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Lie theory of multiplicative tensors

Henrique Bursztyn[1] · **Thiago Drummond**[2]

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Abstract

We study tensors on Lie groupoids suitably compatible with the groupoid structure, called *multiplicative*. Our main result gives a complete description of these objects only in terms of infinitesimal data. Special cases include the infinitesimal counterparts of multiplicative forms, multivector fields and holomorphic structures, obtained through a unifying and conceptual method. We also give a full treatment of multiplicative vector-valued forms, particularly Nijenhuis operators and related structures.

Contents

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✉ Henrique Bursztyn
henrique@impa.br

Thiago Drummond
drummond@im.ufrj.br

1 Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Rio de Janeiro 22460-320, Brazil

2 Departamento de Matemática, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, Rio de Janeiro, RJ 21941-909, Brazil

1 Introduction

Lie groupoids are present in several areas of mathematics, including foliations, group actions and Poisson geometry. In these various contexts, Lie groupoids often come equipped with additional geometric structures, suitably compatible with the groupoid multiplication; such structures, referred to as *multiplicative*, are the main object of interest in this paper.

The study of multiplicative geometric structures on Lie groupoids has by now a long and rich history (see [38] for a recent survey). A basic example of multiplicative structure arises in the definition of complex Lie groups, regarded as real Lie groups endowed with a "compatible" complex structure; here, compatibility means that the group multiplication map is holomorphic. Another class of multiplicative structures on Lie groups arose in the early 80s with the emergence of Poisson-Lie groups, introduced by Drinfel'd [19] (later extended to Poisson groupoids by Weinstein [60]). Around the same time, the first examples of multiplicative differential forms on Lie groupoids appeared with the advent of symplectic groupoids [32,59] (and in their connections with the theory of hamiltonian actions [53] and equivariant cohomology; see [5,7, 62]). Multiplicative structures now abound in the literature, where one finds multi-vector fields [29,46,49,50], differential forms [1,3,8,16], contact and Jacobi structures [15,28,33], holomorphic structures [42,43], as well as distributions and foliations [16,21,27,30], among others (e.g. [23,44,55]).

Any Lie groupoid corresponds to a Lie algebroid, which linearizes it at the units. As in classical Lie theory, a central issue when considering multiplicative geometric structures is identifying their infinitesimal versions, i.e., finding their description solely in terms of Lie-algebroid data. This problem has been studied in numerous settings, through different approaches, leading to various "infinitesimal-global" correspondence results. Examples include the correspondences between symplectic groupoids and Poisson structures [9,13,51], Poisson groups/groupoids and Lie bialgebras/bialgebroids [46,49,51], contact groupoids and Jacobi structures [14,15], presymplectic groupoids and Dirac structures [7], complex Lie groupoids and holomorphic Lie algebroids [43], to mention a few (see also [1,10,29,57]). All these results rely on defining a "Lie functor", taking global to infinitesimal objects, and on an "integration" step, which reconstructs multiplicative structures from infinitesimal data.

In spite of these various results in specific settings, the theory of multiplicative geometric structures still lacked a complete treatment. The very notion of "multiplicativity" seemed to be adapted to each case at hand, and different techniques have been employed to handle seemingly analogous results. In this paper, we introduce the concept of *multiplicative tensor* on Lie groupoids, which agrees with the existing notions of multiplicativity in known situations, and devise a general method to obtain their complete infinitesimal description. As a consequence, all aforementioned "infinitesimal-global" correspondences can be naturally derived from our main result (Theorem 3.19), with a unified proof and conceptual approach, and new applications are obtained. Although we focus on ordinary tensors, our method adapts to more general contexts (such as the study of 1-cocycles on VB-groupoids), including tensors with values in representations (up to homotopy), see [20,22].

Main results Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source and target maps s and t. We denote its Lie algebroid by A, equipped with anchor map ρ and bracket $[\cdot,\cdot]$. Consider a (q,p)-tensor field[1] $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$. Let

$$\mathbb{G} := (\oplus^p T\mathcal{G}) \oplus (\oplus^q T^*\mathcal{G}),$$

which carries a natural groupoid structure over $\mathbb{M} := (\oplus^p TM) \oplus (\oplus^q A^*)$ induced from the tangent and cotangent groupoids of $\mathcal{G}$; see Sect. 2.

Definition 1.1 The tensor $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ is *multiplicative* if the map $c_\tau : \mathbb{G} \to \mathbb{R}$,

$$c_\tau(U_1,\ldots,U_p,\xi_1,\ldots,\xi_q) = \tau(U_1,\ldots,U_p,\xi_1,\ldots,\xi_q), \qquad U_i \in T_g\mathcal{G},\ \xi_j \in T_g^*\mathcal{G},$$

is a groupoid morphism, where $\mathbb{R}$ is viewed as an abelian group; in other words, c_τ is a differentiable *1-cocycle* on $\mathbb{G}$.

To state our main result, consider the action of $\Gamma(A)$ on $\Gamma(\wedge^p T^*M \otimes \wedge^q A)$ by

$$a \cdot (\beta \otimes \mathfrak{X}) = \mathcal{L}_{\rho(a)}\beta \otimes \mathfrak{X} + \beta \otimes [a, \mathfrak{X}],$$

where $[\cdot,\cdot]$ is the Schouten bracket on $\Gamma(\wedge^\bullet A)$. The following theorem gives a full description of the infinitesimal counterparts of multiplicative tensors:

Theorem 1.2 *If $\mathcal{G} \rightrightarrows M$ is a source 1-connected Lie groupoid, then there is a natural one-to-one correspondence between multiplicative (q,p)-tensors τ on $\mathcal{G}$ and triples (D,l,r), where $l : A \to \wedge^{p-1}T^*M \otimes \wedge^q A$ and $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1}A$ are vector bundle maps covering the identity, $D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes \wedge^q A)$ is an $\mathbb{R}$-linear map satisfying the Leibniz-type condition*

$$D(fa) = fD(a) + df \wedge l(a) - a \wedge r(df), \quad f \in C^\infty(M),\ a \in \Gamma(A),$$

[1] The assumption of skew symmetry leads to some simplifications of the results, but it is by no means essential.

and the following equations hold: for $a, b \in \Gamma(A)$ and $\alpha, \beta \in \Omega^1(M)$,

$$D([a,b]) = a \cdot D(b) - b \cdot D(a), \tag{IM1}$$
$$l([a,b]) = a \cdot l(b) - i_{\rho(b)} D(a), \tag{IM2}$$
$$r(\mathcal{L}_{\rho(a)}\alpha) = a \cdot r(\alpha) - i_{\rho^*(\alpha)} D(a), \tag{IM3}$$
$$i_{\rho(a)}\, l(b) = -i_{\rho(b)}\, l(a), \tag{IM4}$$
$$i_{\rho^*\alpha}\, r(\beta) = -i_{\rho^*\beta}\, r(\alpha), \tag{IM5}$$
$$i_{\rho(a)}\, r(\alpha) = i_{\rho^*\alpha}\, l(a). \tag{IM6}$$

We refer to the Eqs. (IM1)–(IM6) as *cocycle equations*, or *IM equations* (where IM stands for "infinitesimally multiplicative"). A more detailed formulation of the previous result can be found in Theorem 3.19 below.

The definition of the "Lie functor", taking multiplicative tensors τ to triples (D, l, r), relies on a useful characterization of multiplicativity proven in Theorem 3.11; it asserts, in particular, that for any $\alpha \in \Omega^1(M)$, $a \in \Gamma(A)$ and $\overrightarrow{a}$ the corresponding right-invariant vector field on $\mathcal{G}$, the tensors $i_{\overrightarrow{a}}\tau$, $i_{\mathsf{t}^*\alpha}\tau$, and $\mathcal{L}_{\overrightarrow{a}}\tau$ lie in the image of the map

$$\mathcal{T} : \Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet A) \to \Gamma(\wedge^\bullet T^*\mathcal{G} \otimes \wedge^\bullet T\mathcal{G}),$$

defined on homogeneous elements by $\mathcal{T}(\alpha \otimes \mathfrak{X}) = \mathsf{t}^*\alpha \otimes \overrightarrow{\mathfrak{X}}$. The maps D, l, and r arise from the equations

$$i_{\overrightarrow{a}}\tau = \mathcal{T}(l(a)), \quad i_{\mathsf{t}^*\alpha}\tau = \mathcal{T}(r(\alpha)), \quad \mathcal{L}_{\overrightarrow{a}}\tau = \mathcal{T}(D(a)).$$

The IM-equations, combined with the 1-connectedness of the source fibers of $\mathcal{G}$, permit the reconstruction of τ out of (D, l, r). Theorem 1.2, when restricted to tensors of types $(0, p)$ or $(q, 0)$, directly recovers the infinitesimal descriptions of multiplicative differential forms and multivector fields proven in [1,5,29], but using other methods. For $(1, 1)$-tensors, it encompasses the correspondence of complex Lie groupoids and holomorphic Lie algebroids of [43]. In these special cases, the operator D takes different guises, codifying k-differentials [29], IM-forms [5] (or the "Spencer operators" of [16]), or flat partial connections defining holomorphic structures.

The content of Theorem 1.2 is discussed in Sect. 3, and its proof is presented in Sect. 4, heavily based on our viewpoint to multiplicative tensors τ on $\mathcal{G}$ as multiplicative *functions* c_τ on the "big" Lie groupoid $\mathbb{G} \rightrightarrows \mathbb{M}$. Multiplicative functions are simple to describe infinitesimally: they correspond to Lie-algebroid 1-cocycles, i.e., sections of the dual of the Lie algebroid which are closed under the Lie-algebroid differential. So the proof follows from a detailed analysis of 1-cocycles of the Lie algebroid $\mathbb{A}$ of $\mathbb{G}$. The key fact that $\mathbb{A} \to \mathbb{M}$ has a natural *VB-algebroid* structure over $A \to M$ allows us to identify a special set of generators of the $C^\infty(\mathbb{M})$-module $\Gamma(\mathbb{A})$, parametrized by $\Gamma(A)$ and $\Omega^1(M)$. We use these generators to describe 1-cocycles of $\mathbb{A}$ by means of triples (D, l, r) as in Theorem 1.2, and we resort to classical lifting operations to realize the cocycle condition as the Eqs. (IM1)–((IM6).

In Sect. 5, we specialize our main result to the case of $(1, p)$-tensors, i.e., *multiplicative vector-valued forms*. As observed in [6], the usual Frölicher–Nijenhuis bracket preserves the multiplicativity condition, so it makes the space of multiplicative vector-valued forms into a graded Lie algebra. One of our key results is the identification of the corresponding graded Lie algebra at the infinitesimal level, in Proposition 5.4. In Sect. 6, we focus on multiplicative vector-valued 1-forms, i.e., (1,1)-tensors. In this case, we obtain an explicit infinitesimal description of their Nijenhuis torsions (Corollary 6.3), which provides a broader viewpoint to the results in [43,57] concerning multiplicative (almost) complex structures and Poisson (quasi-) Nijenhuis structures. We also treat multiplicative projections and product structures, interpreting their infinitesimal versions in terms of matched pairs.

2 Preliminaries

This section reviews some preliminary material, including tangent and cotangent Lie groupoids; see e.g. [47,49]. We also discuss a convenient viewpoint to classical tensor fields, regarded as real-valued functions on Whitney sums of vector bundles.

2.1 Tangent Lie groupoids

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We use the following notation: $\mathsf{s}, \mathsf{t} : \mathcal{G} \to M$ are the source and target maps, $1 : M \to \mathcal{G}$ is the unit map, $\mathsf{i} : \mathcal{G} \to \mathcal{G}$ is the inversion map, and $\mathsf{m} : \mathcal{G}\,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G} \to \mathcal{G}$ is the multiplication map. We will often identify M with its image under the unit map.

The *tangent groupoid* of $\mathcal{G}$ is the Lie groupoid $T\mathcal{G} \rightrightarrows TM$ whose structural maps are all obtained by taking the derivatives of the structural maps of $\mathcal{G}$; e.g., its multiplication map is $T\mathsf{m} : T\mathcal{G}\,{}_{T\mathsf{s}}\!\times_{T\mathsf{t}} T\mathcal{G} \to T\mathcal{G}$, where we have identified $T(\mathcal{G}\,{}_{\mathsf{s}}\!\times_{\mathsf{t}}\mathcal{G}) \cong T\mathcal{G}\,{}_{T\mathsf{s}}\!\times_{T\mathsf{t}} T\mathcal{G}$. We shall denote the multiplication on $T\mathcal{G}$ by $\bullet$.

We denote the Lie algebroid of $\mathcal{G}$ by $A \to M$, or $A\mathcal{G}$ if there is any risk of confusion. We identify A with $\ker(T\mathsf{s})|_M$, so the Lie bracket on $\Gamma(A)$ is induced by right-invariant vector fields on $\mathcal{G}$, and the anchor $\rho : A \to TM$ is given by $T\mathsf{t}|_A$. For $a \in \Gamma(A)$, we denote by $\overrightarrow{a} \in \mathfrak{X}(\mathcal{G})$ the corresponding right-invariant vector field, and by $\overleftarrow{a}$ the left-invariant vector field induced by $a - \rho(a) \in \Gamma(\ker(T\mathsf{t})|_M)$.

Note that each section $a \in \Gamma(A)$ defines a bisection $\mathcal{B}a : TM \to T\mathcal{G}$ of $T\mathcal{G} \rightrightarrows TM$,

$$\mathcal{B}a(X) = T1(X) + a(x), \tag{2.1}$$

for $X \in T_xM$, covering the map $TM \to TM$, $X \mapsto X + \rho(a)(x)$. This bisection splits the exact sequence

$$0 \longrightarrow A \hookrightarrow 1^*T\mathcal{G} \xrightarrow{T\mathsf{s}} TM \longrightarrow 0. \tag{2.2}$$

We refer to $\mathcal{B}a$ as the *translation bisection* associated with a.

2.2 Cotangent Lie groupoids

The cotangent bundle of a Lie groupoid $\mathcal{G} \rightrightarrows M$ also carries a natural Lie groupoid structure, $T^*\mathcal{G} \rightrightarrows A^*$, where A^* is the dual vector bundle to A. The source and target maps $\widetilde{\mathsf{s}}, \widetilde{\mathsf{t}} : T^*\mathcal{G} \to A^*$ are defined by the restriction of covectors to the subspaces tangent to the s- and t-fibers, respectively:

$$\langle \widetilde{\mathsf{s}}(\xi_g), a\rangle = \langle \xi_g, \overleftarrow{a}(g)\rangle \quad \text{and} \quad \langle \widetilde{\mathsf{t}}(\xi_g), b\rangle = \langle \xi_g, \overrightarrow{b}(g)\rangle, \tag{2.3}$$

for $\xi_g \in T^*_g\mathcal{G},\ a \in A_{\mathsf{s}(g)}, b \in A_{\mathsf{t}(g)}$. The unit map $\widetilde{1} : A^* \to T^*\mathcal{G}$ is the vector bundle morphism covering $1 : M \to \mathcal{G}$, determined by

$$\langle \widetilde{1}_\varphi, T1(X) + a\rangle = \langle \varphi, a\rangle,$$

for $(X, \varphi, a) \in TM \times_M A^* \times_M A$.

The multiplication on $T^*\mathcal{G}$ is defined as follows: for $\xi_1 \in T^*_{g_1}\mathcal{G}$ and $\xi_2 \in T^*_{g_2}\mathcal{G}$ such that $\widetilde{\mathsf{s}}(\xi_1) = \widetilde{\mathsf{t}}(\xi_2)$, their product $\xi_1 \bullet \xi_2 \in T^*_{g_1 g_2}\mathcal{G}$ is determined by

$$\langle \xi_1 \bullet \xi_2, U_1 \bullet U_2\rangle = \xi_1(U_1) + \xi_2(U_2), \tag{2.4}$$

for composable $U_1 \in T_{g_1}\mathcal{G}, U_2 \in T_{g_2}\mathcal{G}$.

The source map $\widetilde{\mathsf{s}}$ fits into the following exact sequence of vector bundles over M:

$$0 \longrightarrow T^*M \overset{(T\mathsf{t})^*}{\hookrightarrow} 1^*T^*\mathcal{G} \overset{\widetilde{\mathsf{s}}}{\longrightarrow} A^* \longrightarrow 0. \tag{2.5}$$

Given a differential 1-form $\alpha \in \Omega^1(M)$ on M, the map $\mathcal{B}\alpha : A^* \to 1^*T^*\mathcal{G}$ given by

$$\langle \mathcal{B}\alpha(\varphi), T1(X) + a\rangle = \langle \alpha(x), \rho(a) + X\rangle + \langle \varphi, a\rangle, \tag{2.6}$$

for $\varphi \in A^*_x,\ X \in T_xM,\ a \in A_x$, provides a splitting of the sequence (2.5). It is a bisection of $T^*\mathcal{G} \rightrightarrows A^*$ covering $\varphi \mapsto \varphi + \rho^*(\alpha(x)),\ \varphi \in A^*_x$. We call $\mathcal{B}\alpha$ the *translation bisection* corresponding to α.

2.3 Whitney sums

Tangent and cotangent groupoids satisfy the property that their Whitney sums as vector bundles again carry natural Lie groupoid structures, defined componentwise.[2] In this paper, we will be interested in Lie groupoids of the form $\mathbb{G}^{(p,q)} \rightrightarrows \mathbb{M}^{(p,q)}$, where

$$\mathbb{G}^{(p,q)} = (\oplus^p T\mathcal{G}) \oplus (\oplus^q T^*\mathcal{G}) \quad \text{and} \quad \mathbb{M}^{(p,q)} = (\oplus^p TM) \oplus (\oplus^q A^*), \tag{2.7}$$

for non-negative integers p and q. When there is no risk of confusion, we omit the indices (q, p) and write just $\mathbb{G} \rightrightarrows \mathbb{M}$. We denote the source and target maps by

[2] This property holds, more generally, for *VB-groupoids*, of which tangent and cotangent groupoids are special cases (see e.g. [4]).

$\mathbb{s}, \mathbb{t} : \mathbb{G} \to \mathbb{M}$ and the unit map by $\mathbb{1} : \mathbb{M} \to \mathbb{G}$. We keep the notation $\bullet$ for the multiplication.

We shall denote by $(\underline{X}, \underline{\varphi})$ and $(\underline{U}, \underline{\xi})$ the elements $(X_1, \ldots, X_p, \varphi_1, \ldots, \varphi_q) \in \mathbb{M}$ and $(U_1, \ldots, U_p, \xi_1, \ldots, \xi_q) \in \mathbb{G}$, respectively.

Remark 2.1 An important observation is that each source-fiber of $\mathbb{G}$ is an affine bundle over a source-fiber of $\mathcal{G}$, see e.g. [4, Rem. 3.1.1(a)]; it follows that $\mathbb{G}$ is source connected, or source 1-connected, if and only if so is $\mathcal{G}$.

2.4 Functions on the Whitney sum of vector bundles

A key viewpoint pursued in this work is that tensor fields should be regarded as functions on the Whitney sum of the tangent and cotangent bundles. We now explain this point of view from a general perspective. Let $E_1, \ldots, E_p$ be vector bundles over N and consider their Whitney sum $\pi : E_1 \oplus \cdots \oplus E_p \to N$.

Definition 2.2 A function $F : E_1 \oplus \cdots \oplus E_p \to \mathbb{R}$ is said to be *componentwise linear* if $F_y : (E_1)_y \times \cdots \times (E_p)_y \to \mathbb{R}$ is multi-linear, for each $y \in N$.

Example 2.3 For $p = 1$, componentwise linear functions on a vector bundle $E \to N$ are fiberwise linear functions, i.e., those of the form ℓ_μ, where

$$\ell_\mu(e) := \langle \mu(y), e \rangle, \quad \forall\, e \in E_y,$$

for $\mu \in \Gamma(E^*)$.

Every tensor field $\tau \in \Gamma(E_1^* \otimes \cdots \otimes E_p^*)$ defines a componentwise linear function $c_\tau : E_1 \oplus \cdots \oplus E_p \to \mathbb{R}$ by pulling back the linear function $\ell_\tau : E_1 \otimes \cdots \otimes E_p \to \mathbb{R}$ by the natural map $E_1 \oplus \cdots \oplus E_p \to E_1 \otimes \cdots \otimes E_p$. The letter "*c*" in our notation stands for "componentwise", and it is used to distinguish c_τ from the linear functions on $E_1 \oplus \cdots \oplus E_p$ defined by sections of its dual. In case $\tau = \mu_1 \otimes \cdots \otimes \mu_p$, for $\mu_i \in \Gamma(E_i^*)$, $i = 1, \ldots, p$,

$$c_\tau = \ell_{\mu_1} \circ \mathrm{pr}^1 \ldots \ell_{\mu_p} \circ \mathrm{pr}^p,$$

where $\mathrm{pr}^j : E_1 \oplus \cdots \oplus E_p \to E_j$ is the projection on the j-th component.

The next result is a direct consequence of the properties of tensor products.

Lemma 2.4 *The map $\tau \mapsto c_\tau$ defines a bijection between $\Gamma(E_1^* \otimes \cdots \otimes E_p^*)$ and the space of componentwise linear functions on $E_1 \oplus \cdots \oplus E_p$ satisfying*

$$c_{f\tau} = (f \circ \pi)\, c_\tau, \quad \forall\, f \in C^\infty(N),$$

where $\pi : E_1 \oplus \cdots \oplus E_p \to N$ is the natural projection.

When $E_1 = \cdots = E_p = E$, we say that a function $F \in C^\infty(\oplus_{i=1}^p E)$ is *skew-symmetric* if $F(e_{\sigma(1)}, \ldots, e_{\sigma(p)}) = sgn(\sigma) F(e_1, \ldots, e_p)$, for every permutation $\sigma \in$

$S(p)$. Under the correspondence of Lemma 2.4, the componentwise linear functions on $\oplus^p E$ which are skew-symmetric correspond to $\Gamma(\wedge^p E^*)$. For $\tau = \mu_1 \wedge \cdots \wedge \mu_p$, with $\mu_i \in \Gamma(E^*)$,

$$c_\tau = \sum_{\sigma \in S(p)} sgn(\sigma)(\ell_{\mu_{\sigma(1)}} \circ \mathrm{pr}^1) \cdots (\ell_{\mu_{\sigma(p)}} \circ \mathrm{pr}^p). \tag{2.8}$$

There is a natural projection $C^\infty(\oplus^p E) \to C^\infty(\oplus^p E)$ on the space of skew-symmetric functions, defined by

$$F \mapsto \frac{1}{p!} \sum_{\sigma \in S(p)} sgn(\sigma)\, F \circ \sigma, \tag{2.9}$$

where $\sigma : \oplus^p E \to \oplus^p E$ is given by $\sigma(e_1, \ldots, e_p) = (e_{\sigma(1)}, \ldots, e_{\sigma(p)})$.

Remark 2.5 Functions which are skew-symmetric only on some components can be defined similarly by considering $E_1 = \cdots = E_{p'} = E$, $p' < p$. Extending the previous observations to this case is straightforward.

3 Multiplicative tensors

In this section, we introduce our main object of study, multiplicative tensors, and state our main theorem, which gives their full infinitesimal description. As we follow the idea of regarding tensors as functions on Whitney sums of the tangent and cotangent bundles, we start by discussing multiplicative functions in general.

3.1 Multiplicative functions

Let $\mathcal{H} \rightrightarrows N$ be a Lie groupoid with Lie algebroid $A\mathcal{H}$.

Definition 3.1 A smooth function $f \in C^\infty(\mathcal{H})$ is said to be *multiplicative* if

$$f(h_1 h_2) = f(h_1) + f(h_2), \quad \forall (h_1, h_2) \in \mathcal{H}_{\mathsf{s}} \times_{\mathsf{t}} \mathcal{H}. \tag{3.1}$$

In other words, a multiplicative function $f : \mathcal{H} \to \mathbb{R}$ is a groupoid morphism from $\mathcal{H} \rightrightarrows N$ to the abelian Lie group $\mathbb{R}$. As such, it defines a Lie-algebroid morphism $Af : A\mathcal{H} \to \mathbb{R}$ given by $\langle Af, \chi \rangle = df(\chi)$, for $\chi \in A_y\mathcal{H}$ and $y \in N$; equivalently,

$$\langle Af, \chi(y) \rangle = \left(\mathcal{L}_{\overrightarrow{\chi}} f \right)(y), \tag{3.2}$$

for $\chi \in \Gamma(A\mathcal{H})$. When we view Af as a section of the dual bundle $A^*\mathcal{H}$, the condition for Af to be a Lie-algebroid morphism is expressed by the cocycle equation

$$d_A(Af) = 0,$$

where $d_A : \Gamma(\wedge^\bullet A^*\mathcal{H}) \to \Gamma(\wedge^{\bullet+1} A^*\mathcal{H})$ is the Lie-algebroid differential.

Example 3.2 The function $f = \mathsf{t}^*\psi - \mathsf{s}^*\psi$ is multiplicative, for any $\psi \in C^\infty(N)$. The associated cocycle Af is $d_A\psi \in \Gamma(A^*\mathcal{H})$, which is exact, i.e., a coboundary.

Example 3.3 Let $\pi : E \to N$ be a vector bundle, viewed as a Lie groupoid. A function $f \in C^\infty(E)$ is multiplicative if and only if it is fiberwise linear. Let $\mu \in \Gamma(E^*)$ be such that $f = \ell_\mu$. The cocycle $Af \in \Gamma(A^*E) = \Gamma(E^*)$ is given by

$$\langle Af, e\rangle = df\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \epsilon\, e\right) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \langle \mu, \epsilon\, e\rangle = \langle \mu, e\rangle, \quad \forall\, e \in E.$$

So $Af = \mu$ (which agrees with f itself, if seen as a function $E \to \mathbb{R}$). The equation $d_A\mu = 0$ is trivially satisfied since $d_A = 0$ in this case.

The next result gives a useful formula relating the cocycle Af and the function f.

Lemma 3.4 *For a multiplicative function $f : \mathcal{H} \to \mathbb{R}$, its corresponding cocycle $Af \in \Gamma(A^*\mathcal{H})$ satisfies*

$$\mathcal{L}_{\overrightarrow{\chi}} f = \mathsf{t}^*\langle Af,\ \chi\rangle, \quad \forall\, \chi \in \Gamma(A\mathcal{H}). \tag{3.3}$$

Proof First, by differentiating equation (3.1), one sees that

$$df(U \bullet V) = df(U) + df(V),\ \ \forall (U, V) \in T\mathcal{H}\,_{T\mathsf{s}}\!\times_{T\mathsf{t}} T\mathcal{H}.$$

Also, $\overrightarrow{\chi}(h) = \chi(\mathsf{t}(h)) \bullet 0_h$ (where 0_h is the zero vector field on $\mathcal{H}$ at h). Hence

$$(\mathcal{L}_{\overrightarrow{\chi}} f)(h) = df(\overrightarrow{\chi}(h)) = df(\chi(\mathsf{t}(h)) \bullet 0_h) = df(\chi(\mathsf{t}(h))) = \langle Af,\ \chi\rangle(\mathsf{t}(h)),$$

as we wanted. □

It turns out that when $\mathcal{H}$ is source connected, Eq. (3.3) essentially characterizes multiplicative functions:

Proposition 3.5 *Let $\mathcal{H} \rightrightarrows N$ be a source-connected Lie groupoid. A function $f \in C^\infty(\mathcal{H})$ is multiplicative if and only if $f|_N = 0$ and there exists $\mu \in \Gamma(A^*\mathcal{H})$ such that*

$$\mathcal{L}_{\overrightarrow{\chi}} f = \mathsf{t}^*\langle \mu, \chi\rangle, \quad \forall\, \chi \in \Gamma(A\mathcal{H}). \tag{3.4}$$

In this case, $Af = \mu$.

Proof If f is multiplicative, then $f|_N = 0$ as a consequence of (3.1). Also, setting $\mu = Af$, (3.4) follows from Lemma 3.4.

In the other direction, fix $\chi_1, \chi_2 \in \Gamma(A\mathcal{H})$. Then

$$\begin{aligned} \mathsf{t}^*\langle \mu, [\chi_1, \chi_2] \rangle &= \mathcal{L}_{[\overrightarrow{\chi_1}, \overrightarrow{\chi_2}]} f = \mathcal{L}_{\overrightarrow{\chi_1}} \mathcal{L}_{\overrightarrow{\chi_2}} f - \mathcal{L}_{\overrightarrow{\chi_2}} \mathcal{L}_{\overrightarrow{\chi_1}} f \\ &= \mathsf{t}^*(\mathcal{L}_{\rho(\chi_1)} \mu(\chi_2) - \mathcal{L}_{\rho(\chi_2)} \mu(\chi_1)), \end{aligned}$$

which implies that

$$d_A \mu(\chi_1, \chi_2) = \mathcal{L}_{\rho(\chi_1)} \mu(\chi_2) - \mathcal{L}_{\rho(\chi_2)} \mu(\chi_1) - \mu([\chi_1, \chi_2]) = 0.$$

So μ is a Lie-algebroid cocycle. If $\widetilde{\mathcal{H}}$ is a source 1-connected integration of $A\mathcal{H}$, then there exists a multiplicative function $f_\mu \in C^\infty(\widetilde{\mathcal{H}})$ such that $Af_\mu = \mu$. The Lie groupoids $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ correspond to the same Lie algebroid, so they are related by a groupoid map $\sigma : \widetilde{\mathcal{H}} \to \mathcal{H}$ which is a local diffeomorphism (and restricts to the identity map on N). To check that f is multiplicative, it is enough to check that $\sigma^* f$ is multiplicative. We will verify that $\sigma^* f = f_\mu$.

By (3.3), since σ is a groupoid morphism (hence commutes with structure maps and preserves invariant vector fields), one has that

$$\mathcal{L}_{\overrightarrow{\chi}} \sigma^* f = \sigma^* \mathcal{L}_{\overrightarrow{\chi}} f = \sigma^* \mathsf{t}^* \langle \mu, \chi \rangle = \mathsf{t}^* \langle Af_\mu, \chi \rangle = \mathcal{L}_{\overrightarrow{\chi}} f_\mu,$$

where we have kept the same notation for the structure maps on $\widetilde{\mathcal{H}}$ and $\mathcal{H}$. Since $\widetilde{\mathcal{H}}$ has source-connected fibers, it follows that $\sigma^* f - f_\mu$ is constant along the s-fibers. Finally, since $(\sigma^* f - f_\mu)|_N = 0$, one has that $\sigma^* f - f_\mu = 0$ everywhere. □

3.2 Definition and examples

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider a (q, p)-tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$. Let $c_\tau : \mathbb{G}^{(p,q)} \to \mathbb{R}$ be the corresponding componentwise linear function,

$$c_\tau(U_1, \ldots, U_p, \xi_1, \ldots, \xi_q) = \tau(U_1, \ldots, U_p, \xi_1, \ldots, \xi_q),$$

as in Definition 2.2, where $\mathbb{G}^{(p,q)} \rightrightarrows \mathbb{M}^{p,q)}$ is the Lie groupoid (2.7). In the following, we shall omit the (p, q)-indices.

Definition 3.6 A (q, p)-tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ on $\mathcal{G} \rightrightarrows M$ is *multiplicative* if the function $c_\tau : \mathbb{G} \to \mathbb{R}$ is multiplicative.

Note that the very same definition of multiplicativity makes sense for elements of $\Gamma((\otimes^p T^*\mathcal{G}) \otimes (\otimes^q T\mathcal{G}))$ (or for their symmetric versions). Along the paper, we will make some comments on how to adapt our results to this more general case.

The next examples relate our definition with known notions of multiplicativity for special types of tensors.

Example 3.7 A differential form $\omega \in \Omega^p(\mathcal{G})$ is multiplicative if it satisfies

$$\mathsf{m}^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega,$$

where $\mathrm{pr}_i : \mathcal{G}\,{}_{\mathsf{s}}\times_{\mathsf{t}}\mathcal{G} \to \mathcal{G} \to \mathcal{G}$, $i = 1, 2$, are the natural projections. One can directly check (see e.g. [3]) that this is equivalent to c_ω being multiplicative.

Example 3.8 A multivector field $\Pi \in \mathfrak{X}^q(\mathcal{G})$ is said to be multiplicative [29] if the graph of the multiplication map is coisotropic with respect to $\Pi \oplus \Pi \oplus (-1)^{q+1}\Pi$: for $\xi_i \in \mathrm{Ann}(\mathrm{graph}(\mathsf{m})) \subset T^*(\mathcal{G} \times \mathcal{G} \times \mathcal{G})$, $i = 1, \dots, q$, we have

$$(\Pi \oplus \Pi \oplus (-1)^{q+1}\Pi)(\xi_1, \dots, \xi_q) = 0.$$

It is shown in [29] that this condition is equivalent to c_Π being multiplicative.

The next result shows that our definition of multiplicativity for $(1, p)$-tensor fields $K \in \Gamma(\wedge^p T^*\mathcal{G} \otimes T\mathcal{G})$ agrees with the one given in [43]. Note that K can be seen as a map $\overline{K} : \oplus^p T\mathcal{G} \to T\mathcal{G}$.

Proposition 3.9 *A $(1, p)$-tensor field $K \in \Gamma(\wedge^p T^*\mathcal{G} \otimes T\mathcal{G})$ is multiplicative if and only if there is a vector-bundle map $\overline{r} : \oplus^p TM \to TM$ covering the identity such that*

$$\begin{array}{ccc} \oplus^p T\mathcal{G} & \xrightarrow{\overline{K}} & T\mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \oplus^p TM & \xrightarrow{\overline{r}} & TM \end{array} \tag{3.5}$$

is a groupoid morphism.

Proof It is straightforward to check that if (3.5) is a groupoid morphism, then K is multiplicative. Conversely, let us assume that K is a multiplicative $(1, p)$-tensor field. From Proposition 3.5, one knows that $c_K|_{\mathbb{M}} = 0$. This implies that, for $(X_1, \dots, X_p) \in TM \oplus \cdots \oplus TM$, $\overline{K}(X_1, \dots, X_p) \in T\mathcal{G}|_M$ has zero component on A under the decomposition $T\mathcal{G}|_M = TM \oplus A$. So, define $\overline{r} = \overline{K}|_{\oplus^p TM}$.

It is straightforward to see that once

$$T\mathsf{t} \circ \overline{K} = \overline{r} \circ T\mathsf{t}, \quad T\mathsf{s} \circ \overline{K} = \overline{r} \circ T\mathsf{s},$$

$\overline{K}$ will preserve the multiplication as a direct consequence of the multiplicativity of K. So, let $\alpha \in T^*_{\mathsf{s}(g)}M$.

$$\begin{aligned} \langle T\mathsf{s}(\overline{K}(U_1, \dots, U_p)), \alpha\rangle &= c_K(U_1, \dots, U_p, (d\mathsf{s})^*_g\,\alpha) \\ &= c_K(U_1 \bullet T\mathsf{s}(U_1), \dots, U_p \bullet T\mathsf{s}(U_p), 0_g \bullet (d\mathsf{s})^*_{\mathsf{s}(g)}\,\alpha) \\ &= c_K(U_1, \dots, U_p, 0_g) + c_K(T\mathsf{s}(U_1), \dots, T\mathsf{s}(U_p), (d\mathsf{s})^*_{\mathsf{s}(g)}\,\alpha)) \\ &= \langle \overline{r}(T\mathsf{s}(U_1), \dots, T\mathsf{s}(U_p)), \alpha\rangle. \end{aligned}$$

Here, we have used the equality $(d\mathsf{s})^*_g\,\alpha = 0_g \bullet (d\mathsf{s})^*_{\mathsf{s}(g)}\,\alpha$, which follows from (2.4). The other equality follows similarly. □

We now consider a special class of multiplicative tensor fields on $\mathcal{G}$, analogous to the multiplicative functions in Example 3.2. Let $\mathcal{S},\ \mathcal{T} : \Gamma(\wedge^p T^*M \otimes \wedge^q A) \to \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ be the $\mathbb{R}$-linear maps defined on homogeneous elements as

$$\mathcal{T}(\alpha \otimes \mathfrak{X}) = \mathsf{t}^*\alpha \otimes \overrightarrow{\mathfrak{X}}, \quad \mathcal{S}(\alpha \otimes \mathfrak{X}) = \mathsf{s}^*\alpha \otimes \overleftarrow{\mathfrak{X}}. \tag{3.6}$$

Proposition 3.10 *The maps $\mathcal{S}$ and $\mathcal{T}$ satisfy the following properties:*
(1) $\mathcal{T}(f\Phi) = (t^*f)\,\mathcal{T}(\Phi)$, $\mathcal{S}(f\Phi) = (s^*f)\,\mathcal{S}(\Phi)$,
(2) $\mathbb{t}^*c_\Phi = c_{\mathcal{T}(\Phi)}$,
(3) $\mathbb{s}^*c_\Phi = c_{\mathcal{S}(\Phi)}$,
for $\Phi \in \Gamma(\wedge^p T^*M \otimes \wedge^q A)$ and $f \in C^\infty(M)$. In particular,

$$(\mathcal{T} - \mathcal{S})(\alpha \otimes \mathfrak{X}) = t^*\alpha \otimes \overrightarrow{\mathfrak{X}} - s^*\alpha \otimes \overleftarrow{\mathfrak{X}} \tag{3.7}$$

is a multiplicative (exact) (q, p)-tensor field on $\mathcal{G}$.

Proof The formulas in (1) can be verified directly.

For Φ of the form $\alpha \otimes \mathfrak{X}$ and $\xi_1, \dots, \xi_q \in T^*_g\mathcal{G}$, formulas (2) and (3) reduce to

$$\overrightarrow{\mathfrak{X}}(\xi_1, \dots, \xi_q) = \mathfrak{X}(\widetilde{\mathsf{t}}(\xi_1), \dots, \widetilde{\mathsf{t}}(\xi_q)) \quad \text{and} \quad \overleftarrow{\mathfrak{X}}(\xi_1, \dots, \xi_q) = \mathfrak{X}(\widetilde{\mathsf{s}}(\xi_1), \dots, \widetilde{\mathsf{s}}(\xi_q)),$$

respectively, and these identities are direct consequences of the definitions of the source and target maps of the cotangent groupoid (see (2.3)). The case of arbitrary Φ follows by linearity.

The last assertion follows from the fact that $\mathbb{t}^*c_\Phi - \mathbb{s}^*c_\Phi$ is always a multiplicative function (see Example 3.2). □

3.3 The main results: statements and first examples

We now present a complete infinitesimal characterization of multiplicative tensor fields. Our first main theorem is an analog of Proposition 3.5 for general tensor fields.

Theorem 3.11 *Let $\mathcal{G} \rightrightarrows M$ be a source-connected Lie groupoid. A (q, p)-tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ is multiplicative if and only if*

$$\tau(\underline{X}, \underline{\varphi}) = 0, \quad \forall\, (\underline{X}, \underline{\varphi}) \in \mathbb{M}, \tag{3.8}$$

and there exist vector-bundle maps $l : A \to \wedge^{p-1}T^*M \otimes \wedge^q A$ and $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} A$, covering the identity map on M, and an $\mathbb{R}$-linear map $D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes \wedge^q A)$, such that

$$D(fa) = f D(a) + df \wedge l(a) - a \wedge r(df), \tag{3.9}$$

$$i_{\overrightarrow{a}}\,\tau = \mathcal{T}(l(a)), \quad i_{t^*\alpha}\,\tau = \mathcal{T}(r(\alpha)), \quad \mathcal{L}_{\overrightarrow{a}}\,\tau = \mathcal{T}(D(a)), \tag{3.10}$$

for $a \in \Gamma(A)$, $\alpha \in \Gamma(T^*M)$ and $f \in C^\infty(M)$.

We refer to the triple (D, l, r) as the *infinitesimal components* of the multiplicative tensor τ and to Eq. (3.9) as the Leibniz condition for D.

Remark 3.12 Regarding the notation in Theorem 3.11 above, in (3.9) we view $\Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet A)$ as a left module for the exterior algebras $\Gamma(\wedge^\bullet A)$ and $\Gamma(\wedge^\bullet T^*M)$, and both actions are denoted by $\wedge$: for $\mathfrak{Y} \in \Gamma(\wedge^\bullet A)$, $\eta \in \Gamma(\wedge^\bullet T^*M)$, and $\omega \otimes \mathfrak{X} \in \Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet A)$,

$$\mathfrak{Y} \wedge (\omega \otimes \mathfrak{X}) = \omega \otimes (\mathfrak{Y} \wedge \mathfrak{X}), \qquad \eta \wedge (\omega \otimes \mathfrak{X}) = (\eta \wedge \omega) \otimes \mathfrak{X}.$$

In (3.10), the contraction operators are defined as follows: for $U \in \Gamma(T\mathcal{G})$, $\xi \in \Gamma(T^*\mathcal{G})$ and $\tau = \lambda \otimes W$,

$$i_U \tau = (i_U \lambda) \otimes W, \qquad i_\xi \tau = U \otimes (i_\xi W).$$

We mention some particular cases of interest.

Corollary 3.13 *On a source-connected Lie groupoid $\mathcal{G} \rightrightarrows M$, a p-form $\omega \in \Omega^p(\mathcal{G})$ is multiplicative if and only if there is an $\mathbb{R}$-linear map $D : \Gamma(A) \to \Omega^p(M)$ and a vector-bundle morphism $l : A \to \wedge^{p-1} T^*M$ such that the following holds:*

$$1^*\omega = 0, \quad \mathcal{L}_{\overrightarrow{a}}\omega = t^*D(a), \quad i_{\overrightarrow{a}}\omega = t^*l(a),$$

for all $a \in \Gamma(A)$.

The next result recovers [29, Thm. 2.19] (showing that some conditions there are redundant, cf. [11, Lem. 2.3]).

Corollary 3.14 *Let $\mathcal{G} \rightrightarrows M$ be a source-connected Lie groupoid. A q-vector field $\Pi \in \mathfrak{X}^q(\mathcal{G})$ is multiplicative if and only there is an $\mathbb{R}$-linear map $D : \Gamma(A) \to \Gamma(\wedge^q A)$ and a vector bundle morphism $r : T^*M \to \wedge^{q-1} A$ such that the following holds:*

$$\Pi(\varphi_1, \dots, \varphi_q) = 0, \quad \mathcal{L}_{\overrightarrow{a}}\Pi = \overrightarrow{D(a)}, \quad i_{t^*\alpha}\Pi = \overrightarrow{r(\alpha)},$$

for all $(\varphi_1, \dots, \varphi_q) \in A^* \times_M \cdots \times_M A^*$, $a \in \Gamma(A)$ and $\alpha \in \Omega^1(M)$. (In the terminology of [29,51], the first condition above expresses the fact that M is coisotropic with respect to Π, while the other conditions express the fact that Π is an affine multivector field (*see also* [18,35]).)

In view of the previous corollary, Theorem 3.11 suggests a notion of *affine* tensors on Lie groupoids, in which we retain all properties of multiplicative tensors described in Theorem 3.11 except for (3.8) (cf. [61, Thm. 4.5]).

In the next example, we identify the infinitesimal components of a multiplicative tensor field of type (3.7). To do so, let us consider, for a Lie algebroid $(A, \rho, [\cdot, \cdot])$, the action of the Lie algebra $\Gamma(A)$ on $\Gamma(\wedge^p T^*M \otimes \wedge^q A)$ given by

$$a \cdot (\beta \otimes \mathfrak{X}) = \mathcal{L}_{\rho(a)}\beta \otimes \mathfrak{X} + \beta \otimes [a, \mathfrak{X}], \tag{3.11}$$

where $[\cdot, \cdot]$ is the Schouten bracket on $\Gamma(\wedge^\bullet A)$.

Example 3.15 For $\Phi \in \Gamma(\wedge^p T^*M \otimes \wedge^q A)$, consider the multiplicative tensor given by $\tau = (\mathcal{T} - \mathcal{S})(\Phi)$ (see Proposition 3.10). As particular cases, for $q = 0$, $\tau = \mathsf{t}^*\Phi - \mathsf{s}^*\Phi$, while for $p = 0$, $\tau = \overrightarrow{\Phi} - \overleftarrow{\Phi}$. The infinitesimal components (D, l, r) corresponding to τ are

$$D(a) = a \cdot \Phi, \quad l(a) = i_{\rho(a)}\Phi, \quad r(\alpha) = i_{\rho^*(\alpha)}\Phi.$$

Remark 3.16 For $\tau \in \Gamma((\otimes^p T^*\mathcal{G}) \otimes (\otimes^q T\mathcal{G}))$, there is a result similar to Theorem 3.11 characterizing multiplicativity. In this case we have infinitesimal components $(D, l_1, \ldots, l_p, r_1, \ldots, r_q)$, where $D : \Gamma(A) \to \Gamma((\otimes^p T^*M) \otimes (\otimes^q A))$, $l_i : A \to (\otimes^{p-1} T^*M) \otimes (\otimes^q A)$, and $r_j : T^*M \to (\otimes^p T^*M) \otimes (\otimes^{q-1} A)$, defined by

$$\mathcal{L}_{\overrightarrow{a}}\tau = \mathcal{T}(D(a)), \quad \underbrace{\tau(\ldots, \overrightarrow{a}, \ldots)}_{\text{i-th } T\mathcal{G}\text{-entry}} = \mathcal{T}(l_i(a)), \quad \underbrace{\tau(\ldots, \mathsf{t}^*\alpha, \ldots)}_{\text{j-th } T^*\mathcal{G}\text{-entry}} = \mathcal{T}(r_j(\alpha)). \tag{3.12}$$

The Leibniz equation for D will change accordingly. For instance, for a multiplicative $\tau \in \Gamma(T\mathcal{G} \otimes T\mathcal{G})$, the infinitesimal components (D, r_1, r_2) satisfy

$$D(fa) = f D(a) - a \otimes r_1(df) - r_2(df) \otimes a.$$

We now formulate our main result, which concerns the correspondence between multiplicative tensors τ and their infinitesimal components (D, l, r). The multiplicativity of τ is expressed by a set of equations satisfied by (D, l, r), described in the next definition.

Definition 3.17 Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid. An *IM (q, p)-tensor* on A is a triple (D, l, r), where $l : A \to \wedge^{p-1} T^*M \otimes \wedge^q A$ and $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} A$ are vector-bundle maps covering the identity map on M, $D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes \wedge^q A)$ is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D(fa) = f D(a) + df \wedge l(a) - a \wedge r(df), \quad \forall f \in C^\infty(M),\ a \in \Gamma(A),$$

such that the following equations hold:

$$D([a, b]) = a \cdot D(b) - b \cdot D(a), \tag{IM1}$$
$$l([a, b]) = a \cdot l(b) - i_{\rho(b)} D(a), \tag{IM2}$$
$$r(\mathcal{L}_{\rho(a)}\alpha) = a \cdot r(\alpha) - i_{\rho^*(\alpha)} D(a), \tag{IM3}$$
$$i_{\rho(a)}\, l(b) = -i_{\rho(b)}\, l(a), \tag{IM4}$$
$$i_{\rho^*\alpha}\, r(\beta) = -i_{\rho^*\beta}\, r(\alpha), \tag{IM5}$$
$$i_{\rho(a)}\, r(\alpha) = i_{\rho^*\alpha}\, l(a), \tag{IM6}$$

for $a, b \in \Gamma(A)$ and $\alpha, \beta \in \Omega^1(M)$.

We refer to Eqs. (IM1)–(IM6) as the *IM-equations* of an IM (q, p)-tensor.

Remark 3.18 (Redundancies) We observe that, in many cases, there are some redundancies in the IM-equations. Note first that, if $p > \dim(M) + 1$ or $q > \mathrm{rank}(A) + 1$, then any IM (p, q)-tensor is trivial. On the other hand, when $p < \dim(M) + 1$ and $q < \mathrm{rank}(A) + 1$, we claim that {(IM1), (IM2), (IM6)} and {(IM1), (IM3), (IM6)} are minimal sets of independent IM-equations. Indeed, in this case, (IM1) implies that

$$df \wedge (l([a,b]) - a \cdot l(b) + i_{\rho(b)} D(a)) = b \wedge (r(\mathcal{L}_{\rho(a)}(df)) - a \cdot r(df) + i_{df} D(a)),$$

for all $f \in C^\infty(M)$, $a, b \in \Gamma(A)$. This follows from the Leibniz rule for the Lie bracket $[\cdot,\cdot]$, the Leibniz formula for D, and Proposition 4.3 below. Therefore,

$$(\mathrm{IM1}) + (\mathrm{IM2}) \Rightarrow (\mathrm{IM3}) \text{ and } (\mathrm{IM1}) + (\mathrm{IM3}) \Rightarrow (\mathrm{IM2}).$$

Similarly, one can prove that (IM2) (resp. (IM3)) implies that

$$df \wedge (i_{\rho(a)} l(b) + i_{\rho(b)} l(a)) = a \wedge (i_{df} l(b) - i_{\rho(b)} r(df))$$
$$(\text{resp. } a \wedge (i_{df} r(\alpha) + i_{\rho^*\alpha} r(df)) = df \wedge (i_{\rho(a)} r(\alpha) - i_{\rho^*\alpha} l(a))).$$

As a result, (IM2) + (IM6) ⇒ (IM4) and (IM3) + (IM6) ⇒ (IM5).

Our main theorem can now be stated as follows:

Theorem 3.19 *Let $\mathcal{G} \rightrightarrows M$ be a source 1-connected Lie groupoid, and let A be its Lie algebroid. There is a one-to-one correspondence between multiplicative (q, p)-tensor fields $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ and IM (q, p)-tensors (D, l, r) on A satisfying*

$$i_{\overrightarrow{a}} \tau = \mathcal{T}(l(a))$$
$$i_{t^*\alpha} \tau = \mathcal{T}(r(\alpha))$$
$$\mathcal{L}_{\overrightarrow{a}} \tau = \mathcal{T}(D(a)),$$

where $\mathcal{T}$ is the map given by (3.6).

Remark 3.20 The correspondence in Theorem 3.19 can be naturally extended to multiplicative tensors $\tau \in \Gamma((\otimes^p T^*\mathcal{G}) \otimes (\otimes^q T\mathcal{G}))$; in this case, following Remark 3.16, we have more infinitesimal components $(D, l_1, \ldots, l_p, r_1, \ldots, r_q)$, obtained as in (3.12), satisfying analogous IM-equations.

We now show how Theorem 3.19, when restricted to $(0, p)$ and $(q, 0)$ tensors, directly recovers the infinitesimal descriptions of multiplicative differential forms and multivector fields, proven in [1,5,29]. It also recovers the correspondences in [43,57], but we will leave this discussion to Sect. 6, where we present a more general treatment of multiplicative $(1, 1)$-tensor fields.

Multiplicative multivector fields We start by illustrating our result in the simplest context of multiplicative vectors fields on Lie groupoids, treated in [50] (see also [47, Sec. 3.4]). In this case, our general notion of multiplicativity admits an alternative formulation: a vector field Z on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is *multiplicative* if $Z : \mathcal{G} \to$

$T\mathcal{G}$ is a groupoid morphism; in particular, it covers a map $X : M \to TM$, which is a vector field on M. To describe Z in infinitesimal terms, one considers the Lie-algebroid morphism $Z_A : A \to TA$ obtained by differentiating $Z : \mathcal{G} \to T\mathcal{G}$, and note that Z_A is a *linear* vector field on A; i.e., viewed as a derivation of $C^\infty(A)$, Z_A preserves linear functions on A. By identifying linear functions on A with sections of A^*, we have an induced $\mathbb{R}$-linear map $\Delta^* : \Gamma(A^*) \to \Gamma(A^*)$, which is a *derivation* on A^* in the sense that it satisfies $\Delta^*(f\varphi) = f\Delta^*(\varphi) + (\mathcal{L}_X f)\varphi$, for $\varphi \in \Gamma(A^*)$, $f \in C^\infty(M)$. We can alternatively consider the dual derivation $\Delta : \Gamma(A) \to \Gamma(A)$, defined by $\langle\varphi, \Delta(a)\rangle = \mathcal{L}_X\langle\varphi, a\rangle - \langle\Delta^*(\varphi), a\rangle$. It is then proven in [50] that the fact that Z_A is a morphism of Lie algebroids is equivalent to Δ being a derivation of the Lie bracket on $\Gamma(A)$,

$$\Delta([a, b]) = [\Delta(a), b] + [a, \Delta(b)], \qquad a, b \in \Gamma(A),$$

and that one can obtain Δ directly from Z via $\overrightarrow{\Delta(a)} = [Z, \overrightarrow{a}]$, for $a \in \Gamma(A)$. Comparing to Theorem 3.19, we see that $D = -\Delta : \Gamma(A) \to \Gamma(A)$ and $r : T^*M \to \mathbb{R}$, $r(df) = \mathcal{L}_X f$, are the infinitesimal components of Z. We will now see how this result generalizes to multiplicative multivector fields, as in [29].

Since multiplicative q-vector fields are multiplicative $(q, 0)$-tensor fields, it follows from Theorem 3.19 that their infinitesimal counterparts are IM $(q, 0)$-tensors: on a given Lie algebroid $A \to M$, these are pairs (D, r), where $r : T^*M \to \wedge^{q-1}A$ is a vector-bundle map (covering the identity), $D : \Gamma(A) \to \Gamma(\wedge^q A)$ is $\mathbb{R}$-linear and satisfies

$$D(fa) = fD(a) - a \wedge r(df) = fD(a) + (-1)^q r(df) \wedge a, \tag{3.13}$$

for $a \in \Gamma(A)$ and $f \in C^\infty(M)$, and the following compatibility conditions hold:

$$\begin{aligned} D([a, b]) &= [a, D(b)] - [b, D(a)] = [D(a), b] + [a, D(b)] \\ r(\mathcal{L}_{\rho(a)}\alpha) &= [a, r(\alpha)] - i_{\rho^*\alpha} D(a) \\ i_{\rho^*\alpha} r(\beta) &= -i_{\rho^*\beta} r(\alpha), \end{aligned}$$

for $a, b \in \Gamma(A)$, $\alpha, \beta \in \Omega^1(M)$.

In order to make the connection with the work in [29] (and following its terminology), recall that a *q-differential* on a Lie algebroid A is an $\mathbb{R}$-linear map $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+q-1} A)$ satisfying

$$\delta(\mathfrak{X}_1 \wedge \mathfrak{X}_2) = \delta(\mathfrak{X}_1) \wedge \mathfrak{X}_2 + (-1)^{k_1(q-1)}\mathfrak{X}_1 \wedge \delta(\mathfrak{X}_2) \tag{3.14}$$

$$\delta([\mathfrak{X}_1, \mathfrak{X}_2]) = [\delta(\mathfrak{X}_1), \mathfrak{X}_2] + (-1)^{(k_1-1)(q-1)}[\mathfrak{X}_1, \delta(\mathfrak{X}_2)] \tag{3.15}$$

where $\mathfrak{X}_i \in \Gamma(\wedge^{k_i} A)$, $i = 1, 2$, and $[\cdot, \cdot]$ is the Schouten bracket on $\Gamma(\wedge^\bullet A)$ (which makes it into a Gerstenhaber algebra). In other words, δ is a derivation of degree $(q-1)$ of $\Gamma(\wedge^\bullet A)$ which is also a derivation of the Schouten bracket. We denote the space of q-differentials by $\mathcal{A}_q$. The space $\mathcal{A} = \oplus_{q\geq 0}\mathcal{A}_q$ is naturally a *Gerstenhaber algebra* with respect to the bracket given by the commutator

$$[\delta, \widetilde{\delta}] = \delta \circ \widetilde{\delta} - (-1)^{(q-1)(\widetilde{q}-1)} \widetilde{\delta} \circ \delta,$$

where $\delta \in \mathcal{A}_q$ and $\widetilde{\delta} \in \mathcal{A}_{\widetilde{q}}$.

Note that a q-differential δ is determined by its restrictions

$$\delta_0 : C^\infty(M) \to \Gamma(\wedge^{q-1} A), \qquad \delta_1 : \Gamma(A) \to \Gamma(\wedge^q A).$$

For this reason, we may denote a q-differential by the pair (δ_0, δ_1).

Before studying the relationship between q-differentials and IM $(q, 0)$-tensors, we list here some properties of the Schouten bracket that we need (see e.g. [40,41]):

(1) For $f \in C^\infty(N)$, $\mathfrak{X} \in \Gamma(\wedge^q A)$,

$$[\mathfrak{X}, f] = (-1)^{q-1} i_{\rho^* df} \mathfrak{X}.$$

(2) For $\mathfrak{X}_i \in \Gamma(\wedge^{q_i} A)$, $i = 1, 2$,

$$[\mathfrak{X}_1, \mathfrak{X}_2] = -(-1)^{(q_1-1)(q_2-1)} [\mathfrak{X}_2, \mathfrak{X}_1].$$

(3) For $\mathfrak{X}_i \in \Gamma(\wedge^{q_i} A)$, $i = 1, 2, 3$,

$$[\mathfrak{X}_1, \mathfrak{X}_2 \wedge \mathfrak{X}_3] = [\mathfrak{X}_1, \mathfrak{X}_2] \wedge \mathfrak{X}_3 + (-1)^{(q_1-1)q_2} \mathfrak{X}_2 \wedge [\mathfrak{X}_1, \mathfrak{X}_3].$$

Lemma 3.21 *There is a one-to-one correspondence between q-differentials (δ_0, δ_1) and IM $(q, 0)$-tensors (D, r) via*

$$\delta_1 = D, \qquad \delta_0 = (-1)^q r \circ d, \tag{3.16}$$

where d is the de Rham differential.

Proof Note that (3.14), (3.15) give rise to five equations involving δ_0 and δ_1, to be compared with the four equations characterizing IM $(q, 0)$-tensors.

For $k_1 = k_2 = 0$, (3.14) is equivalent to the existence of a vector-bundle map $r_0 : T^*M \to \wedge^{q-1} A$ such that $r_0(df) = \delta_0(f)$. This guarantees that we can always assume that δ_0 is of the form described in (3.16), i.e., we set $r = (-1)^q r_0$ and $D = \delta_1$.

For $k_1 = 1$ and $k_2 = 0$, (3.14) becomes

$$D(af) = D(a)f + (-1)^{q-1} a \wedge (-1)^q r(df) = f D(a) - a \wedge r(df),$$

which is just the Leibniz rule for (D, r).

Next, when $k_1 = k_2 = 0$ (3.15) reads

$$0 = (-1)^q ([r(df), g] + (-1)^{q-1} [f, r(dg)]).$$

Because the Schouten bracket satisfies $[\mathfrak{X}, f] = (-1)^{k-1} i_{\rho^* df}\mathfrak{X}$, for $f \in C^\infty(M)$ and $\mathfrak{X} \in \Gamma(\wedge^k A)$, we see that this last equation reduces to $i_{\rho^* dg} r(df) = -i_{\rho^* df} r(dg)$, which is equivalent to the condition

$$i_{\rho^*\alpha} r(\beta) = -i_{\rho^*\beta} r(\alpha)$$

for $\alpha, \beta \in \Omega^1(M)$.

It is immediate that, for $k_1 = k_2 = 1$, (3.15) becomes

$$D([a, b]) = [D(a), b] + [a, D(b)].$$

Finally, when $k_1 = 1$, $k_2 = 0$, (3.15) amounts to

$$(-1)^q r(\mathcal{L}_{\rho(a)} df) = (-1)^{q-1} i_{\rho^* df} D(a) + (-1)^q [a, r(df)],$$

which is equivalent to the condition

$$r(\mathcal{L}_{\rho(a)}\alpha) = -i_{\rho^*\alpha} D(a) + [a, r(\alpha)].$$

□

For a multiplicative q-vector Π on a Lie groupoid $\mathcal{G} \rightrightarrows M$, its infinitesimal components are written in terms of the corresponding q-differential as follows:

$$\mathcal{L}_{\overrightarrow{a}}\Pi = \overrightarrow{D(a)} = \overrightarrow{\delta_1(a)},$$
$$i_{\mathsf{t}^* df}\Pi = \overrightarrow{r(df)} = (-1)^q \overrightarrow{\delta_0(f)}.$$

Since $[\overrightarrow{a}, \Pi] = \mathcal{L}_{\overrightarrow{a}}\Pi$ and $[\Pi, \mathsf{t}^* f] = (-1)^{q-1} i_{\mathsf{t}^* df}\Pi$, and using the fact that δ is a q-differential if and only if so is $-\delta$, we see that Theorem 3.19 recovers the following correspondence (which is the central result in [29]; see also [5]):

Corollary 3.22 *For a source 1-connected Lie groupoid $\mathcal{G} \rightrightarrows M$, there is a one-to-one correspondence between multiplicative q-vector fields $\Pi \in \mathfrak{X}^q(\mathcal{G})$ and q-differentials (δ_0, δ_1), given by*

$$\overrightarrow{\delta_0(f)} = [\Pi, t^* f], \qquad \overrightarrow{\delta_1(a)} = [\Pi, \overrightarrow{a}].$$

for $f \in C^\infty(M)$ and $a \in \Gamma(A)$.

Remark 3.23 In [29], it is verified that the space of multiplicative multivector fields on a Lie groupoid $\mathcal{G}$ is closed under the Schouten bracket, so it is a Gerstenhaber subalgebra of the space of all multivector fields on $\mathcal{G}$. The correspondence in the previous corollary is proven to give rise to an isomorphism of Gerstenhaber algebras.

The next example shows how the correspondence between Poisson groupoids and Lie bialgebroids [51] fits into the framework of IM (2, 0)-tensors.

Example 3.24 For the case $q = 2$, there is a one-to-one correspondence between pairs (D, r) satisfying (3.13) and pre-Lie algebroid [3] structures on A^* given as follows: $\rho_* : A^* \to TM$ is the dual map to $r : T^*M \to A$, and the bracket $[\cdot, \cdot]_*$ is determined by the Koszul formula:

$$\langle [\mu_1, \mu_2]_*, a\rangle = \mathcal{L}_{\rho_*(\mu_2)}\langle \mu_1, a\rangle - \mathcal{L}_{\rho_*(\mu_1)}\langle \mu_2, a\rangle - D(a)(\mu_1, \mu_2), \quad (3.17)$$

for $a \in \Gamma(A)$ and $\mu_1, \mu_2 \in \Gamma(A^*)$. This is just another incarnation of the known correspondence between pre-Lie algebroids structures on A^* and linear bivector fields on A [26] (see also Corollary 4.11).

Recall that a **Poisson groupoid** is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a multiplicative 2-vector field $\Pi \in \mathfrak{X}^2(\mathcal{G})$ such that $[\Pi, \Pi] = 0$. The IM $(2, 0)$-tensor (D, r) on its Lie algebroid A corresponding to Π defines a pre-Lie algebroid structure on A^*. It has an associated operator $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$ which is the 2-differential defined by (3.16). Now, the fact that $[\Pi, \Pi] = 0$ implies that $\delta^2 = 0$, which says that the Jacobi identity holds for the pre-Lie bracket $[\cdot, \cdot]_*$; so A^* is a Lie algebroid. Finally, the IM-equation

$$D([a, b]) = [D(a), b] + [a, D(b)], \;\; a, \; b \in \Gamma(A),$$

gives the compatibility condition for (A, A^*) to be a Lie bialgebroid. This is the only relevant IM-equation because of the redundancies explained in Remark 3.18 (see [36]).

Multiplicative differential forms In parallel to what we did for multivector fields, we start by briefly illustrating our general result in the case of multiplicative 1-forms, building on [50]. For a 1-form ω on a Lie groupoid $\mathcal{G} \rightrightarrows M$, being multiplicative is equivalent to the map $\omega : \mathcal{G} \to T^*\mathcal{G}$ being a groupoid morphism; in particular, it covers a section of A^*. By differentiating $\omega : \mathcal{G} \to T^*\mathcal{G}$, we obtain a morphism of Lie algebroids $\omega_A : A \to T^*A$, which is in particular a *linear* 1-form on $A \to M$ (in the sense that it is a bundle map from $A \to M$ to the cotangent prolongation $T^*A \to A^*$, see Sect. 4.2 below). The main observation now is that any linear 1-form on A can be identified with a pair (μ, ν), with $\mu \in \Gamma(A^*)$ (the section of A^* that it covers) and a vector-bundle map $\nu : A \to T^*M$. To prove this, first note that, for a linear 1-form on A, the corresponding map $TA \to \mathbb{R}$ is linear on the fibres of the vector bundle $TA \to TM$, and hence it is given by a section of the bundle dual to TA over TM. But this dual bundle is naturally identified with $T(A^*) \to TM$ [47, Sec. 9.3], so it follows that linear 1-forms on A are in correspondence with sections of $T(A^*) \to TM$ which are linear, i.e., given by vector-bundles maps from $TM \to M$ to $T(A^*) \to TM$. Finally, the space of linear sections of $T(A^*) \to TM$ (which can be also seen as sections of the first jet bundle of A^*) admits a canonical decomposition as $\Gamma(A^*) \oplus \Gamma(\mathrm{Hom}(A, T^*M))$ (see e.g. [16, Example 2.8]). If now ω_A corresponds to the pair (μ, ν), the fact that $\omega_A : A \to T^*A$ is a morphism of Lie algebroids translates into suitable conditions on (μ, ν), described in [3]. By setting $l = \mu$ and

[3] A pre-Lie algebroid structure on a vector bundle $E \to M$ consists of an anchor map $\rho_E : E \to TM$ together with a skew-symmetric bilinear bracket $[\cdot, \cdot]$ on $\Gamma(E)$ such that the Leibniz equation $[u, fv] = f[u, v] + (\mathcal{L}_{\rho_E(u)} f)\, v$ holds (see [26]).

defining D by $D(a) = \nu(a) + d\mu(a)$, we obtain the infinitesimal components given in Theorem 3.19. We now extend the discussion to arbitrary multiplicative forms, as in [3].

Since multiplicative p-forms are $(0, p)$-tensors, it follows from Theorem 3.19 that their infinitesimal counterparts are IM $(0, p)$-tensors. Explicitly, on a given Lie algebroid $A \to M$, IM $(0, p)$-tensors are pairs (D, l), where $D : \Gamma(A) \to \Gamma(\wedge^p T^*M)$, $l : A \to \wedge^{p-1} T^*M$, and these maps satisfy

$$D(fa) = fD(a) + df \wedge l(a),$$

for $a \in \Gamma(A)$ and $f \in C^\infty(M)$, and

$$\begin{aligned} D([a, b]) &= \mathcal{L}_{\rho(a)} D(b) - \mathcal{L}_{\rho(b)} D(a) \\ l([a, b]) &= \mathcal{L}_{\rho(a)} l(b) - i_{\rho(b)} D(a) \\ i_{\rho(a)} l(b) &= -i_{\rho(b)} l(a). \end{aligned}$$

Note that IM $(0, p)$-tensors agree with Spencer operators with values in the trivial representation, as considered in [16].

Example 3.25 On a Poisson manifold (M, Π), the cotangent bundle T^*M has a Lie algebroid structure whose anchor is given by the contraction of covectors with Π, $\Pi^\sharp : T^*M \to TM$, and the Lie bracket is given by

$$[\alpha, \beta]_\Pi = \mathcal{L}_{\Pi^\sharp(\alpha)}\beta - \mathcal{L}_{\Pi^\sharp(\beta)}\alpha - d(i_{\Pi^\sharp(\alpha)}\beta), \;\; \alpha, \; \beta \in \Gamma(T^*M). \tag{3.18}$$

There exists a canonical IM $(0, 2)$-tensor on T^*M given by $D = d$, the de Rham differential, and $l = \mathrm{id}_{T^*M}$.

We have the following alternative way to express IM $(0, p)$-tensors, see [1,3,8]. We consider pairs (μ, ν) with $\mu : A \to \wedge^{p-1} T^*M$, $\nu : A \to \wedge^p T^*M$ bundle maps (covering the identity) satisfying

$$\begin{aligned} \nu([a, b]) &= \mathcal{L}_{\rho(a)} \nu(b) - i_{\rho(b)} d\nu(a), \\ \mu([a, b]) &= \mathcal{L}_{\rho(a)} \mu(b) - i_{\rho(b)} (d\mu(a) + \nu(a)), \\ i_{\rho(a)} \mu(b) &= -i_{\rho(b)} \mu(a), \end{aligned}$$

for all $a, b \in \Gamma(A)$. Such a pair (μ, ν) is called an *IM p-form* in [3]. The equivalence between IM $(0, p)$-tensors (D, l) and IM p-forms (μ, ν) is given by the following explicit relations:

$$\begin{aligned} D(a) &= d\mu(a) + \nu(a), \\ l(a) &= \mu(a). \end{aligned}$$

When (D, l) are the infinitesimal components of a multiplicative $(0, p)$ tensor field (i.e. a differential p-form) $\omega \in \Omega^p(\mathcal{G})$, we have that

$$\mathcal{L}_{\overrightarrow{a}} \omega = \mathsf{t}^*(d\mu(a) + \nu(a)), \qquad i_{\overrightarrow{a}} \omega = \mathsf{t}^* \mu(a)$$

for all $a \in \Gamma(A)$. It follows that

$$\mathsf{t}^*\nu(a) = \mathcal{L}_{\overrightarrow{a}}\omega - \mathsf{t}^* d\mu(a) = i_{\overrightarrow{a}} d\omega + d(i_{\overrightarrow{a}}\omega - \mathsf{t}^*\mu(a)) = i_{\overrightarrow{a}} d\omega.$$

In this way, we can see that Theorem 3.19 immediately recovers the main result of [3], relating multiplicative and IM differential forms:

Corollary 3.26 *For a source 1-connected groupoid $\mathcal{G} \rightrightarrows M$, there is a one-to-one correspondence between multiplicative p-forms $\omega \in \Omega^p(\mathcal{G})$ and IM p-forms (μ, ν), given by*

$$t^*\mu(a) = i_{\overrightarrow{a}}\omega, \qquad t^*\nu(a) = i_{\overrightarrow{a}} d\omega,$$

for $a \in \Gamma(A)$.

For a Poisson manifold (M, Π), if $(T^*M, [\cdot,\cdot]_\Pi)$ is the Lie algebroid of a source 1-connected groupoid $\mathcal{G} \rightrightarrows M$, then $\mathcal{G}$ has a canonical multiplicative 2-form integrating the canonical IM (0, 2)-tensor of Example 3.25. In this case, note that $\nu = 0$, and this implies that $d\omega = 0$. One can also verify that $l = \mu$ being an isomorphism implies that ω is non-degenerate. So $(\mathcal{G}, \omega)$ is the symplectic groupoid integrating T^*M.

Remark 3.27 Multiplicative differential forms on a Lie groupoid $\mathcal{G}$ form a subcomplex of the de Rham complex. On the other hand, if (μ, ν) is an IM p-form, one can directly verify that $(\nu, 0)$ is an IM $(p+1)$-form. So $(\mu, \nu) \mapsto (\nu, 0)$ defines a differential on the space of all IM-forms, in such a way that the correspondence in Corollary 3.26 is an isomorphism of complexes.

4 Proof of the Theorems

Before delving into the proofs of Theorems 3.11 and 3.19, let us briefly sketch the general strategy to obtain the infinitesimal description of multiplicative tensors.

Given a multiplicative (q, p)-tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$, our main object of analysis is formula (3.3) applied to c_τ, the corresponding multiplicative function on the groupoid $\mathbb{G} \rightrightarrows \mathbb{M}$ in (2.7):

$$\mathcal{L}_{\overrightarrow{\chi}}\, c_\tau = \mathbb{t}^*\langle Ac_\tau, \chi\rangle. \tag{4.1}$$

Our goal is to have a concrete description of the infinitesimal cocycle $Ac_\tau \in \Gamma(\mathbb{A}^*)$, which codifies the infinitesimal information of τ.

The first key observation is that it is enough to check the identity (4.1) when χ varies within a special set of generators for the $C^\infty(\mathbb{M})$-module of sections of $\mathbb{A}$. These generators will be parametrized by $\Gamma(A)$ and $\Omega^1(M)$, and their pairing with Ac_τ will give rise to maps from the space of parameters into $C^\infty(\mathbb{M})$; more precisely, we will obtain three maps, D, l, and r, taking values in the subspace $\Gamma(\wedge^p T^*M \otimes \wedge^q A) \subseteq C^\infty(\mathbb{M})$ of componentwise linear functions. These maps, which completely determine Ac_τ, will agree with the infinitesimal components of τ.

Considering the left-hand side of (4.1), we will see that the Lie derivatives of c_τ can be expressed in terms of contraction and Lie-derivative operations on the tensor field τ itself. In this way, the equality (4.1) is re-written as the relations involving τ and (D, l, r) in Theorem 3.19. The last step is expressing the cocycle condition $d_{\mathbb{A}}(Ac_\tau) = 0$, where $d_{\mathbb{A}}$ is the Lie algebroid differential on $\Gamma(\mathbb{A}^*)$, in terms of (D, l, r). This will lead to the IM-equations.

We will need a few technical tools to carry out this strategy, including the study of lifting operations (Sect. 4.1) and an analysis of linear tensor fields (Sect. 4.4).

4.1 Lifting operations

As we now see, classical lifting operations (see e.g. [63]) of vector fields are essential ingredients in relating Lie derivatives of tensor fields τ with Lie derivatives of the corresponding componentwise linear functions c_τ.

Let $\pi_E : E \to N$ be a vector bundle over a smooth manifold N. Given a section $u \in \Gamma(E)$, its *vertical lift* is the vector field $u^{\mathrm{v}} : E \to TE$ on E defined by

$$u^{\mathrm{v}}(e) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (e + \epsilon\, u(y)), \quad y \in N,\ e \in E_y. \tag{4.2}$$

For a section $\psi \in \Gamma(E^*)$, we recall that the Lie derivative of its corresponding linear function $\ell_\psi \in C^\infty(E)$ along u^{v} is given by

$$\mathcal{L}_{u^{\mathrm{v}}}\, \ell_\psi = \langle \psi, u \rangle \circ \pi_E. \tag{4.3}$$

In this paper, we are mostly interested in the cases where $E = TN$ or $E = A^*$, the dual of a Lie algebroid $(A, [\cdot,\cdot], \rho)$ over N. We denote the bundle projections by $\pi : TN \to N$ and $\pi_* : A^* \to N$. In these cases, besides the vertical lifting, there are two other important lifting constructions that we need to recall. For a vector field $X \in \mathfrak{X}(N)$, consider its (local) flow $\phi_\epsilon : N \to N$. The *tangent lift of* X is the vector field X^T on TN with flow given by $\epsilon \mapsto T\phi_\epsilon$. For $\alpha \in \Omega^1(N)$ and $f \in C^\infty(N)$, the Lie derivatives of the functions ℓ_α and $f \circ \pi$ in $C^\infty(TN)$ along X^T are given by

$$\mathcal{L}_{X^T}\, \ell_\alpha = \ell_{\mathcal{L}_X \alpha} \quad \text{and} \quad \mathcal{L}_{X^T}(f \circ \pi) = (\mathcal{L}_X f) \circ \pi. \tag{4.4}$$

The *Hamiltonian lift of* a section $a \in \Gamma(A)$ of a Lie algebroid is the vector field H_a on A^* defined by

$$H_a = \Pi^\sharp_{lin}(d\ell_a), \tag{4.5}$$

where $\Pi_{lin} \in \Gamma(\wedge^2 TA^*)$ is the linear Poisson structure on A^* (dual to the Lie algebroid structure on A) and $\ell_a \in C^\infty(A^*)$ is the linear function corresponding to a. For $b \in \Gamma(A)$ and $f \in C^\infty(N)$,

$$\mathcal{L}_{H_a}\, \ell_b = \ell_{[a,b]} \quad \text{and} \quad \mathcal{L}_{H_a}(f \circ \pi_*) = (\mathcal{L}_{\rho(a)} f) \circ \pi_*. \tag{4.6}$$

Note that (4.4) and (4.6) completely characterize X^T and H_a, respectively. When $A = TN$, the linear Poisson structure on T^*N comes from the canonical symplectic form, and the Hamiltonian lift H_X of a vector field $X \in \mathfrak{X}(N)$ coincides with the *cotangent lift* $X^{T^*} \in \mathfrak{X}(T^*N)$, which is the vector field with flow $\epsilon \mapsto (T\phi_{-\epsilon})^*$.

Our aim is to extend formulas (4.3), (4.4) and (4.6) to elements of the space $\Gamma(\wedge^p T^*N \otimes \wedge^q A)$. Let us first introduce some notation. For $Y \in \mathfrak{X}(M)$, we define vector fields on $\oplus^p TM$ as follows:

$$Y^{T,p}(X_1, \ldots, X_p) = (Y^T(X_1), \ldots, Y^T(X_p)) \tag{4.7}$$

$$Y^{\mathrm{v},p}_{(i)}(X_1, \ldots, X_p) = (0_{X_1}, \ldots, 0_{X_{i-1}}, Y^{\mathrm{v}}(X_i), 0_{X_{i+1}}, \ldots, 0_{X_p}), \quad i = 1, \ldots, p. \tag{4.8}$$

Similarly, for $a \in \Gamma(A)$ and $\mu \in \Gamma(A^*)$, we define vector fields on $\oplus^q A^*$ by

$$H^q_a(\varphi_1, \ldots, \varphi_q) = (H_a(\varphi_1), \ldots, H_a(\xi_q)) \tag{4.9}$$

$$\mu^{\mathrm{v},q}_{(j)}(\varphi_1, \ldots, \varphi_q) = (0_{\varphi_1}, \ldots, 0_{\varphi_{j-1}}, \mu^{\mathrm{v}}(\varphi_j), 0_{\varphi_{j+1}}, \ldots, 0_{\varphi_p}), \quad j = 1, \ldots, q. \tag{4.10}$$

Define

$$\gamma^{(p,q)}_{(i,0)} :(\oplus^p TN) \oplus (\oplus^q A^*) \to (\oplus^{p-1} TN) \oplus (\oplus^q A^*),$$
$$\gamma^{(p,q)}_{(0,j)} :(\oplus^p TN) \oplus (\oplus^q A^*) \to (\oplus^p TN) \oplus (\oplus^{q-1} A^*)$$

to be the projections

$$\gamma^{(p,q)}_{(i,0)}(\underline{X}, \underline{\varphi}) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_p, \varphi_1, \ldots, \varphi_q) \tag{4.11}$$

$$\gamma^{(p,q)}_{(0,j)}(\underline{X}, \underline{\varphi}) = (X_1, \ldots, X_p, \varphi_1, \ldots, \varphi_{j-1}, \varphi_{j+1}, \ldots, \varphi_q), \tag{4.12}$$

for $1 \leq i \leq p$, $1 \leq j \leq q$. When there is no risk of confusion, we simplify the notation by omitting the superscripts (q, p) on the projections.

Proposition 4.1 *Let $\tau \in \Gamma(\wedge^p T^*N \otimes \wedge^q A)$, and consider the corresponding componentwise linear function $c_\tau : (\oplus^p TN) \oplus (\oplus^q A^*) \to \mathbb{R}$. For $a \in \mathfrak{X}(A)$, $\mu \in \Gamma(A^*)$ and $Y \in \Gamma(TN)$, one has that*

$$\mathcal{L}_{(\rho(a)^{T,p}, H^q_a)} c_\tau = c_{a\cdot\tau}$$
$$\mathcal{L}_{(Y^{\mathrm{v},p}_{(i)}, 0)} c_\tau = (-1)^{i-1} c_{i_Y \tau} \circ \gamma_{(i,0)}$$
$$\mathcal{L}_{(0, \mu^{\mathrm{v},q}_{(j)})} c_\tau = (-1)^{j-1} c_{i_\mu \tau} \circ \gamma_{(0,j)},$$

where $\cdot$ is the action (3.11).

Proof Let us consider the case $\tau = \omega \otimes \mathfrak{X}$, for $\omega \in \Omega^p(N)$ and $\mathfrak{X} \in \mathfrak{X}^q(A)$. First note that $c_\tau = (c_\omega \circ \mathrm{pr}_{TN})(c_{\mathfrak{X}} \circ \mathrm{pr}_{A^*})$, where pr_{TN} and pr_{A^*} are the projections of

$(\oplus^p TN) \oplus (\oplus^q A^*)$ onto $\oplus^p TN$ and $\oplus^q A^*$, respectively. Using the Leibniz rule, it suffices to prove that

$$\mathcal{L}_{Y^{T,p}_{(i)}} c_\omega = c_{\mathcal{L}_Y \omega}, \quad \mathcal{L}_{Y^{\mathrm{v},p}_{(i)}} c_\omega = (-1)^{i-1} c_{i_Y\omega} \circ \gamma_{(i,0)}$$
$$\mathcal{L}_{H^q_a} c_{\mathfrak{X}} = c_{[a,\mathfrak{X}]}, \quad \mathcal{L}_{\mu^{\mathrm{v},q}_{(j)}} c_{\mathfrak{X}} = (-1)^{j-1} c_{i_\mu \mathfrak{X}} \circ \gamma_{(0,j)}.$$

Let us simplify matters once more by assuming that $\omega = \alpha_1 \wedge \cdots \wedge \alpha_p$, for $\alpha_1, \ldots, \alpha_p \in \Gamma(T^*N)$. On the one hand, we have that

$$\begin{aligned} c_{i_Y\omega} \circ \gamma_{(i)}(X_1, \ldots, X_p) &= i_Y\omega(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_p) \\ &= \omega(Y, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_p) \\ &= (-1)^{i-1}\omega(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_p). \end{aligned}$$

On the other hand, by (2.8) and (4.3),

$$\begin{aligned} \mathcal{L}_{Y^{\mathrm{v},p}_{(i)}} c_\omega = \sum_{\sigma \in S(p)} & sgn(\sigma) \langle \alpha_{\sigma(i)}, Y \rangle (\ell_{\alpha_{\sigma(1)}} \circ \mathrm{pr}^1_{TN}) \cdots (\widehat{\ell_{\alpha_{\sigma(i)}} \circ \mathrm{pr}^i_{TN}}) \\ & \cdots (\ell_{\alpha_{\sigma(p)}} \circ \mathrm{pr}^p_{TN}), \end{aligned}$$

where $\mathrm{pr}^j_{TN} : \oplus^p TN \to TN$ is the projection on the j-component, for $1 \le j \le p$. Hence,

$$\begin{aligned} \left(\mathcal{L}_{Y^{\mathrm{v},p}_{(i)}} c_\omega \right) (X_1, \ldots, X_p) &= \omega(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_p) \\ &= (-1)^{i-1} c_{i_Y\omega} \circ \gamma_{(i)}(X_1, \ldots, X_p). \end{aligned}$$

The other equations follow similarly using (4.4) and (4.6). The case where τ is arbitrary follows from linearity of the Lie derivative. □

Remark 4.2 When $A = TN$, the action $\cdot$ of $X \in \mathfrak{X}(N)$ on $\tau \in \Gamma(\wedge^p T^*N \otimes \wedge^q TN)$ is the Lie derivative of τ along X, $X \cdot \tau = \mathcal{L}_X \tau$. In this case, Proposition 4.1 says that

$$\mathcal{L}_{(X^{T,p}, X^{T^*,q})} c_\tau = c_{\mathcal{L}_X \tau},$$

where X^T and X^{T^*} are the tangent and cotangent lifts of X.

For our next result, we keep the notation as in Remark 3.12.

Proposition 4.3 *Let $\tau \in \Gamma(\wedge^p T^*N \otimes \wedge^q A)$ and $a \in \Gamma(A)$. For $f \in C^\infty(N)$,*

$$(fa) \cdot \tau = f(a \cdot \tau) + df \wedge i_{\rho(a)}\tau - a \wedge i_{d_A f}\tau,$$

where d_A is the Lie algebroid differential and d is the de Rham differential.

Proof By linearity, it suffices to prove the result for $\tau = \omega \otimes \mathfrak{X}$, where $\omega \in \Gamma(\wedge^p T^*N)$ and $\mathfrak{X} \in \Gamma(\wedge^q A)$. First,

$$\mathcal{L}_{\rho(fa)}\,\omega = f\,\mathcal{L}_{\rho(a)}\omega + df \wedge i_{\rho(a)}\omega$$

and, by the properties of the Schouten bracket,

$$\begin{aligned}[fa, \mathfrak{X}] &= -[\mathfrak{X}, fa] = -[\mathfrak{X}, f] \wedge a - f[\mathfrak{X}, a] \\ &= -(-1)^{q-1}(i_{d_A f}\mathfrak{X}) \wedge a + f\,[a, \mathfrak{X}] \\ &= -a \wedge i_{d_A f}\mathfrak{X} + f\,[a, \mathfrak{X}].\end{aligned}$$

Hence, $(fa) \cdot (\omega \otimes \mathfrak{X}) = f\,(a \cdot \tau) + df \wedge \big((i_{\rho(a)}\omega) \otimes \mathfrak{X}\big) - a \wedge \big(\omega \otimes (i_{d_A f}\mathfrak{X})\big)$, as we wanted. □

4.2 The Lie algebroid of $\mathbb{G} \rightrightarrows \mathbb{M}$

In this subsection, we discuss the Lie algebroid $\mathbb{A} \to \mathbb{M}$ of the Lie groupoid $\mathbb{G} \rightrightarrows \mathbb{M}$ introduced in (2.7) and describe a special set of generators for the $C^\infty(\mathbb{M})$-module $\Gamma(\mathbb{A})$.

Prolongations of vector bundles Given a vector bundle $\pi_E : E \to M$, we may view it as a Lie groupoid whose source and target maps are equal to π_E, the unit map is the zero section $0 : M \to E$, and the multiplication is fiberwise addition. In this case, the tangent Lie groupoid is the vector bundle $T\pi_E : TE \to TM$, called the *tangent prolongation* of E. Similarly, the cotangent Lie groupoid is the vector bundle $\widetilde{\pi}_E : T^*E \to E^*$, called the *cotangent prolongation*. The projection $\widetilde{\pi}_E$ has the following description: for $\xi \in T^*_e E$, $\widetilde{\pi}_E(\xi) \in E^*_{\pi_E(e)}$ is the element defined by

$$\langle \widetilde{\pi}_E(\xi), \dot{e} \rangle = \langle \xi, \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (e + \epsilon\,\dot{e}) \rangle, \quad \forall \dot{e} \in E_{\pi_E(e)}.$$

The Lie algebroid AE is identified with $E \to M$ itself, with the zero anchor and zero bracket; the right-invariant vector field corresponding to $u \in \Gamma(E)$ is the vertical lift $u^{\mathrm{v}} \in \mathfrak{X}(E)$, see (4.2). The exact sequence (2.2) becomes

$$0 \longrightarrow E \hookrightarrow 0^*TE \stackrel{T\pi_E}{\longrightarrow} TM \longrightarrow 0. \tag{4.13}$$

For $e \in E_x$,

$$\overline{e} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\epsilon\, e) \ \in T_{0_x}E$$

defines the inclusion $E \hookrightarrow 0^*TE$ in the exact sequence above. The translation bisection associated to $u \in \Gamma(E)$ is given by

$$\mathcal{B}u(X) = T0(X) + \overline{u(x)}, \quad X \in T_x M. \tag{4.14}$$

In the cotangent prolongation, the exact sequence (2.5) becomes the dual of (4.13),

$$0 \longrightarrow T^*M \overset{(T\pi_E)^*}{\hookrightarrow} 0^*T^*E \overset{\widetilde{\pi}_E}{\longrightarrow} E^* \longrightarrow 0.$$

For $\beta \in T_x^*M$, its image under the first map of the exact sequence is

$$\overline{\beta} = (T_{0_x}\pi_E)^*\beta \in T_{0_x}^*E.$$

For a 1-form $\alpha \in \Omega^1(M)$, its translation bisection $\mathcal{B}\alpha : E^* \to T^*E$ is given by

$$\mathcal{B}\alpha(\varphi) = \widetilde{0}_\varphi + \overline{\alpha(x)}, \ \ \varphi \in E_x^*.$$

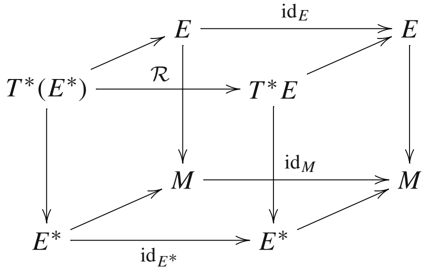
Note that since $\mathsf{t} = \mathsf{s} = \pi_E$, $\mathcal{B}u$ and $\mathcal{B}\alpha$ are sections of $T\pi_E : TE \to TM$ and $\widetilde{\pi}_E : T^*E \to E^*$, respectively. They are known as *core sections* in the general theory of double vector bundles (see e.g. [47]).

Sections of prolongations For a section $u : M \to E$, its derivative $Tu : TM \to TE$ defines a section of the tangent prolongation. There is also an induced section $\mathcal{R}_u : E^* \to T^*E$ of the cotangent prolongation, as we now explain.

Recall the *reversal isomorphism* $\mathcal{R} : T^*(E^*) \to T^*E$ (see e.g. [47] for details): in local coordinates,

$$\mathcal{R}(\varphi, \beta, e) = (e, -\beta, \varphi), \tag{4.15}$$

where we are locally writing $T^*(E^*) \cong E^* \oplus T^*M \oplus E$ and $T^*E \cong E \oplus T^*M \oplus E^*$. Globally, $\mathcal{R}$ is both a vector bundle morphism from the cotangent bundle of E^*, $T^*(E^*) \to E^*$, to the cotangent prolongation of E, $T^*E \to E^*$, and from the cotangent prolongation of E^*, $T^*(E^*) \to E$, to the cotangent bundle of E, $T^*E \to E$. It fits into the following commutative diagram of vector bundle morphisms:



For $u \in \Gamma(E)$ and $\mu \in \Gamma(E^*)$, let $\ell_u \in C^\infty(E^*)$ and $\ell_\mu \in C^\infty(E)$ be the corresponding fiberwise linear functions. Then the composition

$$\mathcal{R}_u := \mathcal{R} \circ d\ell_u : E^* \to T^*E \tag{4.16}$$

defines a section of the cotangent prolongation. Note that the identities

$$\mathcal{R}_u(\mu(x)) = d\ell_\mu(u(x)) - (\pi_E^* d\langle \mu, u\rangle)(u(x)), \tag{4.17}$$

and

$$\mathcal{R}(-\pi_{E^*}^* \alpha) = \mathcal{B}\alpha, \tag{4.18}$$

completely determine $\mathcal{R}$, where $\pi_{E^*} : E^* \to M$ is the projection of the dual bundle.

Let us fix positive integers p, q and consider the Lie groupoid (2.7), with $\mathcal{G} = E$. This is actually a vector bundle

$$\mathbb{E}^{(p,q)} \to \mathbb{M}^{(p,q)}.$$

As before, we will simplify the notation by dropping the superindices.

For a section $u \in \Gamma(E)$, we denote by $(T^p u, \mathcal{R}_u^q) : \mathbb{M} \to \mathbb{E}$ the section given by

$$(T^p u, \mathcal{R}_u^q)(\underline{X}, \underline{\varphi}) = (Tu(X_1), \ldots, Tu(X_p), \mathcal{R}_u(\varphi_1), \ldots, \mathcal{R}_u(\varphi_q)), \tag{4.19}$$

and by $\mathcal{B}u_{(i)} : \mathbb{M} \to \mathbb{E}$ the section given by

$$\mathcal{B}u_{(i)}(\underline{X}, \underline{\varphi}) = (T0(X_1), \ldots, \mathcal{B}u(X_i), \ldots, T0(X_p), \widetilde{0}_{\varphi_1}, \ldots, \widetilde{0}_{\varphi_q}), \tag{4.20}$$

for $i = 1, \ldots, p$. Similarly, for $\alpha \in \Omega^1(M)$, denote by $\mathcal{B}\alpha_{(j)} : \mathbb{M} \to \mathbb{E}$ the section defined by

$$\mathcal{B}\alpha_{(j)}(\underline{X}, \underline{\varphi}) = (T0(X_1), \ldots, T0(X_p), \widetilde{0}_{\varphi_1}, \ldots, \mathcal{B}\alpha(\varphi_j), \ldots, \widetilde{0}_{\varphi_q}), \tag{4.21}$$

for $j = 1, \ldots, q$. The following result is proven in [48].

Proposition 4.4 *The $C^\infty(\mathbb{M})$-module of section $\Gamma(\mathbb{E})$ is generated by $(T^p u, \mathcal{R}_u^q)$, $\mathcal{B}v_{(i)}$ and $\mathcal{B}\alpha_{(j)}$, for $u, v \in \Gamma(E)$ and $\alpha \in \Omega^1(M)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$.*

Remark 4.5 [$C^\infty(M)$-linearity] One can check that

$$T(fu) = (f \circ \pi_E) \cdot Tu +_{\mathfrak{p}} \ell_{df} \cdot \mathcal{B}u, \tag{4.22}$$

where we used the notation $+_{\mathfrak{p}}$ and $\cdot$ for the sum and scalar multiplication on the fibers of the tangent prolongation $TE \to TM$, respectively, and $\ell_{df} \in C^\infty(TM)$ is the linear function corresponding to $df \in \Omega^1(M)$. Similarly,

$$\mathcal{R}_{fu} = (f \circ \pi_{E^*}) \cdot \mathcal{R}_u +_{\mathfrak{p}} \quad \ell_{-u} \cdot \mathcal{B}(df), \tag{4.23}$$

where $+_{\mathfrak{p}}$ and $\cdot$ denote the sum and scalar multiplication on the fibers of the cotangent prolongation $T^*E \to E^*$, respectively.

Brackets and anchors We now recall the main features of the tangent and cotangent Lie algebroids, see e.g. [49] for details.

For a Lie algebroid $(A, [\cdot,\cdot], \rho)$, consider its tangent prolongation $TA \to TM$. It has the structure of a Lie algebroid, where the Lie bracket $[\cdot,\cdot]$ on the space of sections of $TA \to TM$ is determined by the conditions

$$[Ta, Tb] = T\,[a, b]\,, \quad [Ta, \mathcal{B}b] = \mathcal{B}[a, b], \quad [\mathcal{B}a, \mathcal{B}b] = 0,$$

for $a,\ b \in \Gamma(A)$, while the anchor $\rho_T : TA \to T(TM)$ is determined by

$$\rho_T(Ta) = \rho(a)^T, \quad \rho_T(\mathcal{B}a) = \rho(a)^{\mathrm{v}}.$$

where $(\cdot)^T$ and $(\cdot)^{\mathrm{v}}$ are the tangent and vertical lifts, respectively.

The cotangent Lie algebroid is the Lie algebroid structure on $T^*A \to A^*$ defined as follows: the anchor $\rho_{T^*} : T^*(A) \to T(A^*)$ is determined by

$$\rho_{T^*}(\mathcal{R}_a) = H_a, \quad \rho_{T^*}(\mathcal{B}\alpha) = (\rho^*\alpha)^{\mathrm{v}},$$

where $H_a \in \mathfrak{X}(A^*)$ is the Hamiltonian lift of $a \in \Gamma(A)$, see (4.5). The Lie bracket on the space of sections of $T^*A \to A^*$ is determined by

$$[\mathcal{R}_a, \mathcal{R}_b] = \mathcal{R}_{[a,b]}, \quad [\mathcal{R}_a, \mathcal{B}\alpha] = \mathcal{B}\left(\mathcal{L}_{\rho(a)}\alpha\right), \quad [\mathcal{B}\alpha, \mathcal{B}\beta] = 0,$$

where $a, b \in \Gamma(A)$ and $\alpha, \beta \in \Omega^1(M)$.

The Whitney sum

$$\mathbb{A}^{p,q} = (\oplus^p TA) \oplus (\oplus^q T^*A) \to (\oplus^p TM) \oplus (\oplus^q A^*)$$

inherits a Lie algebroid structure which is determined componentwise by the tangent and the cotangent Lie algebroids. We give a detailed description here for convenience. The anchor map is determined by

$$\begin{aligned} \rho_{\mathbb{T}}(T^p a, \mathcal{R}_a^q) &= (\rho(a)^{T,p}, H_a^q) \\ \rho_{\mathbb{T}}(\mathcal{B}a_{(i)}) &= (\rho(a)_{(i)}^{\mathrm{v},p}, 0) \\ \rho_{\mathbb{T}}(\mathcal{B}\alpha_{(j)}) &= (0, \rho^*\alpha_{(j)}^{\mathrm{v},q}), \end{aligned} \tag{4.24}$$

for $\alpha \in \Omega^1(M)$. The Lie bracket on the space of sections of $\mathbb{A} \to M$ is determined by what it does on generators according to the following formulas:

$$
\begin{aligned}
[(T^p a, \mathcal{R}^q_a), (T^p b, \mathcal{R}^q_b)] &= (T^p[a,b], \mathcal{R}^q_{[a,b]}),\ \ [(T^p a, \mathcal{R}^q_a), \mathcal{B}b_{(i)}] = \mathcal{B}[a,b]_{(i)}\\
[(T^p a, \mathcal{R}^q_a), \mathcal{B}\alpha_{(j)}] &= \mathcal{B}(\mathcal{L}_{\rho(a)}\alpha)_{(j)},\ \ [\mathcal{B}a_{(i)}, \mathcal{B}b_{(i')}] = 0\\
[\mathcal{B}a_{(i)}, \mathcal{B}\alpha_{(j)}] &= 0,\ \ [\mathcal{B}\alpha_{(j)}, \mathcal{B}\beta_{(j')}] = 0.
\end{aligned}
\tag{4.25}
$$

For a Lie groupoid $\mathcal{G} \rightrightarrows M$, let $A \to M$ be its Lie algebroid. There are natural Lie-algebroid identifications $A(T\mathcal{G}) \cong TA$ as well as $A(T^*\mathcal{G}) \cong T^*A$ (see [49, §7]). Hence, the Lie algebroid $A(\mathbb{G}^{(p,q)}) \to \mathbb{M}^{(p,q)}$ of the Lie groupoid $\mathbb{G}^{(p,q)} \rightrightarrows \mathbb{M}^{(p,q)}$ is naturally isomorphic to

$$
\mathbb{A}^{p,q} = (\oplus^p TA) \oplus (\oplus^q T^*A) \to (\oplus^p TM) \oplus (\oplus^q A^*). \tag{4.26}
$$

The right-invariant vector fields on $T\mathcal{G}$ corresponding to the sections of type Ta, $\mathcal{B}a$ are given by

$$
\overrightarrow{Ta} = \overrightarrow{a}^{T} \quad \text{and} \quad \overrightarrow{\mathcal{B}a} = \overrightarrow{a}^{\mathrm{v}}, \tag{4.27}
$$

see e.g. [49, §7] for a proof. Similarly, the right-invariant vector fields on $T^*\mathcal{G}$ corresponding to the sections of type $\mathcal{R}_a$, $\mathcal{B}\alpha$ are

$$
\overrightarrow{\mathcal{R}_a} = \overrightarrow{a}^{T^*} \quad \text{and} \quad \overrightarrow{\mathcal{B}\alpha} = (\mathrm{t}^*\alpha)^{\mathrm{v}}. \tag{4.28}
$$

The proof of these last formulas can be found in Appendix B.

It is now a straightforward consequence of (4.27) and (4.28) that the right-invariant vector fields $\overrightarrow{(T^p a, \mathcal{R}^q_a)}$, $\overrightarrow{\mathcal{B}a_{(i)}}$ and $\overrightarrow{\mathcal{B}\alpha_{(j)}} \in \mathfrak{X}(\mathbb{G}^{p,q})$ are given by

$$
\overrightarrow{(T^p a, \mathcal{R}^q_a)} = (\overrightarrow{a}^{T,p}, \overrightarrow{a}^{T^*,q}) \tag{4.29}
$$

$$
\overrightarrow{\mathcal{B}a_{(i)}} = (\overrightarrow{a}^{\mathrm{v}}_{(i)}, 0) \tag{4.30}
$$

$$
\overrightarrow{\mathcal{B}\alpha_{(j)}} = (0, (\mathrm{t}^*\alpha)^{\mathrm{v}}_{(j)}), \tag{4.31}
$$

for $i = 1, \ldots, p$, $j = 1, \ldots, q$.

4.3 Proof of Theorem 3.11

Let us begin with two important lemmas. For the first one, we need to introduce some notation. Define $\pi^{(p,q)}_{(i,0)} : \mathbb{G}^{(p,q)} \to \mathbb{G}^{(p-1,q)}$, $\pi^{(p,q)}_{(0,j)} : \mathbb{G}^{(p,q)} \to \mathbb{G}^{(p,q-1)}$, as the natural projections

$$
\pi^{(p,q)}_{(i,0)}(\underline{U}, \underline{\xi}) = (U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_p, \xi_1, \ldots, \xi_q) \tag{4.32}
$$

$$
\pi^{(p,q)}_{(0,j)}(\underline{U}, \underline{\xi}) = (U_1, \ldots, U_p, \xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_q) \tag{4.33}
$$

for $1 \le i \le p$, $1 \le j \le q$. When there is no risk of confusion, we will omit the superscripts (q, p) on the projections. Observe that $\pi_{(i,0)}$ and $\pi_{(0,j)}$ are groupoid morphisms (covering (4.11) and (4.12), respectively).

Lemma 4.6 *For any multiplicative tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$, there exist vector bundle maps $l : A \to \wedge^{p-1}T^*M \otimes \wedge^q A$ and $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} A$ covering the identity map such that*

$$i_{\overrightarrow{a}}\,\tau = \mathcal{T}(l(a)), \tag{4.34}$$

$$i_{\mathrm{t}^*\alpha}\,\tau = \mathcal{T}(r(\alpha)). \tag{4.35}$$

Proof By formulas (3.3) and (4.30), we have that

$$\mathbb{t}^*\langle Ac_\tau,\, \mathcal{B}a_{(i)}\rangle = \mathcal{L}_{\overrightarrow{\mathcal{B}a_{(i)}}}\, c_\tau = \mathcal{L}_{(\overrightarrow{a}^{\,\mathrm{v}}_{(i)},0)}\, c_\tau = (-1)^{i-1} c_{i_{\overrightarrow{a}}\tau} \circ \pi_{(i,0)}$$

for $a \in \Gamma(A)$, where the last equality is a consequence of Proposition 4.1. Similarly, for $\alpha \in \Omega^1(M)$, we check that

$$\mathbb{t}^*\langle Ac_\tau,\, \mathcal{B}\alpha_{(j)}\rangle = (-1)^{j-1} c_{i_{\mathrm{t}^*\alpha}\,\tau} \circ \pi_{(0,j)}.$$

Note that, for $(\underline{X}, \underline{\varphi}) \in \mathbb{M}$,

$$\langle Ac_\tau, \mathcal{B}a_{(i)}\rangle(\underline{X}, \underline{\varphi}) = (-1)^{i-1}\tau(\overrightarrow{a}, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_p, \varphi_1, \ldots, \varphi_q),$$

which shows that $\langle Ac_\tau, \mathcal{B}a_{(i)}\rangle$ is a componentwise linear function of $\gamma_{(i,0)}(\underline{X}, \underline{\varphi})$. Hence (see Lemma 2.4) there exists $l(a) \in \Gamma(\wedge^{p-1}T^*M \otimes \wedge^q A)$ such that

$$\langle Ac_\tau, \mathcal{B}a_{(i)}\rangle = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i,0)}. \tag{4.36}$$

Now, note that

$$\begin{aligned} c_{i_{\overrightarrow{a}}\tau} \circ \pi_{(i,0)} &= (-1)^{i-1}\mathbb{t}^*\langle Ac_\tau,\, \mathcal{B}a_{(i)}\rangle = \mathbb{t}^*\left(c_{l(a)} \circ \gamma_{(i,0)}\right) \\ &= \left(\mathbb{t}^* c_{l(a)}\right) \circ \pi_{(i,0)} = c_{\mathcal{T}(l(a))} \circ \pi_{(i,0)}. \end{aligned}$$

Formula (4.34) follows from the injectivity of the correspondence between tensors and componentwise linear functions (Lemma 2.4).

To prove that l is $C^\infty(M)$-linear, we use Proposition 3.10 to see that

$$\mathcal{T}(l(fa)) = i_{\overrightarrow{fa}}\tau = (\mathrm{t}^* f)\, i_{\overrightarrow{a}}\tau = (\mathrm{t}^* f)\, \mathcal{T}(l(a)) = \mathcal{T}(f l(a)),$$

for $f \in C^\infty(M)$, so that $C^\infty(M)$-linearity follows from the injectivity of $\mathcal{T}$.

Similarly, we can prove the existence of $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} A$ by checking that $\langle Ac_\tau,\, \mathcal{B}\alpha_{(j)}\rangle$ is componentwise linear, so that (4.35) follows from

$$\langle Ac_\tau, \mathcal{B}\alpha_{(j)}\rangle = (-1)^{j-1} c_{r(\alpha)} \circ \gamma_{(0,j)}. \tag{4.37}$$

□

Lemma 4.7 *For any multiplicative tensor $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$, there exists an $\mathbb{R}$-linear map $D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes \wedge^q A)$ satisfying the Leibniz condition (3.9) and such that*

$$\mathcal{L}_{\overrightarrow{a}} \tau = \mathcal{T}(D(a)). \tag{4.38}$$

Proof From (3.3) and (4.29), we see that

$$\mathbb{t}^* \langle Ac_\tau,\ (T^p a, \mathcal{R}_a^q)\rangle = \mathcal{L}_{\overrightarrow{(T^p a, \mathcal{R}_a^q)}} c_\tau = \mathcal{L}_{(\overrightarrow{a}^{T,p}, \overrightarrow{a}^{T^*,q})} c_\tau = c_{\mathcal{L}_{\overrightarrow{a}} \tau},$$

where the last equality relies on Proposition 4.1. In particular, for $(\underline{X}, \underline{\varphi}) \in \mathbb{M}$,

$$\langle Ac_\tau,\ (T^p a, \mathcal{R}_a^q)\rangle(\underline{X}, \underline{\varphi}) = \mathcal{L}_{\overrightarrow{a}} \tau(\underline{X}, \underline{\varphi}),$$

which proves that $\langle Ac_\tau,\ (T^p a, \mathcal{R}_a^q)\rangle \in C^\infty(\mathbb{M})$ is a componentwise linear function. By Lemma 2.4, there exists $D(a) \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ such that

$$\langle Ac_\tau, (T^p a, \mathcal{R}_a^q)\rangle = c_{D(a)}. \tag{4.39}$$

Now (4.38) follows from Proposition 3.10 (and the injectivity part of Lemma 2.4):

$$c_{\mathcal{L}_{\overrightarrow{a}} \tau} = \mathbb{t}^* \langle Ac_\tau,\ (T^p a, \mathcal{R}_a^q)\rangle = \mathbb{t}^* c_{D(a)} = c_{\mathcal{T}(D(a))}.$$

To prove the Leibniz condition (3.9), we use Proposition 4.3 to see that

$$\begin{aligned}
\mathcal{T}(D(fa)) = \mathcal{L}_{\overrightarrow{fa}} \tau &= \mathcal{L}_{(\mathsf{t}^* f)\overrightarrow{a}} \tau = (\mathsf{t}^* f)\, \mathcal{L}_{\overrightarrow{a}} \tau + \mathsf{t}^* df \wedge i_{\overrightarrow{a}} \tau - \overrightarrow{a} \wedge i_{\mathsf{t}^* df}\, \tau \\
&= (\mathsf{t}^* f)\mathcal{T}(D(a)) + \mathsf{t}^* df \wedge \mathcal{T}(l(a)) - \overrightarrow{a} \wedge \mathcal{T}(r(df)) \\
&= \mathcal{T}(f D(a) + df \wedge l(a) - a \wedge r(df)).
\end{aligned}$$

The conclusion follows from the injectivity of $\mathcal{T}$. □

We are now in position to present the proof of Theorem 3.11.

Proof of Theorem 3.11 If τ is multiplicative, then the existence of the triple (D, l, r) is guaranteed by Lemmas 4.6 and 4.7; condition (3.8) follows from the fact that, since c_τ is a multiplicative function, it must vanish along groupoid units (see Proposition 3.5).

Conversely, assume the existence of (D, l, r). We claim that there is a unique $\mu \in \Gamma(A^*\mathbb{G})$ defined by the following conditions:

$$\langle \mu, \mathcal{B}a_{(i)}\rangle = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i,0)}, \tag{4.40}$$

$$\langle \mu, \mathcal{B}\alpha_{(j)}\rangle = (-1)^{j-1} c_{r(\alpha)} \circ \gamma_{(0,j)}, \tag{4.41}$$

$$\langle \mu, (T^p a, \mathcal{R}_a^q)\rangle = c_{D(a)}, \tag{4.42}$$

for $a \in \Gamma(A)$ and $\alpha \in \Omega^1(M)$. Uniqueness follows from Proposition 4.4, so it remains to verify that μ is indeed well-defined. For local frames $(a^k)_{k=1,\dots,\mathrm{rank}(A)}$ of A and $(\alpha^s)_{s=1,\dots,\dim(M)}$ of T^*M, we observe that the collection of local sections $(T^p a^k, \mathcal{R}^q_{a^k})$, $\mathcal{B}a^k_{(i)}$, and $\mathcal{B}\alpha^s_{(j)}$ form a local frame for $\mathbb{A}$. We first define μ on this local frame using the formulas above (and extend it by linearity). In the following, we use Einstein notation. To show that μ is globally well defined, it suffices to verify that (4.40), (4.41), and (4.42) hold for $a = f_k a^k$ and $\alpha = g_s \alpha^s$, where $f_k, g_s \in C^\infty(M)$. One can check that

$$\mathcal{B}a_{(i)} = (f_k \circ \mathrm{pr})\, \mathcal{B}a^k_{(i)}, \quad \mathcal{B}\alpha_{(j)} = (g_s \circ \mathrm{pr})\, \mathcal{B}\alpha^s_{(j)}$$

where $\mathrm{pr} : \mathbb{M} \to M$ is the bundle projection. Using (4.22) and (4.23), one also verifies that (see Remark 4.5 for notation)

$$(T^p a, \mathcal{R}^q_a) = (f_k \circ \mathrm{pr})(T^p a^k, \mathcal{R}^q_{a^k}) +_{\mathfrak{p}} (\ell_{df_k} \circ \mathrm{pr}^i_{TM})\mathcal{B}a^k_{(i)} +_{\mathfrak{p}} (\ell_{-a_k} \circ \mathrm{pr}^j_{A^*})\mathcal{B}\beta^k_{(j)},$$

where $\beta^k = df_k$, and $\mathrm{pr}^i_{TM} : \mathbb{M} \to TM$, $\mathrm{pr}^j_{A^*} : \mathbb{M} \to A^*$ are given by $\mathrm{pr}^i_{TM}(\underline{X}, \underline{\varphi}) = X_i$, $\mathrm{pr}^j_{A^*}(\underline{X}, \underline{\varphi}) = \varphi_j$. Using Lemma 2.4, we see that

$$\begin{aligned}\langle \mu, \mathcal{B}a_{(i)} \rangle = (f_k \circ \mathrm{pr})\langle \mu, \mathcal{B}a^k_{(i)} \rangle &= (-1)^{i-1}(f_k \circ \mathrm{pr}) c_{l(a^k)} \circ \gamma_{(i,0)} \\ &= (-1)^{i-1} c_{l(a)} \circ \gamma_{(i,0)},\end{aligned}$$

which is (4.40). A similar argument verifies (4.41). For (4.42), note that

$$\begin{aligned}\langle \mu, (T^p a, \mathcal{R}^q_a) \rangle = (f_k \circ \mathrm{pr}) c_{D(a^k)} &+ (-1)^{i-1}(\ell_{df_k} \circ \mathrm{pr}^i_{TM}) c_{l(a^k)} \circ \gamma_{(i,0)} \\ &+ (-1)^{j-1}(\ell_{-a^k} \circ \mathrm{pr}^j_{A^*}) c_{r(df_k)} \circ \gamma_{(0,j)}.\end{aligned}$$

By (2.8), it follows that the right-hand side above agrees with

$$c_{f_k D(a^k)} + c_{df_k \wedge l(a^k)} - c_{a^k \wedge r(df_k)} = c_{D(f_k a^k)},$$

where the Leibniz condition for D is used in the last equality. Hence (4.42) holds and μ is well defined.

Now consider the componentwise linear function $c_\tau \in C^\infty(\mathbb{G})$. It follows from the third equality in (3.10), Lemma 3.10 and Theorem 4.1 that

$$\mathcal{L}_{\overrightarrow{(T^p a, \mathcal{R}^q_a)}} c_\tau = c_{\mathcal{L}_{\vec{a}} \tau} = c_{\mathcal{T}(D(a))} = \mathtt{t}^* c_{D(a)} = \mathtt{t}^* \langle \mu, (T^p a, \mathcal{R}^q_a) \rangle.$$

Similarly, we see that

$$\mathcal{L}_{\overrightarrow{\mathcal{B}b_{(i)}}} c_\tau = \mathtt{t}^* \langle \mu, \mathcal{B}b_{(i)} \rangle, \quad \text{and} \quad \mathcal{L}_{\overrightarrow{\mathcal{B}\alpha_{(j)}}} c_\tau = \mathtt{t}^* \langle \mu, \mathcal{B}\alpha_{(j)} \rangle.$$

By linearity, one has that $\mathcal{L}_{\vec{\chi}} c_\tau = \mathtt{t}^* \langle \mu, \chi \rangle$, for every $\chi \in \Gamma(A\mathbb{G})$. As $\mathbb{G}$ is source connected (because $\mathcal{G}$ is, see Remark 2.1), the result follows from Proposition 3.5. □

4.4 Linear tensor fields

For the proof of Theorem 3.19, we will need to specialize our study of multiplicative tensor fields to Lie groupoids given by vector bundles $\pi_E : E \to M$. As we already saw, in this case the groupoid $\mathbb{E} = (\oplus^p TE) \oplus (\oplus^q T^*E)$ in (2.7) is a vector bundle over $\mathbb{M} = (\oplus^p TM) \oplus (\oplus^q E^*)$, with Lie algebroid $A\mathbb{E}$ given by $\mathbb{E} \to \mathbb{M}$ itself with zero anchor and zero bracket. A tensor field $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ is multiplicative if and only if the associated function $c_\tau : \mathbb{E} \to \mathbb{R}$ is fiberwise linear on $\mathbb{E} \to \mathbb{M}$. For this reason, we refer to multiplicative tensor fields on $E \to M$ also as *linear*. Our goal here is to show how one can reconstruct linear tensor fields on $E \to M$ explicitly from their infinitesimal components.

For a linear tensor $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$,

$$Ac_\tau = c_\tau,$$

noticing that since c_τ is fiberwise linear on $\mathbb{E} \to \mathbb{M}$, it can be seen as a section of $\mathbb{E}^* \to \mathbb{M}$, cf. Example 3.3. In particular, from (4.40), (4.41) and (4.42), we see that the infinitesimal components (D, l, r) of a linear tensor τ satisfy

$$c_{l(u)} \circ \gamma_{(i,0)} = (-1)^{i-1} \langle c_\tau, \mathcal{B}u_{(i)} \rangle, \quad c_{r(\alpha)} \circ \gamma_{(0,j)} = (-1)^{j-1} \langle c_\tau, \mathcal{B}\alpha_{(j)} \rangle, \tag{4.43}$$

$$c_{D(u)} = \langle c_\tau, (T^p u, \mathcal{R}^q_u) \rangle, \tag{4.44}$$

for $u \in \Gamma(E)$, $\alpha \in \Omega^1(M)$, and where $\gamma_{(i,0)}$, $\gamma_{(o,j)}$ are the projections in (4.11) and (4.12).

Before presenting the main result of this subsection, we need two lemmas. We begin with a useful property of multiplicative tensors on Lie groupoids with $\mathsf{s} = \mathsf{t}$, so in particular linear ones.

Lemma 4.8 *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid such that $\mathsf{s} = \mathsf{t}$. If $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$ is a multiplicative (q, p)-tensor field, then*

$$i_{\overrightarrow{a}}\, i_{\overrightarrow{b}}\, \tau = 0, \tag{4.45}$$

$$i_{\mathsf{t}^*\alpha}\, i_{\mathsf{t}^*\beta}\, \tau = 0, \tag{4.46}$$

$$i_{\overrightarrow{a}}\, i_{\mathsf{t}^*\alpha}\, \tau = 0, \tag{4.47}$$

for $a, b \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(T^*M)$.

Proof As $\mathsf{s} = \mathsf{t}$, it follows from (3.6) that $i_{\overrightarrow{a}}\mathcal{T}(\Phi) = 0$ and $i_{\mathsf{t}^*\alpha}\mathcal{T}(\Phi) = 0$, for any $\Phi \in \Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet A)$. The result now follows from Theorem 3.11. □

The following lemma shows how the infinitesimal components of linear tensors can be obtained by means of pointwise evaluation of the tensor on special vectors and covectors .

Lemma 4.9 *Let $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ be a linear (q, p) tensor field. For $(\underline{X}, \underline{\varphi}) \in \mathbb{M}_x$, $u \in \Gamma(E)$ and $\alpha \in \Omega^1(M)$, define*

$$\underline{U}_{(i)} = (T0(X_1), \dots, T0(X_{i-1}), \overline{u(x)}, T0(X_{i+1}), \dots, T0(X_p)) \in \oplus^p T_{0_x}E$$
$$\underline{\xi}_{(j)} = (\widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_{j-1}}, \overline{\alpha(m)}, \ \widetilde{0}_{\varphi_{j+1}}, \dots, \widetilde{0}_{\varphi_q}) \in \ \oplus^q T^*_{0_x}E.$$

The infinitesimal components $l : E \to \wedge^{p-1}T^*M \otimes \wedge^q E$, $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1}E$ and $D : \Gamma(E) \to \Gamma(\wedge^p T^*M \otimes \wedge^q E)$ satisfy

(a) $D(u)(\underline{X}, \underline{\varphi}) = \tau((Tu^p, \mathcal{R}^q)(\underline{X}, \underline{\varphi}))$
(b) $l(u)(\gamma_{(i,0)}(\underline{X}, \underline{\varphi})) = (-1)^{i-1}\tau(\underline{U}_{(i)}, \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q})$
(c) $r(\alpha)(\gamma_{(0,j)}(\underline{X}, \underline{\varphi})) = (-1)^{j-1}\tau(T0(X_1), \dots, T0(X_p), \underline{\xi}_{(j)})$.

Proof From (4.44), it is clear that

$$\begin{aligned} D(u)(\underline{X}, \underline{\varphi}) &= \langle c_\tau, (T^p u, \mathcal{R}^q_u)\rangle|_{(\underline{X}, \underline{\varphi})} \\ &= \tau(Tu(X_1), \dots, Tu(X_p), \mathcal{R}_u(\varphi_1), \dots, \mathcal{R}_u(\varphi_q)), \end{aligned}$$

which proves (a). Similarly, from the first equation in (4.43),

$$\begin{aligned} l(u)(\gamma_{(i,0)}(\underline{X}, \underline{\varphi})) &= \langle c_\tau, \mathcal{B}u_{(i)}\rangle|_{(\underline{X}, \underline{\varphi})} \\ &= (-1)^{i-1}\tau(T0(X_1), \dots, \mathcal{B}u(X_i), \dots, T0(X_p), \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q}). \end{aligned}$$

But since $\mathcal{B}u(X_i) = T0(X_i) + \overline{u(m)}$, we have that this last term equals

$$\begin{aligned} &\tau(T0(X_1), \dots, T0(X_p), \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q}) \\ &\quad + \tau(T0(X_1), \dots, T0(X_{i-1}), \overline{u(m)}, T0(X_{i+1}), \dots, T0(X_p), \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q}). \end{aligned}$$

To conclude that (b) holds, note that $\tau(T0(X_1), \dots, T0(X_p), \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q}) = 0$, as τ is linear on $\mathbb{E} \to \mathbb{M}$. The verification of (c) is similar. □

We can now present the main result regarding linear tensor fields.

Proposition 4.10 *A tensor $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ is linear if and only if there exist vector-bundle maps $l : E \to \wedge^{p-1}T^*M \otimes \wedge^q E$ and $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1}E$ covering the identity, and $D : \Gamma(E) \to \Gamma(\wedge^p T^*M \otimes \wedge^q E)$ satisfying the Leibniz condition (3.9), such that, for $U_1, \dots, U_p \in T_eE$, $\xi_1, \dots, \xi_q \in T^*_eE$, $e \in E_x$,*

$$\begin{aligned} \tau(\underline{U}, \underline{\xi}) = D(u)(\underline{X}, \underline{\varphi}) &+ (-1)^{i-1} l(e_i)(\gamma_{(i,0)}(\underline{X}, \underline{\varphi})) \\ &+ (-1)^{j-1} r(\beta_j)(\gamma_{(0,j)}(\underline{X}, \underline{\varphi})), \end{aligned} \tag{4.48}$$

where $u \in \Gamma(E)$ is any section such that $u(x) = e$, $X_i = T\pi_E(U_i)$, $\varphi_j = \widetilde{\pi_E}(\xi_j)$, and $e_i \in E_x$, $\beta_j \in T^*_xM$ are defined by

$$\langle \psi, e_i\rangle = \langle U_i -_{\mathfrak{p}} Tu(X_i), \widetilde{0}_\psi\rangle, \quad \langle \beta_j, Y\rangle = \langle \xi_j -_{\mathfrak{p}} \mathcal{R}_u(\varphi_j), T0(Y)\rangle,$$

for $\psi \in E_x^*$, $Y \in T_xM$, and $i = 1, \dots, p$, $j = 1, \dots, q$. In this case, (D, l, r) are the infinitesimal components of τ.

Proof First note that the right-hand side of formula (4.48) is well defined, in the sense that it does not depend on the extension u. Indeed, let $(u^k)_{k=1,\dots,\mathrm{rank}(E)}$, $(\alpha^s)_{s=1,\dots,\dim(M)}$ be local frames of E and T^*M respectively. One can write (see notation in Remark 4.5)

$$U_i = t_k \cdot Tu^k(X_i) +_{\mathfrak{p}} \; h_{ik} \cdot \mathcal{B}u^k(X_i), \quad \xi_j = t_k \cdot \mathcal{R}_{u^k}(\varphi_j) +_{\mathfrak{p}} \; g_{js} \cdot \mathcal{B}\alpha^s(\varphi_j),$$

for $t_k, h_{ik}, g_{js} \in \mathbb{R}$, where $e = t_k u^k(x)$. For any section $u = f_k u^k$, $f^k \in C^\infty(M)$, $u(x) = e$ if and only if $f_k(x) = t_k$. Also,

$$e_i = (h_{ik} - \mathcal{L}_{X_i} f_k)u^k, \quad \beta_j = g_{js}\alpha^s + \langle \varphi_j, u^k \rangle df_k.$$

So, by using the Leibniz condition (3.9), one can rewrite (4.48) as

$$\begin{aligned}\tau(\underline{U}, \underline{\xi}) = t_k\, D(u^k)(\underline{X}, \underline{\varphi}) + \sum_{i=1}^{p} (-1)^{i-1} h_{ik}\, l(u^k)(\gamma_{(i,0)}(\underline{X}, \underline{\varphi})) \\ + \sum_{j=1}^{q} (-1)^{j-1} g_{js}\, r(\alpha^s)(\gamma_{(0,j)}(\underline{X}, \underline{\varphi})).\end{aligned}$$

Let us assume that $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ is a linear tensor, and let (D, l, r) be its infinitesimal components. One may directly check that

$$U_i = Tu(X_i) +_{\mathfrak{p}} \underbrace{(T0(X_i) + \overline{e_i})}_{V_i} \quad \text{and} \quad \xi_j = \mathcal{R}_u(\varphi_j) +_{\mathfrak{p}} \underbrace{(\widetilde{0}_{\varphi_j} + \overline{\beta_j})}_{\zeta_j}.$$

Since τ is linear, we have

$$\tau(\underline{U}, \underline{\xi}) = \tau(T^p u(\underline{X}), \mathcal{R}^q(\underline{\varphi})) + \tau(\underline{V}, \underline{\zeta}) = D(u)(\underline{X}, \underline{\varphi}) + \tau(\underline{V}, \underline{\zeta}).$$

Now, using the multilinearity of the tensor τ, one can expand $\tau(\underline{V}, \underline{\zeta})$ as a sum in which every term is τ evaluated on a string involving $T0(X_i)$, $\overline{e_i}$ separatedly on the TE part and $\widetilde{0}_{\varphi_j}$, $\overline{\beta_j}$ separatedly on the T^*E part.

Claim The only non-zero terms on the expansion of $\tau(\underline{V}, \underline{\zeta})$ as a sum are the ones in which the $\overline{(\cdot)}$ terms appear exactly once (counting both the TE and T^*E parts). Indeed, if they do not appear at all, one has $\tau(T0(X_1), \dots, T0(X_p), \widetilde{0}_{\varphi_1}, \dots, \widetilde{0}_{\varphi_q}) = 0$, because τ is linear on the fibers of $\mathbb{E} \to \mathbb{M}$. If they appear twice or more, note that $\overline{e_i} = u_i^{\mathrm{v}}(0_m)$ and $\overline{\beta_j} = (\pi_E^* \alpha_j)(0_m)$, where $u_i \in \Gamma(E)$ and $\alpha_j \in \Gamma(T^*M)$ satisfy $u_i(m) = e_i$ and $\alpha_j(m) = \beta_j$. So, the claim follows from Lemma 4.8. Therefore,

$$\begin{aligned}\tau(\underline{U},\underline{\xi}) &= D(u)(\underline{X},\underline{\varphi})\\ &\quad+\sum_{i=0}^{p}\tau(T0(X_1),\ldots,T0(X_{i-1}),\overline{e_i},T0(X_{i+1}),\ldots,T0(X_p),\widetilde{0}_{\varphi_1},\ldots,\widetilde{0}_{\varphi_q})\\ &\quad+\sum_{j=0}^{q}\tau(T0(X_1),\ldots,T0(X_p),\widetilde{0}_{\varphi_1},\ldots,\widetilde{0}_{\varphi_{j-1}},\overline{\beta_j},\widetilde{0}_{\varphi_{j+1}},\ldots\widetilde{0}_{\varphi_q}).\end{aligned}$$

Formula (4.48) now follows from Lemma 4.9.

Conversely, let us assume $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ is a (q, p)-tensor field for which (4.48) holds. It is straightforward to check that (4.48) is linear on the fibers of $\mathbb{E} \to \mathbb{M}$, so τ is linear. To prove that (D, l, r) are exactly the infinitesimal components of τ, one proceeds as follows: first substitute $(\underline{U}, \underline{\xi})$ with $(T^p u(\underline{X}), \mathcal{R}_u(\underline{\varphi}))$. In this case, $e_i = 0$, $\beta_j = 0$ and formula (4.48) becomes $\tau(T^p u(\underline{X}), \mathcal{R}_u(\underline{\varphi})) = D(u)(\underline{X}, \underline{\varphi})$. By substituting $(\underline{U}, \underline{\xi})$ with $(\overline{e_1}, T0(X_2), \ldots, T0(X_p), \widetilde{0}_{\varphi_1}, \ldots, \widetilde{0}_{\varphi_q})$, one has that $X_1 = 0$, $e_2 = \cdots = e_p = 0$, $\beta_j = 0$ and, therefore, formula (4.48) becomes

$$\tau(\overline{e_1}, T0(X_2), \ldots, T0(X_p), \widetilde{0}_{\varphi_1}, \ldots, \widetilde{0}_{\varphi_q}) = l(e_1)(X_2, \ldots, X_p, \varphi_1, \ldots, \varphi_q).$$

Finally, by substituting $(\underline{U}, \underline{\xi})$ with $(T0(X_1), \ldots, T0(X_p), \overline{\beta_1}, \widetilde{0}_{\varphi_2}, \ldots, \widetilde{0}_{\varphi_q})$, formula (4.48) becomes

$$\tau(T0(X_1), \ldots, T0(X_p), \overline{\beta_1}, \widetilde{0}_{\varphi_2}, \ldots, \widetilde{0}_{\varphi_q}) = r(\beta_1)(X_1, \ldots, X_p, \varphi_2, \ldots, \varphi_q).$$

The result now follows from Lemma 4.9. □

As an immediate consequence, we have

Corollary 4.11 *There is a one-to-one correspondence defined by* (4.48) *between linear tensors* $\tau \in \Gamma(\wedge^p T^*E \otimes \wedge^q TE)$ *and triples* (D, l, r)*, where* $l : E \to \wedge^{p-1}T^*M \otimes \wedge^q E$ *and* $r : T^*M \to \wedge^p T^*M \otimes \wedge^{q-1} E$ *are vector bundle maps covering the identity and* $D : \Gamma(E) \to \Gamma(\wedge^p T^*M \otimes \wedge^q E)$ *satisfies the Leibniz condition* (3.9).

4.5 Proof of Theorem 3.19

We just saw in Corollary 4.11 how linear tensors τ on a vector bundle are described in terms of triples (D, l, r). Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, and consider linear tensors τ on A for which the corresponding fiberwise linear functions $c_\tau : \mathbb{A} \to \mathbb{R}$ are Lie-algebroid cocycles. We now see how to express this additional cocycle property in terms of (D, l, r).

Proposition 4.12 *There is a one-to-one correspondence defined by* (4.48) *between linear* (q, p)*-tensors* $\tau \in \Gamma(\wedge^p T^*A \otimes \wedge^q TA)$ *for which* $c_\tau : \mathbb{A} \to \mathbb{R}$ *is a Lie-algebroid cocycle and IM* (q, p)*-tensors* (D, l, r) *on* A*, as in Definition 3.17.*

Proof By definition, the cocycle condition $d_{\mathbb{A}}\, c_\tau = 0$ is equivalent to the equation

$$\langle c_\tau, [U, V]\rangle = \mathcal{L}_{\rho_{\mathbb{T}}(U)}\langle c_\tau, V\rangle - \mathcal{L}_{\rho_{\mathbb{T}}(V)}\langle c_\tau, U\rangle \tag{4.49}$$

all $U, V \in \Gamma(\mathbb{A})$. In order to prove that this equality holds, it suffices to consider U, V varying on the set of generators given by Proposition 4.4, and this will be shown to be equivalent to the set of IM-equations (IM1)–(IM6) in Definition 3.17. Let us first fix $U = (T^p a, \mathcal{R}^q_a)$, for $a \in \Gamma(A)$. In the following, we shall use repeatedly the anchor and Lie bracket equations (4.24), (4.25) for the Lie algebroid $\mathbb{A} \to \mathbb{M}$.

Equation (IM1) Take $V = (T^p b, \mathcal{R}^q_b)$, for $b \in \Gamma(A)$. It follows from (4.44) that the cocycle equation (4.49) is equivalent to

$$\begin{aligned} c_{D([a,b])} &= \mathcal{L}_{(\rho(a)^{T,p}, H^q_a)}\, c_{D(b)} - \mathcal{L}_{(\rho(b)^{T,p}, H^q_b)}\, c_{D(a)} \\ &= c_{a\cdot D(b) - b\cdot D(a)}, \end{aligned}$$

where the last equality follows from Proposition 4.1. So, in this case (4.49) is equivalent to (IM1).

Equations (IM2) **and** (IM3) Take $V = \mathcal{B}b_{(i)}$, $1 \le i \le p$. From (4.43), it follows that the cocycle equation (4.49) for this pair U, V can be rewritten as

$$\begin{aligned} (-1)^{i-1} c_{l([a,b])} \circ \gamma_{(i,0)} &= (-1)^{i-1} \mathcal{L}_{\rho_{\mathbb{T}}(T^p a,\ \ \mathcal{R}^q_a)} \left(c_{l(b)} \circ \gamma_{(i,0)}\right) - \mathcal{L}_{\rho_{\mathbb{T}}(\mathcal{B}b_{(i)})}\, c_{D(a)} \\ &= (-1)^{i-1} \left(\mathcal{L}_{(\rho(a)^{T,p-1},\ H^q_a)}\, c_{l(b)}\right) \circ \gamma_{(i,0)} - \mathcal{L}_{(\rho(b)^{\mathrm{v},p}_{(i)}, 0)}\, c_{D(a)} \\ &= (-1)^{i-1} \left(\left(c_{a\cdot l(b)}\right) \circ \gamma_{(i,0)} - c_{\,i_{\rho(b)} D(a)} \circ \gamma_{(i,0)}\right) \end{aligned}$$

where $\gamma_{(i,0)}$ is the projection (4.11) and the last equality follows from Proposition 4.1. So, for the given choices of U and V, (4.49) is equivalent to (IM2).

When $V = \mathcal{B}\alpha_{(j)}$, $1 \le j \le q$, for $\alpha \in \Omega^1(M)$, one can prove analogously that (4.49) and (IM3) are equivalent.

Equations (IM4), (IM5) **and** (IM6) Let $U = \mathcal{B}a_{(i)}$ and $V = \mathcal{B}b_{(k)}$, for $1 \le i < k \le p$. As $[\mathcal{B}a_{(i)}, \mathcal{B}b_{(k)}] = 0$, it follows from (4.43) that the cocycle equation (4.49) can be rewritten as

$$\begin{aligned} 0 &= (-1)^{k-1} (\mathcal{L}_{(\rho(a)^{\mathrm{v},\,p-1}_{(i)}, 0)} c_{l(b)}) \circ \gamma_{(k,0)} - (-1)^{i-1} (\mathcal{L}_{(\rho(b)^{\mathrm{v},\,p-1}_{(k-1)}, 0)} c_{l(a)}) \circ \gamma_{(i,0)} \\ &= (-1)^{i+k-2} c_{i_{\rho(b)} l(a)} \circ \gamma^{(p-1,q)}_{(i,0)} \circ \gamma_{(k,0)} - (-1)^{i+k-3} c_{i_{\rho(a)} l(b)} \circ \gamma^{(p-1,q)}_{(k-1,0)} \circ \gamma_{(i,0)} \\ &= (-1)^{i+k-2} \left(c_{i_{\rho(b)} l(a)} \circ \gamma^{(p-1,q)}_{(i,0)} \circ \gamma_{(k,0)} + c_{i_{\rho(a)} l(b)} \circ \gamma^{(p-1,q)}_{(k-1,0)} \circ \gamma_{(i,0)}\right), \end{aligned}$$

where in the second equality we have used Proposition 4.1. One can now directly check that $\gamma^{(p-1,q)}_{(i,0)} \circ \gamma_{(k,0)}$ and $\gamma^{(p-1,q)}_{(k-1,0)} \circ \gamma_{(i,0)}$ are the same projection from $\mathbb{M}^{(p,q)}$ to $\mathbb{M}^{(p-2,q)}$, which forgets the i-th and the k-th components on TM. Hence, for these choices of U and V, (4.49) is equivalent to (IM4).

In a similar way, one checks that, for $U = \mathcal{B}\alpha_{(j)}$, $V = \mathcal{B}\beta_{(k)}$, $1 \leq j < k \leq q$, one obtains the equivalence of (4.49) with (IM5), and for $U = \mathcal{B}a_{(i)}$, $V = \mathcal{B}\alpha_{(j)}$, $1 \leq i \leq p$, $1 \leq j \leq q$, one has the equivalence of (4.49) with (IM6).

These 6 cases cover all possibilities of U, V varying in the set of generators, so the result follows. □

Let now $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \to M$. For a multiplicative (q, p)-tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$, consider the corresponding multiplicative function $c_\tau \in C^\infty(\mathbb{G})$ on the groupoid (2.7). Let $Ac_\tau \in \Gamma(\mathbb{A}^*) \subseteq C^\infty(\mathbb{A})$ be its associated infinitesimal cocycle.

Lemma 4.13 *The following holds:*

(a) *There is a linear tensor field $\tau_A \in \Gamma(\wedge^p T^*A \otimes \wedge^q TA)$ such $Ac_\tau = c_{\tau_A}$.*
(b) *The infinitesimal components (D_A, l_A, r_A) of τ_A satisfy*

$$
\begin{aligned}
D_A(a)(\underline{X}, \underline{\varphi}) &= \left(\mathcal{L}_{\overrightarrow{a}}\tau\right)(\underline{X}, \underline{\varphi}) \\
l_A(a)|_{\gamma_{(i,0)}(\underline{X},\underline{\varphi})} &= \left(i_{\overrightarrow{a}}\tau\right)|_{\pi_{(i,0)}(\underline{X},\underline{\varphi})} \\
r_A(\alpha)|_{\gamma_{(0,j)}(\underline{X},\underline{\varphi})} &= (i_{t^*\alpha}\tau)|_{\pi_{(0,j)}(\underline{X},\underline{\varphi})},
\end{aligned}
$$

where $(\underline{X}, \underline{\varphi}) \in \mathbb{M}$, $a \in \Gamma(A)$, $\alpha \in \Omega^1(M)$ and $\gamma_{(i,0)}$, $\gamma_{(0,j)}$, $\pi_{(i,0)}$ and $\pi_{(0,j)}$ are the forgetful projections (4.11), (4.12), (4.32) and (4.33), respectively.
(c) *The infinitesimal components (D, l, r) of τ coincide with those of τ_A.*

Proof It follows from Proposition A.3 that $Ac_\tau : \mathbb{A} \to \mathbb{R}$ is a componentwise linear function which is antisymmetric on the TA components as well as on the T^*A components. Hence, there exists $\tau_A \in \Gamma(\wedge^p T^*A \otimes \wedge^q TA)$ such that $Ac_\tau = c_{\tau_A}$. This proves (a).

By (3.2), (4.44) and Proposition 4.1,

$$
\begin{aligned}
D_A(a)(\underline{X}, \underline{\varphi}) &= \langle c_{\tau_A}, (T^p a(\underline{X}), \mathcal{R}^q_a(\underline{\varphi}))\rangle = \langle Ac_\tau, (T^p a(\underline{X}), \mathcal{R}^q_a(\underline{\varphi}))\rangle \\
&= (\mathcal{L}_{\overrightarrow{(T^p a, \mathcal{R}^q_a)}} c_\tau)(\underline{X}, \underline{\varphi}) = (\mathcal{L}_{(\overrightarrow{a}^{T,p}, H^q_a)} c_\tau)(\underline{X}, \underline{\varphi}) \\
&= (\mathcal{L}_{\overrightarrow{a}}\tau)(\underline{X}, \underline{\varphi}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(-1)^{i-1} l_A(a)(\gamma_{(i,0)}(\underline{X}, \underline{\varphi})) &= \langle c_{\tau_A}, \mathcal{B}a_{(i)}(\underline{X}, \underline{\varphi})\rangle = \langle Ac_\tau, \mathcal{B}a_{(i)}(\underline{X}, \underline{\varphi})\rangle \\
&= (\mathcal{L}_{\overrightarrow{\mathcal{B}a_{(i)}}} c_\tau)(\underline{X}, \underline{\varphi}) = (\mathcal{L}_{(\overrightarrow{a}^{\mathrm{v},p}_{(i)}, 0)} c_\tau)(\underline{X}, \underline{\varphi}) \\
&= (-1)^{i-1}(i_{\overrightarrow{a}}\tau)(\pi_{(i,0)}(\underline{X}, \underline{\varphi})).
\end{aligned}
$$

The equation involving r_A follows similarly, and we conclude that (b) holds.

If we now let (D, l, r) be the infinitesimal components of τ, the equalities $D = D_A$, $l = l_A$ and $r = r_A$ follow from Theorem 3.11. □

We can now finally proceed to the proof of our main result.

Proof of Theorem 3.19 By Proposition 4.12, there exists a Lie algebroid (q, p) tensor field τ_A having (D, l, r) as its infinitesimal components. The fact that $c_{\tau_A} : A\mathbb{G} \to \mathbb{R}$ is a Lie algebroid cocycle and $\mathbb{G}$ is source 1-connected implies that there exists a unique multiplicative function $F : \mathbb{G} \to \mathbb{R}$ satisfying $AF = c_{\tau_A}$. By Proposition A.3, F is componentwise linear and anti-symmetric on both the $T\mathcal{G}$ and $T^*\mathcal{G}$ components. Therefore, $F = c_\tau$ for a (unique) multiplicative tensor field $\tau \in \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G})$. The fact that the infinitesimal components of τ are (D, l, r) follows from Lemma 4.13. □

5 Multiplicative vector-valued forms

Given a manifold N, by a *vector-valued form* on N we mean an element of $\Omega^\bullet(N, TN) = \Gamma(\wedge^\bullet T^*N \otimes TN)$. The space of vector-valued forms is a graded Lie algebra with respect to the Frölicher–Nijenhuis bracket [24]. On a Lie groupoid, the space of *multiplicative* vector-valued forms is closed under the Frölicher–Nijenhuis bracket [6], so it is also a graded Lie algebra. We now identify its infinitesimal counterpart, in the spirit of Remarks 3.23 and 3.27. Before discussing Lie groupoids, we briefly recall vector-valued forms on manifolds.

5.1 The graded Lie algebra of vector-valued forms

Let $\Omega^\bullet(N)$ be the graded algebra of differential forms on N. A *degree k derivation* of $\Omega^\bullet(N)$ is a linear map $\Delta : \Omega^\bullet(N) \to \Omega^{\bullet+k}(N)$ such that $\Delta(\alpha \wedge \beta) = \Delta(\alpha) \wedge \beta + (-1)^{kj}\alpha \wedge \Delta(\beta)$, for $\alpha \in \Omega^j(N)$. Any vector-valued form $K \in \Gamma(\wedge^p T^*N \otimes TN)$ gives rise to a degree $(p-1)$ derivation of $\Omega^\bullet(N)$ by

$$\begin{aligned} &i_K\omega(X_1, \dots, X_{p+j-1}) \\ &\quad = \frac{1}{p!(j-1)!} \sum_{\sigma \in S_{p+j-1}} sgn(\sigma)\, \omega(K(X_{\sigma(1)}, \dots, X_{\sigma(p)}), X_{\sigma(p+1)}, \dots, X_{\sigma(p+j-1)}), \end{aligned} \tag{5.1}$$

for $\omega \in \Omega^j(N)$, $X_1, \dots, X_{p+j-1} \in TN$. It also gives rise to a degree p derivation of $\Omega^\bullet(N)$ via

$$\mathcal{L}_K = [i_K, d] = i_K d - (-1)^{p-1} d i_K, \tag{5.2}$$

where d is the exterior differential on N.

We extend i_K to a contraction operation $i_K : \Omega^\bullet(N, TN) \to \Omega^{\bullet+p-1}(N, TN)$ by

$$i_K(\omega \otimes X) = (i_K\omega) \otimes X, \qquad \omega \in \Omega(N),\ X \in \mathfrak{X}(N). \tag{5.3}$$

Given $K \in \Omega^p(N, TN)$ and $L \in \Omega^{p'}(N, TN)$, their *Frölicher–Nijenhuis bracket* [24] (see also [34, Ch. 2]) is the vector-valued form $[K, L] \in \Omega^{p+p'}(N, TN)$ uniquely defined by the condition

$$\mathcal{L}_{[K,L]} = [\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K\mathcal{L}_L - (-1)^{pp'}\mathcal{L}_L\mathcal{L}_K. \tag{5.4}$$

When K and L have degree zero (i.e., they are vector fields on N), (5.4) agrees with the usual Lie bracket of vector fields. More generally, for $X \in \mathfrak{X}(N)$,

$$[X, K] = \mathcal{L}_X K.$$

The Frölicher–Nijenhuis bracket makes $\Omega^\bullet(N, TN)$ into a graded Lie algebra. It is a natural bracket in the following sense: for a smooth map $F : N_1 \to N_2$, let $K_i \in \Omega^p(N_i, TN_i)$, $L_i \in \Omega^{p'}(N_i, TN_i)$, $i = 1, 2$, be such that K_1 is F-related to K_2 and L_1 is F-related to L_2.[4] Then $[K_1, L_1]$ is F-related to $[K_2, L_2]$. There are other important properties of the Frölicher–Nijenhuis bracket which will be recalled in subsequent sections.

5.2 Infinitesimal description

Let $\mathcal{G}$ be a Lie groupoid. As seen in Proposition 3.9, a multiplicative vector-valued form $K \in \Omega^p(\mathcal{G}, T\mathcal{G})$ as defined by [43] is exactly a multiplicative $(1, p)$-tensor field. From Theorem 3.19, one obtains a bijective correspondence between multiplicative vector-valued p-forms on a Lie groupoid $\mathcal{G}$ and IM $(p, 1)$-tensors on its Lie algebroid A. For this reason, we will refer to IM $(p, 1)$-tensors also as *IM vector-valued p-forms*.

We denote by $\Omega^\bullet_{mult}(\mathcal{G}, T\mathcal{G})$ the space of multiplicative vector-valued forms on $\mathcal{G}$. In the following, we will also need the following result proven in [6, Thm. 4.3]:

Proposition 5.1 *On a Lie groupoid $\mathcal{G}$, $\Omega^\bullet_{mult}(\mathcal{G}, T\mathcal{G})$ is closed under the Frölicher–Nijenhuis bracket.*

Hence $\Omega^\bullet_{mult}(\mathcal{G}, T\mathcal{G}) \subseteq \Omega^\bullet(\mathcal{G}, T\mathcal{G})$ is a graded Lie subalgebra. We now describe the graded Lie bracket on IM vector-valued forms corresponding to the Frölicher–Nijenhuis bracket on multiplicative vector-valued forms.

For a multiplicative vector-valued form $K \in \Omega^p(\mathcal{G}, T\mathcal{G})$, consider its infinitesimal components $D : \Gamma(A) \to \Gamma(\wedge^p T^*M \otimes A)$, $l : A \to \wedge^{p-1}T^*M \otimes A$ and $r : T^*M \to \wedge^p T^*M$. Note that r can be seen alternatively as an element $r \in \Omega^p(M, TM)$. As such, Proposition 3.9 shows that K is s, t-related to r.

Using the $\Omega^\bullet(M)$-module structure of $\Omega^\bullet(M, A) = \Gamma(\wedge^\bullet T^*M \otimes A)$, we extend l to an operator $l : \wedge^\bullet T^*M \otimes A \to \wedge^{\bullet+p-1}T^*M \otimes A$ by

$$l(\alpha \otimes a) = \alpha \wedge l(a),$$

and D to an operator $D : \Omega^j(M, A) \to \Omega^{p+j}(M, A)$ by

$$D(\alpha \otimes a) = \alpha \wedge D(a) + (-1)^j(d\alpha \wedge l(a) - (-1)^{j(p-1)}\mathcal{L}_r\alpha \otimes a), \tag{5.5}$$

[4] $K_1 \in \Omega^p(N_1, TN_1)$ is *F-related* to $K_2 \in \Omega^p(N_2, TN_2)$ if

$$K_2(TF(X_1), \ldots, TF(X_k)) = TF(K_1(X_1, \ldots, X_k)),$$

for all $X_1, \ldots, X_k \in T_xN$, and $x \in N$.

with $\mathcal{L}_r$ as in (5.2). This extension for D is well-defined as a consequence of the Leibniz rule (3.9). Moreover,

$$D(\alpha \wedge \eta) = \alpha \wedge D(\eta) + (-1)^{i+j}(d\alpha \wedge l(\eta) - (-1)^{(i+j)(p-1)}\mathcal{L}_r\alpha \wedge \eta),$$

for $\alpha \in \Omega^i(M)$, and $\eta \in \Omega^j(M, A)$.

Lemma 5.2 *For $\eta \in \Omega^\bullet(M, A)$, we have $i_{\mathcal{T}(\eta)}K = \mathcal{T}(l(\eta))$.*

Proof For homogeneous $\eta = \alpha \otimes a$, one has that $\mathcal{T}(\eta) = \mathsf{t}^*\alpha \otimes \overrightarrow{a}$. So, by definition of the contraction (5.3),

$$i_{\mathcal{T}(\eta)}K = \mathsf{t}^*\alpha \wedge i_{\overrightarrow{a}}K = \mathsf{t}^*\alpha \wedge \mathcal{T}(l(a)) = \mathcal{T}(\alpha \wedge l(a)) = \mathcal{T}(l(\alpha \otimes a)).$$

□

Let us consider the operation $[\cdot, K] : \Omega^\bullet(\mathcal{G}, T\mathcal{G}) \to \Omega^{\bullet+k}(\mathcal{G}, T\mathcal{G})$, where $[\cdot, \cdot]$ is the Frölicher–Nijenhuis bracket. The following result shows that $[\cdot, K]$ preserves the image of $\mathcal{T}$ inside $\Omega^\bullet(\mathcal{G}, T\mathcal{G})$.

Lemma 5.3 *For $\eta \in \Omega^\bullet(M, A)$, we have $[\mathcal{T}(\eta), K] = \mathcal{T}(D(\eta))$.*

Proof If $\eta = \alpha \otimes a$, for $\alpha \in \Omega^j(M)$, then one has (see [34, Sec. 8.7]):

$$\begin{aligned}
[\mathsf{t}^*\alpha \otimes \overrightarrow{a}, K] &= \mathsf{t}^*\alpha \wedge [\overrightarrow{a}, K] + (-1)^j(\mathsf{t}^*d\alpha \otimes i_{\overrightarrow{a}}K - (-1)^{(j-1)p}\mathcal{L}_K(\mathsf{t}^*\alpha) \otimes \overrightarrow{a})\\
&= \mathsf{t}^*\alpha \wedge \mathcal{T}(D(a)) + (-1)^j(\mathsf{t}^*d\alpha \otimes \mathcal{T}(l(a)) - (-1)^{j(p-1)}\mathsf{t}^*(\mathcal{L}_r\alpha) \otimes \overrightarrow{a}\\
&= \mathcal{T}(\alpha \wedge D(a) + (-1)^j(d\alpha \otimes l(a) - (-1)^{j(p-1)}\mathcal{L}_r\alpha \otimes a))\\
&= \mathcal{T}(D(\eta)).
\end{aligned}$$

In the second equality, we have used the fact that K and r are t-related. □

Proposition 5.4 *Let $K_1 \in \Omega^{p_1}(\mathcal{G}, T\mathcal{G})$, $K_2 \in \Omega^{p_2}(\mathcal{G}, T\mathcal{G})$ be multiplicative vector-valued forms, with infinitesimal components (D_1, l_1, r_1) and (D_2, l_2, r_2), respectively. The infinitesimal components (D, l, r) of their Frölicher–Nijenhuis bracket $[K_1, K_2]$ are*

$$D = [D_2, D_1] = D_2 \circ D_1 - (-1)^{p_1p_2}D_1 \circ D_2 \tag{5.6}$$
$$l = [D_2, l_1] - (-1)^{p_1p_2}[D_1, l_2], \tag{5.7}$$

where the brackets on the right-hand side are the (graded) commutators of endomorphisms of $\Omega^\bullet(M, A)$, and

$$r = [r_1, r_2], \tag{5.8}$$

where the last bracket is the Frölicher–Nijenhuis bracket of $r_i \in \Omega^{p_i}(M, TM)$, $i = 1, 2$.

Proof The equation (5.8) follows from the naturality of the Frölicher–Nijenhuis bracket. As K_1 is t-related to r_1 and K_2 is t-related to r_2, it follows that $[K_1, K_2]$ must be t-related to $[r_1, r_2]$. The identity (5.6) for D follows from the Jacobi equation for the Frölicher–Nijenhuis bracket. Indeed, using Lemma 5.3:

$$\begin{aligned}\mathcal{T}(D(a)) &= [\overrightarrow{a}, [K_1, K_2]] = [[\overrightarrow{a}, K_1], K_2] + [K_1, [\overrightarrow{a}, K_2]]\\ &= [\mathcal{T}(D_1(a)), K_2] + [K_1, \mathcal{T}(D_2(a))]\\ &= \mathcal{T}(D_2(D_1(a))) - (-1)^{p_1p_2}[\mathcal{T}(D_2(a)), K_1]\\ &= \mathcal{T}(D_2(D_1(a)) - (-1)^{p_1p_2}D_1(D_2(a))) = \mathcal{T}([D_2, D_1](a)).\end{aligned}$$

Now, recall the following general property of the Frölicher–Nijenhuis bracket (see Theorem 8.11 in [34]): for $K_i \in \Omega^{p_i}(N, TN)$ and $L \in \Omega^{p'+1}(N, TN)$, $i = 1, 2$, we have

$$\begin{aligned}i_L[K_1, K_2] &= [i_L K_1, K_2] + (-1)^{p_1p'}[K_1, i_L K_2]\\ &\quad - \left((-1)^{p_1p'} i_{[K_1,L]}, K_2 - (-1)^{(p_1+p')p_2} i_{[K_2,L]}K_1\right).\end{aligned}$$

Using this identity and the previous lemmas, we obtain

$$\begin{aligned}i_{\overrightarrow{a}}[K_1, K_2] &= [i_{\overrightarrow{a}} K_1, K_2] + (-1)^{p_1}[K_1, i_{\overrightarrow{a}} K_2]\\ &\quad - ((-1)^{p_1} i_{[K_1, \overrightarrow{a}]} K_2 - (-1)^{(p_1-1)p_2} i_{[K_2, \overrightarrow{a}]} K_1)\\ &= [\mathcal{T}(l_1(a)), K_2] + (-1)^{p_1}\left(-(-1)^{p_1(p_2-1)}[\mathcal{T}(l_2(a)), K_1]\right)\\ &\quad + ((-1)^{p_1} i_{\mathcal{T}(D_1(a))} K_2 - (-1)^{(p_1-1)p_2} i_{\mathcal{T}(D_2(a))} K_1)\\ &= \mathcal{T}\left(D_2(l_1(a)) - (-1)^{p_1p_2}D_1(l_2(a)) + (-1)^{p_1}l_2(D_1(a)\right)\\ &\quad - (-1)^{(p_1-1)p_2}l_1(D_2(a)))\\ &= \mathcal{T}\left([D_2, l_1](a) - (-1)^{p_1p_2}(D_1(l_2(a)) - (-1)^{p_1(p_2-1)}l_2(D_1(a)))\right)\\ &= \mathcal{T}\left([D_2, l_1](a) - (-1)^{p_1p_2}[D_1, l_2](a)\right),\end{aligned}$$

as we wanted to prove. □

Corollary 5.5 *The space of IM vector-valued forms is a graded Lie algebra with the bracket defined by* (5.6),(5.7) *and* (5.8)*:*

$$[(D_1, l_1, r_1), (D_2, l_2, r_2)] = ([D_1, D_2], [l_1, l_2], [r_1, r_2]).$$

The correspondence established by Theorem 3.19 *between multiplicative vector-valued forms and IM vector-valued forms is a graded Lie algebra isomorphism.*

This last result should be regarded as parallel to Remarks 3.27 and 3.23.

6 Multiplicative (1, 1)-tensor fields

We will now focus on multiplicative vector-valued 1-forms, or (1, 1)-tensor fields.

6.1 Infinitesimal components

Let $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ be a multiplicative (1, 1)-tensor field, with infinitesimal components

$$D : \Gamma(A) \to \Gamma(T^*M \otimes A), \quad l : A \to A, \quad r : TM \to TM.$$

Note that we have dualized the r component. For $X \in TM$, we will use the notation

$$D_X : \Gamma(A) \to \Gamma(A), \quad D_X(a) = i_X D(a).$$

The IM-equations satisfied by the triple (D, l, r) take the form

$$D_X([a, b]) = [a, D_X(b)] - [b, D_X(a)] + D_{[\rho(b), X]}(a) - D_{[\rho(a), X]}(b) \qquad \text{(IM1*)}$$

$$l([a, b]) = [a, l(b)] - D_{\rho(b)}(a) \qquad \text{(IM2*)}$$

$$r([\rho(a), X]) = [\rho(a), r(X)] - \rho(D_X(a)) \qquad \text{(IM3*)}$$

$$r \circ \rho = \rho \circ l. \qquad \text{(IM6*)}$$

Proposition 6.1 *Let $K : T\mathcal{G} \to T\mathcal{G}$ be a multiplicative* (1, 1)*-tensor field on the Lie groupoid $\mathcal{G}$, with infinitesimal components (D, l, r). Then K^n is also multiplicative, and its infinitesimal components (D', l', r') satisfy*

$$l' = l^n, \qquad r' = r^n, \qquad D'(a) = \sum_{j=1}^{n} l^{j-1} \circ D(a) \circ r^{n-j}$$

Proof The equations for l' and r' are straightforward to check. As for the equation for D', the proof follows from an induction on n using the recursion formula

$$[\overrightarrow{a}, K^n] = [\overrightarrow{a}, K^{n-1}] \circ K + K^{n-1} \circ [\overrightarrow{a}, K].$$

□

As a result, we obtain infinitesimal descriptions of multiplicative projections and almost complex/product structures.

Corollary 6.2 *Let K be a multiplicative* (1, 1)*-tensor field on a source-connected Lie groupoid $\mathcal{G} \rightrightarrows M$ with infinitesimal components (D, l, r). Then*

(a) *K satisfies $K^2 = K$ if and only if*

$$l \circ D(a) + D(a) \circ r = D(a), \qquad l^2 = l, \qquad r^2 = r.$$

(b) *K satisfies $K^2 = \pm\,\mathrm{id}_{T\mathcal{G}}$ if and only if*

$$l \circ D(a) + D(a) \circ r = 0, \qquad l^2 = \pm\,\mathrm{id}_A, \qquad r^2 = \pm\,\mathrm{id}_{TM}.$$

Proof In (a), the equations for (D, l, r) guarantee that the multiplicative (1, 1)-tensors K^2 and K have the same infinitesimal components. As $\mathcal{G}$ has connected s-fibers, this implies that $K^2 = K$. The same argument holds for almost product and almost complex structures. □

We will now consider an additional integrability condition in terms of the Nijenhuis torsion.

6.2 Nijenhuis torsion

Given a (1,1)-tensor field K on a manifold N, its *Nijenhuis torsion* is the vector-valued 2-form $\mathcal{N}_K \in \Omega^2(N, TN)$ given by

$$\mathcal{N}_K(X, Y) = [K(X), K(Y)] - K([KX, Y] + [KY, X]) + K^2[X, Y],$$

for $X, Y \in TN$. The Nijenhuis torsion has a well-known relation with the Frölicher–Nijenhuis bracket via

$$\frac{1}{2}[K, K] = \mathcal{N}_K. \tag{6.1}$$

For a multiplicative (1,1)-tensor field on a Lie groupoid $\mathcal{G} \rightrightarrows M$, the following description of the infinitesimal components of its Nijenhuis torsion is an immediate consequence of this last formula and Propositions 5.1 and 5.4:

Corollary 6.3 *Let $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ be multiplicative with infinitesimal components (D, l, r). Then $\mathcal{N}_K \in \Omega^2(\mathcal{G}, T\mathcal{G})$ is multiplicative and its infinitesimal components (D', l', r') are*

$$D' = D^2, \qquad l' = [D, l], \qquad r' = \mathcal{N}_r.$$

It will be useful to have a more concrete expression for

$$D^2 : \Gamma(A) \to \Gamma(\wedge^2 TM \otimes A).$$

Recall [34, Cor. 8.12] the following expression for the Frölicher–Nijenhuis bracket of $K_1, K_2 \in \Omega^1(\mathcal{G}, T\mathcal{G})$:

$$
\begin{aligned}
[K_1, K_2](U, V) = &[K_1(U), K_2](V) - [K_1(V), K_2](U)\\
&- K_1([K_2(U), V] - [K_2(V), U])\\
&+ (K_2K_1 + K_1K_2)([U, V]).
\end{aligned}
$$

Since $[\overrightarrow{a}, \frac{1}{2}[K, K]] = [[\overrightarrow{a}, K], K]$, by taking $K_1 = [\overrightarrow{a}, K]$ and $K_2 = K$, and letting $U = X$, $V = Y$ be in TM, one readily obtains that

$$
D^2_{(X,Y)} = D_Y \circ D_X - D_X \circ D_Y - D_{[r(X),Y]} + D_{[r(Y),X]} + l \circ D_{[X,Y]} + D_{r([X,Y])} \tag{6.2}
$$

where both sides are seen as maps $\Gamma(A) \to \Gamma(A)$.

Corollary 6.3 gives a complete infinitesimal description of general multiplicative Nijenhuis operators on Lie groupoids. In the next subsections, we will illustrate how this general result can be specialized to various cases of interest.

6.3 Poisson quasi-Nijenhuis structures

Poisson–Nijenhuis structures [40,52] play a central role in the theory of integrable systems. Their recent connections with Lie groupoids arose in quantization schemes for Poisson manifolds, see e.g. [2]. In this section we revisit the more general Poisson quasi-Nijenhuis structures [57]. We establish a link with the theory of IM (1, 1)-tensors, which leads to an extension of the integration of Poisson quasi-Nijenhuis structures in [57, Thm. 6.2] (originally based on the theory of Lie bialgebroids [37]) as a consequence of Theorem 3.19.

Given a Poisson manifold (M, Π), consider its cotangent bundle T^*M with the Lie bracket $[\cdot, \cdot]_\Pi$ as in (3.18). We say that a (1,1) tensor $r : TM \to TM$ is *compatible with* Π if

$$
\Pi^\sharp \circ r^* = r \circ \Pi^\sharp \tag{6.3}
$$

(equivalently, $r \circ \Pi^\sharp : T^*M \to TM$ is skew-symmetric) and the following equation holds: for all $\alpha, \beta \in \Omega^1(M)$,

$$
C^r_\Pi(\alpha, \beta) := [\alpha, \beta]_{\Pi_r} - ([r^*\alpha, \beta]_\Pi + [\alpha, r^*\beta]_\Pi - r^*([\alpha, \beta]_\Pi)) = 0, \tag{6.4}
$$

where $[\cdot, \cdot]_{\Pi_r}$ is the bracket (3.18) for the bivector field Π_r defined by $r \circ \Pi^\sharp$.

Remark 6.4 The condition $C^r_\Pi = 0$ implies that $[\Pi, \Pi_r] = 0$ (here $[\cdot, \cdot]$ is the Schouten bracket), but the converse does not hold in general; it does if Π is symplectic, see e.g. [58].

The following definition extends [57, Def. 3.3]:

Definition 6.5 A *Poisson quasi-Nijenhuis structure* is a pair (Π, r), where Π is a Poisson bivector field, $r : TM \to TM$ is a $(1, 1)$ tensor, such that Π and r are compatible and the following condition holds:

$$\mathcal{N}_r^*([\alpha,\beta]_\Pi) = \mathcal{L}_{\Pi^\sharp(\alpha)}\mathcal{N}_r^*(\beta) - i_{\Pi^\sharp(\beta)} d\mathcal{N}_r^*(\alpha) \tag{6.5}$$

$$i_{\Pi^\sharp(\alpha)}\mathcal{N}_r^*(\beta) = -i_{\Pi^\sharp(\beta)}\mathcal{N}_r^*(\alpha), \quad \forall\, \alpha,\beta \in \Omega^1(M), \tag{6.6}$$

where $\mathcal{N}_r^* : T^*M \to \wedge^2 T^*M$ is the adjoint of the Nijenhuis torsion of r, given by $\mathcal{N}_r^*(\alpha)(X,Y) = \langle \alpha, \mathcal{N}_r(X,Y)\rangle$. We refer to a *symplectic quasi-Nijenhuis structure* when Π is nondegenerate, i.e., a symplectic structure.

Note that (6.5) and (6.6) say that $(\mathcal{N}_r^*, 0)$ defines an IM 3-form on T^*M.

Remark 6.6 The Poisson quasi-Nijenhuis structures considered in [57, Def. 3.3] are slightly more restricted than in our definition: they are required to satisfy the condition $\mathcal{N}_r^*(\alpha) = -i_{\Pi^\sharp(\alpha)}\phi$, or, equivalently,

$$\mathcal{N}_r(X,Y) = \Pi^\sharp(\phi(X,Y,\cdot)), \tag{6.7}$$

for a given closed 3-form $\phi \in \Omega^3(M)$. One may verify that this implies that $\mathcal{N}_r^*$ automatically satisfies (6.5) and (6.6). This difference in the definitions will become more transparent when we talk about integration, see Theorem 6.11 and Corollary 6.12 below.

Following the previous remark, we shall refer to the structures satisfying (6.7) as *Poisson quasi-Nijenhuis structures relative to* ϕ; we will specify them by triples (Π, r, ϕ) to make the dependence on the closed 3-form ϕ explicit.

The following proposition gives an alternative way to express the compatibility between a (1,1)-tensor r and a Poisson structure Π in terms of IM tensors. Let $D^r : \Gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$ be defined by

$$\langle D_X^r(\alpha), Y\rangle = d\alpha(X, r(Y)) - (\mathcal{L}_r\alpha)(X,Y), \tag{6.8}$$

where $\mathcal{L}_r$ is the operator (5.2). (Note that (D^r, r^*, r) are the infinitesimal components of the cotangent lift of r.)

Proposition 6.7 *The* $(1,1)$*-tensor* r *and the Poisson tensor* Π *are compatible if and only if the triple* (D^r, r^*, r) *is an IM* $(1,1)$*-tensor on the Lie algebroid* T^*M *(where the bracket is defined by* Π*).*

Proof The Leibniz equation for D^r follows from the properties of d and $\mathcal{L}_r$. So one only needs to check that the IM-equations for (D, r, r^*) are equivalent to Π and r being compatible. The IM equations in this case are:

$$D_X^r([\alpha,\beta]_\Pi) = [\alpha, D_X^r(\beta)]_\Pi - [\beta, D_X^r(\alpha)]_\Pi + D^r_{[\Pi^\sharp(\beta),X]}(\alpha) - D^r_{[\Pi^\sharp(\alpha),X]}(\beta), \tag{6.9}$$

$$r^*([\alpha,\beta]_\Pi) = [\alpha, r^*(\beta)]_\Pi - D^r_{\Pi^\sharp(\beta)}(\alpha), \tag{6.10}$$

$$r([\Pi^\sharp(\alpha), X]) = [\Pi^\sharp(\alpha), r(X)] - \Pi^\sharp(D_X^r(\alpha)), \tag{6.11}$$

$$r \circ \Pi^\sharp = \Pi^\sharp \circ r^*. \tag{6.12}$$

The compatibility equation (6.3) is exactly (6.12). Note that (6.10) is equivalent, for $df, dg \in \Omega^1(M)$, to

$$
\begin{aligned}
r^*([df, dg]_\Pi) &= [df, r^*(dg)]_\Pi - i_{\Pi^\sharp(dg)} dr^*(df) \\
&= [df, r^*(dg)]_\Pi - \mathcal{L}_{\Pi^\sharp(dg)} r^*(df) + d i_{\Pi^\sharp(dg)} r^*(df) \\
&= [df, r^*(dg)]_\Pi - \underbrace{(\mathcal{L}_{\Pi^\sharp(dg)} r^*(df) - \mathcal{L}_{\Pi^\sharp(r^*(df))} dg - d i_{\Pi^\sharp(dg)} r^*(df))}_{[dg, r^*(df)]_\Pi} \\
&\quad - \mathcal{L}_{\Pi^\sharp(r^*(df))} dg.
\end{aligned}
$$

Using that $[dg, df]_{\Pi_r} = -\mathcal{L}_{\Pi^\sharp(r^*(df))} dg$, it follows that (6.10) is equivalent to (6.4). So, Π and r are compatible if and only if (6.10) and (6.12) hold. The remaining equations follow from these two. Indeed, a long but straightforward computation shows that (6.9) is equivalent, for $\alpha = df$ and $\beta = dg$, to

$$d(C^r_\Pi(df, dg)) = 0.$$

Finally, the redundancies among the IM equations (see Remark 3.18) guarantee that (6.11) holds. □

From now on, let us assume that the Lie algebroid $(T^*M, \Pi^\sharp, [\cdot,\cdot]_\Pi)$ integrates to a source 1-connected symplectic groupoid $(\mathcal{G}, \omega)$ (see Sect. 3.3). Let Π_ω be the Poisson structure defined by ω, so that

$$\Pi^\sharp_\omega : T^*\mathcal{G} \to T\mathcal{G} \tag{6.13}$$

is the inverse map to $T\mathcal{G} \to T^*\mathcal{G}$, $U \mapsto i_U\omega$. It follows from Theorem 3.19 and Proposition 6.7 that any $(1, 1)$ tensor $r : TM \to TM$ compatible with Π corresponds to a multiplicative $(1, 1)$ tensor $K : T\mathcal{G} \to T\mathcal{G}$ on $\mathcal{G}$ integrating (D^r, r^*, r).

Lemma 6.8 *Let $r : TM \to TM$ be a $(1, 1)$ tensor compatible with Π. One has that $(\mathcal{N}_r^*, 0)$ is an IM 3-form if and only if there exists a closed 3-form $\lambda \in \Omega^3(\mathcal{G})$ such that*

$$\mathcal{N}_K(U, V) = \Pi^\sharp_\omega(\lambda(U, V, \cdot)), \quad \forall\, U, V \in \mathfrak{X}(\mathcal{G}), \tag{6.14}$$

where $K : T\mathcal{G} \to T\mathcal{G}$ is the multiplicative $(1, 1)$ tensor integrating (D^r, r^*, r). In this case, λ is the multiplicative 3-form integrating $(-\mathcal{N}_r^*, 0)$.

Proof Following Corollary 6.3, the infinitesimal components $((D^r)^2, [D^r, r^*], \mathcal{N}_r)$ of the Nijenhuis torsion of K can be explicitly calculated using (6.2) and (6.8). Indeed, using Cartan calculus and the Jacobi identity repeatedly, one may verify that, for $\alpha \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$,

$$
\begin{aligned}
[D^r, r^*](\alpha) &= -\mathcal{N}_r^*(\alpha), \\
\langle (D^r)^2_{(X,Y)}(\alpha), Z\rangle &= -d\mathcal{N}_r^*(\alpha)(X, Y, Z) + d\alpha(Z, \mathcal{N}_r(X, Y)).
\end{aligned}
$$

Similarly, for any multiplicative closed 3-form $\lambda \in \Omega^3(\mathcal{G})$, consider the multiplicative $(1,2)$ tensor field $\tau \in \Omega^2(\mathcal{G}, T\mathcal{G})$ defined by $\tau(U,V) = \Pi^\sharp_\omega(\lambda(U,V,\cdot))$. Its infinitesimal components $\widetilde{D} : \Gamma(T^*M) \to \Omega^2(M, T^*M)$, $\widetilde{l} : T^*M \to T^*M \otimes T^*M$ and $\widetilde{r} : T^*M \to \wedge^2 T^*M$ are:

$$\widetilde{l} = \mu, \quad \widetilde{r} = -\mu, \quad \widetilde{D}_{(X,Y)}(\alpha) = d\mu(\alpha)(X,Y,\cdot) + d\alpha(\underbrace{\mu_{(X,Y)}}_{\in TM}, \cdot),$$

where $\mu : T^*M \to \wedge^2 T^*M$ is such that $(\mu, 0)$ is the IM 3-form corresponding to λ. The result now follows from the fact that (6.14) holds if and only if $\widetilde{r} = \mathcal{N}_r$, $\widetilde{l} = -\mathcal{N}_r$, $\widetilde{D} = (D^r)^2$. □

Our aim now is to prove that Π_ω and K define a symplectic quasi-Nijenhuis structure on $\mathcal{G}$; the previous lemma shows that this structure will be of the more restricted type of Remark 6.6. As Π_ω is symplectic, following Remark 6.4, it suffices to show that

$$\Pi^\sharp_\omega \circ K^* = K \circ \Pi^\sharp_\omega, \tag{6.15}$$

$$[\Pi_\omega, \Pi_K] = 0, \tag{6.16}$$

where Π_K is the bivector field defined by $K \circ \Pi^\sharp_\omega$.

Integration of Poisson quasi-Nijenhuis structures We start by analyzing conditions (6.15) and (6.16) in general. Our setting will be:

- a source 1-connected Poisson groupoid $(\mathcal{H}, \widetilde{\Pi}) \rightrightarrows N$;
- a multiplicative $(1,1)$-tensor $K : T\mathcal{G} \to T\mathcal{G}$ with (D, l, r) as infinitesimal components.

Let $(A^*\mathcal{H}, \rho_*, [\cdot,\cdot]_*)$ be the corresponding Lie algebroid structure on $A^*\mathcal{H}$ and denote by $\delta : \Gamma(\wedge^\bullet A\mathcal{H}) \to \Gamma(\wedge^{\bullet+1} A\mathcal{H})$ the 2-differential associated with $\widetilde{\Pi}$. Define $\delta_K : \Gamma(A\mathcal{H}) \to \Gamma(A^*\mathcal{H} \otimes A^*\mathcal{H})$ by the expression

$$\delta_K(a)(\mu_1, \mu_2) := \delta(a)(\mu_1, l^*(\mu_2)) - \langle D_{\rho_*(\mu_1)}(a), \mu_2\rangle, \quad a \in \Gamma(A\mathcal{H}), \tag{6.17}$$

Lemma 6.9 *One has that*

$$K \circ \widetilde{\Pi}^\sharp = \widetilde{\Pi}^\sharp \circ K^* \iff \begin{cases} r \circ \rho_* = \rho_* \circ l^* \\ \delta_K(\mu_1, \mu_2) = -\delta_K(\mu_2, \mu_1). \end{cases}$$

In this case, the IM $(0,2)$*-tensor on* $A\mathcal{H}$ *corresponding to the multiplicative bivector field* $\widetilde{\Pi}_K$ *is given by* $l \circ (\rho_*)^* : T^*N \to A\mathcal{H}$ *and* $\delta_K : \Gamma(A\mathcal{H}) \to \Gamma(\wedge^2 A\mathcal{H})$.

Proof Let $\tau_L, \tau_R \in \Gamma(T\mathcal{H} \otimes T\mathcal{H})$ be (not necessarily skew-symmetric) tensor fields on $\mathcal{H}$ defined by

$$\tau_L(\xi_1, \xi_2) = \langle \xi_2,\ K(\Pi^\sharp(\xi_1))\rangle, \quad \tau_R(\xi_1, \xi_2) = -\langle \xi_1,\ K(\Pi^\sharp(\xi_2))\rangle,$$

for $\xi_1, \xi_2 \in T^*\mathcal{H}$. Note that $K \circ \widetilde{\Pi}^\sharp = \widetilde{\Pi}^\sharp \circ K^*$ if and only if $\tau_L = \tau_R$. Now τ_L, τ_R are multiplicative and, therefore, $\tau_L = \tau_R$ if and only if

$$\underbrace{\mathcal{L}_{\overrightarrow{a}} \tau_L = \mathcal{L}_{\overrightarrow{a}} \tau_R}_{(i)}, \quad \underbrace{\tau_L(\mathrm{t}^*\alpha, \cdot) = \tau_R(\mathrm{t}^*\alpha, \cdot)}_{(ii)}, \quad \text{and} \quad \underbrace{\tau_L(\cdot\,, \mathrm{t}^*\alpha) = \tau_R(\cdot\,, \mathrm{t}^*\alpha)}_{(iii)}, \quad \alpha \in \Gamma(T^*M),$$

using that $\mathcal{H}$ is source 1-connected (see Remarks 3.16 and 3.20). Since

$$\tau_L(\mathrm{t}^*\alpha, \xi) = \langle \widetilde{\mathrm{t}}(\xi),\, l \circ \rho_*^*(\alpha)\rangle, \quad \tau_R(\mathrm{t}^*\alpha, \xi) = \langle \widetilde{\mathrm{t}}(\xi),\, \rho_*^* \circ r^*(\alpha)\rangle,$$
$$\tau_L(\xi_2, \xi_1) = -\tau_R(\xi_1, \xi_2),$$

it follows that (ii) and (iii) hold if and only if $r \circ \rho_* = \rho_* \circ l^*$. Finally, from

$$(\mathcal{L}_{\overrightarrow{a}} \tau_L)(\xi_1, \xi_2) = \langle \xi_2,\, [\overrightarrow{a}, K](\Pi^\sharp(\xi_1))\rangle + \langle \xi_2,\, K((\mathcal{L}_{\overrightarrow{a}} \Pi)^\sharp(\xi_1)),$$
$$(\mathcal{L}_{\overrightarrow{a}} \tau_R)(\xi_1, \xi_2) = -\langle \xi_1,\, [\overrightarrow{a}, K](\Pi^\sharp(\xi_2))\rangle - \langle \xi_1,\, K((\mathcal{L}_{\overrightarrow{a}} \Pi)^\sharp(\xi_2)),$$

one can substitute $\xi_i = \mu_i \in A^*\mathcal{H} \subset T^*\mathcal{H}$ and use that $\Pi^\sharp|_{A^*\mathcal{H}} = -\rho_*$ to show that (i) holds if and only if δ_K is skew-symmetric. □

Let us assume that $K \circ \widetilde{\Pi}^\sharp = \widetilde{\Pi}^\sharp \circ K^*$. The IM (0, 2)-tensor on $A\mathcal{H}$ associated to the multiplicative bivector field $\widetilde{\Pi}_K$ defines a pre-Lie algebroid structure on $A^*\mathcal{H}$. It follows from (3.17) and (6.17) that the pre-Lie bracket is given by

$$[\mu_1, \mu_2]_K = [l^*(\mu_1), \mu_2]_* + [\mu_1, l^*(\mu_2)]_* - l^*([\mu_1, \mu_2]_*) + \Theta(\mu_1, \mu_2),$$

where $\Theta \in \Gamma(\wedge^2 A\mathcal{H} \otimes A^*\mathcal{H})$ is defined by

$$\langle \Theta(\mu_1, \mu_2), a\rangle = \langle D_{\rho_*(\mu_1)}(a), \mu_2\rangle + \delta(a)(l^*(\mu_1), \mu_2) - \delta(l(a))(\mu_1, \mu_2).$$

For any element $\Omega \in \Gamma(\wedge^k A\mathcal{H} \otimes A^*\mathcal{H})$, one can define a contraction operator $i_\Omega : \Gamma(\wedge^\bullet A\mathcal{H}) \to \Gamma(\wedge^{\bullet+k-1} A\mathcal{H})$ exactly as in (5.1). It is a graded derivation of degree $k-1$ of the graded algebra $\Gamma(\wedge^\bullet A\mathcal{H})$. Note that

$$\delta_K = [i_{l^*}, \delta] + i_\Theta,$$

where $[\cdot, \cdot]$ is the commutator of derivations of $\Gamma(\wedge^\bullet A\mathcal{H})$.

Lemma 6.10 *The bivector fields* $\widetilde{\Pi}$, $\widetilde{\Pi}_K$ *satisfy* $[\widetilde{\Pi}, \widetilde{\Pi}_K] = 0$ *if and only* $[\delta, i_\Theta] = 0$.

Proof It is shown in [57] that $[\widetilde{\Pi}, \widetilde{\Pi}_K] = 0 \Leftrightarrow [\delta, \delta_K] = 0$. The result then follows from $[\delta, [i_{l^*}, \delta]] = 0$. □

We can now conclude the description of the integration of Poisson quasi-Nijenhuis structures using Theorem 3.19:

Theorem 6.11 *Let $(\mathcal{G}, \omega)$ be the source 1-connected symplectic groupoid integrating a Poisson manifold (M, Π). There is a one-to-one correspondence between Poisson quasi-Nijenhuis structures (Π, r) on M and symplectic quasi-Nijenhuis structures (ω, K, λ) relative to λ, where $K : T\mathcal{G} \to T\mathcal{G}$ is the multiplicative $(1, 1)$ tensor integrating (D^r, r^*, r) and $\lambda \in \Omega^3(\mathcal{G})$ is the multiplicative closed 3-form integrating $(-\mathcal{N}_r^*, 0)$.*

Proof We know that Π_ω, the Poisson structure defined by ω, makes $\mathcal{G}$ into a Poisson groupoid. The dual Lie algebroid in this case is $A^* = TM$, with anchor $\rho_* = \mathrm{id}_{\mathrm{TM}}$ and bracket $[\cdot, \cdot]_*$ given by the Lie bracket of vector fields. Note that $\delta = d$ is the de Rham differential.

Let $K : T\mathcal{G} \to T\mathcal{G}$ be a multiplicative $(1, 1)$ tensor on $\mathcal{G}$ and λ a multiplicative closed 3-form. Consider the IM (1,1)-tensor (D, l, r) and the IM 3-form $(\mu, 0)$ associated to K and λ, respectively. From Lemmas 6.9 and 6.10, one knows that K and Π_ω are compatible if and only if $l = r^*$, δ_K in (6.17) is skew-symmetric and, by writing $\delta_K = [i_r, d] + i_\Theta$, the condition $[d, i_\Theta] = 0$ holds (notice that this last condition says that $\Theta = 0$). Hence, using that $[i_r, d] = \mathcal{L}_r$ and (6.17),

$$
\begin{aligned}
\langle D_X(\alpha), Y\rangle = d\alpha(X, r(Y)) - \delta_K(\alpha)(X, Y) &= d\alpha(X, r(Y)) - \mathcal{L}_r(\alpha)(X, Y)\\
&= \langle D^r_X(\alpha), Y\rangle.
\end{aligned}
$$

Therefore K and Π_ω are compatible if and only if Π and r are compatible and, moreover, K is the (1,1) tensor integrating (D^r, r^*, r). Finally, from Lemma 6.8, the equation $\mathcal{N}_K(U, V) = \Pi^\sharp_\omega(\lambda(U, V, \cdot))$ holds if and if $(\mu, 0) = (-\mathcal{N}_r^*, 0)$ is the IM 3-form associated to λ. □

By restricting the previous correspondence to Poisson quasi-Nijenhuis structures on M relative to closed 3-forms, we recover [57, Thm. 6.2] with a different viewpoint:

Corollary 6.12 *Let $(\mathcal{G}, \omega)$ be a source 1-connected symplectic groupoid integrating the Poisson manifold (M, Π). There is one-to-one correspondence between Poisson quasi-Nijenhuis structures (Π, r, ϕ) on M relative to a closed 3-form ϕ and symplectic quasi-Nijenhuis structures (ω, K, λ) on $\mathcal{G}$ relative to $\lambda = \mathsf{t}^*\phi - \mathsf{s}^*\phi$ such that K is multiplicative.*

Proof It follows from the fact that $\mathsf{t}^*\phi - \mathsf{s}^*\phi$ is the multiplicative 3-form integrating the IM 3-form $(\alpha \mapsto i_{\Pi^\sharp(\alpha)}\phi, 0)$ (see Example 3.15). □

Our methods also work, more generally, to describe the infinitesimal counterparts of multiplicative Poisson–Nijenhuis structures on Lie groupoids, not necessarily symplectic; here one obtains compatibilities between the IM (1,1)-tensor corresponding to a Nijenhuis structure and the Lie bialgebroid associated with the Poisson groupoid. We will discuss this case elsewhere.

6.4 Multiplicative (almost) complex structures

Our general results on multiplicative (1,1) tensors in Sects. 6.1 and 6.2 can be readily applied to the study of complex structures on Lie groupoids, giving another viewpoint to results in [43] concerning their infinitesimal versions.

Recall that, for a (real) vector bundle $E \to M$, a holomorphic structure is specified by a triple (J_E, J_M, ∇), where $J_E : E \to E$ is an endomorphism satisfying $J_E^2 = -\mathrm{id}$ (which makes E into a complex vector bundle), J_M is a complex structure on M, and ∇ is a flat $T^{(0,1)}$-connection on E, in such a way that holomorphic sections $u : M \to E$ are characterized by $\nabla u = 0$; see [56]. We shall call ∇ the *Dolbeault connection* on E. More generally, we will be interested in holomorphic structures on Lie groupoids and Lie algebroids.

A *holomorphic structure* on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a multiplicative complex structure; i.e., a multiplicative $J \in \Omega^1(\mathcal{G}, T\mathcal{G})$ such that $J^2 = -\mathrm{id}$ and $\mathcal{N}_J = 0$. One refers to $(\mathcal{G}, J)$ as a *complex* Lie groupoid. A *holomorphic structure* on a Lie algebroid $A \to M$ is a holomorphic structure (J_A, J_M, ∇) on its underlying vector bundle such that the following compatibility conditions are satisfied:

(H1) $[\cdot,\cdot]$ restricts to a Lie bracket $[\cdot,\cdot]_{\mathrm{hol}}$ on the holomorphic sections;
(H2) $[\cdot,\cdot]_{\mathrm{hol}}$ is $\mathbb{C}$-linear.

As we will now see, the following correspondence, proven in [43, Thm. 3.17], is a consequence of Theorem 3.19, along with Corollaries 6.2 and 6.3.

Corollary 6.13 *Let $\mathcal{G} \rightrightarrows M$ be source 1-connected. Then holomorphic structures on $\mathcal{G}$ are in natural one-to-one correspondence with holomorphic structures on its Lie algebroid $A \to M$.*

The remainder of this section proves this result. By Corollaries 6.2 and 6.3, we immediately see that the correspondence in Theorem 3.19 restricts to a bijective correspondence between holomorphic structures J on $\mathcal{G} \rightrightarrows M$ and IM $(1,1)$-tensors (D, l, r) on A satisfying

$$l \circ D(a) + D(a) \circ r = 0, \qquad l^2 = -\mathrm{id}_A, \qquad r^2 = -\mathrm{id}_{TM}. \tag{6.18}$$

and

$$D^2 = 0, \qquad [D, l] = 0, \qquad \mathcal{N}_r = 0. \tag{6.19}$$

So we must verify that an IM $(1,1)$-tensor (D, l, r), for which (6.18) and (6.19) hold, is equivalent to a holomorphic structure on the Lie algebroid A. We start by checking that (6.18) and (6.19) exactly say that the triple (D, l, r) defines a holomorphic structure on the vector bundle underlying A.

From (6.18), it is clear that r is an almost complex structure on M and l is a complex structure on the fibres of A (so we regard A as a complex vector bundle). Defining $\nabla : \Gamma(T^{01}) \times \Gamma(A) \to \Gamma(A)$ by

$$\nabla_{X+ir(X)}(a) := -l(D_X(a)), \tag{6.20}$$

we also see that the first equation in (6.18) says that

$$\nabla_{i(X+ir(X))}a = i(\nabla_{X+ir(X)}a).$$

Moreover, this last property along with the Leibniz rule for D imply that

$$\nabla_{X+ir(X)}fa = f\nabla_{X+ir(X)}a + (\mathcal{L}_{X+ir(X)}f)a,$$

for real functions $f \in C^\infty(M)$.

Let us now consider (6.19). The third condition says that r is a complex structure on M, while the second says that ∇ is complex linear in A. It follows that T^{01} is a (complex) Lie algebroid and ∇ is a (complex) T^{01}-connection. Finally, using (6.2), one can also check that the first equation in (6.19) amounts to ∇ being flat. In conclusion, conditions (6.18) and (6.19) say that (D, l, r) endow A with the structure of a holomorphic vector bundle: $J_A = l$, $J_M = r$ and ∇ given by (6.20). The result in Corollary 6.13 now follows from

Lemma 6.14 *The IM-equations for (D, l, r) are equivalent to conditions (H1) and (H2) above.*

Proof We saw that equations (6.18) and (6.19) say that (D, l, r) defines a holomorphic structure on the vector bundle A; moreover, a section $a \in \Gamma(A)$ is holomorphic if and only if $Da = 0$.

($\Rightarrow$): Note that (IM1*) implies that $[\cdot, \cdot]$ restricts to a Lie bracket $[\cdot, \cdot]_{\mathrm{hol}}$ on the holomorphic sections of A, and (IM2*) implies that $[\cdot, \cdot]_{\mathrm{hol}}$ is $\mathbb{C}$-linear.
($\Leftarrow$:) Assume that $(A, [\cdot, \cdot], \rho)$ is a holomorphic Lie algebroid, and consider

$$\begin{aligned}
E_l(a, b) &:= l([a, b]) - [a, l(b)] + D_{\rho(b)}(a)\\
E_r(a, X) &:= r([\rho(a), X]) - [\rho(a), r(X)] + \rho(D_X(a))\\
E_D(a, b, X) &:= D_X([a, b]) - [a, D_X(b)] + [b, D_X(a)]\\
&\quad - D_{[\rho(b), X]}(a) + D_{[\rho(a), X]}(b).
\end{aligned}$$

One can check that E_l and E_r are $C^\infty(M)$-linear on both components and E_D is anti-symmetric on the $\Gamma(A)$-components. Moreover, for $f \in C^\infty(M)$,

$$E_D(a, fb, X) = fE_D(a, b, X) + (\mathcal{L}_X f)E_l(a, b) - (\mathcal{L}_{E_r(a,X)} f)b. \quad (6.21)$$

Note that $[\cdot, \cdot]_{\mathrm{hol}}$ being $\mathbb{C}$-linear implies that $E_l(a, b) = 0$ for $a, b \in \Gamma(A)$ holomorphic. As $\Gamma(A)$ is generated as a $C^\infty(M)$-module by the holomorphic sections, it follows that $E_l \equiv 0$. The redundancies discussed in Remark 3.18 imply that $\rho \circ l = r \circ \rho$, so ρ is a complex vector-bundle morphism. Furthermore, note that $\rho : A \to TM$ sends holomorphic sections to holomorphic sections. Indeed, if $h \in C^\infty(M, \mathbb{C})$ is a holomorphic function and $u_1, u_2 \in \Gamma(A)$ are arbitrary holomorphic sections,

$$(\mathcal{L}_{\rho(u_2)}h)u_1 = h[u_1, u_2] - [u_1, hu_2] = h[u_1, u_2]_{\mathrm{hol}} - [u_1, hu_2]_{\mathrm{hol}}$$

is a holomorphic section, which implies that $\rho(u_2)$ is a holomorphic section of TM. Using the $C^\infty(M)$-linearity of E_r, one can argue as above to prove that $E_r(a, X) = 0$ for all $a \in \Gamma(A)$ and $X \in \Gamma(TM)$ (use that $r = J_M$, the almost complex structure of M, and that TM with the Lie bracket of vector fields and the identity as anchor is a holomorphic Lie algebroid). Finally, from (6.21) it follows that E_D is tensorial, and the fact that $[\cdot, \cdot]$ restricts to $[\cdot, \cdot]_{\text{hol}}$ on holomorphic sections implies, as before, that $E_D \equiv 0$. ◻

As this section and Sect. 6.3 illustrate, Theorem 3.19 provides tools that can be directly applied to treat multiplicative geometric structures on holomorphic Lie groupoids, including holomorphic symplectic groupoids [42] or more general holomorphic Poisson groupoids, as well as multiplicative generalized complex structures [31], offering complementary information about the latter in terms of infinitesimal components. We will further discuss these cases in a separate work.

6.5 Multiplicative projections

For a Lie groupoid $\mathcal{G} \rightrightarrows M$ with Lie algebroid A, we consider a multiplicative $(1, 1)$ tensor field $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ satisfying $K^2 = K$, referred to as a *multiplicative projection*. In this section we apply our previous results to describe multiplicative projections infinitesimally, making connections with the theory of matched pairs [39, 45,54].

We start by observing that projections can be used to treat other types of multiplicative $(1, 1)$ tensors.

Example 6.15 Suppose that $Q : T\mathcal{G} \to T\mathcal{G}$ satisfies $Q^2 = \text{id}$ and is multiplicative; i.e., Q is a multiplicative *(almost) product structure* on $\mathcal{G}$. Then $K := (Q + \text{id})/2$ is a multiplicative projection.[5]

We know that a multiplicative projection K admits an infinitesimal description by its infinitesimal components (D, l, r). We start by discussing alternative ways to express the operator D, that will be convenient when we consider the Nijenhuis torsion of K.

Since $r^2 = r : TM \to TM$ and $l^2 = l : A \to A$ (by Corollary 6.2), the bundles TM and A decompose as

$$A = A^0 \oplus A^1, \quad TM = T^0 \oplus T^1,$$

where A^0, T^0 are the kernels of the maps l, r, and A^1, T^1 are their images, respectively.

Lemma 6.16 *One has that*

$$\rho(A^0) \subset T^0 \quad \textit{and} \quad \rho(A^1) \subset T^1. \tag{6.22}$$

[5] By working with complexifications, one can also cast (almost) complex structures as projections.

Also, for $a \in \Gamma(A)$*,*

$$D_X(a) \in A_1 = \mathrm{im}(l), \quad \textit{if } X \in \Gamma(T^0),$$
$$D_X(a) \in A_0 = \ker(l), \quad \textit{if } X \in \Gamma(T^1).$$

Proof The conditions in (6.22) follow from (IM6*) (Sect. 6.1), whereas the statements about D follow from Corollary 6.2, part (a). □

So, upon restriction, D gives rise to two operators:

$$D^+ : \Gamma(A) \to \Gamma(T^{0*} \otimes A^1), \quad D^- : \Gamma(A) \to \Gamma(T^{1*} \otimes A^0).$$

Let us consider the operators

$$\Lambda^+ = D^+|_{\Gamma(A^0)}, \quad \nabla^+ = D^+|_{\Gamma(A^1)} \tag{6.23}$$
$$\Lambda^- = D^-|_{\Gamma(A^1)}, \quad \nabla^- = -D^-|_{\Gamma(A^0)}. \tag{6.24}$$

Proposition 6.17 Λ^+, Λ^- *are tensorial, whereas* ∇^+, ∇^- *satisfy*

$$\nabla^+(fa) = f\nabla^+(a) + df|_{T^0} \otimes a,$$
$$\nabla^-(fb) = f\nabla^-(b) + df|_{T^1} \otimes b,$$

for $f \in C^\infty(M)$*,* $a \in \Gamma(A^0)$*,* $b \in \Gamma(A^1)$*.*

Proof This follows immediately from the Leibniz rule for D (3.9). □

Vanishing of the Nijenhuis torsion Let $\mathcal{N}_K \in \Omega^2(\mathcal{G}, T\mathcal{G})$ be the Nijenhuis torsion of K. We say that K is a *flat projection* if $\mathcal{N}_K = 0$. The next result gives an equivalent description of the Nijenhuis vanishing condition.

Proposition 6.18 *Let* $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ *be a multiplicative projection on a source-connected Lie groupoid* $\mathcal{G} \rightrightarrows M$*. Then* $\mathcal{N}_K = 0$ *if and only if*

- $\Lambda^+ = 0$, $\Lambda^- = 0$*;*
- T^0 *and* T^1 *are involutive distributions;*
- ∇^+ *is a flat* T^0*-connection, and* ∇^- *is a flat* T^1*-connection.*

Proof As $\mathcal{G}$ is source connected, the Nijenhuis torsion $\mathcal{N}_K$ vanishes if and only if its infinitesimal components $(D^2, [D, l], N_r)$ are zero (see Corollary 6.3). A direct verification shows that

$$[D, l]_X(a) = D_X(l(a)) - l(D_X(a)) = \begin{cases} \Lambda^+_X(l(a) - a), & \text{if } X \in T^0, \\ \Lambda^-_X(l(a)), & \text{if } X \in T^1. \end{cases}$$

Hence, $[D, l] = 0$ if and only if both Λ^+ and Λ^- vanish. Similarly, $N_r = 0$ is equivalent to T^0 and T^1 being involutive.[6] By Proposition 6.17, ∇^+ is a T^0-connection, and ∇^- is a T^1-connection. Using (6.2), we see that

$$D^2_{(X,Y)}(a) = \begin{cases} \mathrm{Curv}^+_{(Y,X)}(a), & \text{if } X, Y \in \Gamma(T^0),\ a \in \Gamma(A^1) \\ \mathrm{Curv}^-_{(Y,X)}(a), & \text{if } X, Y \in \Gamma(T^1),\ a \in \Gamma(A^0) \\ 0, & \text{otherwise,} \end{cases}$$

where Curv^+ (resp. Curv^-) is the curvature of ∇^+ (resp. ∇^-). Hence, $D^2 = 0$ if and only if both ∇^+, ∇^- are flat. This concludes the proof. □

Remark 6.19 A distribution $\Delta \subset T\mathcal{G}$ is said to be multiplicative if Δ is a Lie subgroupoid. As observed in [6], a multiplicative projection is equivalent to a pair of multiplicative distributions Δ_1, Δ_2 such that $\Delta_1 \oplus \Delta_2 = T\mathcal{G}$. Also, $\mathcal{N}_K = 0$ is equivalent to both distributions being involutive. In this context, Proposition 6.18 agrees with the integrability criteria for multiplicative distributions given in [16,30].

Example 6.20 Following Example 6.15, consider a multiplicative $Q : T\mathcal{G} \to T\mathcal{G}$ satisfying $Q^2 = \mathrm{id}$, and let $K = (Q + \mathrm{id})/2$ be the corresponding multiplicative projection. Let (D, l, r) and (D', l', r') be the infinitesimal components of K and Q, respectively. Then

$$l = (l' + \mathrm{id})/2, \quad r = (r' + \mathrm{id})/2, \quad D = D'.$$

The bundles T^0, A^0 (resp. T^1, A^1) are now the -1 (resp. $+1$) eigenbundles of r' and l'. Also, D' decomposes into tensors $\Lambda^+ \in \Gamma(T^{0*} \otimes A^{0*} \otimes A^1)$, $\Lambda^- \in \Gamma(T^{1*} \otimes A^{1*} \otimes A^0)$ and connections $\nabla^+ : \Gamma(A^1) \to \Gamma(T^{0*} \otimes A^1)$, $\nabla^- : \Gamma(A^0) \to \Gamma(T^{1*} \otimes A^0)$. Noticing that $\mathcal{N}_K = 0 \Leftrightarrow \mathcal{N}_Q = 0$, we see that Proposition 6.18 directly applies to Q instead of K.

We now illustrate the infinitesimal components of a multiplicative projection in the classical example of a projection defined by a connection on a principal bundle.

Example 6.21 Let $P \to M$ be a principal bundle for a Lie group G, and consider its gauge groupoid $\mathcal{G}(P) := (P \times P)/G \rightrightarrows M$ (see e.g. [47, Sec, 1.1]). In [6], it is shown that there is a one-to-one correspondence between principal connections $\theta \in \Omega^1(P, \mathfrak{g})$ and multiplicative projections $K : T\mathcal{G}(P) \to T\mathcal{G}(P)$ such that $\mathrm{im}(K) = \ker(T\mathsf{s}) \cap \ker(T\mathsf{t})$. To explicitly describe this projection it is useful to identify the tangent groupoid $T\mathcal{G}(P)$ with the gauge groupoid $\mathcal{G}(TP) = (TP \times TP)/TG \to TM$ of the principal TG-bundle $TP \to TM$. The quotient map $TP \times TP \to \mathcal{G}(TP)$ is denoted by

[6] In general, for a projection $K \in \Omega^1(M, TM)$ a projection, the Nijenhuis torsion $\mathcal{N}_K$ can be written as $\mathcal{N}_K = R_K + \overline{R}_K$, where $R_K, \overline{R}_K \in \Omega^2(M, TM)$ are the *curvature* and *co-curvature* of K given, respectively, by

$$R_K(X, Y) = K([(\mathrm{id} - K)(X), (\mathrm{id} - K)(Y)])$$
$$\overline{R}_K(X, Y) = (\mathrm{id} - K)([K(X), K(Y)]).$$

In particular, $\mathcal{N}_K = 0$ if and only if both $\ker(K)$ and $\mathrm{im}(K)$ are involutive distributions.

$$(X, Y) \mapsto \overline{(X, Y)}.$$

The projection K is now defined as

$$K(\overline{(X, Y)}) = \overline{(\theta(X)_P, \theta(Y)_P)},$$

where u_P denotes the infinitesimal generator on P corresponding to $u \in \mathfrak{g}$. One can check (see [6]) that the Nijenhuis torsion of K is given by

$$\mathcal{N}_K = \mathcal{S}(F_\theta) - \mathcal{T}(F_\theta),$$

where $F_\theta \in \Omega^2(M, P \times_G \mathfrak{g})$ is the curvature of θ, and the maps $\mathcal{T}$ and $\mathcal{S}$ are defined in (3.6); here $P \times_G \mathfrak{g}$ is the associated bundle with respect to the adjoint representation, and we are using the Atiyah sequence

$$0 \longrightarrow P \times_G \mathfrak{g} \longrightarrow TP/G \longrightarrow TM \longrightarrow 0,$$

to view $P \times_G \mathfrak{g}$ as a subbundle of $A(\mathcal{G}(P)) = TP/G$.

The infinitesimal components (D, l, r) of the multiplicative projection K are given as follows: $r = 0$, while $l : TP/G \to P \times_G \mathfrak{g} \subset TP/G$ is the map induced by $\theta : TP \to \mathfrak{g}$, and $D : \Gamma(TP/G) \to \Gamma(T^*M \otimes TP/G)$ is given by

$$D_X(\overline{Y}) = \theta([X_H, Y]),$$

where $X_H \in \mathfrak{X}(P)$ is the horizontal lift of X. Note that $T^0 = TM$, $T^1 = 0$, $A^0 = H/G \cong TM$, $A^1 = P \times_G \mathfrak{g}$, where $H = \ker(\theta) \subset TP$ is the horizontal distribution. Under the splitting $D = D^+ + D^-$, one may directly check that $D^- = 0$ and $\nabla^+ : \Gamma(P \times_G \mathfrak{g}) \to \Gamma(T^*M \otimes P \times_G \mathfrak{g})$ is the natural connection on the associated bundle $P \times_G \mathfrak{g}$, whereas $\Lambda^+ : \Gamma(TM) \to \Gamma(T^*M \otimes P \times_G \mathfrak{g})$ is

$$\Lambda^+_X(Y) = \theta([X_H, Y_H]) = -F_\theta(X, Y).$$

Proposition 6.18 admits yet another geometric interpretation, that we discuss next.

Characterization via matched pairs In the remainder of this section we provide a characterization of multiplicative flat projections using the theory of matched pairs of Lie algebroids [54].

Definition 6.22 Let $A, B \to M$ be Lie algebroids. We say that (A, B) is a *matched pair* if A has a representation on B, and B has a representation on A such that

$$[\rho_A(a), \rho_B(b)] = -\rho_A(\nabla_b\, a) + \rho_B(\nabla_a\, b), \tag{6.25}$$

$$\nabla_a\, [b_1, b_2] = [\nabla_a\, b_1, b_2] + [b_1, \nabla_a\, b_2] + \nabla_{\nabla_{b_2}\, a}\, b_1 - \nabla_{\nabla_{b_1}\, a}\, b_2, \tag{6.26}$$

$$\nabla_b\, [a_1, a_2] = [\nabla_b\, a_1, a_2] + [a_1, \nabla_b\, a_2] + \nabla_{\nabla_{a_2}\, b}\, a_1 - \nabla_{\nabla_{a_1}\, b}\, a_2. \tag{6.27}$$

Here we denote both representations by ∇.

Definition 6.23 A *morphism of matched pairs* from (A_1, B_1) to (A_2, B_2) is a pair of Lie algebroid morphisms $F_A : A_1 \to A_2$, $F_B : B_1 \to B_2$ such that

$$\nabla_{F_A(a)} F_B(b) = F_B(\nabla_a b), \tag{6.28}$$

$$\nabla_{F_B(b)} F_A(a) = F_A(\nabla_b a). \tag{6.29}$$

A matched pair (A, B) is equivalent to a Lie algebroid structure on the Whitney sum $A \oplus B$ such that A and B are Lie subalgebroids, see [48,54]. From this viewpoint, the representations are determined by the Lie bracket:

$$\nabla_a b = \mathrm{pr}_B([a, b]), \quad \nabla_b a = \mathrm{pr}_A([b, a]),$$

where $\mathrm{pr}_A : A \oplus B \to A$ and $\mathrm{pr}_B : A \oplus B \to B$ are the projections. In this context, a morphism of matched pairs from (A_1, A_2) to (B_1, B_2) is equivalent to a Lie-algebroid morphism $F : A_1 \oplus B_1 \to A_2 \oplus B_2$ which restricts to Lie-algebroid morphisms from A_1 to A_2, and from B_1 to B_2.

For a flat multiplicative projection $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$, Proposition 6.18 implies that the decomposition $TM = T^0 \oplus T^1$ defines a matched pair (T^0, T^1). Also, for $a, b \in \Gamma(A^0)$, the condition

$$l([a, b]) = [a, l(b)] - D_{\rho(b)}(a) = -\Lambda^+_{\rho(b)}(a) = 0$$

implies that $A^0 \subset A$ is a subalgebroid. Similarly, one can check that $A^1 \subset A$ is a subalgebroid. Thus, the decomposition $A = A^0 \oplus A^1$ defines a matched pair (A^0, A^1). For $a \in \Gamma(A^0)$, $b \in \Gamma(A^1)$, the representations are defined by

$$\nabla_a b = l([a, b]), \quad \nabla_b a = [a, b] - l([a, b]).$$

We obtain the following infinitesimal characterization of flat multiplicative projections:

Theorem 6.24 *Let $\mathcal{G}$ be a source 1-connected groupoid. There is a one-to-one correspondence between flat multiplicative projections on $\mathcal{G}$ and decompositions $A = A^0 \oplus A^1$ and $TM = T^0 \oplus T^1$, where*

(i) *$A^0, A^1 \subset A$ and $T^0, T^1 \subset TM$ are Lie subalgebroids;*
(ii) *(A^0, T^1) and (T^0, A^1) are matched pairs;*
(iii) *The sides of the commutative square*

$$\begin{array}{ccc} (A^0, A^1) & \xrightarrow{(\mathrm{id}_{A^0}, \rho)} & (A^0, T^1) \\ {\scriptstyle (\rho, \mathrm{id}_{A^1})}\downarrow & & \downarrow{\scriptstyle (\rho, \mathrm{id}_{T^1})} \\ (T^0, A^1) & \xrightarrow[(\mathrm{id}_{T^0}, \rho)]{} & (T^0, T^1) \end{array}$$

are morphisms of matched pairs.

Proof Consider decompositions $A = A^0 \oplus A^1$ and $TM = T^0 \oplus T^1$ for which (i), (ii) and (iii) hold. We will prove that they define the infinitesimal components of a flat multiplicative projection.

Define $l : A \to A$ (resp. $r : TM \to TM$) to be the projection on A^1 (resp. T^1) along A^0 (resp. T^0). Using that $\mathrm{Ann}(T^1) \cong T^{0*}$ and $\mathrm{Ann}(T^0) \cong T^{1*}$, define $D : \Gamma(A) \to \Gamma(T^*M \otimes A)$ to be the map

$$D(a) = \nabla^+(l(a)) - \nabla^-(a - l(a)), \tag{6.30}$$

where $\nabla^+ : \Gamma(A^1) \to \Gamma(T^{0*} \otimes A^1)$, $\nabla^- : \Gamma(A^0) \to \Gamma(T^{1*} \otimes A^0)$ are the flat connections (i.e., representations) corresponding to the matched pairs (A^0, T^1), (T^0, A^1), respectively. One may directly verify that D satisfies the Leibniz rule (3.9). We now prove that the triple (D, l, r) satisfies the IM-equations (Sect. 6.1).

Equation (IM6*) From (iii), one has that $\rho(A^0) \subset T^0$, $\rho(A^1) \subset T^1$, which is equivalent to $\rho \circ l = r \circ \rho$.

Equation (IM2*) The subbundle $A^0 \subset A$ is a Lie subalgebroid if and only if $l([a, b]) = 0$, for $a, b \in \Gamma(A^0)$. As $l(b) = 0 = D_{\rho(b)}(a)$, this implies that

$$l([a, b]) = [a, l(b)] - D_{\rho(b)}(a), \quad \forall\, a,\, b \in \Gamma(A^0).$$

Similarly, one can check that (IM2*) holds for $a, b \in \Gamma(A^1)$ using that $A^1 \subset A$ is a Lie subalgebroid. It remains to verify that (IM2*) holds for crossed terms. Using (ii), let $\nabla^{+,\,bas} : \Gamma(A^1) \times \Gamma(T^0) \to \Gamma(T^0)$, $\nabla^{-,\,bas} : \Gamma(A^0) \times \Gamma(T^1) \to \Gamma(T^1)$ be the representations of A^1, A^0 on T^0, T^1, respectively. For Eq. (6.25) to hold for the matched pairs (A^0, T^1), (T^0, A^1), one must have that

$$\nabla^{+,\,bas}_a X = [\rho(a), X] + \rho(\nabla^+_X a), \tag{6.31}$$

$$\nabla^{-,\,bas}_b Y = [\rho(b), Y] + \rho(\nabla^-_Y b). \tag{6.32}$$

Now, $(\mathrm{id}_{A^0}, \rho) : (A^0, A^1) \to (A^0, T^1)$ and $(\mathrm{id}_{A^1}, \rho) : (A^0, A^1) \to (T^0, A^1)$ are morphisms of matched pairs if and only if, for all $a \in \Gamma(A^0)$, $b \in \Gamma(A^1)$,

$$\begin{aligned} l([a, b]) &= [a, b] - D_{\rho(b)}(a) = [a, l(b)] - D_{\rho(b)}(a), \\ l([b, a]) &= -D_{\rho(a)}(b) = [b, l(a)] - D_{\rho(a)}(b), \end{aligned}$$

respectively. Altogether, this proves that (IM2*) holds.

Equation (IM3*) Similarly to the previous case, one can check that (IM3*) follows from T^0, T^1 being involutive and $(\rho, \mathrm{id}_{T^1}) : (A^0, T^1) \to (T^0, T^1)$ and $(\mathrm{id}_{T^0}, \rho) : (T^0, A^1) \to (T^0, T^1)$ being morphisms of matched pairs.

Equation (IM1*) We recall (IM1*) for convenience:

$$D_X([a, b]) = [a, D_X(b)] - [b, D_X(a)] + D_{[\rho(b), X]}(a) - D_{[\rho(a), X]}(b).$$

There are 6 cases to check by taking X in T^0 or T^1, and a, b in A^0 or A^1.

(1) $X \in \Gamma(T^0)$ and $a, b \in \Gamma(A^0)$,
(2) $X \in \Gamma(T^0)$ and $a, b \in \Gamma(A^1)$,
(3) $X \in \Gamma(T^0)$ and $a \in \Gamma(A^0)$, $b \in \Gamma(A^1)$.

The other 3 cases where $X \in \Gamma(T^1)$ work analogously. For (1), both sides of (IM1*) are trivially zero. For (2), using that $D|_{T^1}(a) = D|_{T^1}(b) = 0$, one can show that (IM1*) is equivalent to (6.26) for the matched pair (T^0, A^1). Finally, for (3), one has that

$$D_X([a, b]) = -\nabla^+_X\, l([b, a]) \overset{\text{(IM2*)}}{=} -\nabla^+_X \left([b, \cancelto{0}{l(a)]} - D_{\rho(a)}(b) \right) = \nabla^+_X \nabla^+_{\rho(a)}\, b.$$

On the other hand, using that $D|_{T^0}(a) = 0$, the RHS of (IM1*) simplifies to

$$\begin{aligned} &[a, D_X(b)] + D_{r([\rho(b), X])}(a) - D_{[\rho(a), X]}(b) \\ &\qquad \overset{\text{(IM3*)}}{=} [a, D_X(b)] - D_{\rho(D_X(b))}(a) - \nabla^+_{[\rho(a), X]}\, b \\ &\qquad \overset{\text{(IM2*)}}{=} l([a, D_X(b)]) - \nabla^+_{[\rho(a), X]}\, b. \end{aligned}$$

Using (IM2*) once again, one can prove that

$$l([a, D_X(b)]) = -l([D_X(b), a]) = D_{\rho(a)} D_X(b) = \nabla^+_{\rho(a)} \nabla^+_X\, b,$$

so the case (3) follows from the flatness of ∇^+. Putting everything together, we have proved that (D, l, r) defines an IM $(1, 1)$-tensor, so it integrates to a multiplicative $(1, 1)$ tensor $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ by Theorem 3.19. It then follows from Corollary 6.2 and Proposition 6.18 that K is a flat projection.

The converse, i.e., that the infinitesimal components of a multiplicative flat projection give rise to matched pairs as in the statement of the theorem, is proven by similar arguments and is left to the reader. □

Following Example 6.20, an analogous result holds for product structures; see [42, Thm. 4.8] for a parallel result in the context of holomorphic structures.

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Appendix A Lie theory of componentwise linear functions

In this appendix, we study componentwise linear functions on Whitney sums of VB-groupoids. We start by presenting a useful characterization of these functions.

Let $E_i \to M$ be vector bundles, $i = 1, \ldots, p$, and let $E = E_1 \oplus \cdots \oplus E_p$. Consider the multiplication by non-negative scalars $h : \mathbb{R}_{\geq 0} \times E \to E$, $h_\lambda(e) = \lambda e$. We shall refer to h as the *homogeneous structure* on E. The next result gives a characterization of componentwise linear functions on E in terms of its homogeneous structure. This is an extension of the characterization of vector bundle maps in [25], for $p = 1$.

Proposition A.1 *A smooth function $F : E \to \mathbb{R}$ is componentwise linear if and only if*

(1) $F \circ h_\lambda = \lambda^p F$, *for all* $\lambda \geq 0$,
(2) $F \circ 0_i = 0$,

where $0_i : \oplus_{1 \leq j \neq i \leq p} E_j \to \oplus_{1 \leq j \leq p} E_j$ *is the map*

$$(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p) \mapsto (v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_p), \qquad \text{(A.1)}$$

for $i = 1, \ldots, p$.

Proof We will consider the case $p = 2$, the general case being a direct generalization. By restricting F to the fibers of E_1 and E_2, we can assume that $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ satisfies $F(\lambda x, \lambda y) = \lambda^2 F(x, y)$, for all $\lambda \geq 0$, and $F(x, 0) = F(0, y) = 0$. Note that this implies that $F(0, 0) = 0$ and $DF(0, 0) = 0$. Now, one can use Taylor's Theorem and the homogeneity of F to prove that $F(z) = \sum_{1 \leq i, j \leq m+n} \frac{\partial^2 F}{\partial z_i \partial z_j}(0, 0) z_i z_j$, for $z = (x, y)$ satisfying $|z| = 1$. Using the homogeneity once more, it is possible to extend the equality to arbitrary z. The condition that $F(x, 0) = F(0, y) = 0$ implies that the terms $x_i x_j$ and $y_i y_j$ do not appear in the sum. Hence $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is bilinear. This completes the proof. □

We now consider VB-groupoids (see e.g. [47, Sec. 11.2] for details and original references), following the viewpoint of [4].

Definition A.2 A VB-groupoid is a square

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ E & \longrightarrow & M, \end{array}$$

whose horizontal sides are vector bundles, the vertical sides are Lie groupoids, satisfying the following compatibility condition: denoting by h and h^E the homogeneous structures on $\mathcal{V}$ and E, then, for each $\lambda \geq 0$, $h_\lambda : \mathcal{V} \to \mathcal{V}$ is a groupoid morphism over $h^E_\lambda : E \to E$.

The Lie algebroid $A\mathcal{V} \to E$ of a VB-groupoid inherits a vector bundle structure over A by differentiation of h_λ. If h^A is the corresponding homogeneous structure, then $h^A_\lambda : A\mathcal{V} \to A\mathcal{V}$ is a Lie algebroid morphism over h_λ for each λ.

In the following, we consider VB-groupoids $\mathcal{V}_1 \rightrightarrows E_1, \ldots, \mathcal{V}_p \rightrightarrows E_p$ over $\mathcal{G} \rightrightarrows M$ and their Whitney sum $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_p$ (as vector bundles over $\mathcal{G}$). This defines a VB-groupoid $\mathcal{V} \rightrightarrows E$ over $\mathcal{G} \rightrightarrows M$, whose Lie algebroid $A\mathcal{V} \to E$ splits naturally as a Whitney sum $A\mathcal{V}_1 \oplus \cdots \oplus A\mathcal{V}_p$ over $A\mathcal{G}$.

Proposition A.3 *For a source 1-connected Lie groupoid $\mathcal{G} \rightrightarrows M$, a multiplicative function $F : \mathcal{V} \to \mathbb{R}$ is componentwise linear if and only if so is $AF : A\mathcal{V} \to \mathbb{R}$. Moreover, in the case $\mathcal{V}_1 = \cdots = \mathcal{V}_p$, F is skew-symmetric if and only if so is AF.*

Proof We will treat the case $p = 2$, the general case follows similarly. Consider

$$F \circ h_\lambda,\ \lambda^2 F : \mathcal{V} \to \mathbb{R} \text{ and } F \circ 0_{\mathcal{V}_i} : \mathcal{V}_i \to \mathbb{R}, \quad i = 1, 2.$$

These are multiplicative functions and their associated infinitesimal cocycles are: $AF \circ h^A_\lambda$, $\lambda^2 AF : A\mathcal{V} \to \mathbb{R}$ and $AF \circ 0_{A\mathcal{V}_i} : A\mathcal{V}_i \to \mathbb{R}$, where $0_{A\mathcal{V}_i} : A\mathcal{V}_i \to A\mathcal{V}$ is the zero map (A.1) for the $A\mathcal{V} = A\mathcal{V}_1 \oplus A\mathcal{V}_2$, $i = 1, 2$. Note that $0_{\mathcal{V}_i}$ is a Lie groupoid morphism and $A(0_{\mathcal{V}_i}) = 0_{A\mathcal{V}_i}$. The fact that $\mathcal{G}$ is source 1-connected implies that $\mathcal{V}$ is source 1-connected (see Remark 2.1) and, therefore, by uniqueness of integration, one has that

$$F \circ h_\lambda = \lambda^2 F \Leftrightarrow AF \circ h^A_\lambda = \lambda^2 AF$$
$$F \circ 0_{\mathcal{V}_i} = 0 \Leftrightarrow AF \circ 0_{A\mathcal{V}_i} = 0.$$

So, it follows from Proposition A.1 that F is componentwise linear if and only if so is AF.

When $\mathcal{V}_1 = \cdots = \mathcal{V}_p$, each permutation $\sigma \in S(p)$ acts by groupoid morphism on $\mathcal{V}$ via $\sigma(v_1, \ldots, v_p) = (v_{\sigma(1)}, \ldots, v_{\sigma(p)})$. Applying the Lie functor, $A\sigma$ acts on $A\mathcal{V} = A\mathcal{V}_1 \oplus \cdots \oplus A\mathcal{V}_1$ permuting the elements of $A\mathcal{V}_1$ by σ itself. Hence, for a function $F \in C^\infty(\mathcal{V})$,

$$A(\mathfrak{p}_{\mathcal{V}}(F)) = \mathfrak{p}_{A\mathcal{V}}(AF)$$

where $\mathfrak{p}_{\mathcal{V}}$, $\mathfrak{p}_{A\mathcal{V}}$ are the projections (2.9) for $\mathcal{V}$ and $A\mathcal{V}$, respectively. The result now follows exactly as before using the uniqueness of integration. □

Appendix B Right-invariant vector fields on cotangent Lie groupoids.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, with Lie algebroid $\pi_A : A \to M$. Consider the cotangent Lie groupoid $T^*\mathcal{G} \rightrightarrows A^*$. In this appendix, we prove formulas (4.28) describing the right-invariant vector fields on $T^*\mathcal{G}$ corresponding to sections of types $\mathcal{R}_a$ and $\mathcal{B}\alpha$, for $a \in \Gamma(A)$, $\alpha \in \Omega^1(M)$ (see Lemmas B.2 and B.3). In the case of $\mathcal{R}_a$ the formula follows directly from the description of the isomorphism between $A(T^*G)$ and T^*A coming from the symplectic groupoid structure of $T^*\mathcal{G}$ obtained in [49, §7], and that we briefly recall here. The case of $\mathcal{B}\alpha$ is more involved. We note that, unlike for formulas (4.27) for the tangent groupoid, we could not find the analogous formulas (4.28) for the cotangent groupoid in the literature.

Let ω_{can} be the canonical symplectic form on $T^*\mathcal{G}$. It well-known [12] that $(T^*\mathcal{G}, \omega_{can})$ is the symplectic groupoid integrating the linear Poisson structure of A^*. In particular, ω_{can} is a multiplicative 2-form and its infinitesimal component $l_{can} : A(T^*\mathcal{G}) \to T^*(A^*)$, which satisfies

$$\omega_{can}(\overrightarrow{\chi}, \cdot) = \tilde{\mathtt{t}}^* l_{can}(\chi), \quad \forall\, \chi \in \Gamma(A(T^*\mathcal{G})), \tag{B.1}$$

is a Lie algebroid isomorphism, where the Lie algebroid structure on $T^*(A^*)$ comes from the linear Poisson structure Π_{lin} on A^*.

Recall the reversal isomorphism $\mathcal{R} : T^*(A^*) \to T^*A$ (see (4.15)). We shall need the following fact [49, Theorem 7.3]: the map

$$\vartheta := \mathcal{R} \circ l_{can} : A(T^*\mathcal{G}) \longrightarrow T^*A$$

is a Lie algebroid isomorphism from $A(T^*\mathcal{G})$ to the cotangent Lie algebroid.

Remark B.1 The isomorphism ϑ is used implicitly in §4.2 to talk about the right-invariant vector fields corresponding to $\mathcal{R}_a$ and $\mathcal{B}\alpha$, i.e. by $\overrightarrow{\mathcal{R}_a}$ and $\overrightarrow{\mathcal{B}\alpha}$ we mean the right-invariant vector fields of $T^*\mathcal{G} \rightrightarrows A^*$ corresponding to $\vartheta^{-1}(\mathcal{R}_a)$ and $\vartheta^{-1}(\mathcal{B}\alpha)$, for $a \in \Gamma(A)$ and $\alpha \in \Omega^1(M)$, respectively.

Note that, from (4.16) and (4.18), one has that

$$\vartheta^{-1}(\mathcal{R}_a) = l_{can}^{-1}(d\ell_a), \quad \text{and} \quad \vartheta^{-1}(\mathcal{B}\alpha) = l_{can}^{-1}(-\pi_{A^*}^*\alpha),$$

where $\pi_{A^*} : A^* \to M$ is the projection of the dual bundle.

Lemma B.2 *For $a \in \Gamma(A)$, consider $\ell_{\overrightarrow{a}} \in C^\infty(T^*\mathcal{G})$. One has*

$$\omega_{can}(\overrightarrow{\mathcal{R}_a}, \cdot) = d\ell_{\overrightarrow{a}} = \omega_{can}(\overrightarrow{a}^{T^*}, \cdot), \tag{B.2}$$

where $\overrightarrow{a}^{T^*}$ is the cotangent lift of $\overrightarrow{a}$ (4.6). In particular, $\overrightarrow{\mathcal{R}_a} = \overrightarrow{a}^{T^*}$.

Proof It is simple to check that the pull-back of the 1-form $d\ell_a \in \Omega^1(A^*)$ by the target map $\tilde{\mathtt{t}} : T^*\mathcal{G} \to A^*$ is exactly $d\ell_{\overrightarrow{a}}$, i.e., $d\ell_{\overrightarrow{a}} = \tilde{\mathtt{t}}^* d\ell_a$. Then, by (B.1),

$$\omega_{can}(\overrightarrow{\mathcal{R}_a}, \cdot) = \tilde{\mathtt{t}}^* l_{can}(l_{can}^{-1}(d\ell_a)) = d\ell_{\overrightarrow{a}}.$$

The second identity in (B.2) follows from the fact that the cotangent lift of a vector field is exactly its Hamiltonian lift with respect to ω_{can}. The last statement follows from the non-degeneracy of ω_{can}. □

In the following, we shall need the useful relationship between the sum and multiplication on the cotangent bundle known as *interchange law*[7]:

$$(\xi_1 + \eta_1) \bullet (\xi_2 + \eta_2) = (\xi_1 \bullet \xi_2) + (\eta_1 \bullet \eta_2), \tag{B.3}$$

for $\xi_1, \eta_1 \in T_{g_1}^*\mathcal{G}$, $\xi_2, \eta_2 \in T_{g_2}^*\mathcal{G}$ such that (ξ_1, ξ_2), (η_1, η_2) are composable pairs.

7 Interchange laws hold more generally for VB-groupoids, where they express the compatibility between the vector bundle and groupoid structures. See Ref. [47].

Lemma B.3 *Given a 1-form* $\alpha \in \Omega^1(M)$*, we have that*

$$\overrightarrow{\mathcal{B}\alpha} = (t^*\alpha)^{\mathrm{v}},$$

where the right-hand side is the vertical lift of $\mathsf{t}^*\alpha \in \Omega^1(\mathcal{G})$.

Proof To simplify notation, denote $\mathsf{t}^*\alpha$ by η. First note that $\widetilde{\mathsf{s}}(\eta(g)) = 0, \forall g \in \mathcal{G}$. Indeed, for any $a \in A_{\mathsf{s}(g)}$, it follows from the definition of $\widetilde{\mathsf{s}}$ that

$$\langle \widetilde{\mathsf{s}}(\eta(g)), a\rangle = \langle \alpha(\mathsf{t}(g)), T\mathsf{t}(\overleftarrow{a}(g))\rangle = 0.$$

Hence, for $\xi \in T^*_g\mathcal{G}$,

$$T\widetilde{\mathsf{s}}(\eta^{\mathrm{v}}(\xi)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \widetilde{\mathsf{s}}(\xi + \epsilon\eta(g)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\widetilde{\mathsf{s}}(\xi) + \epsilon\,\widetilde{\mathsf{s}}(\eta(g))) = 0.$$

Also, one has that

$$\eta(g) = \eta(\mathsf{t}(g)) \bullet 0_g. \tag{B.4}$$

Indeed, for any $U \in T_g\mathcal{G}$, using (2.4), one obtains

$$\langle \eta(t(g)) \bullet 0_g, U\rangle = \langle \eta(t(g)) \bullet 0_g, T\mathsf{t}(U) \bullet U\rangle = \langle \eta(t(g)), T\mathsf{t}(U)\rangle = \langle \eta(g), U\rangle.$$

Let us now prove that η^{v} is a right-invariant vector field. For $\xi \in T^*_g\mathcal{G}$,

$$\begin{aligned}\eta^{\mathrm{v}}(\xi) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\xi + \epsilon\,\eta(g)) \overset{(B.4)}{=} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\widetilde{1}_{\widetilde{\mathsf{t}}(\xi)} \bullet \xi + \epsilon\,\eta(\mathsf{t}(g)) \bullet 0_g)\\ &\overset{(B.3)}{=} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} [(\widetilde{1}_{\widetilde{\mathsf{t}}(\xi)} + \epsilon\,\eta(\mathsf{t}(g))) \bullet (\xi + 0_g)] = dR_\xi[\eta^{\mathrm{v}}(\widetilde{1}_{\widetilde{\mathsf{t}}(\xi)})].\end{aligned}$$

This proves that η^{v} is right-invariant.

To conclude the proof, we just need to prove that $\omega_{can}(\eta^{\mathrm{v}}(\varphi), \Upsilon) = \omega_{can}(\overrightarrow{\mathcal{B}\alpha}(\varphi), \Upsilon)$ for any $\Upsilon \in T_\varphi A^* \subset T(T^*\mathcal{G})$, $\varphi \in A^*_x$, $x \in M$. Choose any projectable vector field $\widetilde{\Upsilon} \in \mathfrak{X}(T^*\mathcal{G})$ extending Υ, with respect to the cotangent bundle projection $\mathrm{pr} : T^*\mathcal{G} \to \mathcal{G}$. Recall that pr is a groupoid morphism covering $\pi_{A^*} : A^* \to M$. As $\omega_{can} = -d\theta_{can}$ for the tautological 1-form $\theta_{can} \in \Omega^1(T^*\mathcal{G})$, one has that

$$\begin{aligned}\omega_{can}(\eta^{\mathrm{v}}(\varphi), \Upsilon) &= -\mathcal{L}_{\eta^{\mathrm{v}}}\theta_{can}(\widetilde{\Upsilon})|_\varphi = -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \langle \varphi + \epsilon\,\eta(x),\ Tpr(\widetilde{\Upsilon}(\varphi + \epsilon\,\eta(x))\rangle\\ &= -\langle \eta(x), T\mathrm{pr}(\Upsilon)\rangle,\end{aligned}$$

where we have used that $\theta_{can}(\eta^{\mathrm{v}}) = \theta_{can}([\eta^{\mathrm{v}}, \widetilde{\Upsilon}]) = 0$ and $T\mathrm{pr}(\widetilde{\Upsilon}(\varphi + \epsilon\eta(x))) = T\mathrm{pr}(\widetilde{\Upsilon}(\varphi))$. The proof now follows from (B.1)

$$\omega_{can}(\overrightarrow{\mathcal{B}\alpha}, \cdot) = -\widetilde{\mathrm{t}}^{*}(\pi^{*}_{A^{*}}\alpha) = -\mathrm{pr}^{*}(\mathrm{t}^{*}\alpha).$$

□

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