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# Critical regularity of nonlinearities in semilinear classical damped wave equations

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**Abstract**

In this paper we consider the Cauchy problem for the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = h(u), \quad u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x),$$

where $h(s) = |s|^{1+\frac{2}{n}} \mu(|s|)$. Here $n$ is the space dimension and $\mu$ is a modulus of continuity. Our goal is to obtain sharp conditions on $\mu$ to obtain a threshold between global (in time) existence of small data solutions (stability of the zero solution) and blow-up behavior even of small data solutions.

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## 1 Introduction

In [12], the authors proved the global existence of small data energy solutions for the semilinear damped wave equation

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$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \tag{1}$$

in the supercritical range $p > 1+\frac{2}{n}$, by assuming compactly supported small data from the energy space. Under additional regularity the compact support assumption on the data can be removed. By assuming data in Sobolev spaces with additional regularity $L^1(\mathbb{R}^n)$, a global (in time) existence result was proved in space dimensions $n = 1, 2$ in [5], by using energy methods, and in space dimension $n \leq 5$ in [9], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$. Nonexistence of general global (in time) small data solutions is proved in [12] for $1 < p < 1 + \frac{2}{n}$ and in [13] for $p = 1 + \frac{2}{n}$. The exponent $1 + \frac{2}{n}$ is well known as Fujita exponent and it is the critical power for the following semilinear parabolic Cauchy problem (see [2]):

$$v_t - \triangle v = v^p, \quad v(0,x) = v_0(x) \geq 0. \tag{2}$$

If one removes the assumption that the initial data are in $L^1(\mathbb{R}^n)$ and we only assume that they are in the energy space, then the critical exponent is modified to $1 + \frac{4}{n}$ or to $1 + \frac{2m}{n}$ under additional regularity $L^m(\mathbb{R}^n)$, with $m \in [1, 2]$. For the classical damped wave equation, this phenomenon has been investigated in [4].

The diffusion phenomenon between linear heat and linear classical damped wave models (see [3,7,9,10]) explains the parabolic character of classical damped wave models with power nonlinearities from the point of decay estimates of solutions.

In the mathematical literature (see for instance [1]) the situation is in general described as follows: We have a semilinear Cauchy problem

$$L(\partial_t, \partial_x, t, x)u = |u|^p, \quad u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x),$$

where $L$ is a linear partial differential operator. Then the authors would like to find a critical exponent $p_{crit}$ in the scale $\{|u|^p\}_{p>0}$, a threshold between two different qualitative behaviors of solutions. As examples see the models (1) or (2).

*The main concern of this paper is to show by the aid of the model* (1) *that the restriction to the scale* $\{|u|^p\}_{p>0}$ *is too rough to verify the critical non-linearity or the critical regularity of the non-linear right-hand side.*

For this reason we turn to the Cauchy problem for the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = h(u), \quad u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \tag{3}$$

in $[0,\infty) \times \mathbb{R}^n$, where $h(s) = |s|^{1+\frac{2}{n}}\mu(|s|)$. Here $\mu = \mu(s),\ s \in [0,\infty)$, is a modulus of continuity, which provides an additional regularity of the right-hand side $h = h(s)$ for $s \in [0,\infty)$.

**Definition 1** A function $\mu : [0,\infty) \to [0,\infty)$ is called a modulus of continuity, if $\mu$ is a continuous, concave and increasing function satisfying $\mu(0) = 0$.

Our goal is to discuss the influence of the function $\mu$ on the global (in time) existence of small data Sobolev solutions or on statements for blow-up of Sobolev solutions to

(3). In the following result, we assume that the modulus of continuity $\mu$ given in (3) satisfies the following two conditions:

$$s^k|\mu^{(k)}(s)| \leq C\mu(s) \quad \text{for} \quad 1 \leq k \leq n,\ s \in (0, s_0], \qquad \text{and} \qquad \int_0^{C_0} \frac{\mu(s)}{s} ds < \infty, \tag{4}$$

where $C$ is a sufficiently large positive constant, $s_0$ and $C_0$ are sufficiently small positive constants.

***Remark 2*** In the further considerations we need a suitable modulus of continuity satisfying the conditions (4) on a small interval $[0, s_0]$ only. Nevertheless we can assume that the modulus of continuity can be continued to the real line in such a way that the properties from Definition 1 are satisfied.

**Theorem 3** *Let* $n = 1, 2$ *and*

$$(\phi, \psi) \in \mathcal{A} := \left(H^{1+\lfloor\frac{n}{2}\rfloor}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right) \times \left(H^{\lfloor\frac{n}{2}\rfloor}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right),$$

*where we denote by* $\lfloor\cdot\rfloor$ *the floor function. Assume that the modulus of continuity* $\mu$ *satisfies the condition* (4)*. Then, the following statement holds for a sufficiently small* $\varepsilon_0 > 0$*: if*

$$\|(\phi, \psi)\|_{\mathcal{A}} \leq \varepsilon \quad for \quad \varepsilon \leq \varepsilon_0,$$

*then there exists a unique globally (in time) Sobolev solution* $u$ *to* (3) *belonging to the function space*

$$C\left([0, \infty), H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\right),$$

*such that the following decay estimates are satisfied:*

$$\|u(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{2}}\|(\phi, \psi)\|_{\mathcal{A}},$$
$$\|\nabla_x^k u(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{n+2k}{4}}\|(\phi, \psi)\|_{\mathcal{A}}, \quad k = 0, 1.$$

***Remark 4*** The key tool to prove Theorem 3 is to apply estimates for solutions to the parameter-dependent Cauchy problem for the linear classical damped wave equation (Lemma 7). By using more general $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$, derived in [9] for the linear damped wave equation, one can also obtain a global (in time) existence result for higher dimensions $n$, but this aim is beyond the scope of this paper.

***Example 1*** The hypotheses of Theorem 3 hold for the following functions $\mu$ (see also Remark 2) on a small interval $[0, s_0]$:

1. $\mu(s) = s^p,\ p \in (0, 1]$;
2. $\mu(s) = (\log(1+s))^p,\ p \in (0, 1]$;

3. $\mu(0)=0$ and $\mu(s)=\Big(\log\frac{1}{s}\Big)^{-p},\ p>1$;
4. $\mu(0)=0$ and $\mu(s)=\Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\log^k\frac{1}{s}\Big)^{-p},\ p>1,\ k\in\mathbb{N}$.

The next result shows that the integral condition on the function $\mu$ in (4) can not be relaxed.

**Theorem 5** *Consider for $n\geq 1$ the Cauchy problem*

$$\begin{cases} u_{tt}-\Delta u+u_t=|u|^{1+\frac{2}{n}}\mu(|u|), & (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ (u(0,x),u_t(0,x))=(0,g(x)), & x\in\mathbb{R}^n. \end{cases} \tag{5}$$

*Here $\mu=\mu(s),\ s\in[0,\infty)$ is a modulus of continuity which satisfies the condition*

$$\int_0^{C_0}\frac{\mu(s)}{s}\,ds=\infty, \tag{6}$$

*where $C_0$ is a sufficiently small positive constant. Moreover, we assume that the function $h:s\in\mathbb{R}\to h(s):=s^{1+\frac{2}{n}}\mu(s)$ is convex on $\mathbb{R}$. Suppose that the data*

$$g\in\mathcal{A}:=H^{[\frac{n}{2}]}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n),$$

*such that*

$$\int_{\mathbb{R}^n}g(x)\,dx>0.$$

*Then, in general we have no global (in time) existence of Sobolev solutions even if the data are supposed to be very small in the following sense:*

$$\|g\|_{\mathcal{A}}\leq\varepsilon\quad\text{for}\quad\varepsilon\leq\varepsilon_0.$$

To prove Theorem 5 we will follow the approach used in [6] in which the authors get a sharp upper bound for the lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations, by using a test function method.

***Example 2*** The hypotheses of Theorem 5 hold for the following functions $\mu$ (see also Remark 2) on a small interval $[0,s_0]$:

1. $\mu(0)=0$ and $\mu(s)=\Big(\log\frac{1}{s}\Big)^{-p}, 0<p\leq 1$;
2. $\mu(0)=0$ and $\mu(s)=\Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\log^k\frac{1}{s}\Big)^{-p},\ p\in(0,1],\ k\in\mathbb{N}$.

***Remark 6*** Let us discuss the assumption in Theorem 5 that the function

$$h:s\in\mathbb{R}\to h(s):=s^{1+\frac{2}{n}}\mu(s)\quad\text{is convex on}\quad\mathbb{R}.$$

In the case of smooth $\mu$, in a small right-sided neighborhood of $s = 0$, this hypothesis can be replaced by the condition

$$s^k \mu^{(k)}(s) = o(\mu(s)) \quad \text{for} \quad s \to +0, \quad k = 1, 2.$$

Indeed, it is sufficient to verify that on a small interval $(0, s_0]$

$$h''(s) = s^{\frac{2}{n}-1} \left( \frac{2}{n} \left( 1 + \frac{2}{n} \right) \mu(s) + 2 \left( 1 + \frac{2}{n} \right) s\mu'(s) + s^2 \mu''(s) \right) \geq 0.$$

This condition is satisfied in our examples. Outside this interval we can choose a convex continuation of $h$.

In the following we use $f \lesssim g$ for nonnegative $f$ and $g$ if there exists a constant $C$ with $f \leq Cg$. We use $f \sim g$ if $f \leq C_1 g$ and $g \leq C_2 f$ with suitable constants $C_1$ and $C_2$.

## 2 Global existence of small data solutions

In the proof of Theorem 3 we are going to use the following estimates for Sobolev solutions to the parameter-dependent Cauchy problem for the linear classical damped wave equation.

**Lemma 7** (Lemma 1 in [8]) *Let*

$$(\phi, \psi) \in \mathcal{A} := \left( H^{1+\lfloor \frac{n}{2} \rfloor}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \right) \times \left( H^{\lfloor \frac{n}{2} \rfloor}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \right).$$

*Then, the Sobolev solutions to the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0, \quad u(s, x) = \phi_s(x), \quad u_t(s, x) = \psi_s(x), \tag{7}$$

*satisfy the following estimates for $t \geq 0$:*

$$\|u(t, \cdot)\|_{L^\infty} \leq C(1 + t - s)^{-\frac{n}{2}} \left( \|\phi_s\|_{L^1} + \|\phi_s\|_{H^{1+\lfloor \frac{n}{2} \rfloor}} + \|\psi_s\|_{L^1} + \|\psi_s\|_{H^{\lfloor \frac{n}{2} \rfloor}} \right),$$

*and for $k = 0, 1, 1 + \lfloor \frac{n}{2} \rfloor$*

$$\|\nabla_x^k u(t, \cdot)\|_{L^2} \leq C(1 + t - s)^{-\frac{n+2k}{4}} \left( \|\phi_s\|_{L^1} + \|\phi_s\|_{H^k} + \|\psi_s\|_{L^1} + \|\psi_s\|_{H^{k-1}} \right).$$

***Proof of Theorem 3*** The space of Sobolev solutions is $X(t) = C\big([0, t], H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$. Taking into consideration the estimates of Lemma 7 we define on $X(t)$ the norm

$$\|u\|_{X(t)} = \sup_{\tau \in [0,t]} \left\{ \sum_{k=0}^{1} (1 + \tau)^{\frac{n+2k}{4}} \|\nabla^k u(\tau, \cdot)\|_{L^2} + (1 + \tau)^{\frac{n}{2}} \|u(\tau, \cdot)\|_{L^\infty} \right\}.$$

For arbitrarily given data $(\phi, \psi) \in \mathcal{A}$ we introduce the operator

$$N : u \in X(t) \to u^{lin} + \int_0^t \Phi(t, s, \cdot) *_{(x)} h(u(s, \cdot))(x)\, ds$$

in $X(t)$, where by $u^{lin}$ we denote the solution to the linear parameter-dependent Cauchy problem (7) with initial data $(\phi, \psi)$. By

$$\Phi(t, s, \cdot) *_{(x)} h(u(s, \cdot))(x)$$

we denote the Sobolev solution to the Cauchy problem (7) with $\phi_s \equiv 0$ and $\psi_s = h(u(s, \cdot))$. We will prove that

$$\|Nu\|_{X(t)} \leq C_0 \|(\phi, \psi)\|_{\mathcal{A}} + \tilde{C}_{\varepsilon_0} \|u\|_{X(t)}^{1+\frac{2}{n}}, \tag{8}$$

$$\|Nu - Nv\|_{X(t)} \leq C_{\varepsilon_0} \|u - v\|_{X(t)} \left( \|u\|_{X(t)}^{\frac{2}{n}} + \|v\|_{X(t)}^{\frac{2}{n}} \right), \tag{9}$$

where $C_{\varepsilon_0}$ and $\tilde{C}_{\varepsilon_0}$ tend to 0 for $\varepsilon_0$ to 0.

First of all we have after applying Lemma 7 for all $t > 0$ the estimate

$$\|u^{lin}\|_{X(t)} \leq C_0 \|(\phi, \psi)\|_{\mathcal{A}}, \tag{10}$$

where the constant $C_0$ is independent of $t$. Consequently, it remains to estimate

$$G(u)(t, x) := \int_0^t \Phi(t, s, x) *_{(x)} h(u(s, x))\, ds.$$

For $j = 0, 1$ we have

$$\|\nabla^j G(u)(t, \cdot)\|_{L^2} \leq \int_0^t (1 + t - s)^{-\frac{n}{4} - \frac{j}{2}} \|h(u(s, \cdot))\|_{L^1 \cap L^2} ds.$$

It holds

$$\|h(u(s, \cdot))\|_{L^1 \cap L^2} \leq \mu(\|u(s, \cdot)\|_{L^\infty}) \, \| |u(s, \cdot)|^{1+\frac{2}{n}} \|_{L^1 \cap L^2}.$$

Thus, by using that

$$\|u(s, \cdot)\|_{L^\infty} \leq (1 + s)^{-\frac{n}{2}} \|u\|_{X(s)}$$

and the monotonicity of $\mu = \mu(s)$ we get the following estimate:

$$\mu(\|u(s, \cdot)\|_{L^\infty}) \leq \mu \left( (1 + s)^{-\frac{n}{2}} \|u\|_{X(s)} \right). \tag{11}$$

Let us assume $\|u\|_{X(t)} \leq \varepsilon_0$ for all $t > 0$ and some $\varepsilon_0 > 0$ sufficiently small. Then, since the norm in $X(t)$ is increasing with respect to $t$, we can estimate the right-hand side of (11) by

$$\mu\left(\varepsilon_0(1+s)^{-\frac{n}{2}}\right).$$

Moreover, to estimate $\||u(s,\cdot)|^{1+\frac{2}{n}}\|_{L^1\cap L^2}$ we may apply the Gagliardo–Nirenberg inequality and obtain

$$\|u(s,\cdot)\|_{L^{1+\frac{2}{n}}}^{1+\frac{2}{n}} \leq C\|\nabla u(s,\cdot)\|_{L^2}^{1-\frac{n}{2}}\|u(s,\cdot)\|_{L^2}^{\frac{2}{n}+\frac{n}{2}} \leq C(1+s)^{-1}\|u\|_{X(s)}^{1+\frac{2}{n}}, \tag{12}$$

and

$$\|u(s,\cdot)\|_{L^{2+\frac{4}{n}}}^{1+\frac{2}{n}} \leq C\|\nabla u(s,\cdot)\|_{L^2}\|u(s,\cdot)\|_{L^2}^{\frac{2}{n}} \leq C(1+s)^{-1-\frac{n}{4}}\|u\|_{X(s)}^{1+\frac{2}{n}}. \tag{13}$$

Thus, we may conclude

$$\|\nabla^j G(u)(t,\cdot)\|_{L^2} \leq \|u\|_{X(t)}^{1+\frac{2}{n}}\int_0^t (1+t-s)^{-\frac{n}{4}-\frac{j}{2}}(1+s)^{-1}\mu\left(\varepsilon_0(1+s)^{-\frac{n}{2}}\right)ds.$$

To estimate $\|G(u)(t,\cdot)\|_{L^\infty}$, the required regularity to the data increases with $n$, so we split the analysis for $n=1$ and $n=2$. For $n=1$ we may estimate

$$\|G(u)(t,\cdot)\|_{L^\infty} \leq \int_0^t (1+t-s)^{-\frac{1}{2}}\|h(u(s,\cdot))\|_{L^1\cap L^2}ds,$$

and proceed as before to derive

$$\|G(u)(t,\cdot)\|_{L^\infty} \leq \|u\|_{X(t)}^3\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\mu\left(\varepsilon_0(1+s)^{-\frac{1}{2}}\right)ds.$$

For $n=2$, applying Lemma 7 we may estimate

$$\|G(u)(t,\cdot)\|_{L^\infty} \leq \int_0^t (1+t-s)^{-1}\|h(u(s,\cdot))\|_{L^1\cap H^1}ds.$$

Now, we have to deal with a new term $\|\nabla h(u(s,\cdot))\|_{L^2}$. Using (4), we may estimate

$$|\nabla h(u(s,x))| \leq |u(s,x)|\mu(|u(s,x)|)|\nabla u(s,x)|$$

and

$$\begin{aligned}\|\nabla h(u(s,\cdot))\|_{L^2} &\lesssim \|u(s,\cdot)\|_{L^\infty}\mu(\|u(s,\cdot)\|_{L^\infty})\|\nabla u(s,\cdot)\|_{L^2}\\ &\lesssim (1+s)^{-2}\|u\|_{X(s)}^2\mu\left((1+s)^{-1}\|u\|_{X(s)}\right).\end{aligned}$$

Therefore

$$\|G(u)(t,\cdot)\|_{L^\infty} \leq \|u\|_{X(t)}^{1+\frac{2}{n}} \int_0^t (1+t-s)^{-\frac{n}{2}}(1+s)^{-1}\mu\left(\varepsilon_0(1+s)^{-\frac{n}{2}}\right) ds, \quad n=1,2.$$

Now, let $\alpha \leq 1$. On the one hand it holds

$$\int_0^{\frac{t}{2}} (1+t-s)^{-\alpha}(1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\, ds \\ \sim (1+t)^{-\alpha}\int_0^{\frac{t}{2}} (1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\, ds$$

by using $(1+t-s) \sim (1+t)$ on $[0, t/2]$. On the other hand

$$\int_{\frac{t}{2}}^{t} (1+t-s)^{-\alpha}(1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\, ds \\ \lesssim (1+t)^{-\alpha}\int_{\frac{t}{2}}^{t} (1+t-s)^{-\alpha}(1+s)^{-1+\alpha}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\, ds \\ \lesssim (1+t)^{-\alpha}\int_{\frac{t}{2}}^{t} (1+t-s)^{-1}\mu\big(\varepsilon_0(1+t-s)^{-\frac{n}{2}}\big)\, ds,$$

where we used $1+s \sim 1+t$ and $1+s \gtrsim 1+t-s$ on $[t/2, t]$.

By using the change of variables $r = \varepsilon_0(1+s)^{-\frac{n}{2}}$, we get

$$\int_0^{+\infty} (1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\, ds \sim \int_0^{\varepsilon_0} \frac{\mu(r)}{r}\, dr,$$

that is finite, due to assumption (4). Summarizing, we arrive at

$$\|Nu\|_{X(t)} \lesssim C_0\|(\phi,\psi)\|_{\mathcal{A}} + \tilde{C}_{\varepsilon_0}\|u\|_{X(t)}^{1+\frac{2}{n}}, \tag{14}$$

where $\tilde{C}_{\varepsilon_0}$ tends to 0 for $\varepsilon_0$ to 0.

To derive a Lipschitz condition we recall

$$Gu - Gv = \int_0^t \Phi(t,s,x) *_{(x)} \left(|u|^{1+\frac{2}{n}}\mu(|u|) - |v|^{1+\frac{2}{n}}\mu(|v|)\right) ds \\ = \int_0^t \Phi(t,s,x) *_{(x)} \left(\int_0^1 (d_{|u|}H(|u|))(v+\tau(u-v))d\tau\right)(s,x) \\ \times (u-v)(s,x)\, ds,$$

where

$$H : |u| \in \mathbb{R}^+ \to H(|u|) = |u|^{1+\frac{2}{n}}\mu(|u|).$$

By using our assumption to $\mu' = \mu'(s)$ we get

$$\left|d_{|u|}H(|u|)\right| \lesssim |u|^{\frac{2}{n}}\mu(|u|).$$

Here we take into consideration that $|u| \leq s_0$ with $s_0$ from (4) for small data solutions. Applying Minkowski's integral inequality, Lemma 7 and the monotonicity of $d_{|u|}H(|u|)$ for small $|u|$ gives

$$\begin{aligned}
&\left\|\nabla_x^j\left(Gu(t,\cdot)-Gv(t,\cdot)\right)\right\|_{L^2}\\
&\quad\lesssim \int_0^t (1+t-s)^{-\frac{n}{4}-\frac{j}{2}}\left\|\left(\int_0^1 \mu(|v+\tau(u-v)|)|v+\tau(u-v)|^{\frac{2}{n}}\,d\tau\right)\right.\\
&\qquad\left.\times|u-v|(s,\cdot)\right\|_{L^1\cap L^2} ds\\
&\quad\lesssim \int_0^t\int_0^1 (1+t-s)^{-\frac{n}{4}-\frac{j}{2}}\left\|\left(|u|^{\frac{2}{n}}+|v|^{\frac{2}{n}}\right)(u-v)(s,\cdot)\right\|_{L^1\cap L^2}\\
&\qquad\times\left\|\mu(|v+\tau(u-v)|)\right\|_{L^\infty}\,d\tau\,ds.
\end{aligned}$$

By using Hölder's inequality we get

$$\begin{aligned}
&\left\|\left(|u(s,\cdot)|^{\frac{2}{n}}+|v(s,\cdot)|^{\frac{2}{n}}\right)(u-v)(s,\cdot)\right\|_{L^1}\\
&\quad\lesssim \left(\|u(s,\cdot)\|_{L^{1+\frac{2}{n}}}^{\frac{2}{n}}+\|v(s,\cdot)\|_{L^{1+\frac{2}{n}}}^{\frac{2}{n}}\right)\|(u-v)(s,\cdot)\|_{L^{1+\frac{2}{n}}},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\left(|u(s,\cdot)|^{\frac{2}{n}}+|v(s,\cdot)|^{\frac{2}{n}}\right)(u-v)(s,\cdot)\right\|_{L^2}\\
&\quad\lesssim \left(\|u(s,\cdot)\|_{L^{2+\frac{4}{n}}}^{\frac{2}{n}}+\|v(s,\cdot)\|_{L^{2+\frac{4}{n}}}^{\frac{2}{n}}\right)\|(u-v)(s,\cdot)\|_{L^{2+\frac{4}{n}}}.
\end{aligned}$$

Thus, we can apply Gagliardo–Nirenberg as in (12) and (13) to get

$$\begin{aligned}
&\left\|\left(|u(s,\cdot)|^{\frac{2}{n}}+|v(s,\cdot)|^{\frac{2}{n}}\right)(u-v)(s,\cdot)\right\|_{L^1}\\
&\quad\lesssim (1+s)^{-1}\left(\|u\|_{X(s)}^{\frac{2}{n}}+\|v\|_{X(s)}^{\frac{2}{n}}\right)\|u-v\|_{X(s)},\\
&\left\|\left(|u(s,\cdot)|^{\frac{2}{n}}+|v(s,\cdot)|^{\frac{2}{n}}\right)(u-v)(s,\cdot)\right\|_{L^2}\\
&\quad\lesssim (1+s)^{-1-\frac{n}{4}}\left(\|u\|_{X(s)}^{\frac{2}{n}}+\|v\|_{X(s)}^{\frac{2}{n}}\right)\|u-v\|_{X(s)}.
\end{aligned}$$

Now we follow the same ideas presented above to conclude

$$\left\|\nabla_x^j\left(Gu(t,\cdot)-Gv(t,\cdot)\right)\right\|_{L^2}$$

$$\begin{aligned}
&\lesssim \|u-v\|_{X(t)}\left(\|u\|_{X(t)}^{\frac{2}{n}}+\|v\|_{X(t)}^{\frac{2}{n}}\right)\\
&\quad\times\int_0^t\int_0^1(1+t-s)^{-\frac{n}{4}-\frac{j}{2}}(1+s)^{-1}\mu(\|v+\tau(u-v)\|_{L^\infty})\,d\tau\,ds\\
&\lesssim \|u-v\|_{X(t)}\left(\|u\|_{X(t)}^{\frac{2}{n}}+\|v\|_{X(t)}^{\frac{2}{n}}\right)\\
&\quad\times(1+t)^{-\frac{n}{4}-\frac{j}{2}}\int_0^t\int_0^1(1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-\frac{n}{2}}\big)\,d\tau\,ds\\
&\le C'_{\varepsilon_0}(1+t)^{-\frac{n}{4}-\frac{j}{2}}\|u-v\|_{X(t)}\left(\|u\|_{X(t)}^{\frac{2}{n}}+\|v\|_{X(t)}^{\frac{2}{n}}\right),
\end{aligned}$$

where $C'_{\varepsilon_0}$ tends to 0 for $\varepsilon_0$ to 0.

To estimate $\|Gu(t,\cdot)-Gv(t,\cdot)\|_{L^\infty}$, we again split the analysis for $n=1$ and $n=2$. For $n=1$ we may proceed as we did to derive the estimates for $\|\nabla_x^j(Gu(t,\cdot)-Gv(t,\cdot))\|_{L^2}$ to conclude

$$\|Gu(t,\cdot)-Gv(t,\cdot)\|_{L^\infty}\le C'_{\varepsilon_0}(1+t)^{-\frac{1}{2}}\|u-v\|_{X(t)}\left(\|u\|_{X(t)}^{2}+\|v\|_{X(t)}^{2}\right),$$

where $C'_{\varepsilon_0}$ tends to 0 for $\varepsilon_0$ to 0.

For $n=2$, applying Lemma 7 we may estimate

$$\begin{aligned}
\|Gu(t,\cdot)-Gv(t,\cdot)\|_{L^\infty}&\le\int_0^t(1+t-s)^{-1}\\
&\quad\times\left\|\left(\int_0^1(d_{|u|}H(|u|))(v+\tau(u-v))\,d\tau\right)(u-v)(s,\cdot)\right\|_{L^1\cap H^1}ds.
\end{aligned}$$

The only new term to be considered is

$$\big\|(d_{|u|}H(|u|))(v+\tau(u-v))(s,\cdot)(u-v)(s,\cdot)\big\|_{\dot H^1}.$$

Using (4), we may estimate

$$\big|\nabla_x d_{|u|}H(|u|)(v+\tau(u-v))\big|\lesssim(|\nabla u|+|\nabla v|)\mu(|v+\tau(u-v)|)$$

and

$$\begin{aligned}
&\big\|(d_{|u|}H(|u|))(v+\tau(u-v))(s,\cdot)(u-v)(s,\cdot)\big\|_{\dot H^1}\\
&\quad\lesssim\mu(\|v+\tau(u-v)\|_{L^\infty})\big(\|\nabla u(s,\cdot)\|_{L^2}+\|\nabla v(s,\cdot)\|_{L^2}\big)\|(u-v)(s,\cdot)\|_{L^\infty}\\
&\qquad+\mu(\|v+\tau(u-v)\|_{L^\infty})(\|u(s,\cdot)\|_{L^\infty}+\|v(s,\cdot)\|_{L^\infty})\|\nabla(u-v)(s,\cdot)\|_{L^2}\\
&\quad\lesssim(1+s)^{-2}\mu\big(\varepsilon_0(1+s)^{-1}\big)(\|u\|_{X(s)}+\|v\|_{X(s)})\|u-v\|_{X(s)}.
\end{aligned}$$

Hence, we may estimate

$$
\begin{aligned}
&\|Gu(t,\cdot)-Gv(t,\cdot)\|_{L^\infty}\\
&\quad\lesssim \big(\|u\|_{X(t)}+\|v\|_{X(t)}\big)\,\|u-v\|_{X(t)}\\
&\qquad\times\int_0^t (1+t-s)^{-1}(1+s)^{-1}\mu\big(\varepsilon_0(1+s)^{-1}\big)\,ds\\
&\quad\le C'_{\varepsilon_0}(1+t)^{-1}\,\big(\|u\|_{X(t)}+\|v\|_{X(t)}\big)\,\|u-v\|_{X(t)},
\end{aligned}
$$

where $C'_{\varepsilon_0}$ tends to 0 for $\varepsilon_0$ to 0.

Summarizing all the estimates implies

$$
\|Nu-Nv\|_{X(t)}\le C_{\varepsilon_0}\|u-v\|_{X(t)}\left(\|u\|^{\frac{2}{n}}_{X(t)}+\|v\|^{\frac{2}{n}}_{X(t)}\right) \tag{15}
$$

for any $u,v\in X(t)$, where $C_{\varepsilon_0}$ tends to 0 for $\varepsilon_0$ to 0. Due to (14) the operator $N$ maps $X(t)$ into itself if $\varepsilon_0$ is small enough. The existence of a unique global (in time) Sobolev solution $u$ follows by contraction (15) and continuation argument for small data. □

## 3 Non-existence result via test function method

Following the proof of Theorem 3, we obtain a local (in time) Sobolev solution $u\in C\big([0,T),H^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$ to (5). For this reason we restrict ourselves to prove that this solution can not exist globally in time.

***Proof of Theorem 5*** We introduce the following functions:

$$
\eta(s)=\begin{cases}1 & \text{if } s\in[0,\frac12],\\ \text{is decreasing} & \text{if } s\in(\frac12,1),\\ 0 & \text{if } s\in[1,\infty),\end{cases}\qquad
\eta^*(s)=\begin{cases}0 & \text{if } s\in[0,\frac12],\\ \eta(s) & \text{if } s\in[\frac12,\infty),\end{cases}
$$

where the function $\eta=\eta(s)$ is supposed to belong to $C^\infty[0,\infty)$. For $R\ge R_0>0$, where $R_0$ is a large parameter, we define for $(t,x)\in[0,\infty)\times\mathbb{R}^n$ the cut-off functions

$$
\psi_R=\psi_R(t,x)=\eta\left(\frac{|x|^2+t}{R}\right)^{n+2}\quad\text{and}\quad \psi^*_R=\psi^*_R(t,x)=\eta^*\left(\frac{|x|^2+t}{R}\right)^{n+2}.
$$

We note that the support of $\psi_R$ is contained in

$$
Q_R=[0,R]\times B_{\sqrt{R}}\quad\text{with}\quad B_{\sqrt{R}}=\left\{x\in\mathbb{R}^n:\ |x|\le\sqrt{R}\right\}.
$$

The support of $\psi_R^*$ is contained in

$$Q_R^* = Q_R \setminus \left\{ (t,x) : |x|^2 + t \leq \frac{R}{2} \right\}.$$

We suppose that the Sobolev solution $u = u(t,x)$ exists globally in time, that is, the lifespan is $T = T(u) = \infty$. We define the functional

$$I_R = \int_{Q_R} h(|u(t,x)|)\psi_R(t,x)\,d(t,x) \quad \text{with} \quad h(s) := s^{1+\frac{2}{n}}\mu(s).$$

Then, by Eq. (5), after using integration by parts we arrive at

$$I_R = -\int_{\mathbb{R}^n} g(x)\psi_R(0,x)\,dx + \int_{Q_R} u(t,x)\left(\partial_t^2\psi_R - \Delta\psi_R - \partial_t\psi_R\right)d(t,x).$$

It holds

$$\begin{aligned}
\partial_t\psi_R &= \frac{n+2}{R}\eta\left(\frac{|x|^2+t}{R}\right)^{n+1}\eta'\left(\frac{|x|^2+t}{R}\right);\\
\partial_t^2\psi_R &= \frac{(n+2)(n+1)}{R^2}\eta\left(\frac{|x|^2+t}{R}\right)^{n}\eta'\left(\frac{|x|^2+t}{R}\right)^2\\
&\quad + \frac{n+2}{R^2}\eta\left(\frac{|x|^2+t}{R}\right)^{n+1}\eta''\left(\frac{|x|^2+t}{R}\right);\\
\partial_{x_j}^2\psi_R &= \frac{4(n+2)(n+1)x_j^2}{R^2}\eta\left(\frac{|x|^2+t}{R}\right)^{n}\eta'\left(\frac{x^2+t}{R}\right)^2\\
&\quad + \frac{4(n+2)x_j^2}{R^2}\eta\left(\frac{|x|^2+t}{R}\right)^{n+1}\eta''\left(\frac{|x|^2+t}{R}\right)\\
&\quad + \frac{2(n+2)}{R}\eta\left(\frac{|x|^2+t}{R}\right)^{n+1}\eta'\left(\frac{|x|^2+t}{R}\right).
\end{aligned}$$

Thus, since $0 \leq \eta \leq 1$ and $\eta'$, $\eta''$ are bounded on $[0,\infty)$, there exists $C > 0$ such that for each $(t,x) \in \operatorname{supp}\psi_R$ it holds

$$\left|\partial_t^2\psi_R - \Delta\psi_R - \partial_t\psi_R\right| \leq \frac{C}{R}(\psi_R^*(t,x))^{\frac{n}{n+2}}.$$

Thus, we get

$$\begin{aligned}
I_R = \int_{Q_R} h(|u(t,x)|)\psi_R(t,x)\,d(t,x) &\leq -\int_{\mathbb{R}^n} g(x)\psi_R(0,x)\,dx\\
&\quad + \frac{C}{R}\int_{Q_R} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x).
\end{aligned} \tag{16}$$

By applying Lemma 8 from the Appendix with $\alpha \equiv 1$ we get

$$h\left(\frac{\int_{Q_R^*} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x)}{\int_{Q_R^*} 1\,d(t,x)}\right) \le \frac{\int_{Q_R^*} h\big(|u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\big)\,d(t,x)}{\int_{Q_R^*} 1\,d(t,x)}.$$

Taking account of

$$\int_{Q_R^*} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x) = \int_{Q_R} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x),$$
$$\int_{Q_R^*} 1\,d(t,x) = C\int_{Q_R} 1\,d(t,x),$$

we arrive at the estimate

$$h\left(\frac{\int_{Q_R} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x)}{C\int_{Q_R} 1\,d(t,x)}\right) \le \frac{\int_{Q_R} h\big(|u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\big)\,d(t,x)}{C\int_{Q_R} 1\,d(t,x)}.$$

Notice that, since the modulus of continuity $\mu$ is non-decreasing, we can estimate

$$h\big(|u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\big) \le h(|u(t,x)|)\psi_R^*(t,x).$$

Moreover,

$$\int_{Q_R} 1\,d(t,x) = R^{\frac{n+2}{2}}.$$

Thus, thanks again to $\mu$ to be a non-decreasing function, there exists $h^{-1}$ and we may conclude

$$\int_{Q_R} |u(t,x)|(\psi_R^*(t,x))^{\frac{n}{n+2}}\,d(t,x) \le CR^{\frac{n+2}{2}} h^{-1}\left(\frac{\int_{Q_R} h(|u(t,x)|)\psi_R^*(t,x)\,d(t,x)}{CR^{\frac{n+2}{2}}}\right). \tag{17}$$

Let us define the functions

$$y = y(r) = \int_{Q_R} h(|u(t,x)|)\psi_r^*(t,x)\,d(t,x) \quad \text{and} \quad Y = Y(R) = \int_0^R y(r)r^{-1}\,dr.$$

Then, it holds

$$Y(R) = \int_0^R \left(\int_{Q_R} h(|u(t,x)|)\psi_r^*(t,x)\,d(t,x)\right) r^{-1}\,dr$$

$$= \int_{Q_R} h(|u(t,x)|) \left( \int_0^R \eta^* \left( \frac{|x|^2 + t}{r} \right)^{n+2} r^{-1} \, dr \right) d(t,x)$$
$$= \int_{Q_R} h(|u(t,x)|) \left( \int_{\frac{|x|^2+t}{R}}^{\infty} (\eta^*(s))^{n+2} s^{-1} ds \right) d(t,x).$$

Since supp $\eta^* \subset [1/2, 1]$ and $\eta^*$ is a non-increasing function on its support, we obtain the estimate

$$\int_{\frac{|x|^2+t}{R}}^{\infty} (\eta^*(s))^{n+2} s^{-1} \, ds \le \eta \left( \frac{|x|^2 + t}{R} \right)^{n+2} \int_{\frac{1}{2}}^{1} s^{-1} \, ds \le \log(2) \eta \left( \frac{x^2 + t}{R} \right)^{n+2}.$$

Consequently, we may conclude

$$Y(R) \le \log(2) \int_{Q_R} h(|u(t,x)|) \psi_R(t,x) \, d(t,x) = \log(2) \, I_R.$$

Moreover, we notice

$$Y'(R) R = y(R) = \int_{Q_R} h(|u(t,x)|) \psi_R^*(t,x) \, d(t,x).$$

Thus, by (16) and (17), we get

$$\frac{Y(R)}{\log(2)} \le C^2 R^{\frac{n}{2}} h^{-1} \left( \frac{Y'(R)}{C R^{\frac{n}{2}}} \right).$$

It follows

$$h \left( \frac{Y(R)}{C^2 \log(2) R^{\frac{n}{2}}} \right) \le \frac{Y'(R)}{C R^{\frac{n}{2}}}.$$

Thus, we have

$$\left( \frac{Y(R)}{C^2 \log(2) R^{\frac{n}{2}}} \right)^{\frac{n+2}{n}} \mu \left( \frac{Y(R)}{C^2 \log(2) R^{\frac{n}{2}}} \right) \le \frac{Y'(R)}{C R^{\frac{n}{2}}}.$$

For each $R \ge R_0$, since $Y = Y(r)$ is increasing we have $Y(R) \ge Y(R_0)$. Thus, since $\mu$ is non-decreasing, we have

$$\left( \frac{Y(R)}{C^2 \log(2) R^{\frac{n}{2}}} \right)^{\frac{n+2}{n}} \mu \left( \frac{Y(R_0)}{C^2 \log(2) R^{\frac{n}{2}}} \right) \le \frac{Y'(R)}{C R^{\frac{n}{2}}}.$$

Thus, we have

$$\frac{1}{R(C^2\log(2))^{\frac{n+2}{n}}}\mu\left(\frac{Y(R_0)}{C^2\log(2)R^{\frac{n}{2}}}\right)\le\frac{Y'(R)}{CY(R)^{\frac{n+2}{n}}}.$$

By integrating from $R_0$ to $R$, we can conclude that there exist constants $c_1$, $c_2$ such that

$$\int_{R_0}^{R}\frac{1}{s}\,\mu\big(c_2 s^{-\frac{n}{2}}\big)\,ds=c_1\int_{R^{-\frac{n}{2}}}^{R_0^{-\frac{n}{2}}}\frac{\mu(s)}{s}\,ds\lesssim\left[-\frac{1}{Y(s)^{\frac{2}{n}}}\right]_{R_0^{\frac{n}{2}}}^{R^{\frac{n}{2}}}\lesssim\frac{1}{Y(R_0^{\frac{n}{2}})^{\frac{2}{n}}}. \tag{18}$$

Due to the assumption that $u=u(t,x)$ exists globally in time it is allowed to form the limit $R\to\infty$ in (18). But this produces a contradiction, due to the fact that the right-hand side is bounded and the modulus of continuity $\mu$ satisfies condition (6). This completes our proof.

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## Appendix

In this section we include the following generalized version of Jensen inequality [11].

**Lemma 8** *Let $\Phi$ be a convex function on $\mathbb{R}$. Let $\alpha=\alpha(x)$ be defined and non-negative almost everywhere on $\Omega$, such that $\alpha$ is positive in a set of positive measure. Then, it holds*

$$\Phi\left(\frac{\int_\Omega u(x)\alpha(x)\,dx}{\int_\Omega \alpha(x)\,dx}\right)\le\frac{\int_\Omega \Phi(u(x))\alpha(x)\,dx}{\int_\Omega \alpha(x)\,dx}$$

*for all non-negative functions $u$ provided that all the integral terms are meaningful.*

***Proof*** Let $\gamma>0$ be fixed. From the convexity of $\Phi$ it follows that there exists $k\in\mathbb{R}^1$, such that

$$\Phi(t)-\Phi(\gamma)\ge k(t-\gamma)\quad\text{for all}\quad t\ge 0.$$

Putting $t=u(x)$ and multiplying the last inequality by $\alpha(x)$, we get after integration over $\Omega$ that

$$\int_\Omega \Phi(u(x))\alpha(x)\,dx-\Phi(\gamma)\int_\Omega \alpha(x)\,dx\ge k\left(\int_\Omega u(x)\alpha(x)\,dx-\gamma\int_\Omega \alpha(x)\,dx\right).$$

The statement follows by putting

$$\gamma = \frac{\int_\Omega u(x)\alpha(x)\,dx}{\int_\Omega \alpha(x)\,dx}.$$

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