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V-shaped fronts around an obstacle

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Abstract
In this paper, we investigate V-shaped fronts around an obstacle K. We first prove that there exist solutions emanating from any homogeneous transition front including V-shaped front for exterior domains $\Omega = \mathbb{R}^N \backslash K$. By providing the complete propagation of the V-shaped front, we prove that the V-shaped front can recover after passing the obstacle.

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1 Introduction

This paper is concerned with the following reaction-diffusion equation in exterior domains

$$\begin{cases} u_t = \Delta u + f(u), & t \in \mathbb{R},\ x \in \Omega = \mathbb{R}^N \backslash K \subset \mathbb{R}^N, \\ u_\nu = 0, & \text{on } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where the obstacle K is a compact set of $\mathbb{R}^N$ which is the closure of an open set with smooth boundary and Ω is an exterior domain. Here, $\nu = \nu(x)$ is the outward unit normal on the boundary $\partial\Omega$ and $u_\nu = \frac{\partial u}{\partial \nu}$. On the boundary $\partial\Omega$, the homogeneous Neumann boundary condition is imposed.

Throughout of this paper, the reaction term f is assumed to be of bistable type, namely $u = 0$ and $u = 1$ are both stable stationary states. More precisely, we assume

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that f is of $C^1([0,1],\mathbb{R})$ and satisfies

$$f(0)=f(1)=0,\quad f'(0)<0 \quad\text{and}\quad f'(1)<0. \tag{1.2}$$

For mathematical purposes, the function f is extended in $\mathbb{R}$ as a $C^1(\mathbb{R})$ function such that

$$f(s)=f'(0)s>0 \quad \text{for } s\in(-\infty,0)$$

and

$$f(s)=f'(1)(s-1)<0 \quad \text{for } s\in(1,+\infty).$$

A typical example is the cubic nonlinearity $f(s)=s(1-s)(s-\theta)$ with $0<\theta<1$. Notice that the existence of V-shaped front requires the bistable reaction term f being unbalanced. Thus, we assume additionally that

$$\int_0^1 f(s)ds>0, \tag{1.3}$$

(for $\int_0^1 f(s)ds<0$, one can only reverse the roles of 0 and 1). For the balanced case $\int_0^1 f(s)ds=0$, no more V-shaped fronts exist, see [14]. Instead, some fronts with their level sets being exponential shape ($N=2$) or parabolic shape ($N\geq 3$) may exist, see [8].

Since we consider the propagation of homogeneous transition fronts and V-shaped fronts, we assume throughout this paper that if $\Omega=\mathbb{R}$, it admits a unique traveling front $u(t,x)=\phi(x-c_f t)$ such that

$$\begin{cases}\phi''+c_f\phi'+f(\phi)=0 & \text{in } \mathbb{R},\\ \phi(-\infty)=1 \quad\text{and}\quad \phi(+\infty)=0.\end{cases} \tag{1.4}$$

It follows from [9] that the propagation speed c_f is only determined by f and has the sign of $\int_0^1 f(s)ds$. As we consider in this paper, $c_f>0$ by (1.3). We also point out that the existence and nonexistence of traveling fronts relies on conditions of the bistable nonlinearity, see [9].

The first aim of this paper is to prove the existence of entire solutions emanating from any homogeneous transition front, that is, Theorem 1.2. Therefore, we first recall some results in homogeneous case. For the following reaction-diffusion equation in $\mathbb{R}^N$,

$$u_t=\Delta u+f(u),\quad t\in\mathbb{R},\ x\in\mathbb{R}^N, \tag{1.5}$$

it is well known that there are various kinds of entire solutions. The simplest example is the planar front $u(t,x)=\phi(x\cdot e-c_f t)$ where e is a unit vector of $\mathbb{R}^N$. For the existence of planar fronts, one can refer to the existence of one-dimensional traveling fronts. Note that the level sets of a planar front are hyperplanes and a planar front propagates with invariant level sets. More types of non-planar fronts are known to exist in $\mathbb{R}^N$, such as V-shaped fronts, conical shaped fronts, pyramidal fronts and even nonstandard fronts which have no invariant level sets. For the existence, uniqueness, stability and other qualitative properties of these non-planar traveling fronts, we refer to [6,7,14–16,18,19,21–24] and the references therein.

For these types of traveling fronts, their common features, such as they converge to the stable states 0 or 1 far away from their moving or stationary level sets, uniformly in time, led to the introduction of a more general notion of traveling fronts, that is, transition fronts, see [3,4,13] and see [20] in the one-dimensional setting. We here recall the notion of transition fronts for (1.5). First, for any two subsets A and B of $\mathbb{R}^N$ and for $x \in \mathbb{R}^N$, we set

$$d(A, B) = \inf\left\{|y - z|;\ (y, z) \in A \times B\right\}$$

and $d(x, A) = d(\{x\}, A)$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. Consider now two families $\{\Omega_t^-\}_{t\in\mathbb{R}}$ and $\{\Omega_t^+\}_{t\in\mathbb{R}}$ composed of open nonempty subsets of $\mathbb{R}^N$ such that, for any $t \in \mathbb{R}$, Ω_t^+ and Ω_t^- satisfy

$$\begin{cases} \Omega_t^- \cap \Omega_t^+ = \emptyset, \\ \partial\Omega_t^- = \partial\Omega_t^+ =: \Gamma_t, \\ \Omega_t^- \cup \Gamma_t \cup \Omega_t^+ = \mathbb{R}^N, \\ \sup\{d(x, \Gamma_t);\quad x \in \Omega_t^+\} = \sup\{d(x, \Gamma_t);\quad x \in \Omega_t^-\} = +\infty \end{cases} \tag{1.6}$$

and as $r \to +\infty$,

$$\begin{cases} \inf\left\{\sup\left\{d(y, \Gamma_t);\ y \in \Omega_t^+,\ |y - x| \le r\right\};\ t \in \mathbb{R},\ \ x \in \Gamma_t\right\} \to +\infty \\ \inf\left\{\sup\left\{d(y, \Gamma_t);\ y \in \Omega_t^-,\ |y - x| \le r\right\};\ t \in \mathbb{R},\ \ x \in \Gamma_t\right\} \to +\infty. \end{cases} \tag{1.7}$$

Notice that the condition (1.6) implies in particular that the interface Γ_t is not empty for every $t \in \mathbb{R}$. As far as (1.7) is concerned, it says that for any $M > 0$, there is $r_M > 0$ such that for any $t \in \mathbb{R}$ and $x \in \Gamma_t$, there are $y^\pm = y^\pm_{t,x} \in \mathbb{R}^N$ such that

$$y^\pm \in \Omega_t^\pm,\quad |x - y^\pm| \le r_M \quad \text{and} \quad d(y^\pm, \Gamma_t) \ge M, \tag{1.8}$$

that is, $y^\pm \in \overline{B(x, r_M)}$ and $B(y^\pm, M) \subset \Omega_t^\pm$, where $B(y, r)$ denotes the open Euclidean ball of center y and radius $r > 0$. In other words, not too far from any point $x \in \Gamma_t$, the sets $\Omega_t^\pm$ contain large balls. Moreover, the sets Γ_t are assumed to be made of a finite number of graphs: there is an integer $n \ge 1$ such that, for each

$t \in \mathbb{R}$, there are n open subsets $\omega_{i,t} \subset \mathbb{R}^{N-1}$(for $1 \le i \le n$), n continuous maps $\psi_{i,t} : \omega_{i,t} \to \mathbb{R}$ and n rotations $R_{i,t}$ of $\mathbb{R}^N$, such that

$$\Gamma_t \subset \bigcup_{1\le i\le n} R_{i,t}\left(\{x \in \mathbb{R}^N; \quad x' \in \omega_{i,t}, \ \ x_N = \psi_{i,t}(x')\}\right). \tag{1.9}$$

Definition 1.1 ([3,4]) We call $u(t, x)$ a transition front connecting 0 and 1 of (1.5), or simply "transition front", if $u(x, t)$ is a classical solution of (1.5) and there exist some sets $\{\Omega_t^{\pm}\}_{t\in\mathbb{R}}$, $\{\Gamma_t\}_{t\in\mathbb{R}}$ satisfying (1.6), (1.7) and (1.9) such that for any $\varepsilon > 0$, there is a positive constant M_ε satisfying

$$\begin{cases} d(x, \Gamma_t) \ge M_\varepsilon & \text{for } (t, x) \in \mathbb{R} \times \Omega_t^+ \Rightarrow u(t, x) \ge 1 - \varepsilon, \\ d(x, \Gamma_t) \ge M_\varepsilon & \text{for } (t, x) \in \mathbb{R} \times \Omega_t^- \Rightarrow u(t, x) \le \varepsilon. \end{cases} \tag{1.10}$$

Furthermore, u is said to have a global mean speed γ (≥ 0) if

$$\frac{d(\Gamma_t, \Gamma_s)}{|t - s|} \to \gamma \quad \text{as} \ \ |t - s| \to +\infty.$$

It has been proved by [13] that any transition front of (1.5) has a global mean speed which is equal to $c_f > 0$ (by (1.3)), the propagation speed of one-dimensional traveling front.

From the paper of Berestycki, Hamel and Matano [5], they proved the existence of entire solution $u(t, x)$ of (1.1) emanating from a planar front, that is, $u(t, x)$ satisfies

$$u(t, x) \to \phi(x \cdot e - c_f t) \text{ as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

Inspired by [5], we prove that (1.1) admits entire solutions emanating from any transition front of (1.5) defined by Definition 1.1.

Theorem 1.2 *For any transition front $U(t, x)$ solving (1.5), there exists an entire solution $u(t, x)$ of (1.1) such that*

$$u(t, x) \to U(t, x), \quad \textit{as } t \to -\infty \textit{ uniformly in } \overline{\Omega}.$$

Furthermore, $u_t(t, x) > 0$ for $t \in \mathbb{R}$ and $x \in \overline{\Omega}$.

Remark 1.3 One can notice from [5] that (1.1) admits entire solutions emanating from planar fronts even if the obstacle K is unbounded but lying in a half space. However, this can not be true for general transition fronts since the $1/2$ level set of a transition front may always cross with the unbounded obstacle as $t \to -\infty$.

Now, we consider the interaction between a transition front and the obstacle K. From [5], one knows that a planar front coming from somewhere far away from the obstacle can recover to the same planar front under some suitable geometrical conditions on the obstacle K, such as K is star-shaped[1] or directionally convex with respect to some hyperplane.[2] It implies that the perturbation caused by the obstacle will fade out finally. It also implies that the propagation of the entire solution $u(t, x)$ emanating from a planar front is complete in the sense that

$$u(t, x) \to 1, \quad \text{as } t \to +\infty \text{ locally uniformly in } \overline{\Omega}. \tag{1.11}$$

Here we mean the complete propagation of an entire solution $u(t, x)$ or that an entire solution $u(t, x)$ is a complete invasion by (1.11). Another interesting phenomenon in [5] is the blocking phenomenon, that is, the solution $u(t, x)$ might be blocked when the obstacle K contains a small channel, like the neck of a hourglass, in the sense that

$$u(t, x) < 1 \quad \text{for some points } x \quad \text{and any } t \in \mathbb{R}. \tag{1.12}$$

In other words, the perturbation caused by the obstacle remains forever. Such blocking phenomenon has also been studied in [2] for cylinderical domains.

In fact, the above phenomena also hold for more general entire solutions, that is, both phenomena of complete propagation and blocking can occur for the entire solution $u(t, x)$ emanating from not only a planar front but also any homogeneous transition front such as a V-shaped front, depending on the shape of the obstacle K. By applying the arguments used in Step 1 of the proof of [12, Lemma 2.6], there exists a $C^2(\overline{\Omega})$ solution $p : \overline{\Omega} \to (0, 1]$ of

$$\Delta p + f(p) = 0 \text{ in } \Omega, \quad p_\nu = 0 \quad \text{on } \partial\Omega, \quad \text{and} \quad p(x) \to 1 \quad \text{as } |x| \to +\infty, \tag{1.13}$$

such that the entire solution u of (1.1) emanating from any homogeneous transition satisfies

$$\liminf_{t \to +\infty} u(t, x) \geq p(x) > 0 \quad \text{locally uniformly in } x \in \overline{\Omega}. \tag{1.14}$$

It follows from [5, Theorems 6.1 and 6.4] that, if the compact obstacle K is either star-shaped or directionally convex with respect to some hyperplane, then any solution $p : \overline{\Omega} \to [0, 1]$ of (1.13) is identically equal to 1. By (1.14), it means that the propagation of $u(t, x)$ is complete, that is, satisfying (1.11). Besides, a dilated domain $R\Omega_0 = \mathbb{R}^N \backslash (RK_0)$ for large constants R and smooth bounded closed sets K_0 of $\mathbb{R}^N$, can also ensure the complete propagation, refer to [12, Corollary 1.12]. For the blocking phenomenon, the example made in Section 6.3 of [5], where the obstacle K contains a small channel whose width is controlled by a small constant ε, still works

[1] The obstacle K is called star-shaped if either $K = \emptyset$ or there is x in the interior $\mathrm{Int}(K)$ of K such that $x + t(y - x) \in \mathrm{Int}(K)$ for all $y \in \partial K$ and $t \in [0, 1)$.

[2] The obstacle K is called directionally convex with respect to a hyperplane $H = \{x \in \mathbb{R}^N : x \cdot e = a\}$, with $e \in \mathbb{S}^{N-1}$ and $a \in \mathbb{R}$, if for every line Σ parallel to e, the set $K \cap \Sigma$ is either a single line segment or empty and if $K \cap H$ is equal to the orthogonal projection of K onto H.

here. The authors of [5] proved that for any R such that $B(0, R) \supset K$ and small enough ε, the following problem has a solution $\omega \not\equiv 1$

$$\Delta\omega + f(\omega) = 0 \quad \text{in } B(0, R)\backslash K, \quad \omega_\nu = 0 \quad \text{on } \partial K, \quad \text{and} \quad \omega = 1 \quad \text{on } \partial B(0, R).$$

One can easily notice that the function ω extended by 1 outside $B(0, R)$ is actually a supersolution for the entire solution $u(t, x)$. It implies that the propagation of $u(t, x)$ is blocked in the sense of (1.12).

What we are interested in this paper is, for more general situation than planar fronts, whether a transition front coming from somewhere far away from the obstacle can recover to the same transition front provided by the complete propagation of the front (which avoids the blocking). We conjecture that the answer is positive. However, we can not prove this yet. From the arguments in [5], we believe that the global stability of transition front is the key to solve this problem. Nevertheless, the global stability of transition front in general settings is still open. Thus, in this paper, we consider a special nonplanar case, namely, the V-shaped front, to give a positive answer to it.

Before we state our main result, we need to recall some existence results of V-shaped fronts of (1.5). For convenience, we only consider $N = 2$. The result can be extended to high dimensions $N \geq 3$ trivially. We denote points in $\mathbb{R}^2$ by (x_1, x_2). It is known from [15,16,18] that the existence of one-dimensional traveling fronts with nonzero speed guarantees the existence of V-shaped fronts. Without loss of generality, we assume that the V-shaped front propagates towards x_2-direction with speed c denoted by $u(t, x_1, x_2) = V(y, \xi)$ with $y = x_1$ and $\xi = x_2 - ct$. The results of [15,16,18] say that there exists a unique (up to shifts) V-shaped front $V(x_1, x_2 - ct)$ of (1.5) with asymptotic lines

$$x_2 = m_*|x_1| \text{ where } m_* = \frac{\sqrt{c^2 - c_f^2}}{c_f},$$

satisfying

$$-V_{yy} - V_{\xi\xi} - cV_\xi - f(V) = 0 \text{ in } (y, \xi) \in \mathbb{R}^2,$$

where $V_\xi = \partial V/\partial\xi$, $V_{\xi\xi} = \partial^2 V/\partial\xi^2$ and $V_{yy} = \partial^2 V/\partial y^2$.

Furthermore, the V-shaped front $V(x_1, x_2 - ct)$ is known to be asymptotically planar along its asymptotic lines, that is,

$$\lim_{R\to+\infty} \sup_{x_1^2+(x_2-ct)^2>R^2} \left| V(x_1, x_2 - ct) - \phi\left(\frac{c_f}{c}(x_2 - ct - m_*|x_1|)\right)\right| = 0. \quad (1.15)$$

We now state the main result.

Theorem 1.4 *Assume that $u(t, x)$ is an entire solution of (1.1) emanating from a V-shaped front, that is, $u(t, x)$ satisfies*

$$u(t, x) \to V(x_1, x_2 - ct) \quad \text{as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

If $u(t, x)$ is a complete invasion satisfying (1.11)*, then*

$$u(t, x) \to V(x_1, x_2 - ct) \quad \text{as } t \to +\infty \text{ uniformly in } \overline{\Omega}.$$

Remark 1.5 From the above discussion, the entire solution u emanating from a V-shaped front is a complete invasion as the obstacle K is star-shaped or directionally convex with respect to some hyperplane or dilated by $K = RK_0$ for a large constant R and a smooth bounded closed set K_0 of $\mathbb{R}^N$. Thus, the assumption of Theorem 1.4 is not empty. Moreover, from Theorem 1.4, we know that the entire solution emanating from a V-shaped front will recover to the same V-shaped front in such domains. One can easily check that the entire solution $u(t, x)$ in Theorem 1.4 is a transition front connecting 0 and 1 in exterior domains.

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2 and Sect. 3 is devoted to the proof of Theorem 1.4.

2 Entire solutions emanating from transition fronts

In this section, we prove the existence of entire solutions emanating from any homogeneous transition front. In order to follow the idea of [5], we need to prove some additional properties of transition fronts and by which, we can construct supersolutions and subsolutions.

2.1 Properties of homogeneous transition fronts

In this section, we study some properties of general transition fronts of the homogeneous equation (1.5).

Lemma 2.1 *Let $U(t, x)$ be a transition front of* (1.5)*. For any point $x_0 \in \mathbb{R}^N$ and any $R > 0$, there are constants $T_1 < 0$, $\alpha > 0$, $\beta > 0$ and $\eta > 0$ such that it holds that*

$$U(t, x) \le \alpha e^{\eta t}, \quad \text{for } t \le T_1 \text{ and } x \in B(x_0, R),$$

and

$$|\nabla U(t, x)| \le \beta e^{\eta t}, \quad \text{for } t \le T_1 \text{ and } x \in B(x_0, R).$$

Proof Without loss of generality, we assume $x_0 = 0$. Otherwise, one can shift $U(t, x)$ by $\widetilde{U}(t, x) = U(t, x + x_0)$. To obtain our claim, we here make a supersolution of $U(x, t)$ by using the traveling front ϕ of (1.4).

Step 1: Choice of some parameters By (1.2), there is $\sigma > 0$ such that $f(s)$ is nonincreasing in $(-\infty, \sigma]$ and

$$f(s) \le 0 \;\text{ for } s \in [0, \sigma].$$

Since $\lim_{\xi\to+\infty}\phi(\xi)=0$, there is $C>0$ such that

$$\phi(\xi)\le\sigma \quad \text{for } \xi\ge C. \tag{2.1}$$

One can notice that the function ϕ is of class C^3 and ϕ' satisfies

$$(\phi')''+c_f(\phi')'+f'(\phi)\phi'=0,$$

and $\phi'<0$ in $\mathbb{R}$ from [9]. Since $f'(s)$ is bounded, it follows from standard interior estimates and Harnack inequality that the function ϕ''/ϕ' is bounded. Namely, there is $C_1>0$ such that

$$|\phi''(\xi)|\le C_1|\phi'(\xi)| \ \text{ for all } \xi\in\mathbb{R}. \tag{2.2}$$

Take $\mu>0$ such that

$$\sqrt{2\mu C_1}\le\frac{c_f}{2}. \tag{2.3}$$

It is elementary to check that there is a C^2 function $h:[0,+\infty]\to\mathbb{R}$ satisfying the following properties:

$$\begin{cases} 0\le h'\le\sqrt{\frac{\mu}{2C_1}} \ \text{ on } [0,+\infty),\\ h'=0 \ \text{ on a neighborhood of } 0,\\ h(0)>0 \ \text{ and } \ h(r)=\sqrt{\frac{\mu}{2C_1}}r \ \text{ on } [H,+\infty) \quad \text{for some } H>0,\\ \frac{(N-1)h'(r)}{r}+h''(r)\le\frac{\mu}{2} \quad \text{on } [0,+\infty). \end{cases} \tag{2.4}$$

Notice in particular that

$$\sqrt{\frac{\mu}{2C_1}}r\le h(r)\le\sqrt{\frac{\mu}{2C_1}}r+h(0) \quad \text{for all } r\ge 0. \tag{2.5}$$

Step 2: Construction of a supersolution For $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$, we set

$$\overline{u}(t,x)=\phi(\xi(t,x)),$$

where

$$\xi(t,x)=-h(|x|)-\mu t+C+\sqrt{\frac{\mu}{2C_1}}R+h(0).$$

Let

$$E=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^N;\ t\le 0,\ |x|\le-\sqrt{2\mu C_1}t+R\right\}, \tag{2.6}$$

and (t, x) be any point in E. By (2.5), one can get that

$$\xi(t,x) \geq -\sqrt{\frac{\mu}{2C_1}}|x| - h(0) - \mu t + C + \sqrt{\frac{\mu}{2C_1}}R + h(0) \geq C,$$

and hence (2.1) leads to

$$\overline{u}(t,x) \leq \sigma \quad \text{in } E.$$

Thus, $f(\overline{u}(t,x)) \leq 0$ in E. Let us check that $\mathcal{L}\overline{u} := \overline{u}_t - \Delta\overline{u} - f(\overline{u}) \geq 0$ for $(t,x) \in E$. One can easily compute that

$$\begin{aligned}\mathcal{L}\overline{u} = & -\mu\phi'(\xi(t,x)) - \phi''(\xi(t,x))h'^2(|x|) \\ & + \phi'(\xi(t,x))\left(h''(|x|) + \frac{N-1}{|x|}h'(|x|)\right) - f(\overline{u}).\end{aligned}$$

Since $\phi' < 0$ in $\mathbb{R}$ and by (2.2), (2.4), it follows that

$$\mathcal{L}\overline{u} \geq -\mu\phi'(\xi(t,x)) + \frac{\mu}{2}\phi'(\xi(t,x)) + \frac{\mu}{2}\phi'(\xi(t,x)) - f(\overline{u}(t,x)) \geq 0 \quad \text{in } E.$$

Thus $\overline{u}$ is a supersolution of (1.5) in E.

Step 3: Exponentially approaching to 0 Notice that

$$\xi(t,x) \leq -\sqrt{\frac{\mu}{2C_1}}|x| - \mu t + C + \sqrt{\frac{\mu}{2C_1}}R + h(0) = C + h(0)$$

on $\partial E := \{(t,x) \in \mathbb{R} \times \mathbb{R}^N;\ t \leq 0,\ |x| = -\sqrt{2\mu C_1}t + R\}$. Since $\phi' < 0$ in $\mathbb{R}$, one has that

$$\overline{u}(t,x) = \phi(\xi(t,x)) \geq \phi(C + h(0)) \quad \text{on } \partial E.$$

Since $U(t,x) \to 0$ as $t \to -\infty$ locally uniformly for $x \in \mathbb{R}^N$ and $\overline{u}(0,x) > 0$ for $x \in B(0,R)$, there is $T_1 < 0$ such that

$$U(T_1, x) \leq \min\{\overline{u}(0,x), \phi(C + h(0))\}, \quad \text{for all } x \in B(0,R).$$

Since the global mean speed of $U(t,x)$ is c_f, one can decrease T_1 such that

$$U(t,x) \leq \phi(C + h(0)), \quad \text{for } t \leq T_1 \quad \text{and} \quad |x| \leq R - \frac{c_f}{2}(t - T_1).$$

By (2.3), it implies that

$$U(t,x) \leq \phi(C + h(0)) \leq \sigma, \quad \text{for } t \leq T_1 \quad \text{and} \quad |x| \leq R - \sqrt{2\mu C_1}(t - T_1).$$

Thus, $U(t + T_1, x) \leq \sigma$ in E and $U(t + T_1, x) \leq \overline{u}(t,x)$ on ∂E.

Now, define

$$\varepsilon^* = \inf\{\varepsilon > 0;\ U(t+T_1,x) - \varepsilon \le \overline{u}(t,x) \text{ in } E\}.$$

Assume that $\varepsilon^* > 0$. Then, there are sequences $\varepsilon_n \ge \varepsilon^*$ and $(t_n, x_n) \in E$ such that $\varepsilon_n \to \varepsilon^*$ and

$$\|\overline{u}(t_n,x_n) - U(t_n+T_1,x_n) + \varepsilon_n\|_{L^\infty(E)} \to 0, \quad \text{as } n \to +\infty. \tag{2.7}$$

Since $\overline{u}(t_n,x_n) + \varepsilon_n \ge \varepsilon^*$ and $U(t,x) \to 0$ as $t \to -\infty$ uniformly for $x \in E$, it implies that there is $-\infty < t^* \le 0$ such that $t_n \to t^*$ and hence, there is $x^* \in E$ such that $x_n \to x^*$. Thus, by (2.7), one has that

$$\overline{u}(t^*,x^*) - U(t^*+T_1,x^*) + \varepsilon^* = 0.$$

Since $U(t+T_1,x) \le \sigma$ in E and $f(s)$ is nonincreasing in $(-\infty,\sigma]$, one has that

$$(U(t+T_1,x) - \varepsilon^*)_t - \Delta(U(t+T_1,x) - \varepsilon^*) - f(U(t+T_1,x) - \varepsilon^*) \le 0, \ \text{ in } E.$$

Let $z(t,x) = \overline{u}(t,x) - U(t+T_1,x) + \varepsilon^*$. Then, $z(t,x) \ge 0$ in E, $z(t,x) > 0$ on ∂E and $z(t^*,x^*) = 0$. Since $\overline{u}(t,x)$ is a supersolution, one gets that $z_t - \Delta z + b(t,x)z \ge 0$ in E where $b(t,x)$ is a bounded function. Then, by the maximum principle, one gets that $z(t,x) \equiv 0$ in E which contradicts $\overline{u}(t,x) > U(t+T_1,x) - \varepsilon^*$ on ∂E. Therefore, $\varepsilon^* = 0$.

As a consequence, it follows that

$$U(t+T_1,x) \le \overline{u}(t,x) = \phi(\xi(t,x)), \quad \text{in } E.$$

For $x \in B(0,R)$, one has that

$$\xi(t,x) \ge -\sqrt{\frac{\mu}{2C_1}}R - h(0) - \mu t + C + \sqrt{\frac{\mu}{2C_1}}R + h(0) = -\mu t + C \ge 0.$$

By [9], there are positive constants a_0 and λ such that

$$U(t+T_1,x) \le \phi(\xi(t,x)) \le a_0 e^{\lambda\mu t - \lambda C}, \quad \text{for } t \le 0 \quad \text{and} \quad x \in B(0,R),$$

that is, $U(t,x) \le a_0 e^{\lambda\mu(t-T_1) - \lambda C}$ for $t \le T_1$ and $x \in B(0,R)$. By standard interior estimates, there is $a_1 > 0$ such that

$$|\nabla U(t,x)| \le a_1 e^{\lambda\mu(t-T_1)}, \quad \text{for } t \le T_1 \quad \text{and} \quad x \in B(0,R).$$

This completes the proof. □

2.2 Super- and subsolutions before the encounter

Assume without loss of generality that the obstacle K contains 0, namely, $0 \in K$ and there is a positive constant R such that $K \subset B(0, R)$. Otherwise, one can shift $U(t, x)$ by $\widetilde{U}(t, x) = U(t, x + x_0)$ for $x_0 \in K$.

In this section, we construct a supersoluion and subsolution of (1.1) by using the transition front $U(t, x)$. To do so, we prepare an auxiliary function. Let $\widetilde{\zeta}$ be a function of class $C^2(\overline{\Omega})$, with compact support in $\overline{\Omega}$, and such that $\nu \cdot \nabla\widetilde{\zeta} = 1$ on $\partial\Omega$. Assume that there is $R_1 > 0$ such that $supp\{\widetilde{\zeta}\} \in B(0, R_1)$. The functions $\Delta\widetilde{\zeta}$ and $\widetilde{\zeta}$ are continuous and compactly supported in Ω and they are then bounded. For example of a such function $\widetilde{\zeta}$, one can construct a cut-off function satisfying the above conditions by applying the classical distance function in [10] around the boundary $\partial\Omega$. By Lemma 2.1, there are constants $T_1 < 0$, $\beta > 0$ and $\eta > 0$ such that

$$|\nabla U(t+1, x)| \leq \beta e^{\eta t}, \quad \text{for } t \leq T_1 \quad \text{and} \quad x \in B(0, R). \tag{2.8}$$

Take a constant $C_2 > 0$ such that

$$\zeta(x) := \widetilde{\zeta}(x) + C_2 \quad \text{in } \overline{\Omega},$$

and

$$\left\| \frac{\Delta\zeta}{\zeta} \right\|_{L^\infty(\Omega)} \leq \eta. \tag{2.9}$$

By (1.2), there is $\sigma > 0$ such that $f(s)$ is nonincreasing in $(-\infty, 2\sigma]$ and $[1-2\sigma, +\infty)$. By [11], one knows that $U_t(t, x) > 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and there is $k > 0$ such that, if (t, x) satisfies $\sigma \leq U(t, x) \leq 1 - \sigma$, then $U_t(t, x) \geq k$. Take $\omega > 0$ sufficiently large such that

$$k\omega\eta \geq L\beta\|\zeta\|_{L^\infty(\overline{\Omega})}, \tag{2.10}$$

where $L = \max_{s\in\mathbb{R}} |f'(s)|$.

We set

$$U^+(t, x) = U(t + \omega e^{\eta t}, x) + \beta\zeta(x)e^{\eta t}, \tag{2.11}$$

and

$$U^-(t, x) = U(t - \omega e^{\eta t}, x) - \beta\zeta(x)e^{\eta t}. \tag{2.12}$$

Let $T_2 \leq T_1$ such that $\omega e^{\eta t} \leq 1$ for $t \leq T_2$ and

$$\beta\zeta(x)e^{\eta t} \leq \sigma, \quad \text{for } t \leq T_2 \quad \text{and} \quad x \in \overline{\Omega}. \tag{2.13}$$

Lemma 2.2 *$U^+(t,x)$ and $U^-(t,x)$ are a supersolution and a subsolution of (1.1) for $t \le T_2$, respectively.*

Proof We first check the boundary condition on $\partial\Omega$. It follows from (2.8) that

$$\nabla U^+(t,x)\cdot\nu = \nabla U(t+\omega e^{\eta t},x)\cdot\nu + \beta\nabla\zeta\cdot\nu e^{\eta t} \ge 0,$$

and

$$\nabla U^-(t,x)\cdot\nu = \nabla U(t-\omega e^{\eta t},x)\cdot\nu - \beta\nabla\zeta\cdot\nu e^{\eta t} \le 0,$$

for $t \le T_2$ and $x \in \partial\Omega$.

We next check that

$$\mathcal{L}U^+ := U_t^+ - \Delta U^+ - f(U^+) \ge 0,$$

for $t \le T_2$ and $x \in \Omega$. One can easily compute that

$$\begin{aligned}\mathcal{L}U^+ &= \omega\eta e^{\eta t}U_t(t+\omega e^{\eta t},x) + \beta\eta\zeta(x)e^{\eta t} + \beta\Delta\zeta e^{\eta t}\\ &\quad + f(U(t+\omega e^{\eta t},x)) - f(U^+(t,x)).\end{aligned}$$

For $t \le T_2$ and $x \in \Omega$ such that $U(t+\omega e^{\eta t},x) \le \sigma$, it follows from (2.13) that $U^+(t,x) \le 2\sigma$. Since $f(s)$ is nonincreasing in $(-\infty, 2\sigma]$ and by $U_t > 0$, (2.9), one gets that

$$\mathcal{L}U^+ \ge \beta\zeta(x)e^{\eta t}\left(\eta + \frac{\Delta\zeta}{\zeta}\right) \ge 0.$$

For $t \le T_2$ and $x \in \Omega$ such that $U(t+\omega e^{\eta t},x) \ge 1-\sigma$, it follows that $U^+(t,x) \ge 1-\sigma$. Since $f(s)$ is nonincreasing in $[1-2\sigma, +\infty)$ and by $U_t > 0$, (2.9), one gets that $\mathcal{L}U^+ \ge 0$. Finally, if $t \le T_2$ and $x \in \Omega$ such that $\sigma \le U(t+\omega e^{\eta t},x) \le 1-\sigma$, then $U_t(t+\omega e^{\eta t},x) \ge k$. By (2.9) and (2.10), one gets that

$$\mathcal{L}U^+ \ge k\omega\eta e^{\eta t} + \beta\zeta(x)e^{\eta t}\left(\eta + \frac{\Delta\zeta}{\zeta}\right) - L\beta\zeta(x)e^{\eta t} \ge 0.$$

Thus we can confirm that U^+ is a supersolution of (1.1). Similarly, one can easily check that $\mathcal{L}U^- \le 0$ for $t \le T_2$ and $x \in \Omega$, namely, U^- is a subsolution of (1.1). This completes the proof. □

2.3 Proof of Theorem 1.2

We now construct a sequence of solutions defined for $-n \le t < +\infty$ ($n \in \mathbb{N}$). Let $u_n(t,x)$ be the solution of (1.1) for $t \ge -n$ with the initial data

$$u_n(-n,x) = U^+(-n,x).$$

Since $U^{-}(-n,x) \le u_n(-n,x) = U^{+}(-n,x)$, the comparison principle implies

$$U^{-}(t,x) \le u_n(t,x) \le U^{+}(t,x), \quad \text{for } t \in [-n, T_2] \quad \text{and} \quad x \in \Omega. \tag{2.14}$$

Then, it follows that

$$u_n(-n+1,x) \le U^{+}(-n+1,x) = u_{n-1}(-n+1,x).$$

By the comparison principle, one has

$$u_n(t,x) \le u_{n-1}(t,x), \quad \text{for } t \in [-n+1, T_2] \quad \text{and} \quad x \in \Omega.$$

Thus, the sequence $u_n(t,x)$ is monotone decreasing in n. Passing to the limit $n \to +\infty$ and using parabolic estimates, one obtains that this sequence converges to an entire solution $u^*(t,x)$ defined for $t \in \mathbb{R}$ and $x \in \Omega$. By (2.14), it follows that

$$U^{-}(t,x) \le u^*(t,x) \le U^{+}(t,x), \text{ for } t \in (-\infty, T_2] \quad \text{and} \quad x \in \Omega.$$

It also implies that

$$u^*(t,x) \to U(t,x), \quad \text{as } t \to -\infty \text{ uniformly in } \Omega.$$

Finally, we show that $u^*_t(t,x) > 0$ for $t \in \mathbb{R}$ and $x \in \Omega$. One can easily note that $U^{+}(t,x)$ is monotone increasing in t for t sufficiently negative. This means $(u_n)_t(-n,x) > 0$ for all sufficiently large n. By using the maximum principle to u_t, it yields that

$$(u_n)_t(t,x) > 0, \text{ for } t \in (-n, +\infty) \quad \text{and} \quad x \in \Omega.$$

As $n \to +\infty$, we get

$$u^*_t(t,x) \ge 0, \text{ for } t \in \mathbb{R} \quad \text{and} \quad x \in \Omega.$$

It is obviously that u^*_t is not identically equal to 0 and hence, $u^*_t > 0$ for $t \in \mathbb{R}$ and $x \in \Omega$ by the strong maximum principle. This completes the proof of Theorem 1.2.

3 Existence of the almost V-shaped front

This section is devoted to the proof of the existence of the almost V-shaped front, that is, Theorem 1.4.

3.1 Subsolutions and supersolutions

In this section, we construct V-shaped like subsolutions and supersolutions for (1.1) inspired by [18,19]. Let

$$k_1 = \frac{1}{2}\min\{-f'(0), -f'(1)\} > 0 \quad \text{and} \quad L = \max_{s\in[0,1]} |f'(s)|. \tag{3.1}$$

Then, since $f(s) = f'(0)s$ for $s \in (-\infty, 0]$ and $f(s) = f'(1)s$ for $s \in [1, +\infty]$, there exists a positive constant δ_1 $(0 < \delta_1 < 1/4)$ with

$$-f'(s) > k_1 \quad \text{if } s \in (-\infty, 2\delta_1] \text{ and } s \in [1 - 2\delta_1, +\infty). \tag{3.2}$$

Recall that $V(x_1, x_2-ct)$ is the V-shaped front of (1.5) satisfying (1.15). From [18,19], there exist constants τ_1, τ_2 such that

$$\begin{aligned}\phi\left(\tfrac{c_f}{c}(x_2 - ct - m_*|x_1| + \tau_1)\right) &\le V(x_1, x_2 - ct)\\ &\le \phi\left(\tfrac{c_f}{c}(x_2 - ct - m_*|x_1| + \tau_2)\right)\end{aligned} \tag{3.3}$$

for $t \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$. It follows from [18] that $V_\xi < 0$ and there is $k_2 > 0$ such that, if

$$-V_\xi(y, \xi) \ge k_2, \quad \text{for } \delta_1 \le V(y, \xi) \le 1 - \delta_1. \tag{3.4}$$

From [9], one also knows that there are positive constants a_1, a_2, b_1, b_2, α_1, α_2, β_1, β_2 such that

$$a_1 e^{b_1\xi} \le 1 - \phi(\xi) \le a_2 e^{b_2\xi}, \quad \text{for } \xi \le 0, \tag{3.5}$$

and

$$\alpha_1 e^{-\beta_1\xi} \le \phi(\xi) \le \alpha_2 e^{-\beta_2\xi}, \quad \text{for } \xi > 0. \tag{3.6}$$

It implies that we can get the following estimates of derivatives of $V(x_1, x_2 - ct)$.

Lemma 3.1 *There exist positive constants d_1, d_2, λ_1 and λ_2 such that*

$$\|\nabla V(x_1, x_2 - ct)\|_{L^\infty(\mathbb{R}^2)} \le d_1 e^{\lambda_1(x_2 - ct - m_*|x_1|)}, \quad \textit{for } x_2 - ct - m_*|x_1| \le 0,$$

and

$$\|\nabla V(x_1, x_2 - ct)\|_{L^\infty(\mathbb{R}^2)} \le d_2 e^{-\lambda_2(x_2 - ct - m_*|x_1|)}, \quad \textit{for } x_2 - ct - m_*|x_1| > 0.$$

Proof By (3.3), (3.5) and (3.6), one has

$$a_1 e^{b_1 c_f/c(x_2-ct-m_*|x_1|+\tau_2)} \le 1 - V(x_1, x_2-ct) \le a_2 e^{b_2 c_f/c(x_2-ct-m_*|x_1|+\tau_1)},$$

as $x_2 - ct - m_*|x_1| \to -\infty$ and

$$\alpha_1 e^{-\beta_1 c_f/c(x_2-ct-m_*|x_1|+\tau_1)} \le V(x_1, x_2-ct) \le \alpha_2 e^{-\beta_2 c_f/c(x_2-ct-m_*|x_1|+\tau_2)},$$

as $x_2 - ct - m_*|x_1| \to +\infty$. By parabolic interior estimates, one can get that there exist positive constants d_1, d_2, λ_1 and λ_2 such that

$$\|\nabla V(x_1, x_2-ct)\|_{L^\infty} \le d_1 e^{\lambda_1(x_2-ct-m_*|x_1|)}, \quad \text{for } x_2-ct-m_*|x_1| \le 0,$$

and

$$\|\nabla V(x_1, x_2-ct)\|_{L^\infty} \le d_2 e^{-\lambda_2(x_2-ct-m_*|x_1|)}, \quad \text{for } x_2-ct-m_*|x_1| > 0.$$

This completes the proof. □

Next we construct a subsolution of (1.1). As in Sect. 2.2, we prepare an auxiliary function ζ again. From now on, ζ always satisfies the following conditions. Let $\widetilde{\zeta}$ be a nonnegative function of class $C^2(\overline{\Omega})$, with compact support in $\overline{\Omega}$, and such that $\nu \cdot \nabla\widetilde{\zeta} = 1$ on $\partial\Omega$. The function $\Delta\widetilde{\zeta}$ and $\widetilde{\zeta}$ are continuous and compactly supported in $\overline{\Omega}$ and they are then bounded. There exists then a constant $C_3 \ge 1$ such that

$$\zeta := \widetilde{\zeta} + C_3 \ge 1 \quad \text{in } \overline{\Omega} \tag{3.7}$$

and

$$\left\| \frac{\Delta\zeta}{\zeta} \right\|_{L^\infty(\overline{\Omega})} \le \frac{k_1}{2}. \tag{3.8}$$

We remark that, since the obstacle K is bounded,

$$\sqrt{|x_1|^2 + |x_2|^2} < +\infty \quad \text{and} \quad |x_2 - m_*|x_1|| \le \widetilde{C} \tag{3.9}$$

for all $(x_1, x_2) \in \partial\Omega$ and some positive constant $\widetilde{C}$.

Lemma 3.2 *For any fixed $M \in \mathbb{R}$, define*

$$w_1(t,x) := V(x_1, x_2 - c(t+T) + \rho\delta(1-e^{-\beta t}) + M) - \delta\zeta(x)e^{-\beta t}.$$

Then, for any $\delta \in (0, \delta_1/\|\zeta\|_{L^\infty})$, there exist $\beta > 0$, $\rho > 0$ and $T > 0$ (β, ρ are independent of δ) such that $w_1(t,x)$ is a subsolution of (1.1) for $t \ge 0$.

Proof Denote

$$\xi(t) = -c(t+T) + \rho\delta(1-e^{-\beta t}) + M.$$

Then,

$$w_1(t,x) = V(x_1, x_2 + \xi(t)) - \delta\zeta(x)e^{-\beta t}.$$

Take $\beta > 0$ such that

$$\beta \le \min\left(c\lambda_1, \frac{k_1}{2}\right), \tag{3.10}$$

where λ_1 is defined in Lemma 3.1 and k_1 is defined by (3.1). Take $\rho > 0$ sufficiently large such that

$$\rho\beta k_2 \ge \beta\|\zeta\|_{L^\infty(\mathbb{R})} + \|\Delta\zeta\|_{L^\infty(\mathbb{R})} + L\|\zeta\|_{L^\infty(\mathbb{R})}, \tag{3.11}$$

where L and k_2 are defined by (3.1) and (3.4) respectively. By (3.9), one can choose $T > 0$ sufficiently large such that

$$x_2 + \xi(t) - m_*|x_1| = x_2 - c(t+T) + M + \rho\delta(1 - e^{-\beta t}) - m_*|x_1| \le 0, \tag{3.12}$$

for $t \ge 0$ and $x \in \partial\Omega$, and

$$d_1 e^{\lambda_1(\widetilde{C} - cT + M + \rho\delta)} \le \delta, \tag{3.13}$$

where d_1, λ_1 are defined in Lemma 3.1.

Let us first check the boundary conditions. One can compute that

$$\nabla w_1(t,x)\cdot\nu = \nabla V(x_1, x_2+\xi(t))\cdot\nu - \delta\nabla\zeta(x)\cdot\nu e^{-\beta t}.$$

for $x \in \partial\Omega$ and the outer normal unit vector $\nu = \nu(x)$ on $\partial\Omega$. By Lemma 3.1, (3.12) and (3.13), one has

$$\begin{aligned}\|\nabla V(x_1, x_2+\xi(t))\|_{L^\infty(\mathbb{R}^2)} &\le d_1 e^{\lambda_1(x_2 - c(t+T) + M + \rho\delta(1-e^{-\beta t}) - m_*|x_1|)}\\ &\le d_1 e^{-c\lambda_1 t} e^{\lambda_1(\widetilde{C} - cT + M + \rho\delta)}\\ &\le \delta e^{-c\lambda_1 t},\end{aligned}$$

for $x \in \partial\Omega$. Since $\nabla\zeta(x)\cdot\nu = 1$ on $\partial\Omega$ and $\beta \le c\lambda_1$, one gets that

$$\nabla w_1(t,x)\cdot\nu \le \delta e^{-c\lambda_1 t} - \delta e^{-\beta t} \le 0 \quad \text{on } \partial\Omega.$$

Let us now check that

$$N(t,x) := (w_1)_t - \Delta w_1 - f(w_1) \le 0, \quad \text{for } t \ge 0 \quad \text{and} \quad x \in \overline{\Omega}.$$

One can compute that

$$N(t,x) = f(V(x_1, x_2+\xi(t))) - f(V(x_1, x_2+\xi(t)) - \delta\zeta(x)e^{-\beta t})$$
$$+ \rho\delta\beta e^{-\beta t} V_\xi(x_1, x_2+\xi(t)) + \delta\zeta(x)\beta e^{-\beta t} - \delta\Delta\zeta(x)e^{-\beta t}.$$

For $t \geq 0$ and $x \in \overline{\Omega}$ such that $V(x_1, x_2+\xi(t)) \geq 1-\delta_1$, one has that $V(x_1, x_2 + \xi(t)) - \delta\zeta(x)e^{-\beta t} \geq 1 - 2\delta_1$ due to $\delta \in (0, \delta_1/\|\zeta\|_{L^\infty})$ and hence, by (3.2),

$$f(V(x_1, x_2+\xi(t))) - f(V(x_1, x_2+\xi(t)) - \delta\zeta(x)e^{-\beta t}) \leq -k_1\delta\zeta(x)e^{-\beta t} \tag{3.14}$$

Then, it follows from $V_\xi < 0$, (3.8), (3.10) and (3.14) that

$$N(t,x) \leq -k_1\delta\zeta(x)e^{-\beta t} + \delta\zeta(x)\beta e^{-\beta t} - \delta\Delta\zeta(x)e^{-\beta t}$$
$$= \delta\zeta(x)e^{-\beta t}\left(\beta - \frac{\Delta\zeta(x)}{\zeta(x)} - k_1\right) \leq 0.$$

Similarly one can get that $N(t,x) \leq 0$ for $t \geq 0$ and $x \in \overline{\Omega}$ such that $0 \leq V(x_1, x_2 + \xi(t)) \leq \delta_1$. For $t \geq 0$ and $x \in \overline{\Omega}$ such that $\delta_1 \leq V(x_1, x_2+\xi(t)) \leq 1-\delta_1$, one has that, by (3.1) and (3.4),

$$-V_\xi(x_1, x_2+\xi(t)) \geq k_2,$$

and

$$f(V(x_1, x_2+\xi(t))) - f(V(x_1, x_2+\xi(t)) - \delta\zeta(x)e^{-\beta t}) \leq L\delta\zeta(x)e^{-\beta t}.$$

Then, (3.11) leads to

$$N(t,x) \leq L\delta\zeta(x)e^{-\beta t} - \rho\delta\beta k_2 e^{-\beta t} + \delta\zeta(x)\beta e^{-\beta t} - \delta\Delta\zeta(x)e^{-\beta t}$$
$$= \delta e^{-\beta t}(L\zeta(x) - \rho\beta k_2 + \zeta(x)\beta - \Delta\zeta(x)) \leq 0.$$

In conclusion, we have

$$N(t,x) \leq 0, \text{ for } t \geq 0 \quad \text{and} \quad x \in \overline{\Omega}.$$

This completes the proof. □

We here introduce some super- and subsolutions for homogeneous case (1.5) shown in [18,19]. Remember that (1.5) admits V-shaped fronts $V(x_1, x_2 - ct)$ satisfying

$$-V_{yy} - V_{\xi\xi} - cV_\xi - f(V) = 0, t \in \mathbb{R}, (y,\xi) \in \mathbb{R}^2. \tag{3.15}$$

Note that V has asymptotic lines $x_2 = m_*|x_1|$ and a global mean speed $\gamma = c_f$. Define $\psi(\xi)$ by

$$\psi(\xi) := \frac{1}{m_*\gamma} \log\left(1 + \exp(\gamma\xi)\right).$$

Then we obtain the following lemmas and theorem:

Lemma 3.3 [19] *There exist some constants $K_i > 0$ ($i = 1, 2, 3$) and $\gamma > 0$ so that $\psi(\xi)$ satisfies*

$$\begin{array}{ll}
\max\left\{\left|\psi(\xi) - \frac{\xi}{m_*}\right|, \left|\psi'(\xi) - \frac{1}{m_*}\right|\right\} \le K_1 \mathrm{sech}(\gamma\xi) & \textit{for } \xi \ge 0, \\
\max\{|\psi(\xi)|, |\psi'(\xi)|\} \le K_1 \mathrm{sech}(\gamma\xi) & \textit{for } \xi \le 0, \\
\max\{|\psi''(\xi)|, |\psi'''(\xi)|\} \le K_1 \mathrm{sech}(\gamma\xi) & \textit{for } \xi \in \mathbb{R}, \\
c_f - \frac{c\psi'(\xi)}{\sqrt{1+\psi'(\xi)^2}} \ge K_2 \min\{1, \exp(-\gamma\xi)\} & \textit{for } \xi \in \mathbb{R}, \\
0 \le \frac{c}{\sqrt{1+\psi'(\xi)^2}} - c_f m_* \le K_3 \min\{1, \exp(-\gamma\xi)\} & \textit{for } \xi \in \mathbb{R}.
\end{array}$$

Theorem 3.4 [19] *There exist a positive constant ε_0 and a positive function $\alpha_0(\varepsilon)$ so that, for $0 < \varepsilon < \varepsilon_0$ and $0 < \alpha < \alpha_0(\varepsilon)$,*

$$v_2(y, \xi; \varepsilon, \alpha) := \phi\left(\frac{\psi(\alpha\xi) - \alpha y}{\alpha\sqrt{1 + \psi'(\alpha\xi)^2}}\right) - \varepsilon sech(\gamma\alpha\xi)$$

is a subsolution of (3.15). Moreover, there exists a positive constant k_3 such that

$$(v_2)_y \ge k_3, \quad \textit{if } \delta_1 \le v_2 \le 1 - \delta_1.$$

Define $v_3(y, \xi; \varepsilon, \alpha) := v_2(-y, \xi; \varepsilon, \alpha)$. It is also a subsolution of (3.15). In the sequel, we only use $v_2(y, \xi)$, $v_3(y, \xi)$ for short.

Lemma 3.5 [19] *Let $w_j(t, x)$ ($j = 2,\ 3$) be defined by*

$$\begin{aligned}
w_2(t, x) &:= v_2(x_1 - \rho\delta(1 - e^{-\beta t}), x_2 - c(t + T)) - \delta e^{-\beta t}, \\
w_3(t, x) &:= v_3(x_1 + \rho\delta(1 - e^{-\beta t}), x_2 - c(t + T)) - \delta e^{-\beta t}.
\end{aligned}$$

For any $\delta \in (0, \delta_1/2]$ and $T \in \mathbb{R}$, there exist a large positive constant ρ and a small positive constant β such that, w_2 and w_3 are also subsolutions of (1.5) for $t \ge 0$.

Combining three functions w_1, w_2 and w_3, we construct a subsolution which is useful to show Theorem 1.4.

Lemma 3.6 *For any small $\delta > 0$ and any $M > 0$, there exist constants $T > 0$, $\rho > 0$ and $\beta > 0$ such that*

$$w^-(t, x) := \max\{w_1(t, x), w_2(t, x), w_3(t, x)\}$$

is a subsolution of (1.1) for $t \ge 0$ and $x \in \Omega$.

Proof By Lemmas 3.2, 3.5, w_1, w_2, w_2 all satisfy $u_t - \Delta u - f(u) \le 0$. Thus one only have to check the boundary condition for w^-. Here we show that

$$w^-(t,x) = w_1(t,x), \text{ for any } x \in \partial\Omega \text{ and } t \ge 0.$$

If this is true, we immediately know $\partial_\nu w^- \le 0$ on $\partial\Omega$ by the proof of Lemma 3.2.

Notice that $\psi(\xi) > 0$ and $|\psi'(\xi)| < +\infty$ for $\xi \in \mathbb{R}$. Then, by (3.9), there is $0 < \sigma < 1$ such that

$$w_2(t,x) \le v_2(x_1 - \rho\delta(1-e^{-\beta t}), x_2 - c(t+T)) \le 1-\sigma,$$

for any $T \in \mathbb{R}$, $t \ge 0$ and $x \in \partial\Omega$. Similarly, $w_3(t,x) \le 1-\sigma$, for any $T \in \mathbb{R}$, $t \ge 0$ and $x \in \partial\Omega$. On the other hand, $x_2 - c(t+T) + \rho\delta(1-e^{-\beta t}) + M - m_*|x_1| \to -\infty$ for $t \ge 0$ and $x \in \partial\Omega$ as $T \to +\infty$. Thus,

$$w_1(t,x) \ge 1 - \delta\|\zeta\|_{L^\infty} \quad \text{for } t \ge 0 \quad \text{and} \quad x \in \partial\Omega \text{ as } T \to +\infty.$$

Therefore, for sufficiently small δ, there exists a large T such that

$$w_1(t,x) \ge 1-\sigma \ge w_2(t,x),\ w_3(t,x), \quad \text{for } t \ge 0 \quad \text{and} \quad x \in \partial\Omega,$$

and hence,

$$w^-(t,x) = w_1(t,x), \quad \text{for } t \ge 0 \text{ and } x \in \partial\Omega.$$

This completes the proof. □

Next we deal with supersolutions of (1.1). In oder to make it, the traveling curve front of the eikonal-curvature equation is useful. According to the result of [18], there is a unique graph $y = \varphi(\xi; c_f)$ for $\xi \in \mathbb{R}$ with asymptotic lines $y = m_*|\xi|$ such that

$$c_f = \frac{\varphi_{\xi\xi}}{1+\varphi_\xi^2} + c\sqrt{1+\varphi_\xi^2}$$

for $\varphi_{\xi\xi}(\cdot, c) > 0$ in $\mathbb{R}$. As seen in Theorem 2.2 in [18], there exist $\gamma_2 > 0$, $\varepsilon_0 > 0$ and a positive function $\alpha_0(\varepsilon)$ so that, for $0 < \varepsilon < \varepsilon_0$ and $0 < \alpha \le \alpha_0(\varepsilon)$,

$$v^+(y,\xi;\varepsilon,\alpha) := \phi\left(\frac{\alpha\xi - \varphi(\alpha y)}{\alpha\sqrt{1+\varphi'(\alpha y)^2}}\right) + \varepsilon \operatorname{sech}(\gamma_2 \alpha y)$$

is a supersolution of (3.15). Moreover, the supersolution $v^+(y,\xi;\varepsilon,\alpha)$ satisfies $-(v^+)_\xi(y,\xi;\varepsilon,\alpha) > 0$ for $y \in \mathbb{R}$, $\xi \in \mathbb{R}$ and hence there is $k_3 > 0$ such that $-(v^+)_\xi(y,\xi,;\varepsilon,\alpha) \ge k_3$ for $\delta_1 \le v^+(y,\xi;\varepsilon,\alpha) \le 1-\delta_1$. Using this supersolution ν^+, we construct a supersolution of (1.1) in the next lemma.

Lemma 3.7 *For any $\delta \in (0, \delta_1/2]$, there exist constants $\rho > 0$, $\beta > 0$ and $T > 0$ such that*

$$w^+(t,x) = \min\{v^+(x_1, x_2 - c(t+T) - \rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t}, 1\}$$

is a supersolution of (1.1) *for $t \geq 0$.*

Proof By [18], one knows that $v^+(x_1, x_2 - c(t+T) - \rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t}$ is a supersolution of (1.5) for $t \geq 0$. Then, we only has to check that $v^+(x_1, x_2 - c(t + T) - \rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t} \geq 1$ for any $x \in \partial\Omega$ and $t \geq 0$.

By (3.9), one has that $\varepsilon \operatorname{sech}(\gamma_2 \alpha x_1) > 0$ for $x \in \partial\Omega$. Since

$$(\alpha\xi - \varphi(\alpha y)/\alpha\sqrt{1 + \varphi'(\alpha y)^2} \to -\infty$$

as $\xi \to -\infty$ and $\phi(-\infty) = 1$, there is a positive constant T large enough such that

$$v^+(x_1, x_2 - c(t+T) - \rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t} \geq 1,$$

for any $x \in \partial\Omega$ and $t \geq 0$. This completes the proof. □

For any fixed M, let $v_1(y,\xi) := V(y, \xi + M)$ for $(y,\xi) \in \mathbb{R}^2$. We remark that w_1 is written by

$$w_1(t,x) = v_1(x_1, x_2 - c(t+T) + \rho\delta(1-e^{-\beta t}); \varepsilon, \alpha) - \delta\zeta(x)e^{-\beta t}.$$

Define

$$v^-(t,x) := \max\{v_1(x_1, x_2 - ct), v_2(x_1, x_2 - ct), v_3(x_1, x_2 - ct)\},$$

and

$$v^+(t,x) := v^+(x_1, x_2 - ct).$$

From [18,19], one can easily get the following lemma :

Lemma 3.8 *It holds that*

$$\lim_{R\to+\infty} \sup_{x_1^2+(x_2-ct)^2>R^2} \left| v^\pm(t,x) - \phi\left(\frac{c_f}{c}(x_2 - ct - m_*|x_1|)\right)\right| \leq \varepsilon, \text{ for any } t \geq 0,$$

where ε is as defined in v_2, v_3 and v^+.

Remark 3.9 Notice that Lemma 3.8 also means

$$\lim_{R\to+\infty} \sup_{x_1^2+(x_2-ct)^2>R^2} \left| v^\pm(t,x) - V(x_1, x_2 - ct)\right| \leq \varepsilon, \text{ for any } t \geq 0.$$

3.2 Proof of Theorem 1.4

Since V-shaped front is a special transition front of (1.5), we know that, from Theorem 1.2, (1.1) admits a time-increasing entire solution $u(t,x)$ such that

$$u(t,x) \to V(x_1, x_2 - ct), \quad \text{as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

Now we focus on an entire solution emanating from a V-shaped front. In particular, we are interested in the behaviour of this entire solution after passing through the obstacle K. Thus we assume a priori that $u(t,x)$ is a complete invasion, that is, it satisfies

$$u(t,x) \to 1 \text{ as } t \to +\infty \quad \text{locally uniformly in } x \in \overline{\Omega}. \tag{3.16}$$

Before starting the proof of Theorem 1.4, we first introduce some properties of the solution $u(t,x)$ of (1.1). Here we refer to Lemma 5.2 from [5] which is associated to the following initial value problem:

$$\begin{cases} u_t - \Delta u = f(u), & t > 0,\ x \in \overline{\Omega}, \\ u_\nu = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{3.17}$$

with the initial data $u(x,0) = u_0(x)$ satisfying

$$u_0(x) := \begin{cases} 1 - \varepsilon, & \text{if } x \in B(x_0, R) \cap \overline{\Omega}, \\ 0, & \text{if } x \in \overline{\Omega} \backslash B(x_0, R), \end{cases} \tag{3.18}$$

where x_0 is a point of $\mathbb{R}^N$, $B(x_0, R)$ is the open ball of radius R and center x_0 and ε is an arbitrary positive constant such that

$$\max\{0 < \theta < 1;\ f(\theta) = 0\} < 1 - \varepsilon < 1. \tag{3.19}$$

In what follows, $v_{x_0,R}$ denotes the solution of (3.17) with the initial condition (3.18).

Lemma 3.10 [5, Lemma 5.2] *Let ε satisfy (3.19) and $v_{x_0,R}$ be the solution of (3.17) with the initial condition (3.18). Then there exist four positive constants R_1, R_2, R_3 and $\bar{T}$ such that $R_3 > R_2 > R_1 > 0$, $R_2 - R_1 > c_f \bar{T}/4$, and, if $B(x_0, R_3) \subset \Omega$, then*

$$v_{x_0,R_1}(\bar{T}, \cdot) \geq 1 - \varepsilon \quad \text{in } \overline{B(x_0, R_2)}\ (\subset \Omega).$$

Next we show that the level set of $u(t,x)$ can be trapped between two V-shaped curves after passing the obstacle K. In order to show that, the following supersolution of (1.1) is useful. The proof is almost the same as Lemma 3.2 and hence we skip the details of the proof.

Lemma 3.11 *For any fixed $M \in \mathbb{R}$, define*

$$w_1^+(t,x) = V(x_1, x_2 - c(t+T) - \rho\delta(1 - e^{-\beta t}) + M) + \delta\zeta(x)e^{-\beta t}.$$

Then, for any $\delta \in (0, \delta_1/\|\zeta\|_{L^\infty})$, there exist $\beta > 0$, $\rho > 0$ and $T > 0$ such that $w_1^+(t, x)$ is a supersolution of (1.1) for $t \geq 0$.

Now we show the relation between u and V-shaped front.

Lemma 3.12 *For any $\varepsilon > 0$, there are constants $T_1 \in \mathbb{R}$, $M_1 > 0$ and $M_2 > 0$ such that for any $t \geq T_1$, $u(t, x)$ satisfies*

$$u(t, x) \geq 1 - \varepsilon, \quad \text{if } x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 - ct - m_*|x_1| \leq -M_1\}, \tag{3.20}$$

and

$$u(t, x) \leq \varepsilon \quad \text{if } x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 - ct - m_*|x_1| \geq M_2\}. \tag{3.21}$$

Proof Let ε be a positive constant with (3.19) such that Lemma 3.10 holds. Take small δ satisfying

$$0 < \delta < \min\left\{\frac{\varepsilon}{2\|\zeta\|_{L^\infty}}, \frac{\delta_1}{\|\zeta\|_{L^\infty}}\right\} \ (< \varepsilon), \tag{3.22}$$

where ζ is given in (3.7). Then Lemmas 3.2 and 3.11 guarantee that there exist positive constants β_*, ρ_* and T_* such that $w_1(t, x)$ and $w_1^+(t, x)$ with $M = 0$ are a sub- and supersolution of (1.1), respectively.

Recall that, by the monotonicity of $\phi(\xi)$ of (1.4), we can take a constant $R > 0$ such that

$$\phi(\xi) \leq \delta C_3 \quad \text{for } \xi \geq \frac{c_f}{c}(R - cT_* + \tau_2), \tag{3.23}$$

where C_3 and τ_2 satisfy (3.7) and (3.3), respectively. Since $u(t, x) \to V(x_1, x_2 - ct)$ as $t \to -\infty$ uniformly in $\overline{\Omega}$, it implies that there is $\widetilde{T} < 0$ such that

$$|u(\widetilde{T}, x) - V(x_1, x_2 - c\widetilde{T})| \leq \delta/2 \quad \text{for } x \in \overline{\Omega}. \tag{3.24}$$

By (3.24), (3.3) and the monotonicity of $\phi(\xi)$, it follows that there is $\widetilde{M} > 0$ such that

$$\begin{aligned} u(\widetilde{T}, x) &\geq V(x_1, x_2 - c\widetilde{T}) - \delta/2 \\ &\geq 1 - \delta \quad \text{in } \overline{\Omega} \cap \{x \in \mathbb{R};\ x_2 \leq m_*|x_1| + c\widetilde{T} - \widetilde{M}\}. \end{aligned} \tag{3.25}$$

On the other hand, one knows that for any point $y \in \overline{\Omega}$ such that $y_2 \leq m_*|y_1| + R$, $d_\Omega(y, \{x \in \overline{\Omega}; x_2 \leq m^*|x_1| + c\widetilde{T} - \widetilde{M}\}) < +\infty$. Note that there are $x_0 \in \mathbb{R}^2$ and a positive constant L such that $K \subset B(x_0, L)$ because K is compact. By (3.22) and Lemma 3.10, there are positive constants R_1, R_2, R_3 and $\bar{T}$ such that $R_3 > R_2 > R_1 > 0$, $R_2 - R_1 > c_f\bar{T}/4$, and, if $B(x_0, R_3) \subset \Omega$, then

$$v_{x_0, R_1}(\bar{T}, \cdot) \geq 1 - \delta \quad \text{in } \overline{B(x_0, R_2)}.$$

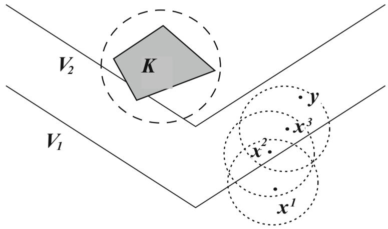

Fig. 1 Example of $\{x^j\}(1 \le j \le k)$ satisfying the above condition for y. Here $V_1 = \{x \in \mathbb{R}^2\,;\ x_2 = m_* x_1 + c\widetilde{T} - \widetilde{M}\}$, $V_2 = \{x \in \mathbb{R}^2\,;\ x_2 = m_* x_1 + R\}$

Then, for any point $y \in \overline{\Omega \backslash B(x_0, L + R_3 - R_2)} \cap \{x \in \mathbb{R}^2\,;\ x_2 \le m_*|x_1| + R\}$, there are k points $x^1, \ldots, x^k$ in $\mathbb{R}^2$ such that (Fig. 1)

$$\begin{cases} B(x^1, R_1) \subset \{x \in \mathbb{R}^2; x_2 \le m_*|x_1| + c\widetilde{T} - \widetilde{M}\}, & \\ B(x^i, R_3) \subset \Omega & \text{for } 1 \le i \le k, \\ B(x^{i+1}, R_1) \subset B(x^i, R_2) & \text{for } 1 \le i \le k-1, \\ y \in B(x^k, R_2). & \end{cases}$$

It follows from Lemma 3.10, (3.25) and the comparison principle that

$$u(\widetilde{T} + \bar{T}, x) \ge v_{x^1, R_1}(\bar{T}, x) \ge 1 - \delta, \quad \text{for } x \in \overline{B(x^1, R_2)}.$$

Since $B(x^2, R_1) \subset B(x^1, R_2)$, one gets that $u(\widetilde{T} + \bar{T}, x) \ge 1 - \delta$ for $x \in B(x^2, R_1)$. Since $B(x^2, R_3) \subset \Omega$, one apply Lemma 3.10 and get that $u(\widetilde{T} + 2\bar{T}, x) \ge 1 - \delta$ for $x \in \overline{B(x^2, R_2)}$. By induction, one has that $u(\widetilde{T} + k\bar{T}, x) \ge 1 - \delta$ for $x \in \overline{B(x^k, R_2)}$. Thus,

$$u(\widetilde{T} + k\bar{T}, x) \ge 1 - \delta \quad \text{in } \overline{\Omega \backslash B(x_0, L + R_3 - R_2)} \cap \{x \in \mathbb{R}^2 \mid x_2 \le m_*|x_1| + R\}. \tag{3.26}$$

By the assumption that $u(t, x)$ is a complete invasion satisfying (3.16), there is $T' \in \mathbb{R}$ such that

$$u(\widetilde{T} + T', x) \ge 1 - \delta, \quad \text{for any } x \in \overline{B(x_0, L + R_3 - R_2) \backslash K}. \tag{3.27}$$

Define $T_1 := \max\{\widetilde{T} + k\bar{T}, \widetilde{T} + T'\}$. Then, from (3.26) and (3.27), it follows from $u_t > 0$ that

$$u(T_1, x) \ge 1 - \delta, \quad \text{for any } x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2 \mid x_2 \le m_*|x_1| + R\}.$$

Then we obtain that

$$u(T_1, x) \ge 1 - \delta \ge V(x_1, x_2 - cT_*) - \delta\zeta(x) = w_1(0, x)$$

in $\overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 \le m_*|x_1| + R\}$ because $0 \le V \le 1$ and $\zeta \ge 1$. For any $x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 \ge m_*|x_1| + R\}$, one has

$$x_2 - cT_* - m_*|x_1| \ge R - cT_*,$$

and by (3.3) and (3.23),

$$\begin{aligned} w_1(0, x) =& V(x_1, x_2 - cT_*) - \delta\zeta(x) \\ \le& \phi\left(\frac{c_f}{c}(x_2 - cT_* - m_*|x_1| + \tau_2)\right) - \delta C_1 \le 0 \le u(T_1, x). \end{aligned}$$

Therefore, $u(T_1, x) \ge w_1(0, x)$ for all $x \in \overline{\Omega}$. By comparison principle, one has that for all $x \in \overline{\Omega}$ and $t \ge 0$,

$$u(T_1 + t, x) \ge w_1(t, x) = V(x_1, x_2 - c(t + T_*) + \rho_*\delta(1 - e^{-\beta_* t})) - \delta\zeta(x)e^{-\beta_* t}.$$

Then there is $M_1 > 0$ such that, for any $t \ge T_1$,

$$u(t, x) \ge 1 - \frac{\varepsilon}{2} - \delta\zeta(x) \ge 1 - \varepsilon \quad \text{in } \overline{\Omega} \cap \{x \in \mathbb{R}^2 \,|\, x_2 - ct - m_*|x_1| \le -M_1\}$$

by (3.3), (3.22) and $\phi(-\infty) = 1$. This implies (3.20).

At last, we show (3.21). By (3.24) and (3.3), it follows that there is $\overline{M} > 0$ such that

$$u(\widetilde{T}, x) \le V(x_1, x_2 - c\widetilde{T}) + \frac{\delta}{2} \le \delta, \quad \text{for } x \in \overline{\Omega} \text{ such that } x_2 \ge m_*|x_1| + c\widetilde{T} + \overline{M}.$$

Then, for any $x \in \overline{\Omega}$ such that $x_2 \ge m_*|x_1| + c\widetilde{T} + \overline{M}$, one has that

$$w_1^+(0, x) = V(x_1, x_2 - cT_*) + \delta\zeta(x) \ge \delta \ge u(\widetilde{T}, x).$$

For any $x \in \overline{\Omega}$ such that $x_2 \le m_*|x_1| + c\widetilde{T} + \overline{M}$, one has that $x_2 - cT_* - m_*|x_1| \le c(\widetilde{T} - T_*)$. Remember that $T_* > 0$ and $\widetilde{T} < 0$. Even if means decreasing $\widetilde{T}$, one can have that $V(x_1, x_2 - cT_*) \ge 1 - \delta$ for $x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 \le m_*|x_1| + c\widetilde{T} + \overline{M}\}$. Therefore,

$$w_1^+(0, x) = V(x_1, x_2 - cT_*) + \delta\zeta(x) \ge 1 \ge u(\widetilde{T}, x),$$

in $\overline{\Omega} \cap \{x \in \mathbb{R}^2;\ x_2 \le m_*|x_1| + c\widetilde{T} + \overline{M}\}$. It leads to

$$w_1^+(0, x) \ge u(\widetilde{T}, x), \quad \text{for } x \in \overline{\Omega}.$$

By the comparison principle, one concludes that

$$u(\widetilde{T} + t, x) \le w_1^+(t, x) = V(x_1, x_2 - c(t + T_*) - \rho_*\delta(1 - e^{-\beta_* t})) + \delta\zeta(x)e^{-\beta_* t}.$$

By (3.3) and $\phi(+\infty) = 0$, there is $M_2 > 0$ such that, for $t \geq \widetilde{T}$,

$$u(t, x) \leq \frac{\varepsilon}{2} + \delta\zeta(x) \leq \varepsilon \quad \text{in } \overline{\Omega} \cap \{x \in \mathbb{R}^2\ ;\ x_2 \geq m_*|x_1| + ct + M_2\}.$$

Therefore the proof is completed. □

Lemma 3.13 *Let $T_1 \in \mathbb{R}$ such that Lemma 3.12 holds. Then, for any $\varepsilon > 0$ and $t \geq T_1$, there is $\widetilde{R} > 0$ such that $u(t, x)$ and V-shaped traveling front $V(x_1, x_2 - ct)$ of (1.5) satisfy*

$$|u(t, x) - V(x_1, x_2 - ct)| \leq \varepsilon, \quad \textit{for } x \in \overline{\Omega \backslash B(\eta(t), \widetilde{R})}.$$

where $\eta(t) = (0, ct) \in \mathbb{R}^2$.

Proof Let $r \in \mathbb{R}$ and take a sequence $\{x_n\}_{n\in\mathbb{N}} = \{(x_{n1}, x_{n2})\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that

$$x_{n1} > 0,\ x_{n2} - m_* x_{n1} = r \quad \text{and} \quad |x_n| \to +\infty \ \text{ as } \ n \to +\infty.$$

Denote $u_n(t, x) = u(t, x + x_n)$ for each $t \in \mathbb{R}$ and $x \in \Omega - \{x_n\}$. Since $0 \leq u \leq 1$ and $K = \mathbb{R}^2 \backslash \Omega$ is bounded, it follows from standard parabolic estimates that, as $n \to +\infty$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converge, up to extraction of a sequence, locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ to a solution $U(t, x)$ of

$$(U)_t - \Delta U = f(U), \quad \text{for } t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^2,$$

with $0 \leq U(t, x) \leq 1$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Notice that $|x_{n1}| \to +\infty$ since $|x_n| \to +\infty$ and $x_{n2} - m_* x_{n1} = r$. It follows that $(x_1 + x_{n1})^2 + (x_2 + x_{n2} - ct)^2 \to +\infty$ as $n \to +\infty$ for $x_1 \geq -x_{n1}/2$. Thus (1.15) leads to

$$\begin{aligned} &V(x_1 + x_{n1}, x_2 + x_{n2} - ct) - \phi\left(\frac{c_f}{c}(x_2 + x_{n2} - ct - m_*(x_1 + x_{n1}))\right) \\ &\quad = V(x_1 + x_{n1}, x_2 + x_{n2} - ct) - \phi\left(\frac{c_f}{c}(x_2 - ct - m_* x_1 + r)\right) \to 0, \end{aligned}$$

as $n \to +\infty$ for $x \in \mathbb{R}^2$. Remember that $u(t, x) - V(x_1, x_2 - ct) \to 0$ as $t \to -\infty$ uniformly in $x \in \overline{\Omega}$. Therefore,

$$U(t, x) \to \phi\left(\frac{c_f}{c}(x_2 - ct - m_* x_1 + r)\right), \quad \text{as } t \to -\infty \quad \text{for } x \in \mathbb{R}^2.$$

By [5], one gets that

$$U(t, x) = \phi\left(\frac{c_f}{c}\,(x_2 - ct - m_* x_1 + r)\right), \quad \text{for } t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^2.$$

Then, one concludes that

$$u(t, x + x_n) \to \phi\left(\frac{c_f}{c}(x_2 - ct - m_* x_1 + r)\right), \tag{3.28}$$

locally uniformly for $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$ as $n \to +\infty$. Similarly, for a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_{n1} < 0,\ x_{n2} - m_* x_{n1} = r \quad \text{and} \quad |x_n| \to +\infty \quad as\ n \to +\infty,$$

one can obtain

$$u(t, x + x_n) \to \phi\left(\frac{c_f}{c}(x_2 - ct + m_* x_1 + r)\right),$$

locally uniformly for $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$ as $n \to +\infty$.

Fix $t_* \geq T_1$. Take a sequence $\{y_n\}_{n\in\mathbb{N}} = \{(y_{n1}, y_{n2})\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that $y_{n2} - m_* y_{n1} = ct_*$ for $y_{n1} \geq 0$, $|y_n| \to \infty$ as $n \to \infty$ and there is $R > 0$ satisfying

$$\left\{x \in \overline{\Omega};\ x_1 > 0,\ -M_1 \leq x_2 - ct_* - m_* x_1 \leq M_2\right\} \subset \bigcup_{n=1}^{\infty} B(y_n, R),$$

where M_1, M_2 are some positive constants. By (3.28) and $y_{n2} - m_* y_{n1} = ct_*$, one has that

$$\left| u(t_*, x + y_n) - \phi\left(\frac{c_f}{c}(x_2 - m_* x_1)\right)\right| < \frac{\varepsilon}{2}, \text{ for large } n \text{ and } x \in B(0, R).$$

This implies that there is $\widetilde{R} > 0$ such that

$$\left| u(t_*, x) - \phi\left(\frac{c_f}{c}(x_2 - ct_* - m_* x_1)\right)\right| < \frac{\varepsilon}{2}, \tag{3.29}$$

for $x \in \overline{\Omega} \cap \{x \in \mathbb{R} \mid |x - (0, ct_*)| \geq \widetilde{R}, x_1 \geq 0, -M_1 \leq x_2 - ct_* - m_* x_1 \leq M_2\}$.

Similarly, there is $\widetilde{R} > 0$ such that

$$\left| u(t, x) - \phi\left(\frac{c_f}{c}(x_2 - ct_* + m_* x_1)\right)\right| < \frac{\varepsilon}{2}, \tag{3.30}$$

for $x \in \overline{\Omega} \cap \{x \in \mathbb{R} \mid |x - (0, ct_*)| \geq \widetilde{R}, x_1 < 0, -M_1 \leq x_2 - ct_* - m_* x_1 \leq M_2\}$. Even if it means increasing $\widetilde{R}$, one can treat that M_1, M_2 are large enough. Note that t_* is an arbitrary fixed point with $t_* \geq T_1$. Thus it follows from Lemma 3.12 and (1.15) that, for $t \geq T_1$,

$$\left| u(t, x) - \phi\left(\frac{c_f}{c}(x_2 - ct - m_* |x_1|)\right)\right| \leq \varepsilon, \text{ for } x \in \overline{\Omega \backslash B(\eta(t), \widetilde{R})},$$

where $\eta = (0, ct)$. Therefore we completes the proof. □

Proof of Theorem 1.4 Take a sufficiently small $\varepsilon > 0$. Let δ be a small constant such that Lemmas 3.6, 3.7 hold and $\delta \geq 2\varepsilon$. Take T such that Lemmas 3.6, 3.7, 3.12 and 3.13 hold for δ and ε. By Lemmas 3.8 and 3.13, one gets that there is $R > 0$ such that

$$u(T,x) \geq \phi\left(\frac{c_f}{c}(x_2 - ct + m_*|x_1|)\right) - \varepsilon \geq v^-(T,x;\varepsilon,\alpha) - \delta\|\zeta\|_{L^\infty} = w^-(0,x),$$

and

$$u(T,x) \leq v^+(T,x;\varepsilon,\alpha) + \delta = w^+(0,x),$$

for $x \in \overline{\Omega} \cap \{x \in \mathbb{R}^2\ ;\ |x-(0,cT)| \geq R\ \}$. From [19], one can make α sufficiently small such that

$$v_2(x_1, x_2 - cT;\varepsilon,\alpha) \leq \delta, \quad \text{for } x \in B((0,cT),R).$$

Then, $w^-(0,x) \leq 0 \leq u(T,x)$ for $x \in B((0,cT),R)$. Thus, $w^-(0,x) \leq u(T,x)$ for $x \in \overline{\Omega}$. From the comparison principle, it follows that

$$w^-(t,x) \leq u(t+T,x), \quad \text{for } x \in \overline{\Omega} \text{ and } t \geq 0.$$

Also, from [18], one knows that one can make α sufficiently small such that

$$v^+(x_1, x_2 - cT;\varepsilon,\alpha) \geq 1 - \delta, \quad \text{for } x \in B((0,cT),R).$$

Then, $w^+(0,x) \geq 1 \geq u(T,x)$ for $x \in B((0,cT),R)$. By the comparison principle, it follows that

$$w^+(t,x) \geq u(T+t,x), \quad \text{for } x \in \overline{\Omega} \text{ and } t \geq 0.$$

In conclusion, one has

$$w^-(t,x) \leq u(T+t,x) \leq w^+(t,x), \quad \text{for } x \in \overline{\Omega} \text{ and } t \geq 0.$$

As $T + t \to +\infty$, one has that

$$\begin{aligned}&\max\{v_1(x_1, x_2 - ct + \rho\delta;\varepsilon,\alpha), v_2(x_1 - \rho\delta, x_2 - ct;\varepsilon,\alpha),\\ &v_3(x_1 + \rho\delta, x_2 - ct;\varepsilon,\alpha)\} \leq u(t,x) \leq v^+(x_1, x_2 - ct - \rho\delta;\varepsilon,\alpha).\end{aligned} \tag{3.31}$$

Take any sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$ as $n \to +\infty$. Let $u_n(t,x) = u(t+t_n, x+(0,ct_n))$. By standard parabolic estimates, $u_n(t,x)$ converge to a solution $U(t,x)$ of $U_t - \Delta U = f(U)$ in $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Since ε and δ could be arbitrary small, it follows from (3.31) and Lemma 3.8 that

$$\lim_{R\to+\infty} \sup_{x_1^2+(x_2-ct)^2>R^2} |U(t,x) - V(x_1, x_2 - ct)| = 0.$$

By stability of V-shaped front due to [18,19], one concludes that

$$U(t, x) \equiv V(x_1, x_2 - ct).$$

This completes the proof. □

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References

1. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusions arising in population genetics. Adv. Math. **30**, 33–76 (1978)
2. Berestycki, H., Bouhours, J., Chapuisat, G.: Front blocking and propagation in cylinders with varying cross section. Calc. Var. Partial Differ. Equations **55**, 1–32 (2016)
3. Berestycki, H., Hamel, F.: Generalized traveling waves for reaction-diffusion equations, In: Perspectives in Nonlinear Partial Differential Equations. In honor of H. Brezis, Amer. Math. Soc., Contemp. Math. , vol. 446, pp. 101–123 (2007)
4. Berestycki, H., Hamel, F.: Generalized transition waves and their properties. Commun. Pure Appl. Math. **65**, 592–648 (2012)
5. Berestycki, H., Hamel, F., Matano, H.: Bistable traveling waves around an obstacle. Commun. Pure Appl. Math. **62**, 729–788 (2009)
6. Bu, Z.-H., Wang, Z.-C.: Stability of pyramidal traveling fronts in the degenerate monostable and combustion equations I. Discrete Contin. Dyn. Syst. **37**, 2395–2430 (2017)
7. Bu, Z.-H., Wang, Z.-C.: Curved fronts of monostable reaction-advection-diffusion equations in space-time periodic media. Commun. Pure Appl. Anal. **15**, 139–160 (2016)
8. Chen, X., Guo, J.-S., Hamel, F., Ninomiya, H., Roquejoffre, J.-M.: Traveling waves with paraboloid like interfaces for balanced bistable dynamics. Ann. Inst. H. Poincaré, Anal. Non Linéaire **24**, 369–393 (2007)
9. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to traveling front solutions. Arch. Ration. Mech. Anal. **65**, 335–361 (1977)
10. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, New York (2001)
11. Guo, H., Hamel, F.: Monotonicity of bistable transition fronts in $\mathbb{R}^N$. J. Elliptic Parabol. Equations **2**, 145–155 (2016)
12. Guo, H., Hamel, F., Sheng, W.J.: On the mean speed of bistable transition fronts in unbounded domains. https://hal.archives-ouvertes.fr/hal-01855979v2 **(preprint)**
13. Hamel, F.: Bistable transition fronts in $\mathbb{R}^N$. Adv. Math. **289**, 279–344 (2016)
14. Hamel, F., Monneau, R.: Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets. Commun. Partial Differ. Equations **25**, 769–819 (2000)
15. Hamel, F., Monneau, R., Roquejoffre, J.-M.: Existence and qualitative properties of multidimensional conical bistable fronts. Disc. Contin. Dyn. Syst. A **13**, 1069–1096 (2005)
16. Hamel, F., Monneau, R., Roquejoffre, J.-M.: Asymptotic properties and classification of bistable fronts with Lipschitz level sets. Disc. Contin. Dyn. Syst. A **14**, 75–92 (2006)
17. Kanel', Y.I.: Stabilization of solution of the Cauchy problem for equations encountered in combustion theory. Mat. Sb. **59**, 245–288 (1962)
18. Ninomiya, H., Taniguchi, M.: Existence and global stability of traveling curved fronts in the Allen–Cahn equations. J. Differ. Equations **213**, 204–233 (2005)
19. Ninomiya, H., Taniguchi, M.: Global stability of traveling curved fronts in the Allen–Cahn equations. Disc. Contin. Dyn. Syst. A **15**, 819–832 (2006)
20. Shen, W.: Traveling waves in diffusive random media. J. Dyn. Differ. Equations **16**, 1011–1060 (2004)
21. Taniguchi, M.: Traveling fronts of pyramidal shapes in the Allen–Cahn equation. SIAM J. Math. Anal. **39**, 319–344 (2007)

22. Taniguchi, M.: The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations. J. Differ. Equations **246**, 2103–2130 (2009)
23. Taniguchi, M.: Multi-dimensional traveling fronts in bistable reaction-diffusion equations. Disc. Contin. Dyn. Syst. A **32**, 1011–1046 (2012)
24. Wang, Z.C., Bu, Z.H.: Nonplanar traveling fronts in reaction-diffusion equations with combustion and degenerate Fisher-KPP nonlinearities. J. Differ. Equations **260**, 6405–6450 (2016)