



$L^p - L^q$ estimates for the circular maximal operator on Heisenberg radial functions

Juyoung Lee¹ · Sanghyuk Lee¹

Received: 9 July 2021 / Revised: 6 February 2022 / Accepted: 9 February 2022 /
Published online: 28 February 2022

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract

L^p boundedness of the circular maximal function $\mathcal{M}_{\mathbb{H}^1}$ on the Heisenberg group \mathbb{H}^1 has received considerable attentions. While the problem still remains open, L^p boundedness of $\mathcal{M}_{\mathbb{H}^1}$ on Heisenberg radial functions was recently shown for $p > 2$ by Beltran et al. (Ann Sc Norm Super Pisa Cl Sci. <https://doi.org/10.2422/2036-2145.202001-006>, 2021). In this paper we extend their result considering the local maximal operator $M_{\mathbb{H}^1}$ which is defined by taking supremum over $1 < t < 2$. We prove $L^p - L^q$ estimates for $M_{\mathbb{H}^1}$ on Heisenberg radial functions on the optimal range of p, q modulo the borderline cases. Our argument also provides a simpler proof of the aforementioned result due to Beltran et al.

Mathematics Subject Classification 42B25 · 22E25 · 35S30

1 introduction

For $d \geq 2$ the spherical maximal function is given by

$$\mathcal{M}_{\mathbb{R}^d} f(x) = \sup_{t>0} \left| \frac{1}{\sigma(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} f(x - ty) d\sigma(y) \right|,$$

where $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ is the $(d - 1)$ -dimensional sphere centered at the origin and $d\sigma$ is the surface measure on \mathbb{S}^{d-1} . When $d \geq 3$, it was shown by Stein [21] that $\mathcal{M}_{\mathbb{R}^d} f$

Communicated by Loukas Grafakos.

✉ Juyoung Lee
ljy219@snu.ac.kr
Sanghyuk Lee
shklee@snu.ac.kr

¹ Department of Mathematical Sciences and RIM, Seoul National University, Seoul 08826, Republic of Korea

is bounded on L^p if and only if $p > \frac{d}{d-1}$. The case $d = 2$ was later settled by Bourgain [5]. An alternative proof of Bourgain’s result was subsequently found by Mockenhaupt, Seeger, Sogge [11], who used a local smoothing estimate for the wave operator. We now consider the local maximal operator

$$M_{\mathbb{R}^d} f(x) = \sup_{1 < t < 2} \left| \int_{\mathbb{S}^{d-1}} f(x - ty) d\sigma(y) \right|.$$

As is easy to see, the maximal operator $\mathcal{M}_{\mathbb{R}^d}$ can not be bounded from L^p to L^q unless $p = q$. However, $M_{\mathbb{R}^d}$ is bounded from L^p to L^q for some $p < q$ thanks to the supremum taken over the restricted range [1, 2]. This phenomenon is called L^p improving. Almost complete characterization of L^p improving property of $M_{\mathbb{R}^2}$ was obtained by Schlag [17] except for the endpoint cases. A different proof which is based on L^p - L^q_α smoothing estimate for the wave operator was also found by Schlag and Sogge [18]. They also proved L^p - L^q boundedness of $M_{\mathbb{R}^d}$ for $d \geq 3$ which is optimal up to the borderline cases. Most of the left open endpoint cases were settled by the second author [8] but there are some endpoint cases where L^p - L^q estimate remains unknown though restricted weak type bounds are available for such cases. There are results which extend the aforementioned results to variable coefficient settings, see [18, 19]. Also, see [1, 4, 14] and references therein for recent extensions of the earlier results.

The analogous spherical maximal operators on the Heisenberg group \mathbb{H}^n also have attracted considerable interests. The Heisenberg group \mathbb{H}^n can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ under the noncommutative multiplication law

$$(x, x_{2n+1}) \cdot (y, y_{2n+1}) = (x + y, x_{2n+1} + y_{2n+1} + x \cdot Ay),$$

where $(x, x_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}$ and A is the $2n \times 2n$ matrix given by

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The natural dilation structure on \mathbb{H}^n is $t(x, x_{2n+1}) = (tx, t^2x_{2n+1})$. Abusing the notation, since there is no ambiguity, we denote by $d\sigma$ the usual surface measure of $\mathbb{S}^{2n-1} \times \{0\}$. Then, the dilation $d\sigma_t$ of the measure $d\sigma$ is defined by $\langle f, d\sigma_t \rangle = \langle f(t \cdot), d\sigma \rangle$. Thus, the average over the sphere is now given by

$$f *_{\mathbb{H}} d\sigma_t(x, x_{2n+1}) = \int_{\mathbb{S}^{2n-1}} f(x - ty, x_{2n+1} - tx \cdot Ay) d\sigma(y).$$

We consider the associated local spherical maximal operator

$$M_{\mathbb{H}^n} f(x, x_{2n+1}) = \sup_{1 < t < 2} |f *_{\mathbb{H}} d\sigma_t(x, x_{2n+1})|.$$

Similarly, the global maximal operator $\mathcal{M}_{\mathbb{H}^n}$ is defined by taking supremum over $t > 0$. As in the Euclidean case, L^p boundedness of $M_{\mathbb{H}^n}$ is essentially equivalent to

that of $\mathcal{M}_{\mathbb{H}^n}$ (for example, see [2] or Section 2.5). The spherical maximal operator on \mathbb{H}^n was first studied by Nevo and Thangavelu [13]. It is easy to see that $M_{\mathbb{H}^n}$ is bounded on L^p only if $p > \frac{2n}{2n-1}$ by using Stein’s example ([21]):

$$f(x, x_{2n+1}) = |x|^{1-2n} \log \frac{1}{|x|} \phi_0(x, x_{2n+1}),$$

where ϕ_0 is a cutoff function supported near the origin. For $n \geq 2$, L^p boundedness of $\mathcal{M}_{\mathbb{H}^n}$ on the optimal range was independently proved by Müller and Seeger [10], and by Narayanan and Thangavelu [12]. Furthermore, for $n \geq 2$, Roos, Seeger and Srivastava[15] recently obtained the complete L^p-L^q estimate for $M_{\mathbb{H}^n}$ except for some endpoint cases. Also see [7] for related results.

However, the problem still remains open when $n = 1$.

Definition We say a function $f : \mathbb{H}^1 \rightarrow \mathbb{C}$ is Heisenberg radial if $f(x, x_3) = f(Rx, x_3)$ for all $R \in \text{SO}(2)$.

Beltran, Guo, Hickman and Seeger [2] obtained L^p boundedness of $\mathcal{M}_{\mathbb{H}^1}$ on the Heisenberg radial functions for $p > 2$. In the perspective of the results concerning the local maximal operators [8, 15, 17, 18], it is natural to consider L^p-L^q estimate for $M_{\mathbb{H}^1}$. The main result of this paper is the following which completely characterizes L^p improving property of $M_{\mathbb{H}^1}$ on Heisenberg radial function except for some borderline cases.

Theorem 1.1 *Let $P_0 = (0, 0)$, $P_1 = (1/2, 1/2)$, and $P_2 = (3/7, 2/7)$, and let \mathbf{T} be the closed region bounded by the triangle $\Delta P_0 P_1 P_2$. Suppose $(1/p, 1/q) \in \{P_0\} \cup (\mathbf{T} \setminus (\overline{P_1 P_2} \cup \overline{P_0 P_2}))$. Then, the estimate*

$$\|M_{\mathbb{H}^1} f\|_q \lesssim \|f\|_{L^p} \tag{1.1}$$

holds for all Heisenberg radial function f . Conversely, if $(1/p, 1/q) \notin \mathbf{T}$, then the estimate fails.

Though the Heisenberg radial assumption significantly simplifies the structure of the averaging operator, the associated defining function of the averaging operator is still lacking of curvature properties. In fact, the defining function has vanishing rotational and cinematic curvatures at some points, see [2] for a detailed discussion. This increases the complexity of the problem. To overcome the issue of vanishing curvatures, Beltran et al. [2] used the oscillatory integral operators with two-sided fold singularities and the variable coefficient version of local smoothing estimate [3] combined with additional localization.

The approach in this paper is quite different from that in [2]. Capitalizing on the Heisenberg radial assumption, we make a change of variables so that the averaging operator on the Heisenberg radial function takes a form close to the circular average, see (2.1) below. While the defining function of the consequent operator still does not have nonvanishing rotational and cinematic curvatures, via a further change of variables we can apply the L^p-L^q local smoothing estimate (see, Theorem 3.1 below)

in a more straightforward manner by exploiting the apparent connection to the wave operator (see (2.2) and (2.3)). Consequently, our approach also provides a simplified proof of the recent result due to Beltran et al. [2]. See Sect. 2.5.

Even though we utilize the local smoothing estimate, we do not need to use the full strength of the local smoothing estimate in $d = 2$ since we only need the sharp L^p-L^q local smoothing estimates for (p, q) near $(7/3, 7/2)$. Such estimates can also be obtained by interpolation and scaling argument if one uses the trilinear restriction estimates for the cone and the sharp local smoothing estimate for some large p (for example, see [9]).

The estimate (1.1) remains open when $(1/p, 1/q) \in (\overline{P_1P_2} \cup \overline{P_0P_2}) \setminus \{P_0, P_1\}$. However, we expect that those borderline cases should be true. Most of the corresponding endpoint estimates for the circular maximal function (in \mathbb{R}^2) are known to be true [8], but to implement the approach in [8] we need the local smoothing estimate without ϵ -loss regularity, which we are not able to establish yet even under the Heisenberg radial assumption.

We close the introduction showing the necessity part of Theorem 1.1.

Optimality of p, q range. We show (1.1) implies $(1/p, 1/q) \in \mathbf{T}$, that is to say,

$$(a) \ p \leq q, \quad (b) \ 1 + 1/q \geq 3/p, \quad (c) \ 3/q \geq 2/p.$$

To see (a), let f_R be the characteristic function of a ball of radius $R \gg 1$, centered at 0. Then, $M_{\mathbb{H}^1} f_R$ is also supported in a ball B of radius $\sim R$ and $M_{\mathbb{H}^1} f_R \gtrsim 1$ on B . Thus, $\sup_{R>1} \|M_{\mathbb{H}^1} f_R\|_q / \|f_R\|_p$ is finite only if $p \leq q$. For (b) let g_r be the characteristic function of a ball of radius $r \ll 1$ centered at 0. Then, $|M_{\mathbb{H}^1} g_r(x, x_3)| \gtrsim r$ when (x, x_3) is contained in a c_0r -neighborhood of $\{(x, x_3) : 1 < |x| < 2, x_3 = 0\}$ for a small constant $c_0 > 0$. Thus, (1.1) implies $r^{1+1/q} \lesssim r^{3/p}$, which gives $1 + 1/q \geq 3/p$ if we let $r \rightarrow 0$. Finally, to show (c) we consider h_r which is the characteristic function of an r -neighborhood of $\{(x, x_3) : |x| = 1, x_3 = 0\}$ with $r \ll 1$. Then, $|M_{\mathbb{H}^1} h_r(x, x_3)| \gtrsim c > 0$ when (x, x_3) is in an r -ball centered at 0. Thus, (1.1) gives $r^{3/q} \lesssim r^{2/p}$, which yields $3/q \geq 2/p$.

The maximal estimate (1.1) for general L^p functions has a smaller range of p, q . Let h_r be a characteristic function of the set $\{(x, x_3) : |x_1 - 1| < r^2, |x_2| < r, |x_3| < r\}$ for a sufficiently small $r > 0$. Then $M_{\mathbb{H}^1} h_r(x, x_3) \sim r$ if $-1 \leq x_1 \leq 0, |x_2| < cr, |x_3| < cr$ for a small constant $c > 0$ independent of r . Thus, (1.1) implies $r^{1+2/q} \lesssim r^{4/p}$. It seems to be plausible to conjecture that (1.1) holds for general f modulo some endpoint cases as long as $1 + 2/q - 4/p \geq 0, 3/q \geq 2/p$, and $1/q \leq 1/p$. The range of p, q is properly contained in \mathbf{T} .

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 while assuming Proposition 2.1 and Proposition 2.2 (see below), which we show in the next section.

2.1 Heisenberg radial function

Since f is a Heisenberg radial function, we have $f(x, x_3) = f_0(|x|, x_3)$ for some f_0 . Let us set

$$g(s, z) = f_0(\sqrt{2s}, z), \quad s \geq 0.$$

Then, it follows $f(x, x_3) = g(|x|^2/2, x_3)$. Since $f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = \int f(r - ty_1, -ty_2, x_3 - tr y_2) d\sigma(y) = \int g(\frac{r^2+t^2}{2} - tr y_1, x_3 - tr y_2) d\sigma(y)$, we have

$$f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = g * d\sigma_{tr}\left(\frac{r^2 + t^2}{2}, x_3\right). \tag{2.1}$$

Let us define an operator \mathcal{A}_t by

$$\mathcal{A}_t g(r, x_3) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2)} \widehat{d\sigma}(tr\xi) \widehat{g}(\xi) d\xi. \tag{2.2}$$

Using Fourier inversion, we have

$$f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = \mathcal{A}_t g(r, x_3). \tag{2.3}$$

Since $f *_{\mathbb{H}} d\sigma_t$ is also Heisenberg radial,¹ $\|M_{\mathbb{H}^1} f\|_q^q = \int |M_{\mathbb{H}^1} f(r, 0, x_3)|^q r dr dx_3$. A computation shows $\|f\|_{L^p_{x,x_3}} = \|g\|_{L^p_{r,x_3}}$. Therefore, we see that the estimate (1.1) is equivalent to

$$\|r^{\frac{1}{q}} \sup_{1 < t < 2} |\mathcal{A}_t g|\|_{L^q_{r,x_3}} \leq C \|g\|_p. \tag{2.4}$$

In what follows we show (2.4) holds for p, q satisfying

$$p \leq q, \quad 3/p - 1/q < 1, \quad 1/p + 2/q > 1. \tag{2.5}$$

Then, interpolation with the trivial L^∞ estimate proves Theorem 1.1.

2.2 Decomposition

Let ϕ denote a positive smooth function on \mathbb{R} supported in $[1 - 10^{-3}, 2 + 10^{-3}]$ such that $\sum_{j=-\infty}^\infty \phi(s/2^j) = 1$ for $s > 0$. We set $\phi_j(s) = \phi(s/2^j)$. To show (2.4) we decompose \mathcal{A}_t as follows:

$$\mathcal{A}_t g(r, x_3) = \sum_{k \in \mathbb{Z}} \phi_k(r) \mathcal{A}_t g(r, x_3).$$

¹ This is true because $SO(2)$ is an abelian group. However, $SO(n)$ is not commutative in general, so the property is not valid in higher dimensions.

We break g via the Littlewood–Paley decomposition and try to obtain estimates for each decomposed pieces. For the purpose we denote $\phi_{<j} = \sum_{\ell < j} \phi_\ell$ and $\phi_{\geq j} = \sum_{\ell \geq j} \phi_\ell$ and define the projection operators

$$\widehat{\mathcal{P}_j g}(\xi) := \phi_j(|\xi|)\widehat{g}(\xi), \quad \widehat{\mathcal{P}_{<j} g}(\xi) := \phi_{<j}(|\xi|)\widehat{g}(\xi).$$

Our proof of (2.4) mainly relies on the following two propositions, which we prove in Sect. 3.

Proposition 2.1 *Let $|k| \geq 2$ and $j \geq -k$. Suppose*

$$p \leq q, \quad 1/p + 1/q \leq 1, \quad 1/p + 3/q \geq 1. \tag{2.6}$$

Then, for $\epsilon > 0$ we have

$$\left\| \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p}} \|g\|_{L^p}, & k \geq 2, \\ 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{2k}{q} - \frac{2k}{p}} \|g\|_{L^p}, & k < -2. \end{cases} \tag{2.7}$$

The estimate (2.7) continues to be valid for the case $k = -1, 0, 1$. However, the range of p, q for which (2.7) holds gets smaller.

Proposition 2.2 *Let $j \geq -1$ and $k = -1, 0, 1$. Suppose $p \leq q, 1/p + 1/q < 1$ and $1/p + 2/q > 1$. Then, for $\epsilon > 0$ we have*

$$\left\| \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \lesssim 2^{\frac{j}{2}(\frac{3}{p} - \frac{1}{q} - 1) + \epsilon j} \|g\|_{L^p}.$$

We frequently use the following elementary lemma (for example, see [8]) which plays the role of the Sobolev imbedding.

Lemma 2.3 *Let I be an interval and let F be a smooth function defined on $\mathbb{R}^n \times I$. Then, for $1 \leq p \leq \infty$,*

$$\left\| \sup_{t \in I} |F(x, t)| \right\|_{L^p(\mathbb{R}^n)} \lesssim |I|^{-\frac{1}{p}} \|F\|_{L^p(\mathbb{R}^n \times I)} + \|F\|_{L^p(\mathbb{R}^n \times I)}^{\frac{(p-1)}{p}} \|\partial_t F\|_{L^p(\mathbb{R}^n \times I)}^{\frac{1}{p}}.$$

2.3 Proof of (2.4)

We prove (2.4) handling the three cases $k \leq -2, |k| \leq 1$, and $k \geq 2$, separately. We first consider a change of variables

$$(r, x_3, t) \rightarrow (y_1, y_2, \tau) := \left(\frac{r^2 + t^2}{2}, x_3, rt \right), \tag{2.8}$$

which plays an important role in what follows. Note that

$$\det \frac{\partial(y_1, y_2, \tau)}{\partial(r, x_3, t)} = r^2 - t^2. \tag{2.9}$$

In order to show (2.4), we shall use the change of variables (2.8) to apply the local smoothing estimate to the averaging operator \mathcal{A}_t (see Sect. 3.1). Since $1 < t < 2$, $|\det \partial(y_1, y_2, \tau)/\partial(r, x_3, t)| = |r^2 - t^2| \sim \max(2^{2k}, 1)$ for $|k| \geq 2$. Thus, the cases $|k| \geq 2$ can be handled directly by using local smoothing estimates for the half wave propagator. However, the determinant of the Jacobian may vanish when $|k| \leq 1$. This requires further decomposition away from the set $\{r = t\}$. See Sect. 3.3. This is why we need to consider the three cases separately.

Let us set $g_k = \mathcal{P}_{<-k} g$ and $g^k = g - \mathcal{P}_{<-k} g$ so that $g = g_k + g^k$. Then, we break

$$\phi_k(r)\mathcal{A}_t g = \phi_k(r)\mathcal{A}_t g_k + \phi_k(r)\mathcal{A}_t g^k. \tag{2.10}$$

We use Propositions 2.1 and 2.2 to obtain the estimate for $\phi_k(r)\mathcal{A}_t g^k$, whereas we show the estimate for $\phi_k(r)\mathcal{A}_t g_k$ by elementary means using (2.2).

Case $k \leq -2$

We claim that

$$\left\| r^{\frac{1}{q}} \sum_{k \leq -2} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{2.11}$$

holds provided that p, q satisfy $2/p < 3/q, 3/p - 1/q < 1$, and (2.6). Thus (2.11) holds for p, q satisfying (2.5).

We first consider $\phi_k(r)\mathcal{A}_t g_k$. We shall show that

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p} \tag{2.12}$$

holds for $1 \leq p \leq q \leq \infty$. We recall (2.2) and note that $\partial_t(\widehat{d\sigma}(tr\xi))$ is uniformly bounded because $|r\xi| \lesssim 1$. Since $\text{supp } \widehat{g}_k \subset \{\xi : |\xi| \leq C2^{-k}\}$ and $\partial_t e^{\frac{r^2+t^2}{2}\xi_1} = t\xi_1 e^{\frac{r^2+t^2}{2}\xi_1}$, we have $\|\phi_k(r)\partial_t \mathcal{A}_t g_k\|_q \lesssim 2^{-k} \|\phi_k(r)\mathcal{A}_t g_k\|_q$ by the Mihlin multiplier theorem. Applying Lemma 2.3 to $\phi_k(r)\mathcal{A}_t g_k$, we see that (2.12) follows if we show

$$\|\phi_k(r)\mathcal{A}_t g_k\|_{L^q_{r,x_3,t}(\mathbb{R}^2 \times [1,2])} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p}. \tag{2.13}$$

We now make use of the change of variables (2.8). Since $k \leq -2$ and $t \in [1, 2]$, we have $|\det \frac{\partial(y_1, y_2, \tau)}{\partial(r, x_3, t)}| \sim 1$. Thus the left hand side of (2.13) is bounded by

$$C \left\| \phi_k(r(y_1, y_2, \tau)) \int e^{iy \cdot \xi} \widehat{g}(\xi) \widehat{d\sigma}(\tau\xi) \phi_{<-k}(\xi) d\xi \right\|_{L^q_{y,\tau}(\mathbb{R}^2 \times [2^{-1}, 2^2])}.$$

Changing variables $\xi \rightarrow 2^{-k}\xi$ and $(y, \tau) \rightarrow (2^k y, 2^k \tau)$ gives

$$\|\phi_k(r)\mathcal{A}_t g_k\|_{L^q_{r,x_3,t}(\mathbb{R}^2 \times [1,2])} \lesssim 2^{\frac{3k}{q}} \left\| \int e^{iy \cdot \xi} \widehat{m(\xi)g(2^k \cdot)}(\xi) d\xi \right\|_{L^q_{y,\tau}(\mathbb{R}^2 \times [2^{-1}, 2^2])},$$

where $m(\xi) = \widehat{d\sigma}(\tau\xi)\phi_{<0}(\xi)$. Since $\tau \sim 1$ and $\phi_{<0}(\xi)$ is a smooth function supported in the set $\{\xi : |\xi| \lesssim 1\}$, $m(\xi)$ is a smooth multiplier whose derivatives are uniformly bounded. So, the multiplier operator given by m is uniformly bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $\tau \in [2^{-1}, 2^2]$. Thus, via scaling we obtain (2.13) and, hence, (2.12).

Using the triangle inequality and (2.12), we have

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} \sum_{k \leq -2} |\phi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim \left(\sum_{k \leq -2} 2^{\frac{3k}{q} - \frac{2k}{p}} \right) \|g\|_p \lesssim \|g\|_p$$

because $2/p < 3/q$. We now consider $\phi_k(r)\mathcal{A}_t g^k$ for which we use Proposition 2.1. Since

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} \sum_{k \leq -2} |\phi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \leq \sum_{k \leq -2} \sum_{j \geq -k} \left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}}$$

and since p, q satisfy $3/p - 1/q < 1, 2/p < 3/q$, and (2.6), using the estimate (2.7), we get

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} \sum_{k \leq -2} |\phi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \lesssim \left(\sum_{k \leq -2} 2^{\frac{3k}{q} - \frac{2k}{p}} \right) \|g\|_p \lesssim \|g\|_p.$$

Combining this with the above estimate for $g \rightarrow \phi_k(r)\mathcal{A}_t g^k$ gives (2.11) and this proves the claim.

Case $k \geq 2$

In this case we show

$$\left\| r^{\frac{1}{q}} \sum_{k \geq 2} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{2.14}$$

if $p \leq q, 3/p - 1/q < 1$, and (2.6) holds. So, we have (2.14) if (2.5) holds.

In order to prove (2.14) we first prove the following.

Lemma 2.4 *Let $k \geq -1$. If $|t| \lesssim 1$ and $0 \leq s \lesssim 2^{2k}$, then*

$$|\mathcal{A}_t \mathcal{P}_{<-k} g|(\sqrt{2s}, x_3) \lesssim \mathcal{E}_k^N * |g|(s, x_3), \tag{2.15}$$

where $\mathcal{E}_\ell^N(y) = 2^{-2\ell}(1 + 2^{-\ell}|y|)^{-N}$.

Proof We note that

$$\mathcal{A}_t \mathcal{P}_{<-k} g(\sqrt{2s}, x_3) = K * g(s + 2^{-1}t^2, x_3),$$

where

$$K(y) = \frac{1}{(2\pi)^2} \int e^{iy \cdot \xi} \phi_{<-k}(\xi) \widehat{d\sigma}(t\sqrt{2s}\xi) d\xi.$$

We note $\partial_\xi^\alpha [\phi_{<-k}(2^{-k}\xi) \widehat{d\sigma}(2^{-k}t\sqrt{2s}\xi)] = O(1)$ since $s \lesssim 2^{2k}$. Thus, changing variables $\xi \rightarrow 2^{-k}\xi$, by integration by parts we have $|K| \lesssim \mathcal{E}_k^N$ for any $N > 0$. Since $|t| \lesssim 1$ and $k \geq -1$, we see $\mathcal{E}_k^N(y_1 + 2^{-1}t^2, y_2) \lesssim \mathcal{E}_k^N(y_1, y_2)$. Therefore, we get (2.15). \square

Proof of 2.14 We begin by observing a localization property of the operator \mathcal{A}_t . From (2.1) we note that

$$\frac{r^2 + t^2}{2} - tr y_1 \subset I_k := [2^{2k-1}(1 - 10^{-2}), 2^{2k+1}(1 + 10^{-2})]$$

for $r \in \text{supp } \phi_k$ if k is large enough, i.e., $2^{-k} \leq 10^{-3}$. Thus, from (2.1) and (2.3) we see that

$$\phi_k(r) \mathcal{A}_t g(r, x_3) = \phi_k(r) \mathcal{A}_t ([g]_k)(r, x_3) \tag{2.16}$$

where $[g]_k(r, x_3) = \chi_{I_k}(r)g(r, x_3)$. Clearly, the intervals I_k are finitely overlapping and so are the supports of ϕ_k . Since $p \leq q$, by a standard localization argument it is sufficient for (2.14) to show

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) \mathcal{A}_t g| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{2.17}$$

for $k \geq 2$.

Using the decomposition (2.10), we first consider $\phi_k(r) \mathcal{A}_t g_k$. Changing variables $r \mapsto \sqrt{2s}$, we have

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) \mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}}^q \lesssim \int \phi_k(\sqrt{2s}) \left(\sup_{1 < t < 2} |\mathcal{A}_t g_k(\sqrt{2s}, x_3)| \right)^q ds dx_3.$$

Since $1 < t < 2$, $k \geq 2$, and $g_k = \mathcal{P}_{<-k} g$, by Lemma 2.4 $|\mathcal{A}_t g_k(\sqrt{2s}, x_3)| \lesssim \mathcal{E}_k^N * |g|(s, x_3)$. Hence,

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) \mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim \|\mathcal{E}_k^N * |g|\|_{L^q_{s,x_3}} \lesssim 2^{2k(1/q-1/p)} \|g\|_p \leq \|g\|_p.$$

The second inequality follows by Young’s convolution inequality and the third is clear because $k \geq 2$ and $p \leq q$. We now handle $\phi_k(r)\mathcal{A}_t g^k$. Since

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \leq \sum_{j \geq -k} \left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \tag{2.18}$$

and since $3/p - 1/q < 1$, $p \leq q$, and (2.6) holds, using the estimate (2.7), we get

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \lesssim 2^{\frac{2k}{q} - \frac{2k}{p}} \|g\|_p \lesssim \|g\|_p.$$

Therefore, we get (2.17). □

2.4 Case $|k| \leq 1$

To complete the proof of (2.4), the matter is now reduced to obtaining

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p}, \quad k = -1, 0, 1$$

if p, q satisfy (2.5). In order to show this we use Proposition 2.2. Using the decomposition (2.10), we first consider $\phi_k(r)\mathcal{A}_t g_k$. Since $1 < t < 2$ and $|k| \leq 1$, by Lemma 2.4 we have $\phi_k(r)|\mathcal{A}_t g_k| \lesssim \mathcal{E}_0^N * |g|$. Hence, it follows that

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_p$$

for $1 \leq p \leq q \leq \infty$.

We now consider $\phi_k(r)\mathcal{A}_t g^k$. Note that (2.6) is satisfied if (2.5) holds. Since $3/p - 1/q < 1$, by (2.18) and Proposition 2.2 we see

$$\left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \lesssim \sum_{j \geq -k} 2^{\frac{j}{2}(\frac{3}{p} - \frac{1}{q} - 1) + \epsilon j} \|g\|_{L^p} \lesssim \|g\|_p$$

taking a small enough $\epsilon > 0$. Therefore we get the desired estimate.

2.5 Global maximal estimate

Using the estimates in this section, one can provide a simpler proof of the result due to Beltran et al. [2], i.e.,

$$\left\| r^{\frac{1}{p}} \sup_{0 < t < \infty} |\mathcal{A}_t g| \right\|_{L^p_{r,x_3}} \leq C \|g\|_p \tag{2.19}$$

for $2 < p \leq \infty$. In order to show this we use the following lemma which is a consequence of Propositions 2.1 and 2.2.

Lemma 2.5 *Let $2 \leq p \leq 4$. Then, for some $c > 0$ we have*

$$\|r^{\frac{1}{p}} \sup_{1 < t < 2} |\mathcal{A}_t \mathcal{P}_j g|\|_{L^p_{r,x_3}} \leq C 2^{-cj} \|g\|_p. \tag{2.20}$$

Proof We briefly explain how one can show (2.20). In fact, similarly as before, we decompose

$$\mathcal{A}_t \mathcal{P}_j g = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 := \sum_{k < -j} \phi_k(r) \mathcal{A}_t \mathcal{P}_j g, \quad S_2 := \sum_{-j \leq k \leq -2} \phi_k(r) \mathcal{A}_t \mathcal{P}_j g, \quad S_3 := \sum_{-1 \leq k \leq 1} \phi_k(r) \mathcal{A}_t \mathcal{P}_j g,$$

and $S_4 = \mathcal{A}_t \mathcal{P}_j g - S_1 - S_2 - S_3$. Then, the estimate (2.20) follows if we show $\|r^{\frac{1}{p}} \sup_{1 < t < 2} |S_\ell|\|_{L^p_{r,x_3}} \leq C 2^{-cj} \|g\|_p$, $\ell = 1, 2, 3, 4$ for some $c > 0$. The estimate for S_1 follows from (2.12) and summation over $k < -j$. Using the estimate of the second case in (2.7), one can easily get the estimate for S_2 . The estimate for S_3 is obvious from Proposition 2.2. By Proposition 2.1 combined with the localization property (2.16) we can obtain the estimate for S_4 . However, due to the projection operator \mathcal{P}_j we need to modify the previous argument slightly.

From (2.1) and (2.3) we see

$$\mathcal{A}_t \mathcal{P}_j g(r, x_3) = \iint g(z_1, z_2) K_j \left(\frac{r^2 + t^2}{2} - z_1 - t r y_1, x_3 - z_2 - t r y_2 \right) d\sigma(y) dz, \tag{2.21}$$

where $K_j = \mathcal{F}^{-1}(\phi(2^{-j} \cdot) \cdot | \cdot |)$. Note that $|K_j| \lesssim E_N^j$ for any N and $k \geq 2$. If $r \in \text{supp } \phi_k$, $\sqrt{2z_1} \notin I_k$, and k is large enough, then we have

$$\begin{aligned} & \left| K_j \left(\frac{r^2 + t^2}{2} - t r y_1 - z_1, x_3 - t r y_2 - z_2 \right) \right| \\ & \lesssim 2^{-(2k+j)N} \left(1 + 2^j |r^2 - 2z_1| + 2^{-k} |x_3 - z_2| \right)^{-N} \end{aligned}$$

for any N since $|2^{-1}r^2 - z_1| \gtrsim 2^{2k}$ and $|r t y| \lesssim 2^k$. Hence it follows that

$$\|r^{\frac{1}{p}} \phi_k(r) \mathcal{A}_t \mathcal{P}_j (1 - \chi_{I_k}) g\|_p \leq C 2^{-(k+j)N} \|g\|_p, \quad 1 \leq p \leq \infty$$

for any N . We break $\mathcal{A}_t \mathcal{P}_j g = \mathcal{A}_t \mathcal{P}_j \chi_{I_k} g + \mathcal{A}_t \mathcal{P}_j (1 - \chi_{I_k}) g$. Using the last inequality and then Proposition 2.1, we obtain

$$\|S_4\|_p \leq \left(\sum_{k \geq 2} \|r^{\frac{1}{p}} \phi_k(r) \mathcal{A}_t \mathcal{P}_j \chi_{I_k} g\|_p^p \right)^{\frac{1}{p}} + \sum_{k \geq 2} 2^{-(k+j)N} \|g\|_p \lesssim 2^{-cj} \|g\|_p$$

for some $c > 0$ by taking an N large enough. □

Once we have (2.20), using a standard argument which relies on the Littlewood–Paley decomposition and rescaling (for example, see [2, 5, 16]) one can easily show (2.19). Indeed, we break the maximal function into high and lower frequency parts:

$$\sup_{0 < t < \infty} |\mathcal{A}_t g| \leq \mathcal{A}_{low} g + \mathcal{A}_{high} g,$$

where

$$\begin{aligned} \mathcal{A}_{low} g &= \sup_l \sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{<-2l} g|, \\ \mathcal{A}_{high} g &= \sum_{k \geq 0} \sup_l \sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{k-2l} g|. \end{aligned}$$

For $\mathcal{A}_{low} g$ we claim

$$\sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1}r^2, x_3). \tag{2.22}$$

This gives $\mathcal{A}_{low} g(r, x_3) \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1}r^2, x_3)$. Since $\mathcal{M}_{\mathbb{R}^2}$ is bounded on L^p for $p > 2$, for $2 < p \leq \infty$ we get

$$\|r^{\frac{1}{p}} \mathcal{A}_{low} g\|_{L_{r,x_3}^p} \leq C \|g\|_p.$$

We now proceed to prove (2.22). Note that $\sum_{j \leq 2l} \phi(2^{-j}|\cdot|) = \phi_{<1}(2^{2l}|\cdot|)$ and $\phi_{<1}$ is a smooth function supported on $[-2^2, 2^2]$. Thus, similarly as in (2.21) we note that $\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3) = \iint g(z_1, z_2) \tilde{K}_l * d\sigma_{tr}(2^{-1}(r^2 + t^2) - z_1, x_3 - z_2) dz$ where $\tilde{K}_l = \mathcal{F}^{-1}(\phi_{<1}(2^{2l}|\cdot|))$. Since $\tilde{K}_l \lesssim \mathcal{E}_{2l}^N$ for any N , for $2^l \leq t < 2^{l+1}$ we see

$$|\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \int |g(z_1, z_2)| \mathcal{E}_{2l}^{2N} * d\sigma_{tr}(2^{-1}r^2 - z_1, x_3 - z_2) dz \tag{2.23}$$

because $2^{2l}t^2 \lesssim 1$ and $\mathcal{E}_{2l}^{2N} = 2^{-4l}(1 + 2^{-2l}|y|)^{-2N}$. Hence, taking an N large enough, we note that

$$\mathcal{E}_{2l}^{2N} * d\sigma_{tr}(x) \lesssim \begin{cases} (2^{2l}tr)^{-1}(1 + 2^{-2l}||x| - tr|)^{-N}, & 2^{2l} \ll tr, \\ 2^{-4l}(1 + 2^{-2l}|x|)^{-N}, & 2^{2l} \gtrsim tr, \end{cases} \tag{2.24}$$

provided that $2^l \leq t < 2^{l+1}$. Indeed, to show this we only have to consider the case $2^{2l} \ll tr$ since the other case is trivial. By scaling $x \rightarrow trx$ we may assume that

$tr = 1$. Thus, it is enough to show $\int L^{-2}(1 + L^{-1}|x - y|)^{-2N}d\sigma(y) \lesssim L^{-1}(1 + L^{-1}||x| - 1|)^{-N}$ for $L \ll 1$ with an N large enough. However, this is easy to see since $|x - y| \geq ||x| - 1|$ and $\int L^{-1}(1 + L^{-1}|x - y|)^{-N}d\sigma(y) \lesssim 1$.

Therefore, combining (2.23) and (2.24), one can see

$$\sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1}r^2, x_3) + \mathfrak{M}_2 g(2^{-1}r^2, x_3).$$

Here \mathfrak{M}_2 denotes the Hardy-Littlewood maximal function on \mathbb{R}^2 . This proves the claim (2.22) since $\mathfrak{M}_2 g \lesssim \mathcal{M}_{\mathbb{R}^2} g$.

So we are reduced to showing $\|r^{\frac{1}{p}} \mathcal{A}_{high} g\|_{L^p_{r,x_3}} \leq C \|g\|_p$ for $p > 2$. For the purpose it is sufficient to show

$$\| \sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{k-2l} g| \|_p \lesssim 2^{-ck} \|g\|_p \tag{2.25}$$

because $\mathcal{A}_{high} g \leq \sum_{k \geq 0} (\sum_l |\sup_{2^l \leq t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{k-2l} g|^p)^{1/p}$ and $(\sum_l \|\mathcal{P}_{k-2l} g\|_p^p)^{1/p} \lesssim \|g\|_p$. By scaling, using (2.2), we can easily see the inequality (2.25) is equivalent to (2.20) while j replaced by k . So, we have (2.25) and this completes the proof of (2.19).

3 Proof of Propositions 2.1 and 2.2

In order to prove Propositions 2.1 and 2.2, we are led by (2.2) to consider $\widehat{d\sigma}(tr\xi)$ for which we use the following well known asymptotic expansion (see, for example, [20]):

$$\widehat{d\sigma}(\xi) = \sum_{j=0}^N C_j^\pm |\xi|^{-\frac{1}{2}-j} e^{\pm i|\xi|} + E_N(|\xi|), \quad |\xi| \gtrsim 1 \tag{3.1}$$

where E_N is a smooth function satisfying

$$|\frac{d^\ell}{dr^\ell} E_N(r)| \lesssim r^{-N} \tag{3.2}$$

for $0 \leq \ell \leq 4$ if $r \gtrsim 1$. The expansion (3.1) relates the operator \mathcal{A}_t to the wave propagator. After changing variables, to prove Propositions 2.1 and 2.2 we can use the local smoothing estimate for the wave operator (see Proposition 3.1 below).

3.1 Local smoothing estimate

Let us denote

$$e^{it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x \cdot \xi + t|\xi|)} \widehat{f}(\xi) d\xi.$$

We make use of L^p - L^q local smoothing estimate for the wave equation in \mathbb{R}^2 .

Theorem 3.1 *Let $j \geq 0$. Suppose (2.6) holds. Then, for $\epsilon > 0$ we have*

$$\left\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \right\|_{L^q_{x,t}(\mathbb{R}^2 \times [1,2])} \lesssim 2^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)j+\epsilon j} \|f\|_{L^p} \tag{3.3}$$

This follows by interpolating the estimates (3.3) with $(p, q) = (2, 2), (1, \infty)$, and $(4, 4)$. The estimate (3.3) with $(p, q) = (2, 2)$ is a straightforward consequence of Plancherel’s theorem and (3.3) with $(p, q) = (1, \infty)$ can be shown by the stationary phase method (for example, see [8]). The case $(p, q) = (4, 4)$ is due to Guth et al. [6].

From Theorem 3.1 we can deduce the following estimate via simple rescaling argument.

Corollary 3.2 *Let $j \geq -\ell$. Suppose (2.6) holds. Then, for $\epsilon > 0$ we have*

$$\left\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \right\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)(\ell+j)+\left(\frac{3}{q}-\frac{2}{p}\right)\ell+\epsilon(\ell+j)} \|f\|_{L^p}.$$

Proof Changing variables $(x, t) \rightarrow 2^\ell(x, t)$, we see

$$\left\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \right\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} = 2^{\frac{3\ell}{q}} \left\| e^{it\sqrt{-\Delta}} \mathcal{P}_{\ell+j} f(2^\ell \cdot) \right\|_{L^q_{x,t}(\mathbb{R}^2 \times [1,2])}.$$

Thus, using (3.3) we have

$$\left\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \right\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3\ell}{q}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)(\ell+j)+\epsilon(\ell+j)} \|f(2^\ell \cdot)\|_{L^p}.$$

So, rescaling gives the desired inequality. □

3.2 Proof of Proposition 2.1

We now recall (2.2) and (3.1). To show Proposition 2.1 we first deal with the contribution from the error part E_N . Let us set

$$\mathcal{E}_t g(r, x_3) = \int e^{i\left(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2\right)} E_N(tr|\xi|) \widehat{g}(\xi) d\xi.$$

Lemma 3.3 *Let $j \geq -k$. Suppose (2.6) holds. Then, we have*

$$\left\| \sup_{1 < t < 2} |\phi_k(r) \mathcal{E}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{-(N-3)(j+k)} 2^{k\left(\frac{1}{q}-\frac{2}{p}\right)} \|g\|_{L^p}, & k \geq -2, \\ 2^{-(N-3)(j+k)} 2^{k\left(\frac{3}{q}-\frac{2}{p}\right)} \|g\|_{L^p}, & k < -2. \end{cases} \tag{3.4}$$

Proof We first consider the case $k \geq -2$. Using Lemma 2.3, we need to estimate $\phi_k(r) \mathcal{E}_t \mathcal{P}_j g$ and $\phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g$ in $L^q_{r,x_3,t}(\mathbb{R}^2 \times [1, 2])$. For simplicity we denote

$L_{r,x_3,t}^q = L_{r,x_3,t}^q(\mathbb{R}^2 \times [1, 2])$. We first consider $\phi_k(r)\mathcal{E}_t\mathcal{P}_jg$. Changing variables $\frac{r^2}{2} \mapsto s$, we note that

$$\phi_k(\sqrt{2s})\mathcal{E}_t\mathcal{P}_jg(\sqrt{2s}, x_3) = \phi_k(\sqrt{2s}) \int \mathcal{K}(s - y_1 + 2^{-1}t^2, x_3 - y_2)g(y_1, y_2)dy,$$

where

$$\mathcal{K}(s, u) = 2^{2j} \int e^{i2^j(s\xi_1+u\xi_2)}\phi_0(\xi)E_N(2^j t\sqrt{2s}|\xi|)d\xi.$$

Since $s \sim 2^{2k}$, using (3.2), we have $|\mathcal{K}(s, u)| \lesssim 2^{2j}(1 + 2^j|(s, u)|)^{-M}2^{-N(j+k)}$ for $1 \leq M \leq 4$ via integration by parts. Thus, we have $\|\phi_k(\sqrt{2s})\mathcal{K}(s + \frac{t^2}{2}, u)\|_{L_{s,u}^q} \leq C2^{-N(j+k)}2^{2j(1-\frac{1}{r})}$ for $1 < t < 2$ with a positive constant C . Young’s convolution inequality gives $\|\phi_k(\sqrt{2s})\mathcal{E}_t\mathcal{P}_jg(\sqrt{2s}, x_3)\|_{L_{s,x_3,t}^q} \lesssim 2^{-N(j+k)}2^{2j(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}$. Thus, reversing $s \rightarrow r^2/2$, after a simple manipulation we get

$$\|\phi_k(r)\mathcal{E}_t\mathcal{P}_jg\|_{L_{r,x_3,t}^q} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{1}{q}-\frac{2}{p})}\|g\|_{L^p} \tag{3.5}$$

for $1 \leq p \leq q \leq \infty$. Indeed, we need only note that $2j(\frac{1}{p} - \frac{1}{q}) - \frac{k}{q} \leq 2(j + k) + k(\frac{1}{q} - \frac{2}{p})$ because $j \geq -k$ and $\frac{1}{p} - \frac{1}{q} - 1 < 0$.

We now consider $\phi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_jg$. Note that

$$\partial_t\mathcal{E}_t g(r, x_3) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2)}(t\xi_1 E_N(tr|\xi|) + r|\xi|E'_N(tr|\xi|))\widehat{g}(\xi)d\xi. \tag{3.6}$$

Using (3.2), we can handle $\phi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_jg$ similarly as before. In fact, since $|t\xi_1| \lesssim 2^j$ and $r|\xi| \sim 2^{k+j}$, we see

$$\|\phi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_jg\|_{L_{r,x_3}^q} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{1}{q}-\frac{2}{p})}(2^{j+k} + 2^j)\|g\|_{L^p}.$$

Hence, combining this and (3.5) with Lemma 2.3, we get (3.4) for $k \geq -2$.

We now consider the case $k < -2$. We first claim that

$$\|\phi_k(r)\mathcal{E}_t\mathcal{P}_jg\|_{L_{r,x_3,t}^q} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{2}{q}-\frac{2}{p})}\|g\|_{L^p}. \tag{3.7}$$

We use the transformation (2.8). By (2.9) we have $|\frac{\partial(y_1, y_2, \tau)}{\partial(r, x_3, t)}| \sim 1$. Therefore,

$$\|\phi_k(r)\mathcal{E}_t\mathcal{P}_jg\|_{L_{r,x_3,t}^q} \lesssim \left(\int \left| \phi_k(r(y, \tau))\widetilde{K}(\cdot, \tau) * g(y) \right|^q dy d\tau \right)^{\frac{1}{q}},$$

where

$$\tilde{K}(y, \tau) = \int e^{iy \cdot \xi} \phi_j(\xi) E_N(\tau|\xi|) d\xi.$$

Note that $\tau \sim 2^k$. Changing $\tau \mapsto 2^k \tau$ and $\xi \mapsto 2^j \xi$, using (3.2) and integration by parts, we have $|\tilde{K}(y, 2^k \tau)| \leq C 2^{2j} (1 + 2^j |y|)^{-M} 2^{-N(j+k)}$ for $1 \leq M \leq 4$ and $1 < \tau < 2$. Young’s convolution inequality gives

$$\|\phi_k(r) \mathcal{E}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-N(j+k)} 2^{2j(\frac{1}{p} - \frac{1}{q})} \|g\|_{L^p}.$$

Thus, we get (3.7). As for $\phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g$, we use (3.6) and repeat the same argument to see $\|\phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-N(j+k)} 2^j 2^{2j(\frac{1}{p} - \frac{1}{q})} \|g\|_{L^p}$ since $|t\xi_1| \lesssim 2^j, r|\xi| \sim 2^{k+j}$, and $k < -2$. Thus, we get

$$\|\phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)} 2^k 2^{k(\frac{2}{q} - \frac{2}{p})} \|g\|_{L^p}.$$

Putting (3.7) and this together, by Lemma 2.3 we obtain (3.4) for $k < -2$. □

By (3.1) and Lemma 3.3, to prove Propositions 2.1 and 2.2 we only have to consider contributions from the remaining $C_j^\pm |tr\xi|^{-\frac{1}{2}-j} e^{\pm i|tr\xi|}$, $j = 0, \dots, N$. To this end, it is sufficient to consider the major term $C_0^\pm |tr\xi|^{-\frac{1}{2}} e^{\pm i|tr\xi|}$ since the other terms can be handled similarly. Furthermore, by reflection $t \rightarrow -t$ it is enough to deal with $|tr\xi|^{-\frac{1}{2}} e^{i|tr\xi|}$ since the estimate (3.3) clearly holds with the interval $[1, 2]$ replaced by $[-2, -1]$.

Let us set

$$\mathcal{U}_t g(r, x_3) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2+tr|\xi|)} |r\xi|^{-\frac{1}{2}} \widehat{g}(\xi) d\xi. \tag{3.8}$$

To complete the proof of Proposition 2.1, we need to show

$$\left\| \sup_{1 < t < 2} |\phi_k(r) \mathcal{U}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p}} \|g\|_{L^p}, & k \geq 2, \\ 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{2k}{q} - \frac{2k}{p}} \|g\|_{L^p}, & k \leq -2. \end{cases} \tag{3.9}$$

Using Lemma 2.3, the matter is reduced to obtaining estimates for $\phi_k(r) \mathcal{U}_t \mathcal{P}_j g$ and $\phi_k(r) \partial_t \mathcal{U}_t \mathcal{P}_j g$ in $L^q_{r,x_3,t}$. Note that

$$\partial_t \mathcal{U}_t \mathcal{P}_j g(r, x_3, t) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2+tr|\xi|)} \widehat{\mathcal{P}_j g}(\xi) \frac{t\xi_1 + r|\xi|}{|r\xi|^{1/2}} d\xi. \tag{3.10}$$

By the Mihlin multiplier theorem one can easily see

$$\|\phi_k(r) \partial_t \mathcal{U}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim \begin{cases} 2^{j+k} \|\phi_k(r) \mathcal{U}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}}, & k \geq 0, \\ 2^j \|\phi_k(r) \mathcal{U}_t \mathcal{P}_j g\|_{L^q_{r,x_3,t}}, & k < 0, \end{cases}$$

where $L_{r,x_3,t}^q$ denotes $L_{r,x_3,t}^q(\mathbb{R}^2 \times [1, 2])$. Therefore, by Lemma 2.3 it is sufficient for (3.9) to prove that

$$\|\phi_k(r)\mathcal{U}_t\mathcal{P}_jg\|_{L_{r,x_3,t}^q} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}+\epsilon)+\frac{k}{q}-\frac{2k}{p}} \|g\|_{L^p}, & k \geq 2, \\ 2^{(j+k)(\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}+\epsilon)+\frac{3k}{q}-\frac{2k}{p}} \|g\|_{L^p}, & k \leq -2. \end{cases}$$

We first consider the case $k \geq 2$. As before, we use the change of variables (2.8). Since $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 2^{2k}$ from (2.9) and since $\tau = rt$ and $1 < t < 2$, we have

$$\|\phi_k(r)\mathcal{U}_t\mathcal{P}_jg\|_{L_{r,x_3,t}^q} \lesssim 2^{-\frac{2k}{q}-\frac{j+k}{2}} \|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_jg\|_{L_{y,\tau}^q(\mathbb{R}^2 \times [2^{k-1}, 2^{k+2}])}$$

since $|r\xi| \sim 2^{j+k}$. Thus, Corollary 3.2 gives the desired estimate (3.9) for $k \geq 2$. The case $k \leq -2$ can be handled in the exactly same manner. The only difference is that $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 1$. Thus, the desired estimate (3.9) immediately follows from Corollary 3.2.

3.3 Proof of Proposition 2.2

As mentioned already, the determinant of the Jacobian $\partial(y_1, y_2, \tau)/\partial(r, x_3, t)$ may vanish when $|k| \leq 1$. So, we need additional decomposition depending on $|r - t|$. We also make decomposition in ξ depending on $|\xi|^{-1}\xi_1 + 1$ to control the size of the multiplier $|t\xi_1 + r|\xi|$ in a more accurate manner (for example, see (3.22)).

For $m \geq 0$ let us set

$$\begin{aligned} \psi_m(\xi) &= \phi(2^m|\xi|^{-1}\xi_1 + 1), \\ \psi^m(\xi) &= 1 - \sum_{0 \leq j < m} \psi_j(\xi), \end{aligned}$$

so that $\sum_{0 \leq k < m} \psi_k + \psi^m = 1$. We additionally define

$$\mathcal{P}_{j,m}g = (\phi_j\psi_m\widehat{g})^\vee, \quad \mathcal{P}_j^m g = (\phi_j\psi^m\widehat{g})^\vee.$$

So it follows that

$$\mathcal{P}_j = \sum_{0 \leq k < m} \mathcal{P}_{j,k} + \mathcal{P}_j^m. \tag{3.11}$$

Proposition 3.4 *Let us set $\phi_{k,l}(r, t) = \phi_k(r)\phi(2^l|r - t|)$. Let $j \geq -1$ and $k = -1, 0, 1$. Suppose (2.6) holds. Then, for $\epsilon > 0$ we have*

$$\|\phi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g\|_{L_{r,x_3,t}^q} \lesssim 2^{-\frac{j}{2}}2^{\frac{l}{q}}2^{(\frac{m-l}{2})(\frac{1}{p}+\frac{3}{q}-1)+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})+\epsilon j} \|g\|_{L^p}. \tag{3.12}$$

In order to prove Proposition 3.4, we make the change of variables (2.8). Since $|k| \leq 1$, we need only to consider (r, t) contained in the set $[2^{-1} - 10^{-2}, 2^2 + 10^2] \times [1, 2]$.

Set

$$S_l = \{(y_1, y_2, \tau) : 2^{-2l-1} \leq |y_1 - \tau| \leq 2^{-2l+1}, y_1, \tau \in [2^{-3}, 2^3]\}.$$

By (2.8) $y_1 - \tau = (r - t)^2/2$. From (2.9) we note $|\det \frac{\partial(y_1, y_2, \tau)}{\partial(r, x_3, t)}| \sim 2^{-l}$ if $(y_1, \tau) \in S_l$. Thus, changing variables $(r, x_3, t) \rightarrow (y_1, y_2, \tau)$ we obtain

$$\|\phi_{k,l} \mathcal{U}_t \mathcal{P}_j h\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{1}{2}j} 2^{\frac{l}{q}} \|e^{i\tau\sqrt{-\Delta}} \mathcal{P}_j h\|_{L^q_{y,\tau}(S_l)}. \tag{3.13}$$

Therefore, for (3.12) it is sufficient to show

$$\|e^{i\tau\sqrt{-\Delta}} \mathcal{P}_{j,m} g\|_{L^q_{y,\tau}(S_l)} \lesssim 2^{(\frac{m}{2}-l)(\frac{1}{p}+\frac{3}{q}-1)+\frac{3l}{2}(\frac{1}{p}-\frac{1}{q})+\epsilon j} \|g\|_{L^p} \tag{3.14}$$

for p, q satisfying (2.6). For the purpose we need the following lemma, which gives an improved L^2 estimate thanks to restriction of the integral over S_l . Indeed, one can remove the localization $y_1, \tau \in [2^{-3}, 2^3]$.

Lemma 3.5 *Let $D_l = \{(x_1, x_2, t) : 2^{-2l} \leq |x_1 - t| \leq 2^{-2l+1}\}$. Then, we have*

$$\left\| \int e^{i(x \cdot \xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \lesssim 2^{\frac{m}{2}-l} \|g\|_{L^2}. \tag{3.15}$$

Proof We write $x \cdot \xi + t|\xi| = x_1(\xi_1 + |\xi|) + x_2\xi_2 + (t - x_1)|\xi|$. Then, changing variables $(x, t - x_1) \rightarrow (x, t)$ and $\xi \rightarrow \eta := \mathcal{L}(\xi) = (\xi_1 + |\xi|, \xi_2)$, we see

$$\left\| \int e^{i(x \cdot \xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \leq \left\| \int e^{i(x \cdot \eta + t|\mathcal{L}^{-1}\eta|)} \frac{\widehat{h}(\mathcal{L}^{-1}\eta)}{|\det J\mathcal{L}(\eta)|} d\eta \right\|_{L^2_{x,t}(\mathbb{R}^2 \times I_l)}$$

where $\widehat{h}(\xi) = \widehat{g}(\xi) \psi_m(\xi)$ and $I_l = [-2^{-2l+1}, -2^{-2l}] \cup [2^{-2l}, 2^{-2l+1}]$. By Plancherel’s theorem in the x -variable and integrating in t , we have

$$\left\| \int e^{i(x \cdot \xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \leq C 2^{-l} \left\| \frac{\widehat{h}(\mathcal{L}^{-1} \cdot)}{|\det J\mathcal{L}|} \right\|_{L^2_x}.$$

A computation shows $\det J\mathcal{L} = 1 + |\xi|^{-1}\xi_1$, so $|\det J\mathcal{L}| \sim 2^{-m}$ on the support of \widehat{h} . Thus, by changing variables and Plancherel’s theorem we get (3.15). \square

We also use the following elementary lemma.

Lemma 3.6 *For any $1 \leq p \leq \infty, j$, and m , we have*

$$\|(\phi_j \psi_m \widehat{g})^\vee\|_{L^p} \lesssim \|g\|_{L^p}, \quad \|(\phi_j \psi^m \widehat{g})^\vee\|_{L^p} \lesssim \|g\|_{L^p}.$$

Proof Since $\psi^m - \psi^{m+1} = \psi_m$, it suffices to prove the second inequality only. By Young’s inequality we need only to show $\|(\phi_j \psi^m)^\vee\|_{L^1} \lesssim 1$. By scaling it is clear that $\|(\phi_j(\xi)\psi^m(\xi))^\vee\|_{L^1} = \|(\phi_0(\xi)\psi^m(\xi))^\vee\|_{L^1}$. Note that $m(\xi) := \phi_0(\xi)\psi^m(\xi)$ is supported in a rectangular box with dimensions $2^{-m} \times 1$. So, $m(\xi_1, 2^{-m}\xi_2)$ is supported in a cube of side length ~ 1 and it is easy to see $\partial_\xi^\alpha(m(\xi_1, 2^{-m}\xi_2))$ is uniformly bounded for any α . This gives $\|(m(\cdot, 2^{-m}\cdot))^\vee\|_1 \lesssim 1$. Therefore, after scaling we get $\|(\phi_0(\xi)\psi^m(\xi))^\vee\|_{L^1} \lesssim 1$. \square

Proof of 3.14 In view of interpolation the estimate (3.14) follows for p, q satisfying (2.6) if we show the next three estimates:

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^2_{y,\tau}(S_t)} \lesssim 2^{\frac{m}{2}-l}\|g\|_{L^2}, \tag{3.16}$$

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^\infty_{y,\tau}(S_t)} \lesssim 2^{\frac{3j}{2}}\|g\|_{L^1}, \tag{3.17}$$

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^4_{y,\tau}(S_t)} \lesssim 2^{\epsilon j}\|g\|_{L^4}.$$

The first estimate follows from Lemma 3.5. Corollary 3.2 and Lemma 3.6 give the other two estimates. \square

It is possible to improve the estimate (3.12) when $j > m$.

Proposition 3.7 *Let $j \geq -1$ and $k = -1, 0, 1$. Suppose $1 \leq p \leq q, 1/p + 1/q \leq 1$, and $j > m$, then*

$$\|\phi_{k,j}\mathcal{U}_t\mathcal{P}_{j,m}g\|_{L^q_{t,x_3,t}} \lesssim 2^{-\frac{j}{2} - \frac{l}{q} - \frac{2}{q}(\frac{m}{2}-l) + \frac{j-m}{2}(1-\frac{1}{p}-\frac{1}{q}) + \frac{3j}{2}(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}.$$

Proof By (3.13) it is sufficient to show

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^q_{y,\tau}(S_t)} \lesssim 2^{\frac{2}{q}(\frac{m}{2}-l) + \frac{j-m}{2}(1-\frac{1}{p}-\frac{1}{q}) + \frac{3j}{2}(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}$$

for p, q satisfying $1 \leq p \leq q, 1/p + 1/q \leq 1$. In fact, by interpolation with the estimates (3.16) and (3.17) we only have to show

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^\infty_{y,\tau}(S_t)} \lesssim 2^{\frac{j-m}{2}}\|g\|_{L^\infty}. \tag{3.18}$$

Let us set

$$K_t^{j,m}(x) = \frac{1}{(2\pi)^2} \int e^{i(x\cdot\xi+t|\xi|)}\phi_j(|\xi|)\psi_m(\xi)d\xi.$$

Then $e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g = K_\tau^{j,m} * g$. Therefore, (3.18) follows if we show

$$\|K_t^{j,m}\|_{L^1_x} \lesssim 2^{\frac{j-m}{2}} \tag{3.19}$$

when $t \sim 1$. Note that $|\xi_2|/|\xi| = \sqrt{1 - \xi_1/|\xi|}\sqrt{1 + \xi_1/|\xi|} \lesssim 2^{-\frac{m}{2}}$ if $\xi \in \text{supp } \psi_m$. So, $\text{supp } \psi_m$ is contained in a conic sector with angle $\sim 2^{-\frac{m}{2}}$. Let \mathcal{S} be a sector centered at the origin in \mathbb{R}^2 with angle $\sim 2^{-\frac{j}{2}}$ and $\phi_{\mathcal{S}}$ be a cut-off function adapted to \mathcal{S} . Then, by integration by parts it follows that

$$\left\| \int e^{i(x \cdot \xi + t|\xi|)} \phi_j(|\xi|) \phi_{\mathcal{S}}(\xi) d\xi \right\|_{L^1_x} \lesssim 1$$

if $t \sim 1$. (See, for example, [8]). Now (3.19) is clear since the support of ψ_m can be decomposed into as many as $C2^{\frac{j-m}{2}}$ such sectors. □

Finally, we prove Proposition 2.2 making use of Propositions 3.4 and 3.7. We recall (2.2) and (3.1). As mentioned before, by Lemma 3.3 we need only to consider \mathcal{U}_t (see (3.8)) and it is sufficient to show

$$\left\| \sup_{1 < t < 2} |\phi_k(r) \mathcal{U}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \lesssim 2^{\frac{1}{2}(\frac{3}{p} - \frac{1}{q} - 1)j + \epsilon j} \|g\|_{L^p} \tag{3.20}$$

for p, q satisfying $p \leq q, 1/p + 1/q < 1$ and $1/p + 2/q > 1$.

Proof of 3.20 Let us set $\phi^l(\cdot) = 1 - \sum_{j=0}^{l-1} \phi(2^j \cdot)$ and $\phi_k^l(r, t) = \phi_k(r) \phi^l(|r - t|)$. Then, we decompose

$$\phi_k(r) = \sum_{0 \leq l \leq j/2} \phi_{k,l}(r, t) + \sum_{j/2 < l < j} \phi_{k,l}(r, t) + \phi_k^j(r, t).$$

Combining this with (3.11) and using $\sum_{\frac{j}{2} < l < j} \phi_{k,l} + \phi_k^j \leq \phi_k^{[j/2]-1}$, by the triangle inequality we have

$$\left\| \sup_{1 < t < 2} |\phi_k(r) \mathcal{U}_t \mathcal{P}_j g| \right\|_{L^q} \leq \sum_{i=1}^5 S_i,$$

where

$$\begin{aligned} S_1 &= \sum_{0 \leq l \leq j/2} \sum_{0 \leq m \leq l-1} \left\| \sup_{1 < t < 2} \phi_{k,l} |\mathcal{U}_t \mathcal{P}_{j,m} g| \right\|_{L^q}, & S_2 &= \sum_{0 \leq l \leq j/2} \left\| \sup_{1 < t < 2} \phi_{k,l} |\mathcal{U}_t \mathcal{P}_j^l g| \right\|_{L^q}, \\ S_3 &= \sum_{\frac{j}{2} < l < j} \sum_{0 \leq m \leq j-1} \left\| \sup_{1 < t < 2} \phi_{k,l} |\mathcal{U}_t \mathcal{P}_{j,m} g| \right\|_{L^q}, & S_4 &= \sum_{0 \leq m \leq j-1} \left\| \sup_{1 < t < 2} \phi_k^j |\mathcal{U}_t \mathcal{P}_{j,m} g| \right\|_{L^q}, \\ S_5 &= \left\| \sup_{1 < t < 2} \phi_k^{[j/2]-1} |\mathcal{U}_t \mathcal{P}_j^j g| \right\|_{L^q}. \end{aligned}$$

The proof of (3.20) is now reduced to showing

$$S_i \lesssim 2^{\frac{1}{2}(\frac{3}{p} - \frac{1}{q} - 1)j + \epsilon j} \|g\|_{L^p}, \quad 1 \leq i \leq 5, \tag{3.21}$$

for p, q satisfying $p \leq q, 1/p + 1/q < 1$ and $1/p + 2/q > 1$.

Before we start the proof of (3.21), we briefly comment on the decomposition $S_i, i = 1, \dots, 5$. As for S_4 and S_5 , which are easier to handle, the sizes of $r - t$ and $|\xi|^{-1}\xi_1 + 1$ are sufficiently small on the supports of the associated multipliers, so we can remove the dependence of t by an elementary argument. For S_1, S_2 , and S_3 , we use Lemma 2.3 combined with (3.10) to control the maximal operators. Different magnitudes of contribution come from $\partial_t \phi_{k,l} = O(2^l)$ and $|t\xi_1 + r|\xi|$, so we need to compare them. Writing $t\xi_1 + r|\xi| = t(|\xi|^{-1}\xi_1 + 1) + (r - t)$, we note

$$|t\xi_1 + r|\xi| \lesssim 2^j \max\{2^{-m}, 2^{-l}\}. \tag{3.22}$$

The decompositions in S_1, S_2 , and S_3 are made according to comparative sizes of $\partial_t \phi_{k,l} = O(2^l)$ and $|t\xi_1 + r|\xi|$ in terms of l, m , and j .

We first consider S_1 . Using Lemma 2.3, we need to estimate $\phi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g$ and $\partial_t(\phi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g)$ in $L^q_{r,x_3,t}(\mathbb{R}^2 \times [1, 2])$. Note that $\partial_t \phi_{k,l} = O(2^l)$ and $2^l \lesssim 2^{j-m}$. Thus, recalling (3.10), we apply Lemma 2.3 and the Mikhlin multiplier theorem to get

$$S_1 \lesssim \sum_{0 \leq l \leq j/2} \sum_{m=0}^{l-1} 2^{\frac{j-m}{q}} \|\phi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g\|_{L^q}.$$

Thus, by Proposition 3.4 it follows that

$$S_1 \lesssim 2^{-\frac{j}{2} + \frac{j}{q} + \frac{3j}{2}(\frac{1}{p} - \frac{1}{q}) + \epsilon j} \sum_{0 \leq l \leq j/2} 2^{l(1 - \frac{1}{p} - \frac{2}{q})} \sum_{m=0}^{l-1} 2^{\frac{m}{2}(\frac{1}{p} + \frac{1}{q} - 1)} \|g\|_{L^p}.$$

Since $1/p + 1/q - 1 < 0$ and $1/p + 2/q > 1$, we obtain (3.21) with $i = 1$.

We can show the estimate (3.21) with $i = 2$ in the same manner. As before, since $\partial_t \phi_{k,l} = O(2^l)$ and $2^l \lesssim 2^{j-l}$, using (3.22), Lemma 2.3, and the Mikhlin multiplier theorem, we have

$$S_2 \lesssim \sum_{0 \leq l \leq j/2} 2^{\frac{j-l}{q}} \|\phi_{k,l} \mathcal{U}_t \mathcal{P}_j^l g\|_{L^q}.$$

Thus, by (3.13) and Theorem 3.1, we have $S_2 \lesssim \sum_{0 \leq l \leq \frac{j}{2}} 2^{-\frac{j}{2} + \frac{j}{q} + \frac{3j}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\epsilon}{2} j} \|g\|_{L^p}$, which gives (3.21) with $i = 2$.

We now consider S_3 , which we handle as before. Since $j < 2l, 2^j \max\{2^{-m}, 2^{-l}\} \leq 2^l$ if $l + m \geq j$. Similarly, $2^{j-m} \geq 2^j \max\{2^{-m}, 2^{-l}\}$ and $2^{j-m} \geq 2^l$ if $l + m < j$. Using (3.22) and (3.10), we see

$$S_3 \lesssim \sum_{j/2 < l < j} \left(\sum_{j-l \leq m \leq j-1} 2^{\frac{l}{q}} \|\phi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g\|_{L^q} + \sum_{0 \leq m < j-l} 2^{\frac{j-m}{q}} \|\phi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g\|_{L^q} \right)$$

Since $1/p + 2/q > 1$, using Proposition 3.7, we get (3.21) for $i = 3$.

We handle S_4 and S_5 in an elementary way without relying on Lemma 2.3. Instead, we can control S_4 and S_5 more directly. Concerning S_4 we claim that

$$S_4 \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q}-1)j} \|g\|_{L^p} \tag{3.23}$$

if $5/q > 1 + 1/p$ and $2 \leq p \leq q \leq \infty$. This clearly gives (3.21) with $i = 4$ for p, q satisfying $p \leq q, 1/p + 1/q < 1$ and $1/p + 2/q > 1$. We note that

$$|\phi_k^j \mathcal{U}_t \mathcal{P}_{j,m} g(r, x_3)| \lesssim 2^{-\frac{1}{2}j} \left| \phi_k^j \int e^{i2^j(r^2\xi_1+x_3\xi_2+r^2|\xi|)} m(\xi)\phi_0(\xi)\psi_m(\xi)\widehat{g(2^{-j}\cdot)}(\xi)d\xi \right|,$$

where

$$m(\xi) = e^{i2^j(\frac{t^2-r^2}{2}\xi_1+(t-r)r|\xi|)} |\xi|^{-\frac{1}{2}} \tilde{\phi}_0(\xi),$$

and $\tilde{\phi}_0$ is a smooth function supported in $[-\pi, \pi]^2$ such that $\tilde{\phi}_0\phi_0 = 1$. If $(r, t) \in \text{supp } \phi_k^j$, then $|t-r| \lesssim 2^{-j}$. Thus, $|\partial_\xi^\alpha m(\xi)| \lesssim 1$ for any α . We remove the dependence of t by using a bound on the coefficient of Fourier series, not the Sobolev embedding. Expanding m into Fourier series on $[-\pi, \pi]^2$ we have $m(\xi) = \sum_{\mathbf{k} \in \mathbb{Z}^2} C_{\mathbf{k}}(r, t) e^{i\mathbf{k}\cdot\xi}$ while $|C_{\mathbf{k}}(r, t)| \lesssim (1 + |\mathbf{k}|)^{-N}$. Since $1 < t < 2$, the estimate (3.23) follows after scaling $\xi \rightarrow 2^j \xi$ if we obtain

$$\|\mathcal{R}\mathcal{P}_{j,m} g\|_{L^q_{r,x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q})j} \|g\|_{L^p},$$

where

$$\mathcal{R}g(r, x_3) = \int e^{i(r^2\xi_1+x_3\xi_2+r^2|\xi|)} \widehat{g}(\xi) d\xi.$$

When $q = 2$, changing variables $r^2 \rightarrow r$ and following the argument in the proof of Lemma 3.5 we have $\|\mathcal{R}\mathcal{P}_{j,m} g\|_{L^2_{r,x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{m/2} \|g\|_{L^2}$. On the other hand, (3.18) gives $\|\mathcal{R}\mathcal{P}_{j,m} g\|_{L^\infty_{r,x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{(j-m)/2} \|g\|_{L^\infty}$. Interpolation between these two estimates gives

$$\|\mathcal{R}\mathcal{P}_{j,m} g\|_{L^q_{r,x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{\frac{m}{q} + \frac{j-m}{2}(1-\frac{2}{q})} \|g\|_{L^q}$$

for $2 \leq q \leq \infty$. Since the support $\widehat{\mathcal{P}_{j,m} g}(\xi)$ is contained in a rectangular region of dimensions $2^j \times 2^{j-\frac{m}{2}}$, by Bernstein’s inequality we have

$$\|\mathcal{R}_m^j g\|_{L^q_{r,x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{j(\frac{2}{p}-\frac{3}{q})+m(\frac{5}{2q}-\frac{1}{2}-\frac{1}{2p})} \|g\|_{L^p}$$

for $2 \leq p \leq q \leq \infty$. Since $5/q > 1 + 1/p$, this proves the claimed estimate (3.23).

Finally, we show (3.21) with $i = 5$. Changing variables $(\xi_1, \xi_2) \rightarrow (2^j \xi_1, \xi_2)$, we observe

$$\phi_k^{[j/2]-1} |\mathcal{U}_t \mathcal{P}_j^j g(r, x_3)| \lesssim 2^{\frac{j}{2}} \phi_k^{[j/2]-1} \left| \int e^{i(\frac{r-t}{2} 2^j \xi_1 + x_3 \xi_2)} \widehat{m(\xi)} \mathcal{P}_j^j g(2^j \xi_1, \xi_2) d\xi \right|,$$

where

$$\widehat{m}(\xi) = e^{i2^j r t (|\xi_1, 2^{-j} \xi_2| - \xi_1)} |(\xi_1, 2^{-j} \xi_2)|^{-\frac{1}{2}} \widetilde{\phi}_0(|(\xi_1, 2^{-j} \xi_2)|) \psi^{j-1}(2^j \xi_1, \xi_2).$$

Note that $\text{supp } \widehat{m} \subset \{\xi_1 \in [2^{-1}, 2^2], |\xi_2| \leq 2^2\}$. Since $|\partial_\xi^\alpha m(\xi)| \lesssim 1$ for any α , expanding \widehat{m} into Fourier series on $[-2\pi, 2\pi]^2$ we have $\widehat{m}(\xi) = \sum_{k \in \mathbb{Z}^2} C_k(r, t) e^{i2^{-1} k \cdot \xi}$ while $|C_k(r, t)| \lesssim (1 + |k|)^{-N}$. Hence, similarly as before, changing variables $(\xi_1, \xi_2) \rightarrow (2^{-j} \xi_1, \xi_2)$, to show (3.21) with $i = 5$ it is sufficient to obtain

$$\left\| \sup_{1 < t < 2} \mathcal{P}_j^j g\left(\frac{(r-t)^2}{2}, x_3\right) \right\|_{L_{r,x_3}^q([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{\frac{1}{2}(\frac{3}{p} - \frac{1}{q})j} \|g\|_{L^p} \tag{3.24}$$

for $1 \leq p \leq q \leq \infty$. Clearly, the left hand side is bounded by $\|\mathcal{P}_j^j g(x_1, x_3)\|_{L_{x_3}^q(L_{x_1}^\infty)}$. The Fourier transform of $\mathcal{P}_j^j g$ is supported on the rectangle $\{\xi_1 \in [2^{j-1}, 2^{j+2}], |\xi_2| \leq 2^{j+2}\}$. Thus, using Bernstein's inequality in x_1 , we get

$$\left\| \sup_{1 < t < 2} \mathcal{P}_j^j g\left(\frac{(r-t)^2}{2}, x_3\right) \right\|_{L_{r,x_3}^q([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{-\frac{j}{2} + \frac{j}{q}} \|\mathcal{P}_j^j g\|_{L^q}$$

for $1 \leq q \leq \infty$. Another use of Bernstein's inequality gives (3.24) for $1 \leq p \leq q \leq \infty$. This completes the proof of (3.20). \square

Acknowledgements Juyoung Lee was supported by the National Research Foundation of Korea (NRF) grant no. 2017H1A2A1043158 and Sanghyuk Lee was supported by NRF grant no. 2021R1A2B5B02001786.

References

1. Anderson, T.C., Hughes, K., Roos, J., Seeger, A.: L^p - L^q bounds for spherical maximal operators. *Math. Z.* **297**, 1057–1074 (2021)
2. Beltran, D., Guo, S., Hickman, J., Seeger, A.: The circular maximal operator on Heisenberg radial functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (2021). <https://doi.org/10.2422/2036-2145.202001-006>
3. Beltran, D., Hickman, J., Sogge, C.D.: Variable coefficient Wolff-type inequalities and sharp local smoothing estimates for wave equations on manifolds. *Anal. PDE* **13**, 403–433 (2020)
4. Beltran, D., Oberlin, R., Roncal, L., Seeger, A., Stovall, B.: Variation bounds for spherical averages. *Math. Ann.* (2021). <https://doi.org/10.1007/s00208-021-02218-2>
5. Bourgain, J.: Averages in the plane over convex curves and maximal operators. *J. Anal. Math.* **47**, 69–85 (1986)
6. Guth, L., Wang, H., Zhang, R.: A sharp square function estimate for the cone in \mathbb{R}^3 . *Ann. Math.* **192**, 551–581 (2020)

7. Kim, J.: Annulus maximal averages on variable hyperplanes. [arXiv:1906.03797](https://arxiv.org/abs/1906.03797)
8. Lee, S.: Endpoint estimates for the circular maximal function. *Proc. Am. Math. Soc.* **131**, 1433–1442 (2003)
9. Lee, S., Vargas, A.: On the cone multiplier in \mathbb{R}^3 . *J. Funct. Anal.* **263**, 925–940 (2012)
10. Müller, D., Seeger, A.: Singular spherical maximal operators on a class of two step nilpotent Lie groups. *Isr. J. Math.* **141**, 315–340 (2004)
11. Mockenhaupt, G., Seeger, A., Sogge, C.: Wave front sets, local smoothing and Bourgain’s circular maximal theorem. *Ann. Math.* **136**, 207–218 (1992)
12. Narayanan, E., Thangavelu, S.: An optimal theorem for the spherical maximal operator on the Heisenberg group. *Isr. J. Math.* **144**, 211–219 (2004)
13. Nevo, A., Thangavelu, S.: Pointwise ergodic theorems for radial averages on the Heisenberg group. *Adv. Math.* **127**, 307–334 (1997)
14. Roos, J., Seeger, A.: Spherical maximal functions and fractal dimensions of dilation sets. [arXiv:2004.00984](https://arxiv.org/abs/2004.00984)
15. Roos, J., Seeger, A., Srivastava, R.: Lebesgue space estimates for spherical maximal functions on Heisenberg groups. *Int. Math. Res. Not.* (2021). <https://doi.org/10.1093/imrn/rnab246>
16. Schlag, W.: $L^p \rightarrow L^q$ estimates for the circular maximal function. Ph.D. Thesis, California Institute of Technology (1996)
17. Schlag, W.: A generalization of Bourgain’s circular maximal theorem. *J. Am. Math. Soc.* **10**, 103–122 (1997)
18. Schlag, W., Sogge, C.D.: Local smoothing estimates related to the circular maximal theorem. *Math. Res. Lett.* **4**, 1–15 (1997)
19. Sogge, C.D.: Propagation of singularities and maximal functions in the plane. *Invent. Math.* **104**, 349–376 (1991)
20. Stein, E.M.: Harmonic analysis: real variable methods, orthogonality and oscillatory integrals. Princeton Univ. Press, Princeton (1993)
21. Stein, E.M.: Maximal functions: spherical means. *Proc. Natl. Acad. Sci. USA* **73**, 2174–2175 (1976)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.