Free boundary problems with long-range interactions: uniform Lipschitz estimates in the radius

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Abstract

Consider the class of optimal partition problems with long range interactions

$$\inf \left\{ \sum_{i=1}^{k} \lambda_1(\omega_i) : \ (\omega_1, \ldots, \omega_k) \in \mathcal{P}_r(\Omega) \right\},$$

where $\lambda_1(\cdot)$ denotes the first Dirichlet eigenvalue, and $\mathcal{P}_r(\Omega)$ is the set of open k-partitions of Ω whose elements are at distance at least r: $\mathrm{dist}(\omega_i, \omega_j) \geq r$ for every $i \neq j$. In this paper we prove optimal uniform bounds (as $r \to 0^+$) in Lip–norm for the associated L^2-normalized eigenfunctions, connecting in particular the nonlocal case $r > 0$ with the local one $r \to 0^+$. The proof uses new pointwise estimates for eigenfunctions, a one-phase Alt–Caffarelli–Friedman and the Caffarelli-Jerison-

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Kenig monotonicity formulas, combined with elliptic and energy estimates. Our result extends to other contexts, such as singularly perturbed harmonic maps with distance constraints.

Keywords Dirichlet integral · Harmonic functions · Laplacian eigenvalues · Lipschitz estimates · Long range interactions · Optimal partition problems · Optimal regularity · Segregation phenomena

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1 Introduction

The purpose of this paper is to investigate uniform regularity estimates for a family of long-range (nonlocal) interaction problems. Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $k \geq 2$ be integers. Given $r \geq 0$, we consider the set of all k-partitions of Ω whose elements are at distance at least r:

$$\mathcal{P}_r(\Omega) = \left\{ (\omega_1, \ldots, \omega_k) \left| \begin{array}{l} \omega_i \subset \Omega \text{ is a nonempty open set for all } i, \\ \omega_i \cap \omega_j = \emptyset \quad \text{and} \quad \text{dist}(\omega_i, \omega_j) \geq r \; \forall i \neq j \end{array} \right. \right\}$$

(notice that the request that $\omega_i \cap \omega_j = \emptyset$ is redundant for $r > 0$, but not for $r = 0$). It is plain that there exists $\bar{r} > 0$ (which depends on Ω and on k) such that $\mathcal{P}_r(\Omega) \neq \emptyset$, for every $r \in [0, \bar{r})$. For any such r, we are concerned with the following optimization problem:

$$c_r := \inf \left\{ \sum_{i=1}^k \lambda_1(\omega_i) : \; (\omega_1, \ldots, \omega_k) \in \mathcal{P}_r(\Omega) \right\}, \tag{1.1}$$

where $\lambda_1(\cdot)$ denotes the first Dirichlet eigenvalue.

The short-range (local) case, corresponding to the choice $r = 0$, is a typical example of optimal partition problem, a very active topic of research since the seminal paper [3]. Existence and properties of minimizers for c_0 are essentially understood: we collect in the following theorem what has been proved in [6, 13, 33] (see also [21] and [26]).

Theorem A *The optimal value c_0 is attained by a minimal partition $(\Omega_{1,0}, \ldots, \Omega_{k,0})$ which exhausts Ω, in the sense that $\bigcup_i \overline{\Omega_{i,0}} = \overline{\Omega}$; moreover, the free boundary $\bigcup_i \partial \Omega_{i,0}$ consists of piece-wise $C^{1,\alpha}$-hypersurfaces of dimension $N - 1$, up to a singular set of dimension $N - 2$ (the singular set is actually discrete in dimension $N = 2$). Finally, the eigenfunctions $u_{i,0}$ associated with $\Omega_{i,0}$ are globally Lipschitz continuous, which is the optimal regularity in this case.*

Finer results for the singular set are proved in the recent paper [1].

Much less is known in the nonlocal case $r > 0$. In a joint paper with S. Terracini [28] (see Theorem 1.2 and Theorem 1.3-(3), (6) therein), we have shown the following properties.

(1) *Existence.* The level c_r is achieved by an open optimal partition $(\Omega_{1,r}, \ldots, \Omega_{k,r})$;
(2) *Exterior sphere condition and exact distance between the optimal sets.* Given $x_0 \in \partial\Omega_{i,r} \setminus \partial\Omega$, there exists $j \neq i$ and $y_0 \in \partial\Omega_{j,r}$ such that $|x_0 - y_0| = r$, and $\Omega_{i,r} \cap B_r(y_0) = \emptyset$; in particular, $\mathrm{dist}(\Omega_{i,r}, \Omega_{j,r}) = r$ and each set $\Omega_{i,r}$ satisfies an exterior sphere condition of radius r at any of its boundary point.

The second statement together with [4, Lemma 6.4] yields:

(3) *Measure of the Free Boundary.* The sets $\partial\Omega_{i,r}$ have locally finite perimeter in Ω.

The approach used both in the local [6, 13, 33] and in the nonlocal case [28] consists in studying the following relaxed formulation of c_r in terms of measurable functions rather than sets:

$$\tilde{c}_r = \inf \left\{ \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 \,\middle|\, \begin{array}{l} u_i \in H_0^1(\Omega), \ \int_\Omega u_i^2 = 1 \ \forall i, \\ \int_\Omega u_i^2 u_j^2 = 0 \ \text{and} \ \mathrm{dist}(\mathrm{supp}\, u_i, \mathrm{supp}\, u_j) \geq r, \ \forall i \neq j, \end{array} \right\}.$$

(1.2)

It is shown that there exists a minimizer $\mathbf{u}_r = (u_{1,r}, \ldots, u_{k,r})$ for $\tilde{c}_r$. Moreover:

(a) *Optimal regularity.* Each $u_{i,r}$ is Lipschitz continuous in $\overline{\Omega}$. In particular, the positivity sets $\Omega_{i,r} := \{u_{i,r} > 0\}$ are open and $(\Omega_{1,r}, \ldots, \Omega_{k,r}) \in \mathcal{P}_r(\Omega)$;
(b) *Equation of $u_{i,r}$.* $-\Delta u_{i,r} = \lambda_1(\Omega_{i,r}) u_{i,r}$ in $\Omega_{i,r}$. The partition $(\Omega_{1,r}, \ldots, \Omega_{2,r})$ achieves c_r, which coincides with $\tilde{c}_r$, and satisfies conditions (1)–(3).

Under an additional regularity assumption of the free boundary $\partial\Omega_i$, we have also derived a free boundary condition, satisfied by the eigenfunctions of the optimal partitions (see [28, Theorem 1.6]). The validity of such a condition remains a crucial open problem in the general setting for optimal partition problems with a distance constraint.

The techniques adopted in the local and nonlocal cases are completely different. Powerful tools typically employed in the former ones, such as monotonicity formulas, free boundary conditions and blow-up methods, cannot be adapted in the context of optimal partitions at distance, due to the nonlocal nature of the interaction between different densities/sets. This is why the free boundary regularity for problem c_0 is settled, while the same problem for c_r is open. However, the common optimal Lipschitz regularity of $\mathbf{u}_r$ suggests that it should be possible to look at both problems, the local and the nonlocal ones, as a 1-parameter family, where the parameter is the distance r between the different supports. The main results of this paper establish that this is possible, at least at the level of the eigenfunctions. More precisely:

Theorem 1.1 *There exists a constant $C > 0$ such that*

$$\|\mathbf{u}_r\|_{\mathrm{Lip}(\overline{\Omega})} := \|\mathbf{u}_r\|_{L^\infty(\Omega)} + \|\nabla\mathbf{u}_r\|_{L^\infty(\Omega)} \leq C,$$

for any $0 < r < \bar{r}$, and any minimizer $\mathbf{u}_r$ of c_r.

Observe that, for each $r > 0$ fixed, Lipschitz regularity is proved via a barrier argument, which is possible due to the exterior sphere condition (see [28, Theorem

3.4]). However the barrier used depends on the radius, and the argument breaks down as $r \to 0^+$. Here we rely on different methods.

Combining this theorem with the information obtained in previous papers about the local case $r = 0$, we have the following.

Theorem 1.2 *There exists $C > 0$ such that*

$$c_0 \leq c_r \leq c_0 + Cr \quad \text{for sufficiently small } r > 0.$$

In particular, $c_r \to c_0$ as $r \to 0$. Moreover, given any minimizer $\mathbf{u}_r$ of c_r for $r > 0$, there exists $\mathbf{u}_0 \in H_0^1(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$, solution to c_0, such that, up to a subsequence,

$$\mathbf{u}_r \to \mathbf{u}_0 \quad \text{strongly in } H_0^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega}), \text{ for every } \alpha \in (0, 1).$$

We believe that these results may pave the way to the development of a common free boundary regularity theory. This will be the object of future investigations.

A closely related problem concerns the regularity of singularly perturbed harmonic maps and of their free boundaries. Under the previous assumptions on Ω, let

$$\Omega_{\bar{r}} = \bigcup_{x \in \Omega} B_{\bar{r}}(x) = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < \bar{r}\},$$

and, given $k \geq 2$ nonnegative nontrivial functions $f_1, \ldots, f_k \in H^1(\Omega_{\bar{r}}) \cap C(\overline{\Omega_{\bar{r}}})$ satisfying

$$\mathrm{dist}(\mathrm{supp} f_i, \mathrm{supp} f_j) \geq \bar{r} \quad \forall i \neq j, \quad \mathrm{supp} f_i \cap (\Omega_{\bar{r}} \setminus \Omega) \neq \emptyset \quad \forall i,$$

let us consider the minimization problems

$$h_r := \inf_{\mathbf{u} \in H_r} \sum_{i=1}^{k} \int_{\Omega} |\nabla u_i|^2, \quad r \in [0, \bar{r}),$$

where

$$H_r = \left\{ \mathbf{u} = (u_1, \ldots, u_k) \in H^1(\Omega_{\bar{r}}, \mathbb{R}^k) \,\middle|\, \begin{array}{l} \int_{\Omega} u_i^2 u_j^2 = 0 \text{ and } \mathrm{dist}(\mathrm{supp} u_i, \mathrm{supp} u_j) \geq r \quad \forall i \neq j \\ u_i = f_i \text{ a.e. in } \Omega_{\bar{r}} \setminus \Omega \end{array} \right\}.$$
(1.3)

As for the optimal partition problems, the local case $r = 0$ is essentially understood (see [7, 33]), while for the nonlocal one $r > 0$, studied in [28], there are still many open questions. However, local and nonlocal cases share the same optimal regularity for the minimizers: if $\mathbf{u}_r$ is a minimizer of h_r, then it is locally Lipschitz continuous in Ω, both for $r = 0$ and $r > 0$. Therefore, it is natural to wonder whether a result similar to Theorem 1.1 holds true or not. We can give an affirmative answer.

Theorem 1.3 *For any compact set $K \subset\subset \Omega$, there exists a constant $C > 0$ (which depends on K, Ω, N and on $\bar{r}$) such that*

$$\|\mathbf{u}_r\|_{\mathrm{Lip}(K)} := \|\mathbf{u}_r\|_{L^\infty(K)} + \|\nabla\mathbf{u}_r\|_{L^\infty(K)} \leq C,$$

for any $0 < r < \bar{r}$, and any minimizer $\mathbf{u}_r$ of h_r. Moreover there exists $\mathbf{u}_0 \in H_0 \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, solution to h_0, such that, up to a subsequence,

$$\mathbf{u}_r \to \mathbf{u}_0 \quad \text{strongly in } H^1_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega), \text{ for every } \alpha \in (0, 1).$$

Problems c_r and h_r are closely related, both for $r = 0$ and $r > 0$. In turn, they are both related to the study of the asymptotic behavior of multi-components system in the limit of strong competition. This topic attracted a lot of attention in the last decades, mainly in the local setting, for which by now a variety of results are available: systems with symmetric quadratic interaction between the different densities were studied in [5, 12, 14, 30]; systems with variational cubic interaction in [7, 10, 11, 15, 23, 27, 30, 31, 39]; analogue problems for systems driven by the fractional Laplacian were addressed in [16, 34, 35, 37, 38]; the fully nonlinear setting was studied in [8, 25]; and systems with asymmetric diffusion or asymmetric interaction were tackled in [29, 36, 39]. See also the references therein.

In contrast, besides [28], the only contributions regarding long range interaction models are [4] and [2]; in [4], the authors analyzed the spatial segregation for systems such as

$$\begin{cases} \Delta u_{i,\beta} = \beta u_{i,\beta} \sum_{j \neq i} (\chi_{B_1} \star |u_j|^p) & \text{in } \Omega \\ u_{i,\beta} = f_i \geq 0 & \text{in } \Omega_1 \setminus \Omega, \end{cases} \tag{1.4}$$

with $1 \leq p < +\infty$. In the above equation, χ_{B_1} denotes the characteristic function of $B_1(0)$, and $\star$ stays for the convolution. The authors proved the equi-continuity and gradient bounds for families of viscosity solutions $\{\mathbf{u}_\beta : \beta > 0\}$ to (1.4), the local uniform convergence to a limit configuration $\mathbf{u}$, and then studied the free-boundary regularity of the positivity sets $\{u_i > 0\}$ in the case $p = 1$ and dimension $N = 2$. In [2], the author proved a uniqueness result.

Notation and structure of the paper

We mainly use standard notation. Whenever a function f is radially symmetric, we write $f(x) = f(|x|)$. We denote by $B_r(x_0)$ the Euclidean ball of radius $r > 0$ and center x_0; whenever $x_0 = 0$, we simply write B_r. In most of the integrals, the volume or surface elements are omitted, for the sake of brevity; the domain of integration suggests the natural choice.

The rest of the paper is devoted to the proof of Theorem 1.1. We focus on the case $N \geq 3$. In Sect. 2, we present some preliminary inequalities regarding eigenfunctions of the Laplacian. Section 3 contains the proof of Theorem 1.1. Concerning the case $N = 2$ in Theorem 1.1, and Theorem 1.3, we shall not present the details. The proof

follows the same sketch of the one of Theorem 1.1, being actually a bit simpler at several points. We will stress the main differences in some remarks whenever necessary.

2 Preliminary results

We devote this section to some inequalities about eigenfunctions of the Laplacian that will be crucial in order to reach the conclusion of Theorem 1.1. Some of these inequalities are already known and are presented here for the sake of clarity. Some others may be of independent interest and are given in a general setting.

2.1 Pointwise estimate of the gradient of eigenfunctions

We show that the maximum of the gradient of a positive eigenfunction is reached at the boundary of its domain, up to a multiplicative constant depending only on the dimension, and in particular not on the domain Ω. The following result can be extended to more general bounded domains (in which case the gradient may be unbounded), but we state and prove it only under the following additional regularity assumption on Ω. Precisely, we recall that Ω is said to satisfy the uniform exterior sphere condition with radius ρ if for every $x_0 \in \partial\Omega$, there exists a ball $B \subset \mathbb{R}^N \setminus \Omega$ with radius ρ such that $x_0 \in \partial B$. Moreover, we say that Ω enjoys the uniform exterior sphere condition if there exists $\rho > 0$ such that Ω enjoys the uniform exterior sphere condition with radius ρ.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded domain that enjoys the uniform exterior sphere condition. Let $\lambda = \lambda_1(\Omega)$ be the first positive eigenvalue of the Laplacian with Dirichlet boundary conditions with eigenfunction $u \in H_0^1(\Omega)$,*

$$\begin{cases} -\Delta u = \lambda u & in \ \Omega \\ u = 0 & on \ \partial\Omega. \end{cases}$$

There exists a universal constant $C = C(N) > 0$ and a sequence $\{x_n\} \subset \Omega$ such that

$$\lim_{n \to +\infty} \mathrm{dist}(x_n, \partial\Omega) = 0 \quad and \quad \liminf_{n \to +\infty} |\nabla u(x_n)| \geq C \|\nabla u\|_{L^\infty(\Omega)}.$$

Proof By classical regularity theory of elliptic equations, we know that the eigenfunction u is a C^∞ function inside of Ω and is Lipschitz continuous up to the boundary [20, Proposition 2.20], and by the maximum principle we can assume that $u > 0$ in Ω. Exploiting the regularity of u inside of Ω, we find that the function $x \mapsto |\nabla u(x)|$ is continuous and bounded in Ω. In order to reach the conclusion, since Ω is bounded, it suffices to show that, if $|\nabla u(x)|$ attains its maximum inside of Ω, then its maximum value is still comparable to the value of the gradient close to a point on the boundary. Hence we can further assume that there exists $y \in \Omega$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} = |\nabla u(y)|.$$

Letting $r = \text{dist}(y, \partial\Omega) > 0$, we consider the function $v \in \text{Lip}(\overline{B_1})$ defined as

$$v(x) := \frac{u(y + rx)}{r|\nabla u(y)|}.$$

Then, by definition, we see that $v > 0$ and $|\nabla v| \leq 1$ in B_1, with $|\nabla v(0)| = 1$ and

$$\begin{cases} -\Delta v = \lambda r^2 v & \text{in } B_1 \\ v(z) = 0 & \text{for some } z \in \partial B_1 \cap \frac{\partial\Omega - y}{r}. \end{cases} \tag{2.1}$$

Observe that, by set inclusion, we find $\lambda r^2 \leq \lambda_1(B_1)$, the first Dirichlet eigenvalue of the unit ball in $\mathbb{R}^N$. Moreover, $|v(x)| = |v(x) - v(z)| \leq |x - z| \leq 2$ for every $x \in B_1$. We want to show that $v(0) \geq m$ for some $m > 0$ that depends only on the dimension N. By elliptic regularity theory [19, Corollary 6.3], we know that there exists a constant $C_N > 0$ that depends only on the dimension N such that

$$\|D^2 v\|_{L^\infty(B_{1/2})} \leq C_N \left(\|v\|_{L^\infty(B_1)} + \|\lambda r^2 v\|_{L^\infty(B_1)} + \|\lambda r^2 \nabla v\|_{L^\infty(B_1)} \right)$$
$$\leq 2C_N \left(1 + \lambda_1(B) \right),$$

where $D^2 v$ is the Hessian matrix of v. Let

$$A_N = \max \left(3, 2C_N \left(1 + \lambda_1(B_1) \right) \right)$$

which, ultimately, depends only on the dimension N. For any $x \in B_{1/2}$ we have

$$v(x) = v(0) + \nabla v(0) \cdot x + R(x),$$

where the remainder verifies $|R(x)| \leq A_N \|x\|^2 / 2$. We now take

$$x_0 = -\frac{1}{A_N} \nabla v(0),$$

which belongs to $B_{1/2}$ since $A_N > 2$ and $|\nabla v(0)| = 1$. Recalling that $v > 0$ in B_1 and using again the fact that $|\nabla v(0)| = 1$, we find

$$0 \leq v(x_0) \leq v(0) - \frac{1}{A_N} |\nabla v(0)|^2 + \frac{1}{2A_N} |\nabla v(0)|^2 = v(0) - \frac{1}{2A_N},$$

that is

$$v(0) \geq \frac{1}{2A_N} > 0.$$

Combining this estimate with the fact that $|\nabla v| \leq 1$ we have that

$$\min \left\{ v(x) : |x| \leq \frac{1}{4A_N} \right\} \geq \frac{1}{2A_N} - \frac{1}{4A_N} = \frac{1}{4A_N} > 0.$$

We now consider the function $\underline{v} \in C^2(B_1 \setminus B_{1/(4A_N)})$ defined as

$$\underline{v}(x) = D_N \left(\frac{1}{|x|^{N-2}} - 1 \right),$$

for a constant $D_N > 0$ defined by the relation $D_N((4A_N)^{N-2} - 1) = 1/(4A_N)$. Therefore, $\underline{v}$ is the solution to the problem

$$\begin{cases} -\Delta\underline{v} = 0 \leq -\Delta v & \text{in } B_1 \setminus B_{1/(4A_N)} \\ \underline{v} = 0 \leq v & \text{on } \partial B_1 \\ \underline{v} = \frac{1}{4A_N} \leq v & \text{on } \partial B_{1/(4A_N)}. \end{cases}$$

Notice that $\underline{v}$ is radially decreasing, $\partial_r \underline{v}$ is radially increasing, and

$$\partial_r \underline{v}(x) \leq \partial_r \underline{v}(z) = (2 - N)D_N =: -\kappa_N < 0 \quad \forall x \in B_1 \setminus B_{1/(4A_N)}.$$

Moreover, by the maximum principle, $\underline{v} \leq v$ in $B_1 \setminus B_{1/(4A_N)}$. We claim that this implies that there exists a sequence $\{z_n\} \subset B_1$ such that

$$z_n \to z \quad \text{and} \quad \liminf_{n \to \infty} |\nabla v(z_n)| \geq \kappa_N.$$

Indeed, let us assume by contradiction that there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(z) \cap B_1$ we have $|\nabla v(x)| < \kappa_N$. We consider the function $f \in \text{Lip}([0, 1])$, defined as $f(t) = v((1-t)z)$ for all $t \in [0, 1]$. We have that $|f'(t)| = |\nabla v((1-t)z) \cdot z| < \kappa_N$ for all $t \in (0, \varepsilon)$, thus

$$f(\varepsilon) = f(0) + \int_0^\varepsilon f'(s)ds \leq \int_0^\varepsilon |f'(s)|ds < \varepsilon\kappa_N \implies v((1-\varepsilon)z) < \varepsilon\kappa_N.$$

On the other hand, by the same reasoning as before we have that

$$\underline{v}((1-\varepsilon)z) = -\int_0^\varepsilon \partial_r \underline{v}((1-s)z)ds \geq \varepsilon\kappa_N,$$

in contradiction with the fact that $\underline{v} \leq v$ in $B_1 \setminus B_{1/(4A_N)}$. The conclusion follows by scaling back to the original function u. □

2.2 Mean-value property for eigenfunctions

We show that the eigenfunctions of the Laplacian and their gradients enjoy a mean-value property similar to harmonic functions. For a given $\bar{\lambda} > 0$, let $\bar{R} = \bar{R}(\bar{\lambda}) > 0$ be such that the ball $B_{2\bar{R}}$ has first Dirichlet eigenvalue equal to $\bar{\lambda}$. We denote by φ the

corresponding positive eigenfunction, with

$$\begin{cases} -\Delta\varphi = \bar{\lambda}\varphi & \text{in } B_{2\bar{R}}, \\ \varphi = 0 & \text{on } \partial B_{2\bar{R}}, \\ \varphi(0) = 1. \end{cases} \tag{2.2}$$

We recall that φ is radially symmetric and radially decreasing, attaining its only maximum at the origin and $\varphi(x) = J_{N/2-1,1}(\alpha|x|)$ where $J_{N/2-1,1}$ is the Bessel function of first kind and index $N/2 - 1$, and $\alpha > 0$ is a suitable scaling parameter.

Lemma 2.2 *Let $R \le \bar{R}$ and assume there exists a nonnegative function $v \in C^\infty(B_R)$ such that*

$$-\Delta v \le \lambda v \qquad \text{in } B_R,$$

for $\lambda \le \bar{\lambda}$. Then for any $r \in (0, R)$ we have

$$\frac{1}{r^N} \int_{B_r} v \le \frac{1}{\varphi(R)R^N} \int_{B_R} v.$$

Proof First we observe that, since $\varphi > 0$ in B_R,

$$-\operatorname{div}\left(\varphi^2 \nabla\left(\frac{v}{\varphi}\right)\right) = -\Delta v\varphi + \Delta\varphi v \le (\lambda - \bar{\lambda})v\varphi \le 0 \quad \text{in } B_R.$$

For $0 < r < R$, integrating the previous inequality in B_r we find

$$0 \le \int_{B_r} \operatorname{div}\left(\varphi^2 \nabla\left(\frac{v}{\varphi}\right)\right) = \varphi^2(r) \int_{\partial B_r} \partial_v\left(\frac{v}{\varphi}\right) \implies \int_{\partial B_r} \partial_v\left(\frac{v}{\varphi}\right) \ge 0.$$

Introduce the smooth function $\Phi : (0, R) \to \mathbb{R}$ as

$$\Phi(r) := \frac{1}{r^{N-1}} \int_{\partial B_r} \frac{v(y)}{\varphi(y)} d\sigma_y = \int_{\partial B_1} \frac{v(rx)}{\varphi(rx)} d\sigma_x.$$

Taking the derivative of Φ yields

$$\Phi'(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} \partial_v\left(\frac{v}{\varphi}\right) \ge 0,$$

that is, the function $r \mapsto \Phi(r)$ is positive and increasing for $r < R$. As a result, for any $0 < s < t < R$ we have

$$t^{N-1} \int_{\partial B_s} \frac{v}{\varphi} \le s^{N-1} \int_{\partial B_t} \frac{v}{\varphi}.$$

Next, for a given $r \in (0, R)$, we integrate the previous inequality for $s \in (0, r)$ and afterwards for $t \in (r, R)$, and deduce that

$$\left(\frac{R^N}{N} - \frac{r^N}{N} \right) \int_{B_r} \frac{v}{\varphi} \le \frac{r^N}{N} \int_{B_R \setminus B_r} \frac{v}{\varphi}.$$

By rearranging the terms we obtain

$$\frac{1}{r^N} \int_{B_r} \frac{v}{\varphi} \le \frac{1}{R^N} \int_{B_R} \frac{v}{\varphi}.$$

To conclude we recall that φ is decreasing in r and that $\varphi(0) = 1$.   □

A direct consequence of the mean-value property is a similar inequality for the gradient of eigenfunctions.

Corollary 2.3 *Let $R \le \bar{R}$ and assume there exists a function $u \in C^\infty(B_R)$ such that*

$$-\Delta u = \lambda u \qquad \text{in } B_R,$$

for $2\lambda \le \bar{\lambda}$. Then for any $r \in (0, R)$ we have

$$\frac{1}{r^N} \int_{B_r} |\nabla u|^2 \le \frac{1}{\varphi(R) R^N} \int_{B_R} |\nabla u|^2$$

and, in particular,

$$|\nabla u(0)|^2 \le \frac{1}{\varphi(R)|B_R|} \int_{B_R} |\nabla u|^2.$$

Proof It suffices to consider Lemma 2.2 with $v = |\nabla u|^2$, since

$$-\Delta |\nabla u|^2 = 2 \left(\lambda |\nabla u|^2 - \sum_{i=1}^{N} |\nabla u_{x_i}|^2 \right) \le 2\lambda |\nabla u|^2 \quad \text{in } B_R$$

and $2\lambda \le \bar{\lambda}$.   □

2.3 Energy estimate of the gradient of eigenfunction

Previously we have shown a mean-value property for the gradient of eigenfunction in the interior of their support. In this section we prove a similar result for points on the boundary. It rests on a monotonicity formula of Alt-Caffarelli-Friedman type for a single function defined in a domain that enjoys the exterior sphere condition. We thus first prove such formula.

As before, we fix $\bar{\lambda} > 0$ and let $\bar{R} = \bar{R}(\bar{\lambda}) > 0$ be such that the ball $B_{2\bar{R}}$ has first Dirichlet eigenvalue equal to $\bar{\lambda}$, with eigenfunction φ normalized in such a way that $\varphi(0) = 1$. Let $\Gamma_\varphi \in C^2(B_{3\bar{R}/2} \setminus \{0\})$ be a positive and radial solution of

$$- \operatorname{div}\left(\varphi^2 \nabla \Gamma_\varphi\right) = \delta \quad \text{in } B_{3\bar{R}/2}, \tag{2.3}$$

where δ is the Dirac delta centered at the origin. A direct computation shows that we can choose

$$\Gamma_\varphi(r) = (N-2)\int_r^{3\bar{R}/2} \frac{s^{1-N}}{\varphi^2(s)}\,ds, \quad r = |x|.$$

With this choice we additionally have that $\Gamma_\varphi(3\bar{R}/2) = 0$, $\Gamma_\varphi(r) > 0$ for any $r \in (0, 3\bar{R}/2)$, and $\Gamma_\varphi'(r) = -\frac{N-2}{\varphi^2(r)r^{N-1}}$. We also define

$$\psi(r) := r^{N-2}\varphi^2(r)\Gamma_\varphi(r). \tag{2.4}$$

which we assume to be extended by continuity for $r = 0$. We have the following.

Lemma 2.4 *The function ψ is Lipschitz continuous in $B_{3\bar{R}/2}$ and radially symmetric. For any $r \in [0, \bar{R}]$, $\psi(r) > 0$, while $\psi(3\bar{R}/2) = 0$ and there exists $C = C(N, \bar{\lambda}) \geq 0$ such that*

$$|\psi(r) - 1| \leq Cr \quad \text{for } r \in (0, 3\bar{R}/2). \tag{2.5}$$

Proof We only need to show (2.5), as the other properties in the statement are direct consequences of the definition of the function ψ. We have

$$\frac{\psi(r) - 1}{r} = \frac{1}{r}\left((N-2)\int_r^{3\bar{R}/2} \frac{s^{1-N}}{r^{2-N}}\frac{\varphi^2(r)}{\varphi^2(s)}\,ds - 1\right)$$

$$= \frac{1}{r}\left((N-2)\int_r^{3\bar{R}/2} \frac{s^{1-N}}{r^{2-N}}\frac{\varphi^2(r)}{\varphi^2(s)}\,ds - (N-2)\int_r^{+\infty} \frac{s^{1-N}}{r^{2-N}}\,ds\right)$$

$$= (N-2)r^{N-3}\left(\int_r^{3\bar{R}/2} s^{1-N}\left(\frac{\varphi^2(r)}{\varphi^2(s)} - 1\right)ds - \int_{3\bar{R}/2}^{+\infty} s^{1-N}\,ds\right)$$

$$= (N-2)r^{N-3}\left(\int_r^{3\bar{R}/2} s^{1-N}\frac{\varphi(r)+\varphi(s)}{\varphi^2(s)}(\varphi(r)-\varphi(s))\,ds - \frac{(3\bar{R})^{2-N}}{(N-2)2^{2-N}}\right).$$

Now observe that, by monotonicity of φ,

$$0 \leq \frac{\varphi(r)+\varphi(s)}{\varphi^2(s)} \leq \frac{2\varphi(0)}{\varphi^2(3\bar{R}/2)} = \frac{2}{\varphi^2(3\bar{R}/2)},$$

and that, since φ is smooth and radial,

$$\varphi(r) = 1 + \varphi''(\xi)\frac{r^2}{2}, \qquad \varphi(s) = 1 + \varphi''(\eta)\frac{s^2}{2}$$

for some $\xi, \eta \in (0, 3\bar{R}/2)$. Therefore, there exists $C > 0$, depending on $\bar{R}$, such that

$$\left| \frac{\psi(r) - 1}{r} \right| \leq Cr^{N-3} \left(r^2 \int_r^{3\bar{R}/2} s^{1-N}\, ds + \int_r^{3\bar{R}/2} s^{3-N}\, ds + C \right) \leq C,$$

since $N \geq 3$. $\square$

We are now in a position to state the monotonicity formula. We work with the family of open domains

$$B_r \setminus \overline{B_1(-e_1)} = \left\{ x \in B_r : (x_1 + 1)^2 + \sum_{i=2}^N x_i^2 > 1 \right\},$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the canonical basis of $\mathbb{R}^N$.

Proposition 2.5 *Let $\lambda \leq \bar{\lambda}$ and let $u \in H^1(B_{\bar{R}})$ be a nonnegative solution to*

$$\begin{cases} -\Delta u \leq \lambda u & \text{in } B_{\bar{R}} \setminus \overline{B_1(-e_1)} \\ u = 0 & \text{in } B_{\bar{R}} \cap \overline{B_1(-e_1)}. \end{cases}$$

Then there exist $C = C(N, \bar{\lambda}) > 0$ and $\tilde{r} = \tilde{r}(N, \bar{\lambda}) > 0$, such that the function

$$\Psi(r) := e^{Cr} \frac{1}{r^2} \int_{B_r} \frac{\psi}{|x|^{N-2}} \left| \nabla\left(\frac{u}{\varphi}\right) \right|^2 = e^{Cr} \frac{1}{r^2} \int_{B_r} \varphi^2 \Gamma_\varphi \left| \nabla\left(\frac{u}{\varphi}\right) \right|^2. \quad (2.6)$$

is nondecreasing in $r \in (0, \tilde{r})$, and

$$\frac{1}{r^N} \int_{B_r} |\nabla u|^2 \leq C\Psi(r) \qquad \forall r \in (0, \tilde{r}). \quad (2.7)$$

Moreover, if $-\Delta u = \lambda u$ in $\{u > 0\}$, and $\{u = 0\}$ has locally finite perimeter, then:

$$\Psi(r) \leq \frac{C}{r^N} \int_{B_r} |\nabla u|^2 \qquad r \in (0, \tilde{r}). \quad (2.8)$$

Remark 2.6 In dimension $N = 2$ we need to change the definition of function Ψ in (2.6) as follows:

$$\Psi(r) = e^{Cr} \frac{1}{r^2} \int_{B_r} \varphi^2 \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2.$$

The proof follows by similar computations.

We start by stating and proving an estimate of the first eigenvalue of spherical caps, and a Poincaré-type inequality.

Lemma 2.7 (Estimates for eigenvalues) *Consider the spherical caps*

$$\omega_r := \partial B_1 \setminus \overline{B_{1/r}(-e_1/r)} = \left\{ y \in \partial B_1 : \left(y_1 + \frac{1}{r} \right)^2 + \sum_{i=2}^{N} y_i^2 > \frac{1}{r^2} \right\}$$

$$= \left\{ y \in \partial B_1 : y_1 > -\frac{r}{2} \right\}$$

and let $\lambda_1(\omega_r)$ stand for the first Dirichlet eigenvalue of the Laplace-Beltrami operator on ω_r:

$$\lambda_1(\omega_r) = \inf_{u \in H_0^1(\omega_r)} \frac{\int_{\omega_r} |\nabla_T u|^2}{\int_{\omega_r} |u|^2},$$

where $\nabla_T u$ is the tangential gradient of u. Then there exist $\bar{r} = \bar{r}(N)$ and $C = C(N) > 0$ such that

$$N - 1 - Cr \le \lambda_1(\omega_r) \le N - 1 \quad \text{for } r \in (0, \bar{r}). \tag{2.9}$$

Proof The sets ω_r are invariant under rotations with respect to the first axis. As a result, the first eigenfunction depends only on $\theta = \arccos\langle y, e_1 \rangle \in [0, \pi]$, the polar angle with e_1 (see [32]). We have

$$\lambda_1(\omega_r) = \inf \left\{ \frac{\int_0^{\theta_r} (\sin\theta)^{N-2} |w'(\theta)|^2 \, d\theta}{\int_0^{\theta_r} (\sin\theta)^{N-2} w^2(\theta) \, d\theta} \; \middle| \; \begin{array}{l} w \in H^1([0, \theta_r]), \\ w(\theta_r) = 0 \end{array} \right\}, \tag{2.10}$$

where $\theta_r > 0$ is

$$\theta_r = \arccos\left(-\frac{r}{2} \right) = \frac{\pi}{2} + \frac{r}{2} + O(r^3)$$

for $r > 0$ small enough. The first eigenvalue of ω_r is simple, and the corresponding eigenfunction is a multiple of the unique positive solution $w = w_r$ of

$$\begin{cases} -((\sin\theta)^{N-2} w')' = \lambda_1(\omega_r)(\sin\theta)^{N-2} w & \text{in } (0, \theta_r), \\ w'(0) = 0, \; w(\theta_r) = 0, \; w(0) = 1. \end{cases}$$

A direct computation shows that when $r = 0$, that is $\theta_r = \pi/2$, we have

$$w_0 = \cos\theta \quad \text{and} \quad \lambda_1(\omega_0) = N - 1.$$

By set inclusion we can deduce that the function $r \mapsto \lambda_1(\omega_r)$ is monotone decreasing in r; moreover, as the first eigenvalue is simple, the function $r \mapsto \lambda_1(\omega_r)$ is differentiable at $r = 0$. Thus the limit for $r \to 0$ exists and we have

$$\lambda_1(\omega_r) = N - 1 - Ar + o(r)$$

for a positive constant $A = A(N)$ that depends only on the dimension.

We can be more precise, by giving an explicit value for the constant in the Taylor expansion of $\lambda_1(\omega_r)$. To this aim, we make use of a shape derivative of the domain ω_r. We introduce the family of smooth diffeomorphisms $\Phi \in C^\infty([0, 1] \times [0, \pi]; [0, \pi])$, defined as

$$\Phi(r, \theta) = \frac{2\theta_r}{\pi}\theta.$$

We observe that $\Phi(0, \theta) = \theta$ (that is, $\Phi(0, \cdot)$ is the identity), while for any $r > 0$, Φ maps the set $[0, \pi/2]$ to the set $[0, \theta_r]$. Moreover we have $\frac{\partial\Phi}{\partial r}(0, \theta) = \frac{\theta}{\pi}$. Applying the theory of domain variation (see e.g. [22, Théorème 5.7.1]) we find that

$$A = \frac{d}{dr}\lambda_1(\omega_r)|_{r=0} = -\frac{\int_{\partial\omega_0}(w_0')^2\partial_r\Phi\left(0, \frac{\pi}{2}\right)}{\int_{\omega_0}(w_0)^2} < 0.$$

$\square$

Next we state and prove an inequality of Poincaré-type for H^1 functions that equal to zero on a ball.

Lemma 2.8 (Poincaré-type inequality) *For any $R > 0$, there exists a constant $C_P = C_P(N, R)$ such that*

$$\frac{1}{r}\int_{\partial B_r} u^2 + \frac{1}{r^2}\int_{B_r} u^2 \leq C_P \int_{B_r} |\nabla u|^2$$

for any $r \in (0, R]$ and $u \in H^1(B_r)$ with $u = 0$ in $B_r \cap \overline{B_1(-e_1)}$.

Proof We start with a change of variable, letting $v(x) = u(rx)$ we find that the statement of the result is equivalent to showing that

$$\int_{\partial B_1} v^2 + \int_{B_1} v^2 \leq C \int_{B_1} |\nabla v|^2$$

for any $v \in H^1(B_1)$ with $v = 0$ in $B_1 \cap \overline{B_{1/r}(-e_1/r)}$, $r \in (0, R)$. Assume, by contradiction, that there exist sequences $\{v_n\} \subset H^1(B)$ and $r_n \to \tilde{r} \in [0, R]$ such

that $v_n = 0$ on $B_1 \cap \overline{B_{1/r_n}(-e_1/r_n)}$, and

$$\int_{\partial B_1} v_n^2 + \int_{B_1} v_n^2 = 1 \quad \text{while} \quad \int_{B_1} |\nabla v_n|^2 \to 0.$$

We conclude that the sequence $\{v_n\}$ converges in $H^1(B_1)$ to a non-zero constant function $v \in H^1(B_1)$. On the other hand, by taking the limit of the sequence of sets $\{B_1 \cap \overline{B_{1/r_n}(-e_1/r_n)}\}$, it must be that $v = 0$ in $B_1 \cap \overline{B_{1/\tilde{r}}(-e_1/\tilde{r})}$ if $\tilde{r} > 0$, or $v = 0$ in $B_1 \setminus \{x_1 \leq 0\}$ ir $\tilde{r} = 0$, a contradiction. □

We state and prove a useful consequence of the previous inequality.

Corollary 2.9 *There exist $C = C(N, \bar{\lambda})$ and $\tilde{r} = \tilde{r}(N, \bar{\lambda}) > 0$, such that*

$$\int_{B_r} |\nabla u|^2 \leq C \int_{B_r} \psi^2 \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2$$

for any $r \in (0, \tilde{r})$ and $u \in H^1(B_r)$, with $u = 0$ in $B_r \cap \overline{B_1(-e_1)}$.

Proof The result follows by a chain of straightforward inequalities. We have, for $r \in (0, \bar{R}]$,

$$\int_{B_r} |\nabla u|^2 = \int_{B_r} \left| \nabla u - u\frac{\nabla \varphi}{\varphi} + u\frac{\nabla \varphi}{\varphi} \right|^2 \leq 2 \int_{B_r} \left| \nabla u - u\frac{\nabla \varphi}{\varphi} \right|^2 + 2 \int_{B_r} \left| u\frac{\nabla \varphi}{\varphi} \right|^2$$

$$\leq 2 \int_{B_r} \varphi^2 \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 + 2 \left\| \frac{\nabla \varphi}{\varphi} \right\|_{L^\infty(B_r)}^2 \int_{B_r} u^2$$

$$\leq 2C(\bar{R}) \int_{B_r} \psi^2 \left| \nabla \left(\frac{u}{\varphi} \right) \right|^2 + 2C_P(N, \bar{R})r^2 \left\| \frac{\nabla \varphi}{\varphi} \right\|_{L^\infty(B_{\bar{R}})}^2 \int_{B_r} |\nabla u|^2,$$

where we used Lemmas 2.4 and 2.8. The result follows by rearranging the terms in the last inequality and choosing $\tilde{r} = \tilde{r}(N, \bar{R}) = \tilde{r}(N, \bar{\lambda}) > 0$ sufficiently small in such a way that

$$2C_P\tilde{r}^2 \left\| \frac{\nabla \varphi}{\varphi} \right\|_{L^\infty(B_{\bar{R}})}^2 \leq \frac{1}{2}.$$

□

Proof of Proposition 2.5 We start by showing the monotonicity of the function Ψ. First of all, let $w := u/\varphi \in H^1(B_{\bar{R}})$, which satisfies

$$\begin{cases} -\operatorname{div}(\varphi^2 \nabla w) \leq 0 & \text{in } B_{\bar{R}} \\ w = 0 & \text{in } B_{\bar{R}} \cap B_1(-e_1). \end{cases}$$

We now show that in this case there exists $C > 0$ such that $\Psi(r)$ defined in (2.6) is monotone nondecreasing in r, for r sufficiently small. To start with, by formally testing the equation for w by $\Gamma_\varphi w$ and integrating in B_r, we see that

$$
\int_{B_r} \varphi^2 \Gamma_\varphi |\nabla w|^2 \leq \int_{\partial B_r} \varphi^2 \Gamma_\varphi w (\partial_\nu w) - \int_{B_r} \varphi^2 w \nabla w \cdot \nabla \Gamma_\varphi
$$
$$
= \int_{\partial B_r} \varphi^2 \Gamma_\varphi w (\partial_\nu w) - \int_{B_r} \varphi^2 \nabla (\frac{w^2}{2}) \cdot \nabla \Gamma_\varphi \qquad (2.11)
$$

(to justify rigorously this computation, it is enough to take a sequence of mollifiers $\{\rho_m\}$, work with the regular function $\rho_m * u \to u$ and $w_m := (\rho_m * u)/\varphi$, integrate by parts in the domain $B_r \setminus B_\varepsilon$ and let first $\varepsilon \to 0$ and then $m \to \infty$). Now, by testing the equation for Γ_φ—(2.3)—by $w^2/2$, and integrating by parts, we find

$$
\int_{B_r} \varphi^2 \nabla (\frac{w^2}{2}) \cdot \nabla \Gamma_\varphi = \int_{\partial B_r} \varphi^2 (\partial_\nu \Gamma_\varphi) \frac{w^2}{2} + \frac{w^2(0)}{2} \geq \int_{\partial B_r} \varphi^2 (\partial_\nu \Gamma_\varphi) \frac{w^2}{2}.
$$
$$
(2.12)
$$

By plugging (2.12) into (2.11) and recalling the definition of Γ_φ and ψ:

$$
\int_{B_r} \frac{\psi}{|x|^{N-2}} |\nabla w|^2 = \int_{B_r} \varphi^2 \Gamma_\varphi |\nabla w|^2
$$
$$
\leq \int_{\partial B_r} \left(\varphi^2 \Gamma_\varphi w (\partial_\nu w) - \frac{1}{2} w^2 \varphi^2 (\partial_\nu \Gamma_\varphi) \right)
$$
$$
= \int_{\partial B_r} \left(\varphi^2 \Gamma_\varphi w (\partial_\nu w) + \frac{N-2}{2 r^{N-1}} w^2 \right)
$$
$$
= \int_{\partial B_r} \left(\frac{\psi}{|x|^{N-2}} w (\partial_\nu w) + \frac{N-2}{2 |x|^{N-1}} w^2 \right). \qquad (2.13)
$$

We now compute the logarithmic derivative of Ψ and find

$$
\frac{d}{dr} \log \Psi(r) = C - \frac{2}{r} + \frac{\displaystyle\int_{\partial B_r} \frac{\psi(x)}{|x|^{N-2}} |\nabla w|^2}{\displaystyle\int_{B_r} \frac{\psi(x)}{|x|^{N-2}} |\nabla w|^2}
$$
$$
\geq C - \frac{2}{r} + \frac{\displaystyle\int_{\partial B_r} \frac{\psi(x)}{|x|^{N-2}} |\nabla w|^2}{\displaystyle\int_{\partial B_r} \left(\frac{\psi}{|x|^{N-2}} w (\partial_\nu w) + \frac{N-2}{2 |x|^{N-1}} w^2 \right)}
$$
$$
= C - \frac{2}{r} + \frac{\dfrac{\psi(r)}{r^{N-2}} \displaystyle\int_{\partial B_r} |\nabla w|^2}{\dfrac{\psi(r)}{r^{N-2}} \displaystyle\int_{\partial B_r} w (\partial_\nu w) + \dfrac{N-2}{2 r^{N-1}} \displaystyle\int_{\partial B_r} w^2}.
$$

Let $v = v^{(r)} := w(rx)$, which by assumption vanishes in the complementary of the set

$$\omega_r = \partial B_1 \setminus \overline{B_{1/r}(-e_1/r)} \subset \partial B_1.$$

Then

$$\frac{d}{dr} \log \Psi(r) \geq C - \frac{2}{r} + \frac{\psi(r)}{r} \frac{\int_{\omega_r} |\nabla v|^2}{\int_{\omega_r} \left(\psi(r) v (\partial_\nu v) + \frac{N-2}{2} v^2 \right)}$$

$$= C - \frac{2}{r} + \frac{1}{r\psi(r)} \frac{\int_{\omega_r} \left(\psi^2(r)(\partial_\nu v)^2 + \psi^2(r)|\nabla_\theta v|^2 \right)}{\int_{\omega_r} \left(\psi(r) v (\partial_\nu v) + \frac{N-2}{2} v^2 \right)}$$

$$\geq C - \frac{2}{r} + \frac{1}{r\psi(r)} \frac{\int_{\omega_r} \left(\psi^2(r)(\partial_\nu v)^2 + \psi^2(r)\lambda_1(\omega_r) v^2 \right)}{\int_{\omega_r} \left(\psi(r) v (\partial_\nu v) + \frac{N-2}{2} v^2 \right)}.$$

Since

$$\int_{\omega_r} \left(\psi(r) v (\partial_\nu v) + \frac{N-2}{2} v^2 \right) \leq \int_{\omega_r} \left(\frac{\psi^2(r)}{2a(N-2)} (\partial_\nu v)^2 + \frac{(N-2)(a+1)}{2} v^2 \right),$$

by choosing $a > 0$ such that

$$\frac{1}{2a(N-2)} = \frac{(N-2)(a+1)}{2\psi^2(r)\lambda_1(\omega_r)} \iff a = \frac{1}{N-2} \sqrt{\left(\frac{N-2}{2} \right)^2 + \psi^2(r)\lambda_1(\omega_r)} - \frac{1}{2}$$

$$= \frac{\gamma(\lambda_1(\omega_r)\psi^2(r))}{N-2},$$

where

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2} \right)^2 + t} - \frac{N-2}{2},$$

we see that

$$\frac{d}{dr} \log \Psi(r) \geq C - \frac{2}{r} + \frac{2}{r\psi(r)} \gamma \left(\psi^2(r)\lambda_1(\omega_r) \right)$$

$$= \frac{2}{r} \left(-1 + \frac{C}{2} r + \frac{1}{\psi(r)} \gamma \left(\psi^2(r)\lambda_1(\omega_r) \right) \right).$$

Since $\gamma(N-1) = 1$ and $\gamma'(N-1) = \frac{1}{N} > 0$, by Lemmas 2.4 and 2.7 we have the existence of constants $C_1, C_2, C_3 > 0$ such that

$$\frac{\gamma\left(\psi^2(r)\lambda_1(\omega_r)\right)}{\psi(r)} \geq \frac{\gamma((1-C_1 r)(N-1-C_2 r))}{1+C_1 r} \geq 1 - C_3 r.$$

for any r sufficiently small. In conclusion, by choosing $C := 2C_3$, we have that Ψ is nondecreasing for small $r > 0$.

Next we show (2.7), which is actually a direct consequence of Lemma 2.4 and Corollary 2.9. Indeed we find

$$\frac{1}{r^N}\int_{B_r}|\nabla u|^2 \leq \frac{C}{r^N}\int_{B_r}\psi^2\left|\nabla\left(\frac{u}{\varphi}\right)\right|^2 \leq \frac{C}{r^2}\int_{B_r}\frac{\psi^2}{|x|^{N-2}}\left|\nabla\left(\frac{u}{\varphi}\right)\right|^2.$$

Finally we show (2.8). Using estimate (2.13), we see that

$$\begin{aligned}
\Psi(r) &\leq \frac{e^{Cr}}{r^2}\int_{\partial B_r}\left(\frac{\psi}{|x|^{N-2}}\left(\frac{u}{\varphi}\right)\partial_\nu\left(\frac{u}{\varphi}\right) + \frac{N-2}{2|x|^{N-1}}\left(\frac{u}{\varphi}\right)^2\right)\\
&= \frac{e^{Cr}\psi(r)}{r^N}\int_{\partial B_r}\left(\frac{u}{\varphi}\right)\partial_\nu\left(\frac{u}{\varphi}\right) + \frac{e^{Cr}(N-2)}{2r^{N+1}\varphi^2(r)}\int_{\partial B_r}u^2\\
&\leq \frac{e^{Cr}\psi(r)}{r^N\varphi^2(r)}\int_{\partial B_r}u\partial_\nu u - \frac{e^{Cr}\psi(r)\varphi'(r)}{r^N\varphi^3(r)}\int_{\partial B_r}u^2 + \frac{e^{Cr}(N-2)}{2r^{N+1}\varphi^2(r)}\int_{\partial B_r}u^2\\
&\leq \frac{e^{Cr}\psi(r)}{r^N\varphi^2(r)}\int_{\partial B_r}u\partial_\nu u + \frac{e^{Cr}}{r^N\varphi^2(r)}\\
&\quad\left(\frac{(N-2)}{2r} + \frac{\psi(r)|\varphi'(r)|}{\varphi(r)}\right)\int_{\partial B_r}u^2.
\end{aligned}$$

Multiplying the equation $-\Delta u = \lambda u$ by u and integrating by parts in $B_r \cap \{u > 0\}$ (since $\{u = 0\}$ has locally finite perimeter, we can apply [17, Section 5.8 - Theorem 1]) yields to the identity

$$\begin{aligned}
\int_{B_r}|\nabla u|^2 &= \int_{\{u>0\}\cap B_r}|\nabla u|^2 = \lambda\int_{\{u>0\}\cap B_r}u^2 + \int_{\partial(\{u>0\}\cap B_r)}u\partial_\nu u\\
&= \lambda\int_{B_r}u^2 + \int_{\partial B_r}u\partial_\nu u
\end{aligned}$$

which in turns give us the estimate

$$\int_{\partial B_r}u\partial_\nu u \leq \int_{B_r}|\nabla u|^2.$$

By Lemma 2.8,

$$\int_{\partial B_r} u^2 \le C_P r \int_{B_r} |\nabla u|^2,$$

and we can conclude that

$$\Psi(r) \le \frac{e^{Cr}}{r^N \varphi(r)^2} \left[\psi(r)(1 + r^2\lambda C_P) + C_P \left(\frac{(N-2)}{2} + \frac{\psi(r)|\varphi'(r)|r}{\varphi(r)} \right) \right] \int_{B_r} |\nabla u|^2$$

finally yielding to

$$\Psi(r) \le C \frac{1}{r^N} \int_{B_r} |\nabla u|^2$$

for any $r \in (0, \tilde{r})$. □

We cite a useful corollary that is a straightforward consequence of Proposition 2.5.

Corollary 2.10 *Let $\Omega \subset \mathbb{R}^N$ be a connected open (and non-empty) set that enjoys the exterior sphere condition at any point of its boundary, which we assume to have locally finite perimeter. Assume, moreover, that at $x_0 \in \partial\Omega$ the exterior sphere has radius at least equal to r_0. Let $\lambda = \lambda_1(\Omega)$ be the first eigenvalue of the Laplacian with Dirichlet boundary conditions, and assume that $\lambda \le \bar{\lambda}$. Let $u \in H_0^1(\Omega)$ be the corresponding eigenfunction. There exist $C = C(N, \bar{\lambda})$ and $\tilde{r} = \tilde{r}(N, \bar{\lambda})$ such that*

$$\frac{1}{r^N} \int_{B_r(x_0)} |\nabla u|^2 \le C \frac{1}{R^N} \int_{B_R(x_0)} |\nabla u|^2$$

for any $0 < r < R \le r_0\tilde{r}$.

Proof By a change of variables, the problems reduces to the one where $r_0 = 1$. In such a case, by Proposition 2.5, we have the existence of $C, C', \tilde{r}$, depending only on N and λ such that, whenever $0 < r < R < \tilde{r}$,

$$\frac{1}{r^N} \int_{B_r(x_0)} |\nabla u|^2 \le C\Psi(r) \le C\Psi(R) \le \frac{C'}{R^N} \int_{B_r} |\nabla u|^2,$$

which concludes the proof. □

3 Uniform bounds

In this section we prove Theorem 1.1. Assume, without loss of generality, that Ω satisfies the uniform exterior sphere condition of radius larger than or equal to 1. Recall that $\bar{r} > 0$ denotes a value such that $\mathcal{P}_r(\Omega) \ne \emptyset$, for every $r \in [0, \bar{r})$. In what follows, for $r \in (0, \bar{r})$, we let $\mathbf{u}_r = (u_{1,r}, \dots, u_{k,r})$ be a nonnegative minimizer for c_r (recall

the characterization (1.2)), with $(\Omega_{1,r}, \ldots, \Omega_{k,r}) := (\{u_{1,r} > 0\}, \ldots, \{u_{k,r} > 0\})$ being an optimal partition. Recall also that properties (1)-(3) and (a)-(b) hold true. The main idea of the proof is to show that, if the eigenfunctions do not have uniformly bounded gradients, then it is possible to construct a competitor for the minimization problem that has a smaller energy, thus contradicting the minimality of $\mathbf{u}_r$.

The starting point is to prove uniform bounds of the eigenfunctions in the H^1 and the L^∞ norms.

Lemma 3.1 *There exist constants C, $\Lambda > 0$ such that*

$$\|\mathbf{u}_r\|_{H_0^1(\Omega)}, \|\mathbf{u}_r\|_{L^\infty(\Omega)} \leq C$$

and

$$\lambda_1(\Omega) \leq \lambda_{i,r} := \lambda_1(\{u_{i,r} > 0\}) \leq \Lambda \quad \forall i = 1, \ldots, k,$$

for every $r \in (0, \bar{r})$.

Proof The lower bound on $\lambda_{i,r}$ follows from the monotonicity of the eigenvalues with respect to domain inclusion. On the other hand, since $r \mapsto \mathcal{P}_r(\Omega)$ is decreasing with respect to domain inclusion, then $r \mapsto c_r$ is monotone increasing and, in particular, $c_r \leq c_{\bar{r}}$ for $0 \leq r < \bar{r}$ and

$$\sum_{i=1}^{k} \int_{\Omega} |\nabla u_{i,r}|^2 = \sum_{i=1}^{k} \lambda_1(\{u_{i,r} > 0\}) \leq c_{\bar{r}}.$$

Since $u_{i,r} \in H_0^1(\Omega)$ is a positive solution to $-\Delta u_{i,r} \leq \lambda_1(\{u_{i,r} > 0\})u_{i,r}$ in Ω, the L^∞-uniform bounds are a standard consequence of the Brezis-Kato iteration technique (see for instance the proof of Corollary 1.6 in [24] for the precise details in this framework). $\square$

We assume from now on, by virtue of a contradiction argument, that the gradient of $\mathbf{u}_r$ is *not* uniformly bounded. That is, there exist a sequence $\{r_n\} \subset (0, \bar{r})$, a sequence of minimizers $\{\mathbf{u}_n\}$ associated with c_{r_n}, and a sequence of indexes $\{i_n\}$ such that

$$M_n := \max_{i=1,\ldots,k} \|\nabla u_{i,n}\|_{L^\infty(\Omega)} = \|\nabla u_{i_n,n}\|_{L^\infty(\Omega)} \to +\infty \quad \text{as } n \to +\infty. \quad (3.1)$$

Up to a subsequence and a relabelling, we can suppose that $i_n = 1$ for every n. In what follows, we work constantly under this assumption.

Notation. In what follows we take $\bar{\lambda}$, the constant appearing in Sect. 2, equal to Λ, the upper bound of the eigenvalues $\lambda_{i,r}$ (see Lemma 3.1). Moreover, without loss of generality, we assume $\tilde{r} = 1$ in Corollary 2.10.

Lemma 3.2 *We have $r_n \to 0$.*

Proof Assume that the thesis is false. Then, up to striking out a subsequence, we have that $r_n \to r_0$ for some $r_0 > 0$. We recall that each $u_{i,n} \in H_0^1(\Omega_{i,r_n})$ solves $-\Delta u_{i,n} = \lambda_{i,r_n} u_{i,n}$ in Ω_{i,r_n}. All of these sets satisfy a $\frac{1}{2} r_0$-uniform exterior sphere condition, for any n sufficiently large. By [28, Theorem 3.4] we have that there exists a constant $C > 0$ such that

$$\|\nabla u_{i,n}\|_{L^\infty(\Omega_{i,r_n})} \le C \left(\|u_{i,n}\|_{L^\infty(\Omega_{i,r_n})} + \|\lambda_{i,n} u_{i,n}\|_{L^\infty(\Omega_{i,r_n})} \right).$$

Since the right hand side is bounded by Lemma 3.1, we obtain a contradiction. $\qquad\square$

Now that we have established the behavior of the sequence $\{r_n\}$, we can introduce the quantities that will guide us in the proof of our main result.

Lemma 3.3 *Let $C > 0$ be the dimensional constant of Lemma 2.1. There exists a sequence $\{x_n\} \subset \{u_{1,n} > 0\}$ such that*

$$C M_n \le |\nabla u_{1,n}(x_n)| \le M_n \qquad (3.2)$$

and, moreover,

$$R_n := \text{dist}(x_n, \partial\{u_{1,n} > 0\}) = o(r_n)$$

as $n \to \infty$.

Proof We can directly apply Lemma 2.1 to each function $u_{1,n}$ (for n fixed) to obtain the desired result. $\qquad\square$

Now, let $y_n \in \partial\{u_{1,n} > 0\}$ be a projection of x_n onto $\partial\{u_{1,n} > 0\}$, so that $R_n = |x_n - y_n|$. We shall analyze the behavior of the sequence $\{x_n\}$ and of $\{y_n\}$. As a first step, we show that the sequence $\{x_n\}$ is very close to the free-boundary $\partial\Omega_{1,n} \cap \Omega$ and not to the fixed boundary of Ω. This is the content of the next result.

Lemma 3.4 *We have that $\text{dist}(x_n, \partial\Omega)/r_n \to +\infty$. In particular $y_n \in \partial\Omega_{1,n} \setminus \partial\Omega$ and, moreover,*

$$M_n^2 \le C \frac{1}{r_n^N} \int_{B_{r_n}(y_n)} |\nabla u_{1,n}|^2$$

for a constant $C = C(N, \bar\lambda) > 0$ and n sufficiently large.

Proof We prove this result by virtue of a contradiction argument. Assume that there exists a constant $\kappa > 0$ and a subsequence (which we shall not relabel) such that

$$\text{dist}(x_n, \partial\Omega) \le \kappa r_n.$$

We assume that n is sufficiently large in such a way that $\tilde r(N, \lambda_{1,n}) \ge 1$.

Case 1) $y_n \in \partial\Omega$. In this case, by joining Corollaries 2.3 and 2.10, and recalling that Ω has the exterior sphere condition with radius at least 1, we have that

$$CM_n \leq |\nabla u_{1,n}(x_n)|^2 \leq C\frac{1}{R_n^N}\int_{B_{R_n}(x_n)}|\nabla u_{1,n}|^2 \leq 2^N C\frac{1}{(2R_n)^N}\int_{B_{2R_n}(y_n)}|\nabla u_{1,n}|^2$$

$$\leq C\int_{B_1(y_n)}|\nabla u_{1,n}|^2 \leq C\|u_{1,n}\|_{H^1}^2,$$

and we find a contradiction with Lemma 3.1, since $M_n \to +\infty$.

Case 2) $y_n \notin \partial\Omega$. In this second case we need an additional step. Let $\rho_n = \mathrm{dist}(y_n, \partial\Omega)$ and $z_n \in \partial\Omega$ such that $|z_n - y_n| = \rho_n$. It is plain that $\rho_n \leq (1+\kappa)r_n$. At first, by using again Corollary 2.3, and Corollary 2.10 on balls centered in y_n, where $\{u_{1,n} > 0\}$ has an exterior sphere of radius $r_n > 2R_n$, we obtain

$$CM_n^2 \leq |\nabla u_{1,n}(x_n)|^2 \leq C\frac{1}{R_n^N}\int_{B_{R_n}(x_n)}|\nabla u_{1,n}|^2$$

$$\leq 2^N C\frac{1}{(2R_n)^N}\int_{B_{2R_n}(y_n)}|\nabla u_{1,n}|^2 \leq \frac{C}{r_n^N}\int_{B_{r_n}(y_n)}|\nabla u_{1,n}|^2. \tag{3.3}$$

At this point, since $B_{r_n}(y_n) \subset B_{r_n+\rho_n}(z_n)$, $\rho_n \leq (1+\kappa)r_n$, and $\{u_{1,n} > 0\}$ has, at z_n, an exterior ball or radius 1, Corollary 2.10 again yields

$$M_n^2 \leq \frac{1}{r_n^N}\int_{B_{r_n}(y_n)}|\nabla u_{1,n}|^2 \leq C\frac{(r_n+\rho_n)^N}{r_n^N}\frac{1}{(r_n+\rho_n)^N}\int_{B_{r_n+\rho_n}(z_n)}|\nabla u_{1,n}|^2$$

$$\leq C(2+\kappa)^N\frac{1}{(r_n+\rho_n)^N}\int_{B_{r_n+\rho_n}(z_n)}|\nabla u_{1,n}|^2 \leq C\int_{B_1(z_n)}|\nabla u_{1,n}|^2 \leq C\|u_{1,n}\|_{H^1}^2,$$

and we find again a contradiction with Lemma 3.1.

This completes the proof of the first part of the statement. To obtain the desired estimate, one can now proceed as in (3.3). $\qquad\square$

To proceed further, we recall the Caffarelli–Jerison–Kenig formula, a fundamental result for free-boundary problems [9]. Let $u, v \in H^1(\mathbb{R}^N)$ be two continuous and non-negative functions such that $u(x)v(x) = 0$ for any $x \in \mathbb{R}^N$ and $\|u\|_{L^2} = \|v\|_{L^2} = 1$. Assume moreover that there exists a constant $M > 0$ such that

$$-\Delta u \leq M, \quad -\Delta v \leq M \quad \text{in } \mathbb{R}^N$$

in the sense of measures. Then there exists $C = C(N, M)$ such that

$$\frac{1}{r^2}\int_{B_r(x)}\frac{|\nabla u|^2}{|x-y|^{N-2}} \cdot \frac{1}{r^2}\int_{B_r(x)}\frac{|\nabla v|^2}{|x-y|^{N-2}} \leq C$$

for any $x \in \mathbb{R}^N$ and $r \in (0, 1)$. We can directly apply the Caffarelli–Jerison–Kenig formula to our setting, since $\{\mathbf{u}_n\}$ is uniformly bounded in $L^\infty(\Omega)$ and the eigenvalues

$\{\lambda_{i,n}\}$ are uniformly bounded as well, see Lemma 3.1. Thus, there exists a constant $C = C(N, \bar{\lambda}) > 0$ such that

$$\frac{1}{r^N} \int_{B_r(y_n)} |\nabla u_{1,n}|^2 \cdot \frac{1}{r^N} \int_{B_r(y_n)} |\nabla u_{j,n}|^2 \leq \frac{1}{r^2} \int_{B_r(y_n)} \frac{|\nabla u_{1,n}|^2}{|x - y_n|^{N-2}}$$

$$\cdot \frac{1}{r^2} \int_{B_r(y_n)} \frac{|\nabla u_{j,n}|^2}{|x - y_n|^{N-2}} \leq C \tag{3.4}$$

for any $0 < r < 1$ and any $j \neq 1$.

Now we introduce the following rescaled functions

$$\mathbf{v}_n(x) := \frac{\mathbf{u}_n(y_n + r_n x)}{r_n M_n}, \quad x \in \Omega_n := \frac{\Omega - y_n}{r_n},$$

extended as 0 to $\mathbb{R}^N \setminus \Omega$.

Clearly, we have $\mathbf{v}_n \in H_0^1(\Omega_n)$, $\|\nabla \mathbf{v}_n\|_{L^\infty(\mathbb{R}^N)} \leq 1$, and $\mathbf{v}_n(0) = 0$, for every n. Each set $\{v_{i,n} > 0\}$ enjoys the exterior sphere condition of the same radius 1. By Lemmas 3.3 and 3.4, the sets Ω_n exhaust $\mathbb{R}^N$ as $n \to \infty$. Moreover,

$$\int_{\mathbb{R}^N} v_{i,n}^2 = \frac{1}{r_n^{N+2} M_n^2} \int_{\mathbb{R}^N} u_{i,n}^2 = \frac{1}{r_n^{N+2} M_n^2},$$

$$\int_{\mathbb{R}^N} |\nabla v_{i,n}|^2 = \frac{1}{r_n^N M_n^2} \int_{\mathbb{R}^N} |\nabla u_{i,n}|^2 = \frac{1}{r_n^N M_n^2} \lambda_{i,n}, \tag{3.5}$$

and $\mathbf{v}_n$ is a minimizer for the following scaled version of problem (1.2):

$$\inf \left\{ J(\mathbf{v}) : v_i \in H_0^1(\Omega_n) \setminus \{0\} \ \forall i, \ \text{dist}(\text{supp } v_i, \text{supp } v_j) \geq 1, \ \forall i \neq j \right\}, \tag{3.6}$$

where

$$J(\mathbf{v}) = \sum_{i=1}^k \frac{\int_{\Omega_n} |\nabla v_i|^2}{\int_{\Omega_n} v_i^2} = \sum_{i=1}^k \frac{\int_{\mathbb{R}^N} |\nabla v_i|^2}{\int_{\mathbb{R}^N} v_i^2}.$$

The asymptotic properties of $\{\mathbf{v}_n\}$ are collected in the following statement.

Lemma 3.5 *There exists a globally Lipschitz function* $\mathbf{v} = (v_1, \dots, v_k)$ *defined in* $\mathbb{R}^N$, *with Lipschitz constant 1, such that:*

(i) $\mathbf{v}_n \to \mathbf{v}$ *in* $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^N)$, *for every* $\alpha \in (0, 1)$, *and strongly in* $H_{\text{loc}}^1(\mathbb{R}^N)$;
(ii) the first component v_1 *is not identically 0 in* B_1, *and moreover, for any* $R \geq 1$ *there exists a constant* $C = C(R)$ *such that*

$$\int_{B_R} |\nabla v_{1,n}|^2 \geq C \frac{r_n^N M_n^2}{\lambda_{1,n}} \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2. \tag{3.7}$$

(iii) *the other components* v_j, $j \neq 1$, *vanish identically in* $\mathbb{R}^N$, *and moreover, for any*
$R > 1$ *there exists* $C = C(R)$ *such that*

$$\int_{B_R} |\nabla v_{j,n}|^2 + v_{j,n}^2 \leq C \frac{r_n^N}{M_n^2 \lambda_{j,n}} \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2. \tag{3.8}$$

Proof The $C_{\mathrm{loc}}^{0,\alpha}$ convergence $\mathbf{v}_n \to \mathbf{v}$ to some $\mathbf{v} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^N)$, with $\|\nabla v\|_{L^\infty(\mathbb{R}^N)} \leq 1$,
follows directly from the uniform gradient bound, and the fact that $\mathbf{v}_n(0) = 0$, via the
Ascoli-Arzelà theorem. To show that the convergence is also strong in $H_{\mathrm{loc}}^1(\mathbb{R}^N)$, it
is not difficult to adapt the argument in [33, Lemma 3.11]: in fact, since $-\Delta v_{i,n} \leq$
$r_n^2 \lambda_{i,n} v_{i,n}$ and $-\Delta v_i \leq 0$ in $\mathbb{R}^N$, there exists (local) nonnegative Radon measures
$\mu_{i,n}, \mu_i \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$-\Delta v_{i,n} = r_n^2 \lambda_{i,n} v_{i,n} + \mu_{i,n}, \qquad -\Delta v_i = \mu_i,$$

and since $v_{i,n} \rightharpoonup v_i$ weakly in $H_{loc}^1(\mathbb{R}^N)$, then $\mu_{i,n} \rightharpoonup \mu_i$ in the sense of measures
$\mathcal{M}_{loc}(\mathbb{R}^N)$. Then, the argument follows by testing $-\Delta(v_{i,n} - v_i) = r_n^2 \lambda_{i,n} v_{i,n} +$
$\mu_{i,n} - \mu_i$ with $(v_{i,n} - v_i)\varphi$, for $\varphi \in C_c^\infty(\mathbb{R}^N)$. This proves (i). Concerning (ii), we
just need to recall that

$$M_n^2 \leq C \frac{1}{r_n^N} \int_{B_{r_n}(y_n)} |\nabla u_{1,n}|^2$$

by Lemma 3.4. This gives, by rescaling and passing to the limit in n, that

$$\int_{B_1} |\nabla v_{1,n}|^2 \geq \frac{1}{C} \quad \Longrightarrow \quad \int_{B_1} |\nabla v_1|^2 \geq \frac{1}{C}$$

and hence $v_1 \not\equiv 0$ in B_1. Furthermore, combining this estimate and (3.5), we obtain
(3.7). It remains to prove the validity of point (iii). By scaling (3.4), we have that for
any $R > 1$ and n sufficiently large,

$$\int_{B_R} |\nabla v_{j,n}|^2 \leq C \int_{B_1} |\nabla v_{1,n}|^2 \int_{B_R} |\nabla v_{j,n}|^2 \leq C \int_{B_R} |\nabla v_{1,n}|^2 \int_{B_R} |\nabla v_{j,n}|^2 \leq C R^{2N} M_n^{-4},$$

that is,

$$\int_{B_R} |\nabla v_{j,n}|^2 \leq C R^{2N} M_n^{-4} \to 0. \tag{3.9}$$

Thus, for any $R > 1$, the sequence $\{v_{j,n}\}$ converges in $H^1(B_R)$ and in $C^{0,\alpha}(B_R)$ (for
any $\alpha \in (0, 1)$) to a constant c. But since $v_{j,n}^2(0) = 0$, necessarily the limit $c = 0$ and,
since $R > 1$ was arbitrarily fixed, $v_j \equiv 0$ in $\mathbb{R}^N$ for every $j = 2, \dots, k$. Moreover,
by Lemma 2.8 we have

$$\int_{B_R} (|\nabla v_{j,n}|^2 + v_{j,n}^2) \leq (1 + C_P) \int_{B_R} |\nabla v_{j,n}|^2$$

for a constant $C_P = C_P(N, R)$ that depends only on the dimension N and on the fixed radius R. On the other hand, recalling (3.5) and (3.9), we find

$$\frac{\int_{B_R} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2} \leq \frac{C R^{2N} M_n^{-4}}{\frac{1}{r_n^N M_n^2} \lambda_{j,n}} \implies \int_{B_R} |\nabla v_{j,n}|^2 \leq C \frac{r_n^N}{M_n^2 \lambda_{j,n}} \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2.$$

Putting these last two inequalities together, estimate (3.8) follows.   □

Point (iii) of the previous lemma establishes that the energy of each $v_{j,n}$, with $j \geq 2$, "escapes to infinity": thus, whenever we remove mass from a fixed ball and distribute it on the remainder of the domain, the H^1-norm should not increase in a significant way. We can be more precise. Let $\rho > 0$ be a fixed large positive number and let η be the defined by

$$\eta(x) := \begin{cases} 1 & \text{if } |x| > 2 + \rho \\ |x| - (1 + \rho) & \text{if } 1 + \rho \leq |x| \leq 2 + \rho \\ 0 & \text{if } |x| < 1 + \rho. \end{cases}$$

We point out that $0 \leq \eta \leq 1$, $|\nabla \eta| \leq 1$. Let also $\bar{v}_{j,n} := \eta v_{j,n}$, for $j \geq 2$. We have that $\bar{v}_{j,n} = v_{j,n}$ in $\mathbb{R}^N \backslash B_{2+\rho}$, while $\bar{v}_{j,n} \leq v_{j,n}$ in $B_{2+\rho}$, and actually the support of $v_{j,n}$ is "cut" by the multiplication with η. In the next lemma we estimate the energy gap between $\bar{v}_{j,n}$ and $v_{j,n}$.

Lemma 3.6 *Let $\delta_n := r_n^N / M_n^2$, which tends to 0 as $n \to \infty$. There exists $C > 0$ such that*

$$\frac{\int_{\mathbb{R}^N} |\nabla \bar{v}_{j,n}|^2}{\int_{\mathbb{R}^N} \bar{v}_{j,n}^2} \leq \left(1 + \frac{C}{\lambda_{j,n}} \delta_n\right) \frac{\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} v_{j,n}^2}$$

for every n sufficiently large.

Proof Recalling that $\|v_{j,n}\|_{H^1(B_R)} \to 0$ for all $R > 0$, we find that

$$\int_{\mathbb{R}^N} v_{j,n}^2 = \int_{\mathbb{R}^N} \eta^2 v_{j,n}^2 + \int_{\mathbb{R}^N} \left(1 - \eta^2\right) v_{j,n}^2 \leq \int_{\mathbb{R}^N} \bar{v}_{j,n}^2 + \int_{B_{2+\rho}}$$

$$v_{j,n}^2 = \int_{\mathbb{R}^N} \bar{v}_{j,n}^2 + \frac{C}{\lambda_{j,n}} \delta_n \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2,$$

where in the last step we used estimate (3.8) (notice that C depends on ρ, which is fixed). Similarly, we find

$$\int_{\mathbb{R}^N} |\nabla \bar{v}_{j,n}|^2 = \int_{\mathbb{R}^N} \left(|\nabla \eta|^2 v_{j,n}^2 + 2\eta v_{j,n} \nabla \eta \cdot \nabla v_{j,n} + \eta^2 |\nabla v_{j,n}|^2 \right)$$

$$= \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + \int_{B_{2+\rho}} \left[(\eta^2 - 1)|\nabla v_{j,n}|^2 + |\nabla \eta|^2 v_{j,n}^2 + 2\eta v_{j,n} \nabla \eta \cdot \nabla v_{j,n} \right]$$

$$\leq \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + \int_{B_{2+\rho}} \left[(\eta^2 - 1)|\nabla v_{j,n}|^2 + 2|\nabla \eta|^2 v_{j,n}^2 + \eta^2 |\nabla v_{j,n}|^2 \right]$$

$$\leq \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + 2\int_{B_{2+\rho}} \left(v_{j,n}^2 + |\nabla v_{j,n}|^2 \right) \leq \left(1 + \frac{C}{\lambda_{j,n}} \delta_n \right) \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2.$$

As a result, combining this with (3.5) and recalling from Lemma 3.1 that the eigenvalues $\lambda_{j,n}$ are bounded from above and away from 0, we obtain that

$$\frac{\int_{\mathbb{R}^N} |\nabla \bar{v}_{j,n}|^2}{\int_{\mathbb{R}^N} \bar{v}_{j,n}^2} \leq \frac{\left(1 + \frac{C}{\lambda_{j,n}} \delta_n \right) \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} v_{j,n}^2 - \frac{C}{\lambda_{j,n}} \delta_n \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2} = \frac{1 + \frac{C}{\lambda_{j,n}} \delta_n}{1 - C \delta_n r_n^2} \frac{\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} v_{j,n}^2}$$

$$\leq \left(1 + \frac{C}{\lambda_{j,n}} \delta_n \right) \frac{\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} v_{j,n}^2}$$

for n sufficiently large, which is the desired result. □

Now the idea is to construct a competitor for $\mathbf{v}_n$ with lower energy J. This will be in contradiction with the fact that $\mathbf{v}_n$ is a minimizer for (3.6), and will complete the proof. The j-th component of the competitor will be $\bar{v}_{j,n}$, for $j \geq 2$. We need to conveniently define the first component $\bar{v}_{1,n}$, and the idea is to enlarge the support of $v_{1,n}$ (taking advantage of the fact that the support of $v_{j,n}$ was previously cut, for $j \geq 2$), in order to substantially lower the Rayleigh quotient of $v_{1,n}$. We present the details in what follows.

We have already established that, in any ball B_R with $R > 1$, the function v_1 is not identically 0. Moreover, $0 \in \partial\{v_{1,n} > 0\}$ for every n, and $\{v_{1,n} > 0\}$ satisfies the exterior sphere condition of radius 1 at 0, and, in the exterior sphere, we have $v_{1,n} \equiv 0$. Up to a rotation, it is not restrictive to suppose that $B_1(e_1)$ is such exterior sphere. We consider a new sequence of functions $\bar{v}_{1,n} \in H_0^1(\Omega_n)$ defined piece-wise as follows:

- for $|x| \geq \rho$, we let $\bar{v}_{1,n}(x) = v_{1,n}(x)$;
- for $|x| < \rho$, we let $\bar{v}_{1,n}$ be such that

$$\bar{v}_{1,n} = \arg\min \left\{ \int_{B_\rho} |\nabla v|^2 : v - v_{1,n} \in H_0^1(B_\rho), \int_{B_\rho} v^2 = \int_{B_\rho} v_{1,n}^2 \right\}. \quad (3.10)$$

Since $v_{1,n} \equiv 0$ in $B_\rho \cap B_1(e_1)$, the support of $\bar{v}_{1,n}$ is strictly larger than the one of $v_{1,n}$, and it is at distance at least 1 from the support of $\bar{v}_{j,n}$, for any $j \geq 2$ (by definition of η). Moreover, we have that

$$\int_{\mathbb{R}^N} \bar{v}_{1,n}^2 = \int_{\mathbb{R}^N} v_{1,n}^2 \quad \text{while} \quad \int_{\mathbb{R}^N} |\nabla \bar{v}_{1,n}|^2 < \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2.$$

Concerning the last inequality, we have to be more precise.

Lemma 3.7 *There exists $\varepsilon \in (0, 1)$ such that*

$$\int_{B_\rho} |\nabla \bar{v}_{1,n}|^2 \le (1 - \varepsilon) \int_{B_\rho} |\nabla v_{1,n}|^2 \quad \forall n. \tag{3.11}$$

Proof The proof is quite long and, for the reader's convenience, we divide it into some intermediate steps. Assume by contradiction that, up to striking out a subsequence,

$$(1 - \varepsilon_n) \int_{B_\rho} |\nabla v_{1,n}|^2 \le \int_{B_\rho} |\nabla \bar{v}_{1,n}|^2 \le \int_{B_\rho} |\nabla v_{1,n}|^2, \tag{3.12}$$

with $\varepsilon_n \to 0^+$. Since $\{\bar{v}_{1,n}\}$ is bounded in $H^1(B_\rho)$, $v_{1,n} = \bar{v}_{1,n}$ on ∂B_ρ, $\|v_{1,n}\|_{L^2(B_\rho)} = \|\bar{v}_{1,n}\|_{L^2(B_\rho)}$, and $v_{1,n} \to v_1 \not\equiv 0$ strongly in $H^1(B_\rho)$ (see Lemma 3.5), we have that up to a subsequence $\bar{v}_{1,n} \rightharpoonup \bar{v}$ weakly in $H^1(B_\rho)$, strongly in $L^2(B_\rho)$ and in $L^2(\partial B_\rho)$ (by compactness of the trace operator $H^1(B_\rho) \to L^2(\partial B_\rho)$), with $v_1 = \bar{v}$ on ∂B_ρ, and $\int_{B_\rho} \bar{v}_1^2 =: c > 0$. Moreover, by minimality and the strong maximum principle

$$-\Delta \bar{v}_{1,n} = \bar{\lambda}_n \bar{v}_{1,n}, \quad \bar{v}_{1,n} > 0 \quad \text{in } B_\rho,$$

for some $\bar{\lambda}_n \in \mathbb{R}$.

Step 1) The sequence $\{\bar{\lambda}_n\}$ is bounded. To prove this claim, we first show that there exists $r \in (0, \rho)$ and a subsequence $n_k \to +\infty$ such that,

$$\int_{B_r} \bar{v}_{1,n_k}^2 \ge \frac{c}{2} \quad \text{for every } k. \tag{3.13}$$

Indeed, if by contradiction this were not true, we would have that

$$\int_{B_r} \bar{v}_{1,n}^2 < \frac{c}{2} \quad \text{for every } r \in (0, \rho), \text{ for every } n \text{ large}.$$

But, in this case, if $r_m \to \rho^-$, with a diagonal selection we could find an increasing sequence $n_m \to \infty$ such that

$$\int_{B_{r_m}} \bar{v}_{1,n_m}^2 < \frac{c}{2} \quad \text{for every } m;$$

then, by strong convergence,

$$\frac{c}{2} > \int_{B_{r_m}} \bar{v}_{1,n_m}^2 = \int_{B_\rho} \bar{v}_{1,n_m}^2 \chi_{B_{r_m}} \to \int_{B_\rho} \bar{v}_1^2 = c > 0,$$

a contradiction. Now, denoting for the sake of simplicity by $\{\bar{v}_{1,n}\}$ the sequence in (3.13), let us take a non-negative $\varphi \in C_c^\infty(B_\rho)$, with $\varphi \equiv 1$ on B_r, and let us test the equation of $\bar{v}_{1,n}$ with $\bar{v}_{1,n}\varphi^2$: we obtain

$$|\bar{\lambda}_n| \int_{B_\rho} \bar{v}_{1,n}^2 \varphi^2 = \left| \int_{B_\rho} |\nabla \bar{v}_{1,n}|^2 \varphi^2 + 2\bar{v}_{1,n}\varphi \nabla \bar{v}_{1,n} \cdot \nabla \varphi \right| \le C \|\bar{v}_{1,n}\|_{H^1(B_\rho)}^2.$$

Since the coefficient of $|\bar{\lambda}_n|$ is bounded from below, by (3.13), and $\{\bar{v}_{1,n}\}$ is bounded in $H^1(B_\rho)$, this implies that $\{\lambda_n\}$ is bounded.

Step 2) $\bar{v}_{1,n} \to \bar{v}$ strongly in $H^1(B_\rho)$. Let $\bar{v}_{1,n} = w_n + v_{1,n}$, by linearity we have that the sequence $\{w_n\} \subset H_0^1(B_\rho)$ converges weakly in $H_0^1(B_\rho)$ and strongly in $L^2(B_\rho)$ to $\bar{v} - v_1$. Moreover, for any $n \in \mathbb{N}$, w_n solves

$$\int_{B_\rho} \nabla w_n \cdot \nabla \varphi - \bar{\lambda}_n w_n \varphi + \int_{B_\rho} \nabla v_{1,n} \cdot \nabla \varphi - \bar{\lambda}_n v_{1,n}\varphi = 0 \qquad \forall \varphi \in H_0^1(\Omega).$$

That is, for any $n, m \in \mathbb{N}$, we have

$$\int_{B_\rho} \nabla(w_n - w_m) \cdot \nabla \varphi + \int_{B_\rho} \nabla(v_{1,n} - v_{1,m}) \cdot \nabla \varphi - \left(\bar{\lambda}_n v_{1,n} - \bar{\lambda}_m v_{1,m} \right) \varphi$$
$$+ \left(\bar{\lambda}_n w_n - \bar{\lambda}_m w_m \right) \varphi = 0,$$

for every $\varphi \in H_0^1(\Omega)$. Taking $\varphi = w_n - w_m \in H_0^1(B_\rho)$ yields

$$\int_{B_\rho} |\nabla(w_n - w_m)|^2 = - \int_{B_\rho} \nabla(v_{1,n} - v_{1,m}) \cdot \nabla(w_n - w_m)$$
$$- \int_{B_\rho} \left(\bar{\lambda}_n w_n - \bar{\lambda}_m w_m - \bar{\lambda}_n v_{1,n} + \bar{\lambda}_m v_{1,m} \right) (w_n - w_m).$$

Now we recall that $v_{1,n} \to v$ in $H^1(B_\rho)$, $\bar{v}_{1,n} \rightharpoonup \bar{v}$ in $H^1(B_\rho)$, and that the sequence $\{\bar{\lambda}_n\}$ is bounded, as proved in Step 1. Thus, we deduce that the right hand side of the previous equation converges to 0 as $m, n \to +\infty$. That is, $\{w_n\}$ is a Cauchy sequence in $H^1(B_\rho)$, and we conclude by linearity that $\bar{v}_{1,n} \to \bar{v}$ strongly in $H^1(B_\rho)$, as claimed.

Step 3) We are ready to prove that (3.12) gives a contradiction, which entails the validity of estimate (3.11). Recall the variational characterization of $\bar{v}_{1,n}$, given in (3.10). Collecting what we proved so far, we have that $\bar{v}_{1,n} \to \bar{v}$ strongly in $H^1(B_\rho)$, where $\bar{v}$ satisfies, for some $\bar{\lambda} \in \mathbb{R}$,

$$\begin{cases} -\Delta \bar{v} = \bar{\lambda}\bar{v}, & \bar{v} > 0 \quad \text{in } B_\rho \\ \bar{v} = v_1 & \text{on } \partial B_\rho; \end{cases}$$

moreover $\|\bar{v}\|_{L^2(B_\rho)} = \|v_1\|_{L^2(B_\rho)}$. We claim that $\bar{v}$ minimizes

$$\inf \left\{ \int_{B_\rho} |\nabla w|^2 : \; w - v_1 \in H_0^1(B_\rho), \; \int_{B_\rho} w^2 = \int_{B_\rho} v_1^2 \right\}.$$

The desired contradiction follows easily from this claim: by (3.12) and the strong convergence $v_{1,n} \to v_1$, we would have that also v_1 is a nonnegative minimizer for the same problem. But any nonnegative minimizer solves

$$-\Delta v = \lambda v, \quad v > 0 \quad \text{in } B_\rho$$

for some $\lambda > 0$, which is in contradiction with the fact that $v_1 \equiv 0$ in $B_\rho \cap B_1(e_1)$. To prove that $\bar{v}$ is a minimizer, we argue again by contradiction, and suppose that there exists $w \in H^1(B_\rho)$ such that

$$w - v_1 \in H_0^1(B_\rho), \quad \int_{B_\rho} w^2 = \int_{B_\rho} v_1^2, \quad \int_{B_\rho} |\nabla w|^2 < \int_{B_\rho} |\nabla \bar{v}|^2.$$

In this case, take

$$z_n = w + (\bar{v}_{1,n} - \bar{v}) + t_n \varphi \quad \text{with } \varphi \in C_c^\infty(B_\rho) : \int_{B_\rho} w\varphi > 0.$$

and $t_n \to 0$ to be chosen later. It is plain that $z_n - \bar{v}_{1,n} \in H_0^1(B_\rho)$, with $z_n \to w$ strongly in $H^1(B_\rho)$. Thus, by strong convergence,

$$\int_{B_\rho} |\nabla w|^2 < \int_{B_\rho} |\nabla \bar{v}|^2 \implies \int_{B_\rho} |\nabla z_n|^2 < \int_{B_\rho} |\nabla \bar{v}_{1,n}|^2$$

for every n large enough. Now we show that we can choose t_n in such a way that $\int_{B_\rho} z_n^2 = \int_{B_\rho} \bar{v}_{1,n}^2$. In fact, to impose such a condition $\int_{B_\rho} z_n^2 = \int_{B_\rho} \bar{v}_{1,n}^2$ amounts to require that

$$\int_{B_\rho} \bar{v}^2 + \int_{B_\rho} (\bar{v}_{1,n} - \bar{v})^2 + t_n^2 \int_{B_\rho} \varphi^2 + 2t_n \left(\int_{B_\rho} w\varphi + (\bar{v}_{1,n} - \bar{v})\varphi \right)$$

$$+ 2 \int_{B_\rho} (\bar{v}_{1,n} - \bar{v})w = \int_{B_\rho} \bar{v}_{1,n}^2.$$

This is an equation of type

$$t_n^2 + a_n t_n + b_n = 0 \quad \text{with } a_n \to a > 0 \text{ and } b_n \to 0,$$

where we used the fact that $\int_{B_\rho} w\varphi > 0$ by assumption, and the convergence of $\bar{v}_{1,n}$ to $\bar{v}$. Such an equation clearly admits a solution $t_n \to 0$. To sum up, we showed that z_n is an admissible competitor for $\bar{v}_{1,n}$ with a lower energy, in contradiction with the minimality of $\bar{v}_{1,n}$. The contradiction shows that w as above cannot exist, that is, $\bar{v}_1$ is a minimizer. As observed, this completes the proof of the lemma. $\qquad \square$

Conclusion of the proof of Theorem 1.1 As consequence of the Lemma 3.7, we can give
a quantitative estimate for the energy gap between $v_{1,n}$ and $\bar{v}_{1,n}$. Indeed, exploiting
also (3.7), we have that

$$
\int_{\mathbb{R}^N} |\nabla \bar{v}_{1,n}|^2 = \int_{B_\rho} |\nabla \bar{v}_{1,n}|^2 + \int_{\mathbb{R}^N \setminus B_\rho} |\nabla \bar{v}_{1,n}|^2 \leq (1-\varepsilon) \int_{B_\rho} |\nabla v_{1,n}|^2 + \int_{\mathbb{R}^N \setminus B_\rho} |\nabla v_{1,n}|^2
$$

$$
\leq \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2 - \varepsilon \int_{B_\rho} |\nabla v_{1,n}|^2
$$

$$
\leq \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2 - \varepsilon C \frac{r_n^N M_n^2}{\lambda_{1,n}} \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2 = \left(1 - \varepsilon C \frac{M_n^4}{\lambda_{1,n}} \delta_n \right) \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2,
$$

where $\delta_n = r_n^N / M_n^2$, as in Lemma 3.6, and $C > 0$ is a positive constant independent of
n. Combining this estimate with Lemma 3.6, we can finally prove that the competitor
$\bar{\mathbf{v}}_n = (\bar{v}_{1,n}, \bar{v}_{2,n}, \ldots, \bar{v}_{k,n})$ has lower energy than $\mathbf{v}_n$: indeed

$$
J(\bar{\mathbf{v}}_n) = \sum_{i=1}^{k} \frac{\int_{\mathbb{R}^N} |\nabla \bar{v}_{i,n}|^2}{\int_{\mathbb{R}^N} \bar{v}_{i,n}^2} = \frac{\int_{\mathbb{R}^N} |\nabla \bar{v}_{1,n}|^2}{\int_{\mathbb{R}^N} \bar{v}_{1,n}^2} + \sum_{j=2}^{k} \frac{\int_{\mathbb{R}^N} |\nabla \bar{v}_{j,n}|^2}{\int_{\mathbb{R}^N} \bar{v}_{j,n}^2}
$$

$$
\leq \left(1 - \varepsilon C \frac{M_n^4}{\lambda_{1,n}} \delta_n \right) \frac{\int_{\mathbb{R}^N} |\nabla v_{1,n}|^2}{\int_{\mathbb{R}^N} v_{1,n}^2} + \sum_{j=2}^{k} \left(1 + \frac{C}{\lambda_{j,n}} \delta_n \right) \frac{\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2}{\int_{\mathbb{R}^N} v_{j,n}^2}
$$

$$
\leq J(\mathbf{v}_n) + \left(-\varepsilon C \frac{M_n^4}{\lambda_{1,n}} \delta_n r_n^2 \lambda_{1,n} + \sum_{j=2}^{N} \frac{C}{\lambda_{j,n}} \delta_n r_n^2 \lambda_{j,n} \right)
$$

$$
\leq J(\mathbf{v}_n) + \delta_n r_n^2 \left(C(k-1) - \varepsilon C M_n^4 \right) < J(\mathbf{v}_n)
$$

for every n large, where in the last step we have exploited the fact that $M_n \to +\infty$, as
by assumption. This is a contradiction with the minimality of $\mathbf{v}_n$, and completes the
proof of the Lipschitz bound in Theorem 1.1.   □

We now pass to the proof of Theorem 1.2. We start with an estimate which implies
the convergence of c_r to c_0 as $r \to 0$.

Lemma 3.8 *There exists a constant $C > 0$ such that*

$$
c_0 \leq c_r \leq c_0 + Cr
$$

for any $r > 0$.

Proof The estimate $c_0 \leq c_r$ is straightforward, so we prove the second one. We will
use the solution of the problem c_0 (that is with distance constraint $r = 0$) to construct
a competitor for c_r with $r > 0$.

Let $\mathbf{u} \in H_0^1(\Omega)$ be any minimizer of the problem c_0, and recall from Theorem A that $\mathbf{u} \in \text{Lip}(\overline{\Omega})$. We denote $K = \max_{i=1,\dots,N} \|\nabla u_i\|_{L^\infty(\Omega)}$. For any $i = 1, \dots, N$ we let $\Omega_i = \{u_i > 0\}$ so that $\Omega_i \cap \Omega_j = \emptyset$, and we have

$$\int_\Omega u_i^2 = 1, \quad \int_\Omega |\nabla u_i|^2 = \lambda_1(\Omega_i).$$

We also recall from Theorem A that the free-boundary $\mathcal{N} := \{x \in \Omega : \mathbf{u}(x) = 0\} = \Omega \backslash (\cup_i \Omega_i)$ is an $(N-1)$-rectifiable set of finite $(N-1)$-Hausdorff measure. In particular, by the rectifiability of $\mathcal{N}$, we have that the Minkowski content of $\mathcal{N}$ coincides with its $(N-1)$-Hausdorff measure ([18, Thm 3.2.39]). More explicitly, if for a given $r > 0$ we denote the r-tubular neighborhood of $\mathcal{N}$ as

$$\mathcal{N}_r = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{N}) < r\},$$

then we have

$$\lim_{r \to 0^+} \frac{|\mathcal{N}_r|}{2r} = \mathcal{H}_{N-1}(\mathcal{N}),$$

where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^n$.

Fix now a constant $C \geq 2$ such that for any $r > 0$ sufficiently small we have

$$|\mathcal{N}_r| \leq Cr\mathcal{H}_{N-1}(\mathcal{N}).$$

We define a cutoff function $\eta \in Lip(\Omega)$ as follows

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \mathcal{N}_{r/2}, \\ \frac{\text{dist}(x,\mathcal{N})-r/2}{r/2} & \text{if } \text{dist}(x, \mathcal{N}) \in [r/2, r], \\ 1 & \text{if } \Omega \backslash \mathcal{N}_r. \end{cases}$$

Observe that $0 \leq \eta(x) \leq 1$ and we have, for a.e. $x \in \Omega$,

$$|\nabla\eta(x)| = \begin{cases} 0 & \text{if } \text{dist}(x, \mathcal{N}) < r/2 \text{ or } \text{dist}(x, \mathcal{N}) > r \\ \frac{2}{r} & \text{if } \text{dist}(x, \mathcal{N}) \in [r/2, r]. \end{cases}$$

We claim that, for $r > 0$ small enough, the function $\mathbf{u}\eta \in H_0^1(\Omega)$ is an admissible competitor for the functional with distance constraint greater than or equal to 0, and moreover that

$$J(\mathbf{u}\eta) \leq J(\mathbf{u}) + Cr\mathcal{H}_{N-1}(\mathcal{N}).$$

Indeed, by construction we immediately see that, for any $i \neq j$,

$$\text{dist}(\text{supp}(u_i\eta), \text{supp}(u_j\eta)) \geq r.$$

Thus $\mathbf{u}\eta$ is an admissible competitor for c_r. In order to estimate the energy

$$J(\mathbf{u}\eta) = \sum_{i=1}^{k} \frac{\displaystyle\int_{\Omega_i} |\nabla(u_i\eta)|^2}{\displaystyle\int_{\Omega_i} |u_i\eta|^2},$$

we proceed separately for each component. We start with the denominator corresponding to the function $u_i\eta$, for which we can write

$$\int_{\Omega_i} |u_i\eta|^2 = \int_{\Omega_i} |u_i|^2 - \int_{\Omega_i \cap \mathcal{N}_r} |u_i|^2(1 - |\eta|^2) = 1 - \int_{\Omega_i \cap \mathcal{N}_r} |u_i|^2(1 - |\eta|^2).$$

Since $|\nabla u_i| \le K$, we find that

$$|u_i(x)| \le \mathrm{dist}(x, \mathcal{N})K \qquad \forall x \in \Omega;$$

thus, recalling that $0 \le \eta(x) \le 1$ for any $x \in \Omega$, we can carry on with the estimate as follows

$$\int_{\Omega_i \cap \mathcal{N}_r} |u_i|^2(1 - |\eta|^2) \le \int_{\Omega_i \cap \mathcal{N}_r} \mathrm{dist}(x, \mathcal{N})^2 K^2 \le K^2 r^2 |\Omega_i \cap \mathcal{N}_r|.$$

Finally we find that, for $r > 0$ small

$$\left(\int_{\Omega_i} |u_i\eta|^2\right)^{-1} \le 1 + 2K^2 r^2 |\Omega_i \cap \mathcal{N}_r|.$$

Concerning the numerator of the Rayleigh quotient, we get

$$\int_{\Omega_i} |\nabla(u_i\eta)|^2 = \int_{\Omega_i} |\nabla u_i|^2 + \int_{\Omega_i \cap \mathcal{N}_r} (|\nabla(u_i\eta)|^2 - |\nabla u_i|^2)$$

$$= \lambda_1(\Omega_i) + \int_{\Omega_i \cap \mathcal{N}_r} (|\nabla u_i|^2(|\eta|^2 - 1) + |u_i|^2|\nabla\eta|^2 + 2u_i\eta\nabla u_i \cdot \nabla\eta)$$

$$\le \lambda_1(\Omega_i) + \int_{\Omega_i \cap \mathcal{N}_r} (|u_i|^2|\nabla\eta|^2 + 2u_i\eta\nabla u_i \cdot \nabla\eta).$$

We estimate the two remaining terms separately. For the first one we have

$$\int_{\Omega_i \cap \mathcal{N}_r} |u_i|^2|\nabla\eta|^2 \le \int_{\Omega_i \cap \{r/2 < \mathrm{dist}(x,\mathcal{N}) < r\}} \mathrm{dist}(x, \mathcal{N})^2 K^2 \frac{4}{r^2} \le 4K^2 |\Omega_i \cap \mathcal{N}_r|.$$

For the second one, in a similar fashion, we obtain

$$\int_{\Omega_i \cap \mathcal{N}_r} 2u_i \eta \nabla u_i \cdot \nabla \eta \le \int_{\Omega_i \cap \mathcal{N}_r} 2u_i |\nabla u_i||\nabla \eta|$$

$$\le \int_{\Omega_i \cap \{r/2 < \mathrm{dist}(x, \mathcal{N}) < r\}} 2 \, \mathrm{dist}(x, \mathcal{N}) K^2 \frac{2}{r} \le 4K^2 |\Omega_i \cap \mathcal{N}_r|.$$

As a result, recollecting the two estimates, we obtain

$$\int_{\Omega_i} |\nabla(u_i \eta)|^2 \le \lambda_1(\Omega_i) + 8K^2 |\Omega_i \cap \mathcal{N}_r|.$$

By combining the previous inequalities, we can control the Rayleigh quotient of $u_i \eta$ as follows:

$$\frac{\displaystyle\int_{\Omega_i} |\nabla(u_i \eta)|^2}{\displaystyle\int_{\Omega_i} |u_i \eta|^2} \le \lambda_1(\Omega_i) + CK^2 |\Omega_i \cap \mathcal{N}_r| + R(r),$$

where C is a constant independent of r and $R(r)$ is a remainder term of higher order. Summing up in i we find

$$J(\mathbf{u}\eta) = \sum_{i=1}^{k} \frac{\displaystyle\int_{\Omega_i} |\nabla(u_i \eta)|^2}{\displaystyle\int_{\Omega_i} |u_i \eta|^2} \le \sum_{i=1}^{k} \lambda_1(\Omega_i) + CK^2 \sum_{i=1}^{k} |\Omega_i \cap \{\mathrm{dist}(x, \mathcal{N}) < r\}|$$

$$\le J(\mathbf{u}) + CK^2 |\mathcal{N}_r| \le J(\mathbf{u}) + C2r\mathcal{H}_{N-1}(\mathcal{N}).$$

To conclude, it suffices to remark that, since $\mathbf{u}\eta$ is an admissible competitor for c_r,

$$c_r \le J(\mathbf{u}\eta) \le J(\mathbf{u}) + C2r\mathcal{H}_{N-1}(\mathcal{N}) = c_0 + C2r\mathcal{H}_{N-1}(\mathcal{N}).$$

$\square$

Proof of Theorem 1.2 Let $\mathbf{u}_r$ be a minimizer for problem c_r, with $r > 0$ sufficiently small. From Theorem 1.1, Lemma 3.8 and the fact that $\mathbf{u}_r|_{\partial \Omega} = 0$, there exists $\mathbf{u} \in \mathrm{Lip}(\overline{\Omega})$ such that, up to a subsequence,

$$\mathbf{u}_r \to \mathbf{u} \quad \text{weakly in } H^1(\Omega), \text{ strongly in } C^{0,\alpha}(\overline{\Omega}).$$

This implies that $\|u_{i,0}\|_{L^2(\Omega)} = 1$ for every i, $\int_\Omega u_{i,0}^2 u_{j,0}^2 = 0$ and dist $(\mathrm{supp}\, u_{i,0}, \mathrm{supp}\, u_{j,0}) = 0$ for every $i \ne j$; thus, $\mathbf{u}_0$ is an admissible competitor for c_0. Moreover, since $\|\nabla u_{i,0}\|_{L^2(\Omega)} \le \liminf_{r \to 0} \|\nabla u_{i,r}\|_{L^2(\Omega)}$ for every i,

$$c_0 = \lim_{r \to 0} c_r = \lim_{r \to 0} \sum_{i=1}^{k} \int_\Omega |\nabla u_{i,r}|^2 \geq \sum_{i=1}^{k} \int_\Omega |\nabla u_{i,0}|^2 \geq c_0,$$

which shows that $\mathbf{u}_0$ achieves c_0, and that $\mathbf{u}_r \to \mathbf{u}_0$ strongly in $H_0^1(\Omega)$. $\qquad \square$

Remark 3.9 The proof of Theorem 1.3 (the case of singularly perturbed harmonic maps with distance constraint) follows by similar arguments, with few differences (for instance the corresponding results in Sect. 2 are much easier to prove). In particular, functions $\mathbf{u}_r$ are not zero on $\partial\Omega$ and we cannot achieve the first conclusion of Lemma 3.4, i.e., the sequence $\{x_n\}$ may accumulate at $\partial\Omega$. To circumvent this issue one can reason with the family of functions $\{\mathbf{u}_r \eta\}$, where $\eta \in C_0^\infty(\Omega)$ is a positive smooth cutoff. We refer to [30] for further details. This is the reason why the uniform estimate in Theorem 1.3 is only of local type (true in any compact $K \subset \Omega$).

Declarations

Conflit of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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