



## Exotic $G_2$ -manifolds

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### Abstract

We exhibit the first examples of closed 7-dimensional Riemannian manifolds with holonomy  $G_2$  that are homeomorphic but not diffeomorphic. These are also the first examples of closed Ricci-flat manifolds that are homeomorphic but not diffeomorphic. The examples are generated by applying the twisted connected sum construction to Fano 3-folds of Picard rank 1 and 2. The smooth structures are distinguished by the generalised Eells–Kuiiper invariant introduced by the authors in a previous paper.

### 1 Introduction

Given a type of special geometric structure, it is often interesting to ask: do there exist manifolds with such structures that are homeomorphic but not diffeomorphic? In this paper we consider the case of Riemannian metrics with holonomy  $G_2$  on closed manifolds of dimension 7. The Lie group  $G_2$  can be described as the automorphism group of the octonion algebra  $\mathbb{O}$ , and its natural action on  $\text{Im } \mathbb{O} \cong \mathbb{R}^7$  appears as an exceptional case in Berger’s classification of Riemannian holonomy [3]. Metrics with holonomy  $G_2$  are always Ricci-flat [5].

A general strategy to address the question of the first paragraph is to apply a smooth classification theorem to a plentiful supply of examples for which the classifying invariants are computable. In this paper we make use of the “twisted connected sum” construction of closed  $G_2$ -manifolds introduced by Kovalev [27]; it was shown in [10] that this construction yields large numbers of closed  $G_2$ -manifolds that are *2-connected* (i.e. the homotopy groups  $\pi_1$  and  $\pi_2$  are trivial) with torsion-free cohomology, and

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how to compute the invariants required to apply homeomorphism classification results of Wilkens [42].

The diffeomorphism classification of 2-connected 7-manifolds was recently completed in [13], which in particular introduced a generalised Eells–Kuiper invariant that distinguishes all the different smooth structures on the same closed 2-connected topological spin 7-manifold. While this invariant can be difficult to compute for interesting examples of manifolds, in the present paper we show how to compute it for twisted connected sums, and use that to identify examples of closed 2-connected manifolds with holonomy  $G_2$  that are homeomorphic but not diffeomorphic.

Using the diffeomorphism classification, the manifolds can be described explicitly as follows. Real vector bundles of rank 4 over  $S^4$  are classified by their Euler class  $e$  and first Pontrjagin class  $p_1$  in  $H^4(S^4) \cong \mathbb{Z}$ . Let  $N$  and  $\Sigma_{M_i}$  be the total space of the unit sphere bundle in the vector bundle with  $(e, p_1) = (0, 16)$  and  $(1, 6)$ , respectively. Then  $\Sigma_{M_i}$  is an exotic 7-sphere; indeed  $\Sigma_{M_i}$  and  $S^7$  were among the first discovered examples of homeomorphic but non-diffeomorphic manifolds (Milnor [31]). Meanwhile for any  $k \geq 1$  the connected sum of  $k$  copies of  $N$  is a closed 2-connected 7-manifold with  $b_3(N^{\#k}) = k$  and torsion-free cohomology. The manifolds  $N^{\#k}$  and  $N^{\#k} \# \Sigma_{M_i}$  are homeomorphic but not diffeomorphic (and in fact these are the only two diffeomorphism types with that underlying homeomorphism type).

**Theorem 1.1** *For  $k = 89$  and  $101$ , both  $N^{\#k}$  and  $N^{\#k} \# \Sigma_{M_i}$  admit a Riemannian metric with holonomy  $G_2$ .*

To the best of our knowledge, these are also the first examples of closed Ricci-flat manifolds (of any dimension) that are homeomorphic but not diffeomorphic.

Since the paper is primarily devoted to the topological analysis of a particular class of examples of  $G_2$ -manifolds, it makes practically no use of results about  $G_2$  geometry in general. Background on the definition of  $G_2$  and Riemannian holonomy can be found e.g. in the books by Salamon [36] or Joyce [26]. The main technical work of the paper is to examine in detail the properties of some examples of Fano 3-folds and their anticanonical divisors.

## 1.1 Twisted connected sums

There are two known sources of examples of closed  $G_2$ -manifolds. The first examples were constructed by Joyce in 1995 by desingularising quotients of flat tori [25]. In this paper we will make use of the later twisted connected sum construction. While this can be used to produce a large number of examples, it is still not known whether there exist infinitely many different topological types of closed 7-manifolds that admit holonomy  $G_2$  metrics.

A *Fano manifold* is a smooth projective variety with ample anticanonical bundle, or in more differential-geometric terms, a closed complex manifold whose first Chern class is a Kähler class. They have been studied extensively, and in complex dimension 3 they have been classified by Iskovskih [20–22] and Mori–Mukai [32,33].

Given a pair of Fano 3-folds  $Y_+$ ,  $Y_-$  with smooth anticanonical K3 divisors  $\Sigma_{\pm} \subset Y_{\pm}$  and a *matching* diffeomorphism  $r : \Sigma_+ \rightarrow \Sigma_-$  (Definition 2.5), the twisted

connected sum construction yields a closed simply-connected 7-manifold  $M$  with metrics of holonomy  $G_2$ . The procedure is summarised in Sects. 2.3 and 4.1. Part of the usefulness of the twisted connected sum construction is that many geometric and topological features of the resulting  $G_2$ -manifolds can be understood in terms of the relatively well-known algebraic input data. On the other hand, the challenge is that a thorough understanding of the algebraic data is required to find any matchings at all.

We categorise the matching as either *perpendicular* or *non-perpendicular*, depending on the action of  $r$  on the images of  $H^2(Y_{\pm})$  in  $H^2(\Sigma_{\pm})$  (Definition 5.1). It is shown in [10] that for most pairs  $\mathcal{Y}_+, \mathcal{Y}_-$  among the 105 deformation types of Fano 3-folds, general deformation theory results make it possible to find a perpendicular matching of some  $Y_{\pm} \in \mathcal{Y}_{\pm}$  resulting in a 2-connected twisted connected sum  $G_2$ -manifold. As we explain below, such perpendicular matchings can never be homeomorphic without being diffeomorphic.

Whether there is any non-perpendicular matching of a pair of members of  $\mathcal{Y}_{\pm}$  is in general a more difficult question, which has not previously been studied systematically. There are necessary conditions of a lattice-arithmetical nature, but, as we discuss in Sect. 5, showing that matchings exist also requires some detailed information about the deformation theory of anticanonical divisors in  $\mathcal{Y}_{\pm}$ , which needs to be worked out separately for each individual deformation type of Fanos.

### 1.2 The classifying invariants

Let us recall the relevant smooth classification results. Given a closed 2-connected 7-manifold  $M$ , two obvious topological invariants are its cohomology ring  $H^*(M)$  and its spin characteristic class  $p_M \in H^4(M)$  (satisfying  $p_1(M) = 2p_M$ ). If  $H^4(M)$  is torsion-free, then this data can be reduced to the third Betti number  $b_3(M)$  and the greatest integer  $d(M)$  dividing  $p_M$  in  $H^4(M)$  (we set  $d = 0$  if  $p_M = 0$ ). In fact, the pair  $(b_3(M), d(M))$  classifies such  $M$  up to homeomorphism (by Wilkens [42], see also Theorem 2.2).

In [13], we introduced the *generalised Eells–Kuiper invariant* of a closed spin 7-manifold  $M$ . If  $H^4(M)$  is torsion-free then this invariant reduces to a constant

$$\mu(M) \in \mathbb{Z}/\widehat{d},$$

and distinguishes between  $\widehat{d} := \gcd(28, \text{Num}(\frac{d}{4}))$  different diffeomorphism classes of smooth structures on the topological manifold underlying  $M$  (where  $\text{Num}(\frac{a}{b}) := \frac{a}{\gcd(a,b)}$ ). If  $p_M = 0$  (so  $\widehat{d} = 28$ ) then  $\mu(M)$  coincides with the invariant introduced by Eells and Kuiper [16], which in particular distinguishes between the 28 classes of smooth structures on  $S^7$ .

In Sect. 2.2 we recall how  $\mu(M)$  can be defined in terms of a  $\text{spin}^c$  coboundary of  $M$ . The challenge with this definition is that while the existence of a suitable coboundary is guaranteed, there is no algorithm for finding one, especially not one with a simple enough description that evaluating the formula (2.1) is tractable. However, we are able to construct explicit  $\text{spin}^c$  coboundaries of twisted connected sums, and in Sect. 3 we use those to express the generalised Eells–Kuiper invariant of a twisted connected sum

**Table 1** Complete diffeomorphism invariants of examples from Theorem 1.2

| $\mathcal{Y}_+$ | $\mathcal{Y}_-$ | $\perp$ | $b_3$ | $d$ | $\mu$ |
|-----------------|-----------------|---------|-------|-----|-------|
| (a)             | (b)             | yes     | 101   | 8   | 0     |
| (f)             | (f)             | no      | 101   | 8   | 1     |
| (b)             | (c)             | yes     | 89    | 8   | 0     |
| (d)             | (f)             | no      | 89    | 8   | 1     |
| (e)             | (f)             | no      | 89    | 8   | 1     |
| (d)             | (d)             | no      | 77    | 8   | 1     |
| (e)             | (d)             | no      | 77    | 8   | 1     |
| (e)             | (e)             | no      | 77    | 24  | 1     |

in terms of data for the Fanos used and the matching. In particular, it turns out that any perpendicularly matched twisted connected sum has  $\mu = 0$  (Corollary 3.7).

### 1.3 The main examples

To have any chance of obtaining homeomorphic but non-diffeomorphic twisted connected sums we must therefore search for non-perpendicular matchings. In that case, both Fanos used must have Picard rank  $\geq 2$ , cf. Remark 5.4. (The Picard group  $\text{Pic } Y$  of a Fano 3-fold  $Y$  coincides with its integral second cohomology  $H^2(Y)$ , so the Picard rank simply means its second Betti number  $b_2(Y)$ .) We are therefore led to study systematically the possible matchings of Fanos with Picard rank 2, and in this setting we can obtain decisive results.

- Theorem 1.2** (i) Any twisted connected sum  $M$  of Fano 3-folds of Picard rank 1 or 2 has  $H^4(M)$  torsion-free.
- (ii) There are precisely six (unordered) pairs  $\mathcal{Y}_+, \mathcal{Y}_-$  of deformation types of Fano 3-folds of Picard rank 2 with members that can be matched in the sense of Definition 2.5 in such a way that the resulting twisted connected sum  $M$  has  $\mu(M) \neq 0$ .
- (iii) Each of those six pairs gives rise to a single diffeomorphism type of  $M$  with  $\mu(M) \neq 0$ ; those  $M$  are all 2-connected.
- (iv) In total, they realise four distinct diffeomorphism types of  $M$  with  $\mu(M) \neq 0$ .
- (v) Precisely two of those are homeomorphic to some twisted connected sum  $M'$  of Fano 3-folds of Picard rank  $\leq 2$  such that  $\mu(M') = 0$ .

In particular, we obtain two pairs  $(M, M')$  of manifolds that are homeomorphic but not diffeomorphic and both admit metrics with holonomy  $G_2$ . These are the manifolds identified in Theorem 1.1.

For each of the examples highlighted in Theorem 1.2, we list in Table 1 the pair of deformation types used, from among the following six. (Recall that a del Pezzo 3-fold is a Fano 3-fold  $Y$  whose anticanonical class  $-K_Y \in \text{Pic } Y$  is even.) (a) Del Pezzo 3-folds of degree 3, i.e. cubic hypersurfaces in  $P^4$  (Picard rank 1) (b) Del Pezzo 3-folds of degree 5 (Picard rank 1) (c) Picard rank 1 Fanos of degree 16 (d) Number 9 in the Mori-Mukai list of Picard rank 2 Fano 3-folds:  $P^3$  blown up in a curve of

degree 7 and genus 5 (e) Number 17 in the Mori-Mukai list of Picard rank 2 Fano 3-folds: a smooth quadric hypersurface in  $P^4$  blown up in an elliptic curve of degree 5 (f) Number 27 in the Mori-Mukai list of Picard rank 2 Fano 3-folds:  $P^3$  blown up in a twisted cubic curve For each example in Table 1 we list whether the matching is perpendicular, and the classifying diffeomorphism invariants of the resulting twisted connected sum. As seen in the table, the two pairs that are homeomorphic but not diffeomorphic.

In Sect. 4 we compute detailed topological data for all 36 types in the Mori-Mukai list of rank 2 Fano 3-folds. In Sect. 7 we identify all pairs that satisfy the necessary arithmetic conditions for existence of a non-perpendicular matching resulting in a twisted connected sum with  $\mu \neq 0$ . The only candidate pairs are among the types (d), (e) and (f) above.

The key difficulty in finding non-perpendicular matchings is to understand precisely which K3 surfaces  $\Sigma$  appear as anticanonical divisors in a given type of Fanos, identifying conditions in terms of the Picard lattice of  $\Sigma$  (i.e.  $\text{Pic } \Sigma = H^2(\Sigma; \mathbb{Z}) \cap H^{1,1}(\Sigma; \mathbb{C})$  equipped with the intersection form). Having at least reduced our list of candidates, we carry out this intricate work only for the types (d), (e) and (f). We find in Theorem 7.8 that non-perpendicular matchings do in fact exist for each of the six pairs of those types, leading to the examples with  $\mu \neq 0$  above.

We then compare the homeomorphism invariants  $(b_3, d)$  of the realised manifolds with the invariants realised by perpendicular matchings of the 1378 pairs of rank 1 and 2 Fanos, listed in Table 4 of Sect. 6. For two of the four twisted connected sums with  $\mu \neq 0$  we can identify some perpendicular matching with the same homeomorphism invariants, and two of those are included in the table above.

In another application of the classification results, we can also exhibit examples of holonomy  $G_2$  metrics on total spaces of smooth  $S^1$ -bundles, while the last entry of Table 1 is an example of a  $G_2$ -manifold that is a topological  $S^1$ -bundle but not a smooth one (Remark 6.3). Further, as a byproduct of our analysis in Sect. 6 we identify all pairs of rank 2 Fanos that can be matched to define twisted connected sums  $M$  with  $H^2(M) \cong \mathbb{Z}$  (Table 5). Such matchings are of interest for the problem of constructing examples of  $G_2$ -instantons by gluing, cf. Sá Earp and Walpuski [37], Walpuski [41] and Menet, the second author and Sá Earp [30].

## 1.4 Context

In dimension 7, the problem of finding special metrics on manifolds that are homeomorphic but not diffeomorphic has been considered for instance in the case of Riemannian metrics with positive sectional curvature (Kreck–Stolz [29]) and 3-Sasakian metrics (Chinburg–Escher–Ziller [7]). In both cases, the examples exhibited have finite  $H^4$ , and the smooth structures can be distinguished by the ordinary Eells–Kuiper invariant.

All known (irreducible) examples of closed Ricci-flat manifolds have special holonomy: they are  $2n$ -manifolds with holonomy  $SU(n)$  ( $n \geq 2$ ),  $4n$ -manifolds with holonomy  $Sp(n)$  ( $n \geq 2$ ), 7-manifolds with holonomy  $G_2$  and 8-manifolds with holonomy  $\text{Spin}(7)$ .

In real dimension 4, the only smooth manifold with holonomy  $SU(2)$  is the K3 surface. In real dimension 6, simply-connected manifolds have unique smooth structures by Zhubr [43, Theorem 6.3], while there is no general classification result for finite but non-trivial fundamental group.

Manifolds with holonomy  $SU(n)$  or  $Sp(n)$  necessarily have  $b_2 \geq 1$ , and in real dimension  $\geq 8$  the case  $b_2 > 1$  is out of reach of current smooth classification results. Complete intersections with  $c_1 = 0$  provide examples with holonomy  $SU(n)$  and  $b_2 = 1$ , many of which have been smoothly classified by Traving [38] (see also [28, Theorem A]); however Traving's results imply that homeomorphic complete intersections with  $c_1 = 0$  are diffeomorphic in real dimension  $\geq 6$ .

Joyce [26, Theorem 15.4.3, 15.5.2, 15.5.6, 15.6.2 & 15.7.2] provides five examples of 8-manifolds of holonomy  $\text{Spin}(7)$  with  $b_2 = 0$ , and four of these also have  $b_3 = 0$ . Computing the torsion in the cohomology of these examples is intricate, and even if they prove to be 3-connected, then the underlying topological manifold admits two diffeomorphism classes of smooth structure if and only if 8 divides the first Pontrjagin class (if 8 does not divide the first Pontrjagin class, then the smooth structure is unique up to diffeomorphism) [11]. However, there is not currently a sufficiently large supply of such examples to be hopeful of finding exotic pairs of 3-connected  $\text{Spin}(7)$  8-manifolds.

Thus 7-manifolds of holonomy  $G_2$  are the only kinds of closed Ricci-flat manifolds where homeomorphic non-diffeomorphic examples are accessible with the current technology.

In the context of non-Ricci-flat holonomy groups, an early application of Donaldson invariants was to give examples of closed manifolds with holonomy  $U(2)$ , i.e. Kähler manifolds of complex dimension 2, that are homeomorphic but not diffeomorphic [15].

## 2 Background

We begin with further explanations of the generalised Eells–Kuiper invariant and the twisted connected sum construction, in order to elucidate the meaning of the main theorem, Theorem 1.2.

### 2.1 Spin and $\text{spin}^c$ characteristic classes

Recall that  $B\text{Spin}$ , the classifying space for stable spin vector bundles, is 3-connected with  $\pi_4(B\text{Spin}) \cong \mathbb{Z}$ . It follows that  $H^4(B\text{Spin}) \cong \mathbb{Z}$  is infinite cyclic. A generator is denoted  $\pm \frac{p_1}{2}$  and the notation is justified since for the canonical map  $\pi : B\text{Spin} \rightarrow BSO$  we have  $\pi^* p_1 = 2 \frac{p_1}{2}$ , where  $p_1$  is the first Pontrjagin class. Given a spin manifold  $X$  we write

$$p_X := \frac{p_1}{2}(TX) \in H^4(X).$$

A  $\text{spin}^c$  structure on a vector bundle  $E$  has an associated complex line bundle  $L$  such that  $E \oplus L$  is spin. Given a  $\text{spin}^c$  structure on the tangent bundle of  $X$  we can therefore define characteristic classes  $z := c_1(L) \in H^2(X)$ , and  $\widehat{p}_X := \frac{p_1}{2}(TX \oplus L) \in H^4(X)$ . Then  $2\widehat{p}_X = p_1(X) + z^2$ . (If  $X$  is spin then the induced  $\text{spin}^c$  structure of course has  $L$  trivial, and  $\widehat{p}_X = p_X$ .)

**Lemma 2.1** (cf. [13, 2.39–2.40])

- (i) If  $TX$  has an almost complex structure then  $\widehat{p}_X = -c_2(X) + c_1(X)^2$ .
- (ii)  $\widehat{p}_X = w_4(X) + w_2(X)^2 \pmod 2$ .
- (iii) Suppose  $X$  is closed.
  - If  $\dim X \leq 7$  then  $\widehat{p}_X$  is even.
  - If  $\dim X = 8$  then  $\widehat{p}_X$  is characteristic for the intersection form of  $X$ , i.e.  $x^2 = x \cup \widehat{p}_X \pmod 2$  for any  $x \in H^4(X)$ .

### 2.2 The generalised Eells–Kuiper invariant

We now describe the invariant  $\mu(M)$  of a closed spin 7-manifold  $M$ , in the special case when  $H^4(M)$  is torsion-free. As in the introduction, let  $d$  be the greatest integer dividing  $p_M$  (so  $d$  is even by Lemma 2.1(iii)), and  $\widehat{d} := \gcd(28, \text{Num}(\frac{d}{4}))$ .

Let  $W$  be a  $\text{spin}^c$  8-manifold with  $\partial W = M$ . To use  $W$  to compute the generalised Eells–Kuiper invariant  $\mu(M)$  we need an element  $n \in H^4(W)$  such that the image of  $dn$  in  $H^4(M)$  equals  $p_M$  [13, §2.7]. Then

$$\mu(M) := \frac{\widehat{\alpha}^2 - \sigma(W)}{8} - \frac{5\bar{z}^2 \widehat{p}_W}{12} + \frac{z^4}{4} \in \mathbb{Z}/\widehat{d}, \tag{2.1}$$

where  $\widehat{\alpha} := \widehat{p}_W - dn \in H^4(W)$  for  $n$  as above,  $\sigma(W)$  denotes the signature of the intersection form of  $W$ , and  $\bar{z} \in H^2(W, M)$  is any pre-image of  $z$  ( $\bar{z}^2 \in H^4(W, M)$  is independent of the choice of  $\bar{z}$ ). The integrals of  $\widehat{\alpha}^2$  and  $z^4$  make sense since they are squares of classes that vanish on the boundary. That  $\mu(M)$  is independent of the choice of  $W$  is a consequence of the index formula for the Dirac operator on a closed  $\text{spin}^c$  8-manifold.

When  $M$  is in addition 2-connected, there are precisely  $\widehat{d}$  different smooth structures on the topological manifold underlying  $M$ , and they are distinguished by  $\mu(M)$ .

**Theorem 2.2** (cf. [13, Theorems 1.2 and 1.3]) *Let  $M_0$  and  $M_1$  be closed 2-connected 7-manifolds such that  $H^4(M_i)$  are torsion-free. Then*

- (i)  $M_0$  and  $M_1$  are homeomorphic if and only if  $b_3(M_0) = b_3(M_1)$  and  $d(M_0) = d(M_1)$ .
- (ii)  $M_0$  and  $M_1$  are diffeomorphic if and only if in addition  $\mu(M_0) = \mu(M_1)$ .

### 2.3 Definition of twisted connected sums

We now explain the construction of the twisted connected sum of a matching pair of building blocks, using the set-up from [10, §3]. Like in the original application of the

construction by Kovalev [27], the present paper uses building blocks obtained from Fano 3-folds (in our case of rank 1 or 2) by a procedure explained in Sect. 4.1. The matching problem is elaborated on in Sect. 5.

**Definition 2.3** Let  $Z$  be a non-singular algebraic 3-fold and  $\Sigma \subset Z$  a non-singular K3 surface. Let  $N$  be the image of  $H^2(Z) \rightarrow H^2(\Sigma)$ . We call  $(Z, \Sigma)$  a *building block* if

- (i) the class in  $H^2(Z)$  of the anticanonical line bundle  $-K_Z$  is indivisible;
- (ii)  $\Sigma \in |-K_Z|$  (i.e.  $\Sigma$  is an anticanonical divisor), and there is a projective morphism  $f: Z \rightarrow \mathbb{P}^1$  with  $\Sigma = f^*(\infty)$ ;
- (iii) the inclusion  $N \hookrightarrow H^2(\Sigma)$  is primitive, that is,  $H^2(\Sigma)/N$  is torsion-free;
- (iv) the group  $H^3(Z)$ —and thus also  $H^4(Z)$ —is torsion-free.

We call  $N$ , equipped with the restriction of the intersection form on  $H^2(\Sigma)$ , the *polarising lattice* of the block. (Because  $H^{2,0}(Z)$  is automatically trivial,  $N \subseteq H^{1,1}(\Sigma)$  [10, Lemma 3.6], so that  $\Sigma$  is ‘ $N$ -polarised’.)

**Remark 2.4** The class  $[\Sigma] = -K_Z \in H^2(Z)$  is always in the kernel of  $H^2(Z) \rightarrow H^2(\Sigma)$ . In this paper we will only consider blocks where  $[\Sigma]$  in fact generates the kernel.

**Definition 2.5** Let  $Z_{\pm}$  be complex 3-folds,  $\Sigma_{\pm} \subset Z_{\pm}$  smooth anticanonical K3 divisors and  $k_{\pm} \in H^2(Z_{\pm}; \mathbb{R})$  Kähler classes. Let  $\Pi_{\pm} \subset H^2(\Sigma_{\pm}; \mathbb{R})$  denote the type  $(2, 0) + (0, 2)$  part. We call a diffeomorphism  $r: \Sigma_+ \rightarrow \Sigma_-$  a *matching* of  $(Z_+, \Sigma_+, k_+)$  and  $(Z_-, \Sigma_-, k_-)$  if  $r^*k_- \in \Pi_+$  and  $(r^{-1})^*k_+ \in \Pi_-$ , while  $r^*\Pi_- \cap \Pi_+$  is non-trivial.

We also say that  $r: \Sigma_+ \rightarrow \Sigma_-$  is a matching of  $Z_+$  and  $Z_-$  if there are Kähler classes  $k_{\pm}$  so that the above holds.

Given a building block  $(Z, \Sigma)$ , let  $\Delta \subset \mathbb{P}^1$  be an open disc that is a trivialising neighbourhood of  $\infty$  for the fibration  $f: Z \rightarrow \mathbb{P}^1$ , and  $U := f^{-1}(\Delta)$ . Then  $V := Z \setminus U$  is a manifold with boundary diffeomorphic to  $S^1 \times \Sigma$ .

**Construction 2.6** (Twisted connected sum) Given a pair of building blocks  $(Z_{\pm}, \Sigma_{\pm})$  with a matching  $r$ , their *twisted connected sum*  $M$  is the smooth 7-manifold defined by gluing  $S^1 \times V_+$  and  $S^1 \times V_-$  by the diffeomorphism

$$S^1 \times S^1 \times \Sigma_+ \rightarrow S^1 \times S^1 \times \Sigma_-,$$

$$(u, v, x) \mapsto (v, u, r(x)) \tag{2.2}$$

of their boundaries.

**Theorem 2.7** *The twisted connected sum  $M$  admits metrics with holonomy  $G_2$ .*

**Proof sketch** (cf [10, Corollary 6.4]) By [19, Theorem D], the interiors of the manifolds  $V_{\pm}$  admit metrics with holonomy  $SU(3)$  that are asymptotically cylindrical, i.e. in a collar neighbourhood  $\cong \mathbb{R} \times S^1 \times \Sigma_{\pm}$  of the boundary they are close to a product cylinder metric. Then  $V_{\pm}$  admit parallel  $SU(3)$ -structures, defining torsion-free product  $G_2$ -structures on  $S^1 \times V_{\pm}$ . The definition of what it means for  $r$  to be a matching



ensures that the  $SU(3)$ -structures can be chosen so that  $r$  is a ‘hyper-Kähler rotation’ of the  $SU(2)$ -structures defining the asymptotic limit. According to Kovalev [27, Theorem 5.34] the map (2.2) can then be used to glue together the product  $G_2$ -structures on  $S^1 \times V_+$  and  $S^1 \times V_-$  to a torsion-free  $G_2$ -structure on  $M$ .  $\square$

### 2.4 Topology of twisted connected sums

Let  $(Z_{\pm}, \Sigma_{\pm})$  be a pair of building blocks with a matching  $r : \Sigma_+ \rightarrow \Sigma_-$ . We think of  $r$  as allowing us to identify both  $\Sigma_+$  and  $\Sigma_-$  with a standard smooth K3 surface  $\Sigma$ . Letting  $L := H^2(\Sigma)$ , we can then think of the polarising lattices  $N_{\pm}$  of the two blocks as a pair of sublattices of  $L$ .

Let  $T_{\pm}$  denote the orthogonal complement of  $N_{\pm}$  in  $L$ . The following result summarises the cohomology of the twisted connected sum  $M$ .

**Theorem 2.8** [10, Theorem 4.8] *Suppose  $M$  is a twisted connected sum of building blocks  $(Z_{\pm}, \Sigma_{\pm})$  such that the kernel of  $H^2(Z_{\pm}) \rightarrow H^2(\Sigma_{\pm})$  is generated by  $[\Sigma_{\pm}]$ . Then*

- (i)  $\pi_1(M) = 0$  and  $H^1(M) = 0$ ;
- (ii)  $H^2(M) \cong N_+ \cap N_-$ ;
- (iii)  $H^3(M) \cong \mathbb{Z} \oplus (L/N_+ + N_-) \oplus (N_- \cap T_+) \oplus (N_+ \cap T_-) \oplus H^3(Z_+) \oplus H^3(Z_-)$ ;
- (iv)  $H^4(M) \cong H^4(\Sigma) \oplus (T_+ \cap T_-) \oplus (L/N_- + T_+) \oplus (L/N_+ + T_-) \oplus H^3(Z_+) \oplus H^3(Z_-)$ .

This implies in particular that if the matching  $r$  is perpendicular (i.e. if  $N_+$  is perpendicular to  $N_-$  in  $L$ , cf. Definition 5.1) then  $H^4(M)$  is torsion-free, and

$$b_2(M) = 0, \quad b_3(M) = b_3(Z_+) + b_3(Z_-) + 23. \tag{2.3}$$

If in addition the perpendicular direct sum  $N_+ \perp N_- \subset L$  is primitive then  $M$  is 2-connected.

We need to devote some extra attention to describing  $H^4(M)$ , since that contains the spin characteristic class  $p_M$ , whose greatest divisor  $d(M)$  is one of our classifying invariants from Theorem 2.2. Let

$$H^4(Z_+) \oplus_0 H^4(Z_-) \subset H^4(Z_+) \oplus H^4(Z_-) \tag{2.4}$$

be the subgroup of pairs with equal image in  $H^4(\Sigma)$ . We define a homomorphism

$$Y : H^4(Z_+) \oplus_0 H^4(Z_-) \rightarrow H^4(M) \tag{2.5}$$

as follows (cf. [10, Definition 4.13]). For  $([\alpha_+], [\alpha_-]) \in H^4(Z_+) \oplus_0 H^4(Z_-)$ , let  $[\beta]$  be their common image in  $H^4(\Sigma)$ . Then we may choose the cocycle  $\alpha_{\pm} \in C^4(Z_{\pm}; \mathbb{Z})$  so that the restriction of  $\alpha_{\pm}$  to  $\partial V_{\pm} \cong S^1 \times \Sigma \subset Z_{\pm}$  is the pull-back of  $\beta$  by projection to the  $\Sigma$  factor. Then the pull-backs of  $\alpha_{\pm}$  to  $S^1 \times V_{\pm}$  patch to a cocycle on  $M$  under the gluing (2.2), and we may set  $Y([\alpha_+], [\alpha_-])$  to be the class represented by that cocycle.

Let

$$b^\pm : N_\mp \rightarrow N_\pm^* \quad (2.6)$$

denote the homomorphism induced by the intersection form of  $L$ , and  $N'_\mp \subseteq N_\pm^*$  the image of  $b^\pm$ . Since  $N_\pm^* \subset H^4(Z_\pm)$ , we can also regard  $N'_\mp$  as a subset of  $H^4(Z_+) \oplus_0 H^4(Z_-)$ .

**Lemma 2.9** [10, Lemma 4.14] *The image of the map  $Y$  defined in (2.5) is the direct summand  $H^4(\Sigma) \oplus (L/N_{-+T_+}) \oplus (L/N_{++T_-})$  of  $H^4(M)$ , and the kernel is  $N'_- \oplus N'_+ \subset H^4(Z_+) \oplus_0 H^4(Z_-)$ .*

**Remark 2.10** The image of  $Y$  contains all the torsion of  $H^4(M)$ , which we can identify as

$$\text{Tor } H^4(M) \cong \text{Tor } N_+^*/N'_- \oplus \text{Tor } N_-^*/N'_+. \quad (2.7)$$

**Proposition 2.11** [10, Proposition 4.20/Remark 4.21]

$$p_M = -Y(c_2(Z_+), c_2(Z_-)).$$

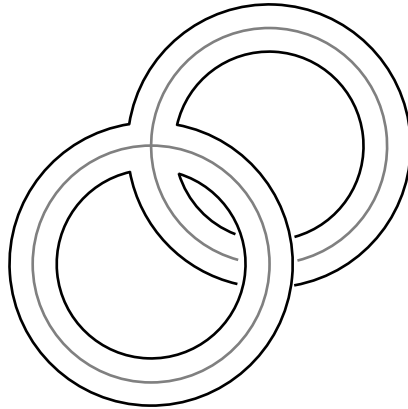
Combining the previous two results, the greatest divisor  $d(M)$  of  $p_M$  can be determined from  $c_2(Z_\pm) \in H^4(Z_\pm)$  (which depends purely on the blocks) and  $N'_\mp \subset H^4(Z_\pm)$  (which depends on the matching).

### 3 The generalised Eells–Kuiper invariant of twisted connected sums

The main result of this section is Theorem 3.6, which gives a formula for the generalised Eells–Kuiper invariant  $\mu(M)$  of a twisted connected sum  $M$  in terms of data for the pair of building blocks used and data for the matching. We compute  $\mu(M)$  using an explicit  $\text{spin}^c$  coboundary.

Throughout this section, let  $M$  be the twisted connected sum of a pair of building blocks  $(Z_\pm, \Sigma_\pm)$ , using a matching  $r: \Sigma_+ \rightarrow \Sigma_-$ . To simplify the description of the cohomology, we assume that the kernel of  $H^2(Z_\pm) \rightarrow H^2(\Sigma_\pm)$  is generated by  $[\Sigma_\pm]$ .

The  $\text{spin}^c$  coboundaries we construct to compute the generalised Eells–Kuiper invariant for twisted connected sum manifolds can be viewed as *parametrised plumbing* of trivial disc bundles. By way of context we recall the plumbing of bundles; see Browder [6, V §2]. If  $\pi_0: W_0 \rightarrow M_0$  is a  $D^p$ -bundle over a  $q$ -manifold and  $\pi_1: W_1 \rightarrow M_1$  is a  $D^q$ -bundle over a  $p$ -manifold, then the *plumbing* of  $\pi_0$  and  $\pi_1$  is the manifold  $W = (W_0 \sqcup W_1)/\sim$  obtained from the disjoint union of  $W_0$  and  $W_1$  by trivialising  $\pi_0$  over  $D^q \subset M_0$  and  $\pi_1$  over  $D^p \subset M_1$ , identifying the resulting spaces  $D^p \times D^q$  and  $D^q \times D^p$  by exchanging co-ordinates and finally smoothing corners. The case of a pair of trivial  $D^1$ -bundles over  $S^1$  is illustrated below.



**Construction 3.1** (Parametric plumbing) As in Construction 2.6, we use the tubular neighbourhood  $U_{\pm} \cong \Delta \times \Sigma_{\pm} \subset Z_{\pm}$  of  $\Sigma_{\pm}$ ; denote the coordinate on the open disc  $\Delta$  by  $z$ . Consider  $B_{\pm} := \Delta \times Z_{\pm}$ , and denote the coordinate on its open disc factor by  $w$ . Form an 8-manifold  $\mathring{W}$  by gluing  $B_+$  and  $B_-$  along  $\Delta \times U_{\pm}$ , using the map

$$\begin{aligned} \Delta \times \Delta \times \Sigma_+ &\rightarrow \Delta \times \Delta \times \Sigma_-, \\ (w, z, x) &\mapsto (z, w, \tau(x)). \end{aligned}$$

We can form a smooth compact manifold  $W$  with boundary by attaching to  $\mathring{W}$  the result of gluing the boundaries  $\partial\bar{\Delta} \times \partial\bar{\Delta} \times \Sigma_{\pm}$  of  $\partial\bar{\Delta} \times (Z_{\pm} \setminus U_{\pm})$  by  $(w, z, x) \mapsto (z, w, \tau(x))$ . Comparing with (2.2), we see that the boundary is precisely the twisted connected sum  $M$ . Hence  $W$  is a coboundary of  $M$ .

Since  $Z_{\pm}$  are not spin,  $B_{\pm}$  and  $W$  are not spin either. However,  $Z_{\pm}$  and  $B_{\pm}$  have complex structures, inducing  $\text{spin}^c$  structures. While the complex structures of  $B_+$  and  $B_-$  do not agree on the overlap region, their  $\text{spin}^c$  structures do agree, so  $W$  is  $\text{spin}^c$  too.

### 3.1 Cohomology

We compute the integral cohomology of the coboundary  $W$  from Construction 3.1 by Mayer–Vietoris. More precisely, we compute  $H^k(\mathring{W}) \cong H^k(W)$  and  $H^k_{cpt}(\mathring{W}) \cong H^k(W, M)$  for  $k \leq 4$ . One can of course recover the remaining cohomology groups from Poincaré duality and universal coefficients, but what we care about is a description for  $k \leq 4$  that lets us determine the characteristic classes and the intersection form.

There are obvious homotopy equivalences  $B_+ \cap B_- \simeq \Sigma$  and  $B_{\pm} \simeq Z_{\pm}$ , so there is a long exact sequence

$$H^{k-1}(\Sigma) \rightarrow H^k(W) \rightarrow H^k(Z_+) \oplus H^k(Z_-) \rightarrow H^k(\Sigma).$$

Hence, using that  $H^2(Z_{\pm}) \rightarrow H^2(\Sigma) = L$  has image  $N_{\pm}$  by definition and is assumed to have kernel  $\mathbb{Z}$ , we find

$$\begin{aligned}
 H^1(W) &= 0 & H^2(W) &\cong (N_+ \cap N_-) \oplus \mathbb{Z}^2 \\
 H^3(W) &\cong L/(N_+ + N_-) \oplus H^3(Z_+) \oplus H^3(Z_-) & H^4(W) &\cong H^4(Z_+) \oplus_0 H^4(Z_-),
 \end{aligned}$$

where  $H^4(Z_+) \oplus_0 H^4(Z_-)$  is defined as in (2.4). There is a natural short exact sequence

$$0 \rightarrow N_+^* \oplus N_-^* \rightarrow H^4(Z_+) \oplus_0 H^4(Z_-) \rightarrow \mathbb{Z} \rightarrow 0. \tag{3.1}$$

We can describe the isomorphism  $H^4(Z_+) \oplus_0 H^4(Z_-) \rightarrow H^4(W)$  as follows: given  $(c_+, c_-) \in H^4(Z_+) \oplus_0 H^4(Z_-)$ , choose a cocycle  $\alpha_0 \in C^4(\Sigma)$  representing the common image of  $c_{\pm}$  in  $H^4(\Sigma)$ , let  $\alpha_{\pm} \in C^4(Z_{\pm})$  be cocycles representing  $c_{\pm}$  that equal the pull-back of  $\alpha_0$  on  $\Delta \times \Sigma_{\pm}$ , and form a cocycle on  $W$  by patching the pull-backs of  $\alpha_{\pm}$  to  $B_{\pm}$ . It is clear that the composition of this isomorphism with the restriction map  $H^4(W) \rightarrow H^4(M)$  equals the map  $Y$  defined in (2.5).

**Remark 3.2** Only the isomorphism for  $H^3(W)$  involves making an arbitrary choice for a splitting of a short exact sequence—below it will turn out to be important that the isomorphisms for  $H^2(W)$  and  $H^4(W)$  are natural.

Similarly, we can compute  $H^*(W, M)$  from a long exact sequence

$$H^{k-4}(\Sigma) \rightarrow H^{k-2}(Z_+) \oplus H^{k-2}(Z_-) \rightarrow H^k(W, M) \rightarrow H^{k-3}(\Sigma),$$

finding

$$\begin{aligned}
 H^1(W, M) &= 0 & H^2(W, M) &\cong \mathbb{Z}^2 \\
 H^3(W, M) &= 0 & H^4(W, M) &\cong (H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z}.
 \end{aligned}$$

Note there is a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow (H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z} \rightarrow N_+ \oplus N_- \rightarrow 0.$$

### 3.2 Characteristic classes

As emphasised in Remark 3.2, the isomorphisms for  $H^2(W)$  and  $H^4(W)$  above are natural. One consequence of this is that we can pin down the characteristic classes of  $W$  by considering their restrictions to the open subsets  $B_+ = \Delta \times Z_+$  and  $B_- = \Delta \times Z_-$ .

The restriction of  $w_2(W) \in H^2(W; \mathbb{Z}/2)$  to  $B_{\pm}$  is  $w_2(B_{\pm})$ , which equals the pull-back of  $w_2(Z_{\pm})$ . That is the mod 2 reduction of  $c_1(Z_{\pm}) = [\Sigma_{\pm}] \in H^2(Z_{\pm})$  and so  $w_2(Z_{\pm})$  is the mod 2 Poincaré dual of  $\Sigma_{\pm}$  in  $Z_{\pm}$ . The  $\text{spin}^c$  structures of  $B_{\pm}$  patch up to a unique  $\text{spin}^c$  structure on  $W$ : this is essentially just saying that we can specify an

integral pre-image  $z \in H^2(W)$  of  $w_2(W)$  uniquely by setting the restriction to  $B_{\pm}$  to be  $c_1(Z_{\pm})$ . Note that the restriction of  $z^2$  to each of  $B_{\pm}$  is 0; hence

$$z^2 = 0 \in H^4(W).$$

Similarly we can pin down the  $\text{spin}^c$  characteristic class  $\widehat{p}_W$ . The restriction to  $B_{\pm}$  is  $\widehat{p}_{B_{\pm}}$ , which by Lemma 2.1(i) equals  $-c_2 + c_1^2$  of the relevant  $U(4)$ -structure. Using again that  $c_1(Z_{\pm})^2 = 0$ , we thus have

**Lemma 3.3** *The image of  $\widehat{p}_W$  in  $H^4(Z_+) \oplus_0 H^4(Z_-)$  is  $(-c_2(Z_+), -c_2(Z_-))$ .*

The restriction of  $\widehat{p}_W$  to  $M$  is  $p_M$ , which recovers Proposition 2.11, and arguably makes for a nicer proof than that in [10, Proposition 4.20].

Since the normal bundle of  $\Sigma \subset W$  is trivial, the image of  $\widehat{p}_W$  in  $H^4(\Sigma) \cong \mathbb{Z}$  is given by  $p_{\Sigma} = -c_2(\Sigma) = -24$ , so in view of the short exact sequence (3.1) most of the interesting information about  $\widehat{p}_W$  is captured by the pre-image of  $\widehat{p}_W \bmod 24$  in  $(N_+^* \oplus N_-^*) \otimes \mathbb{Z}/24$ . Similarly, for any building block  $Z$ , the image of  $c_2(Z)$  under  $H^4(Z) \rightarrow H^4(\Sigma)$  is 24, so  $c_2(Z) \bmod 24$  has a preimage in  $H^4(Z, \Sigma; \mathbb{Z}/24) \cong N^* \otimes \mathbb{Z}/24$ . Denote that by

$$\bar{c}_2(Z) \in N^* \otimes \mathbb{Z}/24. \tag{3.2}$$

The calculation above implies that  $\widehat{p}_W \bmod 24$  is determined by  $\bar{c}_2(Z_{\pm})$ .

**Corollary 3.4**  $\widehat{p}_W \bmod 24 \in H^4(W; \mathbb{Z}/24) \cong H^4(Z_+; \mathbb{Z}/24) \oplus_0 H^4(Z_-; \mathbb{Z}/24)$  is the image of  $(-\bar{c}_2(Z_+), -\bar{c}_2(Z_-))$  under the map in (3.1).

**Remark 3.5** Since  $c_2(Z_{\pm})$  are even, so is  $\widehat{p}_W$ . Therefore Lemma 2.1(iii) implies that the intersection form of  $W$  must be even.

### 3.3 Intersection form

The intersection pairing between  $H^4(W)$  and  $H^4(W, M)$  is simply the natural duality between  $H^4(Z_+) \oplus_0 H^4(Z_-)$  and  $(H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z}$ . To understand the intersection form we also need to describe the map  $H^4(W, M) \rightarrow H^4(W)$ .

First note that the  $\mathbb{Z}$  term in  $H^4(W, M) \cong (H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z}$  is the Poincaré dual to the K3 divisors, and obviously has trivial image in  $H^4(W)$ . Hence  $H^4(W, M) \rightarrow H^4(W)$  factors through the natural map  $(H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z} \rightarrow N_+ \oplus N_-$ . Dually, the composition  $H^4(W) \cong H^4(Z_+) \oplus_0 H^4(Z_-) \rightarrow \mathbb{Z}$  corresponds to restriction to the K3 divisor, which is trivial for any class with compact support in  $\mathring{W}$ . So the map also factors through the inclusion  $N_+^* \oplus N_-^* \hookrightarrow H^4(Z_+) \oplus_0 H^4(Z_-)$ , and is characterised by a homomorphism  $N_+ \oplus N_- \rightarrow N_+^* \oplus N_-^*$ .

The intersection form on each  $B_{\pm}$  is obviously trivial, so there are no diagonal terms. In summary, the map  $H^4(W, M) \rightarrow H^4(W)$  is therefore the composition of  $H^4(W, M) \rightarrow N_+ \oplus N_-$ ,

$$\begin{aligned} N_+ \oplus N_- &\rightarrow N_+^* \oplus N_-^* \\ (x_+, x_-) &\mapsto (b^+(x_-), b^-(x_+)) \end{aligned} \tag{3.3}$$

and  $N_+^* \oplus N_-^* \rightarrow H^4(W)$ , where  $b^\pm$  is the inner product homomorphism as in (2.6). The image equals  $N'_- \oplus N'_+ \subset N_+^* \oplus N_-^*$ , where  $N'_\mp$  is the image of  $b^\pm : N_\mp \rightarrow N_\pm^*$  as before. (This is consistent with the claim from Lemma 2.9 that the kernel of  $Y$  is precisely  $N'_- \oplus N'_+$ , and also with Remark 3.5) Both summands of  $N'_- \oplus N'_+$  are isotropic and so  $\sigma(W)$ , the signature of  $W$  vanishes:

$$\sigma(W) = 0. \tag{3.4}$$

### 3.4 Computing the generalised Eells–Kuiper invariant

Suppose that the twisted connected sum  $M$  has  $H^4(M)$  torsion-free, so that the description of the generalised Eells–Kuiper invariant from Sect. 2.2 is valid. Let  $d$  denote the greatest integer dividing the spin characteristic class  $p_M \in H^4(M)$ . Since  $M$  contains a K3 with trivial normal bundle,  $d \mid 24$  a priori. Hence  $\widehat{d} := \gcd(28, \text{Num}(\frac{d}{4}))$  is either 1 or 2, depending on whether  $d$  is divisible by 8 or not. In particular  $\mu(M) \in \mathbb{Z}/\widehat{d}$  can only possibly be non-trivial if  $d$  is 8 or 24.

We now compute  $\mu(M)$  by evaluating (2.1) for the parametric plumbing coboundary of Construction 3.1. The  $z^4$  term in (2.1) obviously makes no contribution since  $z^2 = 0$ , and the signature term vanishes by (3.4). We can take  $\bar{z}$  to be the sum of the generators of  $H_{cpt}^2(\Delta) \subset H_{cpt}^2(B_\pm)$ , or equivalently the Poincaré duals of  $\{0\} \times Z_\pm$ . This is because the intersection of  $\{0\} \times Z_+$  with  $\{0\} \times Z_-$  is  $\{0\} \times \Sigma_-$ , which is the Poincaré dual of  $z_{|B_-}$ , and similarly for  $z_{|B_+}$ . Thus  $\bar{z}^2$  is twice the Poincaré dual to the K3 divisor (the intersection of  $\{0\} \times Z_+$  and  $\{0\} \times Z_-$ , generating the copy of  $\mathbb{Z}$  in  $(H^2(Z_+) \oplus H^2(Z_-))/\mathbb{Z}$ ). It follows that  $\bar{z}^2 \widehat{p}_W = 2\widehat{p}_{W|\Sigma}$ . As above, the triviality of the normal bundle of  $\Sigma$  implies that  $\widehat{p}_{W|\Sigma} = p_\Sigma = -24$ , and hence  $\frac{5z^2 \widehat{p}_W}{12}$  is divisible by 4. Since the modulus  $\widehat{d}$  is 1 or 2, the only possible non-trivial contribution to the RHS of (2.1) is  $\frac{\widehat{\alpha}^2}{8}$ .

The definition of  $d$  means that  $\widehat{p}_W \in N'_- \oplus N'_+ \text{ mod } d$ , where  $N'_- \oplus N'_+$  is the image of  $H^4(W, M) \rightarrow H^4(W)$  identified in Sect. 3.3. By Corollary 3.4, we can find elements  $[x_\pm] \in N_\pm \otimes \mathbb{Z}/d$  such that  $\widehat{p}_W = (b^+([x_-]), b^-([x_+])) \text{ mod } d$  from  $\bar{c}_2(Z_\pm) \in N_\pm^* \otimes \mathbb{Z}/24$  and the configuration of embeddings  $N_\pm \subset L$  of the matching. We may then take

$$\widehat{\alpha} = (b^+(x_-), b^-(x_+)) \in N'_- \oplus N'_+.$$

One pre-image under (3.3) of this  $\widehat{\alpha}$  in  $N_+ \oplus N_-$  is  $(x_+, x_-)$ , so  $\widehat{\alpha}^2 = 2x_+ \cdot x_-$ . Hence we have

**Theorem 3.6** *Let  $x_\pm \in N_\pm$  such that  $b^\mp(x_\pm) = \bar{c}_2(Z_\mp) \text{ mod } d$ . Then*

$$\mu(M) = \frac{x_+ \cdot x_-}{4} \in \mathbb{Z}/\widehat{d}. \tag{3.5}$$

The elements  $x_\pm$  must be even because  $c_2(Z_\pm)$  is, so the RHS is indeed integral.

If  $N_+ \perp N_-$ , i.e. if the matching used is perpendicular, then  $d$  is the greatest common divisor of  $\bar{c}_2(Z_+)$  and  $\bar{c}_2(Z_-)$ , so we can trivially take  $x_\pm = 0$ . Indeed,

$H^4(Z_+) \oplus_0 H^4(Z_-) \hookrightarrow H^4(M)$ , so since there is no torsion in  $H^3(Z_\pm)$ , there is a well-defined  $n := \frac{1}{d} \widehat{p}_W \in H^4(Z_+) \oplus_0 H^4(Z_-)$ , and that choice gives  $\widehat{\alpha} = 0$ .

**Corollary 3.7**  $\mu(M) = 0$  for all twisted connected sums obtained by perpendicular matching.

But if neither of  $c_2(Z_\pm)$  are divisible by 4, then  $x_+ \cdot x_-$  can be 4 mod 8. For non-perpendicular matchings that can happen even when  $d$  is divisible by 8, in which case  $\mu(M)$  is non-zero.

### 3.5 Quadratic refinement of the torsion linking form

Although in this paper we only consider examples of twisted connected sums with no torsion in  $H^4(M)$ , let us briefly use the coboundary  $W$  to analyse in general the torsion linking form of  $M$  and its quadratic refinement (cf. [13, Definition 2.23]).

Since the blocks  $Z_\pm$  are assumed to have no torsion in  $H^3(Z_\pm)$ ,  $H^4(W)$  resolves all the torsion in  $H^4(M)$ . Corti et al. [10, Remark 4.12] claims that the two summands  $\text{Tor } N_\pm^*/N'_\mp$  of  $\text{Tor } H^4(M)$  in (2.7) are isotropic for the torsion linking form, and naturally dual. This can be checked by considering cocycles representing the classes, but it is nicer to do it by computing cup products on  $W$ . Indeed, this way we can also compute the refinement of the linking form. Because  $\widehat{p}_W$  is even by Remark 3.5, we can characterise the quadratic linking family on  $\text{Tor } H^4(M)$  as follows: it assigns to the image in  $H^4(M)$  of (the unique)  $\frac{1}{2} \widehat{p}_W$  the discriminant form  $q$  of the even lattice  $H^4(W, M)/\text{rad}$ .

**Proposition 3.8**  $\text{Tor } N_\pm^*/N'_\mp \subseteq \text{Tor } H^4(M)$  is isotropic for the torsion linking form, and Lagrangian for  $q$ .

**Proof** We can write any  $c \in \text{Tor } N_\pm^*/N'_\mp \subset H^4(M)$  as the restriction of an element of the form  $(\alpha, 0) \in N_+^* \oplus N_-^* \subset H^4(W)$ , where  $\alpha$  is in the rational span of  $N'_-$ . Then  $(\alpha, 0)$  is the image under (3.3) of some  $(0, y) \in (N_+ \oplus N_-) \otimes \mathbb{Q}$ . This pairs trivially with  $(\alpha, 0)$ , so  $q(c) = 0$ . □

## 4 Fano-type blocks

We now describe how to construct building blocks for twisted connected sums from closed Fano 3-folds, and list detailed data for blocks obtained from Fanos of Picard rank 1 or 2.

### 4.1 Construction of building blocks from Fanos

Recall that a Fano 3-fold is a smooth closed complex 3-fold  $Y$  whose anticanonical bundle  $-K_Y$  is ample, or equivalently,  $c_1(Y)$  can be represented by a Kähler form. Any Fano 3-fold is simply-connected with  $H^{2,0}(Y) = 0$ , so  $\text{Pic } Y \cong H^2(Y)$ . Together with a natural form, this is a key deformation invariant of  $Y$ .

**Table 2** Rank 1 Fano blocks

| $Y$            | $r$ | $-K_Y^3$      | $b_3(Y)$ | $b_3(Z)$ | $\pi_1 c_2(Z)$ |
|----------------|-----|---------------|----------|----------|----------------|
| $\mathbb{P}^3$ | 4   | $4^3$         | 0        | 66       | 22             |
| $Q$            | 3   | $3^3 \cdot 2$ | 0        | 56       | 26             |
| $V_1$          | 2   | $2^3$         | 42       | 52       | 16             |
| $V_2$          | 2   | $2^3 \cdot 2$ | 20       | 38       | 20             |
| $V_3$          | 2   | $2^3 \cdot 3$ | 10       | 36       | 24             |
| $V_4$          | 2   | $2^3 \cdot 4$ | 4        | 38       | 28             |
| $V_5$          | 2   | $2^3 \cdot 5$ | 0        | 42       | 32             |
|                | 1   | 2             | 104      | 108      | 26             |
|                | 1   | 4             | 60       | 66       | 28             |
|                | 1   | 6             | 40       | 48       | 30             |
|                | 1   | 8             | 28       | 38       | 32             |
|                | 1   | 10            | 20       | 32       | 34             |
|                | 1   | 12            | 14       | 28       | 36             |
|                | 1   | 14            | 10       | 26       | 38             |
|                | 1   | 16            | 6        | 24       | 40             |
|                | 1   | 18            | 4        | 24       | 42             |
|                | 1   | 22            | 0        | 24       | 46             |

**Definition 4.1** For a closed complex 3-fold  $Y$ , the *anticanonical form* on  $H^2(Y)$  is the symmetric bilinear form  $(D_1, D_2) \mapsto D_1 \cdot D_2 \cdot (-K_Y)$  (where  $\cdot$  is the cup product on  $H^2(Y)$ ).

If  $\text{Pic } Y \cong H^2(Y)$  we call  $\text{Pic } Y$  equipped with the anticanonical form the *Picard lattice* of  $Y$ .

There are 105 deformation types of Fano 3-folds. Except for two of those, any Fano 3-fold  $Y$  has a pencil in  $|-K_Y|$  with smooth base locus  $C$ .

**Construction 4.2** Given a Fano 3-fold  $Y$  and  $C \subset Y$  a smooth curve that is the base locus of an anticanonical pencil, let  $Z$  be the blow-up of  $Y$  in  $C$ . If  $\Sigma \subset Z$  is the proper transform of a smooth element of said pencil, then we call  $(Z, \Sigma)$  a *Fano-type building block*.

**Proposition 4.3** ([9, Propositions 4.24 and 5.7], cf. [10, Proposition 3.17])  $(Z, \Sigma)$  of Construction 4.2 is indeed a building block. Further

- (i) the image of  $H^2(Z) \rightarrow H^2(\Sigma)$  is isomorphic to  $H^2(Y)$ , and the kernel is generated by  $[\Sigma]$ ;
- (ii) the image in  $H^{1,1}(\Sigma)$  of the Kähler cone of  $Z$  contains the image of the Kähler cone of  $Y$ .

Some of the important data of the block can be read off immediately from well-known data about  $Y$ . For instance (i) implies that the polarising lattice of the block is isometric to the Picard lattice of  $Y$ , and (cf. [9, Lemma 5.6])

$$b_3(Z) = -K_Y^3 + b_3(Y) + 2. \tag{4.1}$$



Meanwhile, the second Chern class can be described as follows. The pull-back  $\pi^* : H^2(Y) \rightarrow H^2(Z)$  of the blow-up map  $\pi : Z \rightarrow Y$  is injective, and  $H^2(Z)$  is the direct sum of  $\pi^*H^2(Y)$  and  $\mathbb{Z}c_1(Z)$ . Thus an element of  $H^4(Z)$  can be characterised in terms of its product with  $c_1(Z)$  and its image under the Poincaré adjoint  $\pi_! : H^4(Z) \rightarrow H^4(Y)$  of  $\pi^*$ , i.e. the map characterised by equality of the intersection products  $(\pi_!x)y$  and  $x(\pi^*y) \in \mathbb{Z}$  for any  $x \in H^4(Z)$  and  $y \in H^2(Y)$ . We have  $c_2(Z)c_1(Z) = 24$  and  $c_1(Z)^2 = 0$  for any building block, so for a Fano-type block  $c_2(Z)$  is completely determined by the following result.

**Lemma 4.4** [9, (5–13)] *For any complex 3-fold  $Y$ ,  $C \subset Y$  a smooth curve contained in a smooth anticanonical divisor and  $\pi : Z \rightarrow Y$  the blow-up of  $Y$  in  $C$ ,*

$$\pi_!(c_2(Z) + c_1(Z)^2) = c_2(Y) + c_1(Y)^2. \tag{4.2}$$

In view of Corollary 3.4, the main interesting information about  $c_2(Z)$  is the mod 24 reduction  $\bar{c}_2(Z) \in N^* \otimes \mathbb{Z}/24$ , which we can thus think of simply as

$$\bar{c}_2(Z) = \pi_!c_2(Z) = c_2(Y) + c_1(Y)^2 \pmod{24}. \tag{4.3}$$

When we tabulate data for the blocks, we will record  $\pi_!c_2(Z)$  rather than  $\bar{c}_2(Z)$  for completeness.

Note that Definition 2.5 makes sense for Fano 3-folds as well as building blocks. Proposition 4.3(ii) implies that a matching of a pair of Fano 3-folds gives rise to a matching of the resulting Fano-type blocks. We use the phrase *twisted connected sum of Fanos* (e.g. in the introduction) to mean the twisted connected sum that arises from such a matching of Fano-type blocks.

**4.2 Rank 1 blocks**

Table 2 summarises the key data of Fano 3-folds of rank 1 and the resulting building blocks (cf. [9, Table 1]). The data included is the index  $r$  (i.e. the largest integer such that  $-K_Y = rH$  for some  $H \in \text{Pic } Y$ ), the anticanonical degree  $-K_Y^3$ ,  $b_3(Y)$ ,  $b_3(Z)$ , and the pairing of  $\pi_!c_2(Z)$  and the positive generator  $H \in \text{Pic } Y$  (equivalently, the product of  $c_2(Z)$  and  $\pi^*H$ ).

$b_3(Z)$  is simply obtained from the preceding data by (4.1). In the rank 1 case,  $\pi_!c_2(Z)$  is also easily determined as follows: For any Fano one has  $c_2(Y)(-K_Y) = 24$ , so if  $-K_Y = rH$  then (4.3) implies that

$$\pi_!c_2(Z)H = \frac{24 - K_Y^3}{r} \pmod{24}. \tag{4.4}$$

$N$  is generated by the image of  $H$ . Its self-intersection (with respect to the quadratic form on  $N$ ) is not included in the table, but it is simply  $\frac{-K_Y^3}{r^2}$ .

### 4.3 The table of rank 2 Fano blocks

In Table 3 we collect the following data for building blocks obtained from rank 2 Fano 3-folds  $Y$ .

- The number of the corresponding entry in the Mori-Mukai list of rank 2 Fanos.
- The anticanonical degree  $-K_Y^3$  of the Fano  $Y$ .
- The quadratic form of the Picard lattice  $N$  presented with respect to a basis  $G, H$  that spans the nef cone (i.e. the ample classes are exactly the linear combinations of  $G$  and  $H$  where both coefficients are positive); we do not know a general reason why the extremal rays should generate all of  $N$  and not just a finite index sublattice, but it does for all entries in the list.
- The absolute value  $\Delta$  of the discriminant of the quadratic form on  $N$ .
- The anticanonical class  $-K_Y$  in terms of the basis  $G, H$ .
- The element  $\pi_1 c_2(Z) = c_2(Y) + c_1(Y)^2 \in N^*$  (whose mod 24 reduction is  $\bar{c}_2(Z) \in N^* \otimes \mathbb{Z}/24$  defined in (3.2)) presented in the dual basis of  $G, H$ . In other words, the entries of the row vector are the pairing of  $c_2(Z)$  with  $\pi^*G$  and  $\pi^*H$ .
- The third Betti number  $b_3(Y)$  of the Fano.
- The third Betti number  $b_3(Z)$  of the resulting block.

The final four columns include data relevant for non-perpendicular matching, about ample  $A \in N$  such that  $A^2$  is not too large compared with  $\Delta$ . In view of Lemmas 6.4 and 7.1, non-perpendicular matchings of a pair of rank 2 Fanos  $Y_+, Y_-$  are only possible if

$$\log_2 \frac{\Delta_+}{A_+^2} + \log_2 \frac{\Delta_-}{A_-^2} \geq 0. \tag{4.5}$$

The largest ratio  $\frac{\Delta}{A^2}$  occurs for #18, which has  $\Delta = 16$  and an ample  $A$  with  $A^2 = 10$ . Accordingly, we list all ample classes with  $A^2 \leq 1.6\Delta$ . (In some examples there is more than one such class. We do not write out  $A$  itself in terms of the basis  $G, H$ , but there is never any ambiguity.) For each such  $A$  we list the following data.

- The result of evaluating  $\pi_1 c_2(Z)$  on  $A$  (equivalently:  $c_2(Z)\pi^*A$ ).
- $A^2$ , the product of  $A$  with itself in the Picard lattice.
- $B^2$ , where  $B$  is a generator for the orthogonal complement to  $A$  in the Picard lattice.
- $h := \log_2 \frac{\Delta}{A^2}$ .

### 4.4 Annotated Mori-Mukai list of rank 2 Fano 3-folds

We now indicate how the data in Table 3 is assembled. The anticanonical degree  $-K_Y^3$  and  $b_3(Y)$  are taken from Iskovskih–Prokhorov [23, Table 12.3], and  $b_3(Z)$  is obtained from (4.1) as before. For the computation of the basis of the nef cone, the quadratic form on the Picard lattice  $N$  and  $\pi_1 c_2(Z)$  we divide the list (except the last two entries) into three groups.

**Table 3** Rank 2 Fano blocks

| MM# | $-K_Y^3$ | $N$  | $\Delta$ | $-K_Y$                                 | $\pi_{ic_2}(Z)$ | $b_3(Y)$ | $b_3(Z)$ | $\pi_{ic_2}(Z)A$ | $A^2$ | $B^2$ | $h$   |
|-----|----------|--|----------|--|-----------------|----------|----------|------------------|-------|-------|-------|
| 1   | 4        | $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ | 1        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | *               | 44       | *        |                  |       |       |       |
| 2   | 6        | $\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$ | 4        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 18)         | 40       | 48       | 30               | 6     | -6    | -0.58 |
| 3   | 8        | $\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$ | 4        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 20)         | 22       | 32       |                  |       |       |       |
| 4   | 10       | $\begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}$ | 9        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 22)         | 20       | 32       | 34               | 10    | -90   | -0.15 |
| 5   | 12       | $\begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}$ | 9        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 24)         | 12       | 26       | 36               | 12    | -12   | -0.42 |
| 6   | 12       | $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ | 12       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 18)         | 18       | 32       | 36               | 12    | -4    | 0     |
| 7   | 14       | $\begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}$ | 16       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 26)         | 10       | 26       | 38               | 14    | -56   | 0.19  |
| 8   | 14       | $\begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$ | 8        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 20)         | 18       | 34       |                  |       |       |       |
| 9   | 16       | $\begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$ | 17       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 22)         | 10       | 28       | 40               | 16    | -272  | 0.09  |
| 10  | 16       | $\begin{pmatrix} 0 & 4 \\ 4 & 8 \end{pmatrix}$ | 16       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 28)         | 6        | 24       | 40               | 16    | -16   | 0     |
|     |          |  |          |  |                 |          |          | 52               | 24    | -24   | -0.58 |
| 11  | 18       | $\begin{pmatrix} 2 & 5 \\ 5 & 6 \end{pmatrix}$ | 13       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 24)         | 10       | 30       | 42               | 18    | -234  | -0.47 |
| 12  | 20       | $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ | 20       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (22 22)         | 6        | 28       | 44               | 20    | -4    | 0     |
| 13  | 20       | $\begin{pmatrix} 2 & 6 \\ 6 & 6 \end{pmatrix}$ | 24       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 26)         | 4        | 26       | 44               | 20    | -30   | 0.26  |

Table 3 continued

| MM# | $-K_Y^3$ | $N$   | $\Delta$ | $-K_Y$                                 | $\pi_{i_2}(Z)$ | $b_3(Y)$ | $b_3(Z)$ | $\pi_{i_2}(Z)A$ | $A^2$ | $B^2$ | $h$   |
|-----|----------|---|----------|--|----------------|----------|----------|-----------------|-------|-------|-------|
| 14  | 20       | $\begin{pmatrix} 0 & 5 \\ 5 & 10 \end{pmatrix}$ | 25       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (12 32)        | 2        | 24       | 44              | 20    | -20   | 0.32  |
| 15  | 22       | $\begin{pmatrix} 6 & 6 \\ 6 & 4 \end{pmatrix}$  | 12       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (24 22)        | 8        | 32       |                 |       |       |       |
| 16  | 22       | $\begin{pmatrix} 2 & 6 \\ 6 & 8 \end{pmatrix}$  | 20       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 28)        | 4        | 28       | 46              | 22    | -110  | -0.14 |
| 17  | 24       | $\begin{pmatrix} 4 & 7 \\ 7 & 6 \end{pmatrix}$  | 25       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (22 26)        | 2        | 28       | 48              | 24    | -600  | 0.06  |
| 18  | 24       | $\begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$  | 16       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (12 18)        | 4        | 30       | 30              | 10    | -40   | 0.68  |
| 19  | 26       | $\begin{pmatrix} 4 & 7 \\ 7 & 8 \end{pmatrix}$  | 17       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (22 28)        | 4        | 32       | 50              | 26    | -442  | -0.61 |
| 20  | 26       | $\begin{pmatrix} 2 & 7 \\ 7 & 10 \end{pmatrix}$ | 29       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (18 32)        | 0        | 28       | 50              | 26    | -754  | 0.16  |
| 21  | 28       | $\begin{pmatrix} 6 & 8 \\ 8 & 6 \end{pmatrix}$  | 28       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (26 26)        | 0        | 30       | 52              | 28    | -4    | 0     |
| 22  | 30       | $\begin{pmatrix} 4 & 8 \\ 8 & 10 \end{pmatrix}$ | 24       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (22 32)        | 0        | 32       | 54              | 30    | -20   | -0.32 |
| 23  | 30       | $\begin{pmatrix} 8 & 8 \\ 8 & 6 \end{pmatrix}$  | 16       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (28 26)        | 2        | 34       |                 |       |       |       |
| 24  | 30       | $\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$  | 21       | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | (18 18)        | 0        | 32       | 36              | 14    | -6    | 0.58  |
|     |          |   |          |  |                |          |          | 54              | 30    | -70   | -0.51 |

Table 3 continued

| MM# | $-K_Y^3$ | $N$   | $\Delta$ | $-K_Y$                                 | $\pi_{i,c_2}(Z)$ | $b_3(Y)$ | $b_3(Z)$ | $\pi_{i,c_2}(Z)A$ | $A^2$ | $B^2$ | $h$   |
|-----|----------|---|----------|--|------------------|----------|----------|-------------------|-------|-------|-------|
| 25  | 32       | $\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$  | 16       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (12 22)          | 2        | 36       | 34                | 12    | -12   | 0.42  |
| 26  | 34       | $\begin{pmatrix} 6 & 9 \\ 9 & 10 \end{pmatrix}$ | 21       | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (26 32)          | 0        | 36       |                   |       |       |       |
| 27  | 38       | $\begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$  | 17       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (18 22)          | 0        | 40       | 40                | 16    | -272  | 0.09  |
| 28  | 40       | $\begin{pmatrix} 18 & 9 \\ 9 & 4 \end{pmatrix}$ | 9        | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (42 22)          | 2        | 44       |                   |       |       |       |
| 29  | 40       | $\begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}$  | 16       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (12 26)          | 0        | 42       | 38                | 14    | -56   | 0.19  |
| 30  | 46       | $\begin{pmatrix} 6 & 6 \\ 6 & 4 \end{pmatrix}$  | 12       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (26 22)          | 0        | 48       |                   |       |       |       |
| 31  | 46       | $\begin{pmatrix} 2 & 5 \\ 5 & 6 \end{pmatrix}$  | 13       | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (18 26)          | 0        | 48       | 44                | 18    | -234  | -0.47 |
| 32  | 48       | $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$  | 12       | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ | (18 18)          | 0        | 50       | 36                | 12    | -4    | 0     |
| 33  | 54       | $\begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}$  | 9        | $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ | (12 22)          | 0        | 56       | 34                | 10    | -90   | -0.15 |
| 34  | 54       | $\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$  | 9        | $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ | (12 18)          | 0        | 56       | 30                | 8     | -72   | 0.17  |
| 35  | 56       | $\begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$  | 8        | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ | (18 22)          | 0        | 58       |                   | 14    | -126  | -0.64 |
| 36  | 62       | $\begin{pmatrix} 2 & 5 \\ 5 & 10 \end{pmatrix}$ | 5        | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | (18 34)          | 0        | 64       | 42                |       |       |       |

No fewer than 27 of the 36 classes of Fanos can be described explicitly as blow-ups in a smooth curve of a rank 1 Fano  $Y_0$  of index  $r \geq 2$  (so  $\text{Pic } Y_0$  is generated by  $-\frac{1}{r}K_{Y_0}$ ). Then one edge of the nef cone of  $Y$  is clearly generated by  $H := \pi^*(-\frac{1}{r}K_{Y_0})$ .

The hypothesis of the next lemma can be read as an elementary formulation of “ $C$  is cut out scheme-theoretically by sections of  $L$ ”, i.e. the tensor product  $\mathcal{I}_C \otimes \mathcal{L}$  of the ideal sheaf of  $C$  and the sheaf of sections of  $L$  being globally generated.

**Lemma 4.5** *Let  $L$  be a line bundle on a closed complex manifold  $Y_0$ , and let  $\pi : Y \rightarrow Y_0$  be the blow-up of a smooth curve  $C$  in  $Y_0$ . Let  $E \subset Y$  be the exceptional divisor of  $\pi$ .*

*Suppose that for every trivialising neighbourhood  $U \subset Y_0$  of  $L$ , the ideal  $\mathcal{I}_C(U) := \{f : f|_{C \cap U} \equiv 0\}$  in the ring  $\mathcal{O}_{Y_0}(U) := \{\text{holomorphic functions } U \rightarrow \mathbb{C}\}$  is generated by coordinate representatives of global sections of  $L$  that vanish identically on  $C$ . Then  $\pi^*L - E$  is basepoint-free.*

**Proof** Note that if  $D \in |L|$  contains  $C$ , then  $\pi^*D - E$  is effective (if  $D$  is smooth then this is simply the proper transform) and belongs to  $|\pi^*L - E|$ . Therefore if  $|\pi^*L - E|$  has a base point  $p \in Y \setminus E$ , then any global section of  $L$  vanishes at the corresponding point  $\pi(p) \in Y_0$ . The hypothesis forces the contradiction  $\pi(p) \in C$ .

Recall that  $E$  can be identified with  $\mathbb{P}(N_{C/Y_0})$ , the projectivisation of the normal bundle of  $C$ . If  $|\pi^*L - E|$  has a basepoint  $p \in E$ , then that corresponds to a non-zero normal vector  $v$  to  $C$  at  $\pi(p) \in C$  such that  $v$  is tangent to every element of  $|L|$  that contains  $C$ . Then the derivative of every local defining function of  $C$  near  $\pi(p)$  vanishes on  $v$ , contradicting that  $C$  is smooth.  $\square$

Returning to the setting of a blow-up in a rank 1 Fano  $Y_0$ , Lemma 4.5 implies that if  $C$  is cut out by sections of  $-\frac{n}{r}K_{Y_0}$  then  $G = \pi^*(-\frac{n}{r}K_{Y_0}) - E$  is nef, so the cone spanned by  $G$  and  $H$  is certainly contained in the nef cone of  $Y$ .

**Lemma 4.6** *If the blow-up  $Y$  of  $Y_0$  in  $C$  is Fano, and  $n$  is the minimal integer such  $C$  is cut out by sections of  $-\frac{n}{r}K_{Y_0}$ , then the nef cone of  $Y$  is exactly the cone spanned by  $G = \pi^*(-\frac{n}{r}K_{Y_0}) - E$  and  $H$ .*

**Proof** This can be verified for each of the 27 rank 2 Fano 3-folds of this type in the classification, aided e.g. by the descriptions of Coates–Corti–Galkin–Kasprzyk [8, §18–§53] of each Fano 3-fold as the zero locus of a general section of a vector bundle (often a complete intersection) of a “key variety”, a GIT quotient of a vector space (often a toric variety) whose nef cone is evident.  $\square$

Note that  $Y$  being Fano means that  $-K_Y = G + (r - n)H$  is in the interior of the nef cone, corresponding to  $1 \leq n \leq r - 1$  in all cases. Indeed, according to [20, Corollary 7.1.2(iii)], the blow-up of a rank 1 Fano 3-fold  $Y_0$  in  $C$  is Fano if and only if  $1 \leq n \leq r - 1$ ; this too is regarded as a consequence of the classification, similarly to how we do not know any uniform proof of Lemma 4.6.

Staying in the context of Lemma 4.6, the matrix representing the quadratic form on  $N$  with respect to the basis  $G, H$  can be computed from

$$\begin{aligned} H^2 &= \frac{(-K_{Y_0})^3}{r^2}, \\ (nH-G).H &= \text{deg } C, \\ (nH-G)^2 &= -\chi(C). \end{aligned}$$

We can apply (4.4) to read off  $\pi_1 c_2(Z)(-K_Y)$  from the other data in the table. To control the other half of  $\pi_1 c_2(Z)$ , we apply Lemma 4.4 to  $\pi_0 : Y \rightarrow Y_0$  to deduce that the product of  $c_2(Y) + c_1(Y)^2$  with  $\pi_0^*(-K_{Y_0})$  equals the product of  $c_2(Y_0) + c_1(Y_0)^2$  with  $-K_{Y_0}$ . We can thus apply (4.4) again to obtain

$$\pi_1 c_2(Z)H = \frac{24 - K_{Y_0}^3}{r} \tag{4.6}$$

(and the RHS is contained in Table 2).

Another four entries in the list are divisors in  $\mathbb{P}^2 \times \mathbb{P}^2$  (including  $\mathbb{P}^1 \times \mathbb{P}^2$ ). Then we can take the basis  $G, H$  of  $N$  to correspond to the restrictions of the hyperplane bundles from the two factors. For a divisor  $Y \in |aG + bH|$ , we can readily compute the quadratic form on  $N$  from  $-K_Y = (3 - a)G + (3 - b)H$ . Further,  $c(Y) = (1 + 3G + 3G^2)(1 + 3H + 3H^2)(1 + [Y])^{-1}$  gives

$$\begin{aligned} c_2(Y) + c_1(Y)^2 &= 3G^2 + 9GH + 3G^2 - (3G + 3H)[Y] + [Y]^2 + (3G + 3H - [Y])^2 \\ &= 12G^2 + 27GH + 12H^2 - (9G + 9H)[Y] + 2[Y]^2. \end{aligned}$$

To evaluate the product with  $G$  we compute

$$(c_2(Y) + c_1(Y)^2)[Y]G = 3((2a - 3)(b - 1)(b - 2) + 6)G^2H^2, \tag{4.7}$$

and the product with  $H$  is analogous.

Another 3 cases are branched double covers over other rank 2 Fanos,  $p : Y \rightarrow X$ . Then we can take  $G$  and  $H$  to be pull-backs of the previously identified edges of the nef cone in  $\text{Pic } X$ . If the branch locus is a smooth divisor in  $|2L|$ , then  $TY \oplus p^*(2L) \cong p^*(TX \oplus L)$  implies  $c_1(Y) = p^*(c_1(X) - [L])$  and  $c_2(Y) = p^*(c_2(X) - c_1(X)[L] + 2[L]^2)$ , so

$$c_2(Y) + c_1(Y)^2 = p^* \left( c_2(X) + c_1(X)^2 + 3[L]([L] - c_1(X)) \right). \tag{4.8}$$

For each entry  $Y$  in the Mori-Mukai list, we repeat below the description from [23, Table 12.3]. For blow-ups of rank 1 Fanos in a smooth curve  $C$  we only indicate in addition the smallest integer  $n$  such that  $C$  is cut out by sections of  $-\frac{n}{r}K_{Y_0}$ . For the remaining 9 cases we provide some additional explanation.

- #1 Blow-up of  $V_1$  (degree 1 del Pezzo 3-fold, degree 6 hypersurface in  $\mathbb{P}^4(3, 2, 1, 1, 1)$ ) in an elliptic curve that is the intersection of two divisors in  $|-\frac{1}{2}K_{V_1}|$  (i.e. the hyperplane class).  $n = 1$ .  
This is the only rank 2 Fano where the linear system  $| -K_Y |$  is not free; the base locus is the pre-image  $\mathbb{P}^1$  over the basepoint of  $|-\frac{1}{2}K_{V_1}|$ . It is therefore the only case where Construction 4.2 does not produce an associated “Fano-type” building block  $Z$  (though one could define a building block by blowing up in several steps [9, Proposition 5.9]).
- #2 Double cover of  $\mathbb{P}^1 \times \mathbb{P}^2$  branched over  $(2, 4)$  divisor.  $G$  and  $H$  are the pull-backs of  $\mathcal{O}_{\mathbb{P}^1}(1)$  and  $\mathcal{O}_{\mathbb{P}^2}(1)$  respectively. Use (4.8) (and result from #34) to compute

$$c_2(Y) + c_1(Y)^2 = 18GH + 12H^2 + 3(G + 2H)(-G - H) = 9GH + 6H^2,$$

and hence  $\pi_1 c_2(Z) = (12 \ 18)$ .

- #3 Blow-up of  $V_2$  (degree 2 del Pezzo 3-fold, a double cover of  $\mathbb{P}^3$  branched over a quartic hypersurface, or equivalently a degree 4 hypersurface in  $\mathbb{P}^3(2, 1, 1, 1)$ ) in an elliptic curve that is the intersection of two hyperplane divisors.  $n = 1$ .
- #4 Blow-up of  $\mathbb{P}^3$  along the intersection of two cubic hypersurfaces.  $n = 3$ .
- #5 Blow-up of  $V_3$  (cubic hypersurface of  $\mathbb{P}^4$ ) along the intersection of two hyperplane divisors (a plane cubic curve).  $n = 1$ .
- #6 Generic members of the family are  $(2,2)$  divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$ , but some are double cover of a  $(1, 1)$  divisor  $W \subset \mathbb{P}^2 \times \mathbb{P}^2$  branched over smooth divisor  $B \in |-K_W|$ .  $G$  and  $H$  are pull-backs of  $\mathcal{O}(1)$  from the two  $\mathbb{P}^2$  factors, and  $\pi_1 c_2(Z)$  can be computed from (4.7) (or from (4.8) and the result in #32).
- #7 Blow-up of  $Q$  (quadric hypersurface in  $\mathbb{P}^4$ ) in intersection of two sections by quadrics.  $n = 2$ .
- #8 Double cover of  $V_7$  ( $\mathbb{P}^3$  blown up in a point, #35) whose branch locus is a divisor  $B \in |-K_{V_7}|$  (for generic members of the intersection  $B \cap E$  with the exceptional divisor of  $V_7 \rightarrow \mathbb{P}^3$  is smooth, but for some members of the family  $B$  is reduced but not smooth).  $G$  and  $H$  are the pull-backs of the respective classes on  $V_7$ .  
By (4.8) we have  $c_2(Y) + c_1(Y)^2 = p^*(c_2(V_7) + c_1(V_7)^2 - 3(-\frac{1}{2}K_{V_7})^2)$ . We compute in #35 that  $c_2(V_7) + c_1(V_7)^2 = (18 \ 22)$  in terms of the basis for  $V_7$ , and we can read off from the Picard lattice that  $(-\frac{1}{2}K_{V_7})^2 = (3 \ 4)$ . Hence  $c_2(Y) + c_1(Y)^2 = p^*(9 \ 10)$ , which in terms of the basis for  $\text{Pic } Y$  is written as  $(18 \ 20)$ .
- #9 Blow-up of  $\mathbb{P}^3$  in a curve  $C$  of degree 7 and genus 5, which is an intersection of a two-parameter family of cubic hypersurfaces.  $n = 3$ .
- #10 Blow-up of  $V_4$  (complete intersection of two quadrics in  $\mathbb{P}^5$ ) in an elliptic curve that is the intersection of two hyperplane sections.  $n = 1$ .
- #11 Blow-up of  $V_3$  (cubic hypersurface in  $\mathbb{P}^4$ ) along a line.  $n = 1$ .
- #12 Blow-up of  $\mathbb{P}^3$  along a curve of degree 6 and genus 3 which is an intersection of cubic hypersurfaces.  $n = 3$ .
- #13 Blow-up of  $Q$  (quadric hypersurface in  $\mathbb{P}^4$ ) along a curve of degree 6 and genus 2.  $n = 2$ .



- #14 Blow-up of  $V_5$  (section of Plücker-embedded Grassmannian  $Gr(2, 5) \subset \mathbb{P}^9$  by a subspace of codimension 3) in an elliptic curve that is the intersection of two hyperplane sections.  $n = 1$ .
- #15 Blow-up of  $\mathbb{P}^3$  along the intersection of a quadric  $A$  and a cubic  $B$  (for generic members of the family  $A$  is smooth, but for some members of the family  $A$  is reduced but not smooth).  $n = 3$ .
- #16 Blow-up of  $V_4$  (complete intersection of two quadrics in  $\mathbb{P}^5$ ) in a conic.  $n = 1$ .
- #17 Blow-up of  $Q$  along an elliptic curve of degree 5.  $n = 2$ .
- #18 Double cover of  $\mathbb{P}^1 \times \mathbb{P}^2$  branched over  $(2, 2)$  divisor. Compute  $\pi_1 c_2(Z) = (12 \ 18)$  using (4.8) like in #2.
- #19 Blow-up of  $V_4$  along a line.  $n = 1$ .
- #20 Blow-up of  $V_5$  along a twisted cubic.  $n = 1$ .
- #21 Blow-up of  $Q$  along a twisted quartic (a rational degree 4 curve, isomorphic to the image of  $(s : t) \mapsto (s^4 : s^3 t : s^2 t^2 : s t^3 : t^4)$ ).  $n = 2$ .
- #22 Blow-up of  $V_5$  along a conic.  $n = 1$ .
- #23 Blow-up of  $Q$  along an intersection of two divisors  $A \in |\mathcal{O}_Q(1)|$  and  $B \in |\mathcal{O}_Q(2)|$  ( $A$  may be smooth or singular).  $n = 2$ .
- #24 A  $(1, 2)$  divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$ . Compute  $\pi_1 c_2(Z)$  by applying (4.7) with  $a = 1, b = 2$ .
- #25 Blow-up of  $\mathbb{P}^3$  along an elliptic curve that is the intersection of two quadrics.  $n = 2$ .
- #26 Blow-up of  $V_5$  along a line.  $n = 1$ .
- #27 Blow-up of  $\mathbb{P}^3$  along a twisted cubic.  $n = 2$ .
- #28 Blow-up of  $\mathbb{P}^3$  along a plane cubic (an elliptic curve).  $n = 3$ .
- #29 Blow-up of  $Q$  along a conic (complete intersection of two hyperplane sections).  $n = 1$ .
- #30 Blow-up of  $\mathbb{P}^3$  along a conic.  $n = 2$ .
- #31 Blow-up of  $Q$  along a line.  $n = 1$ .
- #32 A  $(1, 1)$ -divisor on  $\mathbb{P}^2 \times \mathbb{P}^2$ . Compute  $\pi_1 c_2(Z)$  by applying (4.7) with  $a = b = 1$ .
- #33 Blow-up of  $\mathbb{P}^3$  along a line.  $n = 1$ .
- #34  $Y = \mathbb{P}^1 \times \mathbb{P}^2$ .  $G = \mathcal{O}_{\mathbb{P}^1}(1)$  and  $H = \mathcal{O}_{\mathbb{P}^2}(1)$ . Compute  $\pi_1 c_2(Z) = (12 \ 18)$  by applying (4.7) with  $a = 1, b = 0$ .
- #35  $\mathbb{P}^3$  blown up in one point, or equivalently  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$  over  $\mathbb{P}^2$ .  $G$  is the proper transform of a plane passing through the blow-up point, and  $H$  is a plane not passing through the blow-up point. The product of  $c_2(Y) + c_1(Y)^2$  with  $H$  is 22, just as it is for  $\mathbb{P}^3$ .
- #36  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$  over  $\mathbb{P}^2$ .  $G$  is the pull-back of  $\mathcal{O}_{\mathbb{P}^2}(1)$ , while  $H$  is the dual of the tautological bundle. The intersection form on  $H^2(Y)$  is given by  $G^3 = 0, G^2 H = 1, G H^2 = 2, H^3 = 4$  (use that the section  $\mathbb{P}(\mathcal{O}(2))$  is a divisor representing  $H$ ). Because  $TY$  is stably isomorphic to  $f^* T\mathbb{P}^2 \oplus (H \otimes f^*(\mathcal{O} \oplus \mathcal{O}(-2)))$ , where  $f : Y \rightarrow \mathbb{P}^2$  is the fibration, we find that

$$c(Y) = (1 + 3G + 3G^2)(1 + H)(1 - 2G + H),$$

so  $c_1(Y) = G + 2H$  and  $c_2(Y) = -3G^2 + 4GH + T^2$ . Hence  $c_2(Y) + c_1(Y)^2 = -2G^2 + 8GH + 5H^2$ , which has product 18 with  $G$  and 34 with  $H$ . Thus  $\pi_1 c_2(Z) = (18 \ 34)$ .

**Remark 4.7** In each case we have described a basis for the Picard lattice  $N$ , which is tantamount to specifying an  $N$ -marking in the sense of Definition 5.6. On an elementary level, we could therefore interpret each entry in the list as defining a set  $\mathcal{Y}$  of  $N$ -marked Fano 3-folds.

## 5 The matching problem

Combining the results described in Sects. 2.3 and 4.1, we can produce twisted connected sum  $G_2$ -manifolds from matching pairs of Fano 3-folds. We will apply the methods developed in [10] to the problem of *finding* matchings between Fanos of rank 1 and 2. In this section, we summarise the results from [10, §6] on finding matchings with a prescribed “configuration” of the Picard lattices of a pair of Fano 3-folds, reducing the problem to a combination of problems in lattice arithmetic and deformation theory. The main result here—Proposition 5.8—improves on [10, Proposition 6.18] to deal more clearly with skew configurations.

### 5.1 Configurations and matching

Let  $\Sigma_{\pm} \subset Y_{\pm}$  be smooth anticanonical divisors of a pair of Fanos, and  $r : \Sigma_+ \rightarrow \Sigma_-$  a matching. Let  $h_+$  be a *marking* of  $\Sigma_+$ , i.e. an isometry  $h_+ : H^2(\Sigma_+) \rightarrow L$  where  $L$  is a copy of the K3 lattice (the unique unimodular lattice of signature  $(3,19)$ ). Then  $h_- := h_+ \circ r^*$  is a marking of  $\Sigma_-$ . The images of  $H^2(Y_{\pm}) \subset H^2(\Sigma_{\pm})$  under  $h_{\pm}$  are a pair of primitive sublattices  $N_{\pm} \subset L$ , isometric to the Picard lattices. This pair is well-defined up to the action of the isometry group  $O(L)$ , and plays a crucial role.

**Definition 5.1** Given a pair of Fanos with Picard lattices  $N_+$  and  $N_-$ , call a pair of primitive embeddings  $N_+, N_- \hookrightarrow L$  a *configuration*. Two such pairs of embeddings are considered equivalent if they are related by the action of  $O(L)$ .

We call a configuration *orthogonal* if the reflections of  $L(\mathbb{R})$  in  $N_+$  and  $N_-$  commute. If in addition  $N_+ \cap N_-$  is trivial then we call the configuration *perpendicular*. If the configuration is not orthogonal then we call it *skew*.

We saw in Sect. 2.4 that the homeomorphism invariants of the twisted connected sum  $M$  resulting from the matching depend on the configuration (e.g.  $H^2(M) = N_+ \cap N_-$ ), and in Theorem 3.6 that the generalised Eells–Kuiper invariant  $\mu(M)$  does too. We therefore ask:

*Given a pair  $\mathcal{Y}_+, \mathcal{Y}_-$  of deformation types of Fano 3-folds, which configurations of embeddings  $N_{\pm} \subset L$  of their Picard lattices arise from some matching of elements of  $\mathcal{Y}_+$  and  $\mathcal{Y}_-$ ?*

We see below that it is not too hard to answer this when one of the types has Picard rank 1, and we will be able to say quite a lot when both types have Picard rank 2. In general the question is quite difficult, but in any case a first step in simplifying it is to rephrase it as a problem of finding suitable triples of classes in  $L(\mathbb{R}) := L \otimes \mathbb{R}$ . Recall that the *period* of a marked K3 surface  $(\Sigma, h)$  is an oriented two-plane  $\Pi \subset L(\mathbb{R})$ , the image under  $h : H^2(\Sigma; \mathbb{R}) \rightarrow L(\mathbb{R})$  of the real and imaginary parts of classes in  $H^{2,0}(\Sigma; \mathbb{C})$ .

**Lemma 5.2** *Let  $Y_{\pm}$  be a pair of Fano 3-folds, and let  $N_{\pm} \subset L$  be the images of primitive isometric embeddings of the respective Picard lattices. Then the pair  $(N_+, N_-)$  is the configuration of some matching of  $Y_+$  and  $Y_-$  if and only if there exist*

- an orthonormal triple  $(k_+, k_-, k_0)$  of positive classes in  $L(\mathbb{R})$ ,
- anticanonical divisors  $\Sigma_{\pm} \subset Y_{\pm}$ ,
- markings  $h_{\pm}$  of  $\Sigma_{\pm}$ ,

such that the oriented plane  $\langle k_{\mp}, \pm k_0 \rangle$  is the period of  $(\Sigma_{\pm}, h_{\pm})$ ,  $h_{\pm}^{-1}(k_{\pm})$  is the restriction of a Kähler class on  $Y_{\pm}$ , and  $N_{\pm}$  is the image of the composition  $H^2(Y_{\pm}) \rightarrow H^2(\Sigma_{\pm}) \rightarrow L$ .

**Proof** Necessity is trivial, setting  $k_{\pm} = h_{\pm}(k_{\pm|\Sigma_{\pm}})$  for the Kähler classes  $k_{\pm}$  that appear in Definitions 2.5, and  $k_0$  corresponding to a generator of  $\Pi_+ \cap \Gamma^* \Pi_-$ , all normalised to unit length. Sufficiency relies on the Torelli theorem, cf. [10, Proposition 6.2]. □

To study how the matching problem depends on the choice of configuration, let us first set up some notation for various lattices.

- $W := N_+ + N_-$  (this need not be primitive in  $L$ ),
- $T_{\pm} \subset L$  the perpendicular of  $N_{\pm}$ ,
- $T := T_+ \cap T_-$ , or equivalently the perpendicular of  $W$ ,
- $W_{\pm} := T_{\mp} \cap N_{\pm}$ , and
- $\Lambda_{\pm} \subset L$  the perpendicular to  $T \oplus W_{\mp}$ , or equivalently the perpendicular to  $W_{\mp}$  in the primitive overlattice of  $W$ .

**Remark 5.3**  $N_{\pm} \subseteq \Lambda_{\pm}$ , with equality if and only if  $N_+$  and  $N_-$  “intersect orthogonally”, i.e. when  $W(\mathbb{R}) = W_+(\mathbb{R}) \oplus W_-(\mathbb{R}) \oplus (N_+(\mathbb{R}) \cap N_-(\mathbb{R}))$ ; equivalently the configuration is orthogonal in the sense of Definition 5.1.

### 5.2 Necessary conditions

Note that in Lemma 5.2 we must obviously have  $k_{\pm} \in N_{\pm}(\mathbb{R})$ . On the other hand,  $N_{\pm}$  is contained in the Picard group of the marked K3  $(\Sigma_{\pm}, h_{\pm})$ , which is the subgroup of  $L$  orthogonal to the period; the marked K3 is automatically  $N_{\pm}$ -polarised. Thus  $k_{\mp}$  and  $k_0$  must both lie in  $T_{\pm}(\mathbb{R})$ . Hence

$$k_{\pm} \in W_{\pm}(\mathbb{R}), \quad k_0 \in T(\mathbb{R}). \tag{5.1}$$

Now we come to the heart of how the difficulty of the matching problem depends on the configuration one tries to achieve: (5.1) implies that the period  $\langle k_{\mp}, \pm k_0 \rangle$  is orthogonal to all of  $\Lambda_{\pm}$ , so the marked K3 divisors used in a matching with the given configuration are forced to be  $\Lambda_{\pm}$ -polarised.

The significance is that the Picard group of a generic K3 divisor in a generic member of a deformation type  $\mathcal{Y}_{\pm}$  of Fano 3-folds will be precisely the Picard lattice  $N_{\pm}$  of that type. To find matchings for a configuration where  $\Lambda_{\pm}$  is strictly bigger than  $N_{\pm}$ , we therefore require non-generic K3 divisors in members of  $\mathcal{Y}_{\pm}$  (the moduli space of

$\Lambda_{\pm}$ -polarised marked K3 surfaces forms a subspace of the  $N_{\pm}$ -polarised K3s, whose codimension is  $\text{rk } \Lambda_{\pm} - \text{rk } N_{\pm}$ ).

For configurations where  $\Lambda_{\pm} = N_{\pm}$ , we deduce in Sect. 6 the existence of matchings between some elements of  $\mathcal{Y}_+$  and  $\mathcal{Y}_-$  from a general fact (due to Beauville [2]) that a generic  $N_{\pm}$ -polarised K3 appears as an anticanonical divisor in some member of  $\mathcal{Y}_{\pm}$ . In view of Remark 5.3, this comparatively easy case corresponds to orthogonal configurations. To apply a similar argument for skew configurations (where  $\Lambda_{\pm} \supset N_{\pm}$ ), we first need to show for those specific  $\Lambda_{\pm}$  that generic  $\Lambda_{\pm}$ -polarised K3s appear as anticanonical divisors in members of  $\mathcal{Y}_{\pm}$ . Even when it is true, it is something that we can so far only verify case by case. We refer to this process as ‘handcrafting’.

**Remark 5.4** Before moving on to existence results for matchings with a prescribed configuration, let us point out some necessary conditions.

- (i) Since  $W(\mathbb{R})$  contains a two-dimensional positive-definite subspace (spanned by the orthogonal classes  $k_+$  and  $k_-$ ), while its orthogonal complement in the signature  $(3, 19)$  lattice  $L$  contains a class  $k_0$  with  $k_0^2 > 0$ , the quadratic form on  $W$  must be non-degenerate of signature  $(2, \text{rk}(W) - 2)$ .
- (ii)  $W_{\pm} \subset N_{\pm}$  must contain some ample classes of  $Y_{\pm}$ .
- (iii) Since  $\Lambda_+ \cap \Lambda_- \subset \text{Pic } \Sigma_{\pm}$  and is orthogonal to an ample class of  $\Sigma_{\pm}$ , it cannot contain any  $(-2)$ -classes.

**Remark 5.5** In particular, (ii) implies that any matching involving a Fano with Picard rank  $\text{rk } N = 1$  must be perpendicular. Moreover, for a configuration of lattices  $N_+$  and  $N_-$  where at least one has rank 2, if the intersection  $N_+ \cap N_-$  is non-trivial then (ii) forces the configuration to be orthogonal in the sense of Definition 5.1. For configurations of Picard lattices of Fanos of rank  $\leq 2$  that satisfy the necessary conditions to be realised by a matching, we therefore have the following trichotomy:

- Perpendicular configurations, i.e. every element of  $N_+$  is orthogonal to every element of  $N_-$ .
- Orthogonal configurations with non-trivial intersection. Then  $N_+ \cap N_-$  must have rank 1.
- Skew configurations. Then  $N_+ \cap N_-$  must be trivial, but  $N_+$  is not perpendicular to  $N_-$  (the maps  $N_{\pm} \rightarrow N_{\mp}^*$  must have rank 1).

In Sects. 6 and 7, we will consider these cases in turn.

### 5.3 Sufficient conditions

In order to describe the ‘genericity properties’ we require for anticanonical K3 divisors in families of Fano 3-folds, we recall some further terminology. The *period domain* is the space of oriented positive-definite 2-planes in  $L(\mathbb{R})$ . It can be identified with  $\{\Pi \in \mathbb{P}(L(\mathbb{C})) : \Pi^2 = 0, \Pi \bar{\Pi} > 0\}$  in order to exhibit a natural complex structure. Given  $\Lambda \subset L$ , the period domain of  $\Lambda$ -polarised K3 surfaces is  $D_{\Lambda} := \{\Pi \in \mathbb{P}(\Lambda^{\perp}(\mathbb{C})) : \Pi^2 = 0, \Pi \bar{\Pi} > 0\}$ .

**Definition 5.6** Given a non-degenerate lattice  $N$ , an  $N$ -marking of a closed 3-fold  $Y$  is a surjective homomorphism  $i_Y : H^2(Y) \rightarrow N$  that is isometric for the anticanonical form of Definition 4.1.

We avoid calling  $i_Y$  an “ $N$ -polarisation” since we do not impose any conditions on ample classes. If  $Y$  is Fano then the Picard lattice is non-degenerate so  $i_Y$  is simply an isometry.

**Definition 5.7** Let  $N \subseteq \Lambda \subset L$  be primitive non-degenerate sublattices of  $L$ , and let  $\text{Amp}_Y$  be a non-empty open subcone of the positive cone in  $N(\mathbb{R})$ . We say that a set  $\mathcal{Y}$  of  $N$ -marked 3-folds is  $(\Lambda, \text{Amp}_Y)$ -generic if there is  $U_Y \subseteq D_\Lambda$  with complement a countable union of complex analytic submanifolds of positive codimension with the property that: for any  $\Pi \in U_Y$  and  $k \in \text{Amp}_Y$  there is  $Y \in \mathcal{Y}$ , a smooth anticanonical divisor  $\Sigma \subset Y$  and a marking  $h : H^2(\Sigma) \rightarrow L$  such that  $\Pi$  is the period of  $(\Sigma, h)$ , the composition  $H^2(Y) \rightarrow H^2(\Sigma) \rightarrow L$  equals the marking  $i_Y$ , and  $h^{-1}(k)$  is the image of the restriction to  $\Sigma$  of a Kähler class on  $Y$ .

To be able to prove that a set  $\mathcal{Y}$  of Fano 3-folds satisfies the definition we typically take  $\mathcal{Y}$  to be a deformation type, but to make sense of the definition we do not need to remember any additional structure on  $\mathcal{Y}$  (cf. Remark 4.7).

Meanwhile, when applying the next proposition we typically want all elements of the sets  $\mathcal{Y}_\pm$  to be Fano 3-folds (or building blocks) that are topologically the same, so that we have some control over the topology of the  $G_2$ -manifolds resulting from the matchings produced; essentially this means that all elements of  $\mathcal{Y}_\pm$  should belong to the same deformation type.

**Proposition 5.8** Consider a configuration of primitive non-degenerate sublattices  $N_+, N_- \subset L$ , and let  $\mathcal{Y}_\pm$  be a pair of sets of  $N_\pm$ -marked 3-folds. Define  $W, W_\pm$  and  $\Lambda_\pm$  as above. Suppose that there exist non-empty open cones  $\text{Amp}_{\mathcal{Y}_\pm} \subseteq N_\pm(\mathbb{R})^+$  such that

- (i) the sets  $\mathcal{Y}_\pm$  are  $(\Lambda_\pm, \text{Amp}_{\mathcal{Y}_\pm})$ -generic,
- (ii)  $W_\pm \cap \text{Amp}_{\mathcal{Y}_\pm} \neq \emptyset$ .

Then there is an open dense subcone  $\mathcal{W} \subseteq (W_+(\mathbb{R}) \cap \text{Amp}_{\mathcal{Y}_+}) \times (W_-(\mathbb{R}) \cap \text{Amp}_{\mathcal{Y}_-})$  such that for every  $(k_+, k_-) \in \mathcal{W}$  with  $k_+^2 = k_-^2$  there exist  $Y_\pm \in \mathcal{Y}_\pm$ , anticanonical K3 divisors  $\Sigma_\pm \subset Y_\pm$  and Kähler classes  $k_\pm \in H^2(Y_\pm)$  such that  $k_{\pm|\Sigma_\pm} = k_\pm$ , with a matching  $\tau : \Sigma_+ \rightarrow \Sigma_-$  of  $(Y_+, \Sigma_+, k_+)$  and  $(Y_-, \Sigma_-, k_-)$  whose configuration is the given pair of embeddings  $N_\pm \subset L$ .

**Proof** The argument is essentially the same as [10, Proposition 6.18], even though the conclusion stated here is slightly stronger.

Let  $T = W^\perp$  as before. Denote the ranks of  $W$  and  $W_\pm$  by  $r$  and  $r_\pm$ . Then  $W_\pm(\mathbb{R})$  and  $T(\mathbb{R})$  are real vector spaces of signature  $(1, r_\pm - 1)$  and  $(1, 21 - r)$  respectively

In view of Lemma 5.2 and (5.1), matchings correspond to certain triples of classes  $(k_+, k_-, k_0)$  such that  $k_\pm$  and  $k_0$  belong to the positive cones  $W_\pm(\mathbb{R})^+$  and  $T(\mathbb{R})^+$  respectively. Consider therefore the real manifold

$$D = \mathbb{P}(W_+(\mathbb{R})^+) \times \mathbb{P}(W_-(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+).$$

Below, we need the open subset  $\mathcal{A} = \mathcal{A}_+ \times \mathcal{A}_- \times \mathbb{P}(T(\mathbb{R})^+)$ , where  $\mathcal{A}_\pm := \mathbb{P}(W_\pm(\mathbb{R}) \cap \text{Amp}_{\mathcal{Y}_\pm})$  is non-empty by hypothesis (ii). We have two Griffiths period domains

$$D_{\Lambda_\pm} = \{\text{positive-definite planes } \Pi \subset \Lambda_\pm^\perp(\mathbb{R})\},$$

and projections

$$\text{pr}_\pm : D \rightarrow D_{\Lambda_\pm}, (\ell_+, \ell_-, \ell) \mapsto \langle \ell_\mp, \pm \ell \rangle.$$

As stated before Definition 5.7,  $D_{\Lambda_\pm}$  can be regarded as an open subset of  $\mathbb{P}(C_\pm)$ , where  $C_\pm$  is the null cone in  $\Lambda_\pm^\perp \otimes \mathbb{C}$ ; if  $\alpha, \beta$  is an oriented orthonormal basis of  $\Pi \in D_{\Lambda_\pm}$  then  $\Pi \mapsto \langle \alpha + i\beta \rangle \in \mathbb{P}(C_\pm)$ . Given a choice  $\alpha$  and  $\beta$ , we can identify  $T_\Pi D_{\Lambda_\pm}$  with pairs  $(u, v)$  of vectors in the orthogonal complement of  $\Pi$  in  $\Lambda_\pm^\perp(\mathbb{R})$ . Then the complex structure on  $T_\Pi D_{\Lambda_\pm}$  is given by  $J : (u, v) \mapsto (-v, u)$ .

Observe that the real analytic embedded submanifold  $\mathbb{P}(W_\mp(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+) \hookrightarrow D_{\Lambda_\pm}$  is totally real: for  $w \in W_\mp$  and  $t \in T(\mathbb{R})$ , the tangent space  $\mathcal{T}$  to  $\mathbb{P}(W_\mp(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+)$  at  $\Pi = \langle w, t \rangle$  corresponds to  $(u, v)$  such that  $u \in w^\perp \subseteq W_\mp(\mathbb{R})$  and  $v \in t^\perp \subseteq T(\mathbb{R})$ , so  $J(\mathcal{T})$  is transverse to  $\mathcal{T}$ .

Crucially, this totally real submanifold has *maximal dimension*:

$$\begin{aligned} \dim_{\mathbb{R}} \mathbb{P}(W_\mp(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+) \\ = (r_\mp - 1) + (22 - r - 1) = 20 - r + r_\mp = 20 - \text{rk } \Lambda_\pm = \dim_{\mathbb{C}} D_{\Lambda_\pm} \end{aligned}$$

Consequently, its intersection with any positive-codimensional complex analytic submanifold of  $D_{\Lambda_\pm}$  is a positive-codimensional real analytic submanifold of  $\mathbb{P}(W_\mp(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+)$ . Hence the pre-image in  $\mathbb{P}(W_\mp(\mathbb{R})^+) \times \mathbb{P}(T(\mathbb{R})^+)$  of the subset  $U_{\mathcal{Z}_\pm} \subset D_{\Lambda_\pm}$  from Definition 5.7 is open and dense. Because  $\text{pr}_\pm$  is a projection of a product manifold onto a factor the same is true for  $\text{pr}_\pm^{-1}(U_{\mathcal{Z}_\pm}) \subset D$ . In turn,

$$(\mathcal{A}_+ \times \mathcal{A}_- \times \mathbb{P}(T(\mathbb{R})^+)) \cap \text{pr}_+^{-1}(U_{\mathcal{Z}_+}) \cap \text{pr}_-^{-1}(U_{\mathcal{Z}_-})$$

is open and dense in  $\mathcal{A}_+ \times \mathcal{A}_- \times \mathbb{P}(T(\mathbb{R})^+)$ , and hence the image  $\mathcal{W}'$  of this subset under projection to  $\mathcal{A}_+ \times \mathcal{A}_-$  is open and dense in  $\mathcal{A}_+ \times \mathcal{A}_-$ .

If we let  $\mathcal{W} = \{(k_+, k_-) \in (W_+(\mathbb{R}) \cap \text{Amp}_{\mathcal{Y}_+}) \times (W_-(\mathbb{R}) \cap \text{Amp}_{\mathcal{Y}_-}) : ([k_+], [k_-]) \in \mathcal{W}'\}$ , then for every  $(k_+, k_-) \in \mathcal{W}$  such that  $k_+^2 = k_-^2$  there is a  $k_0 \in T(\mathbb{R})^+$  such that Lemma 5.2 applies to  $(k_+, k_-, k_0)$ .  $\square$

### 6 Orthogonal matching

We now consider the problem of finding matchings of Fano 3-folds of Picard rank 1 or 2, with prescribed configuration that is orthogonal in the sense of Definition 5.1. As pointed out in Remark 5.3, this corresponds to the Picard lattices  $N_\pm$  being equal to the lattices  $\Lambda_\pm$  that are used in the hypothesis of Proposition 5.8. Therefore the following genericity result is enough to let us apply Proposition 5.8 for these configurations.

**Proposition 6.1** ([9, Proposition 6.9], based on Beauville [2]) *Let  $\mathcal{Y}$  be a deformation type of Fano 3-folds, and embed its Picard lattice  $N$  primitively in  $L$ . Then there exists an open cone  $\text{Amp}_{\mathcal{Y}} \subset N(\mathbb{R})$  such that  $\mathcal{Y}$  is  $(N, \text{Amp}_{\mathcal{Y}})$ -generic.*

In the trichotomy of Remark 5.5, orthogonal configurations encompass the cases of perpendicular configurations and orthogonal configurations with non-trivial intersection. Now we study in turn the twisted connected sum  $G_2$ -manifolds that result from matchings of these kinds, using Fano 3-folds of Picard rank 1 or 2.

### 6.1 Perpendicular matching

The simplest way to find a matching between elements of two deformation types  $\mathcal{Y}_{\pm}$  of Fano 3-folds is to consider perpendicular configurations, i.e. where the images of  $N_+$  and  $N_-$  in  $L$  are perpendicular to each other. One reason is that we do not need any genericity results beyond Proposition 6.1, but a further reason is arithmetic: for nearly all pairs  $\mathcal{Y}_{\pm}$  one has  $\text{rk } N_+ + \text{rk } N_- \leq 11$ , in which case Nikulin [34, Theorem 1.12.4] guarantees that there does in fact exist a primitive embedding of the perpendicular direct sum  $N_+ \perp N_-$  into  $L$ .

In particular, we can apply this to find perpendicular matchings among the 53 types of Fanos of Picard rank 1 and 2. However, we ignore #1 in the list of rank 2 Fanos, since that does not have an associated Fano-type building block. Given one of the 1378 unordered pairs  $\mathcal{Y}_+, \mathcal{Y}_-$  among the other 52 deformation types, we carry out the following procedure:

- Apply [34, Theorem 1.12.4] to find a primitive embedding into  $L$  of the perpendicular direct sum  $N_+ \perp N_-$ .
- Apply Proposition 5.8 to find a matching between some  $Y_{\pm} \in \mathcal{Y}_{\pm}$  with the given configuration  $N_+, N_- \subset L$ .
- Apply Proposition 4.3 to produce a pair of Fano-type building blocks with a perpendicular matching.
- Apply Theorem 2.7 to construct a twisted connected sum  $G_2$ -manifold  $M$ .

Now Theorem 2.8 shows that  $M$  is 2-connected, with  $H^4(M)$  torsion-free, and

$$b_3(M) = b_3(Z_+) + b_3(Z_-) + 23.$$

Proposition 2.11 implies that  $d(M)$ , the greatest divisor of  $p_M$ , equals the greatest common divisor of  $\pi_1 c_2(Z_+)$  and  $\pi_1 c_2(Z_-)$ , while Corollary 3.7 shows that  $\mu(M) = 0$ . Thus all the classifying diffeomorphism invariants of  $M$  can be determined from the data in Tables 2 and 3.

Table 4 lists the invariants of the 1378 twisted connected sums obtained this way. A total of 131 different 2-connected manifolds are realised, with 60 different values of  $b_3(M)$ . For comparison, twisted connected sums involving only rank 1 Fanos realise 82 different manifolds, and 46 different values of  $b_3(M)$  [10, Table 3].

**Remark 6.2** According to Theorem 1.7 and Corollary 1.13 of [12], the torsion-free  $G_2$ -structures of diffeomorphic 2-connected twisted connected sums with  $d$  not divisible by 3 are automatically homotopic (if one chooses the diffeomorphism correctly).

**Table 4** Twisted connected sums from perpendicular matching of Fanos with Picard rank 1 or 2

| $b_3$ | #  | $d$ |   |   |   |    |    | $b_3$ | #    | $d$  |    |    |    |    |    |
|-------|----|-----|---|---|---|----|----|-------|------|------|----|----|----|----|----|
|       |    | 2   | 4 | 6 | 8 | 12 | 24 |       |      | 2    | 4  | 6  | 8  | 12 | 24 |
| 71    | 15 | 8   | 5 | 1 | 1 | 0  | 0  | 133   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 73    | 20 | 16  | 3 | 1 | 0 | 0  | 0  | 135   | 10   | 9    | 0  | 1  | 0  | 0  | 0  |
| 75    | 40 | 35  | 3 | 1 | 0 | 1  | 0  | 137   | 12   | 12   | 0  | 0  | 0  | 0  | 0  |
| 77    | 39 | 36  | 0 | 2 | 0 | 1  | 0  | 139   | 4    | 4    | 0  | 0  | 0  | 0  | 0  |
| 79    | 73 | 65  | 3 | 4 | 0 | 1  | 0  | 141   | 2    | 1    | 1  | 0  | 0  | 0  | 0  |
| 81    | 60 | 55  | 1 | 4 | 0 | 0  | 0  | 143   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 83    | 77 | 67  | 3 | 6 | 1 | 0  | 0  | 145   | 7    | 7    | 0  | 0  | 0  | 0  | 0  |
| 85    | 63 | 49  | 8 | 4 | 1 | 1  | 0  | 147   | 2    | 2    | 0  | 0  | 0  | 0  | 0  |
| 87    | 77 | 69  | 4 | 3 | 0 | 1  | 0  | 151   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 89    | 57 | 49  | 5 | 2 | 1 | 0  | 0  | 153   | 2    | 2    | 0  | 0  | 0  | 0  | 0  |
| 91    | 55 | 51  | 2 | 2 | 0 | 0  | 0  | 155   | 8    | 7    | 1  | 0  | 0  | 0  | 0  |
| 93    | 49 | 45  | 4 | 0 | 0 | 0  | 0  | 157   | 4    | 4    | 0  | 0  | 0  | 0  | 0  |
| 95    | 52 | 49  | 0 | 2 | 0 | 0  | 1  | 159   | 6    | 6    | 0  | 0  | 0  | 0  | 0  |
| 97    | 51 | 44  | 3 | 3 | 1 | 0  | 0  | 161   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 99    | 54 | 42  | 7 | 3 | 2 | 0  | 0  | 163   | 8    | 8    | 0  | 0  | 0  | 0  | 0  |
| 101   | 33 | 26  | 1 | 5 | 1 | 0  | 0  | 165   | 2    | 2    | 0  | 0  | 0  | 0  | 0  |
| 103   | 66 | 55  | 3 | 7 | 1 | 0  | 0  | 167   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 105   | 41 | 38  | 0 | 3 | 0 | 0  | 0  | 169   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 107   | 48 | 43  | 1 | 3 | 1 | 0  | 0  | 171   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 109   | 34 | 31  | 0 | 3 | 0 | 0  | 0  | 173   | 2    | 2    | 0  | 0  | 0  | 0  | 0  |
| 111   | 43 | 40  | 0 | 2 | 1 | 0  | 0  | 175   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 113   | 40 | 34  | 5 | 0 | 1 | 0  | 0  | 179   | 4    | 4    | 0  | 0  | 0  | 0  | 0  |
| 115   | 32 | 30  | 1 | 1 | 0 | 0  | 0  | 181   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 117   | 30 | 28  | 1 | 0 | 1 | 0  | 0  | 183   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 119   | 31 | 28  | 0 | 3 | 0 | 0  | 0  | 187   | 3    | 3    | 0  | 0  | 0  | 0  | 0  |
| 121   | 29 | 26  | 1 | 2 | 0 | 0  | 0  | 189   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 123   | 17 | 16  | 0 | 1 | 0 | 0  | 0  | 195   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 125   | 11 | 10  | 1 | 0 | 0 | 0  | 0  | 197   | 2    | 2    | 0  | 0  | 0  | 0  | 0  |
| 127   | 20 | 14  | 3 | 2 | 1 | 0  | 0  | 239   | 1    | 1    | 0  | 0  | 0  | 0  | 0  |
| 129   | 11 | 10  | 0 | 1 | 0 | 0  | 0  |       |      |      |    |    |    |    |    |
| 131   | 9  | 8   | 1 | 0 | 0 | 0  | 0  | Total | 1378 | 1215 | 71 | 72 | 14 | 5  | 1  |

While Table 4 shows that there are also numerous instances of diffeomorphic twisted connected sums with  $d = 6$  (e.g.  $(b_3, d) = (103, 6)$  is realised by 7 different twisted connected sums), Wallis [40, Corollary 6.5.2] builds on the methods of this paper to compute the  $G_2$ -structures'  $\xi$ -invariant introduced in [12, Definition 6.8], showing that the  $G_2$ -structures are homotopic for these examples too. However, [40, Table 8.1] exhibits other examples (involving skew matchings) where the  $G_2$ -structures of twisted connected sums can be distinguished by the  $\xi$ -invariant.



**Remark 6.3** Recall from Wall’s classification of closed simply-connected spin 6-manifolds with torsion-free cohomology [39, Theorem 5] that for  $d$  divisible by 12, there is such a 6-manifold  $X$  with the cohomology ring of  $S^2 \times S^4$  and greatest divisor of  $p_X$  equal to  $d$ .

The total space  $M$  of the circle bundle over  $X\#(S^3 \times S^3)^{\#k}$  with Euler class equal to the generator of  $H^2$  is then a closed 2-connected 7-manifold with  $b_3(M) = 2k + 1$ , torsion-free  $H^4(M)$  and  $d(M) = d$  (and this is the only way to obtain such total spaces). One can use the corresponding closed disc bundle as a coboundary for  $M$  to compute that  $\mu(M) = 0$ . (The more general problem of deciding which 2-connected 7-manifolds can be realised as the total spaces of circle bundles is analysed by Jiang [24].)

Thus the classification Theorem 2.2 implies that the 6 entries in Table 4 that have  $d = 12$  or  $24$  are diffeomorphic to total spaces of circle bundles (in contrast to the result of Baraglia [1, Proposition 6.2.1] that closed  $G_2$ -manifolds cannot be smoothly fibred by 4-manifolds), while the last entry of Table 1 is a topological  $S^1$ -bundle but not a smooth one.

### 6.2 Orthogonal matching with non-trivial intersection

Next we consider matchings with configurations such that  $N_+ \cap N_-$  is non-trivial. Then both Fanos must have Picard rank  $\geq 2$ . If we restrict attention to the case when both Fanos have Picard rank precisely 2, then as pointed out in Remark 5.5 the only configurations with  $N_+ \cap N_-$  non-trivial for which we can possibly find a matching are the ones that are orthogonal, in the sense of Definition 5.1. Such configurations have  $\Lambda_{\pm} = N_{\pm}$ , so to apply Proposition 5.8 we essentially do not need any genericity result beyond Proposition 6.1 that we applied to find perpendicular matchings—the only extra data we need is to actually determine the cone  $\text{Amp}_{\mathcal{Y}}$ , but for rank 2 Fano 3-folds we have done that in Sect. 4.4.

Compared with the perpendicular matching problem, the difficulty of finding matchings of rank 2 blocks with non-trivial intersection  $N_+ \cap N_-$  is therefore one of lattice-arithmetic: there must exist some integral lattice  $W$  of rank 3, containing  $N_+$  and  $N_-$ , such that the orthogonal complement of  $W_{\pm} \subset N_{\pm}$  of  $N_{\mp}$  is non-trivial, and contains a class  $A_{\pm} \in \text{Amp}_{\mathcal{Y}_{\pm}}$ .

**Lemma 6.4** *Let  $N_{\pm}$  be integral lattices of rank 2 with signature  $(1, 1)$ , and let  $A_{\pm} \in N_{\pm}$ . Let  $-\Delta_{\pm}$  be the discriminant of  $N_{\pm}$ , and let  $B_{\pm}$  be a generator of the orthogonal complement. Then there exists a rank 3 integral lattice  $W$  with primitive isometric embeddings  $N_{\pm} \hookrightarrow W$  such that  $A_{\pm} \perp N_{\mp}$  if and only if  $B_+^2 = B_-^2$  and  $\Delta_+ \Delta_- = k^2 A_+^2 A_-^2$  for some integer  $k$ . Then  $k$  is the generator of the image of the product  $N_+ \times N_- \rightarrow \mathbb{Z}$  in  $W$ .*

**Proof** If such a  $W$  exists, then the images of  $B_+$  and  $B_-$  are both primitive vectors, and both perpendicular to both  $A_+$  and  $A_-$ . Thus, up to sign,  $B_+$  and  $B_-$  must have the same image  $B \in W$ , and hence  $B_+^2 = B^2 = B_-^2$ .

Note that for a pair  $(v_+, v_-) \in N_+ \times N_-$ , the product of their images in  $W$  is

$$v_+ \cdot v_- = \frac{(v_+ \cdot B_+)(v_- \cdot B_-)}{B^2}. \tag{6.1}$$

In particular, if we let  $a_{\pm}$  be the positive generator of the image of  $N_{\pm} \rightarrow \mathbb{Z}$ ,  $v \mapsto v \cdot B_{\pm}$ , then  $a_+ a_- = -kB^2$ . Conversely if  $a_+ a_-$  is divisible by  $B^2$ , then we can use (6.1) (together with the given forms on  $N_+$  and  $N_-$ ) to define the desired integral quadratic form on  $W := (N_+ \oplus N_-)/\langle B_+ - B_- \rangle$ .

Now observe that the index of the sublattice  $\langle A_+, B_+ \rangle \subseteq N_+$  is  $\frac{B_+^2}{a_+^2}$ . Letting  $-\Delta_{\pm}$  be the discriminant of  $N_{\pm}$ , we must therefore have  $A_+^2 B^2 = (-\Delta_+) \left(\frac{B_+^2}{a_+^2}\right)^2$ . Hence

$$(a_+ a_-)^2 = \frac{\Delta_+ \Delta_- (B^2)^2}{A_+^2 A_-^2},$$

and  $a_+ a_-$  is divisible by  $B^2$  if and only if  $\frac{\Delta_+ \Delta_-}{A_+^2 A_-^2}$  is a perfect square. □

Next we summarise the relevant topological calculations.

**Lemma 6.5** *Let  $(Z_{\pm}, \Sigma_{\pm})$  be a pair of building blocks whose polarising lattices  $N_{\pm}$  have rank 2, such that the kernel of  $H^2(Z_{\pm}) \rightarrow H^2(\Sigma_{\pm})$  is generated by  $[\Sigma_{\pm}]$ . Let  $r : \Sigma_+ \rightarrow \Sigma_-$  be a matching whose configuration  $N_+, N_- \subset L$  has  $N_+ \cap N_-$  of rank 1, and let  $M$  be the resulting twisted connected sum.*

*Let  $W := N_+ + N_-$ , and let  $A_{\pm} \in N_{\pm}$  be a primitive vector spanning the orthogonal complement of  $N_{\mp}$  in  $N_{\pm}$ . Then*

- (i)  $H^2(M) \cong \mathbb{Z}$ ;
- (ii)  $b_3(M) = b_3(Z_+) + b_3(Z_-) + 22$ ;
- (iii)  $\text{Tor } H^3(M) \cong \text{Tor } L/W$ ;
- (iv)  $\text{Tor } H^4(M) \cong (\mathbb{Z}/k)^2$ , for  $k$  as in Lemma 6.4;
- (v) if  $k = 1$  then  $d(M)$  (which divides 24) is the greatest common divisor of  $\bar{c}_2(Z_+)A_+$  and  $\bar{c}_2(Z_-)A_- \in \mathbb{Z}/24$ , for  $\bar{c}_2(Z_{\pm})$  as in (4.3);
- (vi) if  $k = 1$  and  $d(M)$  is divisible by 8 (so that the Eells–Kuiper invariant  $\mu(M)$  takes values in  $\mathbb{Z}/2$ ) then

$$\mu(M) = \frac{\text{gd}(\bar{c}_2(Z_+)) \text{gd}(\bar{c}_2(Z_-))}{4} \in \mathbb{Z}/2,$$

where  $\text{gd}(\bar{c}_2(Z_{\pm})) \in \{2, 4, 6, 8, 12, 24\}$  is the greatest divisor of  $\bar{c}_2(Z_{\pm})$ .

**Proof** (i)–(iii) are immediate consequences of Theorem 2.8.

The image  $N'_{\mp}$  of the product homomorphism  $b^{\pm} : N_{\mp} \rightarrow N_{\pm}^*$  from (2.6) has rank 1, and Lemma 6.4 implies that it has cotorsion  $\mathbb{Z}/k$  in  $N_{\pm}^*$ . (iv) now follows from (2.7).

When  $k = 1$ , so that  $N'_{\mp} \subset N_{\pm}^*$  is primitive, the isomorphism  $N_{\pm}^*/N'_{\mp} \cong \mathbb{Z}$  is realised by evaluation on  $A_{\pm}$ . Therefore (v) follows from Proposition 2.11 and Lemma 2.9.

**Table 5** Twisted connected sums  $M$  from matchings of rank 2 blocks with non-trivial intersection  $N_+ \cap N_-$

| # <sub>+</sub> | # <sub>-</sub> | $B^2$ | $A_+^2$ | $A_-^2$ | $b_3(M)$ | $d(M)$ |
|----------------|----------------|-------|---------|---------|----------|--------|
| 6              | 6              | -4    | 12      | 12      | 86       | 12     |
| 6              | 12             | -4    | 12      | 20      | 82       | 4      |
| 6              | 21             | -4    | 12      | 28      | 84       | 4      |
| 6              | 32             | -4    | 12      | 12      | 104      | 12     |
| 12             | 12             | -4    | 20      | 20      | 78       | 4      |
| 12             | 21             | -4    | 20      | 28      | 80       | 4      |
| 12             | 32             | -4    | 20      | 12      | 100      | 4      |
| 21             | 21             | -4    | 28      | 28      | 82       | 4      |
| 21             | 32             | -4    | 28      | 12      | 102      | 4      |
| 32             | 32             | -4    | 12      | 12      | 122      | 12     |
| 2              | 24             | -6    | 6       | 14      | 102      | 6      |
| 18             | 24             | -6    | 24      | 14      | 84       | 12     |
| †5             | 25             | -12   | 12      | 12      | 84       | 2      |
| 10             | 10             | -16   | 16      | 16      | 70       | 8      |
| 14             | 22             | -20   | 20      | 30      | 78       | 2      |
| 14             | 25             | -20   | 20      | 20      | 82       | 2      |
| 13             | 14             | -30   | 20      | 30      | 72       | 4      |
| 14             | 18             | -40   | 40      | 10      | 76       | 2      |
| 18             | 34             | -72   | 18      | 8       | 108      | 6      |

† We thank Guio, Jockers, Klemm and Yeh [14] for bringing to our attention that this example was omitted in a previous version of this paper

If  $x_{\pm} \in N_{\pm}$  such that  $\flat^{\pm}(x_{\pm}) = \bar{c}_2(Z_{\mp}) \pmod{d(M)}$ , then the image of  $\frac{x_{\pm}}{2}$  in  $N_{\pm}/\langle A_{\pm} \rangle$  has the same parity as  $\frac{\text{gd}(\bar{c}_2(Z_{\pm}))}{2}$ . Since  $k = 1$ , we find

$$\frac{x_+}{2} \cdot \frac{x_-}{2} = \frac{\text{gd}(\bar{c}_2(Z_+))}{2} \frac{\text{gd}(\bar{c}_2(Z_-))}{2} \pmod{2},$$

and (vi) follows from Theorem 3.6. □

Six examples of matchings of rank 2 Fanos with non-trivial intersection  $N_+ \cap N_-$  are listed in [10, Example No 9]. However, Lemma 6.4 and the data in Table 5 allow us to be more decisive.

**Theorem 6.6** *There are precisely 19 pairs of rank 2 Fanos that can be matched to give twisted connected sums with  $H^2(M) \cong \mathbb{Z}$ . In all cases  $H^4(M)$  is torsion-free and  $\mu(M) = 0$ . For each of the pairs there is at least one matching such that  $\pi_2(M) \cong \mathbb{Z}$ .*

In Table 5 we list the following data about the 19 pairs:

- the numbers  $\#_{\pm}$  of the entries in the Mori-Mukai list used,
- the square of the generator  $B$  of the intersection  $N_+ \cap N_-$ ,
- the squares of the ample classes  $A_{\pm} \in N_{\pm}$ ,

- the topological invariants  $b_3(M)$  and  $d(M)$  of the resulting twisted connected sums.

**Proof of Theorem 6.6** In Table 3 we have listed  $h := \log_2 \frac{\Delta}{A^2}$  and  $B^2$  for all ample classes  $A$  in the Picard lattices of rank 2 Fanos, such that  $h$  is not too small. Therefore it is easy to read off all cases where the criterion of Lemma 6.4 is satisfied, and they are the ones listed in Table 5. Indeed, because  $\Delta$  is never greater than  $2A^2$  for any entry in the table, matchings of rank 2 Fano 3-folds are only possible when  $\Delta_+ \Delta_- = A_+^2 A_-^2$ , or equivalently  $h_+ + h_- = 0$ .

For each of the 19 pairs, we can apply Nikulin [34, Theorem 1.12.4] to embed the rank 3 lattice  $W$  from Lemma 6.4 in  $L$ , thus defining a configuration of primitive embeddings  $N_+, N_- \subset L$ . Because  $\Lambda_{\pm} = N_{\pm}$  we can apply Propositions 5.8 and 6.1 to find a compatible matching.

Since we always have  $k = 1$ , any twisted connected sum arising from matchings with non-trivial  $N_+ \cap N_-$  must have  $\text{Tor } \pi_2(M) \cong \text{Tor } H^3(M) = 0$  by Lemma 6.5(iii). For the one pair (row 14 in Table 5) where  $d(M)$  is divisible by 8, so that  $\mu(M)$  could possibly be non-zero, Lemma 6.5(v) shows that  $\mu(M) = 0$  anyway.  $\square$

There are two reasons why Theorem 6.6 does not claim that there is a unique diffeomorphism type of twisted connected sum arising from each of the 19 pairs. The first is that for some of the pairs we can also embed  $W \subset L$  non-primitively, giving a twisted connected sum  $M$  where  $\text{Tor } \pi_2(M) \cong \text{Tor } H^3(M)$  is non-trivial; we do not study this further at this point.

The second reason is that even if we consider only the matchings with  $\pi_2(M) \cong \mathbb{Z}$ , we cannot automatically deduce that the resulting diffeomorphism type is independent of the matching. This is because we do not have a classification theorem for this type of manifold. We hope to return to the problem of classifying such manifolds elsewhere. We expect that to determine the diffeomorphism type one must further compute the square of a generator of  $H^2(M)$ , and also some generalisation of the invariants used by Kreck and Stolz [29] for 7-manifolds with  $\pi_2(M) \cong \mathbb{Z}$  and finite  $H^4(M)$ .

Similarly, the computations presented here are not sufficient to decide whether the two entries in Table 5 that have the same values of  $b_3(M)$  and  $d(M)$  (rows 2 and 8) must be diffeomorphic or even homeomorphic. However, according to Wallis [40, §B.3] these two manifolds are distinguished by the square of the generator of  $H^2(M)$ .

## 7 Skew matching

Having dealt with matchings of rank 2 Fano 3-folds where the configuration  $N_+, N_- \subset L$  is orthogonal (whether  $N_+ \cap N_-$  is trivial or not), we now consider the skew case. To find skew matchings we have to deal with lattice-arithmetical problems, and also to gain some detailed understanding of the deformation theory of the Fanos involved. Because of the case-by-case checking required, we think of this task as ‘handcrafting’.

### 7.1 Arithmetic conditions for skew matching

For Fano 3-folds of Picard rank 2, at least the arithmetic part of the problem of finding matchings with a skew configuration  $N_+, N_- \subset L$  can be dealt with systematically. As pointed out in Remark 5.5,  $N_+ \cap N_-$  must be trivial in this case, and the subgroup  $W_\pm \subset N_\pm$  orthogonal to  $N_\mp$  has rank exactly 1, and is generated by some ample class  $A_\pm$ . Then  $W$  is isometric to  $W_k$  for some integer  $k$ , where the quadratic form on

$$W_k := N_+ \oplus N_-$$

is characterised as follows:  $A_\pm \perp N_\mp$ , and  $H_+ \cdot H_- = k$  for some  $H_\pm \in N_\pm$  such that  $H_\pm, A_\pm$  is a basis for  $N_\pm$ . (The choice of  $H_\pm$  only affects the sign of  $k$ .)

**Lemma 7.1** *Let  $-\Delta_\pm$  be the discriminant of  $N_\pm$ . Then  $W_k$  has signature  $(2, 2)$  if and only if*

$$k^2 A_+^2 A_-^2 < \Delta_+ \Delta_-.$$

**Proof.** Since  $W_k$  contains the positive definite subspace  $\langle A_+, A_- \rangle$  and some negative elements, its signature is  $(2,2)$  if and only if its discriminant  $D$  is positive. Let  $B_\pm \in N_\pm$  be a generator for the orthogonal complement of  $A_\pm$  in  $N_\pm$ , and let  $n_\pm$  be the index of  $\langle A_\pm, B_\pm \rangle \subseteq N_\pm$ . The index  $n_+ n_-$  sublattice of  $W$  spanned by  $A_+, B_+, A_-, B_-$  has intersection form

$$\begin{pmatrix} A_+^2 & 0 & 0 & 0 \\ 0 & B_+^2 & 0 & kn_+ n_- \\ 0 & 0 & A_-^2 & 0 \\ 0 & kn_+ n_- & 0 & B_-^2 \end{pmatrix}$$

and discriminant  $A_+^2 A_-^2 (B_+^2 B_-^2 - k^2 n_+^2 n_-^2) = n_+^2 n_-^2 D$ . Since  $A_\pm^2 B_\pm^2 = -\Delta_\pm n_\pm^2$ , we find

$$D = \Delta_+ \Delta_- - k^2 A_+^2 A_-^2. \quad \square$$

In particular, a necessary condition for finding a matching of rank 2 Fanos  $Y_+, Y_-$  with skew configuration  $N_+, N_- \subset L$  is that there are ample classes  $A_\pm \in N_\pm$  with

$$h_+ + h_- > 0, \tag{7.1}$$

for  $h_\pm := \log_2 \frac{\Delta_\pm}{A_\pm^2}$ . We can readily identify pairs satisfying this necessary condition from Table 3.

In terms of the notation from Sect. 5.1,  $W_\pm$  is generated by  $A_\pm$ , and if  $W_k \subset L$  is primitive then  $\Lambda_\pm$  is the orthogonal complement of  $A_\pm$  in  $W_k$ . Let us make an observation about the form on  $\Lambda_\pm$  that will prove useful.

**Lemma 7.2** *Let  $N_{\pm}$  be a pair of rank 2 lattices of signature  $(1, 1)$ , and discriminant  $-\Delta_{\pm}$ . Given positive classes  $A_{\pm} \in N_{\pm}$ , define  $W_k$  as in Lemma 7.1, with  $k > 0$  such that  $\Delta_+ \Delta_- > k^2 A_+^2 A_-^2$ . Let  $B_{\pm}$  be a generator for the orthogonal complement of  $A_{\pm}$  in  $N_{\pm}$ , and let  $\Lambda_{\pm}$  be the orthogonal complement of  $A_{\mp}$  in  $W_k$ .*

*Suppose  $H \in N_{\pm}$  has the property that*

$$-H^2 B_{\mp}^2 (\Delta_+ \Delta_- - k^2 A_+^2 A_-^2) \geq \Delta_{\pm} \Delta_+ \Delta_- . \quad (7.2)$$

*Then*

$$(v.H)^2 - v^2 H^2 \geq \Delta_{\pm}$$

*for any  $v \in \Lambda_{\pm}$  linearly independent of  $H$ .*

**Proof.** The inequality certainly holds if  $v \in N_{\pm}$ . If  $H$  is a multiple of  $A := A_{\pm}$ , then the inequality follows easily for any  $v \in \Lambda_{\pm}$ .

$\Lambda_{\pm}$  is generated by  $N_{\pm}$  together with  $B := B_{\mp}$ . If  $H$  is linearly independent of  $A$ , then the projection  $B'$  of  $B$  to the orthogonal complement of  $N_{\pm}$  in  $\Lambda_{\pm}$  can be written as

$$B' = B - \frac{B.H}{(A.H)^2 - A^2 H^2} \left( (A.H)A + A^2 H \right),$$

whose square is

$$(B')^2 = B^2 - \frac{(B.H)^2 A^2}{(A.H)^2 - A^2 H^2}.$$

It suffices to show that

$$-(B')^2 H^2 \geq \Delta_{\pm}.$$

If we let  $m$  denote the index of  $\langle A, H \rangle$  in  $N_{\pm}$ , and let  $n$  be the index of  $\langle A_{\mp}, B \rangle$  in  $N_{\mp}$ , then  $(A.H)^2 - A^2 H^2 = m^2 \Delta_{\pm}$ , while  $B.H = \pm kmn$ . Therefore  $(B.H)^2 \Delta_{\mp} = -k^2 m^2 A_{\mp}^2 B^2$ , and

$$-\Delta_+ \Delta_- H^2 (B')^2 = -H^2 B^2 (\Delta_+ \Delta_- - k^2 A_+^2 A_-^2). \quad \square$$

## 7.2 Topology of skew matchings

We now explain how to determine the topology of twisted connected sums obtained from a skew matching of rank 2 blocks. In particular, we identify all pairs of deformation types of rank 2 Fano 3-folds that could possibly be matched to produce twisted connected sums with non-zero generalised Eells–Kuiper invariant.

**Proposition 7.3** *Let  $(Z_{\pm}, \Sigma_{\pm})$  be a pair of building blocks whose polarising lattices  $N_{\pm}$  have rank 2, such that the kernel of  $H^2(Z_{\pm}) \rightarrow H^2(\Sigma_{\pm})$  is generated by  $[\Sigma_{\pm}]$ . Let  $\tau : \Sigma_+ \rightarrow \Sigma_-$  be a matching whose configuration  $N_+, N_- \subset L$  has  $N_+ \oplus N_-$  isometric to  $W_k$  from Lemma 7.1, and let  $M$  be the resulting twisted connected sum.*

*Let  $A_{\pm} \in N_{\pm}$  be a generator for the orthogonal complement of  $N_{\mp}$  in  $N_{\pm}$ . Then*

- (i)  $H^2(M) = 0$ ;
- (ii)  $b_3(M) = b_3(Z_+) + b_3(Z_-) + 21$ ;
- (iii)  $\text{Tor } H^3(M) \cong L/W_k$ ;
- (iv)  $\text{Tor } H^4(M) \cong (\mathbb{Z}/k)^2$ ;
- (v) if  $k = 1$  then  $d(M)$  (which divides 24) is the greatest common divisor of  $\bar{c}_2(Z_+)A_{\pm}$  and  $\bar{c}_2(Z_-)A_{\pm} \in \mathbb{Z}/24$ , for  $\bar{c}_2(Z_{\pm})$  as in (4.3);
- (vi) if  $k = 1$  and  $d(M)$  is divisible by 8 (so that the Eells–Kuiper invariant  $\mu(M)$  takes values in  $\mathbb{Z}/2$ ) then

$$\mu(M) = \frac{\text{gd}(\bar{c}_2(Z_+)) \text{gd}(\bar{c}_2(Z_-))}{4} \in \mathbb{Z}/2,$$

where  $\text{gd}(\bar{c}_2(Z_{\pm})) \in \{2, 4, 6, 8, 12, 24\}$  is the greatest divisor of  $\bar{c}_2(Z_{\pm})$ .

**Proof** (i)–(iii) are immediate from Theorem 2.8. (iv)–(vi) are entirely analogous to the proof of Lemma 6.5. □

**Remark 7.4** We see in Table 3 that  $h < 1$  (i.e.  $\Delta < 2A^2$ ) for any ample class  $A$  on any rank 2 Fano 3-fold. Therefore Lemma 7.1 implies that  $W_k$  can only have signature  $(2, 2)$  for  $k = 1$ . Therefore, whenever we do find a matching of rank 2 Fanos with a skew configuration, the resulting twisted connected sum  $M$  always has  $H^4(M)$  torsion-free by Proposition 7.3(iv).

**Remark 7.5**  $\mu(M)$  is non-zero if and only if both  $\bar{c}_2(Z_{\pm})A_{\pm}$  are divisible by 8 while neither  $\bar{c}_2(Z_{\pm})$  is divisible by 4. Consulting Table 3, we find that the only rank 2 Fano type blocks with such ample classes  $A$  are

- #9 with  $A := G + H$  ( $h = 0.09$ ),
- #17 with  $A := G + H$  ( $h = 0.06$ ),
- #18 with  $A := G + 2H$  ( $h = -0.58$ ),
- #27 with  $A := G + H$  ( $h = 0.09$ ).

Since #18 has  $h$  quite negative, the condition (7.1) implies that the only ways to match rank 2 Fanos with a skew configuration to construct a twisted connected sum  $M$  with  $\mu(M)$  is to use a pair among #9, #17 and #27.

### 7.3 Handcrafting examples with non-zero generalised Eells–Kuiper invariant

Let  $\mathcal{Y}_+, \mathcal{Y}_-$  be a pair of deformation types of rank 2 Fanos. For a skew configuration  $N_+, N_- \subset L$  of their Picard lattices,  $\Lambda_{\pm}$  has rank 3. Because  $\Lambda_{\pm}$  is strictly bigger than  $N_{\pm}$ , the genericity result Proposition 6.1 does not suffice for applying Proposition 5.8 to find a matching with the prescribed configuration. The laborious part of finding

skew matchings is to prove that generic  $\Lambda_{\pm}$ -polarised K3 surfaces still appear as anticanonical K3 divisors in some members of  $\mathcal{Y}_{\pm}$ , despite being more special than the generic,  $N_{\pm}$ -polarised K3 divisors in the family.

In the present paper we go to that effort only in cases that lead to twisted connected sums with  $\mu(M) \neq 0$ . In Theorem 7.8 we show that all skew configurations of the Picard lattices of rank 2 Fanos identified in Remark 7.5 are in fact realised by a matching. The genericity results needed to find those matchings are provided by Lemma 7.7. To prove that we use

**Lemma 7.6** *Let  $\Sigma$  be a K3 surface, and  $H \in \text{Pic } \Sigma$  a primitive nef class with  $H^2 \geq 4$ . Then  $H$  is a very ample class (i.e. the linear system  $|H|$  defines an embedding  $\Sigma \hookrightarrow \mathbb{P}^{\frac{H^2}{2}+1}$ ) unless there is some  $v \in \text{Pic } \Sigma$  such that*

- (i)  $d = 2$  and  $v^2 = 0$ , or
- (ii)  $d = 0$  and  $v^2 = -2$ ,

for  $d := v.H$ .

**Proof** According to [10, Lemma 7.15] (based on [35, Chapter 3]), the arithmetic conditions rule out all possible ways that  $H$  can fail to be very ample. (i) rules out the existence of any classes with  $d = 1$  and  $v^2 = 0$ , and hence  $|H|$  being monogonal. Therefore  $|H|$  has no fixed part, and defines a morphism.  $H^2 \neq 2$ ,  $H$  primitive and (i) rule out the three different ways that  $|H|$  could be hyperelliptic, so it defines a birational morphism onto its image. (ii) rules out any  $(-2)$ -curves being contracted, so  $|H|$  defines an isomorphism.  $\square$

Using Lemma 7.6 and various results about curves on K3 surfaces, it is for many families  $\mathcal{Y}$  of Fano 3-folds possible to obtain conditions on a lattice  $\Lambda$  (containing the Picard lattice  $N$  of  $\mathcal{Y}$ ) that ensure that any K3 with Picard lattice isometric to  $\Lambda$  can be embedded as an anticanonical divisor in some element of  $\mathcal{Y}$ . The conditions are to exclude the existence in  $\Lambda$  of elements  $v$  with certain combinations of  $v^2$  and inner products of  $v$  with elements of  $N$ . Once these ‘handcrafting’ conditions have been proved for some collection of blocks, then it would be feasible to get a computer to generate candidate configurations involving those blocks, and to verify whether the handcrafting conditions hold.

However, for the purposes of checking a few examples by hand, it is expedient to instead organise the argument by checking that in the examples we are concerned with, the sort of inequality produced by Lemma 7.2 rules out the presence in  $\Lambda$  of vectors  $v$  with the relevant properties.

**Lemma 7.7** *Let  $\mathcal{Y}$  be the deformation type of #27, #9 or #17 in the Mori-Mukai list of rank 2 Fanos. Let  $N \subset L$  be a primitive embedding of its Picard lattice, and let  $\Lambda \subset L$  be a primitive lattice containing  $N$ . Let  $G, H$  be the basis of  $N$  described in Sect. 4.4, and  $\text{Amp}_{\mathcal{Y}} \subset N(\mathbb{R})$  the interior of the cone spanned by  $G$  and  $H$ . Suppose that*

$$(v.H)^2 - H^2 v^2 \geq \Delta \tag{7.3}$$



for all  $v \in \Lambda$  that are linearly independent of  $H$ , where  $-\Delta$  is the discriminant of  $N$ . Then  $\mathcal{Y}$  is  $(\Lambda, \text{Amp}_{\mathcal{Y}})$ -generic.

**Proof** It is enough to prove that any K3 surface  $\Sigma$  with Picard lattice isometric to precisely  $\Lambda$  has a marking  $h : H^2(\Sigma) \rightarrow L$  mapping  $\text{Pic}(\Sigma) \rightarrow \Lambda$  and an embedding  $\Sigma \hookrightarrow Y$  of  $\Sigma$  as an anticanonical divisor in some  $Y \in \mathcal{Y}$ , such that the image of the ample cone of  $Y$  in  $H^2(\Sigma)$  is  $h^{-1}(\text{Amp}_{\mathcal{Y}})$ .

#27 This is the blow-up of  $\mathbb{P}^3$  in a twisted cubic. In this case the Picard lattice is  $\begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$ , so  $H$  is a class of degree 4. (7.3) means that if  $v \in \Lambda$  is linearly independent of  $H$  and  $d := v.H$  then

$$d^2 - 4v^2 \geq 17, \tag{7.4}$$

so neither of the cases in Lemma 7.6 can occur. Since  $H$  is not orthogonal to any  $(-2)$ -class in  $\Lambda$ , we can choose a marking of  $\Sigma$  that maps  $\text{Pic}(\Sigma)$  to  $\Lambda$ , such that  $H$  is the image of a nef class. Lemma 7.6 then implies that  $H$  corresponds to a very ample class, embedding  $\Sigma$  as a quartic in  $\mathbb{P}^3$ .

The class  $2H - G$  has square  $-2$ , so by a standard application of the Riemann–Roch theorem for surfaces either  $2H - G$  or  $-(2H - G)$  is effective [35, Corollary 3.7(i)]. Since  $H.(2H - G) = 3$  is positive, it must be  $2H - G$  that is effective. Since any irreducible effective class  $v$  has  $v^2 \geq -2$ , (7.4) implies that in fact there are no such classes of degree  $d \leq 2$ . Therefore  $2H - G$  is irreducible. A further well-known application of Riemann–Roch (see discussion after [35, Corollary 3.7]) implies that  $2H - G$  is represented by a smooth rational curve  $\Gamma$ . Its image in  $\mathbb{P}^3$  has degree 3, so must be a twisted cubic. By blowing up  $\Gamma$  we obtain a Fano  $Y \in \mathcal{Y}$ .

#9 This is the family of blow-ups of  $\mathbb{P}^3$  in smooth curves of degree 7 and genus 5. The Picard lattice is  $\begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$  in this case too, so we have already proved that  $H$  embeds  $\Sigma$  as a quartic K3 in  $\mathbb{P}^3$ , and that  $2H - G$  is irreducible and represented by a twisted cubic  $\Gamma$ .

We now want to prove that  $3H - G = H + [\Gamma]$  can be represented by a smooth curve  $C$ . Because  $3H - G$  is big and nef, by [35, Theorem 3.8(d) & 3.15] the only way it can fail to be basepoint-free is if it is monogonal, i.e. if  $3H - G = aE + F$  where  $E^2 = 0$  and  $F$  is the fixed part. Since  $H$  is very ample certainly  $H + [\Gamma]$  cannot have any non-trivial fixed part other than  $\Gamma$ , and since  $H^2 \neq 0$  we cannot have  $F = \Gamma$  either.

Hence  $|3H - G|$  is basepoint-free, and a general member  $C$  is smooth by Bertini’s theorem. Now  $C$  has degree 7 and genus 5, so by Blanc–Lamy [4, Theorem 1.1] blowing up  $C$  defines a Fano in the family  $\mathcal{Y}$  unless there is a 4-secant line  $\ell$  to  $C$ .

It thus remains only to show that no such  $\ell$  exists. If it did, then restricting the pencil of hyperplanes containing  $\ell$  to  $C$  defines a pencil of degree 3, a “complete  $g_3^1$ ” in the terminology of Brill–Noether theory. The curve  $C$  therefore has Clifford index 1. By the main theorem of Green–Lazarsfeld [18] this implies the existence of a line bundle  $L \in \Lambda$  with  $L^2 = 0$  and  $L.C = 3$ . Now (7.4) implies  $H.L \geq 5$ . But then the span of  $H + L$  and  $3H - G$  is positive-definite, which is

impossible.

#17 This is the family of blow-ups of smooth quadrics  $Q \subset \mathbb{P}^4$  in smooth elliptic curves of degree 5. In this case the Picard lattice is  $\begin{pmatrix} 4 & 7 \\ 7 & 6 \end{pmatrix}$ , so  $H$  is a degree 6 class. (7.3) means that

$$d^2 - 6v^2 \geq 25 \quad (7.5)$$

for any  $v \in \Lambda$  not a multiple of  $H$ . In particular Lemma 7.6 implies that  $H$  is very ample, and by Riemann–Roch the embedded image of  $\Sigma$  in  $\mathbb{P}^4$  is the intersection of a quadric  $Q$  and a cubic. The quadric must be smooth, for if  $Q$  is singular then it contains planes. The section of  $\Sigma$  by such a plane would be a plane cubic, defining a class with  $v^2 = 0$  and  $d = 3$ , which is ruled out by (7.5) (see Fukuoka [17, Lemma 2.4]).

Now consider the class  $E := 2H - G$ , which has degree 5 and  $E^2 = 0$ . (7.5) rules out the existence of irreducible classes in  $\text{Pic } \Sigma$  with  $d \leq 3$ , so  $E$  is irreducible. In particular  $E$  does not have any  $(-2)$ -curve components, so  $E$  is nef. Therefore [35, Theorem 3.8(b)] implies that  $|E|$  is basepoint-free. A generic  $C \in |E|$  is therefore a smooth elliptic curve of degree 5.

If  $C$  had a 4-secant line  $\ell$ , then the pencil of planes through  $\ell$  would define a morphism from  $C$  to  $\mathbb{P}^1$ , which is impossible. Therefore by Blanc–Lamy [4, Theorem 1.1], blowing up  $Q$  in  $C$  yields a member  $Y \in \mathcal{Y}$ .

□

For the three families of blocks under consideration, the sufficient conditions provided by Lemmas 7.2 and 7.7 turn out to be good enough to prove the existence of skew matching between any pair of families. Moreover, an ad hoc argument rules out any configurations beyond the ones realised.

**Theorem 7.8** *Let  $\mathcal{Y}_+$  and  $\mathcal{Y}_-$  be a pair of deformation types of rank 2 Fanos among #9, #17 and #27 on the Mori–Mukai list. Let  $N_{\pm}$  be their Picard lattices, and let  $A_{\pm} \in N_{\pm}$  be the ample class listed in Table 3. Define  $W_1$  as in Lemma 7.1, embed  $W_1 \subset L$  primitively, and consider the resulting configuration  $N_+, N_- \subset L$ .*

- (i) *There is a matching of some elements of  $\mathcal{Y}_{\pm}$  with that configuration.*
- (ii) *This is the only non-perpendicular configuration of  $N_+$  and  $N_-$  for which a matching exists.*

**Proof** (i) For this configuration, the lattice  $\Lambda_{\pm} \subset L$  defined in Sect. 5.1 is the orthogonal complement of  $A_{\mp}$  in  $W_1$ . Let  $G_{\pm}, H_{\pm} \in N_{\pm}$  be the basis vectors described in Sect. 4.4. Looking up the values of  $H_{\pm}^2, A_{\pm}^2, B_{\pm}^2$  and  $\Delta_{\pm}$  in Table 3, Lemma 7.2 implies that

$$(v \cdot H_{\pm})^2 - H_{\pm}^2 v^2 \geq \Delta_{\pm} \quad (7.6)$$

for any  $v \in \Lambda_{\pm}$  linearly independent of  $H_{\pm}$ . Therefore Lemma 7.7 implies that  $\mathcal{Y}_{\pm}$  is  $(\Lambda_{\pm}, \text{Amp}_{\mathcal{Y}_{\pm}})$ -generic, and the desired matching exists by Proposition 5.8.

- (ii) By Lemma 6.4 there can be no matchings of these types with a configuration such that  $N_+ \cap N_-$  is non-trivial, so a non-perpendicular configuration must be skew. We have explained that for a skew configuration to satisfy the conditions of Remark 5.4,  $N_+ \oplus N_-$  must be isometric to  $W_1$ . Thus it only remains to rule out configurations where  $W_1$  is embedded non-primitively in  $L$ . In Lemma 7.1 we computed that the discriminant of  $W_1$  is

$$D = \Delta_+ \Delta_- - A_+^2 A_-^2.$$

When  $\mathcal{Y}_\pm$  are both among #9 and #27,  $D = 33$  is square-free, so  $W_1$  does not have any integral overlattice. When  $\mathcal{Y}_+$  is one of #9 and #27 while  $\mathcal{Y}_-$  is #17 we get  $D = 41$ , which is also square-free. Hence there are no non-primitive embeddings  $W_1 \subset L$  in these cases.

However, when  $\mathcal{Y}_\pm$  are both #17 we get  $D = 49$ . In the basis  $G_+, A_+, G_-, A_-$ , the quadratic form on  $W_1$  can be written as

$$\begin{pmatrix} 4 & 11 & 1 & 0 \\ 11 & 24 & 0 & 0 \\ 1 & 0 & 4 & 11 \\ 0 & 0 & 11 & 24 \end{pmatrix}.$$

This matrix has rank 3 over  $\mathbb{Z}/7$ , so the discriminant group must be  $\mathbb{Z}/49$  rather than  $(\mathbb{Z}/7)^2$ , and  $W_1$  has a index 7 overlattice  $\tilde{W}$  (which is in fact unimodular). Indeed,  $K := G_+ + A_+ - G_- - A_-$  has  $K^2 = 98$  and its product with any element of  $W$  is divisible by 7. Therefore we can define  $\tilde{W}$  by adjoining  $\frac{1}{7}K$  to  $W$ .

The only possible way to embed  $W_1 \subset L$  non-primitively is via a primitive embedding  $\tilde{W} \subset L$ . We now check that there are no matchings with this configuration. Note that  $\Lambda_\pm$  is spanned by  $G_\pm, H_\pm$  and  $\tilde{B}_\pm := \pm \frac{24}{7}K + 5A_\mp$ . In that basis, the quadratic form on  $\Lambda_\pm$  is represented by

$$\begin{pmatrix} 4 & 7 & 48 \\ 7 & 6 & 72 \\ 48 & 72 & 552 \end{pmatrix}.$$

We find that (7.5) fails for some  $v \in \Lambda_+$  such that  $v^2 = 0$  and  $d = 3$ , e.g.  $v = E' := -9G_+ - H_+ + \tilde{B}_+$ .

Now suppose that  $\Sigma$  is an anticanonical K3 divisor in some rank 2 Fano of type #27, with  $\text{Pic } \Sigma$  isometric to this  $\Lambda_+$ . Then  $H_+$  embeds  $\Sigma$  in a smooth quadric  $Q \subset \mathbb{P}^4$ . The argument from the proof of Lemma 7.7 for the case #17 shows that because  $(E')^2 = 0$  and  $E' \cdot H_+ = 3$ , the linear system  $|E'|$  on  $\Sigma$  is basepoint-free, and represented by a smooth elliptic curve  $C$ . By Riemann–Roch,  $C$  is a plane cubic. Because  $Q$  contains  $C$  it must also contain the plane of  $C$ , contradicting that  $Q$  is non-singular.

□

## 7.4 Proof of main theorem

To prove Theorem 1.2 it now remains only to put together the pieces provided above.

Theorem 7.8 provides exactly one configuration with a matching for each pair of rank 2 Fano types among #9, #17 and #27 (referred to as (d), (e) and (f) in Table 1 in the introduction). Each of those six pairs produces a closed 7-manifold  $M$  with holonomy  $G_2$  by Construction 2.6, whose topology can be computed from Proposition 7.3 and the data in Table 3. These  $M$  are 2-connected with  $H^4(M)$  torsion-free, and  $b_3(M)$  and  $d(M)$  as listed in Table 1. By design  $\mu(M) = 1$ , while all other matchings of rank 2 Fanos give  $\mu = 0$  by Corollary 3.7, Theorem 6.6 and Remark 7.5.

The six matchings realise four different pairs  $(b_3, d)$ , and hence four different diffeomorphism types by Theorem 2.2(ii). We can then consult Table 4 to see that for two of these four smooth manifolds, there exist perpendicular twisted connected sums  $M'$  with the same  $(b_3, d)$ . Then  $M$  and  $M'$  are homeomorphic by Theorem 2.2(i). However,  $\mu(M') = 0$  by Corollary 3.7, so  $M$  and  $M'$  are not diffeomorphic.

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