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A note on the Severi problem for toric surfaces

Lionel Lang[1] · **Ilya Tyomkin[2]**

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Abstract

In this note, we make a step towards the classification of toric surfaces admitting reducible Severi varieties. We provide two families of toric surfaces admitting reducible Severi varieties. The first family is general, and is obtained by a quotient construction. The second family is exceptional, and corresponds to certain narrow polygons, which we call kites. We introduce two types of invariants that distinguish between the components of the Severi varieties, and allow us to provide lower bounds on the numbers of the components. The sharpness of the bounds is verified in some cases, and is expected to hold in general for ample enough linear systems. In the appendix, we establish a connection between the Severi problem and the topological classification of univariate polynomials.

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Contents

✉ Lionel Lang
  lionel.lang@hig.se

  Ilya Tyomkin
  tyomkin@math.bgu.ac.il

[1] Department of Electrical Engineering, Mathematics and Science, University of Gävle, 801 76 Gävle, Sweden

[2] Department of Mathematics, Ben-Gurion University of the Negev, P.O.Box 653, 84105 Be'er Sheva, Israel

1 Introduction

The study of families of curves on algebraic surfaces is a classical problem in algebraic geometry. In the 1920's, Severi considered the locus $V_{g,d}^{\mathrm{irr}}$ of integral, degree d, genus g, planar curves in an attempt to prove the irreducibility of the moduli spaces of algebraic curves. Such loci for different surfaces and linear systems are now called *Severi varieties*. In 1986, Harris proved the irreducibility of the classical Severi varieties $V_{g,d}^{\mathrm{irr}}$ in characteristic zero [11], and very recently Christ, He, and the second author found a characteristic-free proof of Harris' theorem [5]. Over the years, the irreducibility of Severi varieties was proved for other surfaces, such as Hirzebruch surfaces, Del-Pezzo surfaces, and certain toric and K3 surfaces, see, e.g., [1, 6, 24, 25]. Most of the results apply only in the case of small genus, but some are general.

On the negative side, for many surfaces of general type, Severi varieties are known to be reducible, and even non-equidimensional, see [3]. In 2013, the second author found first examples of reducible Severi varieties on toric surfaces, initially in positive characteristic [27], and then in characteristic zero [28]. A different type of examples was discovered recently by the first author in his study of the monodromy action on the set of nodes of rational curves on toric surfaces [18].

The goal of the current paper is to make a step towards a classification of toric surfaces admitting reducible Severi varieties, and towards a description of the corresponding irreducible components. We unify the examples of [18, 27, 28], and introduce certain invariants that distinguish between components of Severi varieties on toric surfaces. As a result, we are able to provide lower bounds on the numbers of the irreducible components.

1.1 The main results

Throughout the paper we work over an algebraically closed field of characteristic zero, which by Lefschetz principle can be assumed to be $\mathbb{C}$.

Let $g \geq 0$ be an integer, $\Delta \subset \mathbb{R}^2$ a lattice polygon, and $(X, \mathcal{L})$ the corresponding polarized toric surface. In the current paper, we discuss two families of polarized toric surfaces admitting reducible Severi varieties. The first family consists of certain toric quotient surfaces, namely the quotients of toric surfaces by finite non-trivial subgroups of the torus that act freely on the complement of the zero-dimensional orbits. For such surfaces, the Severi varieties corresponding to *any* polarization and genus are reducible, as long as the genus is strictly positive and small enough with respect to the polarization. Recall that the Severi varieties $V_{0,\mathcal{L}}^{\mathrm{irr}}$ are irreducible by [25, Proposition 4.1]. The following theorem provides a general lower bound on the number of irreducible components of the Severi variety $V_{g,\mathcal{L}}^{\mathrm{irr}}$ in the case of positive genus.

Theorem A *If $g \geq 1$, then the number of irreducible components of the Severi variety $V_{g,\mathcal{L}}^{\mathrm{irr}}$ is bounded from below by the number of affine sublattices $M \subseteq \mathbb{Z}^2$ for which the following two conditions hold:*

(a) $\partial \Delta \cap M = \partial \Delta \cap \mathbb{Z}^2$, *and*
(b) $|\Delta^\circ \cap M| \geq g$;

where $\partial \Delta$ and Δ° denote the boundary and the interior of Δ, respectively.

Let us now explain the geometric meaning of conditions (a) and (b) above. For an affine sublattice M satisfying condition (a), one can consider the polarized toric surface $(X', \mathcal{L}')$ associated to Δ but with respect to the lattice M. Then the natural toric map $X' \to X$ represents X as a quotient of X' by a finite subgroup of the torus that acts freely on the complement of the zero-dimensional orbits. Furthermore, the pushforward of the curve class $|\mathcal{L}'|$ is $|\mathcal{L}|$. Therefore, if $V_{g,\mathcal{L}'}^{\mathrm{irr}}$ is not empty, then its pushforward is a union of components of $V_{g,\mathcal{L}}^{\mathrm{irr}}$. The non-emptiness of $V_{g,\mathcal{L}'}^{\mathrm{irr}}$ is equivalent to condition (b), because $|\Delta^\circ \cap M|$ is nothing but the arithmetic genus of curves in $|\mathcal{L}'|$. We denote by $\delta_M(\Delta, g)$ the number of nodes of curves in $V_{g,\mathcal{L}'}^{\mathrm{irr}}$, i.e., $\delta_M(\Delta, g) := |\Delta^\circ \cap M| - g$. Then condition (b) reads as follows: $\delta_M(\Delta, g) \geq 0$.

The second family considered in this paper consists of a special type of lattice polygons that we call kites. Let k', k be non-negative integers such that $k' \geq k$ and $k' > 0$. A *kite* $\Delta_{k,k'}$ is the polygon with vertices $(0, 0)$, $(\pm 1, k)$, $(0, k + k')$, see Fig. 1. Let $\Delta = \Delta_{k,k'}$ be a kite, and $g \geq 0$ an integer. We define $\#_{k,k',g}$ to be the number of sublattices $M \subseteq \mathbb{Z}^2$ satisfying the conditions (a) and (b) of Theorem A and counted with the following multiplicities: if the index of M in $\mathbb{Z}^2$ is odd, then the multiplicity is equal to the number of integers $0 \leq \kappa \leq \min\{\delta_M(\Delta, g), g\}$ that are congruent to $\delta_M(\Delta, g)$ modulo 2, and if the index is even, then the multiplicity is one. Notice that
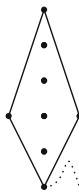

Fig. 1 The kite $\Delta_{2,3}$

even if the only sublattice satisfying conditions (a) and (b) is $M = \mathbb{Z}^2$, its multiplicity may be greater than one. For example, this is the case for the kite $\Delta_{2,3}$, and genus $g = 2$.

Theorem B *If $\Delta = \Delta_{k,k'}$ is a kite, and $(X, \mathcal{L})$ the corresponding polarized toric surface, then for any $g \geq 1$, the number of irreducible components of $V_{g,\mathcal{L}}^{\mathrm{irr}}$ is bounded from below by $\#_{k,k',g}$.*

Plainly the bound in Theorem B is usually strictly bigger than the bound in Theorem A, since one counts the same lattices but with larger multiplicities. Moreover, some of the surfaces in the second family do not belong to the first one, e.g., the surface associated to the kite $\Delta_{2,3}$ is not a quotient of another toric surface by a finite non-trivial subgroup of the torus.

We provide two proofs of the main results. One is tropical and another is topological. The topological approach is based on the study of deformations of simple Harnack curves, and one of its key ingredients is the following generalization of a result of Kenyon and Okounkov [13, Proposition 10], which we believe to be of independent interest.

Let $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}$ be a curve, and $\Upsilon := \{\gamma_1, \ldots, \gamma_g\}$ a collection of oriented simple closed curves in the smooth locus of C that are contained in the open torus orbit $X^\bullet \subset X$ and are contractible in $X^\bullet$. Since $X^\bullet$ is acyclic, each γ_j bounds a smooth disc $M_j \subset X^\bullet$ that is unique up to homotopy. Set $\Omega := \frac{dz \wedge dw}{zw}$, where (z, w) are toric coordinates on $X^\bullet$, and consider the integrals

$$\int_{M_j} \Omega, \quad j = 1, \ldots, g. \tag{1.1}$$

Since Ω is closed, these integrals are independent of the choice of M_j's. Furthermore, since Ω vanishes on C, the integrals are also independent of the choice of γ_j's within their isotopy classes. Finally, any small deformation C_t of C induces a small deformation Υ_t of Υ. Although the latter is not uniquely defined, the isotopy classes of the $\gamma_{j,t}$'s are. Therefore, the integrals (1.1) induce well-defined functions on $V_{g,\mathcal{L}}^{\mathrm{irr}}$ in a neighborhood of $[C]$.

Theorem C *Let $W \subset V_{g,\mathcal{L}}^{\mathrm{irr}}$ be the locus of curves having a given tangency profile with the boundary divisor, $[C] \in W$ any curve, and Υ a collection of loops as above. If Υ generates a g-dimensional subspace of $H_1(C, \mathbb{Z})$, then the integrals (1.1) together with the coordinates of the intersection points of C with the boundary divisor of X provide a local system of coordinates on W.*

We refer to [12, Lemma 3.1] for a similar statement in the case of curves on K3-surfaces.

1.2 Discussion

To prove the irreducibility, or more generally, to classify the irreducible components of a Severi variety, one often follows the following strategy. First, one proves that the closure of any component contains components of Severi varieties of smaller genera,

and then uses a monodromy type argument to get an upper bound on the number of components.

The question, whether the closure of any component of a Severi variety on a toric surface necessarily contains components of Severi varieties of smaller genera, is an interesting open problem. This is known to be the case for $\mathbb{P}^2$ by [11], for Hirzebruch surfaces by [25], and most generally, for toric surfaces associated to h-transverse polygons by [4]. We expect this to be true in general, but we were not able to prove this so far. Let us denote the union of the irreducible components of $V_{g,\mathcal{L}}^{\mathrm{irr}}$ containing $V_{0,\mathcal{L}}^{\mathrm{irr}}$ in their closure by $V_{g,\mathcal{L}}'$.

A monodromy argument allows one to provide an upper bound on the number of components of $V_{g,\mathcal{L}}'$. Indeed, it is not difficult to verify that in a neighborhood of an integral, rational, nodal curve $[C_0]$, the closure of $V_{g,\mathcal{L}}'$ consists of smooth branches, parameterized by subsets of g nodes of C_0. Therefore, the number of such components is bounded from above by the number of orbits of sets of g nodes under the monodromy action on the set of nodes of C_0. For planar curves, the monodromy acts as the full symmetric group by [11]. More generally, this is the case if the toric surface is smooth at (at least) one of its zero-dimensional orbits, and the polarization is ample enough; see [18] for details. Thus, the upper bound in such cases is one.

To get a lower bound on the number of components of a Severi variety, one must find invariants that can distinguish between the components. This is precisely what we do in the current paper. Unfortunately, even if $V_{g,\mathcal{L}}' = V_{g,\mathcal{L}}^{\mathrm{irr}}$, there is often a discrepancy between the upper bound prescribed by the monodromy calculations of [18], and the lower bounds coming from the admissible values of the invariants we construct. One could naively guess that the monodromy action on the nodes of C_0 should prescribe the number of components of $V_{g,\mathcal{L}}'$. However, this turns out to be wrong as the example below shows. Therefore, it is not sufficient to analyze the monodromy action on the sets of nodes of rational curves. We hope that investigating the monodromy actions on the nodes of curves of higher genera could close the gap between the lower and the upper bounds in most cases.

Example. Let Δ be the triangle with vertices $(0, 0)$, $(4, 1)$, $(0, 3)$, and $(X, \mathcal{L})$ the corresponding polarized toric surface. Consider the Severi variety $V_{2,\mathcal{L}}^{\mathrm{irr}}$. By [4, Theorem 4.6], any component of $V_{2,\mathcal{L}}^{\mathrm{irr}}$ contains $V_{0,\mathcal{L}}^{\mathrm{irr}}$ in its closure, and by [18, Lemma 4.1, Theorem 3], the number of orbits of pairs of nodes of C_0 under the monodromy action is two, which suggests that the number of components should also be two. However, one can show that $V_{2,\mathcal{L}}^{\mathrm{irr}}$ is in fact irreducible. To see this, consider the floor decomposed parametrized tropical curve $h \colon \Gamma \to \mathbb{R}^2$ as in Figure 2. By using the techniques of [5], one can show that any irreducible component $V \subseteq V_{2,\mathcal{L}}^{\mathrm{irr}}$ contains a K-point $[C] \in V(K)$ over the field of Puiseux series, whose tropicalization is (Γ, h). On the other hand, the space of tropical Γ-reductions is integral (a single torus), by [26, Proposition 3.21], and the deformation space is unobstructed by [26, Corollary 5.9]. Therefore, all realizations of (Γ, h) belong to the same irreducible component of the Severi variety, and hence $V_{2,\mathcal{L}}^{\mathrm{irr}}$ is irreducible.

Next let us discuss the sharpness of the lower bounds in our main results, and speculate on the expected number of the irreducible components of Severi varieties on toric surfaces in general. First, notice that Theorem B implies that the bound of
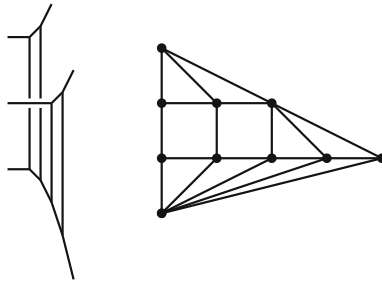
Fig. 2 The floor decomposed curve $h\colon \Gamma \to \mathbb{R}^2$ and its dual subdivision

Theorem A is not sharp for some special toric surfaces. However, we expect the bound to be equal to the actual number of irreducible components of $V_{g,\mathcal{L}}^{\text{irr}}$ at least if the line bundle $\mathcal{L}$ is ample enough. In the case of kites, we have $V'_{g,\mathcal{L}} = V_{g,\mathcal{L}}^{\text{irr}}$ by [4, Theorem 4.6]. However, there is a discrepancy between the upper and the lower bounds, and we expect the lower bound of Theorem B to be the correct one. Proposition 3.1 verifies this guess for curves of genus one.

In the current paper, we discuss two families of toric surfaces admitting reducible Severi varieties. While the first family (quotient surfaces) is rather general, and the Severi varieties on such surfaces are reducible for almost all polarizations, the second family is exceptional. In particular, any surface in the second family that does not belong to the first family admits reducible Severi varieties for only finitely many polarizations by [18]. We believe that the remaining examples, if any, also form finitely many exceptional families, and correspond to some narrow polygons.

Finally, let us mention that the Severi problem for the surfaces associated to kites is closely related to the topological classification of polynomials studied by Zvonkin in [29]. It turns out that the irreducible components of the Severi varieties in this case are in bijective correspondence with the components of the space of Laurent polynomials with given passports. We believe this relation to be fruitful for both topics, and discuss it in the Appendix.

1.3 The idea of the proofs

In order to prove the main results of this paper, we introduce two types of invariants that distinguish between the irreducible components of the Severi variety $V_{g,\mathcal{L}}^{\text{irr}}$. We then describe the admissible values of these invariants, and for each admissible value, construct an irreducible component realizing this value. We provide two proofs of the main theorems, one is topological, and another is tropical. We believe that both approaches will be useful in the ultimate classification of the irreducible components of Severi varieties on toric surfaces.

Our first invariant is a sublattice of the lattice of monomials of the toric surface, that can be described either in topological or in tropical terms. The second invariant makes sense only for polarized toric surfaces associated to kites. The set of nodes of an integral curve on such a surface admits a natural (unordered) partition, that varies

continuously with the curve. Therefore, the absolute value of the difference between the cardinalities of the two blocks of nodes is an invariant of an irreducible component of $V_{g,\mathcal{L}}^{\text{irr}}$, that we call the *signature* of the component.

2 Preliminaries, notation, and conventions

2.1 Toric geometry

We assume that the reader is familiar with the basics of toric geometry, and refer to [8] for details. In particular, the construction of the polarized toric variety associated to a lattice polygon, the notion of the dual fan, the functoriality of toric varieties, and the structure of their orbit decomposition are assumed to be known. For a convex polygon Δ, we denote by Δ° and $\partial\Delta$ the interior and the boundary of Δ, respectively.

Our default pair of dual lattices is $(\mathbb{Z}^2, \mathbb{Z}^2)$ with the standard pairing denoted by $\langle \cdot, \cdot \rangle$. Sometimes we consider different integral structures on $\mathbb{R}^2$ defined by various sublattices. In such cases, we follow the standard convention of toric geometry, and denote the lattice of monomials by M and the lattice of cocharacters by N. For a toric surface X, we denote by $X^\bullet$ the open orbit, and by $\partial X := X \setminus X^\bullet$ the boundary divisor. Then the canonical divisor of X is given by $K_X = -\partial X$.

2.2 Severi varieties

For a projective polarized toric surface $(X, \mathcal{L})$, we denote by $V_{g,\mathcal{L}}^{\text{irr}} \subset |\mathcal{L}|$ the Severi variety, i.e., the locus of integral nodal curves of geometric genus g that contain no zero-dimensional orbits of X.

Severi varieties on toric surfaces are known to be smooth and equidimensional of codimension δ in $|\mathcal{L}|$, where δ is the number of nodes, or equivalently, the difference between the arithmetic genus of $\mathcal{L}$ and g; see, e.g., [15, Theorem 1]. If $\mathcal{L}$ is associated to a lattice polygon Δ in $\mathbb{R}^2$, then its arithmetic genus is given by the number of inner lattice points in Δ. Therefore, the dimension of any component of $V_{g,\mathcal{L}}^{\text{irr}}$ is given by

$$|\partial\Delta \cap \mathbb{Z}^2| + g - 1. \tag{2.1}$$

Furthermore, by [15, Proposition 2], $V_{g,\mathcal{L}}^{\text{irr}}$ is open and dense in the locus of all integral curves of geometric genus g in $|\mathcal{L}|$ that contain no singular points of X.

We say that a curve in $C \subset X$ is *torically strictly transverse* if it intersects the boundary divisor ∂X transversely, i.e., $C \cap X$ is reduced and contains no zero-dimensional orbits of X. It is well known that for a general $[C] \in V_{g,\mathcal{L}}^{\text{irr}}$, the curve C is torically strictly transverse; see e.g., [25, Theorem 2.8].

Recall that in genus zero, Severi varieties on toric surfaces admit natural parameterizations by irreducible rational varieties. Therefore, $V_{0,\mathcal{L}}^{\text{irr}}$ are irreducible for any polarized toric surface, see [25, Proposition 4.1].

2.3 Conventions

Throughout the paper we work over the field of complex numbers. The tropical arguments are applied after a base change to the field of Puiseux series.

For convenience, *we always assume that the lattice polygon, we work with, contains the origin as its vertex.* The latter can plainly be achieved by a translation, and hence does not restrict the generality. The advantage of this assumption is that all sublattices that appear in the statements and in the proofs become *linear* rather than affine.

3 The case of genus one

In this section we show that the bound of Theorem B is sharp in the case of genus one.

Proposition 3.1 *If $\Delta = \Delta_{k,k'}$ is a kite, and $(X, \mathcal{L})$ the corresponding polarized toric surface, then the number of irreducible components of $V_{1,\mathcal{L}}^{\mathrm{irr}}$ is equal to $\#_{k,k',1}$.*

Remark 3.2 Notice that a sublattice $M \subseteq \mathbb{Z}^2$ satisfying the condition (a) of Theorem A for $\Delta = \Delta_{k,k'}$ is uniquely determined by its restriction to the y-axis, and containsre-spectively. the point $(0, 2k)$. Furthermore, associating to a sublattice M its index r, gives rise to a one-to-one correspondence between such sublattices and the positive common divisors of $k' + k$ and $2k$.

Proof By Remark 3.2, the number $\#_{k,k',1}$ counts the positive common divisors r of $k' + k$ and $2k$ for which $\delta_M(\Delta, 1) \geq 0$, with multiplicities. Since $\delta_M(\Delta, 1) = \left(\frac{k'+k}{r} - 1\right) - 1$, the latter inequality is equivalent to $r < k' + k$. We claim that all divisors r are counted with multiplicity one. Indeed, if r is even, then its multiplicity is 1 according to the definition, and if r is odd, then the multiplicity is equal to the number of integers $0 \leq \kappa \leq \min\{\delta_M(\Delta, 1), 1\} \leq 1$ that are congruent to $\delta_M(\Delta, 1)$ modulo two. Since such κ is clearly unique, this proves the claim. Set now $d := \gcd\{k' + k, 2k\} = \gcd\{k' + k, k' - k\}$, and let $\sigma(d)$ be the number of positive divisors of d. Unless $k = k'$ or $k = 0$, the inequality $r < k' + k$ is automatically satisfied, and the number $\#_{k,k',1}$ is therefore equal to $\sigma(d)$. Otherwise, the divisor $r = d$ has to be excluded and $\#_{k,k',1}$ is then equal to $\sigma(d) - 1$.

Let $[C] \in V_{1,\mathcal{L}}^{\mathrm{irr}}$ be a general point, and E the normalization of the curve C. Denote by O, P, Q, R the preimages in E of the toric divisors corresponding to the sides of Δ with outer normals $(-k, -1), (k, -1), (k', 1), (-k', 1)$, respectively. Then $k'(Q - R) + k(P - O)$ and $Q + R - P - O$ are the divisors of the pull-backs of the coordinate functions on the torus, and hence, in the group law of the elliptic curve (E, O) the following holds:

$$P = R + Q \quad \text{and} \quad (k' + k)Q = (k' - k)R. \tag{3.1}$$

Conversely, given an elliptic curve (E, O) and three points $P, Q, R \in E$ satisfying (3.1) such that O, P, Q, R are distinct, pick rational functions x, y such that $\mathrm{div}(x) = k'(Q - R) + k(P - O)$ and $\mathrm{div}(y) = Q + R - P - O$. Then the rational map $(x, y) \colon E \setminus \{O, P, Q, R\} \to (\mathbb{C}^*)^2$ extends to a morphism $\phi \colon E \to X$ and

$[\phi_* E] \in |\mathcal{L}|$. Furthermore, since the pull-back under ϕ of the boundary divisor ∂X is reduced, E is birational onto its image, and hence $[\phi(E)] \in V_{1,\mathcal{L}}^{\mathrm{irr}}$. Notice that the functions x and y as above are unique up-to an action of $\mathbb{C}^*$. Therefore, the number of irreducible components of $V_{1,\mathcal{L}}^{\mathrm{irr}}$ is equal to the number of irreducible components of the locus

$$V := \{[E; O, Q, R] \in \mathcal{M}_{1,3} \mid (k' + k)Q = (k' - k)R\}$$

in the moduli space of genus-one curves with three marked points.

To describe the components of V, set $m := \frac{k'+k}{d} > 0, n := \frac{k'-k}{d} \geq 0$, and let $a, b \in \mathbb{Z}$ be integers such that $am + bn = 1$. For a divisor $d' | d$, let $\mathcal{M}_{1,2}[d']$ be the moduli space of elliptic curves with marked point and level-d' structure, i.e., $\mathcal{M}_{1,2}[d']$ parametrizes elliptic curves with marked points $[E; O, S, T]$ where O is the origin of E, S a marked point, and T a torsion point of order precisely d'. The projection $\mathcal{M}_{1,2}[d'] \to \mathcal{M}_{1,1}[d']$ forgetting the point S has irreducible fibers, and it is well known that $\mathcal{M}_{1,1}[d']$ is irreducible, since it is the quotient of the upper half-plane by the modular group $\Gamma_1(d')$, see, e.g., [23, pp.439-440]. Hence $\mathcal{M}_{1,2}[d']$ is also irreducible.

Pick a general point $[E; O, S, T] \in \mathcal{M}_{1,2}[d']$. Then, the point O is distinct from $mS - bT$ since $m > 0$ and S is not a torsion point. For the same reason, the point O is distinct from $nS + aT$ unless $n = 0$ and $T = O$, that is $k' = k$ and $d' = 1$. Eventually, the points $mS - bT$ and $nS + aT$ are distinct unless $(m - n)S$ is the torsion point $(a + b)T$. The latter occurs only if $m = n$. However, m and n are relatively prime. Thus, $mS - bT = nS + aT$ if and only if $m = n = 1$ (implying that $a + b = 1$) and $T = O$, that is $k = 0$ and $d' = 1$. Besides these particular cases, the three points $O, nS + aT, mS - bT$ are distinct. Since the points are distinct, we can consider the rational map $\mathcal{M}_{1,2}[d'] \to \mathcal{M}_{1,3}$ given by $[E; O, S, T] \mapsto [E; O, nS+aT, mS-bT]$. Its image $V_{d'}$ is irreducible, has the same dimension as V and is contained in V since

$$(k+k')(nS+aT) = dmnS+dmaT = dmnS = dmnS+dnbT = (k - k')(mS - bT).$$

Hence $V_{d'}$ is an irreducible component of V.

Next, let us show that $V = \bigcup_{d'|d} V_{d'}$, which implies that any irreducible component of V is of the form $V_{d'}$ for some d'. Pick a general point $[E; O, Q, R] \in V$. For dimension reasons, Q, R are not torsion points. Let us show that $[E; O, Q, R] \in \bigcup_{d'|d} V_{d'}$. Set $S' := aR + bQ$. Then

$$d(Q - nS') = dQ - danR - dbnQ = dQ - damQ - dbnQ = dQ - dQ = O,$$

since, by definition, $dnR = (k' - k)R = (k' + k)Q = dmQ$ and $am + bn = 1$. Thus, $Q - nS'$ is a d-torsion point. Similarly, $R - mS'$ is a d-torsion point. Set $T_1 := Q - nS'$ and $T_2 := R - mS'$. Since $am + bn = 1$, the vectors (n, m) and $(a, -b)$ form a free basis of $\mathbb{Z}^2$, and hence there exists a unique pair of d-torsion points $T, T' \in E$ such that $(T_1, T_2) = (nT', mT') + (aT, -bT)$. If we set $S := S' + T'$, then $Q = nS + aT$ and $R = mS - bT$ as needed.

Finally, notice that if $d'|d$, $[E; O, S, T] \in \mathcal{M}_{1,2}[d']$, and the tuple $[E; O, Q, R] = [E; O, nS + aT, mS - bT]$ is its image in $V_{d'}$, then d' is precisely the order of $T = mQ - nR \in E$, and hence the components corresponding to different d'-s are distinct. We have proved that the number of irreducible components of V, and hence also of $V_{1,\mathcal{L}}^{\text{irr}}$, is equal to $\sigma(d)$ unless $k' = k$ or $k = 0$, in which case it is equal to $\sigma(d) - 1$. Thus, in both cases, it coincides with the number $\#_{k,k',1}$ as asserted. $\qquad\square$

4 The invariants

In this section, we define the invariants that allow us to obtain the lower bounds in Theorems A and B. The first invariant is a certain sublattice of $\mathbb{Z}^2$, that can be defined either topologically or tropically for any component of the Severi variety $V_{g,\mathcal{L}}^{\text{irr}}$ on a toric surface. The second invariant is an integer, called *the signature*, and is specific to the Severi varieties on the toric surfaces associated to kites.

4.1 The sublattices associated to the irreducible components

Throughout this section, $\Delta \subseteq \mathbb{R}^2$ is a convex lattice polygon one of whose vertices is the origin, Σ its dual fan, and $(X, \mathcal{L})$ the associated polarized toric surface. Denote by x^m the monomial functions of X for $m \in \mathbb{Z}^2$, and by $\{n_i\} \subset \mathbb{Z}^2$ the set of primitive vectors along the rays of Σ.

Let $N \subseteq \mathbb{Z}^2$ be a sublattice containing all n_i's. Since Σ is a complete fan, the n_i's generate $\mathbb{R}^2$ as a vector space, and hence $N \subseteq \mathbb{Z}^2$ has finite index, which we denote by r. Moreover, since N contains primitive vectors, and any primitive vector in $\mathbb{Z}^2$ can be completed to a free basis of $\mathbb{Z}^2$, it follows that the quotient $\mathbb{Z}^2/N$ is a cyclic group of order r. Set

$$M := \{m \in \mathbb{Z}^2 \mid \forall n \in N, \ \langle n, m \rangle \in r\mathbb{Z}\}, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^2$. Then $M \subset \mathbb{Z}^2$ is the sublattice of index r obtained from N by a rotation by $\frac{\pi}{2}$. Furthermore, $M = r \cdot \text{Hom}(N, \mathbb{Z})$.

Lemma 4.1 *In the above notation the following holds:* $\partial\Delta \cap M = \partial\Delta \cap \mathbb{Z}^2$.

Proof Since the primitive integral vector along an edge of Δ is obtained from the primitive integral vector along the corresponding ray of Σ by a rotation by $\frac{\pi}{2}$, it follows that M contains the primitive integral vectors along all edges of Δ, which implies the assertion. $\qquad\square$

4.1.1 The topological point of view

For $[C] \in V_{g,\mathcal{L}}^{\text{irr}}$, set $C^\bullet := C \cap X^\bullet$, and let $\widetilde{C}^\bullet$ be the normalization of the curve $C^\bullet$. We denote by N_C the image of the map $H_1(\widetilde{C}^\bullet, \mathbb{Z}) \to H_1(X^\bullet, \mathbb{Z}) = \mathbb{Z}^2$ induced by the natural map $\widetilde{C}^\bullet \to X^\bullet$. Notice that since the lattice of monomials of X is $\mathbb{Z}^2$, the

open orbit $X^\bullet$ is canonically trivialized, and therefore so is the first homology group $H_1(X^\bullet, \mathbb{Z})$.

Lemma 4.2 *Let $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ be an irreducible component. Then, for a general $[C] \in V$, the sublattice $N_C \subseteq \mathbb{Z}^2$ is independent of $[C]$, and contains all the n_i's.*

Proof Since V parameterizes curves of the same genus, the tautological family $C_V \to V$ is equinormalizable, i.e., the normalization $\widetilde{C}_V \to V$ is a family of smooth curves of genus g, and hence topologically, it is a locally trivial fibration. Furthermore, since generically the family $C_V \to V$ is torically strictly transverse, there exists an open dense subset $B \subseteq V$, over which the fibration $\widetilde{C}_B^\bullet \to B$ is locally trivial. Finally, since B is connected, the image of the first homology group of the fibers of $\widetilde{C}_B^\bullet \to B$ in $H_1(X^\bullet, \mathbb{Z})$ is constant. This proves the first part of the statement.

The second part follows from the fact that the fibers of $C_B \to B$ intersect all components of the boundary divisor, and the intersection is transverse. If p is an intersection point of C with the component corresponding to a ray ρ of the dual fan, then the image of a small loop around p corresponds to the class of the primitive integral vector along ρ in $\mathbb{Z}^2 = H_1(X^\bullet, \mathbb{Z})$, see for instance [17, Lemma 1.12]. □

Set $N(V) := N_C$ for a general $[C] \in V$. By the lemma, it is an invariant of the irreducible component V. Let $M(V)$ be the lattice dual to $N(V)$ in the sense of (4.1), i.e., the lattice obtained from $N(V)$ by a rotation by $\frac{\pi}{2}$. Then $\partial\Delta \cap M(V) = \partial\Delta \cap \mathbb{Z}^2$ by Lemma 4.1. We say that $(N(V), M(V))$ is the pair of sublattices *topologically associated* to the irreducible component V.

4.1.2 The tropical point of view

Let K be the field of Puiseux series over $\mathbb{C}$, and $\nu \colon K \to \mathbb{R} \cup \{\infty\}$ its valuation. Let $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}(K)$ be a K-point. Denote by $\widetilde{C}$ its normalization, and let $h \colon \Gamma \to \mathbb{R}^2$ be the canonical tropicalization of the natural map $f \colon \widetilde{C} \to X$; cf. [26] and [5, § 4.2]. Let us briefly recall how the canonical tropicalization works.

The underlying graph of Γ is the dual graph of the stable reduction of the curve $\widetilde{C}$ with marked points $f^*(\partial X)$: its vertices correspond to the components of the reduction, edges – to the nodes, and legs – to the marked points. The incidence relation among the vertices, edges, and legs is the natural one coming from the incidences between the components of the reduction, the nodes, and the specializations of the marked points. The length of the edge corresponding to a node p of the reduction is defined to be $\nu(\lambda)$, where $\lambda \in K$ is such that, étale locally at p, the stable model of $\widetilde{C}$ is given by $xy = \lambda$. Although λ depends on the neighborhood, its valuation does not, and hence the length is well defined. The lengths of the legs are infinite.

The parametrization h is the piecewise integral affine map uniquely determined by its slopes along the legs and its values at the vertices, which are defined as follows. If l is the leg corresponding to a marked point p, then the slope $\frac{\partial h}{\partial \vec{l}}$ of h along l is given by the order of pole of the monomial functions at p, i.e., $\langle \frac{\partial h}{\partial \vec{l}}, m \rangle = -\mathrm{ord}_p(f^*(x^m))$ for any monomial function x^m on X. The notation $\vec{l}$ here indicates that the leg l is oriented away from the vertex adjacent to it. For a vertex v, corresponding to an irreducible

component $\widetilde{C}_v$ of the reduction, $h(v) \in \mathbb{Q}^2$ is defined to be the vector for which $\langle h(v), m \rangle = \nu(\lambda_m)$ for any $m \in \mathbb{Z}^2$, where $\lambda_m \in K$ is a scalar such that $\lambda_m f^*(x^m)$ restricts to a non-zero rational function on $\widetilde{C}_v$. Once again, although λ_m is not uniquely defined, its valuation is, and hence the definition makes sense.

For a parametrized tropical curve $h \colon \Gamma \to \mathbb{R}^2$, if $\vec{e}$ is an oriented edge or leg of Γ, then the slope $\frac{\partial h}{\partial \vec{e}}$ is an integral vector, whose integral length is the stretching factor of the piecewise integral affine map h. Recall that the *degree* of a parametrized tropical curve $h \colon \Gamma \to \mathbb{R}^2$ is the collection ∇ of slopes of h along the non-contracted legs of Γ. We say that ∇ is *dual* to a polygon Δ if for each side $\partial_i \Delta \subset \Delta$, the sum of vectors in ∇, which are outer normal to $\partial_i \Delta$, is equal to the integral length of $\partial_i \Delta$.

We denote by $N(\Gamma) \subseteq \mathbb{Z}^2$ the sublattice generated by all the slopes $\frac{\partial h}{\partial \vec{e}}$. If the degree of (Γ, h) is dual to Δ, and $\Delta = \cup \Delta_i$ is the Legendre dual subdivision of Δ associated to Γ, then the affine sublattice $M(\Gamma) \subseteq \mathbb{Z}^2$ generated by all vertices of the subdivision is dual to $N(\Gamma)$ in the sense of (4.1), i.e., it is obtained from $N(\Gamma)$ by a rotation by $\frac{\pi}{2}$. If $h \colon \Gamma \to \mathbb{R}^2$ is the tropicalization of $f \colon \widetilde{C} \to X$ corresponding to a point $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}(K)$, then we set $N_C^{\mathrm{tr}} := N(\Gamma)$ and $M_C^{\mathrm{tr}} := M(\Gamma)$.

For an irreducible component $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$, let $B \subseteq V$ be the open dense locus of torically strictly transverse curves. Set $N^{\mathrm{tr}}(V) := \sum_{[C] \in B(K)} N_C^{\mathrm{tr}} \subseteq \mathbb{Z}^2$ and $M^{\mathrm{tr}}(V) := \sum_{[C] \in B(K)} M_C^{\mathrm{tr}} \subseteq \mathbb{Z}^2$. We call this pair of sublattices the sublattices *tropically associated* to the irreducible component V. Notice that since for any $[C] \in B$, the curve C is torically strictly transverse, it follows that the slopes of all legs of its tropicalization $h \colon \Gamma \to \mathbb{R}^2$ are primitive integral vectors belonging to the rays of the dual fan Σ. Therefore, the sublattice N_C^{tr} contains all the n_i's, and $\partial \Delta \cap M^{\mathrm{tr}}(V) = \partial \Delta \cap \mathbb{Z}^2$.

Remark 4.3 One can show that the topological and the tropical lattice invariants coincide, i.e., $(N(V), M(V)) = (N^{\mathrm{tr}}(V), M^{\mathrm{tr}}(V))$. Since we will not use this, we only indicate the main idea and leave the details to an interested reader. The general idea is to compare the lattices using the curves in V defined over the subfield $F \subset K$ of convergent Puiseux series, and to prove that (i) all possible tropicalizations of curves in V are realized by curves defined over F, and (ii) the irreducible component V contains a Mumford curve C defined over F, whose tropicalization is regular. Property (i) allows one to show that $N^{\mathrm{tr}}(V) \subseteq N(V)$, and property (ii) – to deduce the equality.

4.2 The nodal partition and the signature

Let $\Delta = \Delta_{k,k'}$ be a kite, and $0 \le g \le |\Delta^\circ \cap \mathbb{Z}^2|$ an integer. Set $\delta := \delta_{\mathbb{Z}^2}(\Delta, g)$, as defined in Sect. 1.1. Then δ is the number of nodes of the curve C for any $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}$. We claim that the set of δ nodes of C admits a natural unordered partition into two blocks, which we call *the nodal partition* of C. Indeed, let $f \in H^0(X, \mathcal{L})$ be a Laurent polynomial defining C. Since C is torically strictly transverse, all nodes of C belong to $X^\bullet$, and hence are given by the system of equations $f = z\partial_z(f) = w\partial_w(f) = 0$. Furthermore, since Δ is a kite, the polynomial f is of the form $f(z, w) = az^{-1}w^k + p(w) + bzw^k$, where $p(w)$ is a polynomial of degree $k + k'$, and $a, b \in \mathbb{C}^*$. Thus, any node $(z, w) \in C$ satisfies: $-az^{-1} + bz = 0$, and hence $z^2 = \frac{a}{b}$. We conclude, that the set of nodes of C admits a natural partition

into two blocks, and a pair of nodes belongs to the same block if and only if their z-coordinates coincide.

Plainly, the nodal partition varies continuously in the tautological family $C_{V^{\mathrm{irr}}_{g,\mathcal{L}}} \to V^{\mathrm{irr}}_{g,\mathcal{L}}$. Furthermore, the number of nodes in the block corresponding to a given value of z is equal to the number of critical points of $\frac{p(w)}{w^k}$ with critical value $\frac{p(w)}{w^k} = -2bz$. Therefore, the number of nodes in each block is bounded from above by $\lfloor \frac{k+k'}{2} \rfloor = \lfloor \frac{\delta+g+1}{2} \rfloor = \lceil \frac{\delta+g}{2} \rceil$.

Notice that the monodromy acts naturally on the two blocks of nodes of C, and the blocks get interchanged, when $\frac{a}{b}$ travels along a loop around $0 \in \mathbb{C}$. Therefore, the partition is unordered, and the induced integer partition $\delta = \delta_1 + \delta_2$, where $\delta_1 \geq \delta_2$, is an invariant of the irreducible component $V \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}}$ containing $[C]$. The *signature* of C and of V, is defined to be $\kappa(V) := \kappa(C) := \delta_1 - \delta_2$. It is clear from the definition, that for any irreducible component $V \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}}$, its signature satisfies the following properties: $\kappa(V)$ is congruent to δ modulo 2, and $\kappa(V) + \delta = 2\delta_1 \leq 2\lceil \frac{\delta+g}{2} \rceil$. Thus,

$$0 \leq \kappa(V) \leq \min\left\{\delta, 2\left\lceil \frac{\delta + g}{2} \right\rceil - \delta\right\}.$$

In particular, if $k + k'$ is even, then $\delta + g$ is odd, and hence $0 \leq \kappa(V) \leq \min\{\delta, g+1\}$. And if $k + k'$ is odd, then $\delta + g$ is even, and hence $0 \leq \kappa(V) \leq \min\{\delta, g\}$.

5 Tropical proofs of the main results

Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon containing the origin as one of its vertices, and Σ its dual fan. Let $N \subseteq \mathbb{Z}^2$ be a sublattice containing the primitive vectors along all the rays of Σ. Recall that N is a sublattice of finite index r, and the quotient $\mathbb{Z}^2/N$ is isomorphic to the cyclic group $\mathbb{Z}/r\mathbb{Z}$. Let $M \subseteq \mathbb{Z}^2$ be the sublattice obtained from N by a rotation by $\frac{\pi}{2}$. Then $M = r \cdot \mathrm{Hom}(N, \mathbb{Z})$ and $N = r \cdot \mathrm{Hom}(M, \mathbb{Z})$. Since the origin is a vertex of Δ, the sublattice M contains all vertices of Δ. Denote by μ_r the kernel of the natural surjective homomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \to \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^2$.

Lemma 5.1 *Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be the polarized toric varieties associated to the polygon Δ with respect to the lattices $\mathbb{Z}^2$ and M. Then,*

1. *There is a natural μ_r-equivariant projection $\pi\colon X' \to X$, and $X = X'/\mu_r$;*
2. *The action of μ_r on the one-dimensional orbits of X' is free;*
3. *$\pi^*\mathcal{L} \cong (\mathcal{L}')^{\otimes r}$ and $\pi_*|\mathcal{L}'| = |\mathcal{L}|$.*

The assertions of the lemma are well-known in toric geometry. We only include a sketch of its proof for the convenience of the reader.

Proof Since $N = r \cdot \mathrm{Hom}(M, \mathbb{Z})$, the variety X' is canonically isomorphic to the toric variety associated to the fan Σ with respect to the integral structure given by N. By the functoriality of toric varieties, we obtain a natural morphism $\pi\colon X' \to X$ compatible

with the actions of $N \otimes \mathbb{C}^*$ and $(\mathbb{C}^*)^2$. Furthermore, $\pi : X' \to X$ is a Galois cover with Galois group μ_r by, e.g., [8, § 2.6.2].

To verify (2), let ρ be a ray in Σ, and $O \subset X$, $O' \subset X'$ the corresponding one-dimensional orbits. Then $O = \mathrm{Spec}\left(\mathbb{C}[\mathbb{Z}^2 \cap \rho^\perp]\right)$ and $O' = \mathrm{Spec}\left(\mathbb{C}[\frac{1}{r}M \cap \rho^\perp]\right)$. Since N contains the primitive integral vector of ρ, it follows that $\mathbb{Z}^2 \cap \rho^\perp$ has index r in $\frac{1}{r}M \cap \rho^\perp$, and hence the degree of $\pi_{|_{O'}} : O' \to O$ is r, i.e., μ_r acts freely on O'.

If $[C] \in |\mathcal{L}|$, then the intersection of C with the divisor $D_\rho \subset X$ corresponding to a ray ρ is given by the integral length of the dual side of Δ. Therefore, $\pi^*(C).D'_\rho$ is r bigger, and hence coincides with the integral length of Δ with respect to $\frac{1}{r}M$, or equivalently, the integral length of $r\Delta$ with respect to M. We conclude that $\pi^*\mathcal{L}$ and $(\mathcal{L}')^{\otimes r}$ belong to the same class in the Neron-Severi group $\mathrm{NS}(X')$. But $\mathrm{NS}(X') \cong \mathrm{Pic}(X')$, and therefore $\pi^*\mathcal{L} \cong (\mathcal{L}')^{\otimes r}$. The second part of assertion (3) is now clear.   □

Lemma 5.2 *Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be as in* Lemma 5.1. *Denote by $V^{\mathrm{irr}}_{g,\mathcal{L}'}$ the Severi variety of integral genus g curves in the linear system $|\mathcal{L}'|$ on X', and let $V' \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}'}$ be an irreducible component. Denote by V the locus of reduced curves in $\pi_*(V')$. Then,*

1. $\dim(V^{\mathrm{irr}}_{g,\mathcal{L}'}) = \dim(V^{\mathrm{irr}}_{g,\mathcal{L}})$;
2. $V \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}}$ *is an irreducible component;*
3. *If $\Delta = \Delta_{k,k'}$ is a kite, then $\kappa(V') = \kappa(V)$ if r is odd, and $\kappa(V) = g + 1$ if r is even;*
4. $(N^{\mathrm{tr}}(V), M^{\mathrm{tr}}(V)) = (N^{\mathrm{tr}}(V'), M^{\mathrm{tr}}(V'))$.

Proof By [15, Theorem 1], the Severi varieties $V^{\mathrm{irr}}_{g,\mathcal{L}}$ and $V^{\mathrm{irr}}_{g,\mathcal{L}'}$ are equidimensional, and

$$\dim(V^{\mathrm{irr}}_{g,\mathcal{L}}) = |\partial\Delta \cap \mathbb{Z}^2| + g - 1 = |\partial\Delta \cap M| + g - 1 = \dim(V^{\mathrm{irr}}_{g,\mathcal{L}'}),$$

since $|\partial\Delta \cap \mathbb{Z}^2| + g - 1 = |\mathcal{L}| - \delta_{\mathbb{Z}^2}(\Delta, g)$ and $|\partial\Delta \cap M| + g - 1 = |\mathcal{L}'| - \delta_M(\Delta, g)$. This proves (1).

To prove (2), notice that since $\pi_* : |\mathcal{L}'| \to |\mathcal{L}|$ is a finite morphism, we have the equality of dimensions $\dim(\pi_*(V')) = \dim(V^{\mathrm{irr}}_{g,\mathcal{L}})$. Thus, V is dense in an irreducible component of $V^{\mathrm{irr}}_{g,\mathcal{L}}$. On the other hand, if $[C]$ belongs to this component, then $\pi^{-1}(C)$ is a reduced curve in the linear system $|(\mathcal{L}')^{\otimes r}|$. Furthermore, $\pi^{-1}(C)$ is a specialization of a μ_r-orbit of an element of $|\mathcal{L}'|$, and hence it is a μ_r-orbit of some $[C'] \in |\mathcal{L}'|$. Plainly, $\pi : C' \to C$ is a birational map, and hence $[C'] \in V'$. Assertion (2) now follows.

The proof of (3) is a rather long but straight-forward computation. We start with the case when M contains the point $(0, k)$, i.e., Δ is a kite also with respect to the sublattice M. In this case, the action of μ_r on a point $(z, w) \in X^\bullet$ is given by $\xi(z, w) = (\xi z, w)$, and the projection π is given by $\pi(z, w) = (z^r, w)$.

Let $[C'] \in V'$ be general, and set $C := \pi(C')$. Then $[C] \in V$, and hence the curve C is nodal, torically strictly transverse, and has geometric genus g.

Since $\pi^{-1}(C) = \cup_{\xi \in \mu_r} \xi(C')$, each node of C has r preimages, which are either the μ_r-orbit of a node of C', or the μ_r-orbit of a point of intersection of C' with $\xi(C')$ for some $1 \neq \xi \in \mu_r$. Recall that two nodes belong to the same block of the nodal partition if and only if they have the same z-coordinates, and they belong to different blocks if and only if their z-coordinates differ by a sign. Therefore, if r is odd, then a pair of nodes of C' belongs to the same block of the nodal partition if and only if their images do so; and if r is even, then all nodes of C' are mapped to the same block of the nodal partition of C.

Let $p \in C$ be a node, whose preimage on C' is a pair of points $\{p'_1, p'_2\}$. Then there exists a unique $\xi \in \mu_r$ such that $p'_1 \in C' \cap \xi(C')$, and hence $p'_2 \in C' \cap \xi^{-1}(C')$. Let $cz^{-1}w^{k/r} + q(w) + ezw^{k/r}$ be a section of $\mathcal{L}'$ defining C'. Then the intersection $C' \cap \xi(C')$ is given by the system of equations:

$$cz^{-1}w^{k/r} + q(w) + ezw^{k/r} = c\xi^{-1}z^{-1}w^{k/r} + q(w) + e\xi zw^{k/r} = 0,$$

or equivalently $cz^{-1}w^{k/r} + q(w) + ezw^{k/r} = cz^{-1} - e\xi z = 0$. Since $C = \pi(C')$ is nodal, the intersection $C' \cap \xi(C')$ is transverse. Thus, for each solution of $cz^{-1} = e\xi z$, we have $\frac{k+k'}{r}$ solutions of $cz^{-1}w^{k/r} + q(w) + ezw^{k/r} = 0$.

Assume that r is odd. To prove that $\kappa(V') = \kappa(V)$, it remains to show that each block of the nodal partition of C contains the same amount of nodes, whose preimages are pairs of points in C'. Since r is odd, $\xi \neq \xi^{-1}$ for any $1 \neq \xi \in \mu_r$. Therefore, it suffices to show that for any $\xi \neq 1$, the points of intersection $C' \cap \xi(C')$ contribute equally to the two blocks of the nodal partition of C, but the latter is clear because the r-th powers of the two solutions of $cz^{-1} = e\xi z$ differ by a sign, and therefore each one of them contributes $\frac{k+k'}{r}$ nodes to the corresponding block.

Assume now that r is even. In this case, all nodes of C' are mapped to the block of nodes of C for which $z = \left(\frac{c}{e}\right)^{r/2}$. Similarly, for any $\xi \neq 1$, all intersection points of $C' \cap \xi(C')$ are mapped to the block in which $z = \left(\frac{c}{e}\xi\right)^{r/2}$. Thus,

$$\kappa(V) = \left| \frac{k+k'}{r} \sum_{1 \neq \xi \in \mu_r} \xi^{r/2} + \delta_M(\Delta, g) \right| = \left| \frac{k+k'}{r} \sum_{\xi \in \mu_r} \xi^{r/2} - 1 - g \right| = g + 1,$$

since $|C' \cap \xi(C')| = 2\frac{k+k'}{r}$ for all $\xi \neq 1$.

It remains to prove assertion (3) in the case when M does not contain the point $(0, k)$. By Remark 3.2, r is a common divisor of $k + k'$ and $k - k'$. In particular, r is a divisor of $2k$ but it does not divide k since otherwise $(0, k) \in M$. Therefore, r is necessarily even. For the computation, it is more convenient to apply an affine automorphism of $\mathbb{Z}^2$ so that Δ becomes the polygon with vertices $(0, 0), (-1, 0), (1, 2k), (0, k + k')$. Then μ_r again acts by $\xi(z, w) = (\xi z, w)$ and $\pi(z, w) = (z^r, w)$.

Let $az^{-1} + p(w) + bzw^{2k}$ be a section of $\mathcal{L}$ defining the curve C. Then the nodes of C satisfy $-az^{-2} + bw^{2k} = 0$, and hence two nodes of C belong to the same block if and only if they have the same value of zw^k. Similarly, if $cz^{-1} + q(w) + ezw^{2k/r}$ is a section of $\mathcal{L}'$ defining the curve $C' \subset X'$, then the nodes of C' satisfy $z^2 w^{2k/r} = \frac{c}{e}$. Thus, the image (z^r, w) of every node of C' satisfies $z^r w^k = \left(\frac{c}{e}\right)^{r/2}$, and hence belongs to the

same block of the nodal partition of C. Other nodes of C correspond to the points of intersection of C' with $\xi(C')$ for various $1 \neq \xi \in \mu_r$, and the computation, identical to the one we did above, shows that $\kappa(V) = g + 1$.

Finally, let us prove (4). Let K be the field of Puiseux series, and $[C'] \in V^{\mathrm{irr}}_{g,\mathcal{L}'}(K)$ be such that $\pi_*[C'] \in V^{\mathrm{irr}}_{g,\mathcal{L}}(K)$. Set $C := \pi(C')$. Then $C' \to C$ is birational, and hence the normalizations $\widetilde{C}'$ and $\widetilde{C}$ are canonically isomorphic. Let us show that the tropicalizations of $f' \colon \widetilde{C}' \to X'$ and $f \colon \widetilde{C} \to X$ coincide. Under the identification $\widetilde{C}' = \widetilde{C}$, we have $f = \pi \circ f'$. Thus, the abstract tropical curves Γ' and Γ coincide, and for any $m \in \mathbb{Z}^2$, we have $f^*(x^m) = (f')^*(x^m)$. Therefore, the parametrizations h' and h coincide too by their very definition. We conclude that $N^{\mathrm{tr}}_C = N^{\mathrm{tr}}_{C'}$.

We have seen in the proof of assertion (2), that for any $[C] \in V(K)$ there exists $[C'] \in V'(K)$ such that $\pi(C') = C$. Thus, $N^{\mathrm{tr}}_C = N^{\mathrm{tr}}_{C'}$, and hence $N^{\mathrm{tr}}(V) \subseteq N^{\mathrm{tr}}(V')$. Vice versa, for $[C'] \in V'(K)$, consider a general one-parameter family $C'_B \to B \subseteq V'(K)$, and denote the fiber over a point $b \in B$ by C'_b. Set $0 := [C'] \in B$. Then, $C'_0 = C'$, and for any b in a small punctured neighborhood of 0, the pushforward $\pi_*([C'_b])$ belongs to $V(K)$. Hence $N^{\mathrm{tr}}_{C'_b} \subseteq N^{\mathrm{tr}}(V)$. It remains to show that in any punctured neighborhood of 0, there exists b such that the tropicalization of C'_b coincides with that of C'. Indeed, if such b exists, then $N^{\mathrm{tr}}_{C'} \subseteq N^{\mathrm{tr}}(V)$, and hence $N^{\mathrm{tr}}(V') \subseteq N^{\mathrm{tr}}(V)$.

To prove the existence of b as above, notice that since all curves in our family are reduced of geometric genus g, the family is equinormalizable. Therefore, it admits a tropicalization in the sense of [5] without marking the point $0 \in B$, cf. Step 2 in the proof of [5, Theorem 5.1]. By [5, Theorem 4.6], the tropicalization of C'_b depends only on $\mathrm{trop}(b) \in \mathrm{trop}(B)$, which in turn is completely determined by the specialization of b on the stable reduction of B. Plainly, in any punctured neighborhood of 0 there exist points, whose specializations on the stable reduction of B coincide with that of 0, which completes the proof. $\square$

5.1 Proof of Theorem A

Without loss of generality we may assume that Δ contains the origin as one of its vertices, and the sublattices we are interested in are linear. Let $M \subseteq \mathbb{Z}^2$ be a sublattice satisfying the conditions (a) and (b) of the theorem. It is sufficient to prove that for any $1 \leq g' \leq |\Delta^\circ \cap M|$, there exists a convex M-integral triangulation $\Delta = \cup \Delta_i$, whose set of vertices contains $\partial\Delta \cap M$ and generates M, and the number of vertices of the triangulation in Δ° is g'. Indeed, given such a triangulation for $g' = g$, the dual tropical curve $h \colon \Gamma \to \mathbb{R}^2$ is a trivalent irreducible curve of genus g, and hence regular by [20, Proposition 2.23], i.e., its deformation space has the expected dimension. Let $(X', \mathcal{L}')$ be the polarized toric surface corresponding to the polygon Δ with respect to the lattice M. Then $h \colon \Gamma \to \mathbb{R}^2$ is realizable by an irreducible nodal algebraic curve $f \colon C \to X'$ of genus g over the field of Puiseux series; see, e.g., [22, Lemma 3.12] or [26, Theorem 6.2]. Let V' be the component of the Severi variety $V^{\mathrm{irr}}_{g,\mathcal{L}'}$ containing $[C]$, and $V := \pi_*(V')$ the corresponding component of $V^{\mathrm{irr}}_{g,\mathcal{L}}$. Then

$M^{\mathrm{tr}}(V) = M^{\mathrm{tr}}(V') = M$ by Lemma 5.2 (4), which implies the asserted bound on the number of irreducible components of $V_{g,\mathcal{L}}^{\mathrm{irr}}$.

Let us construct the desired M-triangulations of Δ. Since $\partial\Delta \cap M = \partial\Delta \cap \mathbb{Z}^2$, there exists $m \in \Delta^\circ \cap M$ such that $\{m\} \cup (\partial\Delta \cap M)$ generates M. Thus, the triangulation with vertices $\{m\} \cup (\partial\Delta \cap M)$ is the desired convex triangulation in the case $g' = 1$. We proceed by induction. Once a desired triangulation $\Delta = \cup\Delta_i$ is constructed for $1 \leq g' < |\Delta^\circ \cap M|$, we construct the triangulation for $g' + 1$ in the following way: if there is a triangle Δ_i containing a lattice point in its interior, then we add this point as a new vertex of the triangulation. Otherwise there is a pair of triangles containing a lattice point in the interior of their common edge, and we add this point as a new vertex of the triangulation. Plainly, the new triangulation is convex in both cases, and the number of vertices of the new triangulation in the interior of Δ is $g' + 1$. □

5.2 Proof of Theorem B

Let $M \subseteq \mathbb{Z}^2$ be a sublattice of index r. A pair (M, κ) is called *admissible* if and only if M satisfies conditions (a) and (b) of Theorem A, and one of the following conditions holds

– the index r is even and $\kappa = g + 1$, or
– the index r is odd, $0 \leq \kappa \leq \min\{\delta_M(\Delta, g), g\}$, and κ is congruent to $\delta_M(\Delta, g)$ modulo 2.

To prove the theorem, it is sufficient to construct for any admissible pair (M, κ), an irreducible component $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ such that $(M^{\mathrm{tr}}(V), \kappa(V)) = (M, \kappa)$. Pick an admissible pair (M, κ), and let $(X', \mathcal{L}')$ be the polarized toric surface associated to the polygon Δ with respect to the lattice M.

Assume first, that r is even. Then $\kappa = g + 1$. As in the proof of Theorem A, there exists an irreducible component $V' \subseteq V_{g,\mathcal{L}'}^{\mathrm{irr}}$ such that $M^{\mathrm{tr}}(V') = M$. Set $V := \pi_*(V')$. Then $M^{\mathrm{tr}}(V) = M$ and $\kappa(V) = g + 1 = \kappa$ by Lemma 5.2 (3)-(4).

Assume now that r is odd. Then $\kappa \leq \min\{\delta_M(\Delta, g), g\}$, and hence

$$\kappa + g \leq \delta_M(\Delta, g) + g = |\Delta^\circ \cap M| = \frac{k + k'}{r} - 1.$$

Consider the M-integral triangulation of $\Delta = \Delta_{k,k'}$, whose inner vertices are the points

$$k' - 2r, k' - 4r, \ldots, k' - 2\kappa r, k' - 2\kappa r - r, k' - 2\kappa r - 2r, \ldots, k' - 2\kappa r - (g - \kappa)r$$

on the y-axis. Notice that $k' - 2\kappa r - (g - \kappa)r = k' - (\kappa + g)r \geq k' - (k + k' - r) = r - k$, and therefore such a triangulation exists. The triangulation subdivides the interval $[-k, k']$ on the y-axis into $g + 1$ subintervals. The top κ of them have M-integral length 2, the next $g - \kappa$ intervals have M-integral length one, and the bottom interval has M-integral length

$$\frac{k' - 2\kappa r - (g - \kappa)r + k}{r} = \delta_M(\Delta, g) + g + 1 - (\kappa + g) = \delta_M(\Delta, g) - \kappa + 1,$$

which is odd since κ is congruent to $\delta_M(\Delta, g)$ modulo 2. In particular, the vertices of the subdivision generate the affine lattice M.

Let $h\colon \Gamma \to \mathbb{R}^2$ be the trivalent tropical curve dual to the triangulation constructed above. We will prove that it is liftable to an algebraic curve $[C'] \in V^{\mathrm{irr}}_{g,\mathcal{L}'}$ for which $\kappa(C') = \kappa$ by using Shustin's refined tropicalization [22, § 3.5]. Let us start by picking an arbitrary tropical datum as in [22, § 3.7] with nodal amoeba $h(\Gamma)$. Then, we can apply [22, Theorem 5] to lift it to an algebraic curve C' over the field K. Furthermore, since the triangles in the dual subdivision have no inner integral points, the nodes of C' are in natural one-to-one correspondence with the nodes of the z-refinements of the tropicalization of C'. This allows us to control the signature $\kappa(C')$ in the following way.

For each of the top κ intervals on the y-axis, its M-integral length is 2, and hence there are two possible z-refinements as in [22, Lemma 3.9], and each of them contributes a single node to C'. The next $g - \kappa$ intervals have M-integral length one, and hence contribute no nodes at all. Finally, the bottom interval has an odd integral length, and hence contributes an even number of nodes. Furthermore, the nodes of the z-refinement corresponding to the bottom interval contribute equally to the two blocks of nodes of C'. Indeed, if h is the Chebyshev polynomial of an odd degree (or any of its twists as in the proof of [22, Lemma 3.9]), then it has the same number of critical points with critical values 1 and -1, and ± 1 are the only critical values of h. Therefore, half of the nodes of the bottom z-refinement, which is given by the polynomial $az^2 + bzh(w) + c$, have one value of the z-coordinate, and half – the opposite value.

Notice however, that changing the choice of the z-refinement corresponding to one of the top κ intervals, changes the block to which the refining curve contributes its node. Thus, after replacing some of the first κ refining curves with their twists, we can make sure that *all* nodes of the first κ refining curves contribute to *the same* block of nodes of C'. Therefore, by [22, Lemma 3.12], there exists $[C'] \in V^{\mathrm{irr}}_{g,\mathcal{L}'}$ such that $\kappa(C') = \kappa$. Furthermore, since the vertices of the triangulation generate M, we also have: $M^{\mathrm{tr}}(C') = M$.

To finish the proof, let $V' \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}'}$ be the irreducible component containing $[C']$, and set, as usual, $V := \pi_*(V')$. Then $V \subseteq V^{\mathrm{irr}}_{g,\mathcal{L}}$ is an irreducible component, and since r is odd, we have

$$(M^{\mathrm{tr}}(V), \kappa(V)) = (M^{\mathrm{tr}}(V'), \kappa(V')) = (M, \kappa)$$

by Lemma 5.2 (3)-(4). This completes the proof. □

6 Deformation of curves in toric surfaces

The main goal of this section is to introduce local coordinates on $V^{\mathrm{irr}}_{g,\mathcal{L}}$ that we can use to construct deformations to curves of lower genus. In particular, we will prove Theorem C.

For the convenience of the reader, we recollect the relevant material introduced in Sect. 1.1. Fix $[C] \in V^{\mathrm{irr}}_{g,\mathcal{L}}$, and pick a collection $\Upsilon := \{\gamma_1, \ldots, \gamma_g\}$ of oriented simple

closed curves in the smooth locus of $C^\bullet$ such that each γ_j is contractible in $X^\bullet$ and such that Υ generates a g-dimensional subspace of $H_1(C, \mathbb{Z})$. Since $X^\bullet$ is acyclic, each γ_j bounds a smooth disc $M_j \subset X^\bullet$ that is unique up to homotopy. We denote by Ω the holomorphic 2-form $\frac{dz \wedge dw}{zw}$ on $X^\bullet \simeq (\mathbb{C}^*)^2$ and consider the integrals

$$\int_{M_j} \Omega \, , \quad j = 1, \ldots, g. \tag{6.1}$$

Since the 2-form Ω is closed, the above integrals do not depend on the choice of the smooth membranes M_j's. Additionally, the integrals do not depend on the choice of γ_j's within a fixed isotopy class, as the form Ω is identically zero on C.

Denote by $W \subset V_{g,\mathcal{L}}^{\text{irr}}$ the locus of curves having given tangency profile with the boundary divisor. For any $[C] \in W$ and any open neighborhood $U \subset W$ of $[C]$ such that the monodromy action of $\pi_1(U, [C])$ on C acts as the identity on Υ, we can carry the collection Υ to a collection Υ' of simple closed curves in $[C'] \in U$. Therefore, the integrals (6.1) define a map

$$\Phi_\Upsilon : U \to \mathbb{C}^g.$$

At last, denote by $U^C \subset U$ the subset of curves $[C']$ such that $C' \cap \partial X = C \cap \partial X$. The following statement is a slight generalization of Theorem C.

Theorem A *For $[C] \in U$ and Υ as above, the restriction of the map Φ_Υ to U^C is a local diffeomorphism. Moreover, the integrals (6.1) together with the coordinates of the intersection points of C with the boundary divisor of X provide a local system of coordinates on W.*

In order to prove Theorem A, we will need a description of the space of holomorphic 1-forms on the normalization of C. We denote by $\pi : \widetilde{C} \to C$ the normalization map. Recall that the *adjoint polygon* Δ_a is defined by $\Delta_a := \text{conv}(\Delta^\circ)$.

Lemma 6.1 *For any curve $[C] \in V_{g,\mathcal{L}}^{\text{irr}}$ whose set of nodes is contained in $X^\bullet$, the space of holomorphic 1-forms on the normalization $\widetilde{C}$ is isomorphic to the linear subspace $L \subset H^0(X, \mathcal{L})$ consisting of the Laurent polynomials $h(z, w)$ vanishing at the nodes of C and having Newton polygon contained in Δ_a. The isomorphism is given by the map*

$$h(z, w) \mapsto \xi_h := \pi^* \left(\frac{h(z, w)}{\partial_w f(z, w) \cdot zw} dz \right)$$

where $f \in H^0(X, \mathcal{L})$ is a Laurent polynomial defining $C^\bullet$.

Proof Observe that L is isomorphic to the tangent space to U^C at the point $[C]$, which is smooth of dimension g. It follows that L has the expected dimension. Moreover, the two meromorphic 1-forms ξ_h and $\xi_{\tilde{h}}$ are linearly independent provided that h and $\tilde{h}$ are. It remains to show that ξ_h is indeed holomorphic.

Since $h \in L$ vanishes at the nodes of C, the form ξ_h is holomorphic in a neighborhood of the branches of the nodes. This can be seen using a local parametrization of the branches in $X^\bullet$. The fact that ξ_h is holomorphic on the rest of the curve is proven in [7, Lemma 4.3] (the proof is identical except that the charts $\mathbb{C}^2$ are to be replaced with charts $\mathbb{C}^* \times \mathbb{C}$). $\qquad\square$

Proof of Theorem A Assume towards the contradiction that the restriction of Φ_Υ to U^C is not a local diffeomorphism at $[C]$. Then, there exists a non-zero polynomial h as in Lemma 6.1 such that for

$$C_t := \overline{\{(z, w) \in X^\bullet \mid f(z, w) = t \cdot h(z, w)\}},$$

the derivative of the smooth function $t \mapsto \Phi_\Upsilon([C_t])$ vanishes at 0. For any $j = 1, \ldots, g$, fix an arbitrarily small neighborhood $V_j \subset X^\bullet$ of M_j and an arbitrarily small $\varepsilon > 0$. In order to compute the jth coordinate of the derivative of $\Phi_\Upsilon([C_t])$, we introduce a C^∞ complex-valued, time-dependent vector field

$$\chi(z, w, t) = \alpha(z, w, t)\partial z + \beta(z, w, t)\partial w$$

defined on V_j, in time $|t| < \varepsilon$, whose associated flow $\phi_t : V_j \to X^\bullet$ maps $C_0 \cap V_j$ to C_t and such that $\phi_t(M_j)$ is a smooth membrane. In particular, the jth coordinate of $\Phi_\Upsilon([C_t])$ is given by the integral $\int_{\phi_t(M_j)} \Omega$. Any such vector field, if it exists, satisfies the relation

$$\alpha \cdot \partial_z f + \beta \cdot \partial_w f = h + t(\alpha \cdot \partial_z h + \beta \cdot \partial_w h), \tag{6.2}$$

since any integral curve $t \mapsto (z(t), w(t))$ of χ satisfies the relation $f(z(t), w(t)) = t \cdot h(z(t), w(t))$ whose derivative with respect to t is (6.2). By the assumption made on Φ_Υ, we have

$$0 = \frac{d}{dt} \int_{\phi_t(M_j)} \Omega \bigg|_{t=0} = \frac{d}{dt} \int_{M_j} \phi_t^* \Omega \bigg|_{t=0} = \int_{M_j} \frac{d}{dt} \phi_t^* \Omega \bigg|_{t=0}, \tag{6.3}$$

where the second equality is the change of variables formula and the last integrand is the Lie derivative of Ω with respect to χ. According to the Cartan Formula [16, Proposition 5.3.1] and the fact that Ω is closed, the Lie derivative of Ω is the derivative $d(\chi \lrcorner \Omega)$ of the contraction of Ω with χ. Using the relation

$$\partial_z f(z, w) \cdot dz + \partial_w f(z, w) \cdot dw = 0$$

valid on C and the relation (6.2) specialized at $t = 0$, we obtain that

$$\begin{aligned}(\chi \lrcorner \Omega)_{|C} &= \frac{\alpha \cdot dw - \beta \cdot dz}{zw} = \frac{-dz}{zw}\left(\frac{\partial_z f}{\partial_w f}\alpha + \beta\right) \\ &= -\frac{\alpha \cdot \partial_z f + \beta \cdot \partial_w f}{zw\partial_w f}dz = \frac{-h}{zw\partial_w f}dz,\end{aligned} \tag{6.4}$$

whose pullback to $\widetilde{C}$ is the holomorphic form $-\xi_h$, see Lemma 6.1. Putting everything together, we obtain that

$$0 = \int_{M_j} \frac{d}{dt} \phi_t^* \Omega \bigg|_{t=0} = \int_{M_j} d(\chi \lrcorner \Omega) = \int_{\partial M_j} \chi \lrcorner \Omega = - \int_{\gamma_j} \xi_h \quad (6.5)$$

where the penultimate equality follows from Stokes' Theorem and the last equality from (6.4). Since (6.5) is valid for any j, and since the homology classes $[\gamma_j]$ form a basis of $H_1(\widetilde{C}, \mathbb{Z})$, we deduce that ξ_h is zero, and in turn, that $h = 0$. This is a contradiction.

It remains to prove the existence of the vector field χ. As a first approximation, we can define χ to be the meromorphic vector field defined by

$$\alpha := \frac{\mu h}{\partial_z f - t \partial_z h} + \lambda(\partial_w f - t \partial_w h) \quad \text{and} \quad \beta := \frac{(1 - \mu)h}{\partial_w f - t \partial_w h} - \lambda(\partial_z f - t \partial_z h)$$

with $\mu, \lambda \in \mathbb{C}$. Since ε is arbitrarily small, each curve $C_t \cap V_j$ is smooth. In particular, the gradient $(\partial_z f - t \partial_z h, \partial_w f - t \partial_w h)$ of $f - th$ is nowhere vanishing on $\mathscr{C} := \bigcup_{|t| < \varepsilon} C_t \cap V_j$. Therefore, setting μ to either 0 or 1 and $|\lambda|$ large enough, we can ensure that χ is holomorphic and nowhere vanishing on $\mathscr{C}$. Since χ satisfies (6.2), the flow ϕ_t maps C_0 to C_t. A priori, we cannot guarantee that χ is well defined on the whole $V_j \times \{|t| < \varepsilon\}$, since it is only meromorphic and may have poles inside the latter set. To fix this, we can replace χ with $\Psi \cdot \chi$, where $\Psi(z, w, t)$ is a test function supported on an arbitrarily small neighborhood of $\mathscr{C}$ and such that $\Psi_{|\mathscr{C}} = 1$. The resulting vector field is C^∞ and has the required properties. $\square$

7 Simple Harnack curves

In this section, we label the edges of Δ by $\Delta_1, \ldots, \Delta_n$ according to the counterclockwise ordering on $\partial \Delta$. For an edge Δ_j, we denote by $\mathcal{D}_j$ the corresponding toric divisor in X. We also denote by (z, w) the coordinates on $X^\bullet \simeq (\mathbb{C}^*)^2$ induced by the dual lattice $\mathbb{Z}^2$. The complex conjugation on $(\mathbb{C}^*)^2$ extends to an anti-holomorphic involution conj on X. A curve $[C] \in |\mathcal{L}|$ is real if $\text{conj}(C) = C$ and we denote by $\mathbb{R}C$ the fixed locus of $\text{conj}_{|C}$. As we did previously, we denote $C^\bullet := C \cap X^\bullet$. Recall the *amoeba map*

$$\mathcal{A} \colon (\mathbb{C}^*)^2 \to \mathbb{R}^2$$
$$(z, w) \mapsto \big(\log |z|, \log |w|\big).$$

Definition 7.1 A real curve $[C] \in |\mathcal{L}|$ is a (possibly singular) *simple Harnack curve* if the restriction of the amoeba map $\mathcal{A} \colon C^\bullet \to \mathbb{R}^2$ is at most 2-to-1.

The above definition is shown to be equivalent to the original definition [21, Definitions 2 and 3] in [21, Theorem 1]. It is clear from the definition of the map $\mathcal{A}$ that it identifies pairs of complex conjugated points on a real curve C. Therefore, simple

Harnack curves are exactly those curves C for which the restriction of $\mathcal{A}$ to $C^{\bullet}$ realizes the quotient of $C^{\bullet}$ by the involution conj, see e.g., [21, Lemma 5]. In particular, the restriction of $\mathcal{A}$ to $\mathbb{R}C^{\bullet}$ is 1-to-1 onto the boundary of $\mathcal{A}(C^{\bullet})$, see also [19, Lemma 8 and Corollary 9] for smooth curves.

Recall that any smooth simple Harnack curve C satisfies the following properties: C is maximal, i.e., $b_0(\mathbb{R}C) = g+1$, where g is the genus of C, and the real part $\mathbb{R}C$ has a unique component that intersects the boundary divisor ∂X. It follows that $\mathcal{A}(C^{\bullet})$ is a topological disc with g holes bounded by the images of the g remaining components of $\mathbb{R}C$ and with punctures on its boundary. Smooth/singular simple Harnack curves are obtained from each other by contracting/expanding the latter holes, as shown by the following lemma.

Lemma 7.2 *The only singularities of simple Harnack curves are real isolated double points. Moreover, there exists a smooth simple Harnack curve in the neighborhood of any singular simple Harnack curve $[C] \in |\mathcal{L}|$.*

Proof The first property is actually a part of the original definition [21, Definition 3]. With respect to Definition 7.1 above, the latter property corresponds to [21, Lemma 6]. For the second part of the statement, let us fix a defining Laurent polynomial $f(z, w)$ for $C^{\bullet}$. Then, there exists another Laurent polynomial $h(z, w)$ satisfying the following:

– the Newton polygon $\mathrm{New}(h)$ of h equals $\mathrm{New}(f) =: \Delta$;

– for any node $(z, w) \in C^{\bullet}$, we have $h(z, w) = -1$ (resp. $h(z, w) = 1$) if f is positive (resp. negative) in a small neighborhood $\mathcal{U} \subset (\mathbb{R}^*)^2$ of (z, w).

Indeed, the latter conditions are linear in the space $|\mathcal{L}|$, and there are as many conditions as nodes. In particular, the number of conditions is at most $|\Delta^{\circ} \cap \mathbb{Z}^2|$, which in turn is strictly less than the dimension of $|\mathcal{L}|$. The space of polynomials h as above is therefore not empty.

It follows now from the Morse Lemma that the curve defined by $f + \varepsilon h$ is a smooth simple Harnack curve for any arbitrarily small $\varepsilon > 0$. Indeed, for any node (z, w), there exist real analytic coordinates centered at (z, w) such that $f + \varepsilon h$ reads either $z^2 + w^2 - \varepsilon$ or $\varepsilon - z^2 - w^2$, that is the isolated double points of C are deformed into compact ovals in $\mathbb{R}C^{\bullet}$. The resulting curve satisfies therefore the requirements of [21, Definition 2] which is equivalent to Definition 7.1. $\square$

Recall that for a smooth simple Harnack curve C, the order map *ord* of [9] establishes a bijective correspondence between the set of compact connected components of $\mathbb{R}C^{\bullet}$ and the set of lattice points $\Delta^{\circ} \cap \mathbb{Z}^2$, see [19, Corollary 10].

According to [19, Lemma 11], the map *ord* on a smooth simple Harnack curve C can be described as follows. First, assume that the vertex $\Delta_n \cap \Delta_1$ is the origin. Let $c_0 \subset \mathbb{R}C^{\bullet}$ be the unique connected component joining the two consecutive toric divisors $\mathcal{D}_n$ and $\mathcal{D}_1$. For any compact component c of $\mathbb{R}C^{\bullet}$, draw a path $\rho_c \subset \mathcal{A}(C)$ joining $\mathcal{A}(c_0)$ to $\mathcal{A}(c)$. By the 2-to-1 property of the amoeba map, the lift $\gamma_c := \mathcal{A}^{-1}(\rho_c)$ is a loop in $C^{\bullet}$, which is invariant under complex conjugation. There exists a unique orientation of γ_c such that the corresponding homology class $(a, b) \in H_1((\mathbb{C}^*)^2, \mathbb{Z})$ satisfies $(-b, a) \in \Delta^{\circ} \cap \mathbb{Z}^2$ (note the sign mistake in the sixth line of the proof of [19, Lemma 11]). Then, we have $ord(c) = (-b, a)$. Observe that

the homology class of γ_c is independent of the choice of ρ_c. In particular, the map *ord* is well defined.

For a singular simple Harnack curve C, a compact component c of $\mathbb{R}C^{\bullet}$ can be of two types: either c is a topological circle or it is a real isolated double point of C. In the former case, we define γ_c as above. If c is a node of C we can repeat the same construction as above where ρ_c is a path joining $\mathcal{A}(c_0)$ to $\mathcal{A}(c)$ and $\gamma_c := \mathcal{A}^{-1}(\rho_c)$.

Definition 7.3 For a singular simple Harnack curve C, define the *order map*

$$ord\colon \{\text{compact components of } \mathbb{R}C^{\bullet}\} \to \Delta^{\circ} \cap \mathbb{Z}^2$$

by $ord(v) = (-b, a)$ where $(a, b) \in H_1(X^{\bullet}, \mathbb{Z})$ is the homology class of the (properly oriented) loop γ_c constructed above.

The fact that $(-b, a) \in \Delta^{\circ} \cap \mathbb{Z}^2$ (with the appropriate orientation of γ_c) follows from the facts that
  – it holds for smooth curves C;
  – the homology class of γ_c is constant under small perturbations of C;
  – singular Harnack curves can always be perturbed into smooth ones, thanks to Lemma 7.2.
By the same arguments, the statement below follows now from [19, Corollary 10].

Proposition 7.4 *For any singular simple Harnack curve, the order map is a bijection.*

The existence of smooth simple Harnack curves in $|\mathcal{L}|$ is guaranteed by [19, Corollary A4]. For singular ones, the existence is addressed in [13, Theorem 6], [2, Theorems 2 and 10], [7, Theorem 3] in various contexts. The theorem below discusses the existence of singular Harnack curves with prescribed order map.

Theorem 7.5 *For any integer $0 < g < |\Delta^{\circ} \cap \mathbb{Z}^2|$ and any subset $E \subset \Delta^{\circ} \cap \mathbb{Z}^2$ of cardinality $\delta := |\Delta^{\circ} \cap \mathbb{Z}^2| - g$, there exists a one-parameter family of simple Harnack curves $\{[C_t]\}_{t \in [0,1]} \subset |\mathcal{L}|$ such that C_t is smooth for all $t < 1$, the curve C_1 is singular and $ord(\{\text{nodes of } C_1\}) = E$. In particular, any Severi variety $V_{\mathcal{L},g}^{\mathrm{irr}}$ contains a simple Harnack curve.*

In [13], it was observed that the Euclidean area of the g compact holes of the amoeba $\mathcal{A}(C^{\bullet})$ of a smooth simple Harnack curve can be completed into a system of coordinates on the space of all simple Harnack curves for $X = \mathbb{P}^2$, see [13, Theorem 6]. In particular, any sub-collection of these holes can be contracted to points by prescribing the corresponding area to tend to zero.

The proof of Theorem 7.5 is based on the same idea. Below, we give an alternative proof of the fact that the area of the holes of $\mathcal{A}(C^{\bullet})$ provides local coordinates, building on the material of Sect. 6.

Proof of Theorem 7.5 Choose any smooth simple Harnack curve $[C_0] \in |\mathcal{L}|$, and denote by H the space of smooth simple Harnack curves in $|\mathcal{L}|$ that coincide with C_0 on $X \setminus X^{\bullet}$. Note that $H \subset |\mathcal{L}|$ is a smooth subvariety of real dimension $g_{\Delta} := |\Delta^{\circ} \cap \mathbb{Z}^2|$.

Consider the continuous map Area: $H \to \mathbb{R}_{\geq 0}^{\Delta^\circ \cap \mathbb{Z}^2}$ that associates to any curve C the Euclidean area of the holes of $\mathcal{A}(C^\bullet)$, where each hole is indexed by the corresponding point in $\Delta^\circ \cap \mathbb{Z}^2$ via the order map *ord*. We claim that the map Area is a local diffeomorphism. Indeed, the map Area is nothing but the map Φ_Υ, where Υ consists of the g_Δ compact connected components of $\mathbb{R}C^\bullet$ (note that the existence of the map *ord* implies that the monodromy of H acts trivially on Υ). To see this, pick membranes M_j to be the disc in $(\mathbb{R}^*)^2$ bounded by γ_j. Since the 2-form Ω is the pullback under the coordinate-wise complex logarithm of the Euclidean form $dz \wedge dw$, the integrals (6.1) compute the Euclidean area of the holes of $\mathcal{A}(C^\bullet)$, and the claim follows by Theorem A.

Next, notice that the map Area extends continuously to the closure $\overline{H}$. Indeed, by Lemma 7.2, the curves in the boundary of H correspond to the singular curves in which some of the g_Δ real ovals are contracted to real isolated double points. Therefore, we obtain a continuous map Area: $\overline{H} \to \mathbb{R}_{\geq 0}^{\Delta^\circ \cap \mathbb{Z}^2}$. Moreover, by [13, Theorem 6], the latter map is proper in the case of planar curves, and the argument in *loc. cit.* generalizes to arbitrary toric surfaces verbatim.

Finally, set $(a_j)_{j \in \Delta^\circ \cap \mathbb{Z}^2} := \text{Area}([C_0])$, and consider the segment $\sigma \subset \mathbb{R}^{\Delta^\circ \cap \mathbb{Z}^2}$ from the point Area$([C_0])$ to the point $(b_j)_{j \in \Delta^\circ \cap \mathbb{Z}^2}$ defined by $b_j = 0$ if $j \in E$, and $b_j = a_j$ otherwise. We claim that there exists a path in $\overline{H}$ starting from $[C_0]$ and mapping bijectively onto σ. To see this, define $\mathscr{P}$ to be the set of all closed paths ρ starting from $[C_0]$ and mapping injectively into σ. The set $\mathscr{P}$ is totally ordered and contains paths with non-empty relative interior since Area is a covering. Since Area is proper on $\overline{H}$, there exists a unique maximal element $\tilde{\rho}$ in $\mathscr{P}$. Assume towards the contradiction that the end point $[\widetilde{C}]$ of $\tilde{\rho}$ maps to an interior point of σ. Then $\mathcal{A}(\widetilde{C}^\bullet)$ bounds g_Δ compact holes, or equivalently $\mathbb{R}\widetilde{C}^\bullet$ contains g_Δ ovals, that is $\widetilde{C}$ is smooth. Since Area is a local covering on H, the smoothness of $\widetilde{C}$ implies that $\tilde{\rho}$ can be extended into a longer path in $\mathscr{P}$. This is a contradiction. Therefore $\tilde{\rho}$ maps bijectively to σ and provides the sought degeneration of C_0 to C_1. □

8 Topological proof of the main results

In this section, we use the notation Δ_j, $\mathcal{D}_j$ introduced in the previous section. We assume with no loss of generality that $\Delta_n \cap \Delta_1 = 0$.

Lemma 8.1 *Let* $[C] \in V_{g,\mathcal{L}}^{irr}$ *be a torically strictly transverse simple Harnack curve and denote* $E = ord(\{nodes\ of\ C\})$. *If* $V \subset V_{g,\mathcal{L}}^{irr}$ *is the irreducible component containing* $[C]$, *then the lattice* $M(V)$ *of Sect. 4.1 is generated by* $(\Delta \cap \mathbb{Z}^2) \setminus E$.

Proof As in Sect. 7, we denote by c_0 the unique connected component of $\mathbb{R}C^\bullet$ joining $\mathcal{D}_n$ to $\mathcal{D}_1$. We denote by $a_1, \ldots, a_g$ the g ovals of $\mathbb{R}C^\bullet$. For any a_j, we choose a path inside $\mathcal{A}(C^\bullet)$ joining $\mathcal{A}(c_0)$ to $\mathcal{A}(a_j)$, and denote by $a_{g+j} \subset C^\bullet$ its preimage under the map $\mathcal{A}$. By perturbing the paths if necessary, we can make sure that the loops a_{g+j} avoid the nodes of C. Finally, let $\ell_1, \ldots, \ell_k$ be small loops around the punctures of $C^\bullet$. Then, the collection of simple closed curves $\ell_1, \ldots, \ell_k, a_1, \ldots, a_{2g}$ (oriented

arbitrarily) lifts to a basis of $H_1(\widetilde{C}^\bullet, \mathbb{Z})$. In particular, the homology classes of these curves generate the sublattice N_C.

Each ℓ_j is associated to a puncture of $C^\bullet$ corresponding to the intersection of C with one of the toric divisors of X. In turn, such toric divisor corresponds to an edge of Δ directed by some primitive integer vector (a, b). It is a standard fact in toric geometry that the homology class $[\ell_j]$ is then given by $\pm(-b, a)$ (depending on the orientation of ℓ_j), cf. [17, Lemma 1.10]. In particular, the sublattice of N generated by the $[\ell_j]$'s is the rotation by $\frac{\pi}{2}$ of the sublattice $\langle \partial \Delta \cap \mathbb{Z}^2 \rangle$. By the construction and the definition of the order map, the following holds: for any $1 \leq j \leq g$ we have $[a_j] = (0, 0)$ and $[a_{g+j}] = \pm(-b, a)$, where $ord(a_j) = (a, b)$. We deduce that N_C is the rotation by $\frac{\pi}{2}$ of the lattice generated by $(\Delta \cap \mathbb{Z}^2) \setminus E$, which implies the assertion. $\square$

Proof of Theorem A The theorem follows from Lemmata 4.2 and 8.1 and Theorem 7.5.
$\square$

Lemma 8.2 *Let $\Delta_{k,k'} \subset \mathbb{R}^2$ be a kite, g, δ integers such that $\delta + g = k + k' - 1$, and $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}$ a simple Harnack curve. Set $E := ord(\{\text{nodes of } C\})$ and*

$$E^{even} := E \cap (\{0\} \times 2\mathbb{Z}) \quad \text{and} \quad E^{odd} := E \setminus E^{even}.$$

Then, the nodal partition of C is $\{|E^{even}|, |E^{odd}|\}$. In particular, any integer partition $\delta = \delta_1 + \delta_2$ such that $0 \leq \delta_1, \delta_2 \leq \lceil \frac{k+k'-1}{2} \rceil$ is the nodal partition of some curve in $V_{g,\mathcal{L}}^{\mathrm{irr}}$.

Proof By definition, the nodal partition of C is the partition given by the sign of the first coordinate of the nodes. By [19, Lemma 11], the sign corresponding to a node $\nu \in C$ is given by the parity of the second coordinate of $ord(\nu)$. This proves the first part of the statement. In particular, the nodal partition $\delta = \delta_1 + \delta_2$ of C has to satisfy $0 \leq \delta_1, \delta_2 \leq \lceil \frac{k+k'-1}{2} \rceil$, and for each such partition we can find E such that $\{|E^{even}|, |E^{odd}|\} = \{\delta_1, \delta_2\}$. The result now follows from Theorem 7.5. $\square$

Proof of Theorem B As in the tropical proof of the theorem, it is sufficient to construct for any admissible pair (M, κ), an irreducible component $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ such that $(M(V), \kappa(V)) = (M, \kappa)$; see §5.2 for the definition of an admissible pair. By Lemmata 8.1 and 8.2, it suffices to find a subset $E \subseteq \Delta^\circ \cap M$ of cardinality δ such that $(\Delta \cap M) \setminus E$ generates M, and $||E^{even}| - |E^{odd}|| = \kappa$. Such a set E can be taken as the set of interior vertices of the M-integral triangulation of Δ constructed in Sect. 5.2. $\square$

Appendix A

In this section, we establish a connection between the Severi varieties associated to kites and the topological classification of polynomials as studied in [14] and [29].

Let $\Delta := \Delta_{k',k}$ be a kite, and $(X, \mathcal{L})$ the corresponding polarized toric surface. For our purpose, it will be convenient to shift the kite so that its vertices become $(\pm 1, 0)$, $(0, -k)$, and $(0, k')$. For a given integer $g \geq 0$, set $\delta := \delta_{\mathbb{Z}^2}(\Delta, g)$. Recall that a curve $[C] \in |\mathcal{L}|$ is defined by a Laurent polynomial of the form

$$f(z, w) = \frac{a}{z} + p(w) + bz \qquad (A.1)$$

where $a, b \in \mathbb{C}$ and $p(w)$ is a univariate Laurent polynomial. Recall from Sect. 4.2 that any curve $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}$ has an associated nodal partition $\delta = \delta_1 + \delta_2$. Our first observation is that the latter partition determines the *passport* of the polynomial $p(w)$.

Consider a branched cover $p : S \to \mathbb{C}P^1$ of degree d from a compact orientable surface S. For any critical value $b \in \mathbb{C}P^1$ of p with fiber $p^{-1}(b) := \{a_1, \ldots, a_k\}$, we have a partition $d = d_1 + \cdots + d_k$ where d_i is the multiplicity of p at a_i. In turn, the passport Σ of p is the collection of the partitions for all critical values of p other than ∞. In particular, any Laurent polynomial admits a passport.

Denote by $\mathcal{P}^{k',k}$ the space of Laurent polynomials with set of exponents contained in $[-k, k']$, to which the polynomial $p(w)$ from (A.1) belongs. For a given passport Σ, denote by $\mathcal{P}_\Sigma^{k',k}$ the subspace of $\mathcal{P}^{k',k}$ consisting of polynomials with passport Σ. For $k = 0$, the space $\mathcal{P}_\Sigma^{k',k}$ is a central object in the topological classification of polynomials, see for instance [14]. It follows from the proof of [14, Theorem 6] that $\mathcal{P}_\Sigma^{k',k}$ is a smooth variety of complex dimension $m + 2$, where m is the number of distinct finite critical values of Σ. In particular, a path-connected component of $\mathcal{P}_\Sigma^{k',k}$ is irreducible and vice versa.

Let us denote by $\{2^\delta\}$ the partition of d with exactly δ summands "2" and remaining summands "1". Since the case $\delta = 0$ corresponds to a regular value, every time $\{2^0\}$ appears in a passport we disregard it. Then, we have the following simple lemma.

Lemma A.1 *Let* $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ *be an irreducible component with nodal partition* $\delta = \delta_1 + \delta_2$, *then there is a Zariski open subset of curves* $[C] \in V$ *for which the Laurent polynomial* $p(w)$ *of (A.1) has passport* $\left\{ \{2^{\delta_1}\}, \{2^{\delta_2}\}, \{2\}, \cdots, \{2\} \right\}$.

Proof Assume for simplicity that $0 < \delta_2 \leq \delta_1$. The case $\delta_2 = 0$ is similar. The coordinates (z, w) of a node of a curve $[C] \in V_{g,\mathcal{L}}^{\mathrm{irr}}$ defined by (A.1) satisfy the conditions $f(z, w) = \partial_z f(z, w) = \partial_w f(z, w)$ or equivalently

$$p'(w) = 0, \ z^2 = \frac{a}{b} \ \text{and} \ p(w) = -\left(\frac{a}{z} + bz\right).$$

It follows that $p(w) = \pm 2\sqrt{ab}$, and the nodal partition corresponds to the distribution of the w's over the two critical values $\pm 2\sqrt{ab}$ among the δ-many nodes of C. In particular, the number m of distinct finite critical values of the polynomial $p(w)$ is at most $k + k' - 1 - (\delta - 2) = k + k' + 1 - \delta$ and there is equality if and only if the passport of $p(w)$ is $\{\{2^{\delta_1}\}, \{2^{\delta_2}\}, \{2\}, \ldots, \{2\}\}$.

Clearly, the passport of $p(w)$ is constant as long as $[C]$ lies in a certain open Zariski subset of V. We denote this generic passport by Σ. According to the dimension formula of $\mathcal{P}_\Sigma^{k',k}$ and the fact that the 2 critical values of $p(w)$ corresponding to nodes of C are determined by a and b, the locus of polynomials $p(w)$ corresponding to general curves $[C] \in V$ has dimension $m + 2 - 2 = m$. Therefore,

$$k + k' + 2 - \delta = \dim(V) = m + 2 - 1 = m + 1,$$

and hence $m = k + k' + 1 - \delta$. Thus, Σ is the sought passport. $\square$

Lemma A.2 *Let* $\Sigma := \{\{2^{\delta_1}\}, \{2^{\delta_2}\}, \{2\}, \ldots, \{2\}\}$ *be a passport, where* $\delta = \delta_1 + \delta_2$ *is a partition of* δ. *Then, the number of irreducible components* $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ *with nodal partition* $\delta = \delta_1 + \delta_2$ *is equal to the number of irreducible components of* $\mathcal{P}_\Sigma^{k',k}$.

Proof Again, we assume for simplicity that $0 < \delta_2 \leq \delta_1$. The case $\delta_2 = 0$ is similar. Denote by $V_{\{\delta_1, \delta_2\}} \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ the union of the irreducible components with nodal partition $\delta = \delta_1 + \delta_2$, and by $\mathscr{C}_{\{\delta_1, \delta_2\}} \subset H^0(X, \mathcal{L}) \setminus \{0\}$ the cone over it. Then there is a natural bijection between the sets of the irreducible components of $V_{\{\delta_1, \delta_2\}}$ and $\mathscr{C}_{\{\delta_1, \delta_2\}}$. Furthermore, $V_{\{\delta_1, \delta_2\}}$ and $\mathscr{C}_{\{\delta_1, \delta_2\}}$ are disjoint unions of their irreducible components.

By Lemma A.1, there exists a natural map $g \colon \mathscr{C}_{\{\delta_1, \delta_2\}} \rightarrow \mathcal{P}_\Sigma^{k',k}$ mapping $f(z, w) = \frac{a}{z} + p(w) + bz$ to $p(w)$. Plainly, g is continuous, and hence induces an injective map between the sets of irreducible components. We claim that the latter map is surjective. Indeed, any component $W \subseteq \mathcal{P}_\Sigma^{k',k}$ is invariant under affine transformations $p \mapsto \alpha p + \beta$, where $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$. Therefore, W contains p, whose two special critical values are ± 2. Let C be the curve given by the section $f(z, w) = \frac{1}{z} + p(w) + z$. It has nodal partition $\delta = \delta_1 + \delta_2$, and hence W contains the image of the irreducible component of $\mathscr{C}_{\{\delta_1, \delta_2\}}$ containing f. This completes the proof. $\square$

The above lemma can be used to obtain information on Severi varieties from the spaces $\mathcal{P}_\Sigma^{k',k}$ or the other way around, as illustrated by the following.

Corollary A.3 *Assume that* $k + k' \geq 5$ *and that* $\delta = 2$. *Then, the Severi variety* $V_{g,\mathcal{L}}^{\mathrm{irr}}$ *has exactly 2 irreducible components.*

Proof Since $\delta = 2$, there are exactly 2 possible nodal partitions, namely $2 = 1 + 1$ and $2 = 2 + 0$. If Σ_1 and Σ_2 are the corresponding passports as in Lemma A.2, then $\mathcal{P}_{\Sigma_1}^{k',k}$ is the space of Laurent polynomials with distinct critical values. In particular,

it is irreducible. The polynomials in $\mathcal{P}_{\Sigma_2}^{k',k}$ have only simple critical values except one, whose preimage contains 2 critical point. We claim that $\mathcal{P}_{\Sigma_2}^{k',k}$ is also irreducible. Indeed, acting by pre-composition with an affine linear map, we can fix the 2 critical points over the special critical value to be 0 and 1. As the polynomials in $\mathcal{P}_{\Sigma_2}^{k',k}$ satisfying this extra condition form a linear subspace, this proves the claim. The result now follows from Lemma A.2. □

Corollary A.4 *Let* $k, k', g, \delta, \kappa, \delta_1$, *and* δ_2 *be non-negative integers such that* $k' \geq k$, $k' > 0$, $g + \delta = k + k' - 1$, $\delta = \delta_1 + \delta_2$ *and* $\kappa = |\delta_1 - \delta_2|$. *Denote by* $\Sigma := \left\{ \{2^{\delta_1}\}, \{2^{\delta_2}\}, \{2\}, \cdots, \{2\} \right\}$ *the passport of Laurent polynomials in* $\mathcal{P}^{k',k}$. *Then, the number of irreducible components of* $\mathcal{P}_{\Sigma}^{k',k}$ *is bounded from below by the number of lattices* $M \subset \mathbb{Z}^2$ *such that* (M, κ) *is an admissible pair, see* Sect. 5.2.

Proof By Lemma A.2, the number of irreducible components of $\mathcal{P}_{\Sigma}^{k',k}$ is equal to the number of irreducible components of $V_{g,\mathcal{L}}^{\mathrm{irr}}$ with nodal partition $\delta = \delta_1 + \delta_2$ or equivalently with signature κ. According to Sect. 5.2, the latter number is bounded from below by the number of lattices $M \subset \mathbb{Z}^2$ such that (M, κ) is an admissible pair. □

Remark A.5 (i) For a sublattice $M \subseteq \mathbb{Z}^2$ of index r, a component $V \subseteq V_{g,\mathcal{L}}^{\mathrm{irr}}$ for which $M(V) = M$, and any curve $[C] \in V$, the Laurent polynomial $p(w)$ of (A.1) can be written $p(w) = q(w)^r$ if r is odd, and $p(w) = q(w)^r - 2(ab)^{r/2}$ if r is even. Therefore, the existence of a lattice M compatible with κ and with index $r \geq 2$ implies that the passport Σ is decomposable in the sense of [29].

(ii) Let M, V, r, p, and q be as above. By [4], the closure of any component of the Severi variety contains simple rational Harnack curves, and therefore V contains simple Harnack curves. For simple Harnack curves, an explicit computation shows that the polynomial $q(w)$ is of type S_n in the sense of [29, Section 4.1], with $n = k + k'$. Thus, [29, Conjecture 13], if true, implies the sharpness of the bound $\#_{k,k',g}$ of Theorem B in the case $k = 0$.

(iii) One could study the decomposability of more general passports by considering non-complete linear systems associated to support sets $A \subset \mathbb{Z}^2$, following the A-philosophy of [10]. For instance, let Δ be the polygon with vertices $(-m, 0), (1, 0), (0, k'), (0, -k)$ for some $k' \geq k \geq 0$ and $m \geq 2$, and A the set consisting of the vertices of Δ and of the inner integral points of Δ that belong to the y-axis. Consider the associated polarized toric variety $(X, \mathcal{L})$ and the non-complete linear system $|\mathcal{L}_A| \subset |\mathcal{L}|$. Then the curves in $|\mathcal{L}_A|$ are given by polynomials of the form

$$f(z, w) = \frac{a}{z^m} + p(w) + bz.$$

Such curves admit nodal partitions of length m. Thus, in this case the Laurent polynomials $p(w)$ may have up to $m + 1$ special critical values, instead of 2. It would be interesting to study the decomposability of the corresponding passports via the reducibility of the Severi varieties $V_{g,\mathcal{L}_A}^{\mathrm{irr}}$.

References

1. Bourqui, D.: Algebraic points, non-anticanonical heights and the Severi problem on toric varieties. Proc. Lond. Math. Soc. (3) **113**(4), 474–514 (2016)
2. Brugallé, E.: Pseudoholomorphic simple Harnack curves. Enseign. Math. **61**(3–4), 483–498 (2015)
3. Chiantini, L., Ciliberto, C.: On the Severi varieties on surfaces in $\mathbb{P}^3$. J. Algebr. Geom. **8**(1), 67–83 (1999)
4. Christ, K., He, X., Tyomkin, I.: Degeneration of curves on some polarized toric surfaces J. Reine Angew. Math. (2020). https://doi.org/10.1515/crelle-2022-0006 (To appear)
5. Christ, K., He, X., Tyomkin, I.: On the Severi problem in arbitrary characteristic. (2020). arXiv:2005.04134v1
6. Ciliberto, C., Dedieu, T.: On universal Severi varieties of low genus $K3$ surfaces. Math. Z. **271**(3–4), 953–960 (2012)
7. Crétois, R., Lang, L.: The vanishing cycles of curves in toric surfaces. I. Compos. Math. **154**(8), 1659–1697 (2018)
8. Danilov, V.I.: The geometry of toric varieties. Uspekhi Mat. Nauk **33**(2(200)), 85–134, 247 (1978)
9. Forsberg, M., Passare, M., Tsikh, A.: Laurent determinants and arrangements of hyperplane amoebas. Adv. Math. **151**(1), 45–70 (2000)
10. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, Resultants, and Multidimensional Determinants. Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser. x, Boston, p. 523 (2008)
11. Harris, J.: On the Severi problem. Invent. Math. **84**, 445–461 (1986)
12. Itenberg, I., Mikhalkin, G.: Area in real K3-surfaces. EMS Surv. Math. Sci. **8**(1–2), 217–235 (2021)
13. Kenyon, R., Okounkov, A.: Planar dimers and Harnack curves. Duke Math. J. **131**(3), 499–524 (2006)
14. Khovanskii, A.G., Zdravkovska, S.: Branched covers of S^2 and braid groups. J. Knot Theory Ramif. **5**(1), 55–75 (1996)
15. Kleiman, S.L., Shende, V.V.: On the Göttsche threshold. In: A Celebration of Algebraic Geometry, Volume 18 of Clay Math. Proc. Amer. Math. Soc., Providence, pp. 429–449. With an appendix by Ilya Tyomkin (2013)
16. Lang, S.: Fundamentals of Differential Geometry, vol. 191. Springer, New York (1999)
17. Lang, L.: A generalisation of simple Harnack curves (2015). arXiv e-prints, page arXiv:1504.07256
18. Lang, L.: Monodromy of rational curves on toric surfaces. J. Topol. **13**(4), 1658–1681 (2020)
19. Mikhalkin, G.: Real algebraic curves, the moment map and amoebas. Ann. Math. (2) **151**(1), 309–326 (2000)
20. Mikhalkin, G.: Enumerative tropical algebraic geometry in $\mathbb{R}^2$. J. Am. Math. Soc. **18**(2), 313–377 (2005)
21. Mikhalkin, G., Rullgård, H.: Amoebas of maximal area. Int. Math. Res. Not. **2001**(9), 441–451 (2001)
22. Shustin, E.: A tropical approach to enumerative geometry. Algebra i Analiz **17**(2), 170–214 (2005)
23. Silverman, J.H.: The Arithmetic of Elliptic Curves, Volume 106 of Graduate Texts in Mathematics, 2nd edn. Springer, Dordrecht (2009)
24. Testa, D.: The irreducibility of the spaces of rational curves on del Pezzo surfaces. J. Algebr. Geom. **18**(1), 37–61 (2009)
25. Tyomkin, I.: On Severi varieties on Hirzebruch surfaces. Int. Math. Res. Not. **2007**(23), 31 (2007)
26. Tyomkin, I.: Tropical geometry and correspondence theorems via toric stacks. Math. Ann. **353**(3), 945–995 (2012)
27. Tyomkin, I.: On Zariski's theorem in positive characteristic. J. Eur. Math. Soc. (JEMS) **15**(5), 1783–1803 (2013)
28. Tyomkin, I.: An example of a reducible Severi variety. In: Proceedings of the 20th Gökova Geometry-Topology Conference, Gökova, Turkey, May 27–June 1, 2013. International Press, Somerville, pp. 33–40. Gökova: Gökova Geometry-Topology Conferences (GGT) (2014)
29. Zvonkin, A.: Towards topological classification of univariate complex polynomials. In: Formal Power Series and Algebraic Combinatorics. Proceedings of the 12th International Conference, FPSAC'00, Moscow, Russia, June 26–30, 2000. Springer, Berlin, pp. 76–87 (2000)