Mathematische Annalen (2023) 385:575–607
https://doi.org/10.1007/s00208-021-02348-7

Mathematische Annalen

Positivity of holomorphic vector bundles in terms of L^p-estimates for $\bar{\partial}$

Fusheng Deng[1] · Jiafu Ning[2] · Zhiwei Wang[3] · Xiangyu Zhou[4]

Received: 2 September 2021 / Revised: 2 September 2021 / Accepted: 11 December 2021 /
Published online: 17 January 2022

Abstract

In this paper, we introduce a new notion of the Hermitian holomorphic vector bundles satisfying the optimal L^2-estimate, and give a characterization of Nakano positivity for Hermitian holomorphic vector bundles via the notion. As an application, we provide a new method to obtain Nakano positivity of direct image sheaves of twisted relative canonical bundles associated to holomorphic families of complex manifolds. We also present a comprehensive picture about converses of L^p-estimates and L^p-extension for $\bar{\partial}$.

Mathematics Subject Classification 32D15 · 32U05 · 32J25 · 32W05 · 32L05 · 14C30 · 53C55 · 53C56

Communicated by Ngaiming Mok.

✉ Jiafu Ning
jfning@csu.edu.cn

✉ Zhiwei Wang
zhiwei@bnu.edu.cn

Fusheng Deng
fshdeng@ucas.ac.cn

Xiangyu Zhou
xyzhou@math.ac.cn

[1] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

[2] School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, China

[3] Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

[4] Institute of Mathematics, Academy of Mathematics and Systems Sciences, and Hua Loo-Keng Key Labratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China

Contents

1 Introduction

Positivity (e.g., both Nakano and Griffiths positivity) of Hermitian holomorphic vector bundles play an important role in several complex variables and complex geometry. Nakano positivity implies Griffiths positivity. Some important and influential results are obtained under the assumptions of Nakano or Griffiths positivity. For example, an L^2 existence theorem by Hörmander [21] and Demailly [9] asserts that if a hermitian holomorphic vector bundle over a weakly pseudoconvex Kähler manifold is Nakano positive, then the solution for $\bar\partial$ with L^2 estimate exists. Therefore, the characterizations of Griffiths and Nakano positivity for Hermitian holomorphic vector bundles are important. Although there exists characterization of Griffiths positivity for a Hermitian holomorphic vector bundle, the characterization of Nakano positivity for a Hermitian holomorphic vector bundle seems not to be available. In this paper, we find an unexpected way to approach the characterization problem by considering the converse problem of the above famous L^2 existence theorem. In particular, we give an answer to the characterization problem, proving that the converse proposition of the L^2 existence theorem holds. Another purpose of this paper is to present a global picture of various converses of L^p-theory for $\bar\partial$.

To state our results, we first introduce some notions which play a fundamental role throughout the paper.

Definition 1.1 Let (X, ω) be a Kähler manifold of dimension n with a Kähler metric ω which admits a positive Hermitian holomorphic line bundle, (E, h) be a (singular) Hermitian vector bundle (maybe of infinite rank) over X, and $p > 0$.

(1) We call (E, h) satisfies *the optimal L^p-estimate* if for any positive Hermitian holomorphic line bundle (A, h_A) on X, for any $f \in \mathcal{C}^\infty_c(X, \wedge^{n,1}T^*_X \otimes E \otimes A)$

with $\bar\partial f = 0$, there is $u \in L^p(X, \wedge^{n,0}T_X^* \otimes E \otimes A)$, satisfying $\bar\partial u = f$ and

$$\int_X |u|^p_{h\otimes h_A} dV_\omega \le \int_X \langle B^{-1}_{A,h_A} f, f\rangle^{\frac{p}{2}} dV_\omega,$$

provided that the right hand side is finite, where $B_{A,h_A} = [i\Theta_{A,h_A} \otimes Id_E, \Lambda_\omega]$.

(2) We call (E, h) satisfies *the multiple coarse L^p-estimate* if for any $m \ge 1$, for any positive Hermitian holomorphic line bundle (A, h_A) on X, and for any $f \in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^* \otimes E^{\otimes m} \otimes A)$ with $\bar\partial f = 0$, there is $u \in L^p(X, \wedge^{n,0}T_X^* \otimes E^{\otimes m} \otimes A)$, satisfying $\bar\partial u = f$ and

$$\int_X |u|^p_{h^{\otimes m}\otimes h_A} dV_\omega \le C_m \int_X \langle B^{-1}_{A,h_A} f, f\rangle^{\frac{p}{2}} dV_\omega,$$

provided that the right hand side is finite, where C_m are constants satisfying the growth condition $\frac{1}{m}\log C_m \to 0$ as $m \to \infty$.

Definition 1.2 Let (E, h) be a holomorphic vector bundle (maybe of infinite rank) over a domain $D \subset \mathbb{C}^n$ with a singular Finsler metric h, and $p > 0$.

(1) We call (E, h) satisfies *the optimal L^p-extension* if for any $z \in D$, and $a \in E_z$ with $|a| = 1$, and any holomorphic cylinder P with $z + P \subset D$, there is $f \in H^0(z + P, E)$ such that $f(z) = a$ and

$$\frac{1}{\mu(P)} \int_{z+P} |f|^p \le 1,$$

where $\mu(P)$ is the volume of P with respect to the Lebesgue measure. (Here by a holomorphic cylinder we mean a domain of the form $A(P_{r,s})$ for some $A \in U(n)$ and $r, s > 0$, with $P_{r,s} = \{(z_1, z_2, \ldots, z_n) : |z_1|^2 < r^2, |z_2|^2 + \cdots + |z_n|^2 < s^2\}$).

(2) We call (E, h) satisfies *the multiple coarse L^p-extension* if for any $z \in D$, and $a \in E_z$ with $|a| = 1$, and any $m \ge 1$, there is $f_m \in H^0(D, E^{\otimes m})$ such that $f_m(z) = a^{\otimes m}$ and satisfies the following estimate:

$$\int_D |f_m|^p \le C_m,$$

where C_m are constants independent of z and satisfying the growth condition $\frac{1}{m}\log C_m \to 0$ as $m \to \infty$.

(See Sect. 2.4 for the definition of singular Finsler metrics.)

Remark 1.1 Similarly, one can define the optimal (resp. multiple coarse) L^p-extension condition for a Hermitian holomorphic vector bundle (E, h) over a Kähler manifold X. But it is clear that if (E, h) satisfies the optimal (resp. multiple coarse) L^p-extension on X, then it admits the same condition when restricted on any open set D of X. So we just focus on bounded domains in Definition 1.2. However, it is not the case for the optimal (resp. multiple coarse) L^p-estimate condition since a positive Hermitian line bundle over an open domain in X may not extend to X.

The notions defined in Definitions 1.1 and 1.2 for trivial line bundles were studied in [12]. The multiple coarse L^p-extension condition for vector bundles with singular Finsler metrics was introduced in [13], and the multiple coarse L^2-estimate condition for Hermitian vector bundles was introduced in [22], which was named as the twisted Hörmander condition there. A concept called "minimal extension property", similar to the optimal L^2-extension condition, was introduced in [20]. Instead of holomorphic cylinders, embedded holomorphic discs was used in the definition in [20].

The first and the main result is the following characterization of Nakano positivity in terms of the optimal L^2-estimate condition.

Theorem 1.1 *Let (X,ω) be a Kähler manifold of dimension n which admits a positive line bundle, (E,h) be a smooth Hermitian vector bundle over X, and $\theta \in C^0(X, \Lambda^{1,1}T_X^*\otimes End(E))$ such that $\theta^*=\theta$. If for any $f\in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^*\otimes E\otimes A)$ with $\bar\partial f=0$, and any positive Hermitian line bundle (A,h_A) on X with $i\Theta_{A,h_A}\otimes Id_E+\theta>0$ on supp f, there is $u\in L^2(X,\wedge^{n,0}T_X^*\otimes E\otimes A)$, satisfying $\bar\partial u=f$ and*

$$\int_X |u|^2_{h\otimes h_A} dV_\omega \le \int_X \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A} dV_\omega,$$

provided that the right hand side is finite, where $B_{h_A,\theta}=[i\Theta_{A,h_A}\otimes Id_E+\theta, \Lambda_\omega]$, then $i\Theta_{E,h}\ge\theta$ in the sense of Nakano.

On the other hand, if in addition X is assumed to have a complete Kähler metric, the above condition is also necessary for that $i\Theta_{E,h}\ge\theta$ in the sense of Nakano.

In particular, if (E,h) satisfies the optimal L^2-estimate, then (E,h) is Nakano semi-positive.

We prove Theorem 1.1 by connecting $\Theta_{E,h}$ with the optimal L^2-estimate condition through the Bochner–Kodaira–Nakano identity, and then using a localization technique to produce a contradiction if $i\Theta_{E,h}\ge\theta$ is assumed to be not true.

As for the Griffiths positivity for singular hermitian or Finsler holomorphic vector bundles, we have the following results.

Theorem 1.2 *Let (X,ω) be a Kähler manifold, which admits a positive Hermitian holomorphic line bundle, and (E,h) be a holomorphic vector bundle over X with a continuous Hermitian metric h. If (E,h) satisfies the multiple coarse L^p-estimate for some $p>1$, then (E,h) is Griffiths semi-positive.*

Remark 1.2 If X admits a strictly plurisubharmonic function, it is obvious from the proof that, in Theorems 1.1 and 1.2, we can take A to be the trivial bundle (with nontrivial metrics).

The case that $p=2$ and h is Hölder continuous for Theorem 1.2 was proved in [22], by showing that the multiple coarse L^2-estimate condition implies the the multiple coarse L^2-extension condition and then applying [13, Theorem 1.2]. The case that E is a trivial line bundle was proved in [12]. The proof of Theorem 1.2 is based on the technique in [22].

Theorem 1.3 *Let E be a holomorphic vector bundle over a domain $D\subset\mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local* *holomorphic section s of E^*. If (E, h) satisfies the optimal L^p-extension for some $p > 0$, then (E, h) is Griffiths semi-positive.*

A first result in this direction was given by Guan–Zhou in [17], where they showed that Berndtsson's plurisubharmonic variation of the relative Bergman kernels [1] can be deduced from the optimal L^2-extension condition. By developing Guan–Zhou's method, Hacon–Popa–Schnell in [20] proved that a Hermitian vector bundle (E, h) with singular Hermitian metric is Griffiths semi-positive if it satisfies the so called minimal extension property, a notion defined there as mentioned above. Theorem 1.3 is proved by combining the ideas in [17,19,20] and a lemma in [12].

Theorem 1.4 *Let E be a holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local holomorphic section s of E^*. If (E, h) satisfies the multiple coarse L^p-extension for some $p > 0$, then (E, h) is Griffiths semi-positive.*

Theorem 1.4 was originally proved in [13]. In this paper, we give a new proof based on the idea in the proof of [12, Theorem 1.5].

We should emphasize that Theorems 1.1–1.4 are true for vector bundles of infinite rank, as well as for those of finite rank.

Remark 1.3 In applications, it is possible to prove that $(E, e^{\phi}h)$ satisfies the optimal L^p-extension for some $\phi \in C^0(D)$, by Theorem 1.4, which implies that

$$i\Theta_E \geq i\partial\bar{\partial}\phi \otimes Id_E$$

in the sense that $i\partial\bar{\partial}\log|s|^2_{h^*} \geq i\partial\bar{\partial}\phi$ in the sense of currents, for any nonvanishing local holomorphic section s of E^*.

We now explain why Theorems 1.1–1.4 can be roughly viewed as studies of the converses of L^2-estimate for $\bar{\partial}$ and Ohsawa–Takegoshi type L^2-extensions.

Let (X, ω) be a weakly pseudoconvex Kähler manifold, and (E, h) be a Hermitian holomorphic vector bundle over X. If the curvature of (E, h) is Nakano semi-positive, then (E, h) satisfies the optimal L^2-estimate and the multiple coarse L^2-estimate by works of Hörmander [21] and Demailly [9]. Combining Theorem 1.1, we see in this setting that *Nakano positivity for (E, h) is equivalent to satisfying the optimal L^2-estimate*.

Let (E, h) be a Hermitian holomorphic vector bundle over a bounded pseudoconvex domain D. If the curvature of (E, h) is Nakano semi-positive, then (E, h) satisfies the multiple coarse L^2-extension by Ohsawa–Takegoshi [30], Manivel [27] and Demailly [11], and satisfies the optimal L^2-extension by Błocki [6] and Guan–Zhou [15–19]. Theorems 1.3 and 1.4 show that the optimal L^2-extension condition and the multiple coarse L^2-extension condition imply Griffiths positivity of (E, h).

The second part of this paper is to apply the above theorems to study the curvature positivity of direct image sheaves of twisted relative canonical bundles associated to holomorphic fibrations, which is an active topic of extensive study in recent years (see [1–5,7,13,17,20,24,31,33,34]). The main novelty here is that Theorem 1.1 provides a natural explanation and a very simple unified proof of the Nakano positivity of direct image bundles associated to families of both Stein manifolds and compact Kähler manifolds, with an effective estimate of the lower bound of the curvatures. One may expect more new applications of Theorem 1.1.

We first consider a family of bounded domains. Let $\Omega = U \times D \subset \mathbb{C}^n_t \times \mathbb{C}^m_z$ be a bounded pseudoconvex domain and $p : \Omega \to U$ be the natural projection. Let h be a Hermitian metric on the trivial bundle $E = \Omega \times \mathbb{C}^r$ that is C^2-smooth to $\overline{\Omega}$. For $t \in U$, let

$$F_t := \left\{ f \in H^0(D, E|_{\{t\}\times D}) : \|f\|_t^2 := \int_D |f|^2_{h_t} < \infty \right\}$$

and $F := \coprod_{t\in U} F_t$. Since h is continuous to $\overline{\Omega}$, F_t are equal for all $t \in U$ as vector spaces. We may view $(F, \|\cdot\|)$ as a trivial holomorphic Hermitian vector bundle of infinite rank over U.

Theorem 1.5 *Let θ be a continuous real $(1,1)$-form on U such that $i\Theta_E \geq p^*\theta \otimes Id_E$, then $i\Theta_F \geq \theta \otimes Id_F$ in the sense of Nakano. In particular, if $i\Theta_E > 0$ in the sense of Nakano, then $i\Theta_F > 0$ in the sense of Nakano.*

Let $\pi : X \to U$ be a proper holomorphic submersion from a Kähler manifolds X of complex dimension $m+n$, to a bounded pseudoconvex domain U, and (E, h) be a Hermitian holomorphic vector bundle over X, with Nakano semi-positive curvature. From the Ohsawa–Takegoshi extension theorem, the direct image $F := \pi_*(K_{X/U} \otimes E)$ is a vector bundle, whose fiber over $t \in U$ is $F_t = H^0(X_t, K_{X_t} \otimes E|_{X_t})$. There is a hermitian metric $\|\cdot\|$ on F induced by h: for any $u \in F_t$,

$$\|u(t)\|_t^2 := \int_{X_t} c_m u \wedge \bar{u},$$

where $m = \dim X_t$, $c_m = i^{m^2}$, and $u \wedge \bar{u}$ is the composition of the wedge product and the inner product on E. So we get a Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U.

Theorem 1.6 *The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U defined above satisfies the optimal L^2-estimate. Moreover, if $i\Theta_E \geq p^*\theta \otimes Id_E$ for a continuous real $(1,1)$-form θ on U, then $i\Theta_F \geq \theta \otimes Id_F$ in the sense of Nakano.*

Theorem 1.5 in the case that E is a line bundle is a result of Berndtsson [2, Theorem 1.1], the case for vector bundles E without lower bound estimate was proved by Raufi [32, Theorem 1.5] with the same method of Berndtsson. Theorem 1.6 in the case that L is a line bundle is due to Berndtsson [2], and the case for vector bundles was proved in [26] and [25] by developing the method of Berndtsson.

Our method to Theorems 1.5 and 1.6 is very different. In fact, taking Theorem 1.1 for granted, one can clearly see why Theorems 1.5 and 1.6 should be true, since it is obvious that the bundles F in Theorems 1.5 and 1.6 satisfy the optimal L^2-estimate by Hörmander's L^2-estimate for $\bar{\partial}$ and Fubini's theorem.

In this paper, we also provide some new methods to show that $(F, \|\cdot\|)$ is Griffiths semi-positive, via Theorems 1.2, 1.3, and 1.4. By applying the tensor-power technique introduced in [13], we show that $(F, \|\cdot\|)$ satisfies the multiple coarse L^2-estimate; by applying the Ohsawa–Takegoshi extension theorem with optimal estimate for vector bundles ([17,35]), we show that $(F, \|\cdot\|)$ satisfies the optimal L^2-extension; and by applying the tensor-power technique mentioned above and the Ohsawa–Takegoshi extension theorem, we show that $(F, \|\cdot\|)$ satisfies the multiple coarse L^2-extension.

2 Preliminaries

2.1 An extension property of Hermitian metrics on a line bundle

In this section, we present a basic property of Kähler manifolds, which admit positive Hermitian holomorphic line bundles.

Proposition 2.1 *Let X be a Kähler manifold, which admits a positive Hermitian holomorphic line bundle, and (A, h_A) be a positive Hermitian holomorphic line bundle over X. Let $(U \subset X, z = (z_1, \ldots, z_n))$ be a coordinate chart on X, such that $A|_U$ is trivial. Then for any smooth strictly plurisubharmonic function ψ on U, for any point $x \in U$, there is a neighbourhood $V \subset U$ of x, and a positive Hermitian metric $\tilde{h}_A$ on the line bundle A, such that $\tilde{h}_A = e^{-\tilde{\psi}}$ on U with $\tilde{\psi}|_V = \psi|_V$.*

Proof Assume that $h_A|_U = e^{-\phi}$ for some smooth strictly plurisubharmonic function ϕ on U. We may assume that $z(x) = 0$ and the unit ball $B := B_1$ is contained in U. We may assume that $\phi > 0$ on B. Let χ be a cut-off function on B, such that χ is identically equal to 1 on $B_{1/4}$ and vanishes outside $B_{3/4}$. Let $\tilde{\phi} := \phi + \frac{\chi \log(\|z\|^2)}{m} + c$ on B, where $m, c \gg 1$ is an integer such that ϕ_m is strictly p.s.h on B and $\tilde{\phi} > \psi$ on ∂B.

Now we define a function $\tilde{\psi}$ on U as follows:

$$\tilde{\psi} = \begin{cases} \tilde{\phi}, & \text{outside } B; \\ \max_\epsilon\{\tilde{\phi}, \psi\}, & \text{on } B. \end{cases}$$

As $\tilde{\phi}(x) = -\infty < \psi(x)$, and both $\tilde{\phi}$ and ψ are continuous. Then for $0 < \epsilon \ll 1$, there is a neighborhood $V \subset B$ of x, such that $\tilde{\psi} = \psi$ on V, $\tilde{\psi}$ is strictly p.s.h on B. So $\tilde{\psi}$ gives a positive Hermitian metric on $A|_U$ which coincides with $e^{-c}h_A$ on $U \backslash B$. Define $\tilde{h}_A|_U = e^{-\tilde{\psi}}$ and $\tilde{h}_A = e^{-c}h_A$ on $X \backslash U$. $\tilde{h}_A$ is a positive Hermitian metric on A and $\tilde{h}_A|_V = e^{-\psi}$. □

2.2 Basics of Hermitian holomorphic vector bundles

Let (X, ω) be a complex manifold of complex dimension n, equipped with a Hermitian metric ω, and (E, h) be a Hermitian holomorphic vector bundle of rank r over X. In this subsection, we assume $r < \infty$.

Let $D = D' + \bar{\partial}$ be the Chern connection of (E, h), and $\Theta_{E,h} = [D', \bar{\partial}] = D'\bar{\partial} + \bar{\partial}D'$ be the Chern curvature tensor. Denote by $(e_1, \ldots, e_r)$ an orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1, \dots, z_n)$, and

$$i\Theta_{E,h} = i \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_\mu, \quad \bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}.$$

To $i\Theta_{E,h}$ corresponds a natural Hermitian form $\theta_{E,h}$ on $TX \otimes E$ defined by

$$\theta_{E,h}(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(x) u_{j\lambda} \bar{u}_{k\mu}, \qquad u \in T_x X \otimes E_x. \tag{1}$$

Definition 2.1
- E is said to be Nakano positive (resp. Nakano semi-positive) if $\theta_{E,h}$ is positive (resp. semi-positive) definite as a Hermitian form on $TX \otimes E$, i.e. for every $u \in TX \otimes E$, $u \neq 0$, we have

$$\theta(u,u) > 0 \quad (\text{resp.} \geq 0).$$

- E is said to be Griffiths positive (resp. Griffiths semi-positive) if for any $x \in X$, all $\xi \in T_x X$ with $\xi \neq 0$, and $s \in E_x$ with $s \neq 0$, we have

$$\theta(\xi \otimes s, \xi \otimes s) > 0 \quad (\text{resp.} \geq 0).$$

- Nakano negative (resp. Nakano semi-negative) and Griffiths negative (resp. Griffiths semi-negative) are similarly defined by replacing > 0 (resp. ≥ 0) by < 0 (resp. ≤ 0) in the above definitions respectively.

Remark 2.1 The following are basic facts about Griffiths positivity and Nakano positivity.

- It is a well-known fact that, a Hermitian holomorphic vector bundle (E, h) is Griffiths positive (resp. semi-positive) if and only if (E^*, h^*) is Griffiths negative (resp. semi-negative). However, Nakano positivity does not share this duality condition, see [8, Chapter VII, Page 339, Example 6.8] for an example.
- It is a fact that, Griffiths positivity can be explained as a several complex variables property, see Definition 2.5 in Sect. 2.4. However, Nakano positivity does not have such a characterization.

Remark 2.2 Let (E_1, h_1) and (E_2, h_2) be two Hermitian holomorphic vector bundles over a complex n-dimensional manifold X. It is a basic fact that the Chern connection $D_{E_1\otimes E_2}$ of $(E_1 \otimes E_2, h_1 \otimes h_2)$ is just $D_{E_1} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes D_{E_2}$, and we have the following Chern curvature formula

$$\Theta_{E_1\otimes E_2, h_1\otimes h_2} = \Theta_{E_1,h_1} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes \Theta_{E_2,h_2}.$$

From (1) and Remark 2.2, we get the following

Lemma 2.1 *Let (E_1, h_1) and (E_2, h_2) be two Hermitian holomorphic vector bundles over a complex manifold X. Let $(E, h) := (E_1 \otimes E_2, h_1 \otimes h_2)$. Then if (E_1, h_1) and (E_2, h_2) are Nakano positive (resp. Nakano semi-positive), then (E, h) is Nakano positive (resp. Nakano semi-positive).*

Lemma 2.2 *Let $\pi_i : X_i \to Y$ be two holomorphic submersions for $j = 1, 2$. Let $X \subset X_1 \times X_2$ be the fiberwise product of X_1 and X_2 with respect to π_1 and π_2, and $pr_j : X \to X_j$ be the natural projections from X to X_j for $j = 1, 2$. Let (E_1, h_1) and (E_2, h_2) be two Hermitian holomorphic vector bundles over X_1 and X_2, respectively. Denote by (E, h) the Hermitian holomorphic vector bundle $(pr_1^* E_1 \otimes pr_2^* E_2, pr_1^* h_1 \otimes pr_2^* h_2)$ on X. If (E_1, h_1) and (E_2, h_2) are Nakano positive (resp. Nakano semi-positive), then E is also Nakano positive (resp. Nakano semi-positive).*

For any $u \in \Lambda^{p,q} T_X^* \otimes E$, we consider the global L^2-norm

$$\|u\|^2 = \int_X |u|^2_{\omega,h} dV_\omega,$$

where $|u|_{\omega,h}$ is the pointwise Hermitian norm and $dV_\omega = \omega^n/n!$ is the volume form on X. This L^2-norm induces an L^2-inner product on $\Lambda^{p,q} T_X^* \otimes E$, and thus we can define D'^* and $\bar\partial^*$ operators as the (formal) adjoint of D' and $\bar\partial$, respectively. Let

$$\Delta' = D'D'^* + D'^*D', \quad \Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

be the corresponding D' and $\bar\partial$-Laplace operators.

Lemma 2.3 (Bochner–Kodaira–Nakano identity, see [8]) *Let (X, ω) be a Kähler manifold, (E, h) be a Hermitian vector bundle over X. The complex Laplacian operators $\Delta' = D'D'^* + D'^*D'$ and $\Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting on E-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [i\Theta_{E,h}, \Lambda_\omega].$$

Let us say more on the Hermitian operator $[i\Theta_{E,h}, \Lambda_\omega]$. Let $x_0 \in X$ and $(z_1, \ldots, z_n)$ be local coordinates centered at x_0, such that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of TX at x_0. One can write

$$\omega = i \sum dz_j \wedge d\bar z_j + O(\|z\|),$$

and

$$i\Theta_{E,h}(x_0) = i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar z_k \otimes e_\lambda^* \otimes e_\mu,$$

where $(e_1, \ldots, e_r)$ is an orthonormal basis of E_{x_0}. Let $u = \sum u_{K,\lambda} dz \wedge d\bar z_K \otimes e_\lambda \in \Lambda^{n,q} T_X^* \otimes E$, where $dz = dz_1 \wedge \cdots \wedge dz_n$. In [8, Chapter VII, Page 341, (7.1)], it is computed that

$$\langle[i\Theta_{E,h},\Lambda_\omega]u,u\rangle=\sum_{|S|=q-1}\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}u_{jS,\lambda}\bar{u}_{kS,\mu}. \tag{2}$$

In particular, if $q=1$, (2) becomes

$$\langle[i\Theta_{E,h},\Lambda_\omega]u,u\rangle=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}u_{j,\lambda}\bar{u}_{k,\mu}. \tag{3}$$

Comparing (1) and (3), we obtain the following

Lemma 2.4 *Let (X,ω) be a Kähler manifold, (E,h) be a Hermitian vector bundle over X. Then (E,h) is Nakano positive (resp. semi-positive) if and only if the Hermitian operator $[i\Theta_{E,h},\Lambda_\omega]$ is positive definite (resp. semi-positive definite) on $\Lambda^{n,1}T_X^*\otimes E$.*

2.3 Basic concepts and conditions of Hermitian vector bundles of infinite rank

In this subsection, we will briefly discuss some concepts and basic conditions of Hermitian holomorphic vector bundles of infinite rank, and explain why the above mentioned Bochner–Kodaira–Nakano identity also holds in this framework.

Let H be a Hilbert space (separable, say over $\mathbb{C}$) with inner product $(,)$. Let $U\subset\mathbb{R}^n$ be open. Let $f:U\to H$ be a map. If

$$\frac{\partial f}{\partial x_j}=\lim_{\Delta x_j\to 0}\frac{f(x_1,\ldots,x_j+\Delta x_j,\ldots,x_n)-f(x_1,\ldots,x_n)}{\Delta x_j}\in H$$

exists and is continuous on U for any $j=1,2,\ldots,n$, f is called of C^1. We say f is of C^r if all partial derivatives $\frac{\partial^r f}{\partial x_1^{r1}\cdots\partial x_n^{rn}}:U\to H$ of order r exist and continuous, and f is smooth if f is of C^r for any r.

Let $\{e_\lambda\}_{\lambda=1}^\infty$ be an orthonormal basis of H. Then a map $f:U\to H$ can be written as $(f_1,f_2,\ldots)$, where f_λ are functions on U such that

$$\|f(z)\|^2=\sum_\lambda|f_\lambda(z)|^2.$$

If f is continuous, by Dini's theorem, one can see that the series $\sum_\lambda|f_\lambda(z)|^2$ converges uniformly locally on U to $\|f(z)\|^2$; and if f is smooth, then $\sum_\lambda|\frac{\partial^r f_\lambda}{\partial x_1^{r1}\cdots\partial x_n^{rn}}|^2$ locally uniformly converges to $\|\frac{\partial^r f}{\partial x_1^{r1}\cdots\partial x_n^{rn}}\|^2$.

Now assume $U\subset\mathbb{C}^n$ be an open set. A map $f:U\to H$ is called holomorphic if f is smooth and satisfies the Cauchy–Riemann equation

$$\frac{\partial}{\partial\bar{z}_j}f:=\frac{1}{2}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right)f=0,\ \ j=1,\ldots,n.$$

We now consider holomorphic vector bundles of infinite rank with H as the model of the fibers. In the present paper, we will focus on local conditions of holomorphic vector bundles, so we just consider the trivial bundle $E := U \times H \to U$, here we view H as a locally convex topological complex vector space.

Definition 2.2 A Hermitian metric on E is a map

$$h : U \to Herm(H)$$

which satisfies the following conditions:

(1) h is smooth, and
(2) $h(z) \geq \delta(z) Id$ for some positive continuous function δ on U,

where $Herm(H)$ is the space of self-adjoint bounded operators on H.

Given h as above, we get a smooth family of inner products on H as

$$(u, v)_z = (h(z)u, v), \ z \in U.$$

So our definition of the Hermitian metric on E matches to the definition of Hermitian metrics for holomorphic vector bundles of finite rank.

Given a Hermitian metric h on E, we can define a unique connection $D = D' + \bar{\partial}$ on E which is compatible with h and whose (0, 1)-part is $\bar{\partial}$, as in the finite rank case. We view a section of E as a map from U to H. Assume u is a smooth section of E, and $v \in H$ viewed as a constant section of E, then from the condition that

$$\partial(hu, v) = (hD'u, v),$$

we get

$$D'u = h^{-1}\partial(hu).$$

This formula shows that $D'u$ is a smooth section of $\Lambda^{1,0} T^*U \otimes E$.

The curvature operator of (E, h) is give by

$$\Theta_{E,h} = [D', \bar{\partial}],$$

which is an operator that maps smooth sections of E to smooth sections of $\Lambda^{1,1} T^*U \otimes E$. In the same way as in the case of vector bundles with finite rank, Nakano positivity and Griffiths positivity can be defined for (E, h).

We now show that, at any point $z_0 \in U$, the metric h coincides with a flat metric up to order 1. To show this, we may assume $z_0 = 0$ and $h_0 = Id$. Let $h_j = \frac{\partial h}{\partial z_j}(0), \ j = 1, \ldots, n$, and define

$$\tilde{e}_\lambda = e_\lambda - \sum_j z_j h_j(e_\lambda),$$

then

$$(\tilde{e}_\lambda, \tilde{e}_\mu)_z = \delta_{\lambda\mu} + O(\|z\|^2),$$

where the bound for $O(\|z\|^2)$ is uniform for λ, μ.

With the above preparation, following the line of the proof of [8, Theorem 1.1, Theorem 1.2, Chapter VII, §1], we see that the Bochner–Kodaira–Nakano identity also holds for (E, h).

2.4 Singular Finsler metrics on holomorphic vector bundles

In this subsection, we recall the notions of singular Finsler metrics on holomorphic vector bundles and positively curved singular Finsler metrics on coherent analytic sheaves, introduced in [13], see also [14].

Definition 2.3 Let $E \to X$ be a holomorphic vector bundle over a complex manifolds X. A (singular) Finsler metric h on E is a function $h : E \to [0, +\infty]$, such that $|cv|_h^2 := h(cv) = |c|^2 h(v)$ for any $v \in E$ and $c \in \mathbb{C}$.

Definition 2.4 For a singular Finsler metric h on E, its dual Finsler metric h^* on the dual bundle E^* of E is defined as follows. For $f \in E_x^*$, the fiber of E^* at $x \in X$, $|f|_{h^*}$ is defined to be 0 if $|v|_h = +\infty$ for all nonzero $v \in E_x$; otherwise,

$$|f|_{h^*} := \sup\{|f(v)| : v \in E_x, |v|_h \leq 1\} \leq +\infty.$$

Definition 2.5 Let (E, h) be a holomorphic vector bundle over a complex manifold X, equipped with a singular Finsler metric h. We call h is negatively curved (in the sense of Griffiths) if for any local holomorphic section s of E, the function $\log |s|_h^2$ is plurisubharmonic, and we call h is positively curved (in the sense of Griffiths) if its dual metric h^* is negatively curved.

Definition 2.6 Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold X. Let $Z \subset X$ be an analytic subset of X such that $\mathcal{F}|_{X\setminus Z}$ is locally free. A positively curved singular Finsler metric h on $\mathcal{F}$ is a singular Finsler metric on the holomorphic vector bundle $\mathcal{F}|_{X\setminus Z}$, such that for any local holomorphic section s of the dual sheaf $\mathcal{F}^*$ on an open set $U \subset X$, the function $\log |s|_{h^*}$ is plurisubharmonic on $U\setminus Z$, and can be extended to a plurisubharmonic function on U.

Remark 2.3 Suppose that $\log |s|_{h^*}$ is p.s.h. on $U\setminus Z$. It is well-known that if $\mathrm{codim}_{\mathbb{C}}(Z) \geq 2$ or $\log |s|_{H^*}$ is locally bounded above near Z, then $\log |s|_{h^*}$ extends across Z to U uniquely as a p.s.h function. Definition 2.6 matches Definition 2.3 and Definition 2.5 if $\mathcal{F}$ is a vector bundle.

2.5 L^2 theory for $\bar{\partial}$

In this section, we recall Hörmander's L^2-estimate for $\bar{\partial}$ and Ohsawa–Takegoshi type L^2-extension of holomorphic sections of the holomorphic vector bundles.

We first clarify some notions and notations. Let H be a Hilbert space with an inner product $(\cdot,\cdot)$, and $A : H \to H$ be a bounded semi-positive self-adjoint operator with closed range ImA. The we have an orthogonal decomposition

$$H = ImA \oplus \ker A$$

and $A|_{ImA} : ImA \to ImA$ is a linear isomorphism. In the remaining of the paper, we always denote $A|_{ImA}^{-1}$ by A^{-1}, as in general references about complex geometry, and define $(A^{-1}v, v) = +\infty$ if $v \notin ImA$.

Lemma 2.5 (c.f. [8, Theorem 4.5]) *Let (X,ω) be a complete Kähler manifold, with a Kähler metric which is not necessarily complete. Let (E,h) be a Hermitian holomorphic vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta_{E,h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\Lambda^{p,q}T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ satisfying $\bar\partial g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^* \otimes E)$ such that $\bar\partial f = g$ and*

$$\int_X |f|^2 dV_\omega \leq \int_X \langle B^{-1}g, g\rangle dV_\omega.$$

The following L^2-extension theorem for Kähler families is due to Zhou–Zhu [35, Theorem 1.1]. The same result for projective families is due to Guan–Zhou [15–17].

Lemma 2.6 ([35, Theorem 1.1]) *Let $\pi : X \to B$ be a proper holomorphic submersion from a complex n-dimensional Kähler manifold (X,ω) onto a unit ball in $\mathbb{C}^m$. Let $(E, h = h_E)$ be a Hermitian holomorphic vector bundle over X, such that the curvature $i\Theta_{E,h_E} \geq 0$ in the sense of Nakano. Let $t_0 \in B$ be an arbitrarily fixed point. Then for every section $u \in H^0(X_{t_0}, K_{X_{t_0}} \otimes E|_{X_{t_0}})$, such that*

$$\int_{X_{t_0}} |u|^2_{\omega,h} dV_{\omega_{X_{t_0}}} < +\infty,$$

there is a section $\widetilde{u} \in H^0(X, K_X \otimes E)$, such that $\widetilde{u}|_{X_{t_0}} = \widetilde{u} \wedge dt$, with the following L^2-estimate

$$\int_X |\widetilde{u}|^2_{\omega,h} dV_{X,\omega} \leq \mu(B) \int_{X_{t_0}} |u|^2_{\omega,h} dV_{\omega_{X_{t_0}}},$$

where $dt = dt_1 \wedge \cdots \wedge dt_m$, and $t = (t_1, \ldots, t_m)$ are the holomorphic coordinates on $\mathbb{C}^m$, and $\mu(B)$ is the volume of the unit ball in $\mathbb{C}^m$ with respect to the Lebesgue measure on $\mathbb{C}^m$.

Remark 2.4 We take $R(t) = e^{-t}$, $\alpha_0 = \alpha_1 = 0$, and $\psi = m \log \|t - t_0\|^2$ in [35, Theorem 1.1], and from [17, Lemma 4.14], [35, Remark 1.2], we can get the precise form of Theorem 2.6.

3 Positivities of holomorphic vector bundles via L^p-conditions for $\bar\partial$

The aim of this section is to prove Theorems 1.1–1.4.

3.1 Characterizations of Nakano positivity in term of optimal L^2-estimate condition

Theorem 3.1 (= Theorem 1.1) *Let (X, ω) be a Kähler manifold of dimension n with a Kähler metric ω, which admits a positive Hermitian holomorphic line bundle, (E, h) be a smooth Hermitian vector bundle over X, and $\theta \in C^0(X, \Lambda^{1,1}T_X^* \otimes End(E))$ such that $\theta^* = \theta$. If for any $f \in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^* \otimes E \otimes A)$ with $\bar\partial f = 0$, and any positive Hermitian line bundle (A, h_A) on X with $i\Theta_{A,h_A} \otimes Id_E + \theta > 0$ on supp f, there is $u \in L^2(X, \wedge^{n,0}T_X^* \otimes E \otimes A)$, satisfying $\bar\partial u = f$ and*

$$\int_X |u|^2_{h\otimes h_A} dV_\omega \leq \int_X \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A} dV_\omega,$$

provided that the right hand side is finite, where $B_{h_A,\theta} = [i\Theta_{A,h_A} \otimes Id_E + \theta, \Lambda_\omega]$, then $i\Theta_{E,h} \geq \theta$ in the sense of Nakano. On the other hand, if in addition X is assumed to have a complete Kähler metric, the above condition is also necessary for that $i\Theta_{E,h} \geq \theta$ in the sense of Nakano. In particular, if (E, h) satisfies the optimal L^2-estimate, then (E, h) is Nakano semi-positive.

Proof The second statement is a corollary of Theorem 2.5. We now give the proof of the first statement. We give the proof in the case that θ is $\mathcal{C}^1$, and the general case follows the proof by an approximation argument.

To illustrate the main idea more clearly, we may assume that there is a smooth strictly plurisubharmonic function on X, which corresponds to the existence of a positive Hermitian trivial holomorphic line bundle on X. For general case, the same proof goes through by replacing data related to $e^{-\psi}$ by h_A, and using Proposition 2.1.

Let ψ be any smooth strictly plurisubharmonic function on X. By assumption, we can solve the equation $\bar\partial u = f$ for any $\bar\partial$-closed $f \in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^* \otimes E)$, with the estimate

$$\int_X |u|^2 e^{-\psi} dV_\omega \leq \int_X \langle B^{-1}_{\psi,\theta} f, f\rangle e^{-\psi} dV_\omega,$$

where $B_{\psi,\theta} := [i\partial\bar\partial\psi \otimes Id_E + \theta, \Lambda_\omega]$. For any $\alpha \in \mathcal{C}_c^\infty(X, \wedge^{n,1}T_X^* \otimes E)$, we have

$$\begin{aligned} |\langle\langle \alpha, f\rangle\rangle_\psi| &= |\langle\langle \alpha, \bar\partial u\rangle\rangle_\psi| \\ &= |\langle\langle \bar\partial^*\alpha, u\rangle\rangle_\psi| \\ &\leq ||u||_\psi ||\bar\partial^*\alpha||_\psi, \end{aligned}$$

where $\bar\partial^*$ is the adjoint of $\bar\partial$ with respect to ω, $e^{-\psi}h$.

From Lemma 2.3, we obtain

$$
\begin{aligned}
&|\langle\langle \alpha, f\rangle\rangle_\psi|^2 \\
&\quad\le \int_X \langle B_{\psi,\theta}^{-1} f, f\rangle e^{-\psi} dV_\omega \\
&\quad\quad \times \left(||D'\alpha||_\psi^2 + ||D'^*\alpha||_\psi^2 + \langle\langle [i\Theta_{E,h} + i\partial\bar{\partial}\psi \otimes Id_E, \Lambda_\omega]\alpha, \alpha\rangle\rangle_\psi - ||\bar{\partial}\alpha||_\psi^2 \right) \\
&\quad\le \int_X \langle B_{\psi,\theta}^{-1} f, f\rangle e^{-\psi} dV_\omega \times \left(\langle\langle [i\Theta_{E,h} + i\partial\bar{\partial}\psi \otimes Id_E, \Lambda_\omega]\alpha, \alpha\rangle\rangle_\psi + ||D'^*\alpha||_\psi^2 \right),
\end{aligned} \tag{4}
$$

where D' is the $(1, 0)$ part of the Chern connection on E with respect to the metric $e^{-\psi} h$.

Let $\alpha = B_{\psi,\theta}^{-1} f$, i.e., $f = B_{\psi,\theta}\alpha$. Then inequality (4) becomes

$$
\begin{aligned}
&\left(\langle\langle B_{\psi,\theta}\alpha, \alpha\rangle\rangle_\psi\right)^2 \\
&\quad \le \langle\langle B_{\psi,\theta}\alpha, \alpha\rangle\rangle_\psi \left(\langle\langle [i\Theta_{E,h}, \Lambda_\omega]\alpha, \alpha\rangle\rangle_\psi + \langle\langle B_{\psi,0}\alpha, \alpha\rangle\rangle_\psi + ||D'^*\alpha||_\psi^2 \right).
\end{aligned}
$$

Therefore, we can get

$$
\langle\langle [i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha, \alpha\rangle\rangle_\psi + ||D'^*\alpha||_\psi^2 \ge 0. \tag{5}
$$

We argue by contradiction. Suppose that $i\Theta_{E,h} - \theta$ is not Nakano semi-positive on X. By Lemma 2.4, there is $x_0 \in X$ and $\xi_0 \in \Lambda^{n,1} T_{X,x_0}^* \otimes E_{x_0}$ such that $|\xi_0| = 1$ and $\langle [i\Theta_{E,h} - \theta, \Lambda_\omega]\xi_0, \xi_0\rangle = -2c$ for some $c > 0$.

Let $(U; z_1, z_2, \ldots, z_n)$ be a holomorphic coordinate on X centered at x_0 such that $\omega = i \sum dz_j \wedge d\bar{z}_j + O(|z|^2)$, and assume $\{e_1, e_2, \ldots, e_r\}$ is a holomorphic frame of E on U. Let $\xi = \sum \xi_{j\lambda} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_j \otimes e_\lambda$, with constant coefficients such that $\xi(x_0) = \xi_0$. We may assume

$$
\langle [i\Theta_{E,h} - \theta, \Lambda_\omega]\xi, \xi\rangle < -c
$$

on U. Choose $R > 0$ such that $B(0, R) := \{z : |z| < R\} \subset U$, and write $B(0, R)$ as B_R.

Choose $\chi \in \mathcal{C}_c^\infty(B_R)$, satisfying $\chi(z) = 1$ for $z \in B_{R/2}$. Let $f = \bar{\partial}\nu$ with

$$
\nu(z) = (-1)^n \sum_{j,\lambda} \xi_{j\lambda} \bar{z}_j \chi(z) dz_1 \wedge \cdots \wedge dz_n \otimes e_\lambda.
$$

Then

$$
f(z) = \sum \xi_{j\lambda} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_j \otimes e_\lambda
$$

for $z \in B_{R/2}$. From Proposition 2.1, we can construct a smooth strictly plurisubharmonic function ψ on X, such that $\psi|_{B_R}(z) = |z|^2 - \frac{R^2}{4}$. For any integer $m > 0$, set

$$\psi_m(z) = m\psi(z).$$

As before, set $\alpha_m = B^{-1}_{\psi_m,\theta} f = \frac{1}{m} B^{-1}_{\psi,\theta/m} f$. By [8, Chapter VII, Theorem 1.1], we have

$$D'^* B^{-1}_{\psi,0} f(0) = 0.$$

So after shrinking R, we can get $|D'^*\alpha_m(z)| \leq \frac{\sqrt{c}}{2m}$ for $z \in B_{R/2}$ and any m. Since f has compact support in B_R, there is a constant $C > 0$, such that $|\langle[i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha_m, \alpha_m\rangle| \leq \frac{C^2}{m^2}$ and $|D'^*\alpha_m| \leq \frac{C}{m}$ hold for any $m > 0$.

We now estimate both terms in (5) with α and ψ replaced by α_m and ψ_m defined as above.

$$\begin{aligned}
&m^2\left(\langle\langle[i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha_m, \alpha_m\rangle\rangle_{\psi_m} + ||D'^*\alpha_m||^2_{\psi_m}\right)\\
&= m^2\left(\int_{B_{R/2}} \langle[i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha_m, \alpha_m\rangle e^{-\psi_m} dV_\omega + \int_{B_{R/2}} |D'^*\alpha_m|^2 e^{-\psi_m} dV_\omega\right)\\
&\quad + m^2\left(\int_{B_R\setminus B_{R/2}} \langle[i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha_m, \alpha_m\rangle e^{-\psi_m} dV_\omega + \int_{B_R\setminus B_{R/2}} |D'^*\alpha_m|^2 e^{-\psi_m} dV_\omega\right)\\
&\leq -\frac{3c}{4}\int_{B_{R/2}} e^{-\psi_m} dV_\omega + 2C^2\int_{B_R\setminus B_{R/2}} e^{-\psi_m} dV_\omega. \qquad (6)
\end{aligned}$$

Since $\lim_{m\to+\infty} \psi_m(z) = +\infty$ for $z \in B_R\setminus\overline{B}_{R/2}$, and $\psi_m(z) \leq 0$ for $z \in B_{R/2}$ and all m. Therefore, we obtain from (6) that

$$\langle\langle[i\Theta_{E,h} - \theta, \Lambda_\omega]\alpha_m, \alpha_m\rangle\rangle_{\psi_m} + ||D'^*\alpha_m||^2_{\psi_m} < 0$$

for $m >> 1$, which contradicts to the inequality (5). □

Remark 3.1 With the discussion in Sect. 2.3, the above proof holds for vector bundles of infinite rank.

3.2 Griffiths positivity in terms of multiple coarse L^p-estimate condition

Theorem 3.2 (= Theorem 1.2) *Let (X, ω) be a Kähler manifold, which admits a positive Hermitian holomorphic line bundle, and (E, h) be a holomorphic vector bundle over X with a continuous Hermitian metric h. If (E, h) satisfies the multiple coarse L^p-estimate for some $p > 1$, then (E, h) is Griffiths semi-positive.*

Proof We prove the theorem by modifying the idea in [12,22]. For the same reason as in the proof of Theorem 3.1, we may assume that there is a strictly smooth plurisubharmonic function on X.

We will show that (E, h) satisfies the multiple coarse L^p-extension. We assume that $D', z = (z_1, \ldots, z_n)$ is an arbitrary coordinate chart on X, and let D be an arbitrary relatively compact subset of D'. We assume that $E|_{D'} = D' \times \mathbb{C}^r$ is trivial and $\omega|_D \leq C\, i/2 \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ with some $C > 0$.

Fix an integer $m > 0$, $w \in D$ (we identify w with its coordinate $z(w)$) and $a \in E_w$ with $|a|_h = 1$. We will construct $f \in H^0(X, E^{\otimes m})$ such that $f(w) = a^{\otimes m}$ and

$$\int_X |f|^p_{h^{\otimes m}} dV_\omega \leq C'_m,$$

where C'_m are uniform constants independent of w that satisfy

$$\lim_{m\to\infty} \frac{\log C'_m}{m} = 0.$$

Let $\chi = \chi(t)$ be a smooth function on $\mathbb{R}$, such that

- $\chi(t) = 1$ for $t \leq 1/4$,
- $\chi(t) = 0$ for $t \geq 1$, and
- $|\chi'(t)| \leq 2$ on $\mathbb{R}$.

Viewing a as a constant section of $E|_D$, we define an $E^{\otimes m}$-valued $(n, 1)$-form α_ϵ by

$$\begin{aligned}\alpha_\epsilon &:= \bar{\partial}\chi\left(\frac{|z-w|^2}{\epsilon^2}\right) dz \otimes a^{\otimes m}\\ &= \chi'\left(\frac{|z-w|^2}{\epsilon^2}\right)\sum_j \frac{z_j - w_j}{\epsilon^2} d\bar{z}_j \wedge dz \otimes a^{\otimes m},\end{aligned}$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, and from Proposition 2.1, we can choose a smooth strictly plurisubharmonic function ψ_δ on X such that

$$\psi_\delta|_D = |z|^2 + n\log(|z-w|^2 + \delta^2),$$

where $0 < \epsilon, \delta \ll 1$ are parameters. From the multiple coarse L^p-estimate condition, we obtain a smooth section $u_{\epsilon,\delta}$ of $E^{\otimes m}$-valued $(n, 0)$-form on X such that $\bar{\partial} u_{\epsilon,\delta} = \alpha_\epsilon$ and

$$\int_X |u_{\epsilon,\delta}|^p_{h^{\otimes m}} e^{-\psi_\delta} dV_\omega \leq C_m \int_X \langle B^{-1}_{\psi_\delta}\alpha_\epsilon, \alpha_\epsilon\rangle^{\frac{p}{2}} e^{-\psi_\delta} dV_\omega. \tag{7}$$

On D, we have the following estimate:

$$\begin{aligned}\langle B_{\psi_\delta}^{-1}\alpha_\epsilon,\alpha_\epsilon\rangle &= \left|\chi'\left(\frac{|z-w|^2}{\epsilon^2}\right)\right|^2\\ &\quad\cdot\frac{1}{\epsilon^4}\left\langle B_{\psi_\delta}^{-1}\sum_j (z_j-w_j)d\bar{z}_j\wedge dz\otimes a^{\otimes m},\sum_j (z_j-w_j)d\bar{z}_j\wedge dz\otimes a^{\otimes m}\right\rangle\\ &\le C_1\left|\chi'\left(\frac{|z-w|^2}{\epsilon^2}\right)\right|^2\cdot\frac{1}{\epsilon^4}|z-w|^2|a|_{h(z)}^{2m},\end{aligned}$$

where C_1 depends only on ω. Note that

$$\text{supp}\ \chi'\left(\frac{|z-w|^2}{\epsilon^2}\right)\subset\{1/4\le|z-w|^2/\epsilon^2\le 1\}$$

and $\psi_\delta\ge 2n\log|z-w|$, we have

$$\begin{aligned}&(\text{RHS of (7)})\\ &\le C_m C_1^{\frac{p}{2}}\int_{\{\epsilon^2/4\le|z-w|^2\le\epsilon^2\}}\left|\chi'\left(\frac{|z-w|^2}{\epsilon^2}\right)\right|^p\frac{1}{\epsilon^{2p}}|z-w|^p e^{-\psi_\delta}|a|_{h(z)}^{mp}dV_\omega\\ &\le C_m C_1^{\frac{p}{2}}\frac{2^p}{\epsilon^{2p}}\int_{\{\epsilon^2/4\le|z-w|^2\le\epsilon^2\}}|z-w|^p e^{-\psi_\delta}|a|_{h(z)}^{mp}dV_\omega\\ &\le C_m C_1^{\frac{p}{2}}\frac{2^p}{\epsilon^{2p}}\int_{\{\epsilon^2/4\le|z-w|^2\le\epsilon^2\}}\epsilon^p\sup_{B(w,\epsilon)}|a|_{h(z)}^{mp}e^{-2n\log|z-w|}dV_\omega\\ &\le C_2C_m\frac{\sup_{B(w,\epsilon)}|a|_{h(z)}^{mp}}{\epsilon^p},\end{aligned}\tag{8}$$

where $C_2=2^{p+2n}C_1^{\frac{p}{2}}C^n\mu(B_1)$ and $\mu(B_1)$ is the volume of the unit ball B_1 with respect to the Lebesgue measure.

To summarize, we have obtained a smooth section $u_{\epsilon,\delta}$ of $E^{\otimes m}$-valued $(n,0)$-form on X such that

- $\bar\partial u_{\epsilon,\delta}=\alpha_\epsilon$, and
- the following estimate holds:

$$\int_D|u_{\epsilon,\delta}|_{h^{\otimes m}}^p e^{-\psi_\delta}dV_\omega\le C_2C_m\frac{\sup_{B(w,\epsilon)}|a|_{h(z)}^{mp}}{\epsilon^p}.\tag{9}$$

Note that the weight function ψ_δ is decreasing when $\delta\searrow 0$, $e^{-\psi_\delta}$ is increasing when $\delta\searrow 0$. Fix $\delta_0>0$. Then, for $\delta<\delta_0$, we have that

$$\int_D|u_{\epsilon,\delta}|_{h^{\otimes m}}^p e^{-\psi_{\delta_0}}dV_\omega\le\int_D|u_{\epsilon,\delta}|_{h^{\otimes m}}^p e^{-\psi_\delta}dV_\omega\le C_2C_m\frac{\sup_{B(w,\epsilon)}|a|_{h(z)}^{mp}}{\epsilon^p}.$$

Thus $\{u_{\epsilon,\delta}\}_{\delta<\delta_0}$ forms a bounded sequence in $L^p(X, K_X \otimes E^{\otimes m}, e^{-\psi_{\delta_0}})$. Note that $p > 1$, we can choose a sequence $\{u_{\epsilon,\delta(k)}\}_k$ in $L^p(X, e^{-\delta_0})$ which weakly converges to some $u_\epsilon \in L^p(X, K_X \otimes E^{\otimes m}, e^{-\psi_{\delta_0}})$, satisfying

$$\int_D |u_\epsilon|^p_{h^{\otimes m}} e^{-\psi_{\delta_0}} dV_\omega \leq C_2 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p}.$$

Repeating this argument for a sequence $\{\delta_j\}$ decreasing to 0, by diagonal argument, we can select a sequence $\{u_{\epsilon,\delta^k}\}_k$ which weakly converges to u_ϵ in $L^p(X, K_X \otimes E^{\otimes m}, e^{-\psi_{\delta_j}})$ with u_ϵ satisfying

$$\int_D |u_\epsilon|^p_{h^{\otimes m}} e^{-\psi_{\delta_j}} dV_\omega \leq C_2 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p}$$

for all j. By the monotone convergence theorem,

$$\int_D |u_\epsilon|^p_{h^{\otimes m}} e^{-\psi_0} dV_\omega \leq C_2 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p}.$$

Since $\bar\partial$ is weakly continuous, we also have $\bar\partial u_\epsilon = \alpha_\epsilon$.

Since $\frac{1}{|z-w|^{2n}}$ is not integrable near w, $u_\epsilon(w)$ must be 0. Let

$$f_\epsilon := \chi(|z-w|^2/\epsilon^2) dz \otimes a^{\otimes m} - u_\epsilon.$$

Then $f_\epsilon \in H^0(X, \wedge^{(n,0)} T^*_X \otimes E^{\otimes m})$, $f_\epsilon(0) = dz \otimes a^{\otimes m}$ and

$$\begin{aligned}
&\int_D |f_\epsilon|^p_{h^{\otimes m}} dV_\omega \\
&\leq \left(\left(\int_D \left| \chi(|z-w|^2/\epsilon^2) dz \otimes a^{\otimes m} \right|^p_{h^{\otimes m}} dV_\omega \right)^{1/p} + \left(\int_D |u_\epsilon|^p_{h^{\otimes m}} dV_\omega \right)^{1/p} \right)^p \\
&\leq 2^p \left(\int_D \left| \chi(|z-w|^2/\epsilon^2) dz \otimes a^{\otimes m} \right|^p_{h^{\otimes m}} dV_\omega + \int_D |u_\epsilon|^p_{h^{\otimes m}} dV_\omega \right). \qquad (10)
\end{aligned}$$

Since $\chi \leq 1$ and the support of $\chi(|z-w|^2/\epsilon^2)$ is contained in $\{|z-w|^2 \leq \epsilon^2\}$ and $0 < \epsilon \leq 1$, we have

$$\int_D \left| \chi(|z-w|^2/\epsilon^2) dz \otimes a^{\otimes m} \right|^p_{h^{\otimes m}} dV_\omega \leq C^n \mu(B_1) \sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}.$$

We also have

$$\begin{aligned}\int_D |u_\epsilon|^p_{h^{\otimes m}} dV_\omega &\le \sup_{z\in D} e^{\psi_0(z)} \cdot \int_D |u_\epsilon|^p_{h^{\otimes m}} e^{-\psi_0} dV_\omega \\ &\le \sup_{z\in D} e^{\psi_0(z)} \cdot C_2 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p} \\ &\le C_3 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p},\end{aligned}$$

where C_3 is a constant depends only on D. We may assume $C_m \ge 1$. Combining these estimates with (10), we obtain that

$$\int_D |f_\epsilon|^p_{h^{\otimes m}} dV_\omega \le C_4 C_m \frac{\sup_{B(w,\epsilon)} |a|^{mp}_{h(z)}}{\epsilon^p},$$

where C_4 is a constant independent of m and w.

Let

$$O_\epsilon = \sup_{z,w\in D, |z-w|\le\epsilon} \left|\log|a|_{h(z)} - \log|a|_{h(w)}\right|.$$

By the uniform continuity of $\log|a|_{h(z)}$ on D, O_ϵ is finite and goes to 0 as $\epsilon \to 0$. Let $\epsilon := 1/m$. We have $\left|mp\log|a|_{h(z)} - mp\log|a|_{h(w)}\right| \le mpO_{1/m}$ for $|z-w| \le 1/m$. Then

$$\begin{aligned}\int_D |f_{1/m}|^p e^{-m\phi} dV_\omega &\le C_4 C_m m^p e^{mp\log|a|_{h(w)} + mpO_{1/m}} \\ &= C_4 C_m m^p e^{mpO_{1/m}}. \end{aligned} \tag{11}$$

Let $C'_m = C_4 C_m m^p e^{mpO_{1/m}}$, we have

$$\frac{\log C'_m}{m} = \frac{\log(C'' C_m m^p)}{m} + pO_{1/m} \to 0.$$

Considering $f_{1/m}/dz$, we see that (E,h) satisfies the multiple coarse L^p-extension on D, and hence (E,h) is Griffiths semi-positive on D by [13, Theorem 1.2]. Since D is arbitrary, (E,h) is Griffiths semi-positive on X. □

3.3 Griffiths positivity in terms of optimal L^p-extension condition

Theorem 3.3 (= Theorem 1.3) *Let E be a holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local holomorphic section s of E^*. If (E,h) satisfies the optimal L^p-extension for some $p > 0$, then (E,h) is Griffiths semi-positive.*

Proof Let u be a holomorphic section of E^* over D. Let $z \in D$ and P be any holomorphic cylinder such that $z + P \subset D$. Take $a \in E_z$ such that $|a|_h = 1$ and

$|u|_{h^*}(z) = |\langle u(z), a\rangle|$. Since (E, h) satisfies the optimal L^p-extension, there is a holomorphic section f of E on $z + P$, such that $f(z) = a$ and satisfies the estimate

$$\frac{1}{\mu(P)} \int_{z+P} |f|_h^p \leq 1. \tag{12}$$

Note that $|u|_{h^*} \geq |\langle u, f\rangle|/|f|_h$ on $z + P$, it follows that

$$\log |u|_{h^*} \geq \log |\langle u, f\rangle| - \log |f|_h.$$

Taking integration, we get that

$$\begin{aligned} p\left(\frac{1}{\mu(P)} \int_{z+P} \log |u|_{h^*}\right) &\geq p\left(\frac{1}{\mu(P)} \int_{z+P} \log |\langle u, f\rangle|\right) - \frac{1}{\mu(P)} \int_{z+P} \log |f|_h^p \\ &\geq p\left(\frac{1}{\mu(P)} \int_{z+P} \log |\langle u, f\rangle|\right) - \log\left(\frac{1}{\mu(P)} \int_{z+P} |f|_h^p\right) \\ &\geq p \log |\langle u(z), f(z)\rangle| \\ &= p \log |\langle u(z), a\rangle| = p \log |u(z)|_{h^*}, \end{aligned}$$

where the second inequality follows from Jensen's inequality and (12), and the third inequality follows from the fact that $\log |\langle u, f\rangle|$ is a plurisubharmonic function, and from [12, Lemma 3.1]. Dividing by p, we obtain that

$$\log |u(z)|_{h^*} \leq \frac{1}{\mu(P)} \int_{z+P} \log |u|_{h^*}.$$

Again from [12, Lemma 3.1], we see that $\log |u|_{h^*}$ is plurisubharmonic on D. □

3.4 Griffiths positivity in terms of multiple coarse L^p-extension condition

The following theorem was originally given in [13, Theorem 6.4]. In the present paper, we give a new proof of it based on Guan–Zhou's idea [17] about connecting optimal L^2-extension condition to Berndtsson's plurisubharmonic variation of relative Bergman kernels [1].

Theorem 3.4 (= Theorem1.4) *Let E be a holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$, and h be a singular Finsler metric on E, such that $|s|_{h^*}$ is upper semi-continuous for any local holomorphic section s of E^*. If (E, h) satisfies the multiple coarse L^p-extension, then (E, h) is Griffiths semi-positive.*

Proof Let u be a holomorphic section of E^* over D. Then $u^{\otimes m} \in H^0(D, (E^*)^{\otimes m})$.

Let $z \in D$ and P be any holomorphic cylinder such that $z + P \subset D$. Take $a \in E_z$ such that $|a|_h = 1$ and $|u|_{h^*}(z) = |\langle u(z), a\rangle|$. Since (E, h) satisfies the multiple coarse L^p-extension, there is $f_m \in H^0(D, E^{\otimes m})$, such that $f_m(z) = a^{\otimes m}$ and satisfies the following estimate

$$\int_D |f_m|^p \le C_m,$$

where C_m are constants independent of z and satisfy the growth condition $\frac{1}{m}\log C_m \to 0$ as $m \to \infty$. Since $|u^{\otimes m}|_{(h^*)^{\otimes m}} = |u|^m_{h^*} \ge \frac{|\langle u^{\otimes m}, f_m\rangle|}{|f_m|_{h^{\otimes m}}}$, we have that

$$m \log |u|_{h^*} \ge \log |\langle u^{\otimes m}, f_m\rangle| - \log |f_m|.$$

Taking integration, we get that

$$\begin{aligned} m\left(\frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*}\right) &\ge \frac{1}{\mu(P)}\int_{z+P}\log|\langle u^{\otimes m}, f_m\rangle| - \frac{1}{p}\left(\frac{1}{\mu(P)}\int_{z+P}\log|f_m|^p\right)\\ &\ge m\log|u(z)|_{h^*} - \frac{1}{p}\log\left(\frac{1}{\mu(P)}\int_{z+P}|f_m|^p\right)\\ &\ge m\log|u(z)|_{h^*} - \frac{1}{p}\log\left(\frac{1}{\mu(P)}\int_{D}|f_m|^p\right)\\ &\ge m\log|u(z)|_{h^*} - \frac{1}{p}\log(C_m/\mu(P)), \end{aligned}$$

where the first inequality follows from the fact that $\log|\langle u^{\otimes m}, f_m\rangle|$ is a plurisubharmonic function, and [12, Lemma 3.1], and Jensen's inequality, and the second inequality follows from the fact that $z + P \subset D$. Dividing by m in both sides, we obtain that

$$\frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*} \ge \log|u(z)|_{h^*} - \frac{1}{mp}\log(C_m/\mu(P)).$$

Letting $m \to \infty$, we see that $\log|u|_{h^*}$ satisfies the following inequality

$$\log|u(z)|_{h^*} \le \frac{1}{\mu(P)}\int_{z+P}\log|u|_{h^*},$$

since $\frac{1}{m}\log C_m \to 0$ as $m \to \infty$. Then from [12, Lemma 3.1], we get that $\log|u|_{h^*}$ is plurisubharmonic on D. □

4 Positivities of direct images of twisted relative canonical bundles

4.1 Optimal L^2-estimate condition and Nakano positivity

The aim of this subsection is to prove Theorems 1.5 and 1.6.

To avoid some complicated geometric quantities and highlight the main idea, we first consider a simple case of Theorem 1.5 as a warm-up.

Theorem 4.1 *Let U and D be bounded domains in $\mathbb{C}^n_t$ and $\mathbb{C}^m_z$ respectively, and $\phi \in \mathcal{C}^2(\overline{U} \times \overline{D}) \cap PSH(U \times D)$. Assume that D is pseudoconvex. For $t \in U$, let $A^2_t := \{f \in \mathcal{O}(D) : ||f||^2_t := \int_D |f|^2 e^{-\phi(t,\cdot)} < \infty\}$ and $F := \coprod_{t\in U} A^2_t$. We may view F as a Hermitian holomorphic vector bundle on U. Then $(F, ||\cdot||_t)$ is Nakano semi-positive.*

Proof We will first prove that $(F, ||\cdot||_t)$ satisfies the $\bar\partial$ optimal L^2-estimate for pseudoconvex domains contained in U. We may assume U is pseudoconvex.

For any smooth strictly plurisubharmonic function ψ on U, for any $\bar\partial$ closed $f \in \mathcal{C}^\infty_c(T^*_U\Lambda^{(0,1)} \otimes F)$ (We identify $\mathcal{C}^\infty_c(T^*_U\Lambda^{(0,1)} \otimes F)$ with $\mathcal{C}^\infty_c(T^*_U\Lambda^{(n,1)} \otimes F)$). We may write $f = \sum_{j=1}^n f_j(t,z)d\bar t_j$ with $f_j(t,\cdot) \in F_t$ for $t \in U$ and $j = 1, 2, \ldots, n$. Therefore, we may view f as a $\bar\partial$-closed $(0,1)$-form on $U \times D$. By Lemma 2.5, there exists a function u on $U \times D$, satisfying $\bar\partial u = f$ and

$$\begin{aligned}\int_{U\times D} &|u|^2 e^{-(\phi+\psi)}\\ &\le \int_{U\times D} |f|^2_{i\partial\bar\partial(\phi+\psi)} e^{-(\phi+\psi)}\\ &\le \int_{U\times D} |f|^2_{i\partial\bar\partial\psi} e^{-(\phi+\psi)}\\ &= \int_U \sum_{j,k=1}^n \psi^{j\bar k}\langle f_j(t,\cdot), f_k(t,\cdot)\rangle_t e^{-\psi},\end{aligned}$$

where $(\psi^{j\bar k})_{n\times n} := (\frac{\partial^2\psi}{\partial t_j\partial \bar t_k})^{-1}_{n\times n}$. Note that $\int_{U\times D} |u|^2 e^{-(\phi+\psi)} = \int_U ||u||^2_t e^{-\psi} < \infty$ and $\frac{\partial u}{\partial \bar z_j} = 0$ for $j = 1, 2, \ldots, m$, we may view u as a L^2-section of F on U. By Theorem 1.1 and Remark 1.2, $(F, ||\cdot||_t)$ is Nakano semi-positive. □

Let $\Omega = U \times D \subset \mathbb{C}^n_t \times \mathbb{C}^m_z$ be a bounded pseudoconvex domains and $p : \Omega \to U$ be the natural projection. Let h be a Hermitian metric on the trivial bundle $E = \Omega \times \mathbb{C}^r$ that is C^2-smooth to $\overline{\Omega}$. For $t \in U$, let

$$F_t := \left\{ f \in H^0(D, E|_{\{t\}\times D}) : \|f\|^2_t := \int_D |f|^2_{h_t} < \infty \right\}$$

and $F := \coprod_{t\in U} F_t$. Since h is continuous to $\overline{\Omega}$, F_t are equal for all $t \in U$ as vector spaces. We may view $(F, \|\cdot\|)$ as a trivial holomorphic Hermitian vector bundle of infinite rank over U.

Theorem 4.2 (= Theorem 1.5) *Let θ be a continuous real $(1,1)$-form on U such that $i\Theta_E \ge p^*\theta \otimes Id_E$, then $i\Theta_F \ge \theta \otimes Id_F$ in the sense of Nakano. In particular, if $i\Theta_E > 0$ in the sense of Nakano, then $i\Theta_F > 0$ in the sense of Nakano.*

Proof By Theorem 3.1, it suffices to prove that $(F, \|\cdot\|)$ satisfies: for any $f \in \mathcal{C}^\infty_c(U, \wedge^{n,1}T^*_U \otimes F \otimes A)$ with $\bar\partial f = 0$, and any positive Hermitian line bundle

(A, h_A) on U with $i\Theta_{A,h_A} + \theta > 0$ on supp$f$, there is $u \in L^2(U, \wedge^{n,0}T_U^* \otimes F \otimes A)$, satisfying $\bar{\partial}u = f$ and

$$\int_U |u|^2_{h\otimes h_A} dV_\omega \leq \int_U \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A} dV_\omega,$$

provided that the right hand side is finite, where $B_{h_A,\theta} = [(i\Theta_{A,h_A} + \theta) \otimes Id_F, \Lambda_\omega]$.

We may write $f = \sum_{j=1}^n f_j(t,z)dt \wedge d\bar{t}_j$ with $f_j(t,\cdot) \in F_t \otimes A$ for $t \in U$ and $j = 1, 2, \ldots, n$. Therefore, we may view f as a $\bar{\partial}$-closed $E \otimes p^*A$-valued $(n, 1)$-form on Ω. Let $\tilde{f} = f \wedge dz$, then $\tilde{f}$ is a $\bar{\partial}$-closed $E \otimes p^*A$-valued $(m+n, 1)$-form on Ω. By assumption, $i\Theta_E \geq p^*\theta \otimes Id_E$. We get

$$i\Theta_E + ip^*(\Theta_{A,h_A}) \otimes Id_E \geq p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E.$$

Therefore,

$$\begin{aligned} &\langle [i\Theta_E + ip^*(\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A} \\ &\quad \leq \langle [p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A}. \end{aligned}$$

By Lemma 2.5, we can find an $E \otimes p^*A$-valued $(n+m, 0)$-form $\tilde{u}$ on Ω, satisfying $\bar{\partial}\tilde{u} = \tilde{f}$ and

$$\begin{aligned} &\int_\Omega |\tilde{u}|^2_{h\otimes h_A} \\ &\quad \leq \int_\Omega \langle [p^*(\theta + i\Theta_{A,h_A}) \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle_{h\otimes h_A} \\ &\quad = \int_U \langle B^{-1}_{h_A,\theta} f, f\rangle_{h\otimes h_A}, \end{aligned}$$

where the last equality holds by the Fubini theorem. Since $\frac{\partial\tilde{u}}{\partial z_j} = 0$, $\tilde{u}$ is holomorphic along fibers and we may view $u = \tilde{u}/dz$ as a section of $K_U \otimes F \otimes A$. Also by the Fubini theorem, we have

$$\int_\Omega |\tilde{u}|^2_{h\otimes h_A} = \int_U ||u||^2_{h\otimes h_A} < \infty.$$

We also have $\bar{\partial}u = f$. Hence $(F, \|\cdot\|)$ satisfies the optimal L^2-estimate and is Nakano semi-positive by Theorem 3.1. □

Let $\pi : X \to U$ be a proper holomorphic submersion from Kähler manifold X of complex dimension $m+n$, to a bounded pseudoconvex domain $U \subset \mathbb{C}^n$, and (E, h) be a Hermitian holomorphic vector bundle over X, with the Chern curvature Nakano semi-positive. From Lemma 2.6, the direct image $F := \pi_*(K_{X/U} \otimes E)$ is a vector bundle, whose fiber over $t \in U$ is $F_t = H^0(X_t, K_{X_t} \otimes E|_{X_t})$. There is a hermtian metric $\|\cdot\|$ on F induced by h: for any $u \in F_t$,

$$\|u(t)\|_t^2 := \int_{X_t} c_m u \wedge \bar{u},$$

where $m = \dim X_t$, $c_m = i^{m^2}$, and $u \wedge \bar{u}$ is the composition of the wedge product and the inner product on E. So we get a Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U.

Theorem 4.3 (= Theorem 1.6) *The Hermitian holomorphic vector bundle* $(F, \|\cdot\|)$ *over* U *defined above satisfies the optimal* L^2*-estimate. Moreover, if* $i\Theta_E \geq p^*\theta \otimes Id_E$ *for a continuous real* $(1,1)$*-form* θ *on* U*, then* $i\Theta_F \geq \theta \otimes Id_F$ *in the sense of Nakano.*

Proof Similar to the proof of Theorem 4.2, we may assume $\theta = 0$. From Theorem 3.1, it suffices to prove that $(\pi_*(K_{X/Y} \otimes E), \|\cdot\|)$ satisfies the optimal L^2-estimate with the standard Kähler metric ω_0 on $U \subset \mathbb{C}^n$. Let ω be an arbitrary Kähler metric on X.

Let f be a $\bar{\partial}$-closed compact supported smooth $(n,1)$-form with values in F, and let ψ be any smooth strictly plurisubharmonic function on U.

We can write $f(t) = dt \wedge (f_1(t)d\bar{t}_1 + \cdots + f_n(t)d\bar{t}_n)$, with $f_i(t) \in F_t = H^0(X_t, K_{X_t} \otimes E)$. One can identify f as a smooth compact supported $(n+m,1)$-form $\tilde{f}(t,z) := dt \wedge (f_1(t,z)d\bar{t}_1 + \cdots + f_n(t,z)d\bar{t}_n)$ on X, with $f_i(t,z)$ being holomorphic section of $K_{X_t} \otimes E|_{X_t}$. We have the following observations:

- $\bar{\partial}_z f_i(t,z) = 0$ for any fixed $t \in B$, since $f_i(t,z)$ are holomorphic sections $K_{X_t} \otimes E|_{X_t}$.
- $\bar{\partial}_t f = 0$, since f is a $\bar{\partial}$-closed form on B.

It follows that $\tilde{f}$ is a $\bar{\partial}$-closed compact supported $(n+m,1)$-form on X with values in E. We want to solve the equation $\bar{\partial}u = \tilde{f}$ on X by using Lemma 2.5. Now we equipped E with the metric $\tilde{h} := he^{-\pi^*\psi}$, then $i\Theta_{E,\tilde{h}} = i\Theta_{E,h} + i\partial\bar{\partial}\pi^*\psi \otimes Id_E$, which is also semi-positive in the sense of Nakano.

We consider the integration

$$\int_X \langle [i\Theta_{E,h} + i\partial\bar{\partial}\pi^*\psi \otimes Id_E, \Lambda_\omega]^{-1}\tilde{f}, \tilde{f}\rangle e^{-\pi^*\psi} dV_\omega.$$

Note that, acting on $\Lambda^{n+m,1}T_X^* \otimes E$, by Lemma 2.4, we have

$$[i\Theta_{E,h} + i\partial\bar{\partial}\pi^*\psi \otimes Id_E, \Lambda_\omega] \geq [i\partial\bar{\partial}\pi^*\psi \otimes Id_E, \Lambda_\omega].$$

Thus we obtain that

$$[i\Theta_{E,h} + i\partial\bar{\partial}\pi^*\psi \otimes Id_E, \Lambda_\omega]^{-1} \leq [i\partial\bar{\partial}\pi^*\psi \otimes Id_E, \Lambda_\omega]^{-1}.$$

For any $p \in X$, we use Lemma 4.7 to modify ω at p. We take a local coordinate $(t_1, \ldots, t_n, z_1, \ldots, c_m)$ on X near p, where $t_1, \ldots, t_n$ is the standard coordinate on $U \subset \mathbb{C}^n$. Let $\omega' = i\sum_{j=1}^n dt_j \wedge d\bar{t}_j + i\sum_{l=1}^m dz_l \wedge d\bar{z}_l$.

Note that

$$i\partial\bar{\partial}\pi^*\psi=\sum_{j=1}^{n}\frac{\partial^2\psi}{\partial t_j\partial\bar{t}_k}dt_j\wedge d\bar{t}_k,$$

we have

$$[i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega'}]\tilde{f}=\sum_{j,k}\frac{\partial^2\psi}{\partial t_j\partial\bar{t}_k}f_j(t,z)dt\wedge d\bar{t}_k,$$

and

$$[i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega'}]^{-1}\tilde{f}=\sum_{j,k}\psi^{jk}f_j(t,z)dt\wedge d\bar{t}_k$$

at p, where $(\psi^{jk})=(\frac{\partial^2\psi}{\partial t_j\partial\bar{t}_k})^{-1}$. By Lemma 4.7, we have

$$\begin{aligned}&\langle[i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega}]^{-1}\tilde{f},\tilde{f}\rangle_{\omega}dV_{\omega}\\&\quad=\langle[i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega'}]^{-1}\tilde{f},\tilde{f}\rangle_{\omega'}dV_{\omega'}\\&\quad=\sum_{j,k}\psi^{jk}c_m f_j\wedge\bar{f}_k c_n dt\wedge d\bar{t}.\end{aligned}$$

By Fubini's theorem, we get that

$$\begin{aligned}&\int_X\langle[i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega}]^{-1}\tilde{f},\tilde{f}\rangle_{\omega}e^{-\pi^*\psi}dV_{\omega}\\&\quad=\int_X\sum_{j,k}\psi^{jk}c_m f_j\wedge\bar{f}_k e^{-\pi^*\psi}c_n dt\wedge d\bar{t}\\&\quad=\int_U<f_j,f_k>_t\ \psi^{jk}e^{-\psi}c_n dt\wedge d\bar{t}\\&\quad=\int_U\langle[i\partial\bar{\partial}\psi,\Lambda_{\omega_0}]^{-1}f,f\rangle_t e^{-\psi}dV_{\omega_0},\end{aligned}$$

where by $\langle\cdot,\cdot\rangle_t$, we mean that pointwise inner product with respect to the Hermitian metric $\|\cdot\|$ of F.

From Lemma 2.5, there is $\tilde{u}\in\Lambda^{m+n,0}(X,E)$, such that $\bar{\partial}\tilde{u}=\tilde{f}$, and satisfies the following estimate

$$\begin{aligned}&\int_X c_{m+n}\tilde{u}\wedge\bar{\tilde{u}}e^{-\pi^*\psi}\\&\quad\le\int_X\langle[i\Theta_{E,h}+i\partial\bar{\partial}\pi^*\psi\otimes Id_E,\Lambda_{\omega}]^{-1}\tilde{f},\tilde{f}\rangle e^{-\pi^*\psi}dV_{\omega}\end{aligned}$$

$$\leq \int_U \langle [i\partial\bar{\partial}\psi, \Lambda_{\omega_0}]^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0}. \tag{13}$$

We observe that $\bar{\partial}\tilde{u}|_{X_t} = 0$ for any fixed $t \in U$, since $\bar{\partial}\tilde{u} = \tilde{f}$. This means that $\tilde{u}_t := \tilde{u}(t, \cdot) \in F_t$. Therefore we may view $\tilde{u}$ as a section u of F. It is obviously that $\bar{\partial}u = f$.

From Fubini's theorem, we have that

$$\int_X c_{m+n}\tilde{u} \wedge \bar{\tilde{u}} e^{-\pi^*\psi} = \int_U \|u\|_t^2 e^{-\psi} dV_{\omega_0}. \tag{14}$$

Combining (13), we have

$$\int_U \|u\|_t^2 e^{-\psi} dV_{\omega_0} \leq \int_U \langle [i\partial\bar{\partial}\psi, \Lambda_{\omega_0}]^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0}.$$

We have proved that F satisfies the optimal L^2-estimate, thus from Theorem 3.1 (and Remark 1.2), F is Nakano semi-positive. □

4.2 Multiple coarse L^2-estimate condition and Griffiths positivity

We apply Theorem 1.2 and the fiber product technique introduced in [13] to provide a new method to study the Griffiths positivity of direct images.

Theorem 4.4 *The Hermitian holomorphic vector bundle* $(F, \|\cdot\|)$ *over* U *as in Theorem 4.3 satisfies the multiple coarse* L^2*-estimate. In particular,* F *is Griffiths semipositive.*

Proof Let ω_0 be the standard Kähler metric on U and ω be an arbitrary Kähler metric on X. We have the following constructions:

- Let $X_k := X \times_\pi \cdots \times_\pi X$ be the k times fiber product of X with respect to the map $\pi : X \to U$.
- The induced map $X_k \to U$ by π is denoted by $\pi_k : X_k \to U$, and $X_{k,t} := \pi_k^{-1}(t) = X_t^k$ for every $t \in U$.
- There are natural holomorphic projections pr_j from X_k to its j-th factor X.
- The induced Kähler metric $\omega_k := pr_1^*\omega + \cdots + pr_k^*\omega$ on X_k.
- Set $E_j := pr_j^*E$, and $E^k := E_1 \otimes \cdots \otimes E_k$. Then E^k can be equipped with the induced metric $h^k := pr_1^*h \otimes \cdots \otimes pr_k^*h$.

We have the following observations:

- From Lemma 2.2, E^k equipped with the Hermitian metric h^k is Nakano semi-positive.
- From [13, Lemma 9.2], the direct image bundle $F^k := (\pi_k)_*(K_{X_k/U} \otimes E^k) = (\pi_*(K_{X/U} \otimes E))^{\otimes k} = F^{\otimes k}$, as Hermitian holomorphic vector bundles. (In fact, [13, Lemma 9.2] was proved for line bundles, but it is clear that the proof also works for vector bundles.)

Let f be an arbitrarily fixed smooth compactly supported $(m, 1)$-form on U with valued in F^k, such that $\bar\partial f = 0$. Let ψ be an arbitrary smooth strictly plurisubharmonic function on U. To prove that F satisfies the multiple coarse L^2-estimate, we need to show that one can solve the equation $\bar\partial u = f$ on U, with the estimate $\int_U |u|^2_{h^k} e^{-\psi} \leq \int_U \langle B_\psi^{-1} f, f\rangle e^{-\psi}$, where $B_\psi = [i\partial\bar\partial\psi, \Lambda_{\omega_0}]$.

As in the proof of Theorem 4.3, we may consider f as a smooth compactly supported $K_{X_k} \otimes E^k$ valued $(0, 1)$-form $\tilde f$ on X_k. Then it is clear that $\bar\partial \tilde f = 0$. We consider the following integration

$$\int_{X_k} \langle [i\Theta_{E^k,h^k} + i\partial\bar\partial\pi_k^*\psi \otimes Id_{E^k}, \Lambda_{\omega_k}]^{-1}\tilde f, \tilde f\rangle_{\omega_k} e^{-\pi_k^*\psi} dV_{\omega_k}.$$

By the same analysis as in the proof of Theorem 4.3, we can get that

$$\begin{aligned}
&\int_{X_k} \langle [i\Theta_{E^k,h^k} + i\partial\bar\partial\pi_k^*\psi \otimes Id_{E^k}, \Lambda_{\omega_k}]^{-1}\tilde f, \tilde f\rangle_{\omega_k} e^{-\pi_k^*\psi} dV_{\omega_k}\\
&\leq \int_{X_k} \langle [i\partial\bar\partial\pi_k^*\psi \otimes Id_{E^k}, \Lambda_{\pi_k^*\omega_0}]^{-1}\tilde f, \tilde f\rangle_{\omega_k} e^{-\pi_k^*\psi} dV_{\omega_k}\\
&= \int_{X_k} \sum_{j,k} \psi^{jk} c_{km} f_j \wedge \bar f_k e^{-\pi_k^*\psi} c_n dt \wedge d\bar t\\
&= \int_U \langle B_\psi^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0}.
\end{aligned}$$

Now from Lemma 2.5, we can solve the equation $\bar\partial\tilde u = \tilde f$ with the estimate

$$\begin{aligned}
\int_{X_k} |\tilde u|^2_{h^k} e^{-\pi_k^*\psi} dV_{\omega_k} &\leq \int_{X_k} \langle [i\Theta_{E^k,h^k} + i\partial\bar\partial\pi_k^*\psi \otimes Id_{E^k}, \Lambda_{\omega_k}]^{-1}\tilde f, \tilde f\rangle_{\omega_k} e^{-\pi_k^*\psi} dV_{\omega_k}\\
&\leq \int_U \langle B_\psi^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0}.
\end{aligned}$$

Similarly, $\bar\partial\tilde u|_{X_t} = 0$ for any fixed $t \in U$, since $\bar\partial\tilde u = \tilde f$. This means that $\tilde u_t := \tilde u(t, \cdot) \in F_t^k$. Therefore we may view $\tilde u$ as a section u of F^k. It is obviously that $\bar\partial u = f$.

Applying Fubini's theorem to the L.H.S of above inequality, we get that

$$\int_U |u_t|_t^2 e^{-\psi} dV_{\omega_0} \leq \int_U \langle B_\psi^{-1} f, f\rangle_t e^{-\psi} dV_{\omega_0},$$

which implies that $(F, \|\cdot\|)$ satisfies the multiple coarse L^2-estimate on U. □

4.3 Optimal L^2-extension condition and Griffiths positivity

Theorem 4.5 *The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U as in Theorem 4.3 satisfies the optimal L^2-extension. In particular, F is Griffiths semipositive.*

Proof For any $t_0 \in U$, any holomorphic cylinder P such that $t_0 + P \subset U$, and any $a_{t_0} \in F_{t_0}$, which is a holomorphic section of $K_{X_{t_0}} \otimes E|_{X_{t_0}}$ on X_{t_0}. Since E is Nakano semi-positive, from Lemma 2.5, we get a holomorphic extension $a \in H^0(X, K_X \otimes E)$ such that $a|_{X_{t_0}} = a_{t_0} \wedge dt$, and with the estimate

$$\int_{\pi^{-1}(t_0+P)} c_{m+n} a \wedge \bar{a} \leq \mu(P) \int_{X_{t_0}} c_m a_{t_0} \wedge \bar{a}_{t_0} = \mu(P)|a_{t_0}|^2_{t_0},$$

where $\mu(P)$ is the volume of P with respect to the Lebesgue measure $d\mu$ on $\mathbb{C}^m$. Since $a_t := (a/dt)|_{X_t} \in H^0(X_t, K_{X_t} \otimes E|_{X_t})$, a/dt can be seen as a holomorphic section of the direct image bundle F over $t_0 + P$, and from Fubini's theorem, we can obtain that

$$\int_{t_0+P} |a_t|^2_t dV_{\omega_0} \leq \mu(P)|a_{t_0}|^2_{t_0},$$

which is the desired optimal L^2-extension. □

4.4 Multiple coarse L^2-extension condition and Griffiths positivity

In this subsection, we will prove the following

Theorem 4.6 *The Hermitian holomorphic vector bundle $(F, \|\cdot\|)$ over U as in Theorem 4.3 satisfies the multiple coarse L^2-extension. In particular, F is Griffiths semipositive.*

Proof Let $(X_k, \pi_k, \omega_k, F^k)$ be as in the proof of Theorem 4.4.

For any $t_0 \in U$, $a_{t_0} \in F_{t_0}$, $a_{t_0}^{\otimes k}$ is a holomorphic section of $K_{X_{k,t_0}} \otimes E^k$. Since E^k with the induced metric h^k is semi-positive in the sense of Nakano on X_k, by Lemma 2.6, there exists $a \in H^0(X_k, K_{X_k} \otimes E^k)$, such that $a|_{X_{k,t_0}} = a_{t_0}^{\otimes k} \wedge dt$ and satisfies the following estimate

$$\int_{X_k} |a|^2_{h^k} dV_{\omega_k} \leq C|a_{t_0}^{\otimes k}|^2_{t_0},$$

where C is a universal constant which only depends on the diameter and dimension of U. We can view $a_t := (a/dt)|_{X_t}, t \in U$ as a holomorphic section of F^k. From Fubini's theorem, we have that

$$\int_{X_k} |a_t|^2_{h^k} dV_{\omega_k} = \int_U |a_t|^2_t dV_{\omega_0}.$$

In conclusion, we get a holomorphic extension a/dt of $a_{t_0}^{\otimes k}$, with the estimate

$$\int_U |a_t|^2_t dV_{\omega_0} \leq C|a_{t_0}^{\otimes k}|^2_{t_0}.$$

This completes the proof of Theorem 4.6. □

Remark 4.1 Let $\pi : X \to Y$ be a proper holomorphic map between Kähler manifolds which may be not regular. Let (E, h) be a Hermitian holomorphic vector bundle on X whose Chern curvature is Nakano semi-positive. Then the direct image sheaf $\mathcal{F} := \pi_*(K_{X/Y} \otimes E)$ can be equipped with a natural singular metric which is positively curved in the sense of Definition 2.6. In fact, let $Z \subset Y$ be the singular locus of π, then on $X \backslash \pi^{-1}(Z)$, π is a submersion, and $\mathcal{F}$ is locally free and can be viewed as a vector bundle F on $Y' := Y \backslash Z$, with $F_t = H^0(K_{X_t} \otimes E|_{X_t})$. The induced Hermitian metric $\| \cdot \|$ on F is as follows: for any holomorphic section $u \in H^0(Y', F)$,

$$\|u\|_t^2 := \int_{X_t} c_m u \wedge \bar{u}.$$

From one of Theorem 4.4, and Theorem 4.5, Theorem 4.6, we see that $\| \cdot \|_t$ is a Hermitian metric on F with Griffiths semi-positive curvature. Moreover, by similar argument as in [20, Proposition 23.3] (see also [13, Step 3 in the proof of Theorem 9.3]), one can show that the metric on F extends to a positively curved metric on $\mathcal{F}$. In the special case that E is a line bundle, the same conclusion is true if h is singular and pseudoeffective (see [3,13,20,31,34]).

Acknowledgements The first author is partially supported by the NSFC grant (No. 11871451) and by the Fundamental Research Funds for the Central Universities. The second author is partially supported by the National Key R&D Program of China (No. 2021YFA1002600) and by the NSFC (No. 12071485). The third author is partially supported by the National Key R&D Program of China (No.2021YFA1002600), by the Beijing Natural Science Foundation (No. 1202012, Z190003), and by the NSFC grants (No. 12071035). The fourth author is partially supported by the NSFC grant (No. 11688101).

Appendix

We prove a result used in the proof of Theorem 1.6, which seems to be already known.

Lemma 4.7 *Let $U \subset \mathbb{C}^n$ be a domain, ω_1, ω_2 be any two Hermitian forms on U, and $E = U \times \mathbb{C}^r$ be trivial vector bundle on U with a Hermitian metric. Let $\Theta \in C^0(X, \Lambda^{1,1}T_X^* \otimes End(E))$ such that $\Theta^* = -\Theta$. Then*

$$Im[i\Theta, \Lambda_{\omega_1}] = Im[i\Theta, \Lambda_{\omega_2}],$$

and for any E-valued $(n, 1)$ form $u \in Im[i\Theta, \Lambda_{\omega_1}]$,

$$\langle [i\Theta, \Lambda_{\omega_1}]^{-1}u, u\rangle_{\omega_1} dV_{\omega_1} = \langle [i\Theta, \Lambda_{\omega_2}]^{-1}u, u\rangle_{\omega_2} dV_{\omega_2}.$$

Proof For any $z_0 \in U$, after a linearly transformation, we may assume $\omega_1 = i\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and $\omega_2 = i\sum_{j=1}^n \lambda_j^2 dz_j \wedge d\bar{z}_j$ at z_0 with $\lambda_j > 0$. Let $w_j = \lambda_j z_j$

for $j = 1, 2, \ldots, n$, then $\omega_2 = i\sum_{j=1}^n dw_j \wedge d\bar{w}_j$. We may write

$$i\Theta = i\sum_{jk\alpha\beta} c_{jk\alpha\beta} dz_j \wedge d\bar{z}_k \otimes e_\alpha^* \otimes e_\beta = i\sum_{jk\alpha\beta} c'_{jk\alpha\beta} dw_j \wedge d\bar{w}_k \otimes e_\alpha^* \otimes e_\beta \quad (15)$$

with $c'_{jk\alpha\beta} = \frac{c_{jk\alpha\beta}}{\lambda_j\lambda_k}$.

Denote $\lambda = \prod_{j=1}^n \lambda_j$. Let $u = \sum_{j,\alpha} u_{j\alpha} dz \wedge d\bar{z}_j \otimes e_\alpha$, then $u = \sum_{j,\alpha} u'_{j\alpha} dw \wedge d\bar{w}_j \otimes e_\alpha$ with $u'_{j\alpha} = \frac{u_{j\alpha}}{\lambda\lambda_j}$. Note that

$$[i\Theta, \Lambda_{\omega_1}]u = \sum_{jk\alpha\beta} u_{j\alpha} c_{jk\alpha\beta} dz \wedge d\bar{z}_k \otimes e_\beta, \quad (16)$$

and

$$[i\Theta, \Lambda_{\omega_2}]u = \sum_{jk\alpha\beta} u'_{j\alpha} c'_{jk\alpha\beta} dw \wedge d\bar{w}_k \otimes e_\beta. \quad (17)$$

So it is easy to see $Im[i\Theta, \Lambda_{\omega_1}] = Im[i\Theta, \Lambda_{\omega_2}]$. We write

$$[i\Theta, \Lambda_{\omega_1}]^{-1}u = \sum_{jk\alpha\beta} u_{j\alpha} d_{jk\alpha\beta} dz \wedge d\bar{z}_k \otimes e_\beta,$$
$$[i\Theta, \Lambda_{\omega_2}]^{-1}u = \sum_{jk\alpha\beta} u'_{j\alpha} d'_{jk\alpha\beta} dw \wedge d\bar{w}_k \otimes e_\beta,$$

Then from Eqs. (15),(16), (17), we can get

$$d'_{jk\alpha\beta} = \lambda_j\lambda_k d_{jk\alpha\beta}.$$

We now assume that $\{e_\alpha\}$ are orthonormal at z_0. Then

$$\langle [i\Theta, \Lambda_{\omega_1}]^{-1}u, u\rangle_{\omega_1} dV_{\omega_1} = \sum_{jk\alpha\beta} d_{jk\alpha\beta} u_{j\alpha} \bar{u}_{k\beta} c_n dz \wedge d\bar{z},$$
$$\langle [i\Theta, \Lambda_{\omega_2}]^{-1}u, u\rangle_{\omega_2} dV_{\omega_2} = \sum_{jk\alpha\beta} d'_{jk\alpha\beta} u'_{j\alpha} \bar{u'}_{k\beta} c_n dw \wedge d\bar{w}.$$

Note also that

$$c_n dw \wedge d\bar{w} = \lambda^2 c_n dz \wedge d\bar{z},$$

We get

$$\langle [i\Theta, \Lambda_{\omega_1}]^{-1}u, u\rangle_{\omega_1} dV_{\omega_1} = \langle [i\Theta, \Lambda_{\omega_2}]^{-1}u, u\rangle_{\omega_2} dV_{\omega_2}.$$

□

References

1. Berndtsson, B.: Subharmonicity conditions of the Bergman kernel and some other functions associated to pseudoconvex domains. Ann. Inst. Fourier (Grenoble) **56**(6), 1633–1662 (2006)
2. Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. Ann. Math. (2) **169**(2), 531–560 (2009)
3. Berndtsson, B., Păun, M.: Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J. **145**(2), 341–378 (2008)
4. Berndtsson, B.: Complex Brunn–Minkowski theory and positivity of vector bundles. arXiv:1807.05844
5. Berndtsson, B., Păun, M.: Bergman kernels and subadjunction, e-preprint. arXiv:1002.4145
6. Błocki, Z.: Suita conjecture and the Ohsawa–Takegoshi extension theorem. Invent. Math. **193**(1), 149–158 (2013)
7. Cao, J.: Ohsawa–Takegoshi extension theorem for compact Kähler manifolds and applications. In: Complex and Symplectic Geometry, Springer INdAM Ser., vol. 21, pp. 19–38. Springer, Cham (2017)
8. Demailly, J.-P.: Complex analytic and differential geometry. http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf
9. Demailly, J.-P.: Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. Ann. Sci. École Norm. Sup. (4) **15**(3), 457–511 (1982)
10. Demailly, J.-P.: Analytic methods in algebraic geometry, Surveys of Modern Mathematics, vol. 1. International Press, Somerville; Higher Education Press, Beijing (2012)
11. Demailly, J.-P.: On the Ohsawa–Takegoshi–Manivel L^2 extension theorem. In: Complex Analysis and Geometry (Paris, 1997), Progr. Math., vol. 188, pp. 47–82. Birkhäuser, Basel (2000)
12. Deng, F., Ning, J., Wang, Z.: Characterizations of plurisubharmonic functions. Sci. China Math. **64**(9), 1959–1970 (2021)
13. Deng, F., Wang, Z., Zhang, L., Zhou, X.: New characterization of plurisubharmonic functions and positivity of direct image sheaves. arXiv:1809.10371
14. Deng, F., Wang, Z., Zhang, L., Zhou, X.: Linear invariants of complex manifolds and their plurisubharmonic variations. J. Funct. Anal. **279**(1), 108514 (2020)
15. Guan, Q., Zhou, X.: Optimal constant problem in the L^2 extension theorem. C. R. Math. Acad. Sci. Paris **350**(15–16), 753–756 (2012)
16. Guan, Q.A., Zhou, X.Y.: Optimal constant in an L^2 extension problem and a proof of a conjecture of Ohsawa. Sci. China Math. **58**(1), 35–59 (2015)
17. Guan, Q., Zhou, X.: A solution of an L^2 extension problem with an optimal estimate and applications. Ann. Math. (2) **181**(3), 1139–1208 (2015)
18. Guan, Q.A., Zhou, X.Y.: A proof of Demailly's strong openness conjecture. Ann. Math. (2) **182**(2), 605–616 (2015)
19. Guan, Q.A., Zhou, X.Y.: Strong openness of multiplier ideal sheaves and optimal L^2 extension. Sci. China Math. **60**(6), 967–976 (2017)
20. Hacon, C., Popa, M., Schnell, C.: Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun. In: Local and Global Methods in Algebraic Geometry, Contemp. Math., vol. 712, pp. 143–195. Amer. Math. Soc., Providence (2018)
21. Hörmander, L.: L^2 estimates and existence theorems for the $\bar{\partial}$ operator. Acta Math. **113**, 89–152 (1965)
22. Hosono, G., Inayama, T.: A converse of Hörmander's L^2-estimate and new positivity notions for vector bundles. Sci. China Math. **64**(8), 1745–1756 (2021)
23. Huybrechts, D.: Complex geometry. Universitext. Springer, Berlin (2005)
24. Lempert, L., Szőke, R.: Direct images, fields of Hilbert spaces, and geometric quantization. Commun. Math. Phys. **327**(1), 49–99 (2014)
25. Liu, K., Yang, X.: Curvatures of direct image sheaves of vector bundles and applications. J. Differ. Geom. **98**(1), 117–145 (2014)
26. Mourougane, C., Takayama, S.: Hodge metrics and the curvature of higher direct images. Ann. Sci. Éc. Norm. Supér. (4) **41**(6), 905–924 (2008)
27. Manivel, L.: Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien. Math. Z. **212**(1), 107–122 (1993)
28. Ohsawa, T.: On the extension of L^2 holomorphic functions V: effects of generalization. Nagoya Math. J. **161**, 1–21 (2001)
29. Ohsawa, T.: L^2 approaches in several complex variables, development of Oka–Cartan theory by L^2 estimates for the $\bar{\partial}$ operator, Springer Monographs in Mathematics. Springer, Tokyo (2015)

30. Ohsawa, T., Takegoshi, K.: On the extension of L^2 holomorphic functions. Math. Z. **195**(2), 197–204 (1987)
31. Păun, M., Takayama, S.: Positivity of twisted relative pluricanonical bundles and their direct images. J. Algebr. Geom. **27**(2), 211–272 (2018)
32. Raufi, H.: Log concavity for matrix-valued functions and a matrix-valued Prékopa Theorem. arXiv:1311.7343
33. Zhou, X., Zhu, L.: An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds. J. Differ. Geom. **110**(1), 135–186 (2018)
34. Zhou, X., Zhu, L.: Siu's lemma, optimal L^2 extension and applications to twisted pluricanonical sheaves. Math. Ann. **377**(1–2), 675–722 (2020)
35. Zhou, X., Zhu, L.: Optimal L^2 extension of sections from subvarieties in weakly pseudoconvex manifolds. Pac. J. Math. **309**(2), 475–510 (2020). arXiv:1909.08820v1