Mathematische Annalen (2021) 381:575–591
https://doi.org/10.1007/s00208-021-02179-6

Mathematische Annalen

On a conjecture of Montgomery and Soundararajan

Régis de la Bretèche[1] · Daniel Fiorilli[2]

Received: 2 February 2021 / Revised: 31 March 2021 / Accepted: 7 April 2021 / Published online: 5 May 2021

Abstract

We establish lower bounds for all weighted even moments of primes up to X in intervals which are in agreement with a conjecture of Montgomery and Soundararajan. Our bounds hold unconditionally for an unbounded set of values of X, and hold for all X under the Riemann Hypothesis. We also deduce new unconditional Ω-results for the classical prime counting function.

1 Introduction

The goal of this paper is to investigate [24, Conjecture 1]. Let

$$\mu_n := \begin{cases} \frac{(2m)!}{2^m m!} & \text{if } n = 2m \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

be the n-th moment of the Gaussian.

Conjecture 1.1 *(Montgomery, Soundararajan) Fix* $\varepsilon > 0$. *For each fixed* $n \in \mathbb{N}$ *and uniformly for* $\frac{(\log X)^{1+\varepsilon}}{X} \leq \delta \leq \frac{1}{X^{\varepsilon}}$,

$$\frac{1}{X}\int_1^X \frac{(\psi(x+\delta X) - \psi(x) - \delta X)^n}{X^{\frac{n}{2}}} \mathrm{d}x = (\mu_n + o(1))\big(\delta \log(\delta^{-1})\big)^{\frac{n}{2}}. \tag{1.2}$$

Communicated by Kannan Soundararajan.

✉ Régis de la Bretèche
regis.delabreteche@imj-prg.fr

Daniel Fiorilli
daniel.fiorilli@universite-paris-saclay.fr

1 Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université de Paris, Sorbonne Université, CNRS UMR 7586, Case Postale 7012, 75251 Paris Cedex 13, France

2 Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, 91405 Orsay, France

In the range $X^{-1}(\log X)^{1+\varepsilon} \le \delta \le X^{-1+\frac{1}{n}}$, Montgomery and Soundararajan [24, Theorem 3] have shown that (1.2) follows from a strong form of the Hardy-Littlewood prime k-tuple conjecture. For applications on the distribution of gaps between primes, see for instance [2].

Currently, many results towards Conjecture 1.1 are known in the case $n = 2$ (see the remarks following Theorem 1.2 below for a description of the work of Selberg, Goldston, Montgomery, and others on this topic), but little is known for higher moments. This is in contrast with the theory of moments of L-functions, in which we have lower and upper bounds of the correct order of magnitude for higher moments in several different families thanks to the work of Ramachandra [29], Rudnick and Soundararajan [31], Soundararajan [34], Harper [14], Radziwiłł-Soundararajan [28], and others.

In the current paper, we establish lower bounds for a weighted version of (1.2) for all even n, for values of δ that are relatively close to 1. In addition to being the first estimate on higher moments, we believe that our bounds are sharp up to a power-saving error term in δ (cf. [24, Theorem 3]). Prior to our work, the order of magnitude of the left hand side of (1.2) and some variants was known under RH for $n = 2$ and in various ranges of δ. However, the determination of the exact asymptotic size has been shown to be strongly related with deep simplicity and pair-correlation type estimates [1,3,5,8,13,22,23,26].

The key idea which will allow us to circumvent the need to understand spacing statistics and Diophantine properties (for higher moments) of zeros of the zeta function is a positivity argument in the explicit formula. Such an argument in conjunction with Parseval's identity has been successfully used in previous works on the variance (see e.g. [7]), however the novelty in the present paper is to avoid the need for Parseval's identity (in particular for higher moments).

For any fixed $\kappa > 0$, we define the class of test functions $\mathcal{E}_\kappa \subset \mathcal{L}^1(\mathbb{R})$ to be the set of all differentiable[1] even $\eta : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$\eta(t), \eta'(t) \ll \mathrm{e}^{-\kappa|t|}, \tag{1.3}$$

moreover $\widehat{\eta}(0) > 0$ and for all $\xi \in \mathbb{R}$ we have that[2]

$$0 \le \widehat{\eta}(\xi) \ll (|\xi| + 1)^{-1} \log(|\xi| + 2)^{-2-\kappa}. \tag{1.4}$$

We consider the following weighted version of $x^{-\frac{1}{2}}(\psi(x + \delta x) - \psi(x) - \delta x)$. For $\eta \in \mathcal{E}_\kappa$ and $\delta < 2\kappa$, we define

$$\psi_\eta(x, \delta) := \sum_{n\ge 1} \frac{\Lambda(n)}{n^{\frac{1}{2}}} \eta\Big(\delta^{-1} \log\Big(\frac{n}{x}\Big)\Big).$$

Morally, this function counts prime powers in the interval $[x(1-O(\delta)), x(1+O(\delta))]$, in which the weight $n^{-\frac{1}{2}}$ is equal to $x^{-\frac{1}{2}}(1 + O(\delta))$. The expected main term for

[1] One can replace differentiability by a Lipschitz condition if for instance η is compactly supported in $\mathbb{R}$ and monotonic on $\mathbb{R}_{\ge 0}$.

[2] We can take for example $\eta = \eta_0 \star \eta_0$ for some smooth and rapidly decaying η_0.

$\psi_\eta(x,\delta)$ is given by

$$\int_0^\infty \frac{\eta(\delta^{-1}\log(\frac{t}{x}))}{t^{\frac12}}\mathrm{d}t = x^{\frac12}\delta\int_{\mathbb{R}} \mathrm{e}^{\frac{\delta w}{2}}\eta(w)\mathrm{d}w,$$

which we will denote by $x^{\frac12}\delta\mathcal{L}_\eta(\frac{\delta}{2})$ (note that for $\delta<\kappa$, $\mathcal{L}_\eta(\frac{\delta}{2})=\mathcal{L}_\eta(-\frac{\delta}{2})=\widehat{\eta}(0)+O(\delta)$). Subtracting this main term is equivalent to summing $\Lambda(n)-1$ instead of $\Lambda(n)$ (more precisely, it is equivalent to working with the measure $\mathrm{d}(\psi(t)-t)$). We also consider the set $\mathcal{U}$ of non-trivial even integrable functions $\Phi:\mathbb{R}\to\mathbb{R}$ such that $\Phi,\widehat{\Phi}\geq 0$ (in particular, $\Phi(0)>0$). Finally, for $h:\mathbb{R}\to\mathbb{R}$ we define

$$\alpha(h):=\int_{\mathbb{R}} h(t)\mathrm{d}t;\qquad \beta(h):=\int_{\mathbb{R}} h(t)(\log|t|)\mathrm{d}t, \tag{1.5}$$

whenever these integrals converge. Here is our main RH result on the n-th moment

$$M_n(X,\delta;\eta,\Phi):=\frac{1}{(\log X)\int_0^\infty\Phi}\int_1^\infty \Phi\Big(\frac{\log x}{\log X}\Big)\big(\psi_\eta(x,\delta)-x^{\frac12}\delta\mathcal{L}_\eta\big(\tfrac{\delta}{2}\big)\big)^n\frac{\mathrm{d}x}{x}.$$

Theorem 1.2 *Assume RH, and let* $0<\kappa<\frac12$, $\eta\in\mathcal{E}_\kappa$, $\Phi\in\mathcal{U}$. *For* $n\in\mathbb{N}$, $X\in\mathbb{R}_{\geq 2}$, $\delta\in(0,\kappa)$, *and in the range* $n\leq\delta^{-\frac12}(\log(\delta^{-1}+2))^{\frac12}$, *we have that*

$$(-1)^n M_n(X,\delta;\eta,\Phi)\geq \mu_n\delta^{\frac n2}\big(\alpha(\widehat{\eta}^2)\log(\delta^{-1})+\beta(\widehat{\eta}^2)\big)^{\frac n2} \Big(1+O_{\kappa,\eta}\Big(\frac{n^2\delta}{\log(\delta^{-1}+2)}\Big)\Big)+O_\Phi\Big(\delta\frac{(K_\eta\log(\delta^{-1}+2))^n}{\log X}\Big), \tag{1.6}$$

where the implied constants and $K_\eta>0$ *are independent of* n, X *and* δ.

Remark 1.1 (1) For $n=2$ and in the range $X^{-c(\eta,\Phi)}\leq\delta\leq 1$, (1.6) implies a lower bound with the predicted main term as well as a secondary term conjectured in the work of Montgomery and Soundararajan [23, (2)]. Here, $c(\eta,\Phi)>0$ is a constant. Variants of this particular case (with various weights and measures) have attracted a lot of attention since Selberg's foundational work [33]. This includes Goldston and Montgomery's RH upper bound [8] in the whole range $0<\delta\leq 1$, Saffari and Vaughan's unconditional upper bound [32] in the range $x^{-\frac56+\varepsilon}\leq\delta\leq 1$, Goldston's GRH lower bound [4,6] in the range $x^{-1}\leq\delta\leq x^{-\frac34}$ (unconditional for $x^{-1}\leq\delta\leq x^{-1}(\log x)^A$), its generalization by Özlük [27] and Goldston and Yildirim [9,10] to a fixed arithmetic progression, Zaccagnini's unconditional upper bound [36,37] in the range $x^{-\frac56-\varepsilon}\leq\delta\leq 1$ (building on the work of Huxley [17] and Heath-Brown [16]), and others.

(2) For $n=2m$ with $m\geq 2$ and in the interval $(\log X)^{-\frac{1}{m-1}+o(1)}\leq\delta\ll 1$, we obtain a lower bound which is in agreement with Conjecture 1.1.

(3) Goldston and Yildirim [11,12] have computed the first three moments of a related quantity involving a major arcs approximation of $\Lambda(n)$, and deduced that in the

range $X \le x \le 2X$, $X^{-1}(\log X)^{14} \ll \delta \le X^{-\frac{6}{7}-\varepsilon}$ and under GRH, $\psi(x+\delta X) - \psi(x) - \delta X = \Omega_{\pm}((\delta x \log x)^{\frac12})$.

(4) In the function field case, estimates for the variance of $\Lambda(n)$ and more general arithmetic sequences have been established by Keating and Rudnick [19,20] and Rodgers [30]. Moreover, Hast and Matei [15] have given a geometric interpretation for the higher moments.

We now rephrase Theorem 1.2 and state our unconditional results.

Corollary 1.3 *Let $0 < \kappa < \frac12$, $\eta \in \mathcal{E}_\kappa$, and $\Phi \in \mathcal{U}$. Let moreover $f : \mathbb{R}_{\ge 0} \to (0, \frac12]$ be any function such that $\lim_{x\to\infty} f(x) = 0$, and let $\delta \in (0,1)$, $m \in \mathbb{N}$ and $X \in \mathbb{R}_{\ge 2}$ be such that either $m = 1$ and $\delta \in (X^{-f(X)}, f(X)]$, or $2 \le m \le \min(\delta^{-\frac12}(\log(\delta^{-1}+2))^{\frac12} f(X)^{\frac12}, \log\log X)$ and $\delta \in ((\log X)^{-\frac{1}{m-1}}(\log\log X)^4, f(X)]$. Then under RH we have that*

$$M_{2m}(X,\delta;\eta,\Phi) \ge \mu_{2m}\delta^m \big(\alpha(\widehat{\eta}^2)\log(\delta^{-1}) + \beta(\widehat{\eta}^2)\big)^m \Big(1 + O\Big(f(X) + \frac{1}{(\log(\delta^{-1}))^2}\Big)\Big). \tag{1.7}$$

Unconditionally, there exists a sequence $\{X_j\}_{j\ge1}$ tending to infinity such that whenever $X = X_j$, (1.7) holds with $m = 1$ and $\delta \in (X^{-f(X)}, f(X)]$. The same statement holds in the range $2 \le m \le \min(\delta^{-\frac12}, \log\log X)$ and $\delta \in ((\log X)^{-\frac{1}{m-1}}(\log\log X)^4, f(X)]$.

We now state our unconditional Ω-results for the usual prime counting function in short intervals $\psi(x+\delta x) - \psi(x) - \delta x$. Note that this quantity has standard deviation of order $(\delta x \log(\delta^{-1}+2))^{\frac12}$. We will show that $\psi(x+\delta x) - \psi(x) - \delta x$ can be larger than an unbounded constant times this.

Corollary 1.4 *Let $\varepsilon > 0$ be small enough. There exists a sequence $\{(x_j,\delta_j)\}_{j\ge1}$ with $\delta_j \in \Big[\varepsilon \frac{(\log_3 x_j)^{\frac92}}{(\log x_j)^2(\log_2 x_j)^{\frac52}}, 2\frac{(\log_3 x_j)^3}{(\log_2 x_j)^2}\Big]$, $\lim_{j\to\infty} x_j = \infty$, and such that*

$$\big|\psi(x_j+\delta_j x_j) - \psi(x_j) - \delta_j x_j\big| \gg \delta_j^{-\frac14}(\log(\delta_j^{-1}+2))^{\frac14} \cdot \big(\delta_j x_j \log(\delta_j^{-1}+2)\big)^{\frac12}.$$

If instead we require that $\delta_j \in \Big[(\log x_j)^{-\frac72-\frac{3}{2M}}, (\log x_j)^{-\frac{1}{M+1}}\Big]$ for some large fixed $M \in \mathbb{Z}_{\ge2}$, then we can choose the sequence $\{(x_j,\delta_j)\}_{j\ge1}$ in such a way that

$$\big|\psi(x_j+\delta_j x_j) - \psi(x_j) - \delta_j x_j\big| \gg M^{\frac12} \cdot \big(\delta_j x_j \log(\delta_j^{-1}+2)\big)^{\frac12}.$$

2 Proof of Theorem 1.2

Throughout this section, we will denote by $\varrho = \beta + i\gamma$ the non-trivial zeros of the Riemann zeta function. We recall the Riemann-von Mangoldt formula

$$N(T) := \{\varrho : 0 \le \Im m(\varrho) \le T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log(T+2)), \tag{2.1}$$

which is valid for $T \ge 0$.

A major ingredient in our proof is the following explicit formula for $\psi_\eta(x,\delta)$ and a related quantity.

Lemma 2.1 *Let* $0 < \kappa < \frac{1}{2}$ *and*[3] $\eta \in \mathcal{E}_\kappa$. *For* $t \ge 0$ *and* $0 < \delta < \kappa$ *we have the formulas*

$$\psi_\eta(\mathrm{e}^t,\delta) - \mathrm{e}^{\frac{t}{2}}\delta\mathcal{L}_\eta(\tfrac{\delta}{2}) = -\delta\sum_{\varrho}\mathrm{e}^{(\varrho-\frac12)t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big) + O_{\kappa,\eta}(E_{\kappa,\eta}(t,\delta)); \tag{2.2}$$

$$\begin{aligned}\mathrm{e}^{-\frac{t}{2}}\Big(\sum_{n\ge1}\Lambda(n)\eta(\delta^{-1}(\log n - t)) - \mathrm{e}^t\delta\mathcal{L}_\eta(\delta)\Big) &= -\delta\sum_{\varrho}\mathrm{e}^{(\varrho-\frac12)t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho}{i}\Big)\\ &\quad + O_{\kappa,\eta}\Big(\mathrm{e}^{-\frac{t}{2}}(\delta + E_{\kappa,\eta}(t,\delta))\Big),\end{aligned} \tag{2.3}$$

where ϱ *runs over the nontrivial zeros of* $\zeta(s)$, *and*

$$E_{\kappa,\eta}(t,\delta) := \begin{cases}\delta\mathrm{e}^{-\frac{t}{2}} + \log(\delta^{-1}+2)\mathrm{e}^{-\frac{\kappa t}{\delta}} & \text{if } t \ge 1,\\ \frac{\delta}{t} + \log(\delta^{-1}+2)\mathrm{e}^{-\frac{\kappa t}{\delta}} & \text{if } \delta \le t < 1,\\ \log(\delta^{-1}+2) & \text{if } 0 \le t \le \delta.\end{cases} \tag{2.4}$$

Under RH we have the uniform bound

$$\psi_\eta(\mathrm{e}^t,\delta) - \mathrm{e}^{\frac{t}{2}}\delta\mathcal{L}_\eta(\tfrac{\delta}{2}) \ll_{\kappa,\eta} \log(\delta^{-1}+2). \tag{2.5}$$

If in addition to RH we assume that $\widehat{\eta}(s) \ll (1+|s|)^{-2-\varepsilon}$ *for some* $\varepsilon \ge 0$ *and whenever* $|\Im m(s)| \le \frac{1}{2}$, *then we have the estimate*

$$\begin{aligned}\mathrm{e}^{-\frac{t}{2}}\Big(\sum_{n\ge1}\Lambda(n)\eta(\delta^{-1}(\log n - t)) - \mathrm{e}^t\delta\mathcal{L}_\eta(\delta)\Big) &= \psi_\eta(\mathrm{e}^t,\delta) - \mathrm{e}^{\frac{t}{2}}\delta\mathcal{L}_\eta(\tfrac{\delta}{2})\\ &\quad + O_{\kappa,\eta}\Big(\delta^{\frac12+\frac{\varepsilon}{2(2+\varepsilon)}}\log(\delta^{-1}+2) + E_{\kappa,\eta}(t,\delta)\Big).\end{aligned} \tag{2.6}$$

[3] Instead of assuming that η is differentiable, one can assume that it is Lipschitz, compactly supported in $\mathbb{R}$ and monotonic on $\mathbb{R}_{\ge0}$.

Proof To show (2.2) we apply [25, Theorem 12.13] with $F(u) := \eta(\frac{t+2\pi u}{\delta})$, so that $\widehat{F}(\xi) = \mathrm{e}^{i\xi t}\frac{\delta}{2\pi}\widehat{\eta}(\frac{\delta\xi}{2\pi})$. We obtain that

$$\begin{aligned}\psi_\eta(\mathrm{e}^t,\delta) - \mathrm{e}^{\frac{t}{2}}\delta\mathcal{L}_\eta(\tfrac{\delta}{2}) + \delta\sum_{\varrho}\mathrm{e}^{(\varrho-\frac12)t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big)\\ =\mathrm{e}^{-\frac{t}{2}}\delta\int_{\mathbb{R}}\mathrm{e}^{\frac{\delta x}{2}}\eta(x)\mathrm{d}x - \sum_{n\geq 1}\frac{\Lambda(n)}{n^{\frac12}}\eta\Big(\frac{t+\log n}{\delta}\Big) + \Big(\frac{\Gamma'}{\Gamma}\Big(\frac14\Big) - \log\pi\Big)\eta\Big(\frac{t}{\delta}\Big)\\ + \int_0^\infty \frac{\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}}\Big\{2\eta\Big(\frac{t}{\delta}\Big) - \eta\Big(\frac{t+x}{\delta}\Big) - \eta\Big(\frac{t-x}{\delta}\Big)\Big\}\mathrm{d}x.\end{aligned}$$

A careful analysis of the second integral yields the bound (2.4) whenever $\eta \in \mathcal{E}_\kappa$.

The proof of (2.3) is similar, with the choice $F(u) := \mathrm{e}^{-\pi u}\eta(\frac{t+2\pi u}{\delta})$, so that

$$\int_{\mathbb{R}} F(u)\mathrm{e}^{-(\xi-\frac12)2\pi u}\mathrm{d}u = \mathrm{e}^{\xi t}\frac{\delta}{2\pi}\widehat{\eta}\Big(\frac{\delta\xi}{2\pi i}\Big).$$

The uniform bound (2.5) follows from the triangle inequality and a straightforward application of the Riemann-von Mangoldt formula (2.1).

We now move to (2.6). It is sufficient to establish the bound

$$\delta\sum_{\varrho}\mathrm{e}^{\varrho t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho}{i}\Big) - \delta\sum_{\varrho}\mathrm{e}^{\varrho t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big) \ll_{\kappa,\eta} \mathrm{e}^{\frac{t}{2}}\delta^{\frac12+\frac{\varepsilon}{2(2+\varepsilon)}}\log(\delta^{-1}+2).$$

To show this, we first truncate the infinite sums. Our conditions on η imply that

$$\delta\sum_{|\varrho|>\delta^{-\frac{3+\varepsilon}{2+\varepsilon}}}\mathrm{e}^{\varrho t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho}{i}\Big) - \delta\sum_{|\varrho|>\delta^{-\frac{3+\varepsilon}{2+\varepsilon}}}\mathrm{e}^{\varrho t}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big) \ll_{\kappa,\eta} \mathrm{e}^{\frac{t}{2}}\delta^{\frac12+\frac{\varepsilon}{2(2+\varepsilon)}}\log(\delta^{-1}+2).$$

The rest of the sums over ϱ is bounded by combining (2.1) with the bound

$$\begin{aligned}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big) - \widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho}{i}\Big) &= \int_{\mathbb{R}}(\mathrm{e}^{\delta(\varrho-\frac12)\xi} - \mathrm{e}^{\delta\varrho\xi})\eta(\xi)\mathrm{d}\xi\\ &\ll \int_{|\xi|\leq\delta^{-1}}\delta|\xi\eta(\xi)|d\xi + \int_{|\xi|>\delta^{-1}}\mathrm{e}^{\frac{\delta|\xi|}{2}}|\eta(\xi)|d\xi \quad (2.7)\\ &\ll_{\kappa,\eta} \delta + \frac{\mathrm{e}^{-\delta^{-1}\kappa}}{\kappa-\frac{\delta}{2}} \ll_\kappa \delta.\end{aligned}$$

□

The following estimate on a convergent sum over zeros will be helpful in calculating the main terms in our lower bounds on moments.

Lemma 2.2 *Let $0 < \kappa < \frac{1}{2}$, and let $h : \mathbb{R} \to \mathbb{R}$ be a measurable function such that for all $\xi \in \mathbb{R}$, $0 \le h(\xi) \ll (|\xi| + 1)^{-2}(\log(|\xi| + 2))^{-2-\kappa}$, and*[4] *for all $t \in \mathbb{R}$, $\widehat{h}(t), \widehat{h}'(t) \ll \mathrm{e}^{-\kappa|t|}$. For $0 < \delta < 2\kappa$ we have that*

$$\sum_{\varrho} h\Big(\frac{\delta}{2\pi}\frac{\varrho - \frac{1}{2}}{i}\Big) = \alpha(h)\delta^{-1}\log(\delta^{-1}) + \beta(h)\delta^{-1} + O_{\kappa,h}(1), \tag{2.8}$$

where ϱ is running over the non-trivial zeros of the Riemann zeta function, and where h is extended to $\{s \in \mathbb{C} : |\Im m(s)| < \frac{\kappa}{2\pi}\}$ by writing

$$h(z) := \int_{\mathbb{R}} \mathrm{e}^{2\pi i z\xi}\widehat{h}(\xi)\mathrm{d}\xi. \tag{2.9}$$

Proof The claimed estimate can be established with a slightly weaker error term (and a different class of functions h) using the Riemann-von Mangoldt formula (2.1) and the bound

$$h\Big(\frac{\delta}{2\pi}\frac{\varrho - \frac{1}{2}}{i}\Big) - h\Big(\frac{\delta\Im m(\varrho)}{2\pi}\Big) \ll_{\kappa,h} \delta, \tag{2.10}$$

which follows from a calculation similar to (2.7). To obtain the claimed error term, we will use a different technique. Applying the explicit formula [25, Theorem 12.13] with $F(x) := 2\pi\delta^{-1}\widehat{h}(-2\pi\delta^{-1}x)$, we obtain that

$$\sum_{\varrho} h\Big(\frac{\delta}{2\pi}\frac{\varrho - \frac{1}{2}}{i}\Big) = \delta^{-1}\big(b_1(h) + b_2(h) + I(h)\big) + h\Big(\frac{i\delta}{4\pi}\Big) + h\Big(-\frac{i\delta}{4\pi}\Big), \tag{2.11}$$

where

$$b_1(h) := \Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}\Big) - \log\pi\Big)\widehat{h}(0);$$
$$b_2(h) := -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}}\big(\widehat{h}(\delta^{-1}\log n) + \widehat{h}(-\delta^{-1}\log n)\big);$$
$$I(h) := \int_0^{\infty}\frac{\mathrm{e}^{-\frac{x}{2}}}{1 - \mathrm{e}^{-2x}}\big(2\widehat{h}(0) - \widehat{h}(\delta^{-1}x) - \widehat{h}(-\delta^{-1}x)\big)\mathrm{d}x.$$

Integration by parts shows that $b_2(h) \ll_{\kappa} 2^{-\kappa\delta^{-1}}$. We split the integral $I(h)$ into the three ranges $[0, \delta]$, $[\delta, 1]$, $[1, +\infty)$, and denote by $I_1(h)$, $I_2(h)$, $I_3(h)$ the respective integrals. We have that

4 The integrability of $\xi h(\xi)$ implies that $\widehat{h}$ is differentiable (see [21, p. 430]).

$$I_3(h) = \widehat{h}(0)\int_1^\infty \frac{2\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}}\mathrm{d}x + O_h(\mathrm{e}^{-\frac{\kappa}{\delta}}).$$

Moreover,

$$I_2(h) = \widehat{h}(0)\log(\delta^{-1}) + \widehat{h}(0)\int_0^1\Big(\frac{2\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}} - \frac{1}{x}\Big)\mathrm{d}x$$
$$- \int_{\mathbb{R}} h(\xi)\int_1^\infty \cos(2\pi x\xi)\frac{\mathrm{d}x}{x}\mathrm{d}\xi + O_h(\delta).$$

As for $I_1(h)$, we obtain that

$$I_1(h) = \int_{\mathbb{R}} h(\xi)\int_0^1 (1-\cos(2\pi x\xi))\frac{\mathrm{d}x}{x}\mathrm{d}\xi + O_h(\delta).$$

Collecting our estimates for $I_1(h), I_2(h), I_3(h)$ as well as the estimate $h(\pm\frac{i\delta}{4\pi}) = h(0) + O_h(\delta)$, we deduce that

$$\delta\sum_{\varrho} h\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac{1}{2}}{i}\Big) = \widehat{h}(0)\big(\log(\delta^{-1}) + C\big) + \int_{\mathbb{R}} h(\xi)\log|\xi|\mathrm{d}\xi + O_{\kappa,h}(\delta),$$

where

$$C := \int_0^1 \Big(\frac{2\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}} - \frac{1}{x}\Big)\mathrm{d}x + \int_1^\infty \frac{2\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}}\mathrm{d}x + \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}\Big) - \log\pi$$
$$+ \int_0^1 (1-\cos(2\pi x))\frac{\mathrm{d}x}{x} - \int_1^\infty \cos(2\pi x)\frac{\mathrm{d}x}{x}.$$

We will show that $C = 0$, from which the claimed estimate follows. We have the identity [35, §II.0, Exercise 149]

$$\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}\Big) = \int_0^\infty \Big(\frac{\mathrm{e}^{-2x}}{x} - \frac{2\mathrm{e}^{-\frac{x}{2}}}{1-\mathrm{e}^{-2x}}\Big)\mathrm{d}x.$$

We deduce that

$$C = \int_0^\infty \frac{\mathrm{e}^{-2x} - \cos(2x)}{x}\mathrm{d}x,$$

which is readily shown to be equal to zero using the residue theorem. □

We will also need the following combinatorial lemma.

Lemma 2.3 *Let* $0 < \kappa < \frac{1}{2}$, $\eta \in \mathcal{E}_\kappa$, *and assume*[5] *RH. For* $\delta \in (0, \kappa)$, $m \in \mathbb{N}$, *and in the range* $m \le \delta^{-\frac{1}{2}}(\log(\delta^{-1}+2))^{\frac{1}{2}}$, *we have the lower bound*

$$\delta^{2m} \sum_{\substack{\gamma_1,\dots,\gamma_{2m} \\ \gamma_1+\cdots+\gamma_{2m}=0}} \widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\cdots\widehat{\eta}\Big(\frac{\delta\gamma_{2m}}{2\pi}\Big)$$
$$\ge \mu_{2m}\delta^m\big(\alpha(\widehat{\eta}^2)\log(\delta^{-1})+\beta(\widehat{\eta}^2)\big)^m\Big(1+O_{\kappa,\eta}\Big(\frac{m^2\delta}{\log(\delta^{-1}+2)}\Big)\Big),$$

where the γ_j *are running over the imaginary parts of the non-trivial zeros of the Riemann zeta function.*

Proof We will show that

$$M_{2m} := \sum_{\substack{\gamma_1,\dots,\gamma_{2m} \\ \gamma_1+\cdots+\gamma_{2m}=0}} \widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\cdots\widehat{\eta}\Big(\frac{\delta\gamma_{2m}}{2\pi}\Big) \ge \mu_{2m}\big(s_2^m - m(m-1)s_2^{m-2}s_4\big), \tag{2.12}$$

where $s_{2j} := \sum_\gamma |\widehat{\eta}(\frac{\delta\gamma}{2\pi})|^{2j}$. Combining this bound with Lemma 2.2 with $h = |\widehat{\eta}|^2 = \widehat{\eta}^2$ and $h = |\widehat{\eta}|^4 = \widehat{\eta}^4$ implies the claimed bound. One can check that $\eta \in \mathcal{E}_\kappa$ implies that for both those choices of h, we have the bounds $\widehat{h}(t), \widehat{h}'(t) \ll (|t|^3+1)\mathrm{e}^{-\kappa|t|}$.

Now, to establish (2.12), note that this is an equality for $m = 1$, and is clear for $m = 2$. In the general case, we have that

$$M_{2m} \ge M'_{2m} := \sum_{\substack{\gamma_1,\dots,\gamma_{2m} \text{ distinct} \\ \gamma_1+\cdots+\gamma_{2m}=0}} \widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\cdots\widehat{\eta}\Big(\frac{\delta\gamma_{2m}}{2\pi}\Big).$$

Note that $M_2 = M'_2 = s_2$. One can restrict the sum in M'_{2m} to those $2m$-tuples of zeros for which for each $1 \le j \le 2m$, there exists $1 \le i \le 2m$, $i \ne j$, such that $\gamma_i = -\gamma_j$. In other words, for each involution $\pi : \{1,\dots,2m\} \to \{1,\dots,2m\}$ with no fixed points, there exists a subset of $2m$-tuples of zeros $\gamma_1,\dots\gamma_{2m}$ such that for each $1 \le j \le 2m$, $\gamma_j = -\gamma_{\pi(j)}$. Note also that since the γ_j are distinct in M'_{2m}, the sets of $2m$-tuples associated to different involutions π are distinct. Since the total number of such involutions is equal to μ_{2m}, it follows that

$$M'_{2m} = \mu_{2m}\sum_{\gamma_1}\Big|\widehat{\eta}\Big(\tfrac{\delta\gamma_1}{2\pi}\Big)\Big|^2 \sum_{\gamma_3\notin\{\gamma_1,-\gamma_1\}}\Big|\widehat{\eta}\Big(\tfrac{\delta\gamma_3}{2\pi}\Big)\Big|^2\cdots$$
$$\sum_{\gamma_{2m-1}\notin\{\gamma_1,-\gamma_1,\dots,\gamma_{2m-3},-\gamma_{2m-3}\}}\Big|\widehat{\eta}\Big(\tfrac{\delta\gamma_{2m-1}}{2\pi}\Big)\Big|^2.$$

[5] One can obtain a slightly weaker but unconditional lower bound by applying (2.10) at the end of the argument.

Therefore, by symmetry we have that

$$
\begin{aligned}
\frac{M'_{2m}}{\mu_{2m}} &= \sum_{\gamma_1}\Big|\widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\Big|^2 \sum_{\gamma_3\notin\{\gamma_1,-\gamma_1\}}\Big|\widehat{\eta}\Big(\frac{\delta\gamma_3}{2\pi}\Big)\Big|^2\dots\Big\{s_2-2\Big|\widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\Big|^2-\dots-2\Big|\widehat{\eta}\Big(\frac{\delta\gamma_{2m-3}}{2\pi}\Big)\Big|^2\Big\}\\
&= \sum_{\gamma_1}\Big|\widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\Big|^2 \sum_{\gamma_3\notin\{\gamma_1,-\gamma_1\}}\Big|\widehat{\eta}\Big(\frac{\delta\gamma_3}{2\pi}\Big)\Big|^2\dots\Big\{s_2-2(m-1)\Big|\widehat{\eta}\Big(\frac{\delta\gamma_{2m-3}}{2\pi}\Big)\Big|^2\Big\}\\
&\geq \frac{M'_{2m-2}}{\mu_{2(m-1)}}s_2-2(m-1)s_2^{m-2}s_4.
\end{aligned}
$$

The claimed bound follows by induction on m. □

We are ready to prove our main theorem.

Proof of Theorem 1.2 We begin by applying Lemma 2.1. Under RH, we set $T := \log X$ and obtain that

$$
\begin{aligned}
(-1)^n M_n(\mathrm{e}^T,\delta;\eta,\Phi) &= \frac{(-1)^n}{T\int_0^\infty\Phi}\int_0^\infty \Phi\Big(\frac{t}{T}\Big)\big(\psi_\eta(\mathrm{e}^t,\delta)-\mathrm{e}^{\frac{t}{2}}\delta\mathcal{L}_\eta(\tfrac{\delta}{2})\big)^n\mathrm{d}t\\
&= \frac{\delta^n}{\int_0^\infty\Phi}\sum_{\gamma_1,\dots,\gamma_n}\widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\cdots\widehat{\eta}\Big(\frac{\delta\gamma_n}{2\pi}\Big)\int_0^\infty \mathrm{e}^{itT(\gamma_1+\dots+\gamma_n)}\Phi(t)\mathrm{d}t\\
&\quad + O\Big(\frac{\delta(K_\eta\log(\delta^{-1}+2))^n}{T}\Big)\\
&= \frac{\delta^n}{2\int_0^\infty\Phi}\sum_{\gamma_1,\dots,\gamma_n}\widehat{\Phi}\Big(\frac{T(\gamma_1+\dots+\gamma_n)}{2\pi}\Big)\widehat{\eta}\Big(\frac{\delta\gamma_1}{2\pi}\Big)\cdots\widehat{\eta}\Big(\frac{\delta\gamma_n}{2\pi}\Big)\\
&\quad + O\Big(\frac{\delta(K_\eta\log(\delta^{-1}+2))^n}{T}\Big),
\end{aligned}
$$

since both Φ and $\widehat{\Phi}$ are even and real-valued. Here, $\gamma_1,\dots,\gamma_n$ are running over the imaginary parts of the non-trivial zeros of $\zeta(s)$. If n is odd, then the claimed estimate follows from discarding the sum over zeros entirely. If n is even, then by positivity of $\widehat{\eta}$ and $\widehat{\Phi}$ we may only keep the terms for which $\gamma_1+\dots+\gamma_n=0$, and apply Lemma 2.3. The claimed lower bound follows. □

3 Proof of Corollaries 1.3 and 1.4

We first need to establish the following proposition, which is strongly inspired from the work of Kaczorowski and Pintz [18]. We consider

$$
F(x,\delta;\eta) := -\delta\sum_{\varrho}\frac{x^{\varrho-\frac12}}{\varrho-\frac12}\widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac12}{i}\Big),
$$

which is readily shown to be real-valued by grouping conjugate zeros.

Proposition 3.1 *Assume that RH is false, and let $\eta \in \mathcal{E}_\kappa$ with $0 < \kappa < \frac{1}{2}$. Then, there exists an absolute (ineffective) constant $\theta > 0$ and a sequence $\{x_j\}_{j\geq 1}$ tending to infinity such that for each $j \geq 1$ and uniformly for $x_j^{-\theta} \leq \delta \leq \delta_\eta$, where $\delta_\eta > 0$ is small enough, we have that*

$$F(x_j, \delta; \eta) > x_j^\theta.$$

Proof Consider, for $\Theta > 0$, the $(n-1)$-fold average

$$F_n(\mathrm{e}^t, \delta, \Theta; \eta) := -\delta \sum_{\varrho} \frac{\mathrm{e}^{(\varrho-\frac{1}{2})t}}{(\varrho-\frac{1}{2})^n} \widehat{\eta}\Big(\frac{\delta}{2\pi}\frac{\varrho-\frac{1}{2}}{i}\Big) - \delta \frac{\mathrm{e}^{\Theta t}}{\Theta^{n-1}},$$

so that $\frac{\mathrm{d}^{n-1}}{(\mathrm{d}t)^{n-1}} F_n(\mathrm{e}^t, \delta, \Theta; \eta) = F(\mathrm{e}^t, \delta; \eta) - \delta \mathrm{e}^{\Theta t}$. Let $\varrho_e = \beta_e + i\gamma_e$ be a zero of $\zeta(s)$ violating RH, of least positive imaginary part γ_e, and such that there is no other zero of imaginary part equal to γ_e but of greater real part. Let moreover $\varepsilon < \beta_e - \frac{1}{2}$. We will show that $F_n(\mathrm{e}^t, \delta, \Theta; \eta) = 0$ for many values of t (independently of δ), and then apply Rolle's theorem.

We pick $t = cn$, with $n \geq 1$ and $c \in \mathbb{R}$. If $\Theta \leq \varepsilon$ and c is large enough in terms of ε and Θ, say $c \geq c_0(\varepsilon)$ (later we will require that $c_0(\varepsilon) \geq 1$), then

$$\frac{\mathrm{e}^{cn\Theta}}{\Theta^{n-1}} < \left(\frac{\mathrm{e}^{c(\beta_e-\frac{1}{2})}}{2|\varrho_e-\frac{1}{2}|}\right)^n.$$

We will also impose c to be bounded in terms of ε and ϱ_e, say $c \leq c_1(\varepsilon)$. More precisely, we pick $c_1(\varepsilon) = c_0(\varepsilon) + 2$. Then, there exists U_ε large enough so that

$$\sum_{|\Im m(\varrho)|>U_\varepsilon} \frac{\mathrm{e}^{cn(\varrho-\frac{1}{2})}}{(\varrho-\frac{1}{2})^n} \widehat{\eta}\left(\frac{\delta}{2\pi}\frac{\varrho-\frac{1}{2}}{i}\right) \ll_{\kappa,\eta} (\log U_\varepsilon) \frac{\mathrm{e}^{c\frac{n}{2}}}{U_\varepsilon^{n-1}} < \left(\frac{\mathrm{e}^{c(\beta_e-\frac{1}{2})}}{2|\varrho_e-\frac{1}{2}|}\right)^n,$$

whenever $\delta \leq \kappa$, $n > n_0(\varepsilon)$ and $c_0(\varepsilon) < c < c_1(\varepsilon)$. Here we used the bound

$$\begin{aligned}\widehat{\eta}(s) &= \int_{\mathbb{R}} \mathrm{e}^{-2\pi i s x} \eta(x) \mathrm{d}x \\ &\ll \int_0^\infty \mathrm{e}^{2\pi|\Im m(s)|x} \mathrm{e}^{-\kappa x} \mathrm{d}x \ll \frac{1}{\kappa - 2\pi|\Im m(s)|} \qquad (|\Im m(s)| < \kappa/2\pi).\end{aligned}$$

We conclude that under these last two conditions,

$$F_n(\mathrm{e}^{cn}, \delta, \Theta; \eta) = -\delta \sum_{|\Im m(\varrho)|\leq U_\varepsilon} \frac{\mathrm{e}^{cn(\varrho-\frac{1}{2})}}{(\varrho-\frac{1}{2})^n} \widehat{\eta}\left(\frac{\delta}{2\pi}\frac{\varrho-\frac{1}{2}}{i}\right) + O\left(\delta\left(\frac{\mathrm{e}^{c(\beta_e-\frac{1}{2})}}{2|\varrho_e-\frac{1}{2}|}\right)^n\right).$$

For two distinct zeros ϱ_1, ϱ_2 of $\zeta(s)$ of positive imaginary part at most U_ε, consider the function

$$f : (c_0(\varepsilon), c_1(\varepsilon)) \to \mathbb{R}$$
$$c \mapsto c(\Re e(\varrho_1)-\tfrac{1}{2}) - \log|\varrho_1 - \tfrac{1}{2}| - c(\Re e(\varrho_2) - \tfrac{1}{2}) + \log|\varrho_2 - \tfrac{1}{2}|.$$

This linear function is not identically zero and has at most one zero, hence there exists a subset $S_1 \subset (c_0(\varepsilon), c_1(\varepsilon))$ which is a union of two intervals such that for all $c \in S_1$, $|f(c)| \geq \kappa_\varepsilon$, for some fixed and small enough $\kappa_\varepsilon > 0$. By picking κ_ε small enough, we may require that $\lambda(S_1) \geq 2 - 2^{-\#\{\varrho : \zeta(\varrho)=0,\, |\Im m(\varrho)| \leq U_\varepsilon\}}$, where λ is the Lebesgue measure. We may iterate this procedure with all pairs of distinct zeros ϱ_j, ϱ_k such that $0 < \Im m(\varrho_j), \Im m(\varrho_k) \leq U_\varepsilon$, and deduce that there exists a subset $S \subset (c_0(\varepsilon), c_1(\varepsilon))$ of measure ≥ 1 which is a disjoint union of at most $2^{\#\{\varrho : \zeta(\varrho)=0,\, 0<\Im m(\varrho)\leq U_\varepsilon\}} + 1$ intervals (α_j, τ_j) such that for each j and whenever $c \in (\alpha_j, \tau_j)$, there exists a zero $\varrho_j = \beta_j + i\gamma_j$ such that

$$c(\Re e(\varrho_j) - \tfrac{1}{2}) - \log|\varrho_j - \tfrac{1}{2}| - \max\{c(\Re e(\varrho) - \tfrac{1}{2}) - \log|\varrho - \tfrac{1}{2}| : \zeta(\varrho) = 0, 0 < \Im m(\varrho) \leq U_\varepsilon\} \geq \kappa_\varepsilon.$$

Then, denoting by m_j the multiplicity of ϱ_j, for all $c \in (\alpha_j, \tau_j)$ we have that

$$F_n(\mathrm{e}^{cn}, \delta, \Theta; \eta) = -\delta m_j \Re e\bigg(\frac{\mathrm{e}^{cn(\varrho_j - \frac{1}{2})}}{(\varrho_j - \frac{1}{2})^n} \widehat{\eta}\Big(\frac{\delta}{2\pi} \frac{\varrho_j - \frac{1}{2}}{i}\Big)\bigg) + O\bigg(\delta\Big(\frac{K_\varepsilon \mathrm{e}^{c(\beta_j - \frac{1}{2})}}{|\varrho_j - \frac{1}{2}|}\Big)^n\bigg),$$

where $0 < K_\varepsilon < 1$ is absolute. Note that for all small enough δ and for all j, we have that $\widehat{\eta}(\frac{\delta}{2\pi}\frac{\varrho_j - \frac{1}{2}}{i}) = \widehat{\eta}(0) + O(\delta)$. Hence,

$$F_n(\mathrm{e}^{cn}, \delta, \Theta; \eta) = -\delta m_j \Re e\bigg(\frac{\mathrm{e}^{cn(\varrho_j - \frac{1}{2})}}{(\varrho_j - \frac{1}{2})^n}\bigg)\widehat{\eta}(0) + O\bigg(\delta^2 m_j \Big(\frac{\mathrm{e}^{c(\beta_j - \frac{1}{2})}}{|\varrho_j - \frac{1}{2}|}\Big)^n + \delta\Big(\frac{K_\varepsilon \mathrm{e}^{c(\beta_j - \frac{1}{2})}}{|\varrho_j - \frac{1}{2}|}\Big)^n\bigg).$$

For n large enough, this function has at least $(\tau_j - \alpha_j)\Im m(\varrho_j)n/\pi + O(1) \geq 4(\tau_j - \alpha_j)n$ zeros for $c \in (\alpha_j, \tau_j)$. Indeed, this follows from the intermediate value theorem combined with the identity

$$\Re e\bigg(\frac{\mathrm{e}^{cn(\varrho_j - \frac{1}{2})}}{(\varrho_j - \frac{1}{2})^n}\bigg) = \frac{\mathrm{e}^{cn(\beta_j - \frac{1}{2})}}{|\varrho_j - \frac{1}{2}|^n} \cos(\nu_{j,c} n),$$

where $\nu_{j,c} := \Im m(\varrho_j)c - \Im m(\log(\varrho_j - \frac{1}{2}))$. Since this is true for every j, we conclude that $F_n(\mathrm{e}^{cn}, \delta, \Theta; \eta)$ has at least $4n\lambda(S) \geq 4n$ zeros for $c \in S$. In other words, $F_n(\mathrm{e}^t, \delta, \Theta; \eta)$ has at least $4n$ zeros for $t \in [c_0(\varepsilon)n, c_1(\varepsilon)n]$. By Rolle's theorem, we deduce that $F(\mathrm{e}^t, \delta; \eta) - \delta \mathrm{e}^{\Theta t}$ has at least $3n$ zeros on this interval (note that by our conditions on η, $F(\mathrm{e}^t, \delta; \eta)$ is continuous). In the range $\mathrm{e}^{-\theta t} \leq \delta$, the result follows whenever $0 < \theta < \Theta/2$. □

We are ready to prove our first unconditional result.

Proof of Corollary 1.3 If RH is true, then this is a particular case of Theorem 1.2. Let us then assume that RH is false. By Hölder's inequality we have that

$$M_{2m}(X, \delta; \eta, \Phi)^{\frac{1}{2m}} \geq \frac{1}{(\log X) \int_0^\infty \Phi} \int_1^\infty \Phi\Big(\frac{\log x}{\log X}\Big) \big|\psi_\eta(x, \delta) - x^{\frac{1}{2}} \delta \mathcal{L}_\eta(\tfrac{\delta}{2})\big| \frac{\mathrm{d}x}{x}$$
$$\geq \frac{c(\Phi)}{(\log X) \int_0^\infty \Phi} \int_1^{X^{\kappa(\Phi)}} \big(\psi_\eta(x, \delta) - x^{\frac{1}{2}} \delta \mathcal{L}_\eta(\tfrac{\delta}{2})\big) \frac{\mathrm{d}x}{x},$$

where $c(\Phi), \kappa(\Phi) > 0$. By Lemma 2.1, the integral is equal to

$$-\delta \sum_{\varrho} \frac{X^{\kappa(\Phi)(\varrho - \frac{1}{2})}}{\varrho - \frac{1}{2}} \widehat{\eta}\Big(\frac{\delta}{2\pi} \frac{\varrho - \frac{1}{2}}{i}\Big) + O_{\Phi,\eta}\big(\delta (\log(\delta^{-1} + 2))^2\big),$$

by the Riemann-von Mangoldt formula (2.1). The claimed Ω-result then follows from Proposition 3.1. □

In order to prove Corollary 1.4, we will apply Theorem 1.2 with $\eta(u) = \max(0, 1 - |u|)$. This is not an element of $\mathcal{E}_\kappa$ since it is not differentiable. However, as mentioned in its statement, one can go through the proof of Lemma 2.1 and check that it applies when η is Lipschitz, compactly supported, and monotonic on $\mathbb{R}_{\geq 0}$; we deduce that the same is true for Theorem 1.2 (note that the conditions of Lemma 2.2 are satisfied for $h = \widehat{\eta}^2$).

Proof of Corollary 1.4 If RH is false, then the result follows from an adaptation of the proof of Proposition 3.1. Rather than going through the proof, we highlight the two major differences. Firstly, the function we need to study is

$$-\sum_{\varrho} \frac{\mathrm{e}^{\varrho t}}{\varrho^n}((1+\delta)^{\varrho} - 1) - \delta \frac{\mathrm{e}^{(\frac{1}{2}+\Theta)t}}{(\frac{1}{2}+\Theta)^{n-1}},$$

which has the weight $(1+\delta)^{\varrho} - 1$ instead of $\delta \widehat{\eta}(\frac{\delta}{2\pi} \frac{\varrho - \frac{1}{2}}{i})$. However, this weight is $\ll \delta|\varrho|$ uniformly for all $0 < \delta \leq 1$ and $0 < \Re e(\varrho) < 1$. The second major difference is the proof that the existence of two zeros of the continuous and piecewise differentiable function

$$-\sum_{\varrho} \frac{\mathrm{e}^{\varrho t}}{\varrho^2}\big((1+\delta)^{\varrho} - 1\big) - \delta \frac{\mathrm{e}^{(\frac{1}{2}+\Theta)t}}{\frac{1}{2}+\Theta}$$

implies that the piecewise continuous function

$$-\sum_{\varrho}\frac{\mathrm{e}^{\varrho t}}{\varrho}\big((1+\delta)^{\varrho}-1\big)-\delta\mathrm{e}^{(\frac{1}{2}+\Theta)t}$$

has at least one non-negative value between those zeros. This can be done using a straightforward generalization of Rolle's theorem, which states that if f is continuous on $[a, b]$ for which $f(a)=f(b)$ and the one-sided derivatives

$$f^{\pm}(c):=\lim_{x\to c^{\pm}}\frac{f(x)-f(c)}{x-c}$$

exist for all $c\in(a,b)$, then there exists $c_0\in(a,b)$ such that $f^{+}(c_0)f^{-}(c_0)\leq 0$. The rest of the proof is similar.

We now assume RH. Let us also assume that for all large enough x and for all δ' in the range $\frac{\varepsilon_0(\log_3 x)^{\frac{9}{2}}}{4(\log x)^2(\log_2 x)^{\frac{5}{2}}}\leq\delta'\leq 2\frac{(\log_3 x)^3}{(\log_2 x)^2}$ we have that

$$\big|\psi(x+\delta' x)-\psi(x)-\delta' x\big|\leq\varepsilon_0\delta'^{-\frac{1}{4}}(\log(\delta'^{-1}+2))^{\frac{1}{4}}\cdot\big(\delta' x\log(\delta'^{-1}+2)\big)^{\frac{1}{2}},$$

where $\varepsilon_0>0$ is the implied constant in the first error term in (1.6).

Define $\eta(u):=\max(0,1-|u|)$, which is even, non-negative, compactly supported and monotonic for $u\geq 0$. Moreover, $\widehat{\eta}(\xi)=(\sin(\pi\xi)/(\pi\xi))^2\geq 0$. Now, for any $0<\delta\leq 1$, $x\geq 1$ and $x\mathrm{e}^{-\delta}\leq n\leq x\mathrm{e}^{\delta}$, we write $\eta(\delta^{-1}\log(\frac{n}{x}))=1-\delta^{-1}|\int_n^x\frac{\mathrm{d}t}{t}|$ and deduce that

$$\begin{aligned}\sum_{n\geq 1}\Lambda(n)\eta\Big(\delta^{-1}\log\Big(\frac{n}{x}\Big)\Big)-x\delta\mathcal{L}_{\eta}(\delta)&=\psi(x\mathrm{e}^{\delta})-\psi(x\mathrm{e}^{-\delta})\\&\quad-\delta^{-1}\Big(\int_x^{x\mathrm{e}^{\delta}}\Big(\sum_{t<n\leq x\mathrm{e}^{\delta}}\Lambda(n)\Big)\frac{\mathrm{d}t}{t}\\&\quad+\int_{x\mathrm{e}^{-\delta}}^{x}\Big(\sum_{x\mathrm{e}^{-\delta}<n\leq t}\Lambda(n)\Big)\frac{\mathrm{d}t}{t}\Big)-x\delta\int_{\mathbb{R}}\eta(u)\mathrm{e}^{\delta u}\mathrm{d}u\\&=\psi(x\mathrm{e}^{\delta})-\psi(x\mathrm{e}^{-\delta})-2x\sinh(\delta)\\&\quad-\delta^{-1}\Big(\int_x^{x\mathrm{e}^{A}}\Big(\psi(x\mathrm{e}^{\delta})-\psi(t)-(x\mathrm{e}^{\delta}-t)\Big)\frac{\mathrm{d}t}{t}\\&\quad+\int_{x\mathrm{e}^{-A}}^{x}\Big(\psi(t)-\psi(x\mathrm{e}^{-\delta})-(t-x\mathrm{e}^{-\delta})\Big)\frac{\mathrm{d}t}{t}\Big)\\&\quad+O\Big(\delta^{-1}x^{\frac{1}{2}}(\log x)^2(\delta-A)\Big),\end{aligned}$$

for any $0 < A < \delta$; in particular for $A = \delta - \varepsilon_0 \delta^{\frac{5}{4}} (\log(\delta^{-1}+2))^{\frac{3}{4}}/(\log x)^2$. Here we used the (trivial) RH bound

$$\psi(M) - \psi(N) - (M-N) \ll M^{\frac{1}{2}} (\log(M+2))^2 \quad (1 \leq N \leq M).$$

By our hypothesis, we deduce that for X large enough, $m \geq 2$, $\delta = (\log_3 X)^3/(\log_2 X)^2$ and in the range $\exp((\log X)^{\frac{1}{2}}) \leq x \leq X$,

$$x^{-\frac{1}{2}} \Big(\sum_{n \geq 1} \Lambda(n) \eta\Big(\delta^{-1} \log\Big(\frac{n}{x}\Big)\Big) - x\delta \mathcal{L}_\eta(\delta) \Big) \ll \varepsilon_0 \delta^{\frac{1}{4}} \big(\log(\delta^{-1}+2)\big)^{\frac{3}{4}}.$$

Combining this with (2.6) with $\varepsilon = 0$ (since $\widehat{\eta}(s) \ll (1+|s|)^{-2}$ for $|\Im m(s)| \leq \frac{1}{2}$, and recalling that the differentiability condition in Lemma 2.1 can be replaced by one of Lipschitz since η has compact support and is decreasing on $\mathbb{R}_{\geq 0}$), we deduce that

$$\psi_\eta(x,\delta) - x^{\frac{1}{2}} \delta \mathcal{L}_\eta(\tfrac{\delta}{2}) \ll \varepsilon_0 \delta^{\frac{1}{4}} \big(\log(\delta^{-1}+2)\big)^{\frac{3}{4}}.$$

Now, making the choice $\Phi = \eta$, this implies that for X large enough,

$$\begin{aligned} M_n(X,\delta;\eta,\Phi) &= \frac{1}{(\log X)\int_0^\infty \Phi} \int_{\exp((\log X)^{\frac{1}{2}})}^\infty \Phi\Big(\frac{\log x}{\log X}\Big)\Big(\psi_\eta(x,\delta;\eta) \\ &\quad - x^{\frac{1}{2}} \delta \mathcal{L}_\eta(\tfrac{\delta}{2})\Big)^{2m} \frac{\mathrm{d}x}{x} + O\big((\log X)^{-\frac{1}{2}} (K^{\frac{1}{2}} \varepsilon_0 \log(\delta^{-1}+2))^{2m}\big) \\ &\ll \big(K \varepsilon_0^2 \delta^{\frac{1}{2}} (\log(\delta^{-1}+2))^{\frac{3}{2}}\big)^m + (\log X)^{-\frac{1}{2}} \big(K^{\frac{1}{2}} \varepsilon_0 \log(\delta^{-1}+2)\big)^{2m}, \end{aligned}$$

where $K > 0$ is absolute and where we have bounded the part of the integral with $x \leq \exp((\log X)^{\frac{1}{2}})$ using the uniform bound in Lemma 2.1. Recalling that $\delta = (\log_3 X)^3/(\log_2 X)^2$, for $\varepsilon_0^2 \delta^{-\frac{1}{2}} (\log(\delta^{-1}+2))^{\frac{1}{2}} \leq m \leq \varepsilon_0 \delta^{-\frac{1}{2}} (\log(\delta^{-1}+2))^{\frac{1}{2}}$, we have that $(\log X)^{-\frac{1}{m-1}} \leq \delta \leq \delta_0$, and hence Theorem 1.2 implies the lower bound

$$\begin{aligned} M_n(X,\delta;\eta,\Phi) &\geq (1+O(\varepsilon_0^2)) \mu_{2m} \big(\tfrac{2}{3} \delta \log(\delta^{-1}+2)\big)^m \\ &\geq (2\pi)^{\frac{1}{2}} (1+O(\varepsilon_0^2)) \big(\tfrac{2m}{3\mathrm{e}} \delta \log(\delta^{-1}+2)\big)^m. \end{aligned}$$

When ε_0 is small enough, we obtain a contradiction as soon as the range

$$\varepsilon_0^2 K (\tfrac{3}{2}\mathrm{e} + \varepsilon_0) \delta^{-\frac{1}{2}} (\log(\delta^{-1}+2))^{\frac{1}{2}} \leq m \leq \varepsilon_0 \delta^{-\frac{1}{2}} (\log(\delta^{-1}+2))^{\frac{1}{2}}$$

contains an integer; this is clearly the case when ε_0 is small enough and X is large enough. The proof of the first statement follows. The proof of the second is similar. □

References

1. Bui, H.M., Keating, J.P., Smith, D.J.: On the variance of sums of arithmetic functions over primes in short intervals and pair correlation for L-functions in the Selberg class. J. Lond. Math. Soc. II **94**(1), 161–185 (2016)
2. Funkhouser, S., Goldston, D.A., Ledoan, A.H.: Distribution of large gaps between primes. In: Pintz, J., Rassias, M.T. (eds.) Irregularities in the Distribution of Prime Numbers, pp. 45–67. Springer, Cham (2018)
3. Gallagher, P.X., Mueller, J.H.: Primes and zeros in short intervals. J. Reine Angew. Math. **303**(304), 205–220 (1978)
4. Goldston, D.A.: The second moment for prime numbers. Q. J. Math. Oxford Ser. II **35**(138), 153–163 (1984)
5. Goldston, D.A.: On the pair correlation conjecture for zeros of the Riemann zeta-function. J. Reine Angew. Math. **385**, 24–40 (1988)
6. Goldston, D.A.: A lower bound for the second moment of primes in short intervals. Expos. Math. **13**(4), 366–376 (1995)
7. Goldston, D.A., Gonek, S.M.: A note on the number of primes in short intervals. Proc. Am. Math. Soc. **108**(3), 613–620 (1990)
8. Goldston, D.A., Montgomery, H. L.: Pair correlation of zeros and primes in short intervals. In: Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), Progr. Math., vol. 70, pp. 183–203. Birkhäuser Boston, Boston, MA (1987)
9. Goldston, D.A., Yildirim, C.Y.: Primes in short segments of arithmetic progressions. Can. J. Math. **50**(3), 563–580 (1998)
10. Goldston, D.A., Yildirim, C.Y.: On the second moment for primes in an arithmetic progression. Acta Arith. **100**(1), 85–104 (2001)
11. Goldston, D.A., Yildirim, C.Y.: Higher correlations of divisor sums related to primes. I. Triple correlations. Integers **3**(A5), 66 (2003)
12. Goldston, D.A., Yildirim, C.Y.: Higher correlations of divisor sums related to primes. II. Variations of the error term in the prime number theorem. Proc. Lond. Math. Soc. III **95**(1), 199–247 (2007)
13. Goldston, D.A., Gonek, S.M., Montgomery, H.L.: Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals. J. Reine Angew. Math. **537**, 105–126 (2001)
14. Harper, A.J.: Sharp conditional bounds for moments of the Riemann zeta function. arXiv:1305.4618
15. Hast, D.R., Matei, V.: Higher moments of arithmetic functions in short intervals: a geometric perspective. Int. Math. Res. Not. **21**, 6554–6584 (2019)
16. Heath-Brown, D.R.: The number of primes in a short interval. J. Reine Angew. Math. **389**, 22–63 (1988)
17. Huxley, M.N.: On the difference between consecutive primes. Invent. Math. **15**, 164–170 (1972)
18. Kaczorowski, J., Pintz, J.: Oscillatory properties of arithmetical functions. I. Acta Math. Hung. **48**(1–2), 173–185 (1986)
19. Keating, J.P., Rudnick, Z.: The variance of the number of prime polynomials in short intervals and in residue classes. Int. Math. Res. Not. **1**, 259–288 (2014)
20. Keating, J.P., Rudnick, Z.: Squarefree polynomials and Möbius values in short intervals and arithmetic progressions. Algebra Number Theory **10**(2), 375–420 (2016)
21. Kolmogorov, A.N. , Fomin, S. V.: *Элементы теории функций и функционального анализа*, 6 edn, 624 pp. Nauka, Moscow (1989). ISBN: 5-02-013993-9
22. Languasco, A., Perelli, A., Zaccagnini, A.: Explicit relations between pair correlation of zeros and primes in short intervals. J. Math. Anal. Appl. **394**(2), 761–771 (2012)
23. Montgomery, H. L., Soundararajan, K.: Beyond pair correlation. In: Paul Erdős and his Mathematics, I (Budapest, 1999), Bolyai Soc. Math. Stud., vol. 11, pp. 507–514. János Bolyai Math. Soc., Budapest (2002)
24. Montgomery, H.L., Soundararajan, K.: Primes in short intervals. Comm. Math. Phys. **252**(1–3), 589–617 (2004)
25. Montgomery, H.L., Vaughan, R.C.: Multiplicative number theory. I. Classical theory. In: Cambridge Studies in Advanced Mathematics, vol. 97. Cambridge University Press, Cambridge (2007)
26. Mueller, J.: Arithmetic equivalent of essential simplicity of zeta zeros. Trans. Am. Math. Soc. **275**(1), 175–183 (1983)

27. Özlük, A.E.: On the irregularity of distribution of primes in an arithmetic progression over short intervals. İstanbul Tek. Üniv. Bül. **40**(2), 255–264 (1987)
28. Radziwiłł, M., Soundararajan, K.: Moments and distribution of central L-values of quadratic twists of elliptic curves. Invent. Math. **202**(3), 1029–1068 (2015)
29. Ramachandra, K.: Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II. Hardy-Ramanujan J. **3**, 1–24 (1980)
30. Rodgers, B.: Arithmetic functions in short intervals and the symmetric group. Algebra Number Theory **12**(5), 1243–1279 (2018)
31. Rudnick, Z., Soundararajan, K.: Lower bounds for moments of L-functions. Proc. Natl. Acad. Sci. USA **102**(19), 6837–6838 (2005)
32. Saffari, B., Vaughan, R.C.: On the fractional parts of x/n and related sequences. II. Ann. Inst. Fourier (Grenoble) **27**(2), 1–30 (1977)
33. Selberg, A.: On the normal density of primes in small intervals, and the difference between consecutive primes. Arch. Math. Naturvid. **47**(6), 87–105 (1943)
34. Soundararajan, K.: Moments of the Riemann zeta function. Ann. Math. II **170**(2), 981–993 (2009)
35. Tenenbaum, G.: Introduction to analytic and probabilistic number theory. In: Graduate Studies in Mathematics, vol. 163, 3rd edn. American Mathematical Society, Providence, RI (2015)
36. Zaccagnini, A.: On the Selberg integral via Heath-Brown's identity. Riv. Mat. Univ. Parma **5**(5), 205–212 (1996)
37. Zaccagnini, A.: Primes in almost all short intervals. Acta Arith. **84**(3), 225–244 (1998)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.