1)'+k(0)Q_0^1\\&+k^2(0)z_2(Q_0^1)'\Big)+\mathfrak{R}_{26}(\lambda^3)\end{aligned}$$

and

$$\mathfrak{R}_{14}(\lambda^2)=\lambda^2\left(\partial_{z_2y_1}\hat{Q}_1^1\big|_{y_1=0}-k(0)q-k'(0)Q_0^1+k(0)z_2q'\right)+\mathfrak{R}_{27}(\lambda^3).$$

From $\mathfrak{R}_{10}(\lambda^3)$, it follows that

$$\mathfrak{R}_{15}(\lambda^2)=\lambda^2((f_1^2+2f_0f_2)\mathbf{Q}_0+2f_0f_1\mathbf{Q}_1+f_0^2\mathbf{Q}_2)+\mathfrak{R}_{28}(\lambda^3).$$

Comparing with the coefficients of λ^2, we obtain the equations (2.22) for the second order terms.

Proof of Lemma 3.1 *Step 1*. From (3.2), we can see that

$$\mathbf{b}(x,\lambda)=(0,\mathbf{0})\quad\text{for all } x\in\Omega\backslash\sigma_2,$$

where σ_n is defined by (3.1). Then (3.5) holds for $x\in\Omega\backslash\sigma_2$.

Step 2. We show the estimate (3.5) when $x\in\sigma_4$. We consider this problem in a neighborhood $\mathcal{U}$ of $X_0\in\partial\Omega$. We follow the notation used in section 2 and in appendix C.

To obtain $\tilde{\mathbf{b}}$, we replace the expressions of $\tilde{f}$ and $\tilde{\mathbf{Q}}$ by

$$\hat{f}_0(y_1,z_2)+\lambda\hat{f}_1(y_1,z_2)+\lambda^2\hat{f}_2(y_1,z_2)\quad\text{and}\quad\hat{\mathbf{Q}}_0(y_1,z_2)+\lambda\hat{\mathbf{Q}}_1(y_1,z_2)+\lambda^2\hat{\mathbf{Q}}_2(y_1,z_2)$$

under the $z-$ coordinate system in (C.4) respectively.

We first estimate $\tilde{b}_1(z_1,z_2,\lambda)$, where $\tilde{b}_1(z_1,z_2,\lambda)$ is the representation of $b_1(x,\lambda)$ under the z-coordinate system. At the point $(0,z_2)$, from (D.7) and (D.8) we have

$$\begin{aligned}\mathfrak{R}_{24}(\lambda^3) =&\lambda^3\kappa^{-2}\Big(g_0^{-3}k'(0)z_2\left[\partial_{y_1}\hat{f}_0\big|_{y_1=0}+\lambda\partial_{y_1}\hat{f}_1\big|_{y_1=0}+\lambda^2\partial_{y_1}\hat{f}_2\big|_{y_1=0}\right]\\&+g_0^{-2}\left[(1+g_0)k(0)z_2\partial_{y_1^2}\hat{f}_0\big|_{y_1=0}+\partial_{y_1^2}\hat{f}_1\big|_{y_1=0}+\lambda\partial_{y_1^2}\hat{f}_2\big|_{y_1=0}\right]\\&-g_0^{-1}\left[k(0)(f_0)'-\lambda k^3(0)z_2^2(f_1)'-\lambda k^2(0)z_2(f_2)'\right]\\&-k^3(0)z_2^2(f_0)'-k^2(0)z_2(f_1)'-k(0)(f_2)'\Big)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{R}_{25}(\lambda^3) = -\lambda^3\Big(&(f_1+\lambda f_2)\Big[2Q_0^1Q_2^1+2f_0f_2+f_1^2+(Q_1^1)^2+(Q_1^2)^2\Big]\\ &+2f_2(f_0f_1+Q_0^1Q_1^1)+(f_0+\lambda f_1+\lambda^2 f_2)\Big[2Q_1^1Q_2^1+2f_1f_2\\ &+2Q_1^2Q_2^2+\lambda\left(f_2^2+(Q_1^1)^2+(Q_1^2)^2\right)\Big]\Big),\end{aligned}$$

where $k_0=k(0)$ is the curvature of $\partial\Omega$ at the point X_0, $k_0'=\frac{dk}{ds}(0)=\frac{dk}{dy_1}(0)$,

$$g_0=g(0,\lambda z_2)=1-\lambda k(0)z_2.$$

We see that $\mathfrak{R}_{24}(\lambda^3)$ and $\mathfrak{R}_{25}(\lambda^3)$ are the polynomials of $\hat{f}_0,\hat{f}_1,\hat{f}_2,\hat{\mathbf{Q}}_0,\hat{\mathbf{Q}}_1,\hat{\mathbf{Q}}_2$ and their derivatives up to the order 2 at $(0,z_2)$. From Proposition 2.5, Proposition 2.7 and Proposition 2.9, it follows that

$$|\mathfrak{R}_{24}(\lambda^3)|+|\mathfrak{R}_{25}(\lambda^3)|\le C(\Omega,\mathcal{H}^e,\kappa)\lambda^3.$$

For any $x\in\sigma_4$, let ψ be defined by (2.1), $x=\psi(y_1,y_2)$, $z_1=y_1/\lambda$, $z_2=y_2/\lambda$. Then $\tilde{b}_1(0,z_2,\lambda)=\mathfrak{R}_{24}(\lambda^3)+\mathfrak{R}_{25}(\lambda^3)$. Note that $\hat{f}_0,\hat{f}_1,\hat{f}_2,\hat{\mathbf{Q}}_0,\hat{\mathbf{Q}}_1,\hat{\mathbf{Q}}_2$ and their derivatives up to the order 2 are continuously differentiable with respect to the parameter y_1 (by applying the continuous differentiability of solutions with respect to parameters in the theory of ODEs). Therefore, for each $z_1\neq 0$, we also have

$$|\tilde{b}_1(z_1,z_2,\lambda)|\le C(\Omega,\mathcal{H}^e,\kappa)\lambda^3.$$

We now estimate $\tilde{\mathbf{b}}_2=(\tilde{b}_2^1,\tilde{b}_2^2)$. At the point (y_1,z_2) with $y_1=\lambda z_2$, from (D.4) we have

$$\begin{aligned}\tilde{\mathfrak{R}}_{21}(\lambda^3)=&\lambda^3\Big[s_3+\lambda k(y_1)z_2(s_2+\lambda s_3)+\lambda^2k^2(y_1)z_2^2(s_1+\lambda s_2+\lambda^2 s_3)\\ &\quad+\lambda^3k^3(y_1)z_2^3g^{-1}(-\partial_{z_2^2}\hat{Q}_0^1+\lambda s_1+\lambda^2 s_2+\lambda^3 s_3)\Big]\\ &+\lambda^3\Big[k(y_1)s_5+2\lambda^2k^2(y_1)z_2(s_4+\lambda s_5)\\ &\quad+\lambda^3k^3(y_1)z_2^2g^{-2}(2g+1)(-\partial_{z_2}\hat{Q}_0^1+\lambda s_4+\lambda^2 s_5)\Big],\end{aligned}$$

where

$$\begin{aligned}s_1&=k(y_1)\hat{Q}_0^1+k(y_1)z_2\partial_{z_2^2}\hat{Q}_0^1+k(y_1)\partial_{z_2}\hat{Q}_0^1-\partial_{z_2^2}\hat{Q}_1^1,\\ s_2&=\partial_{y_1z_2}\hat{Q}_1^2+k(y_1)\hat{Q}_1^1+k(y_1)z_2\partial_{z_2^2}\hat{Q}_1^1-\partial_{z_2^2}\hat{Q}_2^1+k(y_1)\partial_{z_2}\hat{Q}_1^1,\\ s_3&=\partial_{y_1z_2}\hat{Q}_2^2+k(y_1)\hat{Q}_2^1+k(y_1)z_2\partial_{z_2^2}\hat{Q}_2^1+k(y_1)\partial_{z_2}\hat{Q}_2^1\end{aligned}$$

and

$$
\begin{aligned}
s_4 &= k(y_1)z_2\partial_{z_2}\hat{Q}_0^1 + k(y_1)\hat{Q}_0^1 - \partial_{z_2}\hat{Q}_1^1,\\
s_5 &= \partial_{y_1}\hat{Q}_1^2 + k(y_1)\hat{Q}_1^1 + k(y_1)z_2\partial_{z_2}\hat{Q}_1^1 - \partial_{z_2}\hat{Q}_2^1\lambda\Big(\partial_{y_1}\hat{Q}_2^2 + k(y_1)\hat{Q}_2^1 + k(y_1)z_2\partial_{z_2}\hat{Q}_2^1\Big).
\end{aligned}
$$

From (D.5) we have

$$
\begin{aligned}
\tilde{\mathfrak{R}}_{22}(\lambda^3) = &-\lambda^3\Big[s_7 + 2k(y_1)z_2(s_6+\lambda s_7) + k^2(y_1)z_2^2g^{-2}(2g+1)(-\partial_{y_1z_2}\hat{Q}_0^1 + \lambda s_6 + \lambda^2 s_7)\Big]\\
&-\lambda^3\Big[k'(y_1)z_2s_8 + k(y_1)k'(y_1)z_2^2g^{-3}(1+g+g^2)(-\partial_{z_2}\hat{Q}_0^1 + \lambda s_8)\Big],
\end{aligned}
$$

where g is defined by (2.2),

$$
\begin{aligned}
s_6 =& k(y_1)\partial_{y_1}\hat{Q}_0^1 + k'(y_1)\hat{Q}_0^1 + \partial_{y_1z_2}\hat{Q}_0^1 - \partial_{y_1z_2}\hat{Q}_1^1 + k'(y_1)z_2\partial_{z_2}\hat{Q}_0^1,\\
s_7 =& \partial_{y_1^2}\hat{Q}_1^2 + \lambda\partial_{y_1^2}\hat{Q}_2^2 + k(y_1)\partial_{y_1}\hat{Q}_1^1 + \lambda k(y_1)\partial_{y_1}\hat{Q}_2^1 + k'(y_1)\hat{Q}_1^1 + \lambda k'(y_1)\hat{Q}_2^1\\
&+ k(y_1)z_2\partial_{y_1z_2}\hat{Q}_1^1 - \partial_{y_1z_2}\hat{Q}_2^1 + \lambda k(y_1)z_2\partial_{y_1z_2}\hat{Q}_2^1 + k'(y_1)z_2(\partial_{z_2}\hat{Q}_1^1 + \lambda\partial_{z_2}\hat{Q}_2^1),
\end{aligned}
$$

and

$$
\begin{aligned}
s_8 =& k(y_1)z_2\partial_{z_2}\hat{Q}_0^1 + k(y_1)\hat{Q}_0^1 - \partial_{z_2}Q_1^1\\
&+ \lambda\Big(\partial_{y_1}\hat{Q}_1^2 + k(y_1)\hat{Q}_1^1 + k(y_1)z_2\partial_{z_2}\hat{Q}_1^1 - \partial_{z_2}\hat{Q}_2^1\Big)\\
&+ \lambda^2\Big(\partial_{y_1}\hat{Q}_2^2 + k(y_1)\hat{Q}_2^1 + k(y_1)z_2\partial_{z_2}\hat{Q}_2^1\Big).
\end{aligned}
$$

From (D.6) we have

$$
\begin{aligned}
\tilde{\mathfrak{R}}_{23}^1(\lambda^3) &= \lambda^3\Big[(\hat{Q}_0^1 + \lambda\hat{Q}_1^1 + \lambda^2\hat{Q}_2^1)(2\hat{f}_1\hat{f}_2 + \lambda\hat{f}_2^2) + (\hat{Q}_1^1 + \lambda\hat{Q}_2^1)(\hat{f}_1^2 + 2\hat{f}_0\hat{f}_2) + 2\hat{f}_0\hat{f}_1\hat{Q}_2^1\Big],\\
\tilde{\mathfrak{R}}_{23}^2(\lambda^3) &= \lambda^3\Big[(\lambda\hat{Q}_1^2 + \lambda^2\hat{Q}_2^2)(2\hat{f}_1\hat{f}_2 + \lambda\hat{f}_2^2) + (\hat{Q}_1^2 + \lambda\hat{Q}_2^2)(\hat{f}_1^2 + 2\hat{f}_0\hat{f}_2) + 2\hat{f}_0\hat{f}_1\hat{Q}_2^2\Big].
\end{aligned}
$$

Then

$$
\tilde{b}_2^1 = -\tilde{\mathfrak{R}}_{21} - \tilde{\mathfrak{R}}_{23}^1, \quad \tilde{b}_2^2 = -\tilde{\mathfrak{R}}_{22} - \tilde{\mathfrak{R}}_{23}^2.
$$

We see that $\tilde{b}_2^1$ and $\tilde{b}_2^2$ are the polynomials of $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$, $\hat{\mathbf{Q}}_0$, $\hat{\mathbf{Q}}_1$, $\hat{\mathbf{Q}}_2$ and their derivatives up to the order 2. Using Proposition 2.5, Proposition 2.7 and Proposition 2.9 again, we have

$$
\|\tilde{\mathbf{b}}_2\|_{C^0(\psi^{-1}(\sigma_4))} + \|\operatorname{div}_z\tilde{\mathbf{b}}_2\|_{C^1(\psi^{-1}(\sigma_4))} \le C(\Omega, \mathcal{H}^e, \kappa)\lambda^3.
$$

where σ_4 is defined in (3.1), ψ is defined by (2.1). Thus, we have (3.5) for $x \in \sigma_4$ by the scaling argument.

Step 3. From Proposition 2.5, Proposition 2.7 and Proposition 2.9, we see that, each component of $\tilde{\mathbf{b}}(z, \lambda)$ is a linear combinations of exponentially decaying terms with respect to z_2. Therefore, for $x \in \sigma_2 \backslash \sigma_4$ we also have (3.5) for small λ.

We now finish the proof of (3.5). □

Appendix E: Derivation of the boundary condition (3.8)

To derive the boundary condition of R_f, we use the boundary condition of $\hat{f}_0$ in (5.18), that of $\hat{f}_1$ in (5.19), and that of $\hat{f}_2$ (see (2.22) when $y_1 = 0$), and find

$$\frac{\partial R_f}{\partial \mathbf{n}} = \frac{\partial f}{\partial \mathbf{n}} + \frac{\partial}{\partial z_2}\Big(\hat{f}_0(y_1, z_2) + \lambda \hat{f}_1(y_1, z_2) + \lambda^2 \hat{f}_2(y_1, z_2)\Big)\Big|_{z_2=0} = 0.$$

To derive the boundary condition for $\mathbf{R_Q}$, we first consider the value of $\lambda\,\mathrm{curl}\mathbf{R_Q}$ at $X_0 \in \partial\Omega$. We keep the notation used in section 2. We use equality (C.5) in appendix C with $\mathbf{Q}(x)$ replaced by $\mathbf{Q}_{\mathrm{a}p}(x)$, and use equality (D.3) in appendix D. Then we have

$$\begin{aligned}\lambda\,\mathrm{curl}\,\mathbf{Q}_{\mathrm{a}p}(X_0) = \Big(&-\partial_2 Q_0^1 + \lambda(k_0 Q_0^1 - \partial_2 Q_1^1) + \lambda^2(\partial_{y_1}\hat{Q}_1^2\big|_{y_1=0} + k_0 Q_1^1 - \partial_2 Q_2^1)\\ &+ \lambda^3(k_0 Q_2^1 + \partial_1 Q_2^2)\Big)\Big|_{z_2=0},\end{aligned}$$

where k_0 is the curvature of $\partial\Omega$ at X_0. Then we use the boundary condition of Q_0^1 in (2.14), the boundary condition of (Q_1^1, Q_1^2) in (2.20), and that of (Q_2^1, Q_2^2) in (2.22) to get

$$\lambda\mathrm{curl}\mathbf{Q}_{\mathrm{a}p}(X_0) = \mathcal{H}^e(X_0) + \lambda^3(k_0 Q_2^1 + \partial_1 Q_2^2))\Big|_{z_2=0}.$$

Similarly, for any $x \in \partial\Omega$, we also have

$$\lambda\mathrm{curl}\mathbf{Q}_{\mathrm{a}p}(x) = \mathcal{H}^e(x) - \mathcal{B}_3(x) \quad \text{on } \partial\Omega, \tag{E.1}$$

where

$$\mathcal{B}_3(x) = -\lambda^3\Big(k(x)\hat{Q}_2^1(y_1, z_2) + \partial_1 \hat{Q}_2^2(y_1, z_2)\Big)\Big|_{z_2=0}, \tag{E.2}$$

$k(x)$ is the curvature of $\partial\Omega$ at x, $z_2 = y_2/\lambda$, $x = \psi(y_1, y_2)$ and ψ is defined by (2.1). From Proposition 2.9, we have

$$\|\mathcal{B}_3\|_{C^2(\partial\Omega)} \le C\left(\Omega, \mathcal{H}^e\right)\lambda^3. \tag{E.3}$$

Combining (E.1) with the boundary condition $\lambda\mathrm{curl}\mathbf{Q} = \mathcal{H}^e$ on $\partial\Omega$, we immediately obtain that

$$\lambda\,\mathrm{curl}\mathbf{R_Q} = \mathcal{B}_3 \quad \text{on } \partial\Omega.$$

Now we compute the value of $\mathbf{n}\cdot\mathbf{R_Q}$ on $\partial\Omega$. We first calculate the value of $f_{ap}^2\mathbf{n}\cdot\mathbf{Q}_{ap}(x)$ at $X_0\in\partial\Omega$. From (2.21) and (2.23), we have

$$|f_0|^2Q_1^2\big|_{z_2=0}=\partial_{y_1z_2}\hat{Q}_0^1\big|_{y_1=0,z_2=0}=\hat{\mathcal{H}}_{y_1}^e(0),$$

and

$$\begin{aligned}(|f_0|^2Q_2^2+2f_0f_1Q_1^2)\big|_{z_2=0}&=-(\partial_{z_2y_1}\hat{Q}_1^1-\partial_{y_1}(k\hat{Q}_0^1))\big|_{y_1=0,z_2=0}\\&=-\partial_{y_1}(\partial_{z_2}\hat{Q}_1^1-(k\hat{Q}_0^1))\big|_{y_1=0,z_2=0}=0.\end{aligned}$$

Then

$$\begin{aligned}|f_{ap}|^2\mathbf{n}\cdot\mathbf{Q}_{ap}(X_0)&=-(f_0+\lambda f_1+\lambda^2f_2)^2(\lambda Q_1^2+\lambda^2Q_2^2)\\&=-\lambda\hat{\mathcal{H}}_{y_1}^e(0)+\lambda^3\tilde{\mathfrak{R}}_2(y_1,z_2)\Big|_{y_1=0,z_2=0},\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathfrak{R}}_2(y_1,z_2)=&(f_1^2+2f_0f_2)Q_1^2+2f_0f_1Q_2^2+\lambda\Big(2f_1f_2Q_1^2+(f_1^2+2f_0f_2)Q_2^2\Big)\\&+\lambda^2\Big(f_2^2Q_1^2+2f_1f_2Q_2^2\Big)+\lambda^3f_2^2Q_2^2.\end{aligned}$$

Similarly, for any $x\in\partial\Omega$, we also have

$$f_{ap}^2\mathbf{n}\cdot\mathbf{Q}_{ap}(x)=-\lambda\nabla_{\tan}(\mathcal{H}^e)(x)-\mathcal{B}_4(x),$$

where

$$\mathcal{B}_4(x)=\lambda^3\tilde{\mathfrak{R}}_2(y_1,z_2)\Big|_{z_2=0},\tag{E.4}$$

$z_2=y_2/\lambda$, $x=\psi(y_1,y_2)$, and ψ is defined by (2.1). From Proposition 2.5, Proposition 2.7 and Proposition 2.9, we have

$$\|\mathcal{B}_4\|_{C^2(\partial\Omega)}\le C\left(\Omega,\mathcal{H}^e\right)\lambda^3.\tag{E.5}$$

From the second and the third equations in (1.1), it follows that

$$f^2\mathbf{n}\cdot\mathbf{Q}=-\lambda^2\mathbf{n}\cdot\mathrm{curl}^2\mathbf{Q}=-\lambda\nabla_{\tan}(\lambda\mathrm{curl}\mathbf{Q})=-\lambda\nabla_{\tan}(\mathcal{H}^e).$$

This gives that

$$\nu\cdot\mathbf{R_Q}=f_{ap}^{-2}\left[\mathcal{B}_4+\lambda|f|^{-2}(|f|^2-|f_{ap}|^2)\nabla_{\tan}(\mathcal{H}^e)\right]:=\mathcal{B}_5.\tag{E.6}$$

Summarizing, we obtain the boundary conditions (3.8) for system (3.7).

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