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Green function estimates on complements of low-dimensional uniformly rectifiable sets

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Abstract

It has been recently established in David and Mayboroda (Approximation of green functions and domains with uniformly rectifiable boundaries of all dimensions. arXiv:2010.09793) that on uniformly rectifiable sets the Green function is almost affine in the weak sense, and moreover, in some scenarios such Green function estimates are equivalent to the uniform rectifiability of a set. The present paper tackles a strong analogue of these results, starting with the "flagship" degenerate operators on sets with lower dimensional boundaries. We consider the elliptic operators $L_{\beta,\gamma} = -\operatorname{div} D^{d+1+\gamma-n} \nabla$ associated to a domain $\Omega \subset \mathbb{R}^n$ with a uniformly rectifiable boundary Γ of dimension $d < n-1$, the now usual distance to the boundary $D = D_\beta$ given by $D_\beta(X)^{-\beta} = \int_\Gamma |X - y|^{-d-\beta} d\sigma(y)$ for $X \in \Omega$, where $\beta > 0$ and $\gamma \in (-1, 1)$. In this paper we show that the Green function G for $L_{\beta,\gamma}$, with pole at infinity, is well approximated by multiples of $D^{1-\gamma}$, in the sense that the function $\left| D\nabla \left(\ln \left(\frac{G}{D^{1-\gamma}} \right) \right) \right|^2$ satisfies a Carleson measure estimate on Ω. We underline that the strong and the weak results are different in nature and, of course, at the level

Communicated by Loukas Grafakos.

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of the proofs: the latter extensively used compactness arguments, while the present paper relies on some intricate integration by parts and the properties of the "magical" distance function from David et al. (Duke Math J, to appear).

Mathematics Subject Classification 42B37 · 31B25 · 35J25 · 35J70

Résumé

Dans David and Mayboroda (Approximation of green functions and domains with uniformly rectifiable boundaries of all dimensions. arXiv:2010.09793) il est démontré que pour les domaines à bord uniformément rectifiable, la fonction de Green vérifie des estimations faibles de bonne approximation par des fonctions affines, avec une réciproque vraie dans certains cas encourageants. Ici on part de la rectifiabilité uniforme et on démontre les estimations fortes naturelles d'approximation de la fonction de Green, et aussi des solutions, par des applications affines (ou, de manière équivalente, des multiples de la distance au bord adoucie). L'étude inclut les analogues naturels du Laplacien dans les domaine dont la frontière est de grande co-dimension. On considère les opérateurs elliptiques $L_{\beta,\gamma} = \operatorname{div} D^{d+1+\gamma-n}\nabla$ associés à un domaine $\Omega \subset \mathbb{R}^n$ dont le bord Γ est Ahlfors régulier et uniformément rectifiable de dimension $d < n-1$ et à la distance au bord maintenant usuelle $D = D_\beta$ définie par $D_\beta(X)^{-\beta} = \int_\Gamma |X - y|^{-d-\beta} d\sigma(y)$ pour $X \in \Omega$, où $\beta > 0$ et $\gamma \in (-1, 1)$ sont des paramètres et σ une mesure Ahlfors régulière sur Γ. Les auteurs ont montré précédemment que la mesure elliptique associée à $L_{\beta,\gamma}$ est bien définie et est mutuellement absolument continue par rapport à σ, avec un poids de A_∞. Ici on démontre que la fonction de Green G avec pôle à l'infini associée à $L_{\beta,\gamma}$ est bien approchée par les multiples de D, au sens où la fonction $\left|D\nabla\left(\ln\left(\frac{G}{D^{1-\gamma}}\right)\right)\right|^2$ vérifie une condition de Carleson sur Ω. Ces nouvelles estimations sont différentes en nature. Les estimations de David and Mayboroda (Approximation of green functions and domains with uniformly rectifiable boundaries of all dimensions. arXiv:2010.09793) reposaient sur des arguments de compacité; ici on a besoin d'estimations plus précises, obtenues par intégration par parties et en utilisant les propriétés algébriques de la fonction D_α dans le cas "magique" de David et al. (Duke Math J, to appear).

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1 Introduction

Rectifiable sets are an important notion in geometric measure theory and the calculus of variation, in particular because the sets that minimize an energy often enter this category. In the past decades, many mathematicians worked on finding characteriza-

tions of rectifiability by properties apparently unrelated to geometric measure theory. In the early 90's, the quantifiable version—uniform rectifiability—was introduced in [7, 8] along with many characterizations in terms of geometry (such as big pieces of Lipschitz images, or using Peter Jones' β numbers) and in terms of singular integrals. Later, it was observed that uniformly rectifiable sets may be the right extension of Lipschitz graphs for elliptic boundary value problems, that is, if $\Omega \subset \mathbb{R}^n$ is an open set with uniformly rectifiable boundary and Ω provides enough access its the boundary, then we can control the oscillations of the harmonic functions in Ω. It is even more noteworthy that a criterion for rectifiability can be obtained using harmonic functions. Indeed, Hofmann, Martell, and Uriarte-Tuero proved in [19, 21] that under some conditions regarding the access to the boundary, $\partial\Omega$ is uniformly rectifiable if and only if the harmonic measure on $\partial\Omega$ is absolutely continuous in a quantitative way—called A_∞—with respect to the surface measure (see also [2]). The optimal topological conditions in this regard have been identified in [1]. These were accompanied but a rich array of beautiful and difficult alternative characterizations, exploring Carleson estimates for the solutions, behavior of the singular integral operators, extensions to more general elliptic operators, to mention just a few. Here we do not aim to provide a survey of the related literature; the reader can consult, e.g., [12] for a more detailed presentation of the literature.

A weakness of the above theory is the fact that the harmonic measure on Γ only makes sense when $\Gamma \subset \mathbb{R}^n$ is of dimension $d > n-2$, because lower dimensional sets have probability zero to be hit by a Brownian motion. As a consequence, rectifiability, a notion that exists for all integer dimensions, can only be characterized by means of the harmonic measure for sets $\Gamma \subset \mathbb{R}^n$ of dimension $d = n - 1$. To overcome this obstacle, the authors of the present article developed a theory of degenerate elliptic operators. The idea was to define a 'harmonic measure' on a set Γ of dimension $d < n - 1$ by replacing the Laplacian by an operator on $\Omega := \mathbb{R}^n \setminus \Gamma$ in the form $L = -\operatorname{div} w(x)\nabla$, where $w(x)$ goes to infinity at an appropriate rate when x is approaching Γ, so that the corresponding 'Brownian motion' is attracted by Γ and hits the boundary with probability one. The articles [10, 13] set the elliptic theory for this. In particular an appropriate elliptic measure is constructed for a large class of sets of any dimension $d < n$ (or even mixed dimensions) and operators L as above, with the usual nice properties such as the non-degeneracy and doubling properties for the harmonic measure, the Harnack inequality and the comparison principle for solutions, and estimates for the change of poles.

Then we tested the relevancy of our new elliptic measure. Dahlberg proved in [6] that the classical harmonic measure is absolutely continuous with respect to the surface measure, and even A_∞, whenever the boundary is Lipschitz. In [11] we assumed that Γ is the graph of a Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ with small Lipschitz constant, and proved the same A_∞ property for correctly chosen operators. Yet we had to be careful about the operator we picked because as showed in [12], based on counterexamples from [3, 23], not every operator L will work. We wanted an explicit operator, which could be constructed by a single method for any set of dimension d. We set our choice on

$$L_{\beta,\gamma} := -\operatorname{div}(D_\beta)^{d+1+\gamma-n}\nabla, \qquad (1.1)$$

where $\beta > 0$, $\gamma \in (-1, 1)$, and D_β is defined on Ω as

$$D_\beta(X) := \left(\int_\Gamma |X - y|^{-d-\beta} d\sigma(y) \right)^{-1/\beta}. \tag{1.2}$$

The quantity D_β is equivalent to the distance to the boundary, i.e., there exists $C > 0$ such that

$$C^{-1} \operatorname{dist}(X, \Gamma) \leq D_\beta \leq C \operatorname{dist}(X, \Gamma) \qquad \text{for } X \in \Omega, \tag{1.3}$$

but the advantage of D_β over $\operatorname{dist}(., \Gamma)$ is to always be smooth in a certain quantitative way.

When $\gamma = 0$, the level of the degeneracy D_β^{d+1-n} in the coefficients of the operator $L_{\beta,0} := -\operatorname{div}(D_\beta)^{d+1-n}\nabla$ makes it a perfect analogue of the Laplacian for the sets with a d-dimensional boundary when $d < n - 1$. In particular, we proved that the harmonic measure associated to $L_{\beta,\gamma}$ is A_∞ with respect to the d-dimensional Hausdorff measure. Moreover, recently, in two distinct papers [14, 17] we extended this result to the more general case where $\Gamma \subset \mathbb{R}^n$ is uniformly rectifiable. Studying these operators and the dimension of the support of the corresponding elliptic measure, we were naturally drawn to introducing a parameter γ which seemingly unbalances the situation. Besides, for $d = n - 1$, the operator $L_{\beta,\gamma} := -\operatorname{div}(D_\beta)^\gamma \nabla$ is the celebrated Caffarelli-Silvester extension of the fractional Laplacian operator (cf. [4]). However, we were surprised to realize that the argument in [11] extends to all $\gamma \in (-1, 1)$ rather simply, and the generalization to uniformly rectifiable set is stated for any $\gamma \in (-1, 1)$ in [17]. The proof in [14] relies on geometric arguments, such as corona decompositions and the construction of sawtooth domains, and an extrapolation argument, which allow one to reduce to the case of small Lipschitz graphs. The proof of [17] is substantially simpler and more direct, and is based on a trick unique to the case where the dimension of Γ is at most $n - 2$.

Theorem 1.4 [14, 17] *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d < n - 1$, and let σ be an Ahlfors regular measure that satisfies (2.1). Take $\beta > 0$, $\gamma \in (-1, 1)$, define $L_{\beta,\gamma}$ as in (1.1), and construct the associated harmonic measure $\omega_{\beta,\gamma}^X$ as in Definition 2.20. Then $\omega_{\beta,\gamma}^X$ is A_∞-absolutely continuous with respect to σ. This means that for every choice of $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$, that depends only on C_σ, C_0, ϵ, n, d, β, and γ, such that for each choice of $x \in \Gamma$, $r > 0$, a Borel set $E \subset B(x, r) \cap \Gamma$, and a corkscrew point $X = A_{x,r}$ as in (2.3),*

$$\frac{\sigma(E)}{\sigma(B(x, r) \cap \Gamma)} < \delta \Rightarrow \frac{\omega_{\Omega,L}^X(E)}{\omega_{\Omega,L}^X(B(x, r) \cap \Gamma)} < \epsilon. \tag{1.5}$$

It is known that in the present context where all our measures are doubling, the A_∞ condition also implies that under the assumptions of (1.5),

$$\frac{\omega_{\Omega,L}^X(E)}{\omega_{\Omega,L}^X(B(x, r) \cap \Gamma)} < \delta \Rightarrow \frac{\sigma(E)}{\sigma(B(x, r) \cap \Gamma)} < \epsilon. \tag{1.6}$$

Observe that in the previous theorem, contrary to the case of co-dimension 1, i.e. when $d = n - 1$, we don't assume any topological condition. And that is perfectly natural, since the domain $\Omega := \mathbb{R}^n \setminus \Gamma$ has a lot of paths and ample access to the boundary (see Lemmas 2.1 and 11.6 in [10]). The next big objective would be to prove the reverse implication, meaning that if the harmonic measure on Γ is A_∞ with respect to the Hausdorff measure, then Γ is uniformly rectifiable. Unfortunately (and surprisingly) this fails brutally when $d + \beta = n + 2$ (see [9]), and although we expect this to be the only exception, the methods used to prove the converse in co-dimension 1 do not appear to be adaptable to the higher codimension case.

The purpose of the present paper is to provide different estimates on the harmonic functions, which we hope will ultimately furnish one side of the desired criterion. Indeed, as established in [15], some weak bounds on the Green function are equivalent to the uniform rectifiability even in lower dimensional settings. The general idea is that instead of trying to characterize rectifiable sets using the harmonic measure, we would do so using the property that the Green function behaves like a distance to the boundary. This is not a surprising idea, as the Green functions and harmonic measure are deeply connected, and are both prominent in the analysis of the free boundary problems, and the proof of the properties of the harmonic measure in [10, 13] heavily relies on a comparison between the harmonic measure and Green functions. Yet, the results we are about to prove are not known in the "classical" setting of domains with an $n - 1$ dimensional boundary, except for a few simple situations and certainly not in the generality of the uniformly rectifiable sets which we attack in this paper.

We will use the Green function with a pole at infinity, which is constructed in [9, Definition 6.2, Lemma 6.5] with the following properties.

Proposition 1.7 [9] *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d < n - 1$. Take $\beta > 0$, $\gamma \in (-1, 1)$, and construct $L_{\beta,\gamma}$ on $\Omega := \mathbb{R}^n \setminus \Gamma$ as in (1.1). There exists a continuous function $G^\infty = G^\infty_{\beta,\gamma}$ on $\mathbb{R}^n$ such that*

(i) *$G^\infty > 0$ on Ω,*
(ii) *$G^\infty = 0$ on Γ,*
(iii) *there exists a positive Borel measure $\omega_\infty = \omega^\infty_{\beta,\gamma}$ on Γ such that*

$$\int_\Omega \nabla G^\infty \cdot \nabla \varphi \, D_\beta^{d+1-n+\gamma} dx = \int_\Gamma \varphi(y) d\omega^\infty(y) \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n).$$

We call G^∞ the Green function with pole at infinity, and G^∞ is unique up to multiplication by a scalar constant.

The Green function with pole at infinity for the Laplacian in $\mathbb{R}^n_+ = \{(x, t) \in \mathbb{R}^{n-1} \times (0, +\infty)\}$ is $G^\infty(x, t) = t$. The one for the operator $-\operatorname{div} |t|^{d+1-n}$ in $\mathbb{R}^n \setminus \mathbb{R}^d = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^{n-d}, \, t \neq 0\}$ is $G^\infty(x, t) = |t|$. There is a third case where G^∞ can be computed, which is when Γ is a d-Ahlfors regular set with $d < n - 2$, and we choose the specific operator $L_{n-d-2,0}$; then one can easily compute that $G^\infty = D_{n-d-2}$. In [9] this is called the magic case. The rough idea of what follows is that, when $\gamma = 0$, a good Green function at infinity should behave a bit like the distance to the boundary Γ, or like D_β which is its smooth substitute. When $\gamma \neq 0$, similar homogeneity

considerations lead us to expect that in the good cases, G^∞ will behave like $D_\beta^{1-\gamma}$ instead. Here is our main result.

Theorem 1.8 *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d < n - 1$, and let σ be an Ahlfors regular measure on Γ that satisfies (2.1). For $\beta > 0$ and $\gamma \in (-1, 1)$, let $L_{\beta,\gamma}$ be as in (1.1) and define $G_{\beta,\gamma}^\infty$ as in Proposition 1.7. Then for any $\alpha > 0$, there exists $C > 0$ that depends only on C_σ, C_0, α, β, γ, d, n such that for any ball $B := B(x, r)$ centered on Γ, one has*

$$\int_B \left| \nabla \ln \left(\frac{G_{\beta,\gamma}^\infty}{D_\alpha^{1-\gamma}} \right) \right|^2 D_\alpha^{d+2-n} \, dX \leq C\sigma(B). \tag{1.9}$$

Remarks:

- Remember that $G_{\beta,\gamma}^\infty$ is only defined up to a constant. Yet, the above theorem makes perfect sense, because the value of left-hand side of (1.9) does not change when we replace $G_{\beta,\gamma}^\infty$ by $K G_{\beta,\gamma}^\infty$, where K is a positive constant.
- Theorem 1.8 is just the application of Theorem 2.21 to the Green function at infinity: we shall generalize the estimate of Theorem 1.8 to a large class of solutions that is interesting by itself. However, we decided to highlight the above statement, which was our true purpose in our search for characterizations of rectifiability.
- Here we decided to use the same measure σ for the definitions of $L_{\beta,\gamma}$ and D_α, but a minor modification of the proof would allow us to take two different Ahlfors regular measures σ and $\widetilde{\sigma}$ to define $L_{\beta,\gamma}$ and and D_α.
- As we have pointed out above, these results are not known for $d = n - 1$ in the generality of the uniformly rectifiable sets. In the half-space one can see somewhat similar estimates in [16] and the bounds on the second derivatives of the Green function for a special class of operators were, in disguise, obtained in [20]. However, it is not clear how to deduce from [20] a co-dimension 1 analogue of (1.9), directly or using interpolation.

In the present paper we only show that good geometric properties (the uniform rectifiability of Γ) imply precise approximation properties of G^∞ by D_α, and the proof will rely heavily on the A_∞ property of the harmonic measure and some intermediate results in [17] concerning the uniform rectifiability of Γ. We do not address the issue of the converse in this paper. Yet there are good reasons to believe that it may be easier to prove than for the absolute continuity of the harmonic measure. In a parallel paper [15], the authors study a less precise (weaker) approximation property of the Green function, and show that in some cases (but where $d > n - 2$) it already implies the uniform rectifiability of Γ, while in the case of the present paper it implies another strange property of potentials defined on Γ. But we did not manage to prove that this strange property is impossible to obtain when $d + \alpha \neq n + 2$.

The reader should be aware that even though we think of the Green function estimate (1.9) as a possible alternative to the A_∞ absolute continuity of the harmonic measure for the characterization of uniform rectifiability, we already know that in the general context of elliptic operator in the form $L = -\operatorname{div} A\nabla$, the A_∞ absolute continuity (1.5)—or (1.6)—is not equivalent to (1.9). A counterexample in $\mathbb{R}_+^n$, that can be

extended to any codimensions using the construction (4.6) in [12], is given in the next lines. Denote the running point in $\mathbb{R}^n_+$ as $(x, r) \in \mathbb{R}^{n-1} \times (0, +\infty)$, and observe that in this case $D_\alpha(x, r) = c_\alpha r$. We set $b(r) := 1/(2 + \cos(r))$ and we construct $L_b := -\operatorname{div} b(r)\nabla$. By uniqueness of the Green function with pole at infinity (Proposition 1.7), we have

$$G^\infty(x, r) = \int_0^r \frac{ds}{b(s)} = 2r + \sin(r).$$

On one hand, $r \leq G^\infty(x, r) \leq 3r$ for all $(x, r) \in \mathbb{R}^n_+$, and that is enough, by using a comparison principle with the harmonic measure, to show that the harmonic measure associated to L is equivalent to the surface measure on $\mathbb{R}^{n-1}$ (see for instance Theorem 1.17 in [18] for details), hence it is A_∞ absolutely continuous with respect to the surface measure. On the other hand, we have

$$\left| \nabla \ln \left(\frac{G_\infty}{D_\alpha} \right) \right| = \left| \frac{\nabla G^\infty}{G^\infty} - \frac{\nabla D_\alpha}{D_\alpha} \right| = \left| \frac{2 + \cos(r)}{2r + \sin(r)} - \frac{1}{r} \right| \geq \frac{1}{3r} \left| \cos(r) - \frac{\sin(r)}{r} \right|.$$

That is, for any ball $B(x, R) \subset \mathbb{R}^n$ centered on the boundary $\mathbb{R}^{n-1}$, we have

$$\int_{B(x,R)} \left| \nabla \ln \left(\frac{G_\infty}{D_\alpha} \right) \right|^2 D_\alpha \, dX \geq c_n R^{n-1} \int_0^{R/2} \left| \cos(r) - \frac{\sin(r)}{r} \right|^2 \frac{dr}{r} \geq c'_n R^{n-1} \ln(R),$$

when R is large (e.g. $R \geq 100$), and where c_n, c'_n are constants that depends only on n. As a consequence, (1.9) fails for the elliptic operator L_b. This highlights the difference between Green function and harmonic measure estimates for general elliptic operators.

Coming back to the Green function, the reader might wonder about the comparison with the results in [15]. It is not easy to describe the results in [15] avoiding technicalities, but roughly speaking, we only proved there the weak statement that the set where the Green function behaves like a distance to the boundary is a Carleson-prevalent set. This carries the structural information saying that there are a lot of points and a lot of scales where the desired estimate is true, but it carries no norm control. Respectively, the methods heavily rely on compactness arguments. In the present paper the arguments and the results are completely different, aiming at a strong norm control in the form of (1.9) and using a new idea that the "magical" distance function identified in [9], being an explicit solution to a certain PDE, can be effectively used in the integration-by-parts arguments (cf. [17]). A good comparison is a familiar to many experts integration by parts with the weight t which disappears when the Laplacian hits t, although the details in our case are necessarily considerably more involved.

In the next section, we shall give the definitions that we skipped up to now to lighten the introduction, such as the definition of Ahlfors regular and uniformly rectifiable sets. We also introduce the results from [17] which we will rely upon. The remainder of the article will be devoted to the proof of our main result.

We shall use the notation $A(x) \lesssim B(x)$ when $A(x) \leq C B(x)$ and C is a constant whose dependence into the various parameters will be either recalled or obvious from context. We also write $A(x) \approx B(x)$ when $A(x) \lesssim B(x)$ and $A(x) \gtrsim B(x)$.

2 Definitions and anterior results

For the rest of the article, we take $\Gamma \subset \mathbb{R}^n$ and $\Omega := \mathbb{R}^n \setminus \Gamma$. We assume that Γ is a d-Ahlfors regular set with $d < n - 1$, that is Γ is closed and there exists a measure σ supported on Γ and a constant $C_\sigma \geq 1$ such that

$$C_\sigma^{-1} r^d \leq \sigma(B(x, r)) \leq C_\sigma r^d \qquad \text{for } x \in \Gamma, \ r > 0. \tag{2.1}$$

It is known that if the above property (2.1) is true for some measure σ, then it is also true when σ is replaced by $\mathcal{H}^d|_\Gamma$—the d-dimensional Hausdorff measure restricted to Γ.

The Ahlfors regularity of Γ and the low dimension $d < n - 1$ are sufficient conditions to obtain the aforementioned equivalence (1.3). Indeed, Lemma 5.1 in [11] gives us that

$$C^{-1} \operatorname{dist}(X, \Gamma) \leq D_\beta \leq C \operatorname{dist}(X, \Gamma) \qquad \text{for } X \in \Omega, \tag{2.2}$$

where the constant $C > 0$ above depends only on $\beta > 0$, C_σ and $n - d > 1$.

Moreover, Lemma 11.6 in [10] entails the existence of a constant C that depends only on C_σ and $n - d > 1$ such that for any $x \in \Gamma$ and $r > 0$, we can find a point $A_{x,r}$ such that

$$C^{-1} r \leq \operatorname{dist}(A_{x,r}, \Gamma) \leq |A_{x,r} - x| \leq Cr. \tag{2.3}$$

In other words, when Γ is d-Ahlfors regular with $d < n - 1$, its complement automatically satisfies the interior corkscrew condition.

We shall also assume that Γ is uniformly rectifiable. Equivalent definitions of uniform rectifiability were given in [7, 8], and the reader may use their preferred one, but since we will only use the uniform rectifiability of Γ via results from [17] that rely on the summability properties of Tolsa's α-numbers, we will use these properties as a definition of uniform rectifiability. We need some notation first.

We denote by Ξ the set of affine d-dimensional planes in $\mathbb{R}^n$. Each plane $P \in \Xi$ is associated with a measure μ_P, which is the restriction to P of the d-dimensional Hausdorff measure (i.e. μ_P is the Lebesgue measure on the plane). A **flat measure** is a measure μ that can be written $\mu = c\mu_P$ where c is a positive constant and $P \in \Xi$. The set of flat measure is called $\mathcal{F}$.

We need the following variant of Wasserstein distances to quantify the difference between two measures, and then measure how far σ is from flat measures.

Definition 2.4 For $x \in \mathbb{R}^n$ and $r > 0$, denote by $Lip(x, r)$ the set of 1-Lipschitz functions f supported in $\overline{B(x, r)}$, that is the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(y) = 0$ for $y \in \mathbb{R}^n \setminus B(x, r)$ and $|f(y) - f(z)| \leq |y - z|$ for $y, z \in \mathbb{R}^n$. The normalized Wasserstein distance in B(x,r) between two measures σ and μ is

$$\operatorname{dist}_{x,r}(\mu, \sigma) = r^{-d-1} \sup_{f \in Lip(x,r)} \left| \int f \, d\sigma - \int f \, d\mu \right|. \tag{2.5}$$

The distance to flat measures is then defined by

$$\alpha_\sigma(x, r) = \inf_{\mu \in \mathcal{F}} \text{dist}_{x,r}(\mu, \sigma). \tag{2.6}$$

One can easily check that when (2.1) holds, the quantity α_σ is uniformly bounded, i.e. there exists a constant C that depends only on d, n, and C_σ such that $\alpha_\sigma(x, r) \leq C$ for $x \in \Gamma$ and $r > 0$.

Let Γ be a d-Ahlfors regular set, and σ a measure that satisfies (2.1). Tolsa's characterization of uniform rectifiability, Theorem 1.2 in [25], is as follows[1]: Γ is uniformly rectifiable if and only if there exists a constant $C_0 > 0$ such that

$$\int_0^r \int_{\Gamma \cap B(x,r)} |\alpha_\sigma(y, s)|^2 \, d\sigma(y) \, \frac{ds}{s} \leq C_0 \sigma(B(x, r)) \qquad \text{for } x \in \Gamma \text{ and } r > 0. \tag{2.7}$$

Here we will only use the fact that (2.7) holds when Γ is uniformly rectifiable. That is, we will only use (2.7) and do not need to know other properties of uniformly rectifiable sets. The property (2.7) will allow us to obtain additional estimates on the smooth distance D_β. The presentation of those bounds will be easier after the following definition.

Definition 2.8 Let the function f be defined on Ω. We say that f satisfies the Carleson measure condition when $f \in L^\infty(\Omega)$ and $|f(X)|^2 \text{dist}(X, \Gamma)^{d-n} dX$ is a Carleson measure, that is,

$$\int_{B(x,r)} |f(X)|^2 \text{dist}(X, \Gamma)^{d-n} dX \leq C\sigma(B(x, r)) \tag{2.9}$$

for $x \in \Gamma$ and $r > 0$, with a constant C that does not depend on x or r.

Thus this is actually a quadratic Carleson condition. For short, we shall write $f \in CM$, or $f \in CM(C)$ when we want to refer to the constant in (2.9).

Due to (2.2), we can replace $\text{dist}(X, \Gamma)^{d-n}$ with $D_\beta^{d-n}(X)$, and even choose β to fit our purposes; we shall often do this without additional explanations. We shall rely strongly on Lemma 1.24 in [17], which says the following.

Lemma 2.10 *Let Γ be uniformly rectifiable, so that (2.1) and (2.7) hold. Let $\beta > 0$. Then there exist a scalar function b and a vector function $\mathcal{V}$, both defined on Ω, such that*

$$\int_\Gamma |X - y|^{-n}(X - y) d\sigma(y) = (b\nabla D_\beta + \mathcal{V}) D_\beta^{d+1-n} \qquad \text{for } X \in \Omega \tag{2.11}$$

and a constant C_1 that depends only on C_σ, C_0, β, n, and d, such that

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$$C_1^{-1} \le b \le C_1, \tag{2.12}$$

$$D_\beta \nabla b \in CM(C_1), \tag{2.13}$$

$$|\mathcal{V}| \le C_1, \tag{2.14}$$

and

$$\mathcal{V} \in CM(C_1). \tag{2.15}$$

Observe that the left-hand side of (2.11) is divergence free. So if we use (2.11) and we write the divergence free condition in weak terms, we obtain

$$\int_\Omega (b\nabla D_\beta + \mathcal{V}) \cdot \nabla\varphi \, D_\beta^{d+1-n} \, dX = 0 \qquad \text{for } \varphi \in C_0^\infty(\Omega). \tag{2.16}$$

We also need the following, which is Lemma 1.26 in [17].

Lemma 2.17 *Let Γ be uniformly rectifiable, i.e., assume that (2.1) and (2.7) hold. Let $\alpha, \beta > 0$. Then $D_\alpha \nabla[D_\beta/D_\alpha]$ satisfies the Carleson measure condition with a constant that depends only on C_σ, C_0, α, β, n, and d.*

We are now finished with the geometric background and turn to the elliptic theory. Pick $\beta > 0$ and $\gamma \in (-1, 1)$. We whall use the operator $L_{\beta,\gamma}$ constructed in (1.1); hence $L_{\beta,\gamma}$ enters the scope of the theory developed in [13], and in particular there is an elliptic measure ω^X which we shall describe now.

We first need a Hilbert space W_γ, which is the same as in [13], Definition 3.1, but is more easily defined as

$$W_\gamma = \left\{ u \in L^1_{loc}(\mathbb{R}^n), \|u\|_\gamma := \int_\Omega |\nabla u(X)|^2 \, \text{dist}(X, \Gamma)^{d+1+\gamma-n} dX < +\infty \right\}; \tag{2.18}$$

the equivalence between the two definition is proved as in [10, Lemma 3.3 and Lemma 5.21].

Each $f \in W_\gamma$ has a trace on Γ, which lies in a corresponding Sobolev space H_γ (which is equal to $H^{1/2}(\Gamma)$ when $\gamma = 0$); then we denote by $W_{\gamma,0}$ the set of functions in W_γ with zero trace; $W_{\gamma,0}$ is also the completion of $C_0^\infty(\Omega)$ under the norm $\|.\|_\gamma$. And for any open set $E \subset \mathbb{R}^n$, we write that $u \in W_\gamma(E)$ [respectively, $u \in W_{\gamma,0}(E)$] if $u\varphi \in W_\gamma$ [respectively, $u\varphi \in W_{\gamma,0}$] for any $\varphi \in C_0^\infty(E)$.

Then there is a notion of weak solution for $L_{\beta,\gamma} = 0$, which the reader may also find in [13], such that in particular

$$\int_\Omega (\nabla u \cdot \nabla\varphi) \, D_\beta^{d+1+\gamma-n} = 0 \qquad \text{for } \varphi \in C_0^\infty(\Omega). \tag{2.19}$$

when u is a weak solution for $L_{\beta,\gamma} = 0$ (in Ω). Here and below, we remove the variable X and the integration symbol dX from the notation when they are not entirely needed;

unless otherwise specified, all our integrals on Ω will be against the Lebesgue measure dX.

With all this notation, the main properties of our elliptic measures ω^X are as follows.

Definition 2.20 For each $X \in \Omega$, we can define a unique probability measure $\omega^X :=$ $\omega^X_{\beta,\gamma}$ on Γ with the following properties. For any $g \in C_0(\Gamma)$ (i.e., continuous function on Γ and compactly supported), the function u_g defined as

$$u_g(X) = \int_\Gamma g(y) d\omega^X(y)$$

is a weak solution to $L_{\beta,\gamma}$ and, if in addition g lies in the Sobolev space H_γ, then $u_g \in W_\gamma$ and the trace of u_g is equal to g.

The space $H_\gamma \cap C_0(\Gamma)$ is dense in $C_0(\Gamma)$ (with the sup norm), so the last condition is our way of solving a Dirichlet problem. We are now ready to state the general version of our main theorem.

Theorem 2.21 *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d <$ $n - 1$, and let σ be an Ahlfors regular measure on Γ that satisfies (2.1). For $\beta > 0$ and $\gamma \in (-1, 1)$, define $L_{\beta,\gamma}$ as in (1.1). Then for any $\alpha > 0$, there exists $C > 0$ that depends only on C_σ, C_0, α, β, γ, d, n, such that for any ball $B := B(x, r)$ centered on Γ and any non-negative non identically zero weak solution u of $L_{\beta,\gamma} u = 0$ in $\Omega \cap 3B$ which lies in $W_{\gamma,0}(3B)$, one has*

$$\int_B \left| \nabla \ln \left(\frac{u}{D_\alpha^{1-\gamma}} \right) \right|^2 D_\alpha^{d+2-n} \leq C\sigma(B). \tag{2.22}$$

The proof of the Theorem will use the uniform rectifiability of Γ via Theorem 1.4 and Lemma 2.10. The Lemma will be used to estimate the left-hand side of (2.22) via simple integration techniques, except for one more complicated term in the form

$$\int \left| \nabla \ln \left(\frac{u}{D_\alpha^{1-\gamma}} \right) \right| |\nabla \phi_{B,\epsilon}| D_\alpha^{d+1-n},$$

where $\phi_{B,\epsilon}$ is a well chosen cut off function, which is related to the integral of the logarithm of the Poisson kernel and that will be estimated using Theorem 1.4, that is, the A_∞ absolute continuity of the harmonic measure.

3 Proof of Theorem 2.21

Let us recall

Lemma 3.1 *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d < n - 2$, and let σ be an Ahlfors regular measure on Γ that satisfies (2.1). Define $L_{\beta,\gamma}$ as in (1.1). Then there exist $C > 0$ and $\theta \in (0, 1]$, that depend only on C_σ, C_0, β, γ, n and*

d, such that for each choice of $x \in \Gamma$, $r > 0$, any Borel set $E \subset B(x, r) \cap \Gamma$, and any corkscrew point $X = A_{x,r}$ as in (2.3), one has

$$\frac{\omega_{\beta,\gamma}^X(E)}{\omega_{\beta,\gamma}^X(B(x, r))} \leq C \left(\frac{\sigma(E)}{\sigma(B(x, r))} \right)^\theta \tag{3.2}$$

and

$$\frac{\sigma(E)}{\sigma(B(x, r))} \leq C \left(\frac{\omega_{\beta,\gamma}^X(E)}{\omega_{\beta,\gamma}^X(B(x, r))} \right)^\theta. \tag{3.3}$$

Here and below, we assume implicitly that the constant C in (2.3) (the definition of corkscrew points) is chosen to depend on C_σ, n and d only. If we allow a larger C in (2.3), the constants in (3.2) and (3.3) depend on C as well.

Proof The conditions (3.2) and (3.3) are another characterizations of the fact that $\omega^X \in A_\infty(\sigma)$, which is true by Theorem 1.4. The fact that (3.2)–(3.3) is implied by (1.6) can be found in [22, Theorem 1.4.13] and its proof in [5, Lemma 5]. □

For our next result, we want to establish that the logarithm of the Poisson kernel, that is, $\ln(\frac{d\omega^X}{d\sigma})$, is integrable. We want a quantitative version, and moreover, we shall state this in a form that is more directly applicable when we need it (in the proof of Proposition 3.8).

Lemma 3.4 *Let Γ, σ, and $L_{\beta,\gamma}$ as in Lemma 3.1.*
Take $B := B(x, r)$, a ball centered on Γ, and $X = A_{x,r}$, a corkscrew point as in (2.3). If $\{Q_i\}_{i \in \mathcal{I}}$ is a finitely overlapping collection of Borel subsets of $B \cap \Gamma$, then

$$\sum_{i \in \mathcal{I}} \left| \ln \left(\frac{\omega_{\beta,\gamma}^X(Q_i)}{\sigma(Q_i)} \frac{\sigma(B)}{\omega_{\beta,\gamma}^X(B)} \right) \right| \sigma(Q_i) \leq C\sigma(B), \tag{3.5}$$

where C depends only on C_σ, C_0, β, γ, n, d, and the maximal number of overlaps in the collection $\{Q_i\}_{i \in \mathcal{I}}$.

Proof We introduce for $k \in \mathbb{Z}$,

$$\mathcal{I}_k := \left\{ i \in \mathcal{I}, \, 2^k \leq \frac{\omega_{\beta,\gamma}^X(Q_i)}{\sigma(Q_i)} \frac{\sigma(B)}{\omega_{\beta,\gamma}^X(B)} \leq 2^{k+1} \right\}.$$

Then we define

$$E_k := \bigcup_{i \in \mathcal{I}_k} Q_i.$$

Since $\{Q_i\}$ is finitely overlapping, we have $\sum_{i \in \mathcal{I}_k} \sigma(Q_i) \leq C\sigma(E_k)$, and we can thus write

$$\sum_{i \in \mathcal{I}} \ln\left(\frac{\omega_{\beta,\gamma}^X(Q_i)}{\sigma(Q_i)} \frac{\sigma(B)}{\omega_{\beta,\gamma}^X(B)}\right) \sigma(Q_i) \lesssim \sum_{k \in \mathbb{Z}} (|k| + 1)\sigma(E_k). \qquad (3.6)$$

Yet, since $\{Q_i\}_{i \in \mathcal{I}_k}$ is a finitely overlapping covering of E_k,

$$\omega_{\beta,\gamma}^X(E_k) \approx \sum_{i \in \mathcal{I}_k} \omega_{\beta,\gamma}^X(Q_i) \approx 2^k \frac{\omega_{\beta,\gamma}^X(B)}{\sigma(B)} \sum_{i \in \mathcal{I}_k} \sigma(Q_i) \approx 2^k \frac{\omega_{\beta,\gamma}^X(B)}{\sigma(B)} \sigma(E_k),$$

which means that

$$\frac{\omega_{\beta,\gamma}^X(E_k)}{\sigma(E_k)} \frac{\sigma(B)}{\omega_{\beta,\gamma}^X(B)} \approx 2^k. \qquad (3.7)$$

The use of (3.7) in (3.2) leads to

$$\sigma(E_k) \lesssim 2^{-k/(1-\theta)} \sigma(B)$$

while the use of (3.7) in (3.3) gives

$$\sigma(E_k) \lesssim 2^{k\theta/(1-\theta)} \sigma(B).$$

We use the first of the last two estimate to bound $\sigma(E_k)$ when $k \geq 0$ and the second one to bound $\sigma(E_k)$ when k is negative. Combined with (3.6), we deduce

$$\sum_{i \in \mathcal{I}} \ln\left(\frac{\omega_{\beta,\gamma}^X(Q_i)}{\sigma(Q_i)} \frac{\sigma(B)}{\omega_{\beta,\gamma}^X(B)}\right) \sigma(Q_i) \lesssim \sum_{k \geq 0} (k+1) 2^{-\frac{k}{1-\theta}} \sigma(B) + \sum_{k < 0} (1-k) 2^{-\frac{k\theta}{1-\theta}} \sigma(B)$$

$$\lesssim \sigma(B).$$

The lemma follows. $\qquad \qquad \qquad \square$

Proposition 3.8 *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular uniformly rectifiable set with $d < n - 2$, and let σ be an Ahlfors regular measure that satisfies (2.1). Take $\beta > 0$ and $\gamma \in (-1, 1)$, define $L_{\beta,\gamma}$ as in (1.1). Then for any ball B centered on Γ and any non-negative non identically zero weak solution u to $L_{\beta,\gamma}u = 0$ in $3B \cap \Omega$, with $u \in W_{\gamma,0(3B)}$,*

$$\int_B \left| \nabla \ln\left(\frac{u}{D_\beta^{1-\gamma}}\right) \right|^2 D_\beta^{d+2-n} \leq C\sigma(B). \qquad (3.9)$$

with a constant $C > 0$ that depends only on C_σ, C_0, β, γ, d, and n.

Observe that Proposition 3.8 is the special case of Theorem 2.21 where we take $\alpha = \beta$. We shall now check that conversely, Theorem 2.21 follows from Proposition 3.8 and Lemma 2.17.

Proof of Theorem 2.21 from Proposition 3.8 with the help of Lemma 2.17. Let $\alpha > 0$. Notice that

$$\left| \nabla \ln \left(\frac{u}{D_\alpha^{1-\gamma}} \right) \right| = \left| \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) + (1-\gamma) \ln \left(\frac{D_\beta}{D_\alpha} \right) \right] \right|$$
$$\leq \left| \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \right| + (1-\gamma) \frac{D_\alpha}{D_\beta} \left| \nabla \left[\frac{D_\beta}{D_\alpha} \right] \right|.$$

So Proposition 3.8, Lemma 2.17, and (2.2) easily implies (3.9). □

Proof of Proposition 3.8. The proof of the lemma will be divided into 4 steps. The core step is step 4, where we use the properties of the solutions. Step 3 treats in advance the most complicated term that we met in Step 4. In this step, we will compare a solution u to a Green function, then to the harmonic measure, and finally we use Lemma 3.4. In Step 2, we construct the finitely overlapping covering $\{Q_i\}_{i \in \mathcal{I}}$ that will be needed in order to invoke Lemma 3.4. At last, Step 1 introduces the cut-off function $\phi_{B,\epsilon}$ used to bound the left-hand side of (2.22) and shows that $D_\beta \nabla \phi_{B,\epsilon}$ satisfies the Carleson measure condition.

Step 1: Introduction of the cut-off function $\phi_{B,\epsilon}$. Let $B = B(x,r)$ be a ball in $\mathbb{R}^n$ centered on the boundary Γ and $\epsilon > 0$ small. The proof of Theorem 2.21 is a local one and thus as usual will involve cut off functions. Take $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi \equiv 1$ on $[-1, 1]$, ψ is compactly supported in $(-2, 2)$, $0 \leq \psi \leq 1$, and $|\psi'| \leq 2$. We define the function $\phi_{B,\epsilon}$ on Ω by

$$\phi_{B,\epsilon}(X) := \psi \left(\frac{\text{dist}(X, B)}{10\,\text{dist}(X, \Gamma)} \right) \psi \left(\frac{2\,\text{dist}(X, B)}{r} \right) \psi \left(\frac{\epsilon}{\text{dist}(X, \Gamma)} \right). \quad (3.10)$$

The support of $\phi_{B,\epsilon}$ is thus contained in

$$E_0 = \left\{ X \in 2B, \ \text{dist}(X, B) \leq 20\,\text{dist}(X, \Gamma) \text{ and } \text{dist}(X, \Gamma) \geq \epsilon/2 \right\}. \quad (3.11)$$

The gradient of $\phi_{B,\epsilon}$ comes from 3 regions; the first one (associated to the first cut-off) is

$$\left\{ X \in E_0, \ 10\,\text{dist}(X, \Gamma) \leq \text{dist}(X, B) \leq 20\,\text{dist}(X, \Gamma) \right\}$$
$$\subset E_1 := \left\{ X \in 2B, \ 10\,\text{dist}(X, \Gamma) \leq \text{dist}(X, B) \leq 20\,\text{dist}(X, \Gamma) \right\}, \quad (3.12)$$

the second one is

$$\left\{ X \in E_0, \ r \leq 2\,\text{dist}(X, B) \leq 2r \right\} \subset E_2 := \left\{ X \in 2B, \ r/40 \leq \text{dist}(X, \Gamma) \leq 2r \right\} \quad (3.13)$$

because $\text{dist}(X, \Gamma) \leq 2r$ when $X \in 2B$ and similarly $\text{dist}(X, \Gamma) \geq \frac{1}{20} \text{dist}(X, B) \geq \frac{r}{20}$, and the third region is contained in

$$E_3 := \left\{ X \in 2B, \; \epsilon/2 \leq \text{dist}(X, \Gamma) \leq \epsilon \right\}. \tag{3.14}$$

Then it is easy to check that the gradient of $\phi_{B,\epsilon}$ satisfies

$$|\nabla \phi_{B,\epsilon}| \leq \frac{100}{\text{dist}(X, \Gamma)} \left[\mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \mathbb{1}_{E_3} \right]. \tag{3.15}$$

We claim that $\mathbb{1}_{E_1}$, $\mathbb{1}_{E_2}$, and $\mathbb{1}_{E_3}$ all satisfy the Carleson measure condition. That is, we have

$$\int_{B(y,s)} \mathbb{1}_{E_j}^2 \, \text{dist}(X, \Gamma)^{d-n} \leq C s^d \tag{3.16}$$

for $1 \leq j \leq 3$, $y \in \Gamma$, and $0 < s < +\infty$. Of course the square can be removed, and it will follows from (3.15), (3.16) (as soon as we prove it), and (2.2) that for any $\beta > 0$, $y \in \Gamma$ and $s \in (0, +\infty)$, we have

$$\int_{B(y,s)} |D_\beta \nabla \phi_{B,\epsilon}| D_\beta^{d-n} + \int_{B(y,s)} |D_\beta \nabla \phi_{B,\epsilon}|^2 D_\beta^{d-n} \leq C s^d, \tag{3.17}$$

where the constant $C > 0$ depends only on β, n, d, and C_σ.

We will prove the claim (3.16) in the end of the next Step.

Step 2: Construction of the collection $\{Q_i\}_{i \in \mathcal{I}}$. We would like to have a collection of boundary cubes $Q_i \subset \Gamma$ and the corresponding Whitney cubes $R_i \subset \Omega$ satisfying rather usual nice properties: bounded overlap, control of the size and and the distance to the boundary for R_i's, reasonable placement of the corkscrew points. There are plenty of papers that present various versions of this construction, but here, in fact, we need something simpler than usual as we do not need to control the cones or the related Harnack chains. For these reasons, let us simply carry out the construction by hands.

We need a family of Whitney cubes $\mathcal{W}$, as constructed in [24]. We record the basic properties of $\mathcal{W}$ that we shall need. The collection $\mathcal{W}$ is the family of maximal dyadic cubes $R \subset \Omega$ such that $20R \subset \Omega$, different cubes R_i and R_j, $i \neq j$, in $\mathcal{W}$ have disjoint interiors (by maximality), their union covers $\Omega = \mathbb{R}^n \setminus \Gamma$, and for $Q \in \mathcal{W}$

$$20Q \subset \Omega \text{ but } 60R \cap \Gamma \neq \emptyset. \tag{3.18}$$

Moreover, given $R \in \mathcal{W}$, the number of cubes $\widetilde{R} \in \mathcal{W}$ such that

$$\text{dist}(R, \Gamma) \approx \text{dist}(\widetilde{R}, \Gamma) \text{ and } \text{dist}(R, \widetilde{R}) \lesssim \text{dist}(R, \Gamma) \tag{3.19}$$

is uniformly bounded by a constant that depends only on the dimensions and the constants involved in (3.19). Also, the cubes $\widetilde{R} \in \mathcal{W}$ such that $3R \cap 3\widetilde{R}$, all satisfy (3.19).

We will use the cubes of $\mathcal{W}$ to cover the region $E_1 \cup E_2 \cup E_3$. Consider the subset $\mathcal{W}_0 \subset \mathcal{W}$ of cubes $R \in \mathcal{W}$ that meet $E_1 \cup E_2 \cup E_3$, and label these cubes with a set I, so that $\mathcal{W}_0 = \{R_i \; ; \; i \in I\}$. We also want to associate a boundary ball $Q_i = \Gamma \cap B(x_i, r_i)$ to each R_i, $i \in I$. This is a classical thing to do, but we shall do it by hand to get a slightly better control. We shall choose the Q_i so that

$$Q_i \subset 3B \cap \Gamma \tag{3.20}$$

(we can do this and this will be helpful because we shall consider solutions in $3B$),

$$\{Q_i\}_{i \in \mathcal{I}} \text{ has bounded overlap,} \tag{3.21}$$

where the bound for the overlap depends only on C_σ, n, and d, and

$$r_i \approx \operatorname{dist}(R_i, \Gamma) \approx \operatorname{dist}(R_i, x_i), \tag{3.22}$$

also with constants depend only on C_σ, n, and d. As a consequence, we will be able to use any point $X_i \in R_i$ as a corkscrew point for the pair (x_i, Cr_i) (see (2.3)).

So let us construct the Q_i. First write $I = I_1 \cup I_2 \cup I_3$, a disjoint union where $I_2 = \{i \in I \; ; \; R_i \cap E_2 \neq \emptyset\}$, then $I_3 = \{i \in I \setminus I_2 \; ; \; R_i \cap E_3 \neq \emptyset\}$, and finally $I_1 = \{i \in I \setminus (I_2 \cup I_3) \; ; \; R_i \cap E_1 \neq \emptyset\}$.

We start with $i \in I_2$. This is the simplest case because $E_2 \subset 2B = B(x, 2r)$, and $\operatorname{dist}(X, \Gamma) \geq r/40$ on E_2. By definition of $\mathcal{W}$, I_2 has at most C elements (see near (3.19)), and for each $i \in I_2$ we take $x_i = x$ and $r_i = 3r$ (and hence $Q_i = \Gamma \cap 3B$); the constraints (3.20), (3.21), (3.22) for I_2 are easily checked.

Next consider $i \in I_3$; thus R_i meets E_3, where $\varepsilon/2 \leq \operatorname{dist}(X, \Gamma) \leq \varepsilon$. Pick $x_i \in \Gamma$ such that $\operatorname{dist}(x_i, E_3 \cap R_i) = \operatorname{dist}(\Gamma, E_3 \cap R_i)$, and take $r_i = \varepsilon$. Then $Q_i \subset 3B$ if ε is small enough, because $E_3 \subset 2B$ and $\operatorname{diam}(R_i) \approx \operatorname{dist}(R_i, \Gamma) \approx \operatorname{dist}(E_3 \cap R_i, \Gamma) \approx \varepsilon$, (3.22) holds for the same reasons, and the Q_i, $i \in I_3$ have bounded overlap by the property near (3.19).

Finally, for $i \in I_1$, take $X_i \in R_i \cap E_1$ and $x_i \in \Gamma$ such that $|X_i - x_i| = \operatorname{dist}(R_i \cap E_1, \Gamma)$, and set $Q_i = \Gamma \cap B(x_i, r_i)$, with $r_i = |X_i - x_i|$. Since $X_i \in E_1$, the definition (3.12) yields

$$\operatorname{dist}(X_i, B) \leq 20 \operatorname{dist}(X, \Gamma) = 20|X_i - x_i| \leq 2 \operatorname{dist}(X_i, B) \leq 2r, \tag{3.23}$$

and in particular $x_i \in \frac{5}{2}B$. Also, $X_i \notin E_2$ (because $i \notin I_2$), so $r_i = \operatorname{dist}(R_i \cap E_1, \Gamma) \leq \operatorname{dist}(X_i, \Gamma) \leq r/40$, and hence $Q_i \subset 3B$, as needed for (3.20). The Whitney property (3.22) holds essentially by definition (and because R_i is a Whitney cube), so we just need to bound the overlap of the Q_i.

Assume that $Q_j \cap Q_i \neq \emptyset$ for two indices $i, j \in \mathcal{I}_1$, and let X_i, X_j, r_i, and r_j be as above. Let us first check that $r_i \approx r_j$. By (3.23)

$$\operatorname{dist}(x_i, B) \leq |X_i - x_i| + \operatorname{dist}(X_i, B) \leq 21|X_i - x_i| = 21r_i$$

and

$$\text{dist}(x_i, B) \geq \text{dist}(X_i, B) - |X_i - x_i| \leq 9|X_i - x_i| = 9r_i,$$

which can be summarized as

$$9r_i \leq \text{dist}(x_i, B) \leq 21r_i. \tag{3.24}$$

Similarly,

$$9r_j \leq \text{dist}(x_j, B) \leq 21r_j. \tag{3.25}$$

We can assume without loss of generality that $\text{dist}(x_j, B) \leq \text{dist}(x_i, B)$, which, together with (3.24)–(3.25), leads to $9r_j \leq 21r_i$. Moreover, since $Q_i \cap Q_j \neq \emptyset$,

$$r_i + r_j \geq |x_j - x_i| \geq \text{dist}(x_i, B) - \text{dist}(x_j, B) \geq 9r_i - 21r_j,$$

hence $22r_j \geq 8r_i$. Recall that $9r_j \leq 21r_i$, so the two radii are equivalent. The bounded overlap property follows, because $|x_j - x_i| \leq r_i + r_j$ when $Q_i \cap Q_j \neq \emptyset$.

This completes our construction of Whitney cubes R_i and associated surface balls Q_i, with the properties (3.20)–(3.22) (notice that the overlap constant for the $Q_i, i \in I$, is less than the sum of the overlap constants for the I_j).

With this at hand, let us prove (3.16). This is now quite easy as for any $y \in \Gamma$ and $0 < s < +\infty$ we have

$$\int_{B(y,s)} \mathbb{1}_{E_1 \cup E_2 \cup E_3}^2 \text{dist}(X, \Gamma)^{d-n}$$

$$\leq \sum_{R_i : R_i \cap B(y,s) \neq \emptyset} \int_{R_i} \text{dist}(X, \Gamma)^{d-n} \lesssim \sum_{R_i : R_i \cap B(y,s) \neq \emptyset} r_i^d \lesssim s^d, \tag{3.26}$$

using the bounded overlap property (3.21).

Step 3: Estimates for the integral S. Take $u \in W_{\gamma,0}(3B)$, a weak solution to $L_{\beta,\gamma} u = 0$ in $3B$. Let $X_0 \in 2B$ be a corkscrew point for the boundary ball $2B \cap \Gamma$, and also choose another corkscrew point $X_1 \in \Omega \cap 8B \setminus 4B$, so that $\text{dist}(X_0, \Gamma) \geq cr$ and $\text{dist}(X_1, \Gamma) \geq cr$ for a small constant c that depends only on C_σ, d, and n. In this Step 3, we prove that

$$S := \int_{E_1 \cup E_2 \cup E_3} \left| \ln \left(\frac{u}{D_\beta^{1-\gamma}} \frac{D_\beta^{1-\gamma}(X_0)}{u(X_0)} \right) \right| D_\beta^{d-n} \leq Cr^d, \tag{3.27}$$

where C depends only on C_σ, C_0, β, γ, n and d.

We shall use the comparison principle to compare u to the Green function, and then estimate the Green function in terms of harmonic measure. We define the Green function g on $\Omega \times \Omega$ as in [13, Section 11]. The precise definition is not relevant for the

present proof, and the properties that we care about are the fact that $X \to g(X, X_1)$ lies in $W_{\gamma,0}(3B)$ and is a solution to $L_{\beta,\gamma} u = 0$ in $3B$, and that $g(X, Y) = g(Y, X)$ (which is true because the operator $L_{\beta,\gamma}$ is selfadjoint). Theorem 15.64 in [13] (the comparison theorem) yields that

$$\frac{u(X)}{u(X_0)} \approx \frac{g(X, X_1)}{g(X_0, X_1)} \qquad \text{for } X \in 2B, \tag{3.28}$$

with constants that depend only on C_σ, C_0, n, d, β and γ. Actually, Theorem 15.64 in [13] requires the solutions (in our case u and $g(., X_1)$) to be solutions in a larger ball $2KB \cap \Omega$, and not in only $3B \cap \Omega$. This condition is only needed because [13] also allows sets Γ of codimension 1 or less, and then we need to ensure that we can connect every component of $2B \cap \Omega$ by Harnack chains that stays in $2KB$. Here Γ is of dimension $d < n - 1$, so $2B \cap \Omega$ is very well connected in the first place (see Lemma 2.1 in [10]) , and assuming that u and $g(., X_1)$ are solutions in $3B$ is enough. Using the fact that $g(X, Y)$ is symmetric, we deduce that

$$\frac{u(X)}{u(X_0)} \approx \frac{g(X_1, X)}{g(X_1, X_0)} \qquad \text{for } X \in 2B. \tag{3.29}$$

Recall that X_0 is a corkscrew point associated to $2B$ and that if $X \in R_i$, then $\text{dist}(X, \Gamma) \geq C^{-1} r_i$ and $|X - x_i| \leq C r_i$ by (3.22), so X can be used as a corkscrew point for the boundary ball Q_i. As a consequence, Lemma 15.28 in [13], (where here $m(B \cap \Omega)$ is the mass of B for the measure $\text{dist}(X, \Gamma)^{d+1+\gamma-n} dX$ on Ω, with $\gamma \in (-1, 1)$, so $m(B \cap \Omega) \approx r^{d+1+\gamma}$; see the discussion in Section 3.2 of [13]), and the doubling property of harmonic measure (Lemma 15.43 of [13]) give that

$$g(X_1, X_0) \approx \text{dist}(X_0, \Gamma)^{1-d-\gamma} \omega^{X_1}(3B)$$

and for each $i \in I$

$$g(X_1, X) \approx \text{dist}(X, \Gamma)^{1-d-\gamma} \omega^{X_1}(Q_i) \qquad \text{for } X \in R_i,$$

where the constants depend only on C_σ, C_0, n, d, β and γ. Using the equivalence (2.2) and the Ahlfors regularity of σ, we deduce that

$$g(X_1, X_0) \approx D_\beta(X_0)^{1-\gamma} \frac{\omega^{X_1}(3B)}{\sigma(3B)} \tag{3.30}$$

and for each $i \in I$,

$$g(X_1, X) \approx D_\beta(X)^{1-\gamma} \frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \qquad \text{for } X \in R_i. \tag{3.31}$$

We gather (3.29), (3.30), and (3.31) to obtain that, for every $i \in I$ and every $X \in R_i$,

$$\frac{u(X)}{D_\beta^{1-\gamma}(X)} \frac{D_\beta^{1-\gamma}(X_0)}{u(X_0)} \approx \frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \frac{\sigma(3B)}{\omega^{X_1}(3B)}, \tag{3.32}$$

where the constants depend only on C_σ, C_0, n, d, β and γ. This immediately implies that for $i \in I$ and $X \in R_i$,

$$\left| \ln \left(\frac{u(X)}{D_\beta^{1-\gamma}(X)} \frac{D_\beta^{1-\gamma}(X_0)}{u(X_0)} \right) \right| \leq C + \left| \ln \left(\frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \frac{\sigma(3B)}{\omega^{X_1}(3B)} \right) \right|$$

and then

$$\begin{aligned}
S &\leq \sum_{i \in I} \int_{R_i} \left| \ln \left(\frac{u}{D_\beta^{1-\gamma}} \frac{D_\beta^{1-\gamma}(X_0)}{u(X_0)} \right) \right| D_\beta^{d-n} \\
&\leq \sum_{i \in I} \left[C + \left| \ln \left(\frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \frac{\sigma(3B)}{\omega^{X_1}(3B)} \right) \right| \right] \int_{R_i} D_\beta^{d-n}.
\end{aligned} \tag{3.33}$$

By(2.2) and the fact that $R_i \in \mathcal{W}$ is a Whitney cube with the property (3.22) (by construction), we have $D_\beta \approx r_i$ on R_i and $|R_i| \lesssim r_i^n$. Hence

$$\int_{R_i} D_\beta^{d-n} \lesssim r_i^d \approx \sigma(Q_i)$$

by (2.1). Using this observation in (3.33), we infer that

$$\begin{aligned}
S &\leq \sum_{i \in I} \left[C + \left| \ln \left(\frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \frac{\sigma(3B)}{\omega^{X_1}(3B)} \right) \right| \right] \sigma(Q_i) \\
&= C \sum_{i \in I} \sigma(Q_i) + \sum_{i \in I} \left| \ln \left(\frac{\omega^{X_1}(Q_i)}{\sigma(Q_i)} \frac{\sigma(3B)}{\omega^{X_1}(3B)} \right) \right| \sigma(Q_i).
\end{aligned}$$

The first term of the right-hand side is bounded by $C\sigma(3B)$ since the Q_i, $i \in I$, are contained in $\Gamma \cap 3B$ (by (3.20)) and have bounded overlap (by (3.21)). The second term is also less than $C\sigma(3B)$, by Lemma 3.4. We conclude that

$$S \lesssim \sigma(3B) \lesssim r^d$$

by (2.1). The claim 3.27 follows.

Step 4: Core of the proof. Let us turn to the main and last step of the proof. Set

$$T := \int_\Omega \left| \nabla \ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right|^2 \phi_{B,\epsilon}^2 D_\beta^{d+2-n}. \tag{3.34}$$

We aim to prove that

$$T \le Cr^d + Cr^{d/2}T^{1/2}, \tag{3.35}$$

where C depends only on C_σ, C_0, β, γ, n and d. The solution u and the smooth distance D_β are uniformly bounded from above and from below by a positive constant on the support of the cut-off function $\phi_{B,\epsilon}$. Thanks to this fact, the quantities of both sides of (3.35) are finite, and (3.35) self-improves into

$$\int_\Omega \left| \nabla \ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right|^2 \phi_{B,\epsilon}^2 D_\beta^{d+2-n} = T \le Cr^d, \tag{3.36}$$

where $C > 0$ depends on the same parameters as in (3.35). Once we are there, the left-hand side is uniformly bounded in ϵ, so taking $\epsilon \to 0$ leads to the desired result (3.9).

Keep in mind that

$$\nabla \ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) = \frac{\nabla u}{u} - \frac{\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} = \frac{D_\beta^{1-\gamma} \nabla u - u \nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma} u}. \tag{3.37}$$

We shall use Lemma 2.10 to obtain the existence of a scalar function b and a vector function $\mathcal{V}$ such that

$$H_{n-d-1} := \int_\Gamma |X - y|^{-n}(X - y)d\sigma(y) = (b\nabla D_\beta + \mathcal{V})D_\beta^{d+1-n} \tag{3.38}$$

as in (2.11), with the bounds (2.12)–(2.15).

Before we start to bound T, let us comment on H_{n-d-1} and ∇D_β. First, the vector function H_{n-d-1} is smooth and divergence free in Ω, and the formulation of this fact in the weak sense is that for any compactly supported $\varphi \in W^{1,1}(\Omega)$,

$$\int_\Omega H_{n-d-1} \cdot \nabla \varphi = 0. \tag{3.39}$$

Set $H_\alpha(X) := \int_\Gamma |X - y|^{-d-1-\alpha}(X - y)d\sigma(y)$ for $\alpha > 0$. Since $|H_\alpha(X)| \le \int_\Gamma |X - y|^{-d-\alpha}d\sigma(y) = D_\alpha(X)^{-\alpha}$, (2.2) implies that

$$|H_\alpha| \lesssim D_\alpha^{-\alpha} \lesssim D_\beta^{-\alpha}. \tag{3.40}$$

Moreover, observe that $\nabla(D_\beta^{-\beta}) = -(d + \beta)H_{\beta+1}$, directly by (1.2); since also $\nabla(D_\beta^{-\beta}) = -\beta D_\beta^{-\beta-1}\nabla D_\beta$, we deduce from (3.40) that

$$|\nabla D_\beta| \lesssim 1. \tag{3.41}$$

We turn to the proof of (3.35). Write ϕ instead of $\phi_{B,\epsilon}$ to lighten the notation. By (2.12) and (3.37),

$$T \lesssim \int_\Omega \left| \nabla \ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right|^2 \phi^2 b \, D_\beta^{d+2-n} = \int_\Omega \frac{b\nabla u}{u} \cdot \left(\frac{D_\beta^{1-\gamma} \nabla u - u \nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma} u} \right) \phi^2 D_\beta^{d+2-n}$$

$$- \int_\Omega \frac{b\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} \cdot \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \phi^2 D_\beta^{d+2-n}$$

$$:= T_1 - T_2.$$

Let us start with T_2. We simplify the factors D_β and we invoke the relation (3.38) to get

$$T_2 = (\gamma - 1) \int_\Omega \frac{b\nabla D_\beta}{D_\beta^{n-d-1}} \cdot \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \phi^2$$

$$= (\gamma - 1) \int_\Omega H_{n-d-1} \cdot \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \phi^2$$

$$- (\gamma - 1) \int_\Omega \mathcal{V} \cdot \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \phi^2 D_\beta^{d+1-n}$$

$$:= T_{21} + T_{22}.$$

We use the Cauchy–Schwarz inequality to bound T_{22}, as follows:

$$|T_{22}| \lesssim \left(\int_\Omega |\mathcal{V}|^2 \phi^2 D_\beta^{d-n} \right)^{\frac{1}{2}} \left(\int_\Omega \left| \nabla \left[\ln \left(\frac{u}{D_\beta^{1-\gamma}} \right) \right] \right|^2 \phi^2 D_\beta^{d+2-n} \right)^{\frac{1}{2}}$$

$$\lesssim T^{1/2} \left(\int_{2B} |\mathcal{V}|^2 D_\beta^{d-n} \right)^{\frac{1}{2}} \lesssim r^{d/2} T^{1/2}$$

by (2.15). This fits with (3.35). As for T_{21}, notice that its value will not be changed if we replace u by Ku, where K is a constant. Hence

$$T_{21} = (\gamma - 1) \int_\Omega H_{n-d-1} \cdot \nabla \left[\ln \left(\frac{Ku}{D_\beta^{1-\gamma}} \right) \right] \phi^2,$$

where K is a constant to be chosen later. We force ϕ^2 into the gradient, then use the fact that H_{n-d-1} is divergence free (see (3.39)), and obtain

$$T_{21} = (\gamma - 1) \int_\Omega H_{n-d-1} \cdot \nabla \left[\phi^2 \ln \left(\frac{Ku}{D_\beta^{1-\gamma}} \right) \right] - 2(\gamma - 1) \int_\Omega H_{n-d-1} \cdot \nabla \phi \, \phi \ln \left(\frac{Ku}{D_\beta^{1-\gamma}} \right)$$

$$= 0 - 2(\gamma - 1) \int_\Omega H_{n-d-1} \cdot \nabla \phi \, \phi \ln \left(\frac{Ku}{D_\beta^{1-\gamma}} \right).$$

Recall that $|H_{n-d-1}| \lesssim D_\beta^{d+1-n}$ by (3.40), and that $|\nabla\phi| \lesssim \mathbb{1}_{E_1 \cup E_2 \cup E_3}/D_\beta$ by (3.15) and (2.2), so

$$|T_{21}| \lesssim \int_{E_1 \cup E_2 \cup E_3} \left| \ln\left(\frac{Ku}{D_\beta^{1-\gamma}} \right) \right| D_\beta^{d-n}.$$

We choose $K = D_\beta^{1-\gamma}(X_0)/u(X_0)$, so that the right-hand side above is what we called S in (3.27). Hence $T_{21} \lesssim r^d$ by (3.27), as needed for (3.35).

We switch to the estimation of T_1. We want to use the fact that u is a solution, and for this we write

$$
\begin{aligned}
T_1 &= \int_\Omega \nabla u \cdot \frac{D_\beta^{1-\gamma}}{u} \left(\frac{D_\beta^{1-\gamma} \nabla u - u\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma} u} \right) b\phi^2 D_\beta^{d+1+\gamma-n} \\
&= -\int_\Omega \nabla u \cdot \nabla\left[\frac{D_\beta^{1-\gamma}}{u} \right] b\phi^2 D_\beta^{d+1+\gamma-n} \\
&= -\int_\Omega \nabla u \cdot \nabla\left[b\phi^2 \frac{D_\beta^{1-\gamma}}{u} \right] D_\beta^{d+1+\gamma-n} + \int_\Omega \nabla u \cdot \nabla b\left(\phi^2 \frac{D_\beta^{1-\gamma}}{u} \right) D_\beta^{d+1+\gamma-n} \\
&\quad + 2\int_\Omega \nabla u \cdot \nabla\phi\left(\phi b \frac{D_\beta^{1-\gamma}}{u} \right) D_\beta^{d+1+\gamma-n} \\
&:= T_{11} + T_{12} + T_{13}.
\end{aligned}
\tag{3.42}
$$

Notice that $T_{11} = 0$ because u is a weak solution to $L_{\beta,\gamma} u = 0$ on $\Omega \cap 2B$ and $\phi^2 \dfrac{D_\beta^{1-\gamma}}{u}$ lies in $W_{loc}^{1,2}(\Omega)$ and compactly supported in Ω. The terms T_{12} and T_{13} are morally similar. In both case, we don't like the terms with u because we don't know so much about it, so we use (3.37) to replace u by the nice function $D_\beta^{1-\gamma}$, and the difference will be controlled with the help of $T^{1/2}$, the square root of our initial integral. We start with

$$
\begin{aligned}
T_{13} &= 2\int_\Omega \frac{\nabla u}{u} \cdot \nabla\phi\, \phi\, b\, D_\beta^{d+2-n} \\
&= 2\int_\Omega \left(\frac{\nabla u}{u} - \frac{\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} \right) \cdot (\nabla\phi)\, \phi\, b\, D_\beta^{d+2-n} - 2(\gamma-1)\int_\Omega \frac{b\nabla D_\beta}{D_\beta^{n-d-1}} \cdot (\nabla\phi)\phi \\
&:= T_{131} + T_{132}.
\end{aligned}
\tag{3.43}
$$

We use (2.12) and the fact that $D_\beta \nabla \phi$ satisfies the Carleson measure property to get

$$|T_{131}| \lesssim \left(\int_\Omega |\nabla \phi|^2 D_\alpha^{d+2-n} \right)^{1/2} \left(\int_\Omega \left| \frac{\nabla u}{u} - \frac{\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} \right|^2 \phi^2 D_\alpha^{d+2-n} \right)^{1/2}$$

$$\lesssim r^{d/2} \left(\int_\Omega \left| \nabla \ln \left(\frac{u}{D_\alpha^{1-\gamma}} \right) \right|^2 \phi^2 D_\alpha^{d+2-n} \right)^{1/2}$$

by (3.17), and then (3.37). The bound of T_{131} that we just obtained appears in the right-hand side of (3.35), as desired. As for T_{132}, we invoke (3.41), (2.12), and then (3.17) to write

$$|T_{132}| \lesssim \int_{2B} |D_\beta \nabla \phi| D_\beta^{n-d} \lesssim r^d \quad .$$

Similarly to T_{13}, we treat T_{12} as follows:

$$T_{12} = \int_\Omega \frac{\nabla u}{u} \cdot \nabla b \, \phi^2 D_\beta^{d+2-n}$$

$$= \int_\Omega \left(\frac{\nabla u}{u} - \frac{\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} \right) \cdot \nabla b \, \phi^2 D_\beta^{d+2-n} - (\gamma - 1) \int_\Omega \nabla D_\beta \cdot \nabla b \, \phi^2 D_\beta^{d+1-n}$$

$$:= T_{121} + T_{122}.$$

$$(3.44)$$

Thanks to the definition (3.34) and then (2.13),

$$|T_{121}| \lesssim \left(\int_\Omega \left| \frac{\nabla u}{u} - \frac{\nabla D_\beta^{1-\gamma}}{D_\beta^{1-\gamma}} \right|^2 \phi^2 D_\beta^{d+2-n} \right)^{1/2} \left(\int_\Omega |\nabla b|^2 \phi^2 D_\beta^{d+2-n} \right)^{1/2}$$

$$(3.45)$$

$$\lesssim T^{1/2} \left(\int_{2B} |D_\beta \nabla b|^2 D_\beta^{d-n} \right)^{\frac{1}{2}} \lesssim r^{d/2} T^{1/2}.$$

Now, we want to use (3.38) again, so we force the function b to appear and we place all the remaining terms in the second gradient. Then T_{122} becomes

$$T_{122} = -(\gamma - 1) \int_\Omega b \nabla D_\beta \cdot \frac{\nabla b}{b} \phi^2 D_\beta^{d+1-n}$$

$$= -(\gamma - 1) \int_\Omega b \nabla D_\beta \cdot \nabla[\phi^2 \ln(b)] D_\beta^{d+1-n}$$

$$- 2(1 - \gamma) \int_\Omega b \nabla D_\beta \cdot \nabla \phi \, \phi \ln(b) D_\beta^{d+1-n}$$

$$:= T_{1221} + T_{1222}.$$

We start with T_{1222}. Since $b \approx 1$, we have that $|b \ln(b)| \lesssim 1$. Besides, $|\nabla D_\beta| \lesssim 1$ by (3.41). Therefore,

$$|T_{1222}| \lesssim \int_{2B} |D_\beta \nabla \phi| D_\beta^{d-n},$$

and then $|T_{1222}| \lesssim r^d$ by (3.17). We use (3.38) to write T_{1221} as

$$T_{1221} = (1 - \gamma) \int_\Omega H_{n-d-1} \cdot \nabla[\phi^2 \ln(b)] + (\gamma - 1) \int_\Omega \mathcal{V} \cdot \nabla[\phi^2 \ln(b)] D_\beta^{d+1-n}$$

$$= 0 + (\gamma - 1) \int_\Omega \mathcal{V} \cdot \left[2\phi \nabla \phi \ln(b) + \phi^2 \frac{\nabla b}{b} \right] D_\beta^{d+1-n}$$

by (3.39). Hence by (2.12) and then the Cauchy–Schwarz inequality,

$$|T_{1221}| \lesssim \int_{2B} |\mathcal{V}| (|\nabla \phi| + |\nabla b|) D_\beta^{d+1-n}$$

$$\lesssim \left(\int_{2B} |\mathcal{V}|^2 D_\beta^{n-d} \right)^{\frac{1}{2}} \left(\int_{2B} (|D_\beta \nabla \phi|^2 + |D_\beta \nabla b|^2) D_\beta^{d-n} \right)^{\frac{1}{2}}.$$

But since $\mathcal{V}$, $D_\beta \nabla b$, and $D_\beta \nabla \phi$ all satisfies the Carleson measure condition, we conclude that $|T_{1221}| \lesssim r^d$ as desired.

We bounded each term derived from T by either r^d or $r^{d/2} T^{1/2}$, and consequently proved the claim (3.35). As was observed before, (3.9), Proposition 3.8, and then Theorems 2.21 and 1.8 follow. $\qquad\square$

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