Mathematische Annalen (2023) 387:133–143
https://doi.org/10.1007/s00208-022-02459-9

Mathematische Annalen

On extra-special Enriques surfaces

Gebhard Martin[1] · Giacomo Mezzedimi[2] · Davide Cesare Veniani[3]

Received: 24 January 2022 / Revised: 26 April 2022 / Accepted: 3 August 2022 /
Published online: 30 August 2022

Abstract
We refine Cossec and Dolgachev's classification of extra-special Enriques surfaces, providing a complete and concise proof.

Mathematics Subject Classification 14J28 · 14C20

1 Introduction

Throughout the paper, X will denote an Enriques surface defined over an algebraically closed field of arbitrary characteristic p.

A *half-fiber* on X is a divisor F such that $2F$ is a fiber of a genus one fibration. A *c-sequence* on X is a sequence of half-fibers $(F_1, \dots, F_c)$ such that $F_i.F_j = 1 - \delta_{ij}$. As observed by Enriques himself for classical Enriques surfaces, and later extended to non-classical Enriques surfaces by Bombieri and Mumford [1, Theorem 3], every Enriques surface admits a 1-sequence. The maximal length of a c-sequence is 10.

It is a natural question to ask whether every c-sequence can be extended to a c'-sequence with $c' > c$. In this context, *extra-special* Enriques surfaces play a fundamental role.

✉ Giacomo Mezzedimi
mezzedimi@math.uni-hannover.de

Gebhard Martin
gmartin@math.uni-bonn.de

Davide Cesare Veniani
davide.veniani@mathematik.uni-stuttgart.de

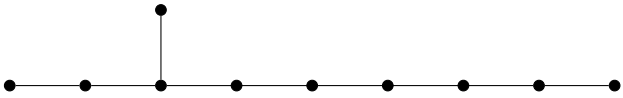
1 Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

2 Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

3 Institut für Diskrete Strukturen und Symbolisches Rechnen, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

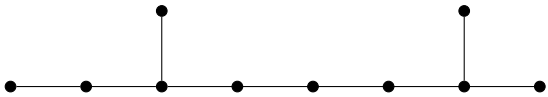
Definition 1.1 An Enriques surface X is called

- *extra-special of type* $\tilde{E}_8$ if the dual graph of all (-2)-curves on X is



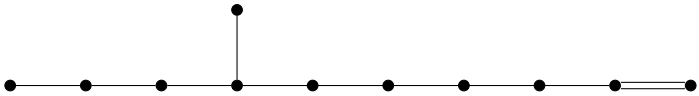
(A)

- *extra-special of type* $\tilde{D}_8$ if the dual graph of all (-2)-curves on X is



(B)

- *extra-special of type* $\tilde{E}_7$ if the dual graph of all (-2)-curves on X is



(C)

An Enriques surface is called *extra-special* if it is extra-special of type $\tilde{E}_8$, $\tilde{D}_8$, or $\tilde{E}_7$.

Extra-special Enriques surfaces do exist. Salomonsson [13] gave equations for all extra-special surfaces of type $\tilde{E}_8$ and $\tilde{E}_7$. An alternative construction of these surfaces as well as examples of extra-special surfaces of type $\tilde{D}_8$ were given by Katsura, Kondō and Martin [9, §§10, 11, 12]. A description of all genus one fibrations on these surfaces can be found in [7, Proposition 6.2.7].

The classification of Enriques surfaces on which every 1-sequence can be extended to a 2-sequence is well-known and is contained in the following theorem due to Cossec and Dolgachev.

Theorem 1.2 ([3, Theorem 3.4.1] or [4, Theorem 6.1.10]) *An Enriques surface X is not extra-special of type $\tilde{E}_8$ if and only if every 1-sequence on X extends to a 2-sequence.*

More precisely, on extra-special surfaces of type $\tilde{E}_8$ there exists only one genus one fibration, so the only 1-sequence is non-extendable.

The main result of this paper is the following theorem, whose proof is obtained by combining Theorem 3.4 and Corollary 4.3.

Theorem 1.3 *An Enriques surface X is not extra-special if and only if every c-sequence with $c \leq 2$ on X extends to a 3-sequence.*

More precisely, on extra-special surfaces of type $\tilde{D}_8$ there are exactly three genus one fibrations, giving rise to two distinct 2-sequences, both non-extendable. On extra-special surfaces of type $\tilde{E}_7$ there are exactly two genus one fibrations, giving rise to a non-extendable 2-sequence.

Remark 1.4 In their first book about Enriques surfaces, Cossec and Dolgachev proved that X admits a 3-sequence if X is not extra-special [3, Theorem 3.5.1]. In a similar vein, Cossec proved that if $p \neq 2$, then every 1-sequence on X extends to a 3-sequence [2, Theorem 3.5]. Note that our result is strictly stronger than both of these results, since it asserts the extendability of any given 2-sequence in all characteristics. Big parts of the proof of the result of Cossec are left to the 'patient reader'. Moreover, the proof of the result of Cossec–Dolgachev occupies 32 pages and is described as 'lengthy' in the new book on Enriques surfaces, with 'no guarantee that it is correct' (see [7, Remark 6.1.14 and Theorem 6.2.6]). The main ingredient that allows us to give a substantially shorter proof of a stronger result is the observation that in a non-extendable 2-sequence one of the genus one fibrations is special with a fiber of type II^* (Theorem 3.4), and that Enriques surfaces with such a fibration can be easily classified in all characteristics (Theorem 4.2).

Remark 1.5 Every extra-special surface admits a genus one fibration with a half-fiber of additive type, and an extra-special surface of type $\tilde{E}_7$ even admits a quasi-elliptic fibration with two half-fibers [7, Proposition 6.2.7]. Consequently (cf. Lemma 2.2), extra-special Enriques surfaces can exist only in characteristic 2 and they are either classical or supersingular. Additionally, extra-special surfaces of type $\tilde{E}_7$ can only be classical.

We infer an immediate corollary from Remark 1.5.

Corollary 1.6 *If one of the following conditions holds:*

1. $p \neq 2$,
2. $p = 2$ *and* X *is ordinary,*
3. $p = 2$ *and* X *is not extra-special,*

then every c-sequence on X *with* $c \leq 2$ *extends to a* 3*-sequence.*

Let us describe one of the geometric consequences of the extendability of 1- and 2-sequences to 3-sequences. By applying [4, Theorem 3.5.1 and the following discussion] and [5, Section 7.8.1] to a 3-sequence that extends a given 2-sequence (F_1, F_2), we obtain the following strong version of the Enriques–Artin Theorem for non-extra-special, classical Enriques surfaces.

Corollary 1.7 *If* X *is a classical Enriques surface which is not extra-special, then* X *is the minimal resolution of a sextic* $S \subseteq \mathbb{P}^3$ *given by an equation of the form*

$$x_1x_2x_3LQ + x_1^2x_2^2L^2 + x_1^2x_3^2L^2 + x_2^2x_3^2L^2 + x_1^2x_2^2x_3^2,$$

where L *is linear and* Q *is a quadratic form. Moreover, for each half-fiber* F_1 *(resp.* 2*-sequence* (F_1, F_2)*) on* X*, there is a sextic model as above such that* F_1 *(resp.* (F_1, F_2)*) maps to a line (resp. a pair of lines) in the non-normal locus of* S*.*

Extending 3-sequences to 4-sequences is a much more difficult problem, even in characteristic $p \neq 2$. The classification of non-extendable 3-sequences is the subject of our follow-up article [11].

2 Preliminaries

Recall that an *Enriques surface* is a smooth and proper surface X of Kodaira dimension 0 with $b_2(X) = 10$ over an algebraically closed field of arbitrary characteristic p. An Enriques surface X is called *classical* if its canonical divisor is not linearly equivalent to 0. A non-classical Enriques surface X (which can only exist if $p = 2$) is called *ordinary* (or *singular* or a μ_2-*surface*) if the absolute Frobenius morphism is bijective on $H^1(X, \mathcal{O}_X)$, and *supersingular* (or an α_2-*surface*) otherwise.

A *genus one fibration* on X is a fibration $f\colon X \to \mathbb{P}^1$ such that the generic fiber X_η is a regular genus one curve. A genus one fibration is called *elliptic* if X_η is smooth and *quasi-elliptic* otherwise. We will use Kodaira's notation for singular fibers. Singular fibers of type I_n are called of *multiplicative* type, while all others are called of *additive* type. Any genus one fibration on X admits at least one *half-fiber*, that is a curve F of arithmetic genus one such that $2F$ is a fiber of the fibration. In particular, the degree of a multisection of a genus one fibration f on X is divisible by 2 and if f admits a (-2)-curve as a bisection, both f and the bisection are called *special*. A non-multiple fiber of a genus one fibration is called *simple*.

By a *component* of a divisor D on X, we always mean an irreducible component and we call a component *simple*, if it is reduced, that is, if it has multiplicity 1 in D. We call D *primitive* if its class in $\mathrm{Num}(X)$ spans a primitive sublattice.

Lemma 2.1 ([4, Corollary 2.2.9]) *An effective divisor D on X is a half-fiber of a genus one fibration on X if and only if D is nef, primitive, and $D^2 = 0$.*

We recall here the behaviour of genus one fibrations on Enriques surfaces.

Lemma 2.2 ([4, Theorem 4.10.3]) *Let $f\colon X \to \mathbb{P}^1$ be a genus one fibration on X.*

- *If $p \neq 2$, then f is an elliptic fibration with two half-fibers, and each of them is either non-singular or singular of multiplicative type.*
- *If $p = 2$ and X is classical, then f is an elliptic or quasi-elliptic fibration with two half-fibers, and each of them is either an ordinary elliptic curve or a singular curve of additive type.*
- *If $p = 2$ and X is ordinary, then f is an elliptic fibration with one half-fiber, which is either a non-singular ordinary elliptic curve or a singular curve of multiplicative type.*
- *If $p = 2$ and X is supersingular, then f is an elliptic or quasi-elliptic fibration with one half-fiber, which is either a supersingular elliptic curve or a singular curve of additive type.*

The symbol $\sim$ denotes linear equivalence, and W_X^{nod} denotes the Weyl group generated by reflections in classes of (-2)-curves on X.

Lemma 2.3 ([3, Theorem 3.2.1] or [4, Theorem 2.3.3]) *If D is an effective divisor on X with $D^2 \geq 0$, then there exist non-negative integers a_i and (-2)-curves R_i such that*

$$D \sim D' + \sum_i a_i R_i,$$

where D' is the unique nef divisor in the W_X^{nod}-orbit of D. In particular, $D^2 = D'^2$.

The following bound is an immediate consequence of the Shioda–Tate formula.

Lemma 2.4 *The number of irreducible curves contained in s fibers of any genus one fibration on X is at most $8 + s$.*

A *c-degenerate (canonical isotropic) n-sequence on X* is an n-tuple of the form

$$\Big(F_1, F_1 + R_{1,1}, \ldots, F_1 + \sum_{j=1}^{m_1} R_{1,j}, F_2, F_2 + R_{2,1}, \ldots, F_c + \sum_{j=1}^{m_c} R_{c,j}\Big),$$

where the F_i are half-fibers of genus one fibrations on X and the $R_{i,j}$ are (-2)-curves satisfying the conditions

1. $F_i.F_j = 1 - \delta_{ij}$.
2. $R_{i,j}.R_{i,j+1} = 1$.
3. $R_{i,j}.R_{k,l} = 0$ unless $(k,l) = (i,j)$ or $(k,l) = (i, j \pm 1)$.
4. $F_i.R_{i,1} = 1$ and $F_i.R_{k,l} = 0$ if $(k,l) \neq (i,1)$.

If $c = n$, we simply call the above sequence a *c-sequence*.

We say that a c-degenerate n-sequence *extends* to a c'-degenerate n'-sequence if the former is contained in the latter, disregarding the ordering. The following fundamental theorem is due to Cossec and holds also in positive characteristic (cf. [7, Proposition 6.1.7]).

Theorem 2.5 ([2, Lemma 1.6.1, Theorem 3.3]) *If $n \neq 9$, then every c-degenerate n-sequence on X can be extended to a c'-degenerate 10-sequence for some $c' \geq c$.*

A generic Enriques surface does not contain any (-2)-curve. Thus, by Theorem 2.5 every c-sequence with $c \neq 9$ can be extended to a 10-sequence.

Lemma 2.6 ([4, Lemma 2.6.3] or [8, Lemma 3.5]) *If (F_1, F_2) is a 2-sequence, then F_1 and F_2 do not have common irreducible components.*

3 Non-extendable 2-sequences

In this section, we investigate the extendability of 2-sequences and the properties of non-extendable 2-sequences.

Proposition 3.1 *Let (F_1, F_2) be a 2-sequence on an Enriques surface X. If there are two simple fibers $G_1 \in |2F_1|$ and $G_2 \in |2F_2|$ with a common irreducible component, then there is a half-fiber F_3 extending (F_1, F_2) to a 3-sequence.*

Proof By [4, Proposition 2.6.1], the linear system $|F_1 + F_2|$ is a pencil of curves without fixed components. Let R be a common component of G_1 and G_2. Then, R is a (-2)-curve with $R.(F_1 + F_2) = 0$. Hence, there exists a curve $C \in |F_1 + F_2|$ containing R. Set $C' := C - R$. Then, $C'.F_i = (F_1 + F_2 - R).F_i = 1$ and $C'^2 = (C - R)^2 =$

$(F_1+F_2)^2-2(F_1+F_2).R+R^2=0$. Using Lemma 2.3, we write $C'=C''+\sum_i a_i R_i$ with $a_i \geq 0$, C'' nef and $C''^2=C'^2=0$. Then, $C''.F_i=C'.F_i-\sum a_j R_j.F_i \leq 1$.

Since C'' is nef, $C''.F_i=0$ for at most one i and then $C''.F_j=1$ for $j \neq i$. This implies that C'' is primitive, and therefore C'' is a half-fiber of some genus one fibration on X by Lemma 2.1. Hence, if $C''.F_1=C''.F_2=1$, then (F_1, F_2, C'') is a 3-sequence.

Thus, let us exclude the possibility that $C''.F_1=0$ (the case $C''.F_2=0$ is analogous). In this case, C'' is a half-fiber of $|2F_1|$. If $C''=F_1$, then

$$F_2 \sim \left(F_1+F_2-C''\right) \sim (F_1+F_2-C)+\left(R+\sum a_i R_i\right) \sim \left(R+\sum a_i R_i\right).$$

Since $|F_2|$ is 0-dimensional, this shows that R is a component of F_2. But this is impossible, since we assumed that R is a component of a simple fiber $G_2 \in |2F_2|$. If $C'' \neq F_1$, then the above argument applied to $|F_2+K_X|$ shows that R is a component of the other half-fiber of $|2F_2|$, again contradicting the fact that R is contained in the simple fiber $G_2 \in |2F_2|$. □

Remark 3.2 Let us explain what happens geometrically in the situation of Proposition 3.1. We will restrict ourselves to characteristic $p \neq 2$ and leave it to the motivated reader to formulate the analogous picture in characteristic $p=2$. We will use the notation of Proposition 3.1 and refer to [4, Section 3.3] for details on bielliptic models of Enriques surfaces.

The linear system $|2F_1+2F_2|$ induces morphisms $X \xrightarrow{\varphi_1} X' \xrightarrow{\varphi_2} \mathsf{D}$, where φ_1 is birational, and φ_2 is 2-to-1 onto a 4-nodal quartic symmetroid del Pezzo surface D. The surface D contains a quadrangle of lines whose vertices are the nodes, and each of the four half-fibers maps to one of the lines. Since R is not contained in a half-fiber and $R.(2F_1+2F_2)=0$, the curve R is contracted by φ_1 and maps to a singular point P of the branch curve of φ_2 outside the quadrangle of lines. There are four pencils of conics on D and their preimages on X are the pencils $|2F_1|$, $|2F_2|$, $|F_1+F_2|$, and $|F_1+F_2+K_X|$. The proof of Proposition 3.1 shows that the preimage $|F_1+F_2|$ of one of the pencils of conics that contains two lines, which is a pencil of curves of genus 2 on X, has a reducible member that contains the half-fiber F_3.

Another way of seeing this more geometrically was suggested by the referee: the linear system $|2F_1+2F_2-R|$ induces morphisms $X \xrightarrow{\psi_1} X' \xrightarrow{\psi_2} \mathsf{C}$, where ψ_1 is birational, ψ_2 is 2-to-1, and C is the blow-up of D in P. Since P does not lie on a line, C is isomorphic to a cubic surface with four nodes, hence it is the Cayley cubic surface. The surface C contains exactly nine lines, and we can describe them explicitly in terms of D: four of them are the strict transforms of the four lines on D, four of them are the strict transforms of the four conics on D that pass through P, and one of them is the exceptional curve E over P. Six of the nine lines form a tetrahedron, and the other three lie on a plane that intersects the tetrahedron away from the vertices. The three pencils of residual conics that we obtain from the latter three lines give rise to the genus one fibrations $|2F_1|$, $|2F_2|$, and $|2F_3|$, and the preimages of the former six lines are the six half-fibers. We refer the reader to our follow-up article [11] for a closer study of the so-called *special* 3-sequences that give rise to this geometric situation.

Corollary 3.3 *If (F_1, F_2) is a 2-sequence on X that does not extend to a 3-sequence, then one of the half-fibers of $|2F_1|$ or $|2F_2|$ is reducible and has at least 4 irreducible components.*

Proof By Theorem 2.5, after possibly interchanging F_1 and F_2, we may assume that there exist (-2)-curves $R_{1,1}, \ldots, R_{1,m}, R_{2,1}, \ldots, R_{2,n}$ with $m + n = 8$, $m \geq 4$, and such that $(F_1, F_1 + R_{1,1}, \ldots, F_2, F_2 + R_{2,1}, \ldots)$ is a 2-degenerate 10-sequence. Since $R_{1,2}.F_1 = R_{1,2}.F_2 = 0$, $R_{1,2}$ is a component of two fibers $G_1 \in |2F_1|$ and $G_2 \in |2F_2|$. By Proposition 3.1, G_1 and G_2 cannot both be simple. Therefore, $R_{1,2}$ is a component of a reducible half-fiber, which also contains all the other $R_{1,j}$ with $j \neq 1$. Hence, one of the reducible half-fibers has at least 4 components. □

The bound of Corollary 3.3 on the number of irreducible components is not sharp, but it is enough to prove the following theorem, which is the key result of this paper.

Theorem 3.4 *If (F_1, F_2) is a 2-sequence on X that does not extend to a 3-sequence, then either $|2F_1|$ or $|2F_2|$ is a special genus one fibration with a fiber of type* II*.

Proof By Corollary 3.3 we can suppose that F_1 is reducible and has at least 4 components. Let $R_{2,1}$ be the component of F_1 meeting F_2. Since $R_{2,1}$ is a simple component of F_1, we can find two more components $R_{2,2}$, $R_{2,3}$ of F_1 forming a chain with $R_{2,1}$.

Applying Theorem 2.5, we extend the 2-degenerate 5-sequence $(F_1, F_2, \ldots, F_2 + R_{2,1} + R_{2,2} + R_{2,3})$ to a 2-degenerate 10-sequence $(F_1, \ldots, F_1 + \sum_{i=1}^{m} R_{1,i}, F_2, \ldots, F_2 + \sum_{i=1}^{n} R_{2,i})$ with $n \geq 3$. Clearly, all the $R_{2,i}$ are contained in F_1.

Note that $R_{2,1}$ is a special bisection of $|2F_2|$. Since $F_2.F_1 = F_2.R_{2,1} = 1$ and $F_1 - R_{2,1}$ is connected ($R_{2,1}$ being a simple component), all components of F_1 except $R_{2,1}$ are contained in a fiber $G_2 \in |2F_2|$. We want to prove that G_2 is of type II*.

The fiber G_2 is necessarily simple because of Lemma 2.6. Let Λ be the sublattice of $\mathrm{Num}(X)$ generated by the components of G_2 and $R_{2,1}$ and let $\Lambda' = \Lambda[F_2]$ be the sublattice of $\mathrm{Num}(X)$ obtained from Λ by adjoining $F_2 = \frac{1}{2}G_2 \in \frac{1}{2}\Lambda$.

If $m \geq 1$, then $R_{1,1}$ meets a component of F_1 distinct from $R_{2,1}$, hence $R_{1,1}$ meets a component of G_2. But $R_{1,1}.G_2 = 2R_{1,1}.F_2 = 0$, so $R_{1,1}$ is contained in G_2 and then so are all the $R_{1,j}$. This shows that Λ has rank 10, because it contains all $R_{i,j}$, the class of F_1 and the class of $G_2 \equiv 2F_2$. Since Λ' contains all the 10 divisors of the 10-sequence and since these divisors generate a lattice of rank 10 and discriminant 9, Λ' has index 1 or 3 in $\mathrm{Num}(X)$.

By [2, Lemma 1.6.2], there exists a vector $e \in \mathrm{Num}(X)$ with $e^2 = 0$, $e.F_1 = e.F_2 = 1$, $e.R_{i,j} = 0$ for all (i, j) except for $e.R_{2,n-1} = 1$, and such that e and the components of the 10-sequence generate $\mathrm{Num}(X)$. (In the notation of [2, Lemma 1.6.2], we can choose $e = e_{9,10} = d - f_9 - f_{10}$, since $n \geq 3$.)

Note that it suffices to show that e is contained in the sublattice Λ'. Indeed, if this holds, then $\Lambda' = \mathrm{Num}(X)$ and therefore Λ has index at most 2 in $\mathrm{Num}(X)$. Then, consider the basis of Λ given by G_2, $R_{2,1}$ and all the components of G_2 except a simple one. The intersection matrix of Λ with respect to this basis is

$$\left(\begin{array}{cc|c} 0 & 2 & 0 \\ 2 & -2 & * \\ \hline 0 & * & L \end{array}\right),$$

where L is a root lattice of rank 8 depending on the type of G_2. More precisely, L is the root lattice associated to the Dynkin diagram obtained by removing a simple component from G_2. By reducing along the first row and first column, we obtain that $\det(\Lambda) = -4\det(L)$. Since Λ has index at most 2 in the unimodular lattice $\mathrm{Num}(X)$, we have $|\det(\Lambda)| \leq 4$, and hence $\det(L) = 1$. This forces L to be isometric to the lattice E_8, and in turn G_2 to be of type II*.

It remains to show that $e \in \Lambda'$. Let D be an effective lift of e to $\mathrm{Pic}(X)$. Write $D \sim D' + \sum_i a_i R_i$ as in Lemma 2.3. Since $D'^2 = D^2 = 0$ and $D'.F_i \leq 1$ with equality for at least one i, D' is nef and primitive, hence equal to a half-fiber by Lemma 2.1. If D' is not a half-fiber of $|2F_1|$ or $|2F_2|$, then (F_1, F_2, D') is a 3-sequence, contradicting our assumption that (F_1, F_2) does not extend to a 3-sequence. Therefore, we can suppose that D' is a half-fiber of $|2F_k|$ for some $k \in \{1, 2\}$.

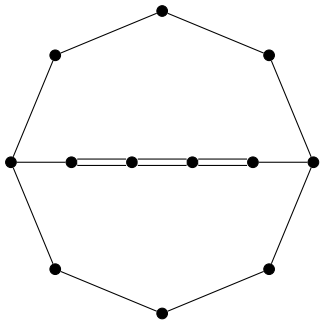
Take $j \in \{1, 2\}$ with $j \neq k$. Then, $D'.F_j = 1$ and $D'.F_k = 0$, while $D.F_j = D.F_k = 1$, so $(\sum_i a_i R_i)^2 = (D-D')^2 = (D-F_k)^2 = -2$ and $(\sum_i a_i R_i).F_j = (D-D').F_j = 0$. Thus, $\sum_i a_i R_i$ is a connected configuration of (-2)-curves contained in a single fiber of $|2F_j|$. But $(\sum_i a_i R_i).R_{2,n-1} = (D - D').R_{2,n-1} = D.R_{2,n-1} = 1$, since $R_{2,n-1}.F_1 = R_{2,n-1}.F_2 = 0$, hence $\sum_i a_i R_i$ is contained in the fiber of $|2F_j|$ containing $R_{2,n-1}$. If $j = 1$, this fiber is F_1, whereas if $j = 2$ this fiber is G_2. In both cases, we obtain that $e \in \Lambda'$. □

4 Special genus one fibrations with a fiber of type II*

By Theorem 3.4, one of the two fibrations in a non-extendable 2-sequence is special with a fiber of type II*. It turns out that admitting such a fibration is such a restrictive property that all Enriques surfaces satisfying this property can be classified. We will carry out this classification in this section, thereby proving Theorem 1.3 (see Corollary 4.3).

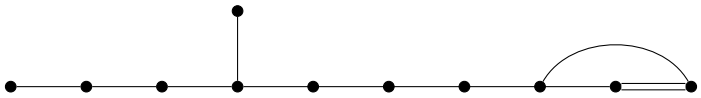
Definition 4.1 An Enriques surface X is called

- *of type* I if the dual graph of all (-2)-curves on X is



(D)

- *of type* $\tilde{E}_7^{(2)}$ if the dual graph of all (-2)-curves on X is



(E)

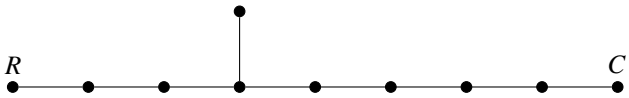
Theorem 4.2 *If an Enriques surface X admits a special genus one fibration with a fiber of type* II**, then one of the following holds:*

1. *X is of type* I*,*
2. *X is of type $\tilde{E}_7^{(2)}$,*
3. *X is extra-special.*

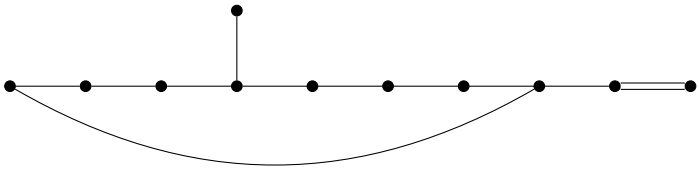
Proof First, note that it suffices to show that X contains a configuration of (-2)-curves whose dual graph is one of the Graphs (A), (B), (C), (D), or (E). Indeed, by [14, Theorem 2.3] and [9, Remark 2.15], the group generated by reflections in the (-2)-curves in the dual graph of one of these surfaces has finite index in the orthogonal group of $\mathrm{Num}(X)$. By [12, Proposition 6.9], this implies that the (-2)-curves in the graph are in fact all the (-2)-curves on a surface containing such a configuration.

Assume first that the fiber of type II* is a half-fiber. Its special bisection must meet the only component of the half-fiber of multiplicity 1, giving rise to Graph (A).

Assume instead that the fiber of type II* is simple. Since this fiber has only one simple component, the special bisection R intersects transversely one of the two components of multiplicity 2 or it intersects doubly the component of multiplicity 1. If R intersects the component of multiplicity 2 next to the simple component, then we obtain Graph (B). If R intersects the other component of multiplicity 2, then we obtain a new half-fiber of type III*:



The component C is orthogonal to the fiber of type III*, hence it belongs to another reducible fiber G of this new fibration. By Lemma 2.4, G must be of type I_2 or III. If G is a half-fiber, then we obtain Graph (C). If instead it is a simple fiber, we obtain Graph (E). Finally, if R intersects doubly the simple component of the fiber of type II*, then we obtain a new half-fiber G of type I_2 or III. The seven components orthogonal to G are contained in a fiber of type III* by Lemma 2.4, which can be a half-fiber or a simple fiber. In the former case, we obtain Graph (C). In the latter, the bisection of the III* fiber also intersects the other simple component:



There is a half-fiber of type I_8, hence, by Lemma 2.2, either $p \neq 2$ or X is ordinary. Thus, it follows from [10, Theorem 3.1] that X is of type I. □

Corollary 4.3 *If X is an Enriques surface with a special genus one fibration with a fiber of type II^*, then X satisfies the following properties:*

1. *X contains finitely many (-2)-curves,*
2. *X has finite automorphism group,*
3. *if $p \neq 2$ or X is ordinary, then X is of type* I,
4. *X is not extra-special if and only if every c-sequence with $c \leq 2$ extends to a 3-sequence.*

Proof Claim (1) follows from the classification in Theorem 4.2, since all the surfaces listed there contain only finitely many (-2)-curves.

By the first paragraph of the proof of Theorem 4.2, the Weyl group W_X^{nod} has finite index in the orthogonal group of $\mathrm{Num}(X)$. Since $\mathrm{Aut}(X)$ acts with finite kernel on $\mathrm{Num}(X)/W_X^{\mathrm{nod}}$ by [8, Theorem], this implies Claim (2) (see also [6, Corollary 3.4]).

For Claim (3), by Lemma 2.2 it suffices to observe that if X is not of type I, then there exists an additive half-fiber on X.

Finally, for Claim (4), one can simply list all genus one fibrations on X. For the surfaces of type I, we refer to [7, Proposition 8.9.6]. For the extra-special surfaces and for the surfaces of type $\tilde{E}_7^{(2)}$, we refer to [7, Proposition 6.2.5]. □

Remark 4.4 Enriques surfaces of type I and of type $\tilde{E}_7^{(2)}$ also display a special behaviour with respect to c-sequences: on both types of surfaces, there exist 3-sequences that cannot be extended to 4-sequences. In fact, there are no 4-sequences at all on type $\tilde{E}_7^{(2)}$ [7, Proposition 6.2.5].

Acknowledgements The authors would like to thank Igor Dolgachev, Matthias Schütt, and the referee for helpful comments on a first draft of this paper.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Bombieri, E., Mumford, D.: Enriques' classification of surfaces in char, p, III. Invent. Math. **35**, 197–232 (1976)
2. Cossec, F.: On the Picard group of Enriques surfaces. Math. Ann. **271**(4), 577–600 (1985)
3. Cossec, F., Dolgachev, I.: Enriques Surfaces. I, Progress in Mathematics. Birkhäuser Boston, Inc, Boston (1989)
4. Cossec, F., Dolgachev, I., Liedtke, C.: Enriques surfaces I, with an appendix by Shigeyuki Kondō. www.math.lsa.umich.edu/idolga/EnriquesOne.pdf (2021)
5. Cossec, F.R.: Projective models of Enriques surfaces. Math. Ann. **265**(3), 283–334 (1983)
6. Dolgachev, I.: On automorphisms of Enriques surfaces. Invent. Math. **76**(1), 163–177 (1984)
7. Dolgachev, I., Kondō, S.: Enriques surfaces II. www.math.lsa.umich.edu/idolga/EnriquesTwo.pdf (2021)
8. Dolgachev, I., Martin, G.: Numerically trivial automorphisms of Enriques surfaces in characteristic 2. J. Math. Soc. Jpn. **71**(4), 1181–1200 (2019)
9. Katsura, T., Kondō, S., Martin, G.: Classification of Enriques surfaces with finite automorphism group in characteristic 2. Algebr. Geom. **7**(4), 390–459 (2020)
10. Martin, G.: Enriques surfaces with finite automorphism group in positive characteristic. Algebr. Geom. **6**(5), 592–649 (2019)
11. Martin, G., Mezzedimi, G., Veniani, D.C.: Enriques surfaces of non-degeneracy 3. arXiv:2203.08000 (2022)
12. Namikawa, Y.: Periods of Enriques surfaces. Math. Ann. **270**(2), 201–222 (1985)
13. Salomonsson, P.: Equations for some very special Enriques surfaces in characteristic two. arXiv:math/0309210 (2003)
14. Vinberg, È.B.: Some Arithmetical Discrete Groups in Lobačevskiĭ Spaces, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), pp. 323–348. Oxford University Press, Bombay (1975)