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Non-uniqueness for an energy-critical heat equation on $\mathbb{R}^2$

Slim Ibrahim[1] · Hiroaki Kikuchi[2] · Kenji Nakanishi[3] · Juncheng Wei[4]

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Abstract
We construct a singular solution of a stationary nonlinear Schrödinger equation on $\mathbb{R}^2$ with square-exponential nonlinearity having linear behavior around zero. In view of Trudinger-Moser inequality, this type of nonlinearity has an energy-critical growth. We use this singular solution to prove non-uniqueness of mild solutions for the Cauchy problem of the corresponding semilinear heat equation. The proof relies on explicit computation showing a regularizing effect of the heat equation in an appropriate functional space.

1 Introduction

In this paper, we consider the Cauchy problem for the following semilinear heat equation

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✉ Slim Ibrahim
ibrahims@uvic.ca

Hiroaki Kikuchi
hiroaki@tsuda.ac.jp

Kenji Nakanishi
kenji@kurims.kyoto-u.ac.jp

Juncheng Wei
jcwei@math.ubc.ca

1 Department of Mathematics and Statistics, University of Victoria, 3800 Finnerty Road, Victoria, BC V8P 5C2, Canada

2 Department of Mathematics, Tsuda University, 2-1-1 Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan

3 Research Institute for Mathematical Sciences, Kyoto University, Oiwake Kita-Shirakawa, Sakyo, Kyoto 606-8502, Japan

4 Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada

$$\begin{cases} \dot{u} - \Delta u = f(u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u(t,x) : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$, $d \geq 2$ is the unknown function, f is the nonlinearity and $u_0 \in L^q(\mathbb{R}^d)$ with $1 \leq q \leq \infty$ is the given initial data.

It is well known that when f is $C^1(\mathbb{R},\mathbb{R})$, the Cauchy problem (1.1) possesses a unique classical solution if the initial data $u_0 \in L^\infty(\mathbb{R}^d)$. For unbounded initial data, this equation has been studied intensively since the pioneering work of Weissler [18]. For instance, in the pure power case i.e. $f(u) = |u|^{p-1}u$ $(p > 1)$, equation (1.1) becomes scale invariant, that is, if $u(t,x)$ satisfies (1.1), then so does

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

for any $\lambda > 0$. Moreover, if one defines the index

$$q_c = \frac{d(p-1)}{2},$$

then the Lebesgue space $L^{q_c}(\mathbb{R}^d)$ becomes invariant and we have $\|u_\lambda\|_{L^{q_c}} = \|u\|_{L^{q_c}}$ for all $\lambda > 0$. The *critical* exponent q_c then plays an important role for the well-posedness of the Cauchy problem. Indeed, first recall Weissler's result [18] concerning the subcritical case $q > q_c$ or the critical case $q = q_c > 1$. Weissler [18] proved that for any $u_0 \in L^q(\mathbb{R}^d)$, there exists a local time $T = T(u_0) > 0$ and a solution $u \in C([0,T); L^q(\mathbb{R}^d)) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty(\mathbb{R}^d))$ to (1.1). After that, Brezis and Cazenave [1] proved the *unconditional* uniqueness of Weissler's solutions i.e. solution is unique in $C([0,T); L^q(\mathbb{R}^d))$ when the subcritical case $q > q_c$, $q \geq p$ or the critical case $q = q_c > p$. In the *supercritical* case $q < q_c$, Weissler [18], and Brezis and Cazenave [1] indicated that, for a specific initial data, there is no local solution in any reasonable weak sense. Moreover, Haraux and Weissler [7] proved non-uniqueness of the trivial solution in $C([0,T); L^q(\mathbb{R}^d)) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty(\mathbb{R}^d))$ when $1 + 2/d < p < (d+2)/(d-2)$.

In the critical case $q = q_c$ and $d \geq 3$, when $q_c > p$, Weissler [18] proved the existence of solutions to (1.1) and Brezis and Cazenave [1] obtained the *unconditional* uniqueness of the solutions. In the double critical case of $q = q_c = p$ $(= d/(d-2))$, Weissler [19] proved the conditional well-posedness (uniqueness in a subspace of $C([0,T); L^q(\mathbb{R}^d))$). In the case where the underlying space is the ball of $\mathbb{R}^d$ with Dirichlet boundary condition, Ni and Sacks [15] showed that the unconditional uniqueness fails, while Terraneo [17] and Matos and Terraneo [12] extended this result to the entire space $\mathbb{R}^d$ $(d \geq 3)$.

We note that the critical exponent q_c is also important for the blow-up problem (1.1). Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $T(u_0)$ be the maximal existence of the time of the classical solution u. It is known that if $T(u_0) < \infty$, the solution u satisfies $\lim_{t\to T}\|u(t)\|_{L^\infty} = \infty$ and we say that the solution u blows up in finite time and $T(u_0)$ is called the blow-up time of u. In particular, the *critical L^{q_c} norm blow-up problem* has attracted attention for a long time. Namely, it is a question whether the solution satisfies

$$\sup_{t\in(0,T)} \|u\|_{L^{q_c}} = \infty$$

or not when $T(u_0) < \infty$. Concerning this problem, Giga and Kohn [6] showed that if u is a positive radially decreasing blow-up solution to (1.1), Then, for any neighborhood N of 0 in $\mathbb{R}^d$, $\lim_{t\to T} \|u(t)\|_{L^{q_c,\infty}(N)} = \infty$, where

$$\|u\|_{L^{q_c,\infty}(N)} = \sup_{\sigma>0} \sigma |\{x \in N \mid |u(x)| > \sigma\}|^{\frac{1}{q_c}}$$

and $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^d$. Since $\|u\|_{L^{q_c,\infty}(N)} \le \|u\|_{L^{q_c}}$, it follows that $\lim_{t\to T} \|u(t)\|_{L^{q_c}} = \infty$. Recently, Mizoguchi and Souplet [14] gave light on this problem and showed that if u is a type I blow-up solution of (1.1), that is, $\limsup_{t\to T}(T-t)^{1/(p-1)}\|u(t)\|_{L^\infty} < \infty$, then we have $\lim_{t\to T} \|u(t)\|_{L^{q_c}} = \infty$.

Now, let us pay our attention to the two dimensional case. When we consider the two space dimension, we see that any exponent $1 < p < \infty$ is subcritical, and thanks to the result of Weissler [18], we have the local well-posedness of the Cauchy problem in $L^q(\mathbb{R}^2)$ $(1 < p \le q < \infty)$. However, for exponential type nonlinearities, like $f(u) = \pm u(e^{u^2} - 1)$, the result of Weissler [18] is not applicable for any Lebesgue space $L^q(\mathbb{R}^2)(1 < q < \infty)$. On the other hand, we can show the local well-posedness for $u_0 \in L^\infty(\mathbb{R}^2)$, as we mentioned first. However, $L^\infty(\mathbb{R}^2)$ is the subcritical space for the problem (1.1) with $d = 2$ and exponential type nonlinearities. Therefore, one can wonder if there is any notion of criticality in two space dimensions. In this regard, Ibrahim, Jrad, Majdoub and Saanouni [8] have considered the following problem in two space dimensions,

$$\begin{cases} \dot{u} - \Delta u = f_0(u) := \pm u(e^{u^2} - 1) & \text{in } (0,\infty)\times\mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

The nonlinearity $f_0(u)$ has an energy-critical growth in view of Trudinger-Moser inequality. In [8], the authors proved the local existence and uniqueness in $C([0,T], H^1(\mathbb{R}^2))$ of the solution to (1.2) with the initial data $u_0 \in H^1(\mathbb{R}^2)$.

On the other hand, it is expected that the problem (1.2) for the heat equation can be solved in spaces which are defined by an integrability of functions such as the Orlicz space as an extension of the class of Lebesgue spaces. Ruf and Terraneo [16] showed the local existence of a solution to (1.2) for small initial data in the Orlicz space $\exp L^2$ defined by

$$\exp L^2(\mathbb{R}^2) := \left\{ u \in L^1_{loc}(\mathbb{R}^2) : \text{ such that } \int_{\mathbb{R}^2} \exp(u^2/\lambda^2) - 1\, dx < \infty, \right.$$
$$\left. \text{for some } \lambda > 0 \right\}$$

endowed with Luxemburg norm

$$\|u\|_{\exp L^2} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^2} \exp(u^2/\lambda^2) - 1\, dx \le 1 \right\}.$$

Subsequently, Ioku [9] has shown that these local solutions are indeed global-in time, and the behavior of $f_0(u) \sim u^m$ near $u \sim 0$ with $m \geq 3$ was important in his result. Later on, Ioku, Ruf and Terraneo [10] proved that there is no local solution do not exist for the specific initial data

$$u_*(r) = a(-\log r)^{\frac{1}{2}} \text{ for } r \leq 1, \quad a \gg 1, \text{ and } u_*(r) = 0, \text{ elsewhere}$$

that belongs to the Orlicz space, while they showed the local well-posedness (local existence and uniqueness) for initial data in the subclass of the Orlicz space

$$\exp L_0^2(\mathbb{R}^2) := \left\{ u \in L^1_{loc}(\mathbb{R}^2) : \text{ such that } \int_{\mathbb{R}^2} \exp(\alpha u^2) - 1 \, dx < \infty \text{ for every } \alpha > 0 \right\}$$

The aim of this paper is to construct an explicit initial data, with neither small nor too large norm in Orlicz space, for which two corresponding distinct solutions exist. The idea is to use the concept of *singular solutions* as in Ni and Sacks [15]. Before stating our result, let us recall the strategy of the proof of [15]. Ni and Sacks first constructed a singular static solution ϕ_* to (1.1) in the unit ball. Namely, ϕ_* satisfies the following:

$$\begin{cases} -\Delta \phi = f(\phi) & \text{in } B_1, \\ \phi = 0 \quad \text{on } \partial B_1, & \lim_{x \to 0} \phi(x) = \infty, \end{cases} \tag{1.3}$$

where B_1 is the unit ball in $\mathbb{R}^d$ $(d \geq 3)$ and $f(\phi) = |\phi|^{p-1}\phi$. Then, they showed that there exists a regular solution u_{R} to the Dirichlet problem corresponding to (1.1) with $u(0, x) = \phi_*$. Setting $u_{\mathrm{S}} = \phi_*$, we see that both of u_{S} and u_{R} are solutions to (1.1), but $u_{\mathrm{S}} \neq u_{\mathrm{R}}$ because for any $t > 0$, $u_{\mathrm{R}}(t, \cdot) \in L^\infty(B_1)$ while $u_{\mathrm{S}} \notin L^\infty(B_1)$ (the subscripts S and R indicating 'singular' and 'regular' solutions). For the entire space $\mathbb{R}^d$, Terraneo [17] constructed a solution $\phi \in C^2(\mathbb{R}^d \backslash \{-x_0, x_0\})$ to the equation

$$\begin{cases} -\Delta \phi = f(\phi) & \text{in } \mathbb{R}^d, \\ \lim_{|x| \to \infty} \phi(x) = 0 \end{cases} \tag{1.4}$$

such that $\limsup_{x \to x_0} \phi(x) = \infty$ and $\liminf_{x \to -x_0} \phi(x) = -\infty$, where $d \geq 3$, $x_0 = (1, \ldots, 1)$ and $f(\phi) = |\phi|^{p-1}\phi$. However, we cannot apply the method of [15] nor of [17] to two dimensional entire space $\mathbb{R}^2$ for $f_0(u) = (e^{u^2} - 1)u$ directly. Actually, there is no positive solution to the equation (1.4) with $f(u) = (e^{u^2} - 1)u$ (see the proof of Theorem 2.1 below). For this reason, we consider

$$\begin{cases} \dot{u} - \Delta u = f_m(u) & \text{in } (0, \infty) \times \mathbb{R}^2, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{1.5}$$

where our nonlinearity f_m, which depends on $m > 0$, satisfies

$$\lim_{u \to \infty} \frac{f_m(u)}{(e^{u^2} - 1)u} = 1, \qquad \lim_{u \to 0} \frac{f_m(u)}{mu} = -1. \tag{1.6}$$

See (2.17) below for the precise form of f_m. Here, we would like to stress that the essential characterization of the asymptotic form of our nonlinearity f_m at infinity and 0 is just given by (1.6). Let X be the Fréchet space defined as the intersection of the Lebesgue Banach spaces

$$X := \bigcap_{1 \le p < \infty} L^p(\mathbb{R}^2),$$

endowed with the metric

$$\mathrm{dist}(u, v) = \sum_{p=1}^{\infty} 2^{-p} \frac{\|u - v\|_{L^p}}{\|u - v\|_{L^p} + 1}.$$

Recall that the continuity of any map defined on X is equivalent to its continuity on all L^p spaces with $1 \le p < \infty$.

Definition 1 (*Mild solution*) We call u a *mild solution* to (1.5) if the corresponding integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f_m(u(s)) ds$$

holds in $C([0, T), X)$, where $e^{t\Delta}$ denotes the heat semigroup on $\mathbb{R}^2$.

Then, our main result is the following:

Theorem 1.1 *There exist a positive mass $m_* > 0$, a nonlinearity f_{m_*} satisfying (1.6), an initial condition $\varphi_* \in X$ and a time $T = T(\varphi_*) > 0$ such that the Cauchy problem (1.5) with $u_0 = \varphi_*$ has two distinct mild solutions $u_S, u_R \in C([0, T), X)$.*

Remark 1 (i) To prove Theorem 1.1, we construct a singular stationary solution φ_* to (1.5). Here, by a *singular stationary solution*, we mean a time independent function which satisfies the equation (1.5) in the distribution sense on the whole domain and diverges at some points. The result seems to be of independent interest.

(ii) We essentially use the condition $\lim_{u \to 0} f_m(u)/mu = -1$ only for the decay of a singular stationary soliton φ_* to (1.5), that is, $\lim_{|x| \to \infty} \varphi_*(x) = 0$. The other argument in our proof works even without the condition. It is a challenging problem to study whether non-uniqueness still hold without the condition or not.

Remark 2 After completing this paper, it has been brought to our attention that Ioku, Ruf and Terraneo [11] obtained a similar non-uniqueness result for the following two dimensional heat equation on a ball with Dirichlet boundary condition:

$$\begin{cases} \dot{u} - \Delta u = f(u) & \text{in } (0, \infty) \times B_\rho \\ u = 0 & \text{on } (0, \infty) \times \partial B_\rho \end{cases}, \tag{1.7}$$

for some $\rho > 0$, and a nonlinearity f satisfying

$$f(u) := \begin{cases} \frac{1}{u^3} e^{u^2} & \text{if } |u| > \beta, \\ \alpha u^2 & \text{if } |u| \le \beta, \end{cases}$$

with $\alpha = e^{5/2}(5/2)^{-5/2}$ and $\beta = (5/2)^{1/2}$. First, the authors constructed a stationary singular solution φ_* of (1.7). Then, they solved the Cauchy problem (1.7) with the initial data $u_0(x) = \mu\varphi_*$ for $\mu > 0$. They proved the following interesting result:

(i) If $\mu < 1$, the equation (1.7) has a unique regular local-in-time solution
(ii) If $\mu = 1$, the equation (1.7) has not only a singular solution $u_S (= \varphi_*)$ but also a regular solution u_R with the same initial data,
(iii) If $\mu > 1$, the equation has no non-negative solution on any positive time interval.

See [11, Theorem 2.1] for a more precise statement of their trichotomy result. From (ii), we see that non-uniqueness of a solution holds since there exists two distinct solutions u_S and u_R with the same initial data. Thus, the strategy of the proof for the non-uniqueness is same as that of our result. However, in addition to the fact that the settings are different, the ways of constructing the singular and regular solutions are not similar either. For example, their proof of the constructing of the singular stationary solution φ_* depended on the fact that the following equation

$$-U'' - \frac{1}{r}U' = \frac{1}{U^3} e^{U^2}, \qquad 0 < r < 1$$

has an explicit solution $U = \sqrt{-2\log r}$. On the other hand, the singular solution we constructed is not explicit. We construct the singular solution in a ball by using the contraction mapping theorem and extend it to the entire space by the shooting method. Moreover, for the construction of the regular solution, they employed the sub-super-solution method, while we employ the contraction mapping theorem.

Remark 3 The problem of uniqueness of solutions for PDEs is a classical and old issue that can be tricky sometimes. It has caught a special attention in the last few years. Among many others, one can refer to the pioneer works of De Lellis and Székelyhidi [4] showing non-uniqueness of very weak solutions to the Euler problem. Their proof relies on convex integration techniques, which more recently, have been upgraded to show non-uniqueness of weak solutions of the Navier-Stokes system thanks to the work of Buckmaster and Vicol [2]. Davila, Del Pino and Wei [3] constructed non-unique weak solutions for the two-dimensional harmonic map, by attaching reverse bubbling. In [5], Germain, Ghoul and Miura investigated the genericity of the non-unique solutions of the supercritical heat flow map.

This paper is organized as follows. In Sect. 2, we construct a singular static soliton φ_* to (1.5). To this end, we first prove the existence of a singular soliton ϕ_* to (1.3) with $f(\phi) = e^{\phi^2}(\phi - 1)$ in the ball in $\mathbb{R}^2$, following Merle and Peletier [13]. Then, we seek the singular stationary soliton φ_* to (1.5) by the shooting method. In Sect. 3, we shall show the existence of a regular solution to (1.5) with $u|_{t=0} = \varphi_*$ by the heat iteration and give a proof of Theorem 1.1. In Appendix A, we show a monotonicity property of solution to the linear heat equation.

Notation

Throughout the paper, C denotes a positive constant, that does not depend on the parameters, unless otherwise noted and may change from line to line.

2 Construction of singular soliton

2.1 Singular stationary solution on some disk

In this section, we construct a singular solution to the following elliptic equation on a disk

$$\begin{cases} -\Delta u = u(e^{u^2} - 1) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{2.1}$$

where $R > 0$. More precisely, we shall show the following:

Theorem 2.1 *There exist $R_\infty > 0$ and a singular solution $u_\infty \in C^\infty(B_{R_\infty} \backslash \{0\})$ to (2.1) with $R = R_\infty$ satisfying*

$$u_\infty(x) = (-2\log|x| - 2\log(-\log|x|) - 2\log 2)^{\frac{1}{2}} + O((-\log|x|)^{-\frac{3}{2}} \log(-\log|x|)) \quad \text{as } x \to 0. \tag{2.2}$$

To prove Theorem 2.1, we look for radially symmetric solutions to (2.1). We first pay our attention for $0 < r = |x| \ll 1$ and employ the following Emden-Fowler transformation:

$$y(\rho) = u(x), \qquad \rho = 2|\log r|.$$

Then, we see that the equation in (2.1) is equivalent to the following:

$$-\frac{d^2 y}{d\rho^2} = \frac{e^{-\rho}}{4} y(e^{y^2} - 1). \tag{2.3}$$

We shall show the following:

Proposition 2.2 *There exists $\Lambda_\infty > 0$ and a solution $y_\infty(\rho)$ to (2.3) for $\rho \in [\Lambda_\infty, \infty)$ satisfying*

$$y_\infty(\rho) = (\rho - 2\log\rho)^{\frac{1}{2}} + O(\rho^{-\frac{3}{2}} \log\rho) \qquad \text{as } \rho \to \infty.$$

In order to prove Proposition 2.2, we write

$$y(\rho) = \phi(\rho) + \eta(\rho), \qquad \phi(\rho) := (\rho - 2\log\rho)^{\frac{1}{2}}, \tag{2.4}$$

and look for the ODE satisfied by η. We have

$$\frac{d^2y}{d\rho^2} = -\frac{\rho^{-\frac{3}{2}}}{4} - \frac{1}{4}(\phi^{-3} - \rho^{-\frac{3}{2}}) + \phi^{-3}\left\{\frac{1}{\rho} - \frac{1}{\rho^2}\right\} + \phi^{-1}\rho^{-2} + \frac{d^2\eta}{d\rho^2}. \tag{2.5}$$

Now, since

$$y = \rho^{\frac{1}{2}}\left(1 - 2\frac{\log\rho}{\rho}\right)^{\frac{1}{2}} + \eta = \rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \eta,$$

and

$$e^{y^2-\rho} = \rho^{-2}e^{y^2-\phi^2} = \rho^{-2}\left\{1 + (y^2 - \phi^2) + \left(e^{y^2-\phi^2} - 1 - (y^2 - \phi^2)\right)\right\},$$

we obtain

$$\begin{aligned}
ye^{y^2-\rho} &= \rho^{-2}\left\{\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \eta\right\} \\
&\quad \times \left\{1 + (y^2 - \phi^2) + \left(e^{y^2-\phi^2} - 1 - (y^2 - \phi^2)\right)\right\} \\
&= \rho^{-2}\left\{\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \eta\right\} \\
&\quad + \rho^{-2}(y^2 - \phi^2)\left\{\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \eta\right\} \\
&\quad + \rho^{-2}\left(e^{y^2-\phi^2} - 1 - (y^2 - \phi^2)\right)\left\{\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \eta\right\} \\
&= \rho^{-\frac{3}{2}} + \rho^{-\frac{3}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \rho^{-2}\eta \\
&\quad + \rho^{-\frac{3}{2}}(y^2 - \phi^2) + \rho^{-\frac{3}{2}}(y^2 - \phi^2)\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \rho^{-2}(y^2 - \phi^2)\eta \\
&\quad + \rho^{-2}\left(e^{y^2-\phi^2} - 1 - (y^2 - \phi^2)\right)\left\{\rho^{\frac{1}{2}}(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} + \eta\right\}.
\end{aligned}$$

In addition, since $y^2 - \phi^2 = 2\phi\eta + \eta^2 = 2\rho^{\frac{1}{2}}\eta + \eta^2 + 2(\phi - \rho^{\frac{1}{2}})\eta$, we have

$$(y^2 - \phi^2)\rho^{-\frac{3}{2}} = \frac{2\eta}{\rho} + \rho^{-\frac{3}{2}}\eta^2 + 2\rho^{-\frac{3}{2}}(\phi - \rho^{\frac{1}{2}})\eta.$$

All this yields that

$$\begin{aligned} ye^{y^2-\rho} =& \rho^{-\frac{3}{2}} + \rho^{-\frac{3}{2}}\left\{(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \rho^{-2}\eta \\ &+ 2\frac{\eta}{\rho} + \rho^{-\frac{3}{2}}\eta^2 + 2\rho^{-\frac{3}{2}}(\phi - \rho^{\frac{1}{2}})\eta \\ &+ (y^2-\phi^2)\rho^{-\frac{3}{2}}\left\{(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \rho^{-2}(y^2-\phi^2)\eta \\ &+ \rho^{-2}\left(e^{y^2-\phi^2} - 1 - (y^2-\phi^2)\right)\left\{\rho^{\frac{1}{2}}(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} + \eta\right\}. \end{aligned} \tag{2.6}$$

From (2.3), (2.5) and (2.6), we have

$$\begin{aligned} &\frac{\rho^{-\frac{3}{2}}}{4} + \frac{1}{4}(\phi^{-3} - \rho^{-\frac{3}{2}}) - \phi^{-3}\left\{\frac{1}{\rho} - \frac{1}{\rho^2}\right\} - \phi^{-1}\rho^{-2} - \frac{d^2\eta}{d\rho^2} \\ &= \frac{\rho^{-\frac{3}{2}}}{4} + \frac{\rho^{-\frac{3}{2}}}{4}\left\{(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \frac{\rho^{-2}}{4}\eta \\ &\quad + \frac{1}{2\rho}\eta + \frac{\rho^{-\frac{3}{2}}\eta^2}{4} + \frac{1}{2}\rho^{-\frac{3}{2}}(\phi - \rho^{\frac{1}{2}})\eta \\ &\quad + \frac{1}{4}(y^2-\phi^2)\rho^{-\frac{3}{2}}\left\{(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\} + \frac{1}{4}\rho^{-2}(y^2-\phi^2)\eta \\ &\quad + \frac{\rho^{-2}}{4}\left(e^{y^2-\phi^2} - 1 - (y^2-\phi^2)\right)\left\{\rho^{\frac{1}{2}}(1-2\frac{\log\rho}{\rho})^{\frac{1}{2}} + \eta\right\} - \frac{e^{-\rho}y}{4}. \end{aligned}$$

Namely, η satisfies the following:

$$0 = \frac{d^2\eta}{d\rho^2} + \frac{1}{2\rho}\eta + f(\rho) + \sum_{i=1}^{6} g_i(\rho,\eta) - \frac{e^{-\rho}y}{4}, \tag{2.7}$$

where

$$\begin{aligned} f(\rho) =& -\frac{1}{4}(\phi^{-3} - \rho^{-\frac{3}{2}}) + \phi^{-3}\left\{\frac{1}{\rho} - \frac{1}{\rho^2}\right\} + \phi^{-1}\rho^{-2} \\ &+ \frac{\rho^{-\frac{3}{2}}}{4}\left\{\left(1-2\frac{\log\rho}{\rho}\right)^{\frac{1}{2}} - 1\right\}, \end{aligned}$$

$$g_1(\rho,\eta) = \frac{\rho^{-2}}{4}\eta, \quad g_2(\rho,\eta) = \frac{\rho^{-\frac{3}{2}}}{2}(\phi - \rho^{\frac{1}{2}})\eta, \quad g_3(\rho,\eta) = \frac{\rho^{-\frac{3}{2}}}{4}\eta^2,$$

$$g_4(\rho,\eta) = \frac{1}{4}(y^2-\phi^2)\rho^{-\frac{3}{2}}\left\{\left(1-2\frac{\log\rho}{\rho}\right)^{\frac{1}{2}} - 1\right\}, \quad g_5(\rho,\eta) = \frac{\rho^{-2}}{4}(y^2-\phi^2)\eta,$$

$$g_6(\rho,\eta)=\frac{\rho^{-2}}{4}\left(e^{y^2-\phi^2}-1-(y^2-\phi^2)\right)\left\{\rho^{\frac{1}{2}}\left(1-2\frac{\log\rho}{\rho}\right)^{\frac{1}{2}}+\eta\right\}.$$

To solve equation (2.7), we first consider the following linear equation: [1]

$$\frac{d^2\eta}{d\rho^2}+\left(\frac{1}{2\rho}+\frac{3}{16\rho^2}\right)\eta=0. \tag{2.8}$$

Any solution η to (2.8) can be written explicitly as follows: we have

$$\eta(\rho)=A\Phi(\rho)+B\Psi(\rho)$$

for some $A, B\in\mathbb{R}$, where

$$\Phi(\rho)=\rho^{\frac{1}{4}}\sin((2\rho)^{\frac{1}{2}}),\qquad \Psi(\rho)=\rho^{\frac{1}{4}}\cos((2\rho)^{\frac{1}{2}}).$$

Namely, Φ and Ψ are form the fundamental system of solutions to (2.8). For a given function F, we seek a solution to the following problem:

$$\begin{cases}\frac{d^2\eta}{d\rho^2}+(\frac{1}{2\rho}+\frac{3}{16\rho^2})\eta+F=0, & \rho\gg 1,\\ \lim_{\rho\to\infty}\eta(\rho)=0.\end{cases} \tag{2.9}$$

By the variation of parameters, we see that (2.9) is equivalent to the following integral equation:

$$\begin{aligned}\eta(\rho)&=\int_\rho^\infty(\Phi(s)\Psi(\rho)-\Phi(\rho)\Psi(s))F(s)ds\\ &=\int_\rho^\infty(\rho s)^{\frac{1}{4}}\sin((2\rho)^{\frac{1}{2}}-(2s)^{\frac{1}{2}})F(s)ds.\end{aligned}$$

By integrating by parts, we can obtain the following:

Lemma 2.3 *Let $\sigma>1$. Then, we have*

$$\begin{aligned}\int_\rho^\infty\sin((2\rho)^{\frac{1}{2}}-(2s)^{\frac{1}{2}})s^{-\sigma}\log s\,ds&=-\sqrt{2}\rho^{-\sigma+\frac{1}{2}}\log\rho+O(\rho^{-\sigma-\frac{1}{2}}\log\rho),\\ \int_\rho^\infty\sin((2\rho)^{\frac{1}{2}}-(2s)^{\frac{1}{2}})s^{-\sigma}ds&=-\sqrt{2}\rho^{-\sigma+\frac{1}{2}}+O(\rho^{-\sigma-\frac{1}{2}}).\end{aligned}$$

[1] Equation (2.8) is obtained by removing the better decaying terms in right hand of (2.7) and by adding $3/(16\rho^2)$ in the linear potential. By adding the term, we can write the solution to (2.8) just by the trigonometric functions. Otherwise, we need to use the Bessel functions of order 1. Thus, adding the extra term makes the proof a bit simpler.

Let $g_7(\rho,\eta) = -\frac{3}{16\rho^2}\eta$. In order to prove Theorem 2.1, we seek a solution to the following problem:

$$\begin{cases} \frac{d^2\eta}{d\rho^2} + (\frac{1}{2\rho} + \frac{3}{16\rho^2})\eta + f(\rho) + \sum_{i=1}^{7} g_i(\rho,\eta) - \frac{e^{-\rho}y}{4} = 0, & \rho \gg 1, \\ \lim_{\rho\to\infty}\eta(\rho) = 0. \end{cases} \tag{2.10}$$

To this end, we prepare several estimates, which are needed later. First, note that

$$\phi(\rho) = \rho^{\frac{1}{2}} + O(\rho^{-\frac{1}{2}}\log\rho). \tag{2.11}$$

By (2.11) and elementary calculations, we can obtain the following:

Lemma 2.4 *Let ϕ be the function given by (2.4). Then, for sufficiently large $\rho > 0$, we have*

$$\begin{aligned} \phi^{-3} - \rho^{-3/2} &= 3\rho^{-5/2}\log\rho + O(\rho^{-7/2}\log^2\rho), \\ \left|\rho^{-\frac{3}{2}}\left\{(1 - 2\frac{\log\rho}{\rho})^{\frac{1}{2}} - 1\right\}\right| &= -\rho^{-5/2}\log\rho + O(\rho^{-7/2}\log^2\rho), \\ \phi^{-1}\rho^{-2} &= \rho^{-5/2} + O(\rho^{-7/2}\log\rho), \\ \phi^{-3}(\rho^{-1} - \rho^{-2}) &= \rho^{-5/2} + O(\rho^{-7/2}\log\rho). \end{aligned}$$

In addition, we have the following estimates on the terms g_j of equation (2.10). More specifically, we have

Lemma 2.5 *Suppose that $\eta_1(\rho), \eta_2(\rho) = O(\rho^{-\frac{3}{2}}\log\rho)$. Then, we have*

$$\begin{aligned} &|g_1(\rho,\eta_1) - g_1(\rho,\eta_2)|, |g_7(\rho,\eta_1) - g_7(\rho,\eta_2)| \le C\rho^{-2}|\eta_1 - \eta_2|, \\ &|g_2(\rho,\eta_1) - g_2(\rho,\eta_2)|, |g_4(\rho,\eta_1) - g_4(\rho,\eta_2)|, |g_6(\rho,\eta_1) - g_6(\rho,\eta_2)| \\ &\quad \le C\rho^{-2}(\log\rho)|\eta_1 - \eta_2|, \\ &|g_3(\rho,\eta_1) - g_3(\rho,\eta_2)|, |g_5(\rho,\eta_1) - g_5(\rho,\eta_2)| \le C\rho^{-3}(\log\rho)|\eta_1 - \eta_2|. \end{aligned}$$

Proof From the definitions of g_i and (2.11), we have

$$\begin{aligned} &|g_1(\rho,\eta_1) - g_1(\rho,\eta_2)|, |g_7(\rho,\eta_1) - g_7(\rho,\eta_2)| \le C\rho^{-2}|\eta_1 - \eta_2|, \\ &|g_2(\rho,\eta_1) - g_2(\rho,\eta_2)| \le C\rho^{-\frac{3}{2}}|\phi - \rho^{\frac{1}{2}}||\eta_1 - \eta_2| \le C\rho^{-2}(\log\rho)|\eta_1 - \eta_2|, \\ &|g_3(\rho,\eta_1) - g_3(\rho,\eta_2)| \le C\rho^{-\frac{3}{2}}|\eta_1^2 - \eta_2^2| = \rho^{-\frac{3}{2}}|\eta_1 + \eta_2||\eta_1 - \eta_2| \\ &\quad \le C\rho^{-3}(\log\rho)|\eta_1 - \eta_2|. \end{aligned}$$

Let $y_1(\rho) = \phi(\rho) + \eta_1(\rho)$ and $y_2(\rho) = \phi(\rho) + \eta_2(\rho)$. Then, we obtain

$$\begin{aligned}|g_4(\rho,\eta_1) - g_4(\rho,\eta_2)| &\le C\rho^{-\frac{5}{2}}(\log\rho)|(y_1^2(\rho) - \phi^2(\rho)) - (y_2^2(\rho) - \phi^2(\rho))| \\ &\le C\rho^{-\frac{5}{2}}(\log\rho)(2\phi|\eta_1 - \eta_2| + |\eta_1 + \eta_2||\eta_1 - \eta_2|) \\ &\le C\rho^{-2}(\log\rho)|\eta_1 - \eta_2|.\end{aligned}$$

Since $y_i^2 - \phi^2 = 2\phi\eta_i + \eta_i^2$ for $i = 1, 2$, we have

$$|y_i^2 - \phi^2| \le |2\phi + \eta_i||\eta_i| \le C\rho^{-1}\log\rho \qquad \text{for } i = 1, 2.$$

It follows that

$$\begin{aligned}|g_5(\rho,\eta_1) - g_5(\rho,\eta_2)| &\le C\rho^{-2}|(y_1^2 - \phi^2)\eta_1 - (y_2^2 - \phi^2)\eta_2| \\ &\le C\rho^{-2}|(y_1^2 - \phi^2)(\eta_1 - \eta_2)| + \rho^{-2}|y_1^2 - y_2^2||\eta_2| \\ &\le C\rho^{-3}(\log\rho)|\eta_1 - \eta_2| + \rho^{-2}\rho^{-\frac{3}{2}}(\log\rho)\rho^{\frac{1}{2}}|\eta_1 - \eta_2| \\ &\le C\rho^{-3}(\log\rho)|\eta_1 - \eta_2|. \\ |g_6(\rho,\eta_1) - g_6(\rho,\eta_2)| &\le C\rho^{-2}|e^{y_1^2-\phi^2} - 1 - (y_1^2 - \phi^2)||\eta_1 - \eta_2| \\ &\quad + C\rho^{-2}|(e^{y_1^2-\phi^2} - 1 - (y_1^2 - \phi^2)) \\ &\quad - (e^{y_2^2-\phi^2} - 1 - (y_2^2 - \phi^2))||\rho^{\frac{1}{2}} + \eta_2| \\ &\le C\rho^{-2}|e^{y_1^2-\phi^2} - 1 - (y_1^2 - \phi^2)||\eta_1 - \eta_2| \\ &\quad + C\rho^{-\frac{3}{2}}|(e^{y_1^2-\phi^2} - 1 - (y_1^2 - \phi^2)) \\ &\quad - (e^{y_2^2-\phi^2} - 1 - (y_2^2 - \phi^2))| \\ &=: I + II.\end{aligned}$$

From (2.1), we have

$$I \le C\rho^{-2}|y_1^2 - \phi^2|^2|\eta_1 - \eta_2| \le C\rho^{-4}(\log\rho)^2|\eta_1 - \eta_2|.$$

Using the mean value theorem and (2.1), we have

$$\begin{aligned}&|(e^{y_1^2-\phi^2} - 1 - (y_1^2 - \phi^2)) - (e^{y_2^2-\phi^2} - 1 - (y_2^2 - \phi^2))| \\ &\quad \le C|\exp[\theta(y_1^2 - \phi^2) + (1-\theta)(y_2^2 - \phi^2)] - 1||y_1^2 - y_2^2| \\ &\quad \le C|\theta(y_1^2 - \phi^2) + (1-\theta)(y_2^2 - \phi^2)||y_1^2 - y_2^2| \\ &\quad \le C\rho^{-1}(\log\rho)(2\phi|\eta_1 - \eta_2| + |\eta_1 + \eta_2||\eta_1 - \eta_2|) \\ &\quad \le C\rho^{-\frac{1}{2}}(\log\rho)|\eta_1 - \eta_2|.\end{aligned}$$

This yields that $II \leq C\rho^{-2}(\log\rho)|\eta_1 - \eta_2|$. Thus, we see that

$$|g_6(\rho,\eta_1) - g_6(\rho,\eta_2)| \leq C\rho^{-2}(\log\rho)|\eta_1 - \eta_2|,$$

as desired. □

We are now in a position to prove Proposition 2.2.

Proof of Proposition 2.2 Note that (2.7) is equivalent to the following integral equation:

$$\eta(\rho) = \mathcal{T}[\eta](\rho),$$

in which

$$\mathcal{T}[\eta](\rho) = \int_\rho^\infty (\rho s)^{\frac{1}{4}} \sin((2\rho)^{\frac{1}{2}} - (2s)^{\frac{1}{2}})F(s,\eta)ds,$$

where

$$F(\rho,\eta) = f(\rho) + \sum_{i=1}^{7} g_i(\rho,\eta) - \frac{e^{-\rho}}{4}y.$$

Fix $\Lambda > 0$ sufficiently large and let X be a space of continuous functions on $[\Lambda, \infty)$ equipped with the following norm:

$$\|\xi\| = \sup\left\{|\rho|^{\frac{3}{2}}(\log\rho)^{-1}|\xi(\rho)| \mid \rho \geq \Lambda\right\}.$$

Lemmas 2.3 and 2.4 insure that there exists a constant $C_* > 0$ such that

$$\left|\int_\rho^\infty (\rho s)^{\frac{1}{4}} \sin((2\rho)^{\frac{1}{2}} - (2s)^{\frac{1}{2}}) f(s)ds\right| \leq C_*\rho^{-\frac{3}{2}}\log\rho. \tag{2.12}$$

For this constant $C_* > 0$, define the space

$$\Sigma = \{\xi \in X \mid \|\xi\| \leq 3C_*\},$$

and we shall first show that $\mathcal{T}$ maps Σ to itself. From Lemma 2.5, we have

$$\left|\sum_{i=1}^{7} g_i(\rho,\eta)\right| \leq C\rho^{-2}\log\rho|\eta| \leq C\rho^{-\frac{7}{2}}(\log\rho)^2,$$

which yields that

$$\left|\int_\rho^\infty (\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})\sum_{i=1}^7 g_i(\rho,\eta)ds\right|\le C\int_\rho^\infty \rho^{\frac14}s^{\frac14}s^{-\frac72}(\log s)^2ds$$
$$\le C\rho^{\frac14}\int_\rho^\infty s^{-3}ds\le C\rho^{-\frac74}.$$

Therefore, we can take $\Lambda>0$ sufficiently large so that

$$\left|\int_\rho^\infty (\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})\sum_{i=1}^7 g_i(\rho,\eta)ds\right|\le C_*\rho^{-\frac32}\log\rho \tag{2.13}$$

for $\rho\ge\Lambda$. We can easily find that for sufficiently large $\Lambda>0$, we have

$$\left|\int_\rho^\infty (\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})e^{-s}y(s)ds\right|\le Ce^{-\frac{\rho}{2}}\le C_*\rho^{-\frac32}\log\rho \tag{2.14}$$

for $\rho\ge\Lambda$. By (2.12), (2.13) and (2.14), we obtain

$$|\mathcal{T}[\eta](\rho)|\le 3C_*\rho^{-\frac32}\log\rho$$

for $\eta\in\Sigma$. This yields that $\mathcal{T}[\eta]\in\Sigma$.

Next, we shall show that $\mathcal{T}$ is a contraction mapping. For $\eta_1,\eta_2\in\Sigma$, we have

$$|\mathcal{T}[\eta_1](\rho)-\mathcal{T}[\eta_2](\rho)|\le\sum_{i=1}^7\int_\rho^\infty |(\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})||g_i(s,\eta_1)-g_i(s,\eta_2)|ds$$
$$+\int_\rho^\infty |(\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})|e^{-s}|\eta_1-\eta_2|ds.$$

From Lemma 2.5 follows the estimate

$$\begin{aligned}|\mathcal{T}[\eta_1](\rho)-\mathcal{T}[\eta_2](\rho)|&\le C\int_\rho^\infty |(\rho s)^{\frac14}\sin((2\rho)^{\frac12}-(2s)^{\frac12})|s^{-2}\log s|\eta_1-\eta_2|ds\\&\quad+C\|\eta_1-\eta_2\|e^{-\rho/2}\\&\le C\int_\rho^\infty s^{\frac14}\rho^{\frac14}s^{-\frac72}(\log s)^2\|\eta_1-\eta_2\|ds+C\|\eta_1-\eta_2\|e^{-\rho}\\&\le C\|\eta_1-\eta_2\|\rho^{-2}(\log\rho)^2.\end{aligned}$$

This yields that

$$\|\mathcal{T}[\eta_1](\rho)-\mathcal{T}[\eta_2](\rho)\|\le C\rho^{-\frac14}\|\eta_1-\eta_2\|<\frac12\|\eta_1-\eta_2\|$$

for sufficiently large $\rho > 0$. Thus, we find that $\mathcal{T}$ is a contraction mapping. This completes the proof. □

Proof of Theorem 2.1 Let $u_\infty(r) = y_\infty(\rho)$, where $y_\infty(\rho)$ is the solution to (2.3) obtained by Proposition 2.2. We note that u_∞ satisfies the following:

$$-\frac{d^2u_\infty}{dr^2} - \frac{1}{r}\frac{du_\infty}{dr} = u_\infty(e^{u_\infty^2} - 1) \qquad \text{for } r \in (0, R_\infty), \tag{2.15}$$

where $R_\infty = e^{-\Lambda_\infty/2}$. Since u_∞ is a solution to the ordinary differential equation (2.15), we can extend u_∞ in the positive direction of r as long as u_∞ remains bounded. We claim that u_∞ has a zero at some point. Suppose the contrary that $u_\infty(r) > 0$ for all $0 < r < \infty$. Then, we see that u_∞ is monotone decreasing. Indeed, if not, there exists a local minimum point $r_* \in (0, \infty)$. It follows that $\partial_r^2 u_\infty(r_*) \geq 0$ and $\partial_r u_\infty(r_*) = 0$. Then, from the equation (2.15), we obtain

$$0 \leq \frac{d^2u_\infty}{dr^2}(r_*) = -u_\infty(r_*)(e^{u_\infty^2(r_*)} - 1) < 0,$$

which is a contradiction.

Since u_∞ is positive and monotone decreasing, there exists a constant $C_\infty \geq 0$ such that $u_\infty(r) \to C_\infty$ as $r \to \infty$. To prove that $C_\infty = 0$, suppose the contrary that $C_\infty > 0$. This together with (2.15) yields that

$$0 = \lim_{r\to\infty}\left(\frac{d^2u_\infty}{dr^2}(r) + \frac{1}{r}\frac{du_\infty}{dr}(r)\right) = -\lim_{r\to\infty}(e^{u_\infty^2(r)} - 1)u_\infty(r) < 0,$$

which is absurd. Therefore, we see that $C_\infty = 0$, that is, $\lim_{r\to\infty} u_\infty(r) = 0$. Multiplying (2.15) by r and integrating the resulting equation from 0 to r yields that

$$-r\frac{du_\infty}{dr}(r) = \int_0^r su_\infty(e^{u_\infty^2} - 1)ds > 0.$$

This yields that for any $R > 0$, there exists a constant $C_1 > 0$ such that $-du_\infty/dr(r) \geq C_1/r$ for all $r > R$. It follows that

$$u_\infty(r) - u_\infty(R) = \int_R^r \frac{du_\infty}{ds}(s)ds \leq -C_1\int_R^r \frac{1}{s}ds.$$

Letting $r \to \infty$, we have

$$-u_\infty(R) = \lim_{r\to\infty}(u_\infty(r) - u_\infty(R)) \leq -C_1 \lim_{r\to\infty}\int_R^r \frac{1}{s}ds = -\infty,$$

which is a contradiction. Therefore, there exists $R_\infty > 0$ such that $u_\infty(r)$ has a zero at $r = R_\infty$.

Finally, we shall show that u_∞ is a solution (2.1) in a distribution sense. Let $\eta(\rho)$ be the solution to (2.10), obtained in the proof of Proposition 2.2. Since $\eta(\rho) = O(\rho^{-3/2}\log\rho)$, we see that u_∞ satisfies

$$u_\infty(r) = (-2\log r - 2\log(-\log r) - 2\log 2)^{\frac{1}{2}} + O((-\log r)^{-\frac{3}{2}}\log(-\log r)) \quad \text{as } r \to 0.$$

This yields that there exists a constant $C > 0$ such that

$$u_\infty e^{u_\infty^2} \le Cr^{-2}(-\log r)^{-\frac{3}{2}} \qquad \text{for sufficiently small } r > 0. \tag{2.16}$$

This together with the monotonicity implies that $u_\infty e^{u_\infty^2} \in L^1_{\mathrm{loc}}(B_{R_\infty})$. Therefore, we see that u_∞ is a distributional solution to (2.1). This completes the proof. □

2.2 Singular soliton by the shooting

Let $R \in (0, R_\infty)$ be the unique point such that $u_\infty(R) = 2$. For each $m \ge 0$, we put

$$f_m(s) := s(e^{s^2} - 1) - m\chi(s)s, \tag{2.17}$$

where $\chi \in C^\infty(\mathbb{R})$ is a cut-off function satisfying $\chi(t) = 1$ for $|t| \le 1$, $\chi(t) = 0$ for $|t| \ge 2$, and $t\chi'(t) \le 0$ for all $t \in \mathbb{R}$. Consider a family of radial ODE's with a parameter $m \ge 0$:

$$\begin{cases} -\varphi_*'' - \frac{\varphi_*'}{r} = f_m(\varphi_*), & (r > R) \\ \varphi_*(R) = u_\infty(R) = 2, \ \varphi_*'(R) = u_\infty'(R), \end{cases} \tag{2.18}$$

where the prime mark denotes the differentiation with respect to r. Let ϕ_m be the unique solution of the above. We shall show that there exists $m_* > 0$ such that $\phi_{m_*}(r) \searrow 0$ as $r \to \infty$. To this end, we show the following:

Proposition 2.6 *Let $m \ge 0$ and ϕ_m be the solution to (2.18). There exists $m_S > 0$ and $m_L > 0$ such that if $m \in [0, m_S)$, ϕ_m has a zero in (R, ∞), and if $m \in (m_L, \infty)$, $\phi_m(r)$ is positive for all $r \ge R$.*

First, we define an energy density function $E_m : [R, \infty) \to \mathbb{R}$ by

$$E_m(r) := \frac{(\phi_m'(r))^2}{2} + F_m(\phi_m(r)),$$

where F_m is the nonlinear potential energy defined by

$$F_m(u) := \int_0^u f_m(s)ds = \frac{e^{u^2} - 1 - u^2}{2} - m\int_0^u \chi(s)sds,$$

which enjoys the standard superquadratic condition:

$$0 \le G_m(u) := uF_m'(u) - 2F_m(u) = \sum_{k=2}^{\infty} \frac{(k-1)}{k!} u^{2k} + 2m \int_0^u (\chi(s) - \chi(u)) s ds, \tag{2.19}$$

and, thanks to the monotonicity of χ,

$$G_m(u) = 0 \quad \text{if and only if} \quad u = 0. \tag{2.20}$$

It follows from (2.18) that

$$E_m'(r) = -\frac{(\phi_m'(r))^2}{r} \le 0. \tag{2.21}$$

Thus, $E_m(r)$ is a non-increasing function of r. Using (2.21), we shall show the following:

Lemma 2.7 *Let $m > 0$ and ϕ_m be a solution to (2.18). Let $s \ge R$, $E_m(s) < 0$ and $\phi_m(s) > 0$. Then, we have $\phi_m(r) > 0$ for all $r \in (s, \infty)$.*

Proof Suppose the contrary. Then there exists $z \in (s, \infty)$ such that $\phi_m(z) = 0$. This together with (2.21) yields that

$$0 > E_m(s) \ge E_m(z) = \frac{(\phi_m'(z))^2}{2} \ge 0,$$

which is a contradiction. □

We are now in a position to prove Proposition 2.6.

Proof of Proposition 2.6 We note that $m = 0$, $\phi_0(r) = u_\infty$ has a zero at $r = R_\infty$. Then, by the continuity of ϕ_m with respect to m, we see that $\phi_m(r)$ still has a zero if $m > 0$ is sufficiently small.

On the other hand, we have

$$E_m(R) = \frac{(u_\infty'(R))^2}{2} + \frac{e^4 - 5}{2} - m \int_0^2 \chi(s) s ds < 0$$

for large $m > 0$, then Lemma 2.7 implies that $\phi_m(r) > 0$ for all $r > R$. □

We put

$$m_* = \inf \{ m > 0 \mid \phi_m(r) > 0 \quad \text{on } r > R \}. \tag{2.22}$$

We see from Proposition 2.6 that $0 < m_* < \infty$. We extend ϕ_{m_*} by defining $\phi_{m_*}(r) = u_\infty(r)$ for $r \in (0, R)$ (still denoted by the same symbol). Then, we have the following:

Theorem 2.8 *Let $m_* > 0$ be the number defined by (2.22). Then, ϕ_{m_*} is a singular positive radial solution to the following elliptic equation*

$$\begin{cases} -\Delta\phi = f_{m_*}(\phi) & \text{in } \mathbb{R}^2, \\ \lim_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{2.23}$$

where $f_m(s)$ is defined by (2.17). Moreover, ϕ_{m_*} is strictly decreasing in the radial direction. Moreover, for any $m \in (0, m_*)$ there exists $C_m \in (0, \infty)$ such that

$$\phi_{m_*}(r) + |\phi'_{m_*}(r)| \le C_m e^{-\sqrt{m}r} \quad \text{for all } r \ge R. \tag{2.24}$$

Proof To prove Theorem 2.8, it suffices to show that $\phi_{m_*}(r) > 0$, $\phi'_{m_*}(r) < 0$ for all $r > 0$, $\lim_{r\to\infty} \phi_{m_*}(r) = 0$ and (2.24).

First, we shall show that $\phi_{m_*}(r) > 0$ on $r \in (R, \infty)$. By definition of m_*, there exists a sequence $m_n \searrow m_*$ such that $\phi_{m_n}(r) > 0$ for all $r > R$ and $n \in \mathbb{N}$. Then $\phi_{m_*}(r) = \lim_{n\to\infty} \phi_{m_n}(r) \ge 0$ for all $r > R$. If $\phi_{m_*}(r) = 0$ at some $r > R$, then $\phi'_{m_*}(r) = 0$ and so $\phi_{m_*} \equiv 0$ by the ODE, a contradiction. Hence $\phi_{m_*}(r) > 0$ for all $r > R$.

Next, we claim that

$$E_{m_*}(r) \ge 0 \quad \text{for all } r \in (R, \infty). \tag{2.25}$$

Suppose the contrary that there exists $R_* > 0$ such that $E_{m_*}(R_*) < 0$. Then the continuity for m yields that $\phi_m(r) > 0$ on $R \le r \le R_*$ and $E_m(R_*) < 0$ when $m \in (0, m_*)$ is close enough to m_*. Then Lemma 2.7 implies $\phi_m(r) > 0$ for $r \ge R_*$, hence for all $r > R$, contradicting the definition of m_*. Hence we have (2.25).

Next, we shall show that $\phi'_{m_*}(r) < 0$ for all $r > R$. Suppose the contrary and let $s > R$ be the first zero of ϕ'_{m_*}. Then we have $0 = \phi'_{m_*}(s) \le \phi''_{m_*}(s) = -f_{m_*}(\phi_{m_*}(s))$, $0 \le E_{m_*}(s) = F_{m_*}(\phi_{m_*}(s))$, and so $G_{m_*}(\phi_{m_*}(s)) \le 0$, contradicting (2.19).

Therefore $\phi'_{m_*}(r) < 0 < \phi_{m_*}(r)$ for all $r > R$, so $\phi_{m_*}(r) \searrow \exists C_* \in [0, 2)$ and $\phi'_{m_*}(r) \to 0$ as $r \to \infty$. Then we have $0 \le \lim_{r\to\infty} E_{m_*}(r) = F_{m_*}(C_*)$ and

$$\phi''_{m_*}(r) = -\phi'_{m_*}(r)/r - f_{m_*}(\phi_{m_*}(r)) \to -f_{m_*}(C_*) \quad (r \to \infty),$$

which has to be 0 because $\phi'_{m_*}(r) \to 0$. Hence $G_{m_*}(C_*) \le 0$ and so $C_* = 0$ by (2.19).

Next, let $m \in (0, m_*)$. Then $\phi_{m_*}(r) \searrow 0$ together with the definition of f_m implies

$$\phi''_{m_*}(r) = -\phi'_{m_*}(r)/r - f_{m_*}(\phi_{m_*}(r)) > m\phi_{m_*}(r) > 0$$

for sufficiently large $r > R$. Hence $e^{\sqrt{m}r}(\sqrt{m} - \partial_r)\phi_{m_*}(r)$ is decreasing for large r, which implies the desired exponential decay.

Finally, it follows from (2.16) and (2.24) that $f_{m_*} \in L^1(\mathbb{R}^2)$. Thus, we see that ϕ_{m_*} is a distributional solution to (2.23). This completes the proof. □

Remark 4 Let u_∞ be a solution to (2.1), which is obtained in Theorem 2.1. Since $\phi_{m_*}(r) = u_\infty(r)$ for $r \in (0, R]$, we see from (2.2) that ϕ_{m_*} satisfies

$$\begin{aligned}\phi_{m_*}(r) = &(-2\log r - 2\log(-\log r) - 2\log 2)^{\frac{1}{2}}\\ &+O((-\log r)^{-\frac{3}{2}}\log(-\log r)) \quad \text{as } r \to 0.\end{aligned} \tag{2.26}$$

Moreover, we have, by (2.10), Lemmas 2.3 and 2.4, that $|d^2\eta/d\rho^2(\rho)| \le C\rho^{-5/2}\log\rho$. Thus, by integrating, we see that $|d\eta/d\rho(\rho)| \le C\rho^{-3/2}\log\rho$. Thus, ϕ'_{m_*} satisfies

$$\begin{aligned}\phi'_{m_*}(r) = &-(-2\log r - 2\log(-\log r) - 2\log 2)^{-\frac{1}{2}}\left(\frac{1}{r} + \frac{1}{r\log r}\right)\\ &+O((-\log r)^{-3/2}\log(-\log r)) \quad \text{as } r \to 0.\end{aligned} \tag{2.27}$$

3 Regular solution by the heat iteration

In what follows, we denote ϕ_{m_*}, which is obtained in Theorem 2.8, by φ_*. Now we are ready to construct a regular solution to (1.5) with initial data φ_*.

Theorem 3.1 *Let $u_0 := e^{t\Delta}\varphi_*$. Then for any $t > 0$, $u_0(t)$ is bounded on $\mathbb{R}^2$. Moreover, there exists a small $T > 0$, and a solution u_R to (1.5) with $u_R|_{t=0} = \varphi_*$ satisfying*

$$|\log t|^{1/2}(u_R - u_0) \in L^\infty([0, T) \times \mathbb{R}^2).$$

Note that $\varphi_* \in C^\infty(\mathbb{R}^2\backslash\{0\})$ is a positive radial function satisfying the asymptotic form,

$$\begin{aligned}\varphi_*(r) &= (\rho - 2\log\rho)^{1/2} + O(\rho^{-3/2}\log\rho)\\ &= (\rho - 2\log\rho + O(\rho^{-1}\log\rho))^{1/2} \quad (\rho \to \infty),\end{aligned} \tag{3.1}$$

where, as before, $\rho := |\log r^2| = 2|\log r| \gg 1$. The above two equivalent expressions of remainder will be frequently switched in the following computations.

We need a precise estimate or asymptotic behavior around $t, r \to 0$ of the iteration sequence. Consider the first (or zeroth) iteration

$$u_0 := e^{t\Delta}\varphi_* = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}}\varphi_*(y)dy = \int_{\mathbb{R}^2}\frac{e^{-|z|^2/4}}{4\pi}\varphi_*(x - \sqrt{t}z)dz.$$

We shall show the following:

Lemma 3.2 *There exists $\varepsilon > 0$ such that if $\max\{t, |x|^2\} < \varepsilon^2$, we have*

$$u_0(t, x) \le \min\{\varphi_*(\sqrt{t}), \varphi_*(x)\} + O(|\log t|^{-\frac{1}{2}}). \tag{3.2}$$

Proof We take $\varepsilon > 0$ sufficiently small so that for any $r < \varepsilon$, $\varphi_*(r)$ satisfies (3.1), and consider the region $(t, x) \in (0, \infty) \times \mathbb{R}^2$ satisfying $\max\{t, |x|^2\} < \varepsilon^2$. Put

$$\ell := -\log t = |\log t| \gg 1, \quad \nu := |x|^2/t \in [0, \infty).$$

We shall estimate u_0 by dividing the space-time region $\max\{t, |x|^2\} < \varepsilon^2$ into the three subregions: inside the parabolic cylinder $\nu \le 1$, $\nu > 8\log\ell \gg 1$ and $1 \le \nu \le 8\log\ell$.

First, we consider the region $\nu \le 1$. It follows from Lemma A.1 that

$$\|u_0(t,\cdot)\|_{L^\infty_x} = u_0(t,0) = \int_{\mathbb{R}^2} \frac{e^{-|z|^2/4}}{4\pi}\varphi_*(\sqrt{t}z)dz.$$

Thus, by (3.1), we obtain

$$\begin{aligned}
\|u_0(t,\cdot)\|_{L^\infty_x} &= \int_{\mathbb{R}^2} \frac{e^{-|z|^2/4}}{4\pi}\varphi_*(\sqrt{t}z)dz \\
&= \int_0^t e^{-s/4}(-\log(ts) - 2\log|\log(ts)| + O(1))^{1/2} ds/4 \\
&\quad + \int_t^{\frac{\varepsilon^2}{t}} e^{-s/4}(-\log(ts) - 2\log|\log(ts)| + O(1))^{1/2} ds/4 \\
&\quad + \int_{\frac{\varepsilon^2}{t}}^\infty e^{-\frac{s}{4}}\varphi_*(\sqrt{ts})ds/4 \\
&=: \mathrm{I} + \mathrm{J} + \mathrm{K}.
\end{aligned} \tag{3.3}$$

For $0 < s < t$, we have $ts > s^2$. This yields via an integration by parts

$$\mathrm{I} \le C\int_0^t |2\log s|^{1/2} ds \le Ct\ell^{1/2} \le C\ell^{-1/2}. \tag{3.4}$$

For $t < s < \frac{\varepsilon^2}{t}$, we have $|\log s| < \ell$, so the integrand is expanded

$$\begin{aligned}
(-\log(ts) - 2\log|\log(ts)| + O(1))^{1/2} &= (\ell - 2\log\ell - \log s + O(1))^{1/2} \\
&= (\ell - 2\log\ell)^{1/2} + O(\ell^{-1/2}\langle\log s\rangle),
\end{aligned} \tag{3.5}$$

where we set $\langle\log s\rangle = \sqrt{1 + |\log s|^2}$. Since $\varphi_*(r)$ is monotone decreasing in $r > 0$, we have, by (3.1) and (3.5), that

$$\varphi_*(\sqrt{ts}) \le (\ell - 2\log\ell)^{1/2} + O(\ell^{-1/2}\langle\log s\rangle)$$

for $s \ge \frac{\varepsilon^2}{t}$. Integration against $e^{-s/4}$ yields

$$\mathrm{J}+\mathrm{K} \leq \int_t^\infty e^{-s/4}((\ell-2\log\ell)^{1/2}+O(\ell^{-1/2}\langle\log s\rangle)ds/4$$
$$\leq(\ell-2\log\ell)^{1/2}+O(\ell^{-1/2}). \tag{3.6}$$

Estimates (3.4) and (3.6) together with (3.3) imply

$$u_0(t,0)-(\ell-2\log\ell)^{1/2}\leq C\ell^{-1/2}.$$

Namely, we have

$$u_0\leq\varphi_*(\sqrt{t})+O(\ell^{-1/2}). \tag{3.7}$$

This is enough in the region $\nu\leq 1$.

Before focusing on the two remaining regions, let $0<\delta\leq\varepsilon$ be a small parameter and set

$$B:=\{z\in\mathbb{R}^2 \mid |x-\sqrt{t}z|<\delta|x|\}.$$

Decomposing u_0 as

$$u_0(t,x)=\int_B\frac{e^{-|z|^2/4}}{4\pi}\varphi_*(x-\sqrt{t}z)dz+\int_{B^c}\frac{e^{-|z|^2/4}}{4\pi}\varphi_*(x-\sqrt{t}z)dz=:u_0^I+u_0^X, \tag{3.8}$$

and writing

$$u_0^X=\int_{B^C}\frac{e^{-|z|^2/4}}{4\pi}[\varphi_*(x)+\varphi_*(x-\sqrt{t}z)-\varphi_*(x)]dz. \tag{3.9}$$

First, we consider $z\in B^C$ satisfying $|x-\sqrt{t}z|\leq|x|$. One can apply the mean value theorem, and use (2.27) to write

$$|\varphi_*(|x-\sqrt{t}z|)-\varphi_*(|x|)|\leq\sup_{\delta|x|\leq|y|\leq|x|}|\varphi_*'(|y|)|\big||x-\sqrt{t}z|-|x|\big|\leq C_\delta\frac{\sqrt{t}|z|}{r|\log r|^{1/2}} \tag{3.10}$$

for $z\in B^C$ satisfying $|x-\sqrt{t}z|\leq|x|$. Here, C_δ is a positive constant which depends on δ. Next, we consider $z\in B^C$ satisfying $|x-\sqrt{t}z|\geq|x|$. Since φ_* is monotone decreasing in $r=|x|$, we see that

$$\varphi_*(x-\sqrt{t}z)\leq\varphi_*(x). \tag{3.11}$$

for $z\in B^C$ satisfying $|x-\sqrt{t}z|\geq|x|$. From (3.9)–(3.11), we conclude that

$$
\begin{aligned}
u_0^X &\le \int_{B^C} \frac{e^{-|z|^2/4}}{4\pi}\left[\varphi_*(x) + C_\delta \frac{\sqrt{t}|z|}{r|\log r|^{1/2}} dz\right] \\
&= \varphi_*(x)(1-\theta_B) + C_\delta \sqrt{t/r^2}|\log r|^{-1/2},
\end{aligned} \tag{3.12}
$$

where we set

$$
\theta_B := \int_B \frac{e^{-|z|^2/4}}{4\pi} dz \in (0,1).
$$

Now, for $z \in B$, it follows that

$$
|\sqrt{t}z| \ge |x| - |x - \sqrt{t}z| \ge (1-\delta)|x|. \tag{3.13}
$$

Thus, we have $|z|^2 \ge \nu(1-\delta)^2$, and so

$$
u_0^I = \int_B \frac{e^{-|z|^2/4}}{4\pi}\varphi_*(x-\sqrt{t}z)dz \le \frac{e^{-\nu(1-\delta)^2/4}}{4\pi t}\int_{|y|^2 < t\delta^2\nu} \varphi_*(y)dy. \tag{3.14}
$$

Second, we estimate u_0^I in the region $\nu > 8\log\ell \gg 1$. For $0 < a \ll 1$ we have

$$
\begin{aligned}
\frac{1}{\pi}\int_{|y|^2<a} \varphi_*(y)dy &= \int_0^a (-\log s - 2\log|\log s| + O(|\log s|^{-1}\log|\log s|))^{1/2} ds \\
&= \int_{|\log a|}^\infty (t - 2\log t + O(t^{-1}\log t))^{1/2} e^{-t} dt \\
&= a\Big[(|\log a| - 2\log|\log a|)^{1/2} \\
&\quad + \frac{1}{2}|\log a|^{-1/2} + O(|\log a|^{-3/2}\log|\log a|)\Big] \\
&= a\left[|\log a| - 2\log|\log a| + 1 + O(|\log a|^{-1}\log|\log a|)\right]^{1/2},
\end{aligned} \tag{3.15}
$$

by partial integration on e^{-t}. Note that $t\delta^2\nu = \delta|x|^2 \le \delta\varepsilon^2 \ll 1$. Plugging the formula (3.15) into (3.14) yields

$$
u_0^I \le \frac{e^{-\nu(1-\delta)^2/4}}{4\pi}\delta^2\nu(|\log(t\delta^2\nu)| - 2\log|\log(t\delta^2\nu)| + O(1))^{1/2}. \tag{3.16}
$$

In the region $\nu > 8\log\ell \gg 1$, we have $\ell = |\log t| < e^{\nu/8}$, yielding

$$
\begin{aligned}
|\log(t\delta^2\nu)| - 2\log|\log(t\delta^2\nu)| + O(1) &\le 2|\log(t\delta^2\nu)| \\
&\le 2\left(|\log t| + 2|\log\delta| + |\log\nu|\right)
\end{aligned}
$$

$$\leq 2\left(e^{\nu/8} + 2|\log\delta| + |\log\nu|\right)$$
$$\leq 4\left(e^{\nu/8} + 2|\log\delta|\right). \tag{3.17}$$

If we take $\delta > 0$ sufficiently small so that $-(1-\delta)^2/4 + 1/16 < -1/6$, we have, by (3.16) and (3.17), that

$$u_0^I \leq \frac{e^{-\nu(1-\delta)^2/4}}{\pi}\delta^2 \nu e^{\nu/16}(1 + 2e^{-\nu/8}|\log\delta|)^{1/2}$$
$$\leq \nu e^{-\nu/6}\delta^2(1 + 2|\log\delta|)^{1/2}$$
$$\leq \nu e^{-\nu/6}.$$

Thus, we get the first intermediate estimate $u_0^I \leq \nu e^{-\nu/6}$. Moreover, using the fact that $|\log t| < e^{\nu/8}$ again, we obtain

$$|2\log r| = -\log r^2 = -\log(\nu t) = -\log\nu + |\log t| \leq e^{\nu/8},$$

and therefore,

$$u_0^I \ll |\log r|^{-1/2}\nu^{-1/2}. \tag{3.18}$$

Finally, observe that in the same region $\nu > 8\log\ell \gg 1$, we have $r^2 \gg t$ and

$$r^2|\log r| = -\frac{r^2\log r^2}{2} \gg -\frac{t\log t}{2} = \frac{t\ell}{2}.$$

This together with (3.8), (3.12) and (3.18) yields that for $\nu > 8\log\ell \gg 1$, we have obtained

$$u_0 - \varphi_*(x) \leq C_\delta|\log r|^{-1/2}\nu^{-1/2} \ll \ell^{-1/2}. \tag{3.19}$$

Thus it remains to estimate u_0^I in the region

$$1 \leq \nu = \frac{|x|^2}{t} < 8\log\ell.$$

Let $\zeta := z - x/\sqrt{t}$. On $z \in B$, we have $|\zeta| < \delta\sqrt{\nu} \ll \ell$. We shall show that

$$\varphi_*(x - \sqrt{t}z) = \varphi_*(\sqrt{t}\zeta) \leq (\ell - 2\log\ell)^{1/2} + C\ell^{-1/2}\langle\log|\zeta|\rangle \tag{3.20}$$

for some constant $C > 0$, which is independent of ν and δ. We first consider the region $1 < |\zeta|$. Since $1 < |\zeta| \ll \ell$, we have $\ell^{-1}\log|\zeta| < \ell^{-1}\log\ell \ll 1$. Thus, by (3.1) and the Taylor expansion, we have

$$\begin{aligned}\varphi_*(x-\sqrt{t}z) &= \varphi_*(\sqrt{t}\zeta)\\ &\le (-\log t - 2\log|\log t| - \log|\zeta|^2 - 2\log|1-\ell^{-1}\log|\zeta|^2| + C)^{1/2}\\ &\le (\ell - 2\log\ell + C\langle\log|\zeta|\rangle)^{1/2}\\ &\le (\ell-2\log\ell)^{1/2}(1+C\ell^{-1}\langle\log|\zeta|\rangle)^{1/2}\\ &\le (\ell-2\log\ell)^{1/2}(1+C\ell^{-1}\langle\log|\zeta|\rangle)\\ &\le (\ell-2\log\ell)^{1/2} + C\ell^{-1/2}\langle\log|\zeta|\rangle. \end{aligned} \tag{3.21}$$

Next, we consider the region $|\zeta| \le 1$. Since $-\log|\zeta|^2 \ge 0$, we have

$$-\log|\zeta|^2 - 2\log|1-\ell^{-1}\log|\zeta|^2| \le -2\log|\zeta|^2.$$

Using this together with (3.1), we obtain

$$\begin{aligned}\varphi_*(x-\sqrt{t}z) &\le (-\log t - 2\log|\log t| - \log|\zeta|^2 - 2\log|1-\ell^{-1}\log|\zeta|^2| + C)^{1/2}\\ &\le (\ell - 2\log\ell + C(-\log|\zeta|+1))^{1/2}\\ &\le (\ell-2\log\ell)^{1/2} + C(\ell-2\log\ell)^{-1/2}(-\log|\zeta|+1)\\ &\le (\ell-2\log\ell)^{1/2} + C\ell^{-1/2}\langle\log|\zeta|\rangle. \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we see that (3.20) holds.

Moreover, it follows from (3.13) that $|z|^2 \ge \nu(1-\delta)^2$. This yields that

$$\begin{aligned}\int_B e^{-\frac{|z|^2}{4}}\langle\log|\zeta|\rangle dz &\le \int_{|\zeta|\le\delta\sqrt{\nu}} e^{-\nu(1-\delta)^2/4}\langle\log|\zeta|\rangle d\zeta \le e^{-\nu(1-\delta)^2/4}\delta^2\nu\langle\log\delta^2\nu\rangle\\ &\le C \end{aligned} \tag{3.23}$$

for some constant $C > 0$, which is independent of ν and δ. It follows from (3.23) and (3.20) that

$$\begin{aligned} u_0^I &= \int_B \frac{e^{-|z|^2}/4}{4\pi}\varphi_*(x-\sqrt{t}z)dz\\ &\le (\ell-2\log\ell)^{1/2}\int_B \frac{e^{-|z|^2/4}}{4\pi}dz + O(\ell^{-1/2})\\ &\le \varphi_*(\sqrt{t})\int_B \frac{e^{-|z|^2/4}}{4\pi}dz + O(\ell^{-1/2}). \end{aligned} \tag{3.24}$$

Hence, by (3.8), (3.12) and (3.24), we have

$$u_0 \le \varphi_*(x) + (\varphi_*(\sqrt{t}) - \varphi_*(x))\int_B \frac{e^{-|z|^2/4}}{4\pi}dz + O(\ell^{-1/2}),$$

where the remainder on B is estimated by

$$\varphi_*(\sqrt{t}) - \varphi_*(x) \le C\frac{-\log\sqrt{t}+\log r}{\ell^{1/2}} \le C\frac{\log \nu}{\ell^{1/2}},$$
$$\int_B e^{-|z|^2/4}dz \le e^{-\nu(1-\delta)^2/4}\nu \le Ce^{-\nu/8}.$$

Here, we have used (3.13) in the second inequality. This yields that

$$u_0 \le \varphi_*(x) + O(\ell^{-\frac{1}{2}}). \tag{3.25}$$

From (3.7), (3.19) and (3.25), we see that (3.2) holds. □

From Lemma 3.2, we have obtained, denoting $\rho := |\log r^2|$,

$$\begin{aligned}
\max(t,|x|^2) < \varepsilon^2 &\implies u_0 \le \min\left(\varphi_*(\sqrt{t}), \varphi_*(x)\right) + O(\ell^{-1/2})\\
&\implies u_0^2 \le \min\left(\ell - 2\log\ell, \rho - 2\log\rho\right) + O(1)\\
&\implies e^{u_0^2} \le C\min\left(\frac{1}{t\ell^2}, \frac{1}{r^2\rho^2}\right)\\
&\implies u_0 e^{u_0^2} \le C\min\left(\frac{1}{t\ell^{3/2}}, \frac{1}{r^2\rho^{3/2}}\right),\\
u_0^2 e^{u_0^2} &\le C\min\left(\frac{1}{t\ell}, \frac{1}{r^2\rho}\right).
\end{aligned} \tag{3.26}$$

One can use the radial monotonicity of u_0 (cf. Lemma A.1) to extend, to all $x \in \mathbb{R}^2$, the bounds of the functions $u_0e^{u_0^2}$ and $u_0^2e^{u_0^2}$ as follows:

$$\begin{aligned}
t < \varepsilon^2 \implies u_0 \le \sqrt{\ell},\quad & u_0e^{u_0^2} \le C[(t+r^2)^{-1} + \varepsilon^{-2}]|\log\min\{t+r^2, \varepsilon^2\}|^{-3/2} =: F_0,\\
& u_0^2e^{u_0^2} \le C[(t+r^2)^{-1} + \varepsilon^{-2}]|\log\min\{t+r^2, \varepsilon^2\}|^{-1} =: F_0'.
\end{aligned} \tag{3.27}$$

We shall show the bound of the function $u_0e^{u_0^2}$. From the monotonicity of u_0, it is enough to consider the region $0 < r < \varepsilon^2$ only.

Note that we have $\ell^{-3/2}, \rho^{-3/2} < |\log(t+r^2)|^{-3/2}$ for $\max(t, r^2) < \varepsilon^2$. Moreover, it follows that $\min(\frac{1}{t}, \frac{1}{r^2}) < 2(t+r^2)^{-1}$. These together with (3.26) yield that

$$u_0e^{u_0^2} \le C\min\left(\frac{1}{t}, \frac{1}{r^2}\right)|\log(t+r^2)|^{-3/2} \le C(t+r^2)^{-1}|\log(t+r^2)|^{-3/2}. \tag{3.28}$$

We first consider the region $t + r^2 < \varepsilon^2$. It follows from (3.28) that

$$\begin{aligned}
u_0e^{u_0^2} &\le C(t+r^2)^{-1}|\log\min\{t+r^2, \varepsilon^2\}|^{-3/2}\\
&\le C[(t+r^2)^{-1} + \varepsilon^{-2}]|\log\min\{t+r^2, \varepsilon^2\}|^{-3/2}.
\end{aligned} \tag{3.29}$$

Next, we consider the region $\varepsilon^2 \le t + r^2 < 2\varepsilon^2$. Since the function $s^{-1}|\log s|^{-3/2}$ is decreasing for sufficiently small $s > 0$, we have, by (3.28), that

$$\begin{aligned} u_0 e^{u_0^2} &\le C(t+r^2)^{-1}|\log(t+r^2)|^{-3/2} \le C\varepsilon^{-2}|\log \varepsilon^2|^{-3/2} \\ &\le C[(t+r^2)^{-1} + \varepsilon^{-2}]|\log \min\{t+r^2, \varepsilon^2\}|^{-3/2}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we see that $u_0 e^{u_0^2} \le CF_0$ for $t < \varepsilon^2$. We can obtain the bound of the function $u_0^2 e^{u_0^2}$ similarly. Thus, (3.27) holds.

Hence by the mean value theorem, for any functions v_0 and v_1, we have, for $t < \varepsilon^2$, that

$$\begin{aligned} \sqrt{\ell}(|v_0| + |v_1|) \le C \implies & |f_0(u_0+v_0) - f_0(u_0+v_1)| \\ &\le Cu_0^2 e^{u_0^2}|v_0 - v_1| \\ &\le CF_0'|v_0 - v_1|, \end{aligned} \tag{3.31}$$

where $f_0(u) = u(e^{u^2} - 1)$.

To estimate the second iteration, we prepare the following:

Lemma 3.3 *Let $\alpha > 0$ and $0 < \varepsilon < 1$. For any $(t, r) \in (0, \varepsilon^2) \times (0, \infty)$, there exists a positive constant C_* such that*

$$\int_0^t e^{(t-s)\Delta} \frac{1}{s+r^2} |\log \min\{s+r^2, \varepsilon^2\}|^{-\alpha} ds \le C_* \ell^{-\alpha}, \tag{3.32}$$

$$\int_0^t e^{(t-s)\Delta} |\log \min\{s+r^2, \varepsilon^2\}|^{-\alpha} ds \le C_* t \ell^{-\alpha} \tag{3.33}$$

Proof From Lemma A.1, we have

$$\begin{aligned} &\int_0^t e^{(t-s)\Delta} \frac{1}{s+r^2} |\log \min\{s+r^2, \varepsilon^2\}|^{-\alpha} ds \\ &\le \int_0^t \int_0^\infty \frac{re^{-\frac{r^2}{4(t-s)}}}{2(t-s)(s+r^2)} |\log \min\{s+r^2, \varepsilon^2\}|^{-\alpha} dr ds. \end{aligned}$$

Then, the integral is estimated using the following formula

$$\begin{aligned} &\int_0^t e^{(t-s)\Delta} \frac{1}{s+r^2} |\log \min\{s+r^2, \varepsilon^2\}|^{-\alpha} ds \\ &\le \int_0^t \int_0^\infty \frac{e^{-\frac{\sigma}{4s}}}{4s(t-s+\sigma)} |\log \min\{t-s+\sigma, \varepsilon^2\}|^{-\alpha} d\sigma ds \\ &= \int_0^\infty \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} |\log \min\{t-s+s\eta, \varepsilon^2\}|^{-\alpha} ds \frac{d\eta}{4} \end{aligned}$$

where the variables are changed by $s \to t-s$, $r^2 \to \sigma \to s\eta$. Then, we split the integral as follows.

$$\begin{aligned}
&\int_0^\infty \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} |\log \min\{t-s+s\eta, \varepsilon^2\}|^{-\alpha} ds \frac{d\eta}{4} \\
&= \int_0^{1/\sqrt{t}} \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} |\log \min\{t-s+s\eta, \varepsilon^2\}|^{-\alpha} ds \frac{d\eta}{4} \\
&\quad + \int_{1/\sqrt{t}}^\infty \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} |\log \min\{t-s+s\eta, \varepsilon^2\}|^{-\alpha} ds \frac{d\eta}{4} =: I + II.
\end{aligned}$$

We first estimate I. Since $t-s+s\eta \le \sqrt{t}$ for $0<s<t$ and $0<\eta<1/\sqrt{t}$, we have

$$\begin{aligned}
I &\le \int_0^{1/\sqrt{t}} \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} |\log \min\{\sqrt{t}, \varepsilon^2\}|^{-\alpha} ds \frac{d\eta}{4} \\
&\le C\ell^{-\alpha} \int_0^{1/\sqrt{t}} \int_0^t \frac{e^{-\eta/4}}{t-s+s\eta} ds d\eta = C\ell^{-\alpha} \int_0^{1/\sqrt{t}} e^{-\eta/4} \frac{\log \eta}{\eta - 1} d\eta \le C\ell^{-\alpha}.
\end{aligned} \tag{3.34}$$

Next, we estimate II. We note that $|\log \min\{t-s+s\eta, \varepsilon^2\}|^{-\alpha} \le (-\log \varepsilon^2)^{-\alpha}$ and $t-s+s\eta \ge t$ for $0<s<t$ and $\eta \ge 1/\sqrt{t}$. Therefore, we have

$$II \le C(-\log \varepsilon^2)^{-\alpha} \int_{1/\sqrt{t}}^\infty e^{-\eta/4} \int_0^t \frac{1}{t} ds d\eta \le C e^{-\frac{1}{4\sqrt{t}}}. \tag{3.35}$$

From (3.34) and (3.35), we obtain (3.32). We can obtain (3.33) by the similar argument above. This completes the proof. □

Using Lemma 3.3, we shall show the following:

Lemma 3.4 *Let $\varepsilon > 0$ be given by Lemma 3.3. For any space-time function v on $(0, \varepsilon^2) \times \mathbb{R}^2$, let*

$$D[v] := \int_0^t e^{(t-s)\Delta} f_0((u_0 + v)(s)) ds.$$

Then, there exists a positive constant C_0 such that

$$|D[0]| \le C_0 \ell^{-3/2}. \tag{3.36}$$

Moreover, for any $C_1 \in (0, \infty)$, there exists $C_2 \in (0, \infty)$ such that for any v_0 and v_1 satisfying

$$\sup_{0<t<\varepsilon} |\log t|^{1/2} \|v_j(t)\|_{L^\infty} \le C_1 \quad (j = 0, 1),$$

and for any $\alpha > -1$, we have

$$|D[v_0] - D[v_1]| \leq C_2 \ell^{-1-\alpha} \sup_{0<s<t} |\log s|^{\alpha} \|v_0(s) - v_1(s)\|_{L^\infty}. \tag{3.37}$$

Proof (3.27) and Lemma 3.3 yield

$$\begin{aligned}|D[0]| &\leq \int_0^t e^{(t-s)\Delta} |f_0(u_0(s))| ds \leq C \int_0^t e^{(t-s)\Delta} F_0(s) ds \\ &\leq C\ell^{-3/2} + \varepsilon^{-2} t \ell^{-3/2} \leq C\ell^{-3/2}.\end{aligned}$$

Similarly, (3.31) and Lemma 3.3 yield

$$\begin{aligned}|D[v_0] - D[v_1]| &\leq \int_0^t e^{(t-s)\Delta} |f_0(u_0 + v_0)) - f_0(u_0 + v_1)| ds \\ &\leq C \int_0^t e^{(t-s)\Delta} F_0'(s) \|v_1(s) - v_0(s)\|_\infty ds \\ &\leq C \int_0^t e^{(t-s)\Delta} [(s + r^2)^{-1} + \varepsilon^{-2}] |\log \min\{s + r^2, \varepsilon^2\}|^{-1-\alpha} ds \\ &\quad \times \sup_{0<s<t} |\log s|^{\alpha} \|v_0(s) - v_1(s)\|_{L^\infty} \\ &\leq C\ell^{-1-\alpha} \sup_{0<s<t} |\log s|^{\alpha} \|v_0(s) - v_1(s)\|_{L^\infty}\end{aligned}$$

for any $\alpha > -1$. Noting that the constants are independent of v_0, v_1, α, t, we arrive at the desired conclusion. □

We are now in a position to prove Theorem 3.1

Proof of Theorem 3.1 We put

$$E[v] := \int_0^t e^{(t-s)\Delta} L(u_0 + v)(s) ds, \qquad L(u) := m_* \chi(u) u$$

and $I[v] := D[v] - E[v]$. Then, we are naturally led to consider the mapping $v \mapsto I[v]$ for v in the following set

$$B_T^{1/2} := \{v \in C([0, T] \times \mathbb{R}^2) \mid \|v\|_{X_T^{1/2}} := \sup_{0<t<T} \ell^{1/2} \|v(t)\|_{L^\infty} \leq 1\}$$

for some constant $T \in (0, \varepsilon^2)$ to be determined, which is a closed ball of a Banach space with the norm $X_T^{1/2}$. The estimates on $D[\cdot]$ in (3.36) and (3.37) with $C_1 = 1$ imply

$$\begin{aligned}&\|D[v_0] - D[v_1]\|_{X_T^{1/2}} \leq C_2 |\log T|^{-1} \|v_0 - v_1\|_{X_T^{1/2}}, \\ &\|D[v_0]\|_{X_T^{1/2}} \leq \|D[0]\|_{X_T^{1/2}} + \|D[0] - D[v_0]\|_{X_T^{1/2}} \leq (C_0 + C_2) |\log T|^{-1}\end{aligned}$$

for any $v_0, v_1 \in B_T^{1/2}$. Moreover, we can easily obtain

$$\|E[v]\|_{L^\infty} \leq t\|L\|_{L^\infty},$$
$$\left\| \int_0^t e^{(t-s)\Delta}(L(v_0) - L(v_1))(s)ds \right\|_{L^\infty} \leq t\|L\|_{Lip}\|v_0 - v_1\|_{L^\infty}.$$

Hence if $T > 0$ is small enough then $v \mapsto I[v]$ is a contraction mapping on $B_T^{1/2}$, so there is a unique fixed point $v \in B_T^{1/2}$. Then $u = u_0 + v$ is a local (mild) solution on $0 < t < T$ of

$$\dot{u} - \Delta u + m_* \chi(u)u = f_0(u), \quad u(0) = \varphi_*.$$

This completes the proof. □

We can prove Theorem 1.1 immediately from Theorems 2.8 and 3.1 .

Proof of Theorem 1.1 We also denote ϕ_{m_*}, which is the stationary singular soliton to (1.5) obtained in Theorem 2.8, by φ_*. Let $u_S(t) = \varphi_*$ and $u_R(t)$ be the regular solution to (1.5), obtained in Theorem 3.1. We see that $u_S(0) = u_R(0) = \varphi_*$.

We shall show that u_S belongs to $C([0, T), X)$ and becomes a mild solution for each $T > 0$. As we mentioned in the proof of Theorem 2.8, φ_* satisfies (2.23) in a distributional sense and $f_{m_*}(\varphi_*) \in L^1(\mathbb{R}^2)$. It follows from the $L^p - L^q$ estimate of the heat kernel (see e.g. [19, Proposition 1]) that for $1 \leq p < \infty$, we obtain

$$\left\| \int_0^t e^{\Delta(t-s)} f_{m_*}(\varphi_*)ds \right\|_{L^p} \leq \int_0^t \|e^{\Delta(t-s)} f_{m_*}(\varphi_*)\|_{L^p} ds$$
$$\leq C \int_0^t (t-s)^{-(1-\frac{1}{p})} \|f_{m_*}(\varphi_*)\|_{L^1} ds \leq C\|f_{m_*}(\varphi_*)\|_{L^1} < \infty.$$

Thus $u_S(t) = \varphi_* \in C([0, T), X)$ satisfies the Duhamel formula:

$$u(t) = e^{t\Delta}\varphi_* + \int_0^t e^{(t-s)\Delta} f_{m_*}(u(s))ds \quad \text{in } C([0, T), X). \tag{3.38}$$

Next, we shall show that u_R belongs to $C([0, T), X)$ for sufficiently small $T > 0$ and becomes a mild solution. The fixed point $v \in B_T^{1/2}$ of I can be obtained as the limit of $v_n = I[v_{n-1}]$ starting from $v_0 = 0$. Then for $T > 0$ small enough, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} |\log t|^{1/2}\|v_n(t)\|_{L^\infty} \leq M$ and $\lim_{n\to\infty} v_n(t) = v(t)$ in $L^\infty(\mathbb{R}^2)$ for $0 < t < T$. Let $u_n = u_0 + v_n$. From the estimate (3.26) on u_0 together with Lemma A.1 and that $\|v_n(t)\|_{L^\infty} \leq M\ell^{-1/2}$, we obtain

$$|u_n|^2 \leq (|u_0| + |v_n|)^2 \leq \ell - \frac{3}{2}\log \ell$$

uniformly for $0 < t < T \ll 1$, $x \in \mathbb{R}^2$ and $n \in \mathbb{N}$. Hence, there exists a constant $C_{m_*} > 0$

$$|f_{m_*}(u_n)| \le C_{m_*}|u_n|e^{u_n^2} \le C_{m_*}t^{-1}\ell^{-3/2}|u_n|, \tag{3.39}$$

where f_m is defined by (2.17). Suppose that $\sup_{0<s<T} \|u_n(s)\|_{L^p} < \infty$ for any $p \in [1, \infty)$. Using (3.39) in the Duhamel formula

$$u_{n+1}(t) = u_0(t) + v_{n+1}(t) = u_0(t) + \int_0^t e^{(t-s)\Delta} f_{m_*}(u_n(s))ds,$$

we obtain

$$\begin{aligned} \|u_{n+1}(t)\|_{L^p} &\le \|\varphi_*\|_{L^p} + \int_0^t \|f_{m_*}(u_n(s))\|_{L^p}ds \\ &\le \|\varphi_*\|_{L^p} + C_{m_*}\int_0^t s^{-1}|\log s|^{-3/2}\|u_n(s)\|_{L^p}ds \\ &\le \|\varphi_*\|_{L^p} + 2C_{m_*}|\log T|^{-1/2} \sup_{0<s<T} \|u_n(s)\|_{L^p} \end{aligned} \tag{3.40}$$

for $0 < t < T \ll 1$. Here, we have used the L^p contraction estimate of the heat kernel (see e.g. [19, Proposition 1]) in the first inequality. Thus, we obtain by the induction argument

$$\sup_{0<s<T} \|u_n(s)\|_{L^p} \le \|\varphi_*\|_{L^p}(1 + 3C_{m_*}|\log T|^{-1/2}) \tag{3.41}$$

for $n \in \mathbb{N}$. Since, for $0 < t < T \ll 1$,

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty}(u_0 + v_n(t)) = u_0 + v = u_R(t) \qquad \text{in } L^\infty(\mathbb{R}^2),$$

the above uniform bound (3.41) together with Fatou's lemma implies that,

$$\|u_R(t)\|_{L^p} \le \liminf_{n\to\infty} \|u_n(t)\|_{L^p} \le \|\varphi_*\|_{L^p}(1 + 3C_{m_*}|\log T|^{-1/2})$$

for $0 < t < T \ll 1$. In addition, the same estimate as in (3.40) implies that $f_{m_*}(u_R) \in L^1([0, T), L^p(\mathbb{R}^2))$ and by the Duhamel formula, $u_R \in C([0, T), L^p(\mathbb{R}^2))$. Thus, we conclude that $u_R \in C([0, T), L^p(\mathbb{R}^2))$ for all $p \in [1, \infty)$. That is $u_R \in C([0, T), X)$. Then, we see that u_R also satisfies (3.38).

For all $t \in (0, T)$, we have $u_S(t) = \varphi_* \notin L^\infty(\mathbb{R}^2)$ while $u_R(t) \in L^\infty(\mathbb{R}^2)$. This implies that $u_S(t) \neq u_R(t)$ for all $t \in (0, T)$. This completes the proof. □

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Appendix A: Maximum point of solutions to the linear Heat equation

In this appendix, we shall give a proof of the fact, which is used in Sect. 3. More precisely, we show the following:

Lemma A.1 *Let ϕ be a radially decreasing function. Set $u(t) = e^{t\Delta}\phi$. Then, $u(t)$ is also radially decreasing and*

$$\|u(t, \cdot)\|_{L_x^\infty} = u(t, 0).$$

Proof Note that u is also radial. Setting $v = \partial_r u$, we see that v satisfies the following:

$$\dot{v} - \Delta v + \frac{1}{r^2} v = 0. \tag{A.1}$$

We put $v_+ = \max\{v, 0\}$. Multiplying (A.1) by v_+ and integrating the resulting equation over $\mathbb{R}^2$, we have

$$\partial_t \|v_+\|_{L^2}^2 = -\|\nabla v_+\|_{L^2}^2 - \int \frac{1}{r^2} |v_+|^2 dx \leq 0. \tag{A.2}$$

From the assumption, we infer that $v_+(0) = 0$. This together with (A.2) yields that $v_+ \equiv 0$ in $(0, \infty) \times \mathbb{R}^2$. This completes the proof. □

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