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Global analytic hypoellipticity of involutive systems on compact manifolds

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Abstract

Given M a compact, connected and orientable, real-analytic manifold, and closed, real-valued, real-analytic 1-forms $\omega_1, \ldots, \omega_m$ on M, we characterize the global analytic hypoellipticity of the first operator featuring in the differential complex over $M \times \mathbb{T}^m$ naturally associated to an involutive system of vector fields determined by them. Global Gevrey hypoellipticity is determined simultaneously.

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1 Introduction

One of the foremost models of systems of linear PDEs is that of the so-called *tube structures*, whose global properties have long attracted the attention of several researchers; see e.g. [1–6, 8, 9, 11] as well as further references therein and subsequent works.

A straightforward way to define a corank m tube structure goes as follows: given a compact manifold M and closed 1-forms $\omega_1, \ldots, \omega_m$ on M, we look at the product

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manifold $M \times \mathbb{T}^m$ and the sub bundle $\mathcal{V} \subset \mathbb{C}T(M \times \mathbb{T}^m)$ annihilated by all the forms $\zeta_k \doteq \mathrm{d}x_k - \omega_k$, $k = 1, \ldots, m$; here $(x_1, \ldots, x_m)$ denote standard angular coordinates on the torus $\mathbb{T}^m$. Such a bundle is then involutive (in the sense of Frobenius) and therefore gives rise to a complex of first-order differential operators on $M \times \mathbb{T}^m$ [7, 17], whose first operator $\mathbb{L}$ maps (complex-valued) functions to 1-forms via the expression

$$\mathbb{L} f \doteq \mathrm{d}_t f + \sum_{k=1}^{m} \omega_k \wedge (\partial_{x_k} f), \tag{1.1}$$

where $t \in M$, $x \in \mathbb{T}^m$ and d_t stands for the exterior derivative on M.

The question of determining *global hypoellipticity* of $\mathbb{L}$ is then of interest:

$$u \in \mathcal{D}'(M \times \mathbb{T}^m) \text{ and } \mathbb{L}u \in \Lambda^1 C^\infty(M \times \mathbb{T}^m) \Longrightarrow u \in C^\infty(M \times \mathbb{T}^m), \tag{1.2}$$

which was previously investigated mainly when $m = 1$, or for general corank when M is itself a torus. In the former situation, a classical result [1, Theorem 2.4] yields a complete characterization when $\omega = \omega_1$ is *real*: $\mathbb{L}$ is globally hypoelliptic in $M \times \mathbb{T}^1$ if and only if ω is neither rational nor Liouville – a Diophantine condition that aims to describe how ω can be approximated by rational forms in the Fréchet topology of $\Lambda^1 C^\infty(M)$. This is our motivation and starting point.

In this work, we take M a compact (and for simplicity also connected and orientable) real-analytic manifold and real-analytic 1-forms $\omega_1, \ldots, \omega_m$ which we assume to be real-valued and closed, aiming to characterize, instead, *global analytic hypoellipticity* of $\mathbb{L}$:

$$u \in \mathcal{D}'(M \times \mathbb{T}^m) \text{ and } \mathbb{L}u \in \Lambda^1 C^\omega(M \times \mathbb{T}^m) \Longrightarrow u \in C^\omega(M \times \mathbb{T}^m).$$

Besides some necessary conceptual adjustments for treating the case of arbitrary corank m, the actual difficulties arise from the more delicate nature of the spaces of real-analytic functions and forms. In Sect. 2, we define properly their natural (locally convex) topologies on a general compact, real-analytic manifold: from a functional analytic perspective, such topologies turn them in what one calls DFS spaces, which are well-understood [12] but have properties rather diverse than, say, Fréchet spaces. This fact is crucial to understand the structure of their bounded sets – a notion that is instrumental in the very definition of the number theoretic conditions (Definition 3.2) we impose on the system $\boldsymbol{\omega} \doteq (\omega_1, \ldots, \omega_m)$.

The advantage of such an abstract approach is that we can abstain from adding any further hypotheses to M (such as the existence of global frames) and also make our definitions inherently coordinate-free; nevertheless, all of them admit characterizations by means of local data computed via suitable norms (which we actually needed in our proofs).

We prove our results in the more general framework of Gevrey classes of order $s \geq 1$ (the real-analytic case corresponds to $s = 1$) whose definition and essential properties we recall in Sect. 2 and along the way as needed; we refer the reader to [15] for more details. Next, we investigate the *global s-hypoellipticity* (3.1) of $\mathbb{L}$, whose characterization is the content of our main result (Theorem 3.4). For that matter, we can relax our assumptions and suppose that the 1-forms $\omega_1, \ldots, \omega_m$ are just Gevrey of order s (even though our base manifold M is always assumed real-analytic, mainly for simplicity). Actually, one is tempted to conjecture that our strategy can be carried out in ultradifferentiable settings of Roumieu type.

It must be pointed out that our results were previously obtained when M is a torus $\mathbb{T}^n$ [9, Theorem 8.3], where its strong geometric properties were used: for instance, its parallelizability and the possibility of doing "total" Fourier series (whilst in our case only a partial Fourier series in the x-variable makes sense, see Sect. 5), which in turn enables one to effectively reduce $\mathbb{L}$ to an operator with constant coefficients in $\mathbb{T}^n \times \mathbb{T}^m$; such properties make many technical issues a lot simpler. In the present work, however, we prove our results for a general M—in particular, we do not make use of symmetries or assume the existence of a global frame of vector fields for $\mathcal{V}$.

In [9], even the definition of the correct number theoretic condition is clearer: theirs and ours turn out, however, to be the same (Proposition 4.1); this is achieved through a concrete realization of our abstract Diophantine conditions by means of the so-called matrix of periods of $\boldsymbol{\omega}$ (Sect. 4), which also happens to be critical in obtaining estimates throughout the proofs. Curiously, what plays a role in our conditions is the dimension d of the homology space $H_1(M; \mathbb{R})$, and not the dimension n of M as a manifold, thus revealing their true nature (when $M = \mathbb{T}^n$ these parameters are of course equal).

Moreover, although at a first glance there is no relationship between the definitions of *s-exponential Liouville systems* (Definition 3.2) for different values of s, the fact that these conditions can be read off as inequalities involving the matrix of periods of $\boldsymbol{\omega}$ allows us to compare them, and conclude, for instance, that when $s_1 > s_2$ the global s_1-hypoellipticity of $\mathbb{L}$ implies its global s_2-hypoellipticity (provided the latter makes sense).

Finally, a similar condition can be obtained in classifying (smooth) global hypoellipticity of $\mathbb{L}$, thus generalizing [1, Theorem 2.4] to arbitrary corank. We state it in Sect. 7 (the proofs are omitted since can be easily obtained using our framework). This condition can be encoded in an inequality (7.1) involving the matrix of periods of $\boldsymbol{\omega}$ as well (of "polynomial flavor", as in [8]); which, in turn, imply the Gevrey ones (4.2) by simple comparison. A corollary of this reasoning is the following: if $\mathbb{L}$ is globally hypoelliptic in $M \times \mathbb{T}^m$ then it is also globally s-hypoelliptic in $M \times \mathbb{T}^m$ for every $s \geq 1$ for which $\omega_1, \ldots, \omega_m$ are G^s.

2 Spaces of Gevrey forms on compact manifolds and their topologies

The space $G^s(U)$ of Gevrey functions of order $s \geq 1$ over an open set $U \subset \mathbb{R}^n$ consists of all functions $f \in C^\infty(U)$ such that for each compact set $K \subset U$ one can find constants $C, h > 0$ for which

$$\sup_K |\partial^\alpha f| \leq C h^{|\alpha|} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n.$$

Given $K \subset \mathbb{R}^n$ a regular compact set (i.e. K is the closure of a bounded open set with smooth boundary) and $h > 0$ we define

$$G^{s,h}(K) \doteq \left\{ f \in C^\infty(K)\ ;\ \|f\|_{s,h,K} \doteq \sup_{\alpha \in \mathbb{Z}^n_+} h^{-|\alpha|} \alpha!^{-s} \sup_K |\partial^\alpha f| < \infty \right\}.$$

This is a Banach space with respect to the norm $\| \cdot \|_{s,h,K}$, and for $h_+ > h$ the natural inclusion map $G^{s,h}(K) \hookrightarrow G^{s,h_+}(K)$ is compact, meaning that

$$G^s(K) \doteq \varinjlim_{h>0} G^{s,h}(K)$$

is a so-called *DFS space*.

Now let Ω be a real-analytic manifold. A function $f \in C^\infty(\Omega)$ is said to belong to $G^s(\Omega)$ if given an analytic atlas $\{(U_i, \chi_i)\}_{i \in I}$ of Ω we have that $f \circ \chi_i^{-1} \in G^s(\chi_i(U_i))$ for every $i \in I$; this is a meaningful definition since Gevrey regularity is preserved by composition with real-analytic diffeomorphisms. It is also independent of our choice of the atlas $\{(U_i, \chi_i)\}_{i \in I}$.

When Ω is further assumed to be compact (and, for simplicity, also connected) we endow $G^s(\Omega)$ with a locally convex topology as follows. We select a *finite* analytic atlas $\{(U_i, \chi_i)\}_{i \in I}$ of Ω and regular compact sets $K_i \subset U_i$ whose interiors still cover Ω, and endow $G^s(\Omega)$ with the coarsest topology which makes continuous each one of the linear maps

$$f \in G^s(\Omega) \longmapsto f \circ \chi_i^{-1} \in G^s(\chi_i(K_i)), \quad i \in I.$$

Or, equivalently, the coarsest topology that makes continuous their direct sum

$$f \in G^s(\Omega) \longmapsto (f \circ \chi_i^{-1})_{i \in I} \in \bigoplus_{i \in I} G^s(\chi_i(K_i)), \tag{2.1}$$

where we endow the right-hand side with the (finite) direct sum topology; this is also a DFS space, actually [12, Theorems 9 and 10]

$$\bigoplus_{i \in I} G^s(\chi_i(K_i)) = \varinjlim_{h>0} \bigoplus_{i \in I} G^{s,h}(\chi_i(K_i)). \tag{2.2}$$

Notice that the map (2.1) is injective since the family $\{K_i\}_{i \in I}$ covers Ω, and also has closed range (as one easily checks using (2.2) and [12, Theorem 6']), being therefore a topological isomorphism onto its range (since closed subspaces of DFS spaces are also DFS [12, Theorem 7'] and the Open Mapping Theorem [13, p. 59] applies).

This device allows one to recast the topology on $G^s(\Omega)$ as follows: for each $h > 0$ we define

$$G^{s,h}(\Omega) \doteq \{ f \in G^s(\Omega)\ ;\ f \circ \chi_i^{-1} \in G^{s,h}(\chi_i(K_i)),\ \forall i \in I \}$$

and endow it with the norm

$$\|f\|_{s,h,\Omega} \doteq \sum_{i \in I} \|f \circ \chi_i^{-1}\|_{s,h,\chi_i(K_i)};$$

then $G^s(\Omega) = \varinjlim G^{s,h}(\Omega)$ as the direct limit of an injective sequence of Banach spaces with compact inclusion maps. By [12, Lemma 3 and Theorem 6'] we conclude:

Proposition 2.1 *A subset $B \subset G^s(\Omega)$ is bounded if and only if B is contained in some $G^{s,h}(\Omega)$ and is bounded there. A sequence $\{f_\nu\}_{\nu \in \mathbb{N}}$ converges to zero in $G^s(\Omega)$ if and only if there exists $h > 0$ such that either one of the following equivalent conditions hold:*

(1) $\{f_\nu\}_{\nu \in \mathbb{N}} \subset G^{s,h}(\Omega)$ *and* $\|f_\nu\|_{s,h,\Omega} \to 0$*;*
(2) $\{f_\nu \circ \chi_i^{-1}\}_{\nu \in \mathbb{N}} \subset G^{s,h}(\chi_i(K_i))$ *and* $\|f_\nu \circ \chi_i^{-1}\|_{s,h,\chi_i(K_i)} \to 0$ *for every* $i \in I$.

It follows that the inclusion map $G^s(\Omega) \hookrightarrow C^\infty(\Omega)$ is continuous. Moreover, the topology we endowed $G^s(\Omega)$ with is clearly independent of the coverings employed. Indeed, denote temporarily by τ the topology on $G^s(\Omega)$ defined above. Pick any analytic chart (U_0, χ_0) in Ω and $K_0 \subset U_0$ a compact set, and let $I_0 \doteq I \cup \{0\}$. Then $\{(U_i, \chi_i)\}_{i \in I_0}$ is a new analytic atlas and $\{\mathring{K}_i\}_{i \in I_0}$ is an open covering of Ω. By definition, the topology induced on $G^s(\Omega)$ by this new choice is the coarsest one to make each assignment

$$f \in G^s(\Omega) \longmapsto f \circ \chi_i^{-1} \in G^s(\chi_i(K_i)), \quad i \in I_0,$$

continuous. Denote it by τ_0: since $I \subset I_0$ we conclude that $\tau \subset \tau_0$ by definition, i.e. the identity map $(G^s(\Omega), \tau_0) \to (G^s(\Omega), \tau)$ is continuous, hence a homeomorphism (by the Open Mapping Theorem), meaning that $\tau_0 = \tau$. Proceeding inductively, any finite refinement of our choices yields that same topology τ; since any two initial choices admit a common refinement (namely, their union) we are done.

The same basic construction works on the space of Gevrey sections of any real-analytic vector bundle over Ω, but here we will only deal with the space of *Gevrey 1-forms* $\Lambda^1 G^s(\Omega)$. A smooth 1-form $f \in \Lambda^1 C^\infty(\Omega)$ can be written on each coordinate patch U_i as

$$f = \sum_{j=1}^{n} f_{ij} \, \mathrm{d}\chi_{ij}, \quad f_{ij} \in C^\infty(U_i),$$

where $\chi_i = (\chi_{i1}, \dots, \chi_{in}) : U_i \to \mathbb{R}^n$. Since each χ_i is a real-analytic map, we have that $f \in \Lambda^1 G^s(\Omega)$ if and only if $f_{ij} \circ \chi_i^{-1} \in G^s(\chi_i(U_i))$ for each $j \in \{1, \dots, n\}$ and $i \in I$.If

$$f_i \doteq (f_{i1}, \dots, f_{in}) : U_i \longrightarrow \mathbb{R}^n$$

then the condition above reduces to: $f_i \circ \chi_i^{-1} \in G^s(\chi_i(U_i))^n$ for every $i \in I$. We put on $G^s(\chi_i(K_i))^n$ the (finite) product topology, which turns it into a DFS space in the same manner as above; actually

$$G^s(\chi_i(K_i))^n = \varinjlim_{h>0} G^{s,h}(\chi_i(K_i))^n,$$

where $G^{s,h}(\chi_i(K_i))^n$ is a Banach space with norm

$$(g_1, \dots, g_n) \in G^{s,h}(\chi_i(K_i))^n \longmapsto \sum_{j=1}^{n} \|g_j\|_{s,h,\chi_i(K_i)}.$$

We endow $\Lambda^1 G^s(\Omega)$ with the coarsest topology that makes the linear map

$$f \in \Lambda^1 G^s(\Omega) \longmapsto (f_i \circ \chi_i^{-1})_{i \in I} \in \bigoplus_{i \in I} G^s(\chi_i(K_i))^n$$

continuous: again, this is the locally convex injective limit of the Banach spaces

$$\Lambda^1 G^{s,h}(\Omega) \doteq \{f \in \Lambda^1 G^s(\Omega)\ ;\ f_i \circ \chi_i^{-1} \in G^{s,h}(\chi_i(K_i))^n,\ \forall i \in I\}, \quad h > 0,$$

where the norm is defined, say, by

$$\|f\|_{s,h,\Omega} \doteq \sum_{i \in I} \sum_{j=1}^{n} \|f_{ij} \circ \chi_i^{-1}\|_{s,h,\chi_i(K_i)},$$

hence turning $\Lambda^1 G^s(\Omega)$ into a DFS space. One then easily derives the following criteria for boundedness and convergence of sequences there.

Proposition 2.2 *A subset $B \subset \Lambda^1 G^s(\Omega)$ is bounded if and only if there exists constants $C, h > 0$ such that*

$$\|f_{ij} \circ \chi_i^{-1}\|_{s,h,\chi_i(K_i)} \leq C,$$

for every $f \in B$, every $i \in I$ and $j \in \{1, \dots, n\}$. A sequence $\{f_\nu\}_{\nu \in \mathbb{N}}$ converges to zero in $\Lambda^1 G^s(\Omega)$ if and only if there exists $h > 0$ such that for every $i \in I$ and $j \in \{1, \dots, n\}$ we have $\{(f_\nu)_{ij} \circ \chi_i^{-1}\}_{\nu \in \mathbb{N}} \subset G^{s,h}(\chi_i(K_i))$ and $\|(f_\nu)_{ij} \circ \chi_i^{-1}\|_{s,h,\chi_i(K_i)} \to 0$.

Again, one deduces that the inclusion map $\Lambda^1 G^s(\Omega) \hookrightarrow \Lambda^1 C^\infty(\Omega)$ is continuous and that the topology on $\Lambda^1 G^s(\Omega)$ just introduced is independent of the coverings chosen.

3 A class of real differential operators

Let M be a compact real-analytic manifold, which for simplicity we further assume to be connected and oriented. Given a system $\omega_1, \dots, \omega_m$ of real, closed 1-forms belonging to $\Lambda^1 G^s(M)$, our main purpose is to study *global s-hypoellipticity* of the differential operator $\mathbb{L}$ as defined in (1.1), by which we mean

$$u \in \mathcal{D}'(M \times \mathbb{T}^m) \text{ and } \mathbb{L}u \in \Lambda^1 G^s(M \times \mathbb{T}^m) \Longrightarrow u \in G^s(M \times \mathbb{T}^m). \quad (3.1)$$

Our classification will be in terms of properties of the system $\boldsymbol{\omega} \doteq (\omega_1, \dots, \omega_m)$. Let us first recall [1, Definition 2.1]:

Definition 3.1 A real 1-form $\alpha \in \Lambda^1 C^\infty(M)$ is *integral* if $\mathrm{d}\alpha = 0$ and $\int_\sigma \alpha \in 2\pi\mathbb{Z}$ for every 1-cycle σ in M. It is otherwise *rational* if $q\alpha$ is integral for some $q \in \mathbb{Z}\backslash\{0\}$.

Definition 3.2 We say that $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ is:

(1) a *rational system* if there exists$\xi \in \mathbb{Z}^m\backslash\{0\}$ such that

$$\xi \cdot \boldsymbol{\omega} \doteq \sum_{k=1}^{m} \xi_k \omega_k$$

is an integral 1-form i.e.

$$\frac{1}{2\pi} \int_\sigma \xi \cdot \boldsymbol{\omega} \in \mathbb{Z} \quad (3.2)$$

for every 1-cycle σ in M.

(2) an *s-exponential Liouville system* if $\boldsymbol{\omega}$ is not rational and there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 G^s(M; \mathbb{R})$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m$ such that $|\xi_\nu| \to \infty$ and

$$\{e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu)\}_{\nu\in\mathbb{N}} \text{is bounded in} \Lambda^1 G^s(M). \quad (3.3)$$

Remark 3.3 The reader should notice that $\boldsymbol{\omega}$ can be a rational system even if no ω_k is a rational form in the sense of Definition 3.1. Moreover, the conditions set forth in Definition 3.2 depend only on the cohomology classes of $\omega_1, \dots, \omega_m$ in $H^1(M; \mathbb{R})$. In fact, suppose that $\boldsymbol{\omega}^\bullet \doteq (\omega_1^\bullet, \dots, \omega_m^\bullet)$ is another m-tuple of real, closed 1-forms in $\Lambda^1 G^s(M)$ such that $[\omega_k^\bullet] = [\omega_k]$ in $H^1(M; \mathbb{R})$ for every $k \in \{1, \dots, m\}$, i.e. there exist $g_k \in C^\infty(M; \mathbb{R})$ such that $\omega_k^\bullet = \omega_k + \mathrm{d}g_k$ (hence g_k are *a posteriori* G^s). It is then clear that $\xi \cdot \boldsymbol{\omega}^\bullet$ and $\xi \cdot \boldsymbol{\omega}$ are in the same cohomology class for every $\xi \in \mathbb{Z}^m$, in particular their integrals over an 1-cycle are the same; moreover, $\xi_\nu \cdot \boldsymbol{\omega}^\bullet - \theta_\nu^\bullet = \xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu$ provided we let

$$\theta_\nu^\bullet \doteq \theta_\nu - \sum_{k=1}^{m} (\xi_\nu)_k \, \mathrm{d}g_k$$

which is obviously integral if so is θ_ν.

We are ready to state our main result. The next sections are dedicated to prove it.

Theorem 3.4 *Let $\omega_1, \ldots, \omega_m \in \Lambda^1 G^s(M)$ be real and closed. The operator $\mathbb{L}$ defined in (1.1) is globally s-hypoelliptic if and only if $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ is neither a rational system nor an s-exponential Liouville system.*

4 The matrix of periods

The notions established in Definition 3.2 admit a more concrete characterization. Fix

$$\begin{aligned} &\sigma_1, \ldots, \sigma_d \ 1-\text{cycles in } M \\ &\quad \text{whose classes in } H_1(M;\mathbb{Z}) \text{ form a basis of its free part} \end{aligned} \tag{$\sharp$}$$

and regard them as a real basis of $H_1(M;\mathbb{R})$. We may assume that these cycles are smooth (or even real-analytic [16, Theorem 5]). To $\boldsymbol{\omega}$ (as defined in the previous section) we then assign a *matrix of periods* as follows: define $\mathsf{A}(\boldsymbol{\omega}) \in \mathsf{M}_{d\times m}(\mathbb{R})$ by

$$\mathsf{A}(\boldsymbol{\omega})_{\ell k} \doteq \frac{1}{2\pi} \int_{\sigma_\ell} \omega_k, \quad \ell \in \{1, \ldots, d\},\ k \in \{1, \ldots, m\},$$

that is[1]

$$\mathsf{A}(\boldsymbol{\omega})\xi = \frac{1}{2\pi} \left(\int_{\sigma_1} \xi \cdot \boldsymbol{\omega}, \ldots, \int_{\sigma_d} \xi \cdot \boldsymbol{\omega} \right), \quad \xi \in \mathbb{Z}^m. \tag{4.1}$$

Again, the definition of $\mathsf{A}(\boldsymbol{\omega})$ clearly depends only on the classes $[\omega_1], \ldots, [\omega_m] \in H^1(M;\mathbb{R})$ and we therefore have a linear map

$$\mathsf{A} : H^1(M;\mathbb{R})^m \longrightarrow \mathsf{M}_{d\times m}(\mathbb{R}).$$

As in [9, Section 3], we say that a $d \times m$ matrix with real entries $\mathbf{A}$ satisfies *condition* $(\mathrm{DC})^2_s$ if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}, \quad \forall(\kappa, \xi) \in (\mathbb{Z}^d \times \mathbb{Z}^m)\backslash\{(0,0)\}; \tag{4.2}$$

or, equivalently, if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall(\kappa, \xi) \in (\mathbb{Z}^d \times \mathbb{Z}^m)\backslash\{(0,0)\}.$$

[1] About the notation: ξ in (4.1) is to be regarded as a column vector, so it should, more properly, be transposed. We keep however this notation for simplicity and use it consistently along the text.

This condition implies [9, Lemma 8.1] that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\max_{\ell} |e^{2\pi i a_\ell \cdot \xi} - 1| \geq C_\epsilon e^{-\epsilon |\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m \backslash \{0\}, \tag{4.3}$$

where $a_\ell \in \mathbb{R}^m$ denotes the ℓ-th row of $\mathbf{A}$.

Proposition 4.1 *The system $\boldsymbol{\omega}$ is rational if and only if $\mathsf{A}(\boldsymbol{\omega})(\mathbb{Z}^m \backslash \{0\}) \cap \mathbb{Z}^d \neq \emptyset$. It is an s-exponential Liouville system if and only if it is not rational and $\mathsf{A}(\boldsymbol{\omega})$ does not satisfy* $(\mathrm{DC})^2_s$.

Proof If $\boldsymbol{\omega}$ is a rational system then it follows from (4.1) that $\mathsf{A}(\boldsymbol{\omega})\xi \in \mathbb{Z}^d$ for some $\xi \in \mathbb{Z}^m \backslash \{0\}$. Conversely, if $\mathsf{A}(\boldsymbol{\omega})\xi \in \mathbb{Z}^d$ then

$$\frac{1}{2\pi} \int_{\sigma_\ell} \xi \cdot \boldsymbol{\omega} \in \mathbb{Z}, \quad \forall \ell \in \{1, \dots, d\},$$

which thanks to (♯) is enough to ensure (3.2) for every 1-cycle σ in M.

For the second statement, we need some preliminary remarks. By de Rham's Theorem, one can identify $H^1(M; \mathbb{R})$ with the dual space of $H_1(M; \mathbb{R})$, via the pairing

$$([\alpha], [\sigma]) \in H^1(M; \mathbb{R}) \times H_1(M; \mathbb{R}) \longmapsto \frac{1}{2\pi} \int_\sigma \alpha \in \mathbb{R},$$

and hence consider

$$\begin{gathered} \vartheta_1, \dots, \vartheta_d \in \Lambda^1 C^\infty(M; \mathbb{R}) \text{ closed} \\ \text{whose classes form a basis of } H^1(M; \mathbb{R}) \text{ dual to } [\sigma_1], \dots, [\sigma_d]. \end{gathered} \tag{♭}$$

We can assume without loss of generality that each ϑ_ℓ is actually real-analytic. Indeed, by endowing M with a real-analytic Riemannian metric (which is always possible thanks to Grauert's embedding theorem [10]), the Laplace–Beltrami operator

$$\Delta \doteq \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \Lambda^1 C^\infty(M) \longrightarrow \Lambda^1 C^\infty(M)$$

is an elliptic, real-analytic operator, and by Hodge theory every cohomology class in $H^1(M)$ has a representative $f \in \Lambda^1 C^\infty(M)$ such that $\Delta f = 0$; such an f is therefore real-analytic thanks to the ellipticity of Δ.

We may write for each $k \in \{1, \dots, m\}$

$$\omega_k = \sum_{\ell=1}^d \lambda_{\ell k} \vartheta_\ell + \mathrm{d}v_k,$$

where each $\lambda_{\ell k} \in \mathbb{R}$ is uniquely determined by

$$\lambda_{\ell k} = \frac{1}{2\pi} \int_{\sigma_\ell} \omega_k = \mathsf{A}(\boldsymbol{\omega})_{\ell k},$$

and $v_k \in G^s(M;\mathbb{R})$ by ellipticity of the exterior derivative on M. Hence, for $\xi \in \mathbb{Z}^m$,

$$\xi \cdot \boldsymbol{\omega} = \sum_{\ell=1}^{d} \left(\sum_{k=1}^{m} \mathsf{A}(\boldsymbol{\omega})_{\ell k} \xi_k \right) \vartheta_\ell + \sum_{k=1}^{m} \xi_k \mathrm{d}v_k. \tag{4.4}$$

We assume from here until the end of the proof that $\boldsymbol{\omega}$ is not a rational system; hence, as we have seen, $\mathsf{A}(\boldsymbol{\omega})\xi \notin \mathbb{Z}^d$ for every $\xi \in \mathbb{Z}^m \backslash \{0\}$.

Suppose first that $\mathsf{A}(\boldsymbol{\omega})$ does not satisfy $(\mathrm{DC})^2_s$. Then there exists $\epsilon > 0$ and a sequence $\{(\kappa_\nu, \xi_\nu)\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^d \times \mathbb{Z}^m \backslash \{(0,0)\}$ such that

$$\lim_{\nu\to\infty} e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} |\kappa_\nu + \mathsf{A}(\boldsymbol{\omega})\xi_\nu| = 0. \tag{4.5}$$

Suppose by contradiction that $\{\xi_\nu\}_{\nu\in\mathbb{N}}$ is bounded: the same cannot hold for $\{\kappa_\nu\}_{\nu\in\mathbb{N}}$ (otherwise the sequence $\{(\kappa_\nu, \xi_\nu)\}_{\nu\in\mathbb{N}}$ would attain at most finitely many values, contradicting (4.5)). Moreover, in this case,

$$e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} |\kappa_\nu + \mathsf{A}(\boldsymbol{\omega})\xi_\nu| \geq |\kappa_\nu + \mathsf{A}(\boldsymbol{\omega})\xi_\nu| \geq |\kappa_\nu| - |\mathsf{A}(\boldsymbol{\omega})\xi_\nu| \geq |\kappa_\nu| - C,$$

for some $C > 0$, now contradicting the unboundedness of $\{\kappa_\nu\}_{\nu\in\mathbb{N}}$. We can therefore assume that $|\xi_\nu| \to \infty$ as $\nu \to \infty$. Now define

$$\theta_\nu \doteq -\sum_{\ell=1}^{d} (\kappa_\nu)_\ell \vartheta_\ell + \sum_{k=1}^{m} (\xi_\nu)_k \mathrm{d}v_k. \tag{4.6}$$

Clearly each θ_ν is an integral 1-form. Furthermore, using (4.4) and (4.6) we obtain

$$\rho_\nu \doteq e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} (\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu) = e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} \sum_{\ell=1}^{d} \left(\sum_{k=1}^{m} \mathsf{A}(\boldsymbol{\omega})_{\ell k} (\xi_\nu)_k + (\kappa_\nu)_\ell \right) \vartheta_\ell.$$

We fix a coordinate chart $(U; t_1, \dots, t_n)$ in M and a compact set $K \subset U$. Hence, there exist $C_1, h_1 > 0$ such that

$$\sup_K |\partial_t^\alpha \vartheta_\ell| \leq C_1 h_1^{|\alpha|} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}^n_+, \ \forall \ell \in \{1, \dots, d\}.$$

Thus on K,

$$\begin{aligned} |\partial_t^\alpha \rho_\nu| &\leq e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} \sum_{\ell=1}^{d} \left(\sum_{k=1}^{m} |\mathsf{A}(\boldsymbol{\omega})_{\ell k} (\xi_\nu)_k + (\kappa_\nu)_\ell| \right) |\partial_t^\alpha \vartheta_\ell| \\ &\leq C_1 h_1^{|\alpha|} \alpha!^s e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} \sum_{\ell=1}^{d} \sum_{k=1}^{m} |\mathsf{A}(\boldsymbol{\omega})_{\ell k} (\xi_\nu)_k + (\kappa_\nu)_\ell| \end{aligned}$$

$$\leq C_2 h_1^{|\alpha|} \alpha!^s e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} |\mathsf{A}(\boldsymbol{\omega})\xi_\nu + \kappa_\nu|$$
$$\leq C_3 h_1^{|\alpha|} \alpha!^s,$$

for some constant $C_3 > 0$ independent of ν thanks to (4.5). This proves that $\{\rho_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $\Lambda^1 G^s(M)$; hence, $\boldsymbol{\omega}$ is an s-exponential Liouville system.

Conversely, suppose that there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 G^s(M;\mathbb{R})$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m$, such that $|\xi_\nu| \to \infty$ and (3.3) holds. Again, we may write for every $\nu \in \mathbb{N}$

$$\theta_\nu = \sum_{\ell=1}^{d} \beta_{\nu\ell}\vartheta_\ell + \mathrm{d}g_\nu, \tag{4.7}$$

where $\beta_{\nu\ell} \in \mathbb{Z}$ and $g_\nu \in G^s(M;\mathbb{R})$. By associating (4.4) with (4.7) we have

$$\begin{aligned}\rho_\nu &\doteq e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu\cdot\boldsymbol{\omega} - \theta_\nu)\\ &= e^{\epsilon|\xi_\nu|^{\frac{1}{s}}} \sum_{\ell=1}^{d}\left(\sum_{k=1}^{m} \mathsf{A}(\boldsymbol{\omega})_{\ell k}(\xi_\nu)_k - \beta_{\nu\ell}\right)\vartheta_\ell + e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}\left(\sum_{k=1}^{m}(\xi_\nu)_k \mathrm{d}v_k - \mathrm{d}g_\nu\right).\end{aligned}$$

As a consequence of the exactness of both $\mathrm{d}v_k$ and $\mathrm{d}g_\nu$ we have

$$\frac{1}{2\pi}\int_{\sigma_\ell}\rho_\nu = e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}\left(\sum_{k=1}^{m}\mathsf{A}(\boldsymbol{\omega})_{\ell k}(\xi_\nu)_k - \beta_{\nu\ell}\right), \quad \forall \ell \in \{1,\dots,d\},\ \forall\nu\in\mathbb{N}.$$

Since by hypothesis $\{\rho_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $\Lambda^1 G^s(M)$ we can find a constant $C > 0$ such that

$$\left|\int_{\sigma_\ell}\rho_\nu\right| \leq C, \quad \forall \ell \in \{1,\dots,d\},\ \forall\nu\in\mathbb{N}.$$

Indeed, only the sup norms of the local coefficients of the ρ_ν added across a finite covering of M play a role in the estimation of such integrals (no derivatives are required). Therefore, by setting $\kappa_\nu \doteq -(\beta_{\nu 1},\dots\beta_{\nu d}) \in \mathbb{Z}^d$ one obtains

$$|\mathsf{A}(\boldsymbol{\omega})\xi_\nu + \kappa_\nu| \leq C_1 e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}}, \quad \forall\nu\in\mathbb{N},$$

for some $C_1 > 0$. Since $|\xi_\nu| \to \infty$, by possibly extracting a subsequence we may assume that $e^{-\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}} < (C_1\nu)^{-1}$, for every $\nu\in\mathbb{N}$, which allows us to obtain

$$|\mathsf{A}(\boldsymbol{\omega})\xi_\nu + \kappa_\nu| \leq \frac{1}{\nu} e^{-\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}}, \quad \forall\nu\in\mathbb{N};$$

therefore $\mathsf{A}(\boldsymbol{\omega})$ does not satisfy $(\mathrm{DC})_s^2$. □

Remark 4.2 If $d \doteq \dim H^1(M;\mathbb{R})$ then any $\mathbf{A} \in \mathsf{M}_{d\times m}(\mathbb{R})$ is the matrix of periods of some system of closed 1-forms $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ on M. Indeed, defining

$$\omega_k \doteq \sum_{\ell=1}^{d} \mathbf{A}_{\ell k} \vartheta_\ell, \quad k \in \{1, \ldots, m\},$$

where $\vartheta_1, \ldots, \vartheta_d$ are as in (♭), yields $\mathsf{A}(\boldsymbol{\omega}) = \mathbf{A}$ by previous computations. We can use this fact to provide examples of systems $\boldsymbol{\omega}$ on M that satisfy the number theoretic conditions in Definition 3.2, as these can be read directly from $\mathsf{A}(\boldsymbol{\omega})$ by Proposition 4.1. For instance, in [9] many examples of matrices $\mathbf{A}$ that do (or do not) satisfy condition $(\mathrm{DC})^2_s$ are discussed.

5 Partial Fourier series

Let $U \subset \mathbb{R}^n$ be an open set. Given $f \in C^\infty(U \times \mathbb{T}^m)$ we define for each $\xi \in \mathbb{Z}^m$ a function $\hat{f}_\xi \in C^\infty(U)$ by

$$\hat{f}_\xi(t) \doteq \int_{\mathbb{T}^m} e^{-ix\cdot\xi} f(t,x)\mathrm{d}x, \quad t \in U;$$

more generally, if $f \in \mathcal{D}'(U \times \mathbb{T}^m)$ we define $\hat{f}_\xi \in \mathcal{D}'(U)$ by the rule

$$\phi \in C_c^\infty(U) \longmapsto \langle f, \phi \otimes e^{-ix\cdot\xi} \rangle \in \mathbb{C}.$$

It is easy to see that this construction is local in U; that is

$$V \subset U \text{ open} \Longrightarrow (\widehat{f|_{V\times\mathbb{T}^m}})_\xi = \hat{f}_\xi|_V.$$

A related issue is the following formula that one checks at once:

$$\widehat{(\phi f)}_\xi = \phi \hat{f}_\xi, \quad \forall \phi \in C^\infty(U). \tag{5.1}$$

Another important feature is its invariance under changes of variables. We take $\chi : U' \to U$ a diffeomorphism between open sets in $\mathbb{R}^n$ and define

$$X \doteq \chi \times \mathrm{id}_{\mathbb{T}^m} : U' \times \mathbb{T}^m \longrightarrow U \times \mathbb{T}^m, \tag{5.2}$$

that is, $X(t', x) = (\chi(t'), x)$ for $(t', x) \in U' \times \mathbb{T}^m$. One checks easily that

$$\widehat{(X^* f)}_\xi = \chi^* \hat{f}_\xi, \quad \forall \xi \in \mathbb{Z}^m, \tag{5.3}$$

whatever $f \in \mathcal{D}'(U \times \mathbb{T}^m)$.

Finally, we are able to define $\hat{f}_\xi \in \mathcal{D}'(M)$ for $f \in \mathcal{D}'(M \times \mathbb{T}^m)$, where M is now a smooth manifold. On a coordinate domain $U \subset M$ we must define $\hat{f}_\xi|_U \in \mathcal{D}'(U)$ through the following steps:

(1) take $\chi_1 : U_1' \to U$ a diffeomorphism where $U_1' \subset \mathbb{R}^n$ is an open set;
(2) define $X_1 \doteq \chi_1 \times \mathrm{id}_{\mathbb{T}^m}$;
(3) let $f_1 \doteq X_1^*(f|_{U\times\mathbb{T}^m}) \in \mathcal{D}'(U_1' \times \mathbb{T}^m)$;
(4) take its Fourier coefficient $\widehat{(f_1)}_\xi \in \mathcal{D}'(U_1')$ (using the former definition);
(5) define $\hat{f}_\xi|_U \doteq (\chi_1^{-1})^*\widehat{(f_1)}_\xi \in \mathcal{D}'(U)$.

This definition is independent of our choice of parametrization on U: if $\chi_2 : U_2' \to U$ is another such diffeomorphism then we let, in accordance with (5.2),

$$\chi \doteq \chi_2^{-1} \circ \chi_1 : U_1' \longrightarrow U_2' \Longrightarrow X \doteq \chi \times \mathrm{id}_{\mathbb{T}^m} = X_2^{-1} \circ X_1 : U_1' \times \mathbb{T}^m \longrightarrow U_2' \times \mathbb{T}^m$$

so that, by (5.3),

$$\chi^*\widehat{(f_2)}_\xi = \widehat{(X^* f_2)}_\xi = \widehat{(f_1)}_\xi,$$

where we have used that

$$X^* f_2 = X^* X_2^*(f|_{U\times\mathbb{T}^m}) = (X_2 \circ X)^*(f|_{U\times\mathbb{T}^m}) = X_1^*(f|_{U\times\mathbb{T}^m}) = f_1,$$

and from which it follows that

$$(\chi_1^{-1})^*\widehat{(f_1)}_\xi = (\chi_1^{-1})^*\chi^*\widehat{(f_2)}_\xi = (\chi_2^{-1})^*\widehat{(f_2)}_\xi.$$

Notice that this procedure yields, for $f \in L^1_{\mathrm{loc}}(M \times \mathbb{T}^m)$:

$$f_1(t', x) = f(\chi_1(t'), x) \Longrightarrow \widehat{(f_1)}_\xi(t') = \int_{\mathbb{T}^m} e^{-ix\cdot\xi} f(\chi_1(t'), x)\mathrm{d}x$$
$$\Longrightarrow \hat{f}_\xi(t) = \int_{\mathbb{T}^m} e^{-ix\cdot\xi} f(t, x)\mathrm{d}x$$

as expected.

Let again $U \subset \mathbb{R}^n$ be an open set.

Lemma 5.1 *An $f \in \mathcal{D}'(U \times \mathbb{T}^m)$ is zero if and only if $\hat{f}_\xi = 0$ for every $\xi \in \mathbb{Z}^m$.*

Proof Suppose all the partial Fourier coefficients of f are zero. Thanks to (5.1) we may assume that $f \in \mathcal{E}'(U \times \mathbb{T}^m)$, which we regard as a continuous linear functional on $C^\infty(U \times \mathbb{T}^m)$, whose vanishing we proceed to check. By $\mathbb{C}$-linearity, it is sufficient to show that it vanishes on the space of real-valued functions $C^\infty(U \times \mathbb{T}^m; \mathbb{R})$.

Let $\psi \in C^\infty(\mathbb{T}^m;\mathbb{R})$. Using Fourier series we can write

$$\psi = \frac{1}{(2\pi)^m}\sum_{\xi\in\mathbb{Z}^m}\hat{\psi}_\xi\, e^{ix\cdot\xi}, \quad \hat{\psi}_\xi \in \mathbb{C},$$

with convergence in $C^\infty(\mathbb{T}^m)$. We have that

$$\langle f, \phi\otimes\psi\rangle = \frac{1}{(2\pi)^m}\sum_{\xi\in\mathbb{Z}^m}\hat{\psi}_\xi\,\langle f, \phi\otimes e^{ix\cdot\xi}\rangle = \frac{1}{(2\pi)^m}\sum_{\xi\in\mathbb{Z}^m}\hat{\psi}_\xi\,\langle \hat{f}_{(-\xi)}, \phi\rangle = 0,$$

whatever $\phi \in C^\infty(U;\mathbb{R})$; hence, by passing to finite sums of simple tensors we prove that f vanishes on $\mathcal{A} \doteq C^\infty(U;\mathbb{R})\otimes C^\infty(\mathbb{T}^m;\mathbb{R})$. This is a real subalgebra of $C^\infty(U\times\mathbb{T}^m;\mathbb{R})$ that satisfies:

(1) given distinct $(t,x),(t',x') \in U\times\mathbb{T}^m$ there exists $g\in\mathcal{A}$ such that $g(t,x)\neq g(t',x')$;
(2) given $(t,x)\in U\times\mathbb{T}^m$ there exists $g\in\mathcal{A}$ such that $g(t,x)\neq 0$; and
(3) given $(t,x)\in U\times\mathbb{T}^m$ and a non-zero $(v,w)\in T_tU\oplus T_x\mathbb{T}^m \cong T_{(t,x)}(U\times\mathbb{T}^m)$ there exists $g\in\mathcal{A}$ such that $\mathrm{d}g_{(t,x)}(v,w)\neq 0$.

These happen to be the hypotheses of Nachbin's extension of the Stone-Weierstrass Theorem [14, Theorem 1.2.1], by virtue of which $\mathcal{A}$ is dense in $C^\infty(U\times\mathbb{T}^m;\mathbb{R})$. Continuity of f entails our conclusion. □

5.1 Gevrey type estimates

Back to an open set $U\subset\mathbb{R}^n$, for an $f\in\mathcal{D}'(U\times\mathbb{T}^m)$ we take a closer look at the following couple of properties:

(1) for each $\xi\in\mathbb{Z}^m$ we have that $\hat{f}_\xi\in C^\infty(U)$ and
(2) for each compact set $K\subset U$ there exist constants $C,h,\epsilon>0$ such that

$$\sup_K|\partial_t^\alpha\hat{f}_\xi| \leq Ch^{|\alpha|}\alpha!^s e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall\alpha\in\mathbb{Z}_+^n,\ \forall\xi\in\mathbb{Z}^m. \tag{5.4}$$

Let us investigate how they behave under a change of variables $\chi : U'\to U$. First, concerning condition 1, it is clear from (5.3) that $\hat{f}_\xi$ is smooth in U if and only if $\widehat{(X^*f)}_\xi$ is smooth in U'. As for condition 2, we will be interested only in the case when χ is a *real-analytic* diffeomorphism, in which case the same is true for its associated diffeomorphism X defined in (5.2). Keeping in mind (5.3), a careful inspection in the proof of [15, Proposition 1.4.6] shows that if (5.4) holds for some constants $C,h,\epsilon>0$ (provided of course each $\hat{f}_\xi$ is smooth) then

$$\sup_{K'} \left| \partial_{t'}^{\alpha} \widehat{(X^* f)}_{\xi} \right| \leq C'(h')^{|\alpha|} \alpha!^s e^{-\epsilon |\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m,$$

where $K' \doteq \chi^{-1}(K) \subset U'$ and $C', h' > 0$ depend only on C, h and χ. Notice that every compact set $K' \subset U'$ is of that form. We summarize our conclusions in the first statement of the next result.

Proposition 5.2 *Conditions 1–2 are invariant by real-analytic changes of variables. They hold if and only if $f \in G^s(U \times \mathbb{T}^m)$.*

Proof We prove the equivalence stated above. If $f \in G^s(U \times \mathbb{T}^m)$ then given $K \subset U$ a compact set there exist $C, h > 0$ such that

$$\sup_{K \times \mathbb{T}^m} |\partial_t^{\alpha} f| \leq C h^{|\alpha|} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n;$$

hence for any $t \in K$ and $\alpha \in \mathbb{Z}_+^n$,

$$|\partial_t^{\alpha} \hat{f}_{\xi}(t)| = \left| \int_{\mathbb{T}^m} e^{-ix\cdot\xi} \partial_t^{\alpha} f(t,x) \mathrm{d}x \right| \leq \int_{\mathbb{T}^m} |\partial_t^{\alpha} f(t,x)| \mathrm{d}x \leq C(2\pi)^m h^{|\alpha|} \alpha!^s.$$

For the converse, conditions 1–2 ensure that the series

$$\frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} \hat{f}_{\xi}(t) e^{ix\cdot\xi}$$

converges uniformly on compact sets to a continuous function $g : U \times \mathbb{T}^m \to \mathbb{C}$, which is actually smooth and, moreover, satisfies

$$\partial_t^{\alpha} \partial_x^{\beta} g(t,x) = \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} \partial_t^{\alpha} \hat{f}_{\xi}(t) (i\xi)^{\beta} e^{ix\cdot\xi},$$

with uniform convergence on compact sets for every $(\alpha, \beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^m$ thanks to 2. In particular, for $t \in K$ we have

$$|\partial_t^{\alpha} \partial_x^{\beta} g(t,x)| \leq \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} |\partial_t^{\alpha} \hat{f}_{\xi}(t)| |\xi|^{|\beta|} \leq \frac{1}{(2\pi)^m} C h^{|\alpha|} \alpha!^s \sum_{\xi \in \mathbb{Z}^m} |\xi|^{|\beta|} e^{-\epsilon |\xi|^{\frac{1}{s}}},$$

which easily yields $g \in G^s(U \times \mathbb{T}^m)$. Since we are allowed to integrate under the summation sign we obtain

$$\hat{g}_{\eta}(t) = \frac{1}{(2\pi)^m} \sum_{\xi \in \mathbb{Z}^m} \int_{\mathbb{T}^m} e^{-ix\cdot\eta} \hat{f}_{\xi}(t) e^{ix\cdot\xi} \mathrm{d}x = \hat{f}_{\eta}(t), \quad \forall \eta \in \mathbb{Z}^m;$$

hence, by Lemma 5.1 we conclude that $f = g$ is G^s. □

5.2 Forms of type (0, 1)

Let M be a smooth manifold. Given $U \subset M$ the domain of a coordinate system $(t_1, \dots, t_n)$ we denote by $\Lambda^{0,1}C^\infty(U \times \mathbb{T}^m)$ the space of 1-forms f on $U \times \mathbb{T}^m$ with no $\mathrm{d}x$ component i.e.

$$f = \sum_{j=1}^{n} f_j \, \mathrm{d}t_j, \tag{5.5}$$

where $f_j \in C^\infty(U \times \mathbb{T}^m)$. We then define $\hat{f}_\xi \in \Lambda^1 C^\infty(U)$ by

$$\hat{f}_\xi \doteq \sum_{j=1}^{n} \widehat{(f_j)}_\xi \, \mathrm{d}t_j. \tag{5.6}$$

One can prove that these definitions are independent of the choice of coordinates on U (recall (5.1)), which allows us to define the space $\Lambda^{0,1}C^\infty(M \times \mathbb{T}^m)$ of all $f \in \Lambda^1 C^\infty(M \times \mathbb{T}^m)$ such that $f|_{U \times \mathbb{T}^m} \in \Lambda^{0,1}C^\infty(U \times \mathbb{T}^m)$ for every coordinate open set $U \subset M$, as well as their partial Fourier coefficients $\hat{f}_\xi \in \Lambda^1 C^\infty(M)$. We also let $\Lambda^{0,1}G^s(U \times \mathbb{T}^m) \doteq \Lambda^{0,1}C^\infty(U \times \mathbb{T}^m) \cap \Lambda^1 G^s(U \times \mathbb{T}^m)$.

More generally, for $U \subset M$ we define $\Lambda^{0,1}\mathcal{D}'(U \times \mathbb{T}^m)$ as the space of currents $f \in \Lambda^1\mathcal{D}'(U \times \mathbb{T}^m)$ which can be written as (5.5), where now $f_j \in \mathcal{D}'(U \times \mathbb{T}^m)$ for each $j \in \{1, \dots, n\}$, in which case we define $\hat{f}_\xi \in \Lambda^1\mathcal{D}'(U)$ by (5.6). Again, this is independent of the coordinates $(t_1, \dots, t_n)$ so we can define the space of currents $\Lambda^{0,1}\mathcal{D}'(M \times \mathbb{T}^m)$ and their partial Fourier coefficients, which are elements of $\Lambda^1\mathcal{D}'(M)$. One can apply the results in the previous section to each local coefficient f_j in order to retrieve Gevrey regularity of f from local estimates on $\hat{f}_\xi$; more precisely, on their local coefficients $\widehat{(f_j)}_\xi$.

Concerning our operator $\mathbb{L}$ defined in (1.1), notice that for $u \in C^\infty(M \times \mathbb{T}^m)$ we have that $\mathbb{L}u \in \Lambda^{0,1}C^\infty(M \times \mathbb{T}^m)$ and, moreover,

$$\widehat{(\mathbb{L}u)}_\xi = \mathrm{d}\hat{u}_\xi + i\hat{u}_\xi(\xi \cdot \boldsymbol{\omega}) \doteq \mathbb{L}_\xi \hat{u}_\xi, \tag{5.7}$$

thus defining a differential operator $\mathbb{L}_\xi = \mathrm{d} + i(\xi \cdot \boldsymbol{\omega}) \wedge \cdot : C^\infty(M) \to \Lambda^1 C^\infty(M)$; identity (5.7) also holds for $u \in \mathcal{D}'(M \times \mathbb{T}^m)$. It is enough to check this locally: we reason in a coordinate chart $(U; t_1, \dots, t_n)$, where

$$\omega_k = \sum_{j=1}^{n} \omega_{jk} \, \mathrm{d}t_j, \quad \omega_{jk} \in C^\infty(U),$$

hence, by (1.1), we have that

$$\mathbb{L}u = \sum_{j=1}^{n} \left(\partial_{t_j} u + \sum_{k=1}^{m} \omega_{jk} \partial_{x_k} u \right) \mathrm{d}t_j \doteq \sum_{j=1}^{n} (\mathrm{L}_j u) \, \mathrm{d}t_j \doteq \sum_{j=1}^{n} f_j \, \mathrm{d}t_j$$

belongs to $\Lambda^{0,1}\mathcal{D}'(U \times \mathbb{T}^m)$, where L_j is a complex vector field with smooth coefficients on $U \times \mathbb{T}^m$ for each $j \in \{1, \ldots, n\}$. We take as a parametrization of U the inverse of the chart map $\chi \doteq (t_1, \ldots, t_n)^{-1} : U' \to U$ where $U' \subset \mathbb{R}^n$ is an open set and let $X \doteq \chi \times \mathrm{id}_{\mathbb{T}^m}$. Notice that

$$X^* f_j = X^*(\mathrm{L}_j u) = (X^*\mathrm{L}_j)(X^*u) = \partial_{t_j}(X^*u) + \sum_{k=1}^{m}(\omega_{jk} \circ \chi)\partial_{x_k}(X^*u),$$

where by abuse of notation $(t_1, \ldots, t_n)$ also denotes the standard Euclidean coordinates on U' so that ∂_{t_j} is simply a partial derivative. Using the local definition of the Fourier coefficients we take an arbitrary $\phi \in C_c^\infty(U')$ and evaluate

$$\begin{aligned}
\langle \widehat{(X^* f_j)}_\xi, \phi\rangle &= \langle X^* f_j, \phi \otimes e^{-ix\cdot\xi}\rangle \\
&= \langle \partial_{t_j}(X^*u), \phi \otimes e^{-ix\cdot\xi}\rangle + \sum_{k=1}^{m}\langle (\omega_{jk} \circ \chi)\partial_{x_k}(X^*u), \phi \otimes e^{-ix\cdot\xi}\rangle \\
&= -\langle X^*u, (\partial_{t_j}\phi) \otimes e^{-ix\cdot\xi}\rangle - \sum_{k=1}^{m}\langle X^*u, (\omega_{jk} \circ \chi)\phi \otimes (-i\xi_k e^{-ix\cdot\xi})\rangle \\
&= -\langle \widehat{(X^*u)}_\xi, \partial_{t_j}\phi\rangle + i\sum_{k=1}^{m}\xi_k\langle \widehat{(X^*u)}_\xi, (\omega_{jk} \circ \chi)\phi\rangle \\
&= \langle \partial_{t_j}\widehat{(X^*u)}_\xi, \phi\rangle + i\sum_{k=1}^{m}\xi_k\langle (\omega_{jk} \circ \chi)\widehat{(X^*u)}_\xi, \phi\rangle,
\end{aligned}$$

which implies that

$$\widehat{(X^* f_j)}_\xi = \partial_{t_j}\widehat{(X^*u)}_\xi + i\sum_{k=1}^{m}\xi_k(\omega_{jk} \circ \chi)\widehat{(X^*u)}_\xi$$

as elements of $\mathcal{D}'(U')$. Finally, pulling everything back to U via χ^{-1} we obtain, by definition,

$$\widehat{(f_j)}_\xi = \partial_{t_j}\hat{u}_\xi + i\sum_{k=1}^{m}\xi_k\omega_{jk}\hat{u}_\xi \quad \text{in } U,$$

for each $j \in \{1, \ldots, n\}$; hence, by (5.6),

$$\hat{f}_\xi = \sum_{j=1}^{n}\left(\partial_{t_j}\hat{u}_\xi + i\hat{u}_\xi\sum_{k=1}^{m}\xi_k\omega_{jk}\right)\mathrm{d}t_j = \mathrm{d}\hat{u}_\xi + i\hat{u}_\xi\sum_{k=1}^{m}\xi_k\omega_k = \mathrm{d}\hat{u}_\xi + i\hat{u}_\xi(\xi \cdot \boldsymbol{\omega}).$$

6 Proof of Theorem 3.4

Before we start, we state a technical lemma whose proof follows closely that of [9, Lemma 4.3].

Lemma 6.1 *Let $U \subset \mathbb{R}^n$ be an open set and $\boldsymbol{\phi} = (\phi_1, \dots, \phi_m) : U \to \mathbb{R}^m$ be a smooth map satisfying the following condition: for some compact set $K \subset U$ there exist $C_1, h_1 > 0$ such that*

$$\sup_K |\partial^\alpha \phi_k| \leq C_1 h_1^{|\alpha|} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall k \in \{1, \dots, m\}.$$

Then for every $\epsilon > 0$ we can find $h_2 > 0$ depending on C_1, h_1, m and ϵ such that

$$\sup_K |\partial^\alpha e^{i\xi\cdot\boldsymbol{\phi}}| \leq h_2^{|\alpha|} \alpha!^s e^{\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m.$$

We will make use of the universal covering space $\Pi : \widetilde{M} \to M$ of M. One can prove that since M is a real-analytic manifold then so is $\widetilde{M}$, and that Π is a real-analytic map that satisfies the following property: a map $f : M \to N$ is real-analytic (resp. G^s) – N being an arbitrary real-analytic manifold – if and only if the same holds for $f \circ \Pi : \widetilde{M} \to N$.

It is also helpful to endow M with a real-analytic Riemannian metric, $\mathbb{T}^m$ with the standard (flat) metric and $M \times \mathbb{T}^m$ with the product metric, whose volume forms we denote by $\mathrm{d}\mu$, $\mathrm{d}x$ and $\mathrm{d}\mu \wedge \mathrm{d}x$, respectively. We assume without loss of generality that $\int_M \mathrm{d}\mu = 1$, and for $\Omega \in \{M, \mathbb{T}^m, M \times \mathbb{T}^m\}$ we consider the space of square integrable functions $L^2(\Omega)$.

Step 1

Suppose that $\boldsymbol{\omega}$ is a rational system. Then there exists $\eta \in \mathbb{Z}^m \backslash \{0\}$ such that $\eta \cdot \boldsymbol{\omega}$ is integral. It is well-known that if we take a $\psi \in C^\infty(\widetilde{M}; \mathbb{R})$ such that $\mathrm{d}\psi = \Pi^*(\eta \cdot \boldsymbol{\omega})$ (recall that integral forms on M are by definition closed, and therefore exact on $\widetilde{M}$) then

$$\text{for every } p, q \in \widetilde{M} \text{ such that } \Pi(p) = \Pi(q) \text{ we have } \psi(p) - \psi(q) \in 2\pi\mathbb{Z},$$

which is the condition one needs to descend $e^{i\psi}$ to M via Π to a function $g \in C^\infty(M)$. Notice that

$$\Pi^*(\mathrm{d}g) = \mathrm{d}\Pi^* g = \mathrm{d}e^{i\psi} = ie^{i\psi}\mathrm{d}\psi = \Pi^*[ig(\eta \cdot \boldsymbol{\omega})] \Longrightarrow \mathrm{d}g = ig(\eta \cdot \boldsymbol{\omega}) \quad \text{on } M,$$

thanks to injectivity of Π^* (since Π is a submersion). By ellipticity of the exterior derivative on $\widetilde{M}$ we have $\psi \in G^s(\widetilde{M}; \mathbb{R})$, which in turn yields $g \in G^s(M)$. We define

$$u \doteq \frac{1}{(2\pi)^m} \sum_{\nu=1}^{\infty} \nu^{-2} g^\nu e^{-ix\cdot(\nu\eta)} \in L^2(M \times \mathbb{T}^m),$$

which satisfies $\hat{u}_\xi = \nu^{-2} g^\nu$ if $\xi = -\nu\eta$ and $\hat{u}_\xi = 0$ otherwise. Notice that u does not belong to $G^s(M \times \mathbb{T}^m)$: indeed, if it did (Proposition 5.2) there would exist constants $C, \epsilon > 0$ such that

$$\sup_M |\hat{u}_\xi| \leq C e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m,$$

contradicting that $|\hat{u}_{(-\nu\eta)}(t)| = \nu^{-2}|g(t)|^\nu = \nu^{-2}$, for every $\nu \in \mathbb{N}$. However,

$$\mathbb{L}_{(-\nu\eta)}\hat{u}_{(-\nu\eta)} = \mathrm{d}(\nu^{-2}g^\nu) + i(\nu^{-2}g^\nu)(-\nu\eta \cdot \boldsymbol{\omega}) = \nu^{-1}g^{\nu-1}[\mathrm{d}g - ig(\eta \cdot \boldsymbol{\omega})].$$

Hence, $\widehat{(\mathbb{L}u)}_\xi = \mathbb{L}_\xi \hat{u}_\xi = 0$, for every $\xi \in \mathbb{Z}^m$; i.e. $\mathbb{L}u = 0$ (which follows by applying Lemma 5.1 locally), showing that $\mathbb{L}$ is not globally s-hypoelliptic.

Step 2

We proceed to the case where $\boldsymbol{\omega}$ is an s-exponential Liouville system. By hypothesis there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 G^s(M;\mathbb{R})$ and a sequence $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m$ such that $|\xi_\nu| \to \infty$ and (3.3) holds. For each $\nu \in \mathbb{N}$ we select $\psi_\nu \in C^\infty(\widetilde{M};\mathbb{R})$ such that $\mathrm{d}\psi_\nu = \Pi^*\theta_\nu$, and once again integrality of θ_ν allows us to descend $e^{i\psi_\nu}$ to a function $g_\nu \in C^\infty(M)$ that satisfies $\mathrm{d}g_\nu = ig_\nu\theta_\nu$ on M; as before, ψ_ν and g_ν are *a posteriori* G^s on their respective domains.

Assume without loss of generality $\xi_\nu \neq \xi_{\nu'}$ if $\nu \neq \nu'$, and set

$$u \doteq \frac{1}{(2\pi)^m}\sum_{\nu=1}^{\infty} \nu^{-2} g_\nu e^{-ix\cdot\xi_\nu};$$

by similar arguments as in the previous step we have $u \in L^2(M \times \mathbb{T}^m) \backslash G^s(M \times \mathbb{T}^m)$. We will prove that $f \doteq \mathbb{L}u \in \Lambda^1 G^s(M \times \mathbb{T}^m)$. Notice that

$$\begin{aligned}\mathbb{L}_{(-\xi_\nu)}\hat{u}_{(-\xi_\nu)} &= \mathrm{d}(\nu^{-2}g_\nu) + i(\nu^{-2}g_\nu)(-\xi_\nu \cdot \boldsymbol{\omega}) = i\nu^{-2}g_\nu(\theta_\nu - \xi_\nu \cdot \boldsymbol{\omega}) \\ &= -i\nu^{-2}e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}} g_\nu \rho_\nu,\end{aligned}$$

where $\{\rho_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $\Lambda^1 G^s(M)$; hence,

$$\hat{f}_\xi = \begin{cases} -i\nu^{-2}e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}} g_\nu\rho_\nu, & if \xi = \xi_\nu; \\ 0, & \text{otherwise.}\end{cases} \tag{6.1}$$

We proceed to estimate the G^s norms of these terms.

We address g_ν first, focusing our attention on a coordinate domain $V \subset M$ so small that $\Pi : \widetilde{V} \to V$ is a real-analytic diffeomorphism for some open set $\widetilde{V} \subset \widetilde{M}$. Let $\phi_\nu \in G^s(V;\mathbb{R})$ be given by $\phi_\nu \circ \Pi = \psi_\nu|_{\widetilde{V}}$; hence,

$$g_\nu = e^{i\phi_\nu} \quad \text{on } V \tag{6.2}$$

and therefore

$$\Pi^*(\mathrm{d}\phi_\nu) = \mathrm{d}\Pi^*\phi_\nu = \mathrm{d}\psi_\nu = \Pi^*\theta_\nu \Longrightarrow \mathrm{d}\phi_\nu = \theta_\nu \text{ on } V. \tag{6.3}$$

Since DFS spaces are regular injective limits of Banach spaces, one can reduce the property of boundedness in $\Lambda^1 G^s(M)$ to boundedness in some normed space (Proposition 2.2): this one piece of information (the actual definition of the norm is irrelevant in this argument) can be used to prove that, in the topology of $\Lambda^1 G^s(M)$,

$$e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu) \text{ bounded} \Longrightarrow |\xi_\nu|^{-1}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu) \text{ bounded} \Longrightarrow |\xi_\nu|^{-1}\theta_\nu \text{ bounded},$$

where we have used that $|\xi_\nu| \to \infty$. Hence, by Proposition 2.2, given a compact set $K \subset V$ there exist $C_1, h_1 > 0$ such that

$$\sup_K \left|\partial_t^\alpha(|\xi_\nu|^{-1}\theta_\nu)\right| \leq C_1 h_1^{|\alpha|}\alpha!^s, \quad \sup_K \left|\partial_t^\alpha \rho_\nu\right| \leq C_1 h_1^{|\alpha|}\alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \nu \in \mathbb{N}, \tag{6.4}$$

with the abuse of notation of treating a 1-form as a n-tuple of functions on V, on which the partial derivatives act, and whose indices we omit. It follows from the former inequality, together with identity (6.3), that there exist $C_2, h_2 > 0$ such that

$$\sup_K \left|\partial_t^\alpha(|\xi_\nu|^{-1}\phi_\nu)\right| \leq C_2 h_2^{|\alpha|}\alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \nu \in \mathbb{N};$$

hence, by the scalar version of Lemma 6.1 applied to (6.2) we conclude that one can find $h_3 > 0$ (depending only on C_2, h_2, ϵ) such that

$$\sup_K |\partial_t^\alpha g_\nu| = \sup_K \left|\partial_t^\alpha e^{i\phi_\nu}\right| = \sup_K \left|\partial_t^\alpha e^{i|\xi_\nu|(|\xi_\nu|^{-1}\phi_\nu)}\right| \leq h_3^{|\alpha|}\alpha!^s e^{\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}}, \tag{6.5}$$

for every $\alpha \in \mathbb{Z}_+^n$ and $\nu \in \mathbb{N}$. From (6.4) and (6.5) one then deduces that

$$\sup_K |\partial_t^\alpha(g_\nu \rho_\nu)| \leq C_1 e^{\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} h_3^{|\alpha|-|\beta|}(\alpha-\beta)!^s h_1^{|\beta|}\beta!^s \leq C_4 h_4^{|\alpha|}\alpha!^s e^{\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}},$$

for some constants $C_4, h_4 > 0$. This ultimately proves, in view of (6.1), that

$$\sup_K |\partial_t^\alpha \hat{f}_\xi| \leq C_4 h_4^{|\alpha|}\alpha!^s e^{-\frac{\epsilon}{2}|\xi_\nu|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m,$$

ensuring that $f \in \Lambda^1 G^s(M \times \mathbb{T}^m)$ by Proposition 5.2.

Step 3

Finally, we consider the case when the system $\boldsymbol{\omega}$ is neither rational nor s-exponential Liouville. In this situation, take $u \in \mathcal{D}'(M\times\mathbb{T}^m)$ such that $f \doteq \mathbb{L}u \in \Lambda^1 G^s(M\times\mathbb{T}^m)$. Then, for every $\xi \in \mathbb{Z}^m$ we have

$$\hat{f}_\xi = \mathbb{L}_\xi \hat{u}_\xi = \mathrm{d}\hat{u}_\xi + i\hat{u}_\xi(\xi\cdot\boldsymbol{\omega}).$$

For each $k \in \{1,\dots,m\}$ we take a function $\psi_k \in G^s(\widetilde{M};\mathbb{R})$ such that $\mathrm{d}\psi_k = \Pi^*\omega_k$. It follows that

$$\begin{aligned}\mathrm{d}[e^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\hat{u}_\xi)] &= e^{i\xi\cdot\boldsymbol{\psi}}\mathrm{d}\Pi^*(\hat{u}_\xi) + \Pi^*(\hat{u}_\xi)\mathrm{d}e^{i\xi\cdot\boldsymbol{\psi}}\\ &= e^{i\xi\cdot\boldsymbol{\psi}}\{\Pi^*(\hat{f}_\xi) - i\Pi^*[\hat{u}_\xi(\xi\cdot\boldsymbol{\omega})]\} + \Pi^*(\hat{u}_\xi)ie^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\xi\cdot\boldsymbol{\omega}) = e^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\hat{f}_\xi)\end{aligned} \tag{6.6}$$

on $\widetilde{M}$, where we employed the notation $\boldsymbol{\psi} \doteq (\psi_1,\dots,\psi_m)$. For each $\ell \in \{1,\dots,d\}$ we denote by $\tilde{\sigma}_\ell : [0,2\pi] \to \widetilde{M}$ a lift of the 1-cycle σ_ℓ described in (♯) to $\widetilde{M}$. If we fix a base point $t_0 \in M$ and some $\tilde{t}_0 \in \Pi^{-1}(t_0)$ we can assume that $\sigma_\ell(0) = t_0 = \sigma_\ell(2\pi)$ and that $\tilde{\sigma}_\ell(0) = \tilde{t}_0$ for every $\ell \in \{1,\dots,d\}$. Then, by (6.6),

$$\begin{aligned}\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\hat{f}_\xi) &= \int_{\tilde{\sigma}_\ell}\mathrm{d}[e^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\hat{u}_\xi)]\\ &= e^{i\xi\cdot\boldsymbol{\psi}(\tilde{t}_\ell)}\hat{u}_\xi(\Pi(\tilde{t}_\ell)) - e^{i\xi\cdot\boldsymbol{\psi}(\tilde{t}_0)}\hat{u}_\xi(\Pi(\tilde{t}_0)) = (e^{i\xi\cdot\boldsymbol{\psi}(\tilde{t}_\ell)} - e^{i\xi\cdot\boldsymbol{\psi}(\tilde{t}_0)})\hat{u}_\xi(t_0),\end{aligned}$$

where $\tilde{t}_\ell \doteq \tilde{\sigma}_\ell(2\pi) \in \Pi^{-1}(t_0)$, which allows us to deduce that

$$\hat{u}_\xi(t_0) = e^{i\xi\cdot[\boldsymbol{\psi}(\tilde{t}_0)-\boldsymbol{\psi}(\tilde{t}_\ell)]}\hat{u}_\xi(t_0) + e^{-i\xi\cdot\boldsymbol{\psi}(\tilde{t}_\ell)}\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}}\Pi^*(\hat{f}_\xi). \tag{6.7}$$

Observe that

$$\psi_k(\tilde{t}_\ell) - \psi_k(\tilde{t}_0) = \int_{\tilde{\sigma}_\ell}\mathrm{d}\psi_k = \int_{\tilde{\sigma}_\ell}\Pi^*\omega_k = \int_{\Pi\circ\tilde{\sigma}_\ell}\omega_k = \int_{\sigma_\ell}\omega_k = 2\pi\mathsf{A}(\boldsymbol{\omega})_{\ell k};$$

hence,

$$\mathsf{A}(\boldsymbol{\omega})\xi = \frac{1}{2\pi}\left(\xi\cdot[\boldsymbol{\psi}(\tilde{t}_1) - \boldsymbol{\psi}(\tilde{t}_0)],\dots,\xi\cdot[\boldsymbol{\psi}(\tilde{t}_d) - \boldsymbol{\psi}(\tilde{t}_0)]\right).$$

Due to our assumptions on $\boldsymbol{\omega}$, we have by Proposition 4.1 that $\mathsf{A}(\boldsymbol{\omega})$ satisfies condition $(\mathrm{DC})^2_s$. Hence, by (4.3) for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\max_\ell\left|1 - e^{i\xi\cdot[\boldsymbol{\psi}(\tilde{t}_0)-\boldsymbol{\psi}(\tilde{t}_\ell)]}\right| = \max_\ell\left|1 - e^{-2\pi i(\mathsf{A}(\boldsymbol{\omega})\xi)_\ell}\right| \geq C_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall\xi \in \mathbb{Z}^m\backslash\{0\}.$$

Therefore, for a given $\xi \in \mathbb{Z}^m \backslash \{0\}$ we pick that $\ell \in \{1, \dots, d\}$ at which the maximum in the left-hand side is attained; in particular, that term does not vanish, allowing us to conclude from (6.7) that

$$|\hat{u}_\xi(t_0)| = \frac{|e^{-i\xi\cdot\boldsymbol{\psi}(\tilde{t}_\ell)}|}{\left|1 - e^{i\xi\cdot[\boldsymbol{\psi}(\tilde{t}_\ell) - \boldsymbol{\psi}(\tilde{t}_0)]}\right|} \left|\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}} \Pi^*(\hat{f}_\xi)\right| \le C_\epsilon^{-1} e^{\epsilon|\xi|^{\frac{1}{s}}} \left|\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}} \Pi^*(\hat{f}_\xi)\right|. \tag{6.8}$$

We claim that there exist $C, \delta > 0$ such that

$$\left|\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}} \Pi^*(\hat{f}_\xi)\right| \le C e^{-\delta|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m. \tag{6.9}$$

Indeed, first of all we have, by definition,

$$\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}} \Pi^*(\hat{f}_\xi) = \int_0^{2\pi} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\, \tilde{\sigma}_\ell^* \Pi^*(\hat{f}_\xi) = \int_0^{2\pi} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\, \sigma_\ell^*(\hat{f}_\xi).$$

As in Sect. 2, we fix a finite family $\{K_i\}_{i\in I}$ of coordinate compact subsets of M whose interiors form an open covering of M, and a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = 2\pi$ such that for each $r \in \{1, \dots, N\}$ (that will remain fixed until (6.11) and upon which our choices will depend without explicit mention) $\sigma_\ell([\tau_{r-1}, \tau_r])$ is contained in the interior of a single K_i (depending on r). Denoting by $(t_1, \dots, t_n)$ a fixed set of real-analytic coordinates in a neighborhood of that K_i, we write f as (5.5) and $\hat{f}_\xi$ as (5.6), and conclude that on $[\tau_{r-1}, \tau_r] \subset \mathbb{R}$ we have

$$\sigma_\ell^*(\hat{f}_\xi) = \sum_{j=1}^n (\widehat{(f_j)_\xi} \circ \sigma_\ell)\, \mathrm{d}(t_j \circ \sigma_\ell) = \sum_{j=1}^n (\widehat{(f_j)_\xi} \circ \sigma_\ell) g_j\, \mathrm{d}\tau, \tag{6.10}$$

for some continuous function[2] g_j in $[\tau_{r-1}, \tau_r]$.

Hence,

$$\int_{\tau_{r-1}}^{\tau_r} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\, \sigma_\ell^*(\hat{f}_\xi) = \sum_{j=1}^n \int_{\tau_{r-1}}^{\tau_r} e^{i\xi\cdot(\boldsymbol{\psi}(\tilde{\sigma}_\ell(\tau)))} \widehat{(f_j)_\xi}(\sigma_\ell(\tau)) g_j(\tau)\, \mathrm{d}\tau.$$

Now, since $f \in \Lambda^1 G^s(M \times \mathbb{T}^m)$ one can apply Proposition 5.2 to find constants $C', \delta > 0$ independent of r such that

$$\sup_{K_i} |\widehat{(f_j)_\xi}| \le C' e^{-\delta|\xi|^{\frac{1}{s}}}, \quad \forall j \in \{1, \dots, n\},\ \forall \xi \in \mathbb{Z}^m,$$

[2] We stress that the expression (6.10) can only be obtained because $\sigma_\ell([\tau_{r-1}, \tau_r]) \subset K_i$ lies within a coordinate patch of M. One cannot expect that the whole image of the cycle σ_ℓ be contained in such a patch, and this is the reason to introduce the partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = 2\pi$.

and, therefore,

$$\left|\int_{\tau_{r-1}}^{\tau_r} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\,\sigma_\ell^*(\hat{f}_\xi)\right| \leq \sum_{j=1}^{n}\int_{\tau_{r-1}}^{\tau_r} |\widehat{(f_j)}_\xi(\sigma_\ell(\tau))||g_j(\tau)|\,\mathrm{d}\tau \leq C'' e^{-\delta|\xi|^{\frac{1}{s}}}, \tag{6.11}$$

for some $C'' > 0$ independent of $\xi \in \mathbb{Z}^m$ and $r \in \{1, \dots, N\}$. Since

$$\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}}\,\Pi^*(\hat{f}_\xi) = \int_0^{2\pi} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\,\sigma_\ell^*(\hat{f}_\xi) = \sum_{r=1}^{N}\int_{\tau_{r-1}}^{\tau_r} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\,\sigma_\ell^*(\hat{f}_\xi),$$

we conclude that

$$\left|\int_{\tilde{\sigma}_\ell} e^{i\xi\cdot\boldsymbol{\psi}}\,\Pi^*(\hat{f}_\xi)\right| \leq \sum_{r=1}^{N}\left|\int_{\tau_{r-1}}^{\tau_r} e^{i\xi\cdot(\boldsymbol{\psi}\circ\tilde{\sigma}_\ell)}\,\sigma_\ell^*(\hat{f}_\xi)\right| \leq C'' N e^{-\delta|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m,$$

thus proving (6.9).

It follows from (6.8) and (6.9), by taking $\epsilon \doteq \delta/2$ in the former and $C_1 \doteq C_\epsilon^{-1}C$, that

$$|\hat{u}_\xi(t_0)| \leq C_1 e^{-\frac{\delta}{2}|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m. \tag{6.12}$$

Next we must estimate the derivatives of $\hat{u}_\xi$. Once again we take $V \subset M$ a coordinate ball centered at t_0, so small that $\Pi : \widetilde{V} \to V$ is a real-analytic diffeomorphism for an open set $\widetilde{V} \subset \widetilde{M}$; this time, for each $k \in \{1, \dots, m\}$ there exists $\phi_k \in G^s(V; \mathbb{R})$ so that $\phi_k \circ \Pi = \psi_k$ on $\widetilde{V}$; notice that $\mathrm{d}\phi_k = \omega_k$ on V, and thus

$$\begin{aligned}\mathrm{d}(e^{i\xi\cdot\boldsymbol{\phi}}\hat{u}_\xi) &= e^{i\xi\cdot\boldsymbol{\phi}}\mathrm{d}\hat{u}_\xi + \hat{u}_\xi\mathrm{d}e^{i\xi\cdot\boldsymbol{\phi}} = e^{i\xi\cdot\boldsymbol{\phi}}[\hat{f}_\xi - i\hat{u}_\xi(\xi\cdot\boldsymbol{\omega})] + \hat{u}_\xi[ie^{i\xi\cdot\boldsymbol{\phi}}(\xi\cdot\boldsymbol{\omega})]\\ &= e^{i\xi\cdot\boldsymbol{\phi}}\hat{f}_\xi.\end{aligned}$$

Therefore, by integrating over the segment $\{\tau t_0 + (1-\tau)t\ ;\ \tau \in [0,1]\} \subset V$ we deduce that

$$e^{i\xi\cdot\boldsymbol{\phi}(t)}\hat{u}_\xi(t) - e^{i\xi\cdot\boldsymbol{\phi}(t_0)}\hat{u}_\xi(t_0) = \int_{t_0}^{t} e^{i\xi\cdot\boldsymbol{\phi}}\hat{f}_\xi,$$

which implies that

$$\hat{u}_\xi(t) = \underbrace{e^{i\xi\cdot[\boldsymbol{\phi}(t_0)-\boldsymbol{\phi}(t)]}\hat{u}_\xi(t_0)}_{\doteq\Upsilon_\xi^1(t)} + \underbrace{e^{-i\xi\cdot\boldsymbol{\phi}(t)}\int_{t_0}^{t} e^{i\xi\cdot\boldsymbol{\phi}}\hat{f}_\xi}_{\doteq\Upsilon_\xi^2(t)}, \quad \forall t \in V,\ \forall \xi \in \mathbb{Z}^m. \tag{6.13}$$

It follows from (6.12) and Lemma 6.1 that on a given compact set $K \subset V$

$$|\partial_t^\alpha \Upsilon_\xi^1| \leq C_1 e^{-\frac{\delta}{2}|\xi|^{\frac{1}{s}}} |\partial_t^\alpha e^{-i\xi\cdot\boldsymbol{\phi}}| \leq C_1 h_2^{|\alpha|}\alpha!^s e^{-\frac{\delta}{4}|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \xi \in \mathbb{Z}^m, \tag{6.14}$$

for some $h_2 > 0$. We further write

$$\Upsilon_\xi^2(t) = e^{-i\xi\cdot\boldsymbol{\phi}(t)} \int_{t_0}^t e^{i\xi\cdot\boldsymbol{\phi}} \hat{f}_\xi \doteq e^{-i\xi\cdot\boldsymbol{\phi}(t)} Y_\xi(t), \tag{6.15}$$

so that

$$\mathrm{d}Y_\xi = e^{i\xi\cdot\boldsymbol{\phi}} \hat{f}_\xi.$$

It follows from the hypothesis that $f \in \Lambda^1 G^s(M \times \mathbb{T}^m)$ and Proposition 5.2 that there exist $C_3, h_3, \theta > 0$ such that

$$\sup_K |\partial_t^\alpha \hat{f}_\xi| \leq C_3 h_3^{|\alpha|}\alpha!^s e^{-\theta|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \xi \in \mathbb{Z}^m, \tag{6.16}$$

where again we treat 1-forms on V as n-tuples of functions. Therefore, it follows from (6.16) and Lemma 6.1 that for some $h_4 > 0$ we have on K:

$$\begin{aligned}
|\partial_t^\alpha(\mathrm{d}Y_\xi)| &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |\partial_t^{\alpha-\beta} e^{i\xi\cdot\boldsymbol{\phi}}||\partial_t^\beta \hat{f}_\xi| \\
&\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |\partial_t^{\alpha-\beta} e^{i\xi\cdot\boldsymbol{\phi}}| C_3 h_3^{|\beta|}\beta!^s e^{-\theta|\xi|^{\frac{1}{s}}} \\
&\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} h_4^{|\alpha|-|\beta|}(\alpha-\beta)!^s C_3 h_3^{|\beta|}\beta!^s e^{-\frac{\theta}{2}|\xi|^{\frac{1}{s}}} \\
&\leq C_3 h_5^{|\alpha|}\alpha!^s e^{-\frac{\theta}{2}|\xi|^{\frac{1}{s}}},
\end{aligned}$$

whatever $\alpha \in \mathbb{Z}_+^n$, for some $h_5 > 0$. By possibly increasing C_3 and h_5, one obtains

$$\sup_K |\partial_t^\alpha Y_\xi| \leq C_3 h_5^{|\alpha|}\alpha!^s e^{-\frac{\theta}{2}|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \xi \in \mathbb{Z}^m. \tag{6.17}$$

Finally, we return to (6.15); one repeats the previous argument using Lemma 6.1 and now (6.17) to conclude that

$$\sup_K |\partial_t^\alpha \Upsilon_\xi^2| \leq C_7 h_7^{|\alpha|}\alpha!^s e^{-\frac{\theta}{4}|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \xi \in \mathbb{Z}^m, \tag{6.18}$$

for some constants $C_7, h_7 > 0$. Using (6.13), (6.14) and (6.18), we deduce the existence of C, h and $\epsilon > 0$ such that

$$\sup_K |\partial_t^\alpha \hat{u}_\xi| \leq C h^{|\alpha|} \alpha!^s e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m.$$

Since M can be covered by finitely many V with the aforementioned properties and $K \subset V$ is arbitrary, this last estimate shows that u is G^s in $M \times \mathbb{T}^m$ by a final application of Proposition 5.2.

7 Final remarks

Using the tools developed above, one can derive the following characterizations of (smooth) global hypoellipticity of the operator $\mathbb{L}$, thus extending [1, Theorem 2.4] to arbitrary corank. Of course, in that case M and the 1-forms $\omega_1, \dots, \omega_m$ can be assumed just smooth.

In order to properly state them, we need some preliminary definitions.

Definition 7.1 We say that $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ is a *Liouville system* if $\boldsymbol{\omega}$ is not rational and there exist a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 C^\infty(M; \mathbb{R})$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m$ such that $|\xi_\nu| \to \infty$ and

$$\{|\xi_\nu|^\nu (\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu)\}_{\nu\in\mathbb{N}} \text{ is bounded in } \Lambda^1 C^\infty(M).$$

Definition 7.2 We say that a matrix $\mathbf{A} \in \mathsf{M}_{d\times m}(\mathbb{R})$ satisfies *condition* (DC)^2 if there exist $C, \rho > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C(|\kappa| + |\xi|)^{-\rho}, \quad \forall (\kappa, \xi) \in (\mathbb{Z}^d \times \mathbb{Z}^m) \backslash \{(0, 0)\}. \tag{7.1}$$

Notice that the latter condition implies condition (DC) used in [8] to study global solvability of corank m tube structures when $M = \mathbb{T}^d$. It also implies condition (DC)^2_s for every $s \geq 1$.

Theorem 7.3 *The following are equivalent:*

(1) $\mathbb{L}$ *is globally hypoelliptic; i.e.* (1.2) *holds.*
(2) $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ *is neither a rational system nor a Liouville system.*
(3) *The matrix of periods* $\mathsf{A}(\boldsymbol{\omega})$ *satisfies condition* (DC)^2.

We omit the proof. As a consequence, using the theorem above together with Theorem 3.4 and Proposition 4.1 we conclude that when $\omega_1, \dots, \omega_m$ are G^s we have:

Corollary 7.4 *If* $\mathbb{L}$ *is globally hypoelliptic then it is globally* s*-hypoelliptic.*

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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