



Kähler–Ricci flow on blowups along submanifolds

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Abstract

In this short note, we study the behavior of Kähler–Ricci flow on Kähler manifolds which contract divisors to smooth submanifolds. We show that the Kähler potentials are Hölder continuous and the flow converges sequentially in Gromov–Hausdorff topology to a compact metric space which is homeomorphic to the base manifold.

1 Introduction

The Ricci flow, introduced by Hamilton [6] in 1982, has been a powerful tool in solving problems in geometry and analysis. It deforms any metric with positive Ricci curvature in real 3-dimensional manifold to a metric with constant curvature [6]. By performing surgery through singular times, Perelman [9] used Ricci flow to solve the geometrization conjecture for 3-dimensional manifolds. On the complex aspect, the Ricci flow preserves the Kähler condition [1] and is reduced to a scalar equation with Monge–Ampère type, which after suitable normalization converges to a solution of the Calabi conjecture [1,30]. The non-Kähler analogue of Ricci flow also generates much interest recently, among them are the Hermitian curvature flows and the pluriclosed flow [24,25], the Chern–Ricci flow [27], the Anomaly flow [14] and etc, and we refer to [13] for a survey on the recent development of non-Kähler geometric flows.

The analytic minimal model program, laid out in [19], predicts how the Kähler–Ricci flow behaves on a projective variety. It is conjectured that the Kähler–Ricci flow will either collapse in finite time, or deform any projective variety to its minimal model after finitely many divisorial contractions or flips in the Gromov–Hausdorff (GH) topology. There are various results on the finite time collapsing of Kähler–Ricci flow, see for example [12,17,23,28,29] and references therein. The behavior of Kähler–Ricci flow on some small contractions is studied in [16,22] and it is shown that the

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flow forms a continuous path in GH topology. In [20,21], Song and Weinkove study the divisorial contractions when the divisor is contracted to discrete points, and it is shown that the flow converges in GH topology to a metric space which is isometric to the metric completion of the base manifold with the smooth limit of the flow outside the divisor, and the flow can be continued on the new space. The main purpose of this note is to generalize their results to divisorial contractions when the divisor is contracted to a higher dimensional subvariety.

Let Y be a Kähler manifold and $N \subset Y$ be a complex submanifold of codimension $k \geq 1$. Let X be the Kähler manifold obtained by blowing up Y along N , $\pi : X \rightarrow Y$ be the blown-down map and $E = \pi^{-1}(N)$ be the exceptional divisor in X . We consider the (unnormalized) Kähler–Ricci flow on X :

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega), \\ \omega(0) = \omega_0, \end{cases} \tag{1.1}$$

for a suitable fixed Kähler metric ω_0 on X . We assume the limit cohomology class satisfies $[\omega_0] + TK_X = [\pi^*\omega_Y]$ for some Kähler metric ω_Y on Y , where the maximal existence time (see [26]) of the flow (1.1) is given by

$$T = \sup\{t > 0 : [\omega_0] + tK_X \text{ is Kähler}\} < \infty.$$

We define the reference metrics along the flow

$$\hat{\omega}_t = \frac{T-t}{T}\omega_0 + \frac{t}{T}\pi^*\omega_Y.$$

In the following for notational simplicity we shall denote $\hat{\omega}_Y = \pi^*\omega_Y$, which is a nonnegative $(1, 1)$ -form on X .

It is well-known that the flow (1.1) is equivalent to the following parabolic complex Monge–Ampère equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + i\partial\bar{\partial}\varphi)^n}{\Omega}, \\ \varphi(0) = 0, \end{cases} \tag{1.2}$$

where $\omega = \hat{\omega}_t + i\partial\bar{\partial}\varphi$ satisfies (1.1) and Ω is a smooth volume form satisfying $i\partial\bar{\partial} \log \Omega = \frac{1}{T}(\hat{\omega}_Y - \omega_0)$.

Our main theorem is on the behavior of the metrics $\omega(t)$ as $t \rightarrow T^-$.

Theorem 1.1 *Let $\pi : X \rightarrow Y$ and $\omega_t = \omega_0 + i\partial\bar{\partial}\varphi_t$ be as above, then the following hold: there exists a uniform constant $C = C(n, \omega_0, \omega_Y, \pi) > 0$*

- (1) φ_t is uniformly Hölder continuous in (X, ω_0) , i.e. $|\varphi_t(p) - \varphi_t(q)| \leq Cd_{\omega_0}(p, q)^\delta$, for any $p, q \in X$ and some $\delta \in (0, 1)$, and $\varphi_t \xrightarrow{C^\delta(X, \omega_0)} \varphi_T \in PSH(X, \pi^*\omega_Y) \cap C^\delta(X, \omega_0)$. Moreover, φ_T descends to a function $\bar{\varphi}_T \in PSH(Y, \omega_Y) \cap C^{\delta_0}(Y, \omega_Y)$ for some $\delta_0 \in (0, 1)$.

- (2) ω_t converge weakly to $\omega_T := \pi^*\omega_Y + i\partial\bar{\partial}\varphi_T$ as $(1, 1)$ -currents on X and the convergence is smooth and uniform on any compact subset $K \Subset X \setminus E$.
- (3) $\text{diam}(X, \omega_t) \leq C$ for any $t \in [0, T)$.
- (4) for any sequence $t_i \rightarrow T^-$, there exists a subsequence $\{t_{i_j}\}$ such that $(X, \omega_{t_{i_j}})$ (as compact metric spaces) converge in Gromov–Hausdorff topology to a compact metric space (Z, d_Z) .
- (5) the metric completion of $(Y \setminus N, \omega_T)$ is isometric to (Y, d_T) , where the distance function d_T is induced from ω_T and defined in (3.12). And there exists an open dense subset $Z^\circ \subset Z$ such that $(Y \setminus N, d_T)$ and (Z°, d_Z) are homeomorphic and locally isometric. Furthermore (Z, d_Z) is homeomorphic to (Y, d_T) .

The item (2) is known to hold for Kähler–Ricci flow for more general holomorphic maps $\pi : X \rightarrow Y$ with $\dim Y = \dim X$ (see e.g. [11,20,26]). We include it in the theorem just for completeness. We remark that Theorem 1.1 also holds if the base Y has some mild singularities, for example, if the analytic subvariety N is locally of the form $\mathbb{C}^k \times (\mathbb{C}^{n-k}/\mathbb{Z}_p)$, where \mathbb{Z}_p denotes the S^1 -action $\{e^{2l\pi i/p}\}_{l=1}^p$ on \mathbb{C}^{n-k} by

$$e^{2l\pi i/p} \cdot (z_{k+1}, \dots, z_n) \rightarrow (e^{2l\pi i/p} z_{k+1}, \dots, e^{2l\pi i/p} z_n).$$

The proof is by combining the techniques of [21] and this note, so we omit the details.

Lastly we mention that under the same set-up as in Theorem 1.1, the same and even stronger results hold for Kähler metrics along continuity method. More precisely, let $u_t \in PSH(X, \hat{\omega}_Y + t\omega_0)$ be the solution to the complex Monge–Ampère equations

$$\omega_t^n = (\hat{\omega}_Y + t\omega_0 + i\partial\bar{\partial}u_t)^n = c_t e^F \omega_0^n, \quad \sup u_t = 0, \quad t \in (0, 1], \quad (1.3)$$

where F is a given smooth function on X and c_t is a normalizing constant so that the integral of both sides are the same. It has been shown in [3] that $\text{diam}(X, \omega_t)$ is bounded by a constant $C = C(n, \omega_0, \hat{\omega}_Y, F) > 0$ and the Ricci curvature of ω_t is uniformly bounded below. We can repeat almost identically the proof of Theorem 1.1 to the Eq. (1.3) to get the same conclusions for u_t as the φ_t in Theorem 1.1. Furthermore, along the continuity method (1.3), we can improve the Gromov–Hausdorff convergence in Theorem 1.1 in the sense that the full sequence (without the need of passing to a subsequence) (X, ω_t) converges in GH topology to a compact metric space (Z, d_Z) which is isometric to the metric completion of $(Y \setminus N, \hat{\omega}_0)$, where $\hat{\omega}_0$ is the smooth limit of ω_t on $X \setminus \pi^{-1}(N) = Y \setminus N$. The main advantage in this case is that the Ricci curvature has uniform lower bound so we can apply the argument in [2], in particular the Gromov’s lemma to find an almost geodesic connecting any two points away from the singular set $\pi^{-1}(N)$.

2 Preliminaries

The following estimates are well-known [11,20,26,30], so we just state the results and omit the proofs.

Lemma 2.1 *There exists a constant $C > 0$ depending only on $(X, \omega_0), (Y, \omega_Y)$ such that*

- (i) $\|\varphi\|_{L^\infty(X)} \leq C$ for all $t \in [0, T)$,
- (ii) $\dot{\varphi} := \frac{\partial \varphi}{\partial t} \leq C$ and this is equivalent to $\omega^n \leq C\Omega$ from the Eq. (1.2).
- (iii) as $t \rightarrow T^-$, φ converge to a bounded $\hat{\omega}_Y$ -PSH function φ_T and ω converge weakly to $\omega_T := \hat{\omega}_Y + i\partial\bar{\partial}\varphi_T$ as $(1, 1)$ -currents on X .

Lemma 2.2 *There exists a uniform constant $C > 0$ such that*

- (i) $\hat{\omega}_Y \leq C\omega$ for all $t \in [0, T)$,
- (ii) for any compact subset $K \Subset X \setminus E$, there exists a constant $C_{j,K} > 0$ such that $\|\varphi\|_{C^j(K, \omega_0)} \leq C_{j,K}$. Therefore the convergence $\omega_t \rightarrow \omega_T$ and $\varphi \rightarrow \varphi_T$ is smooth on $X \setminus E$, so ω_T and φ_T are both smooth on $X \setminus E$.

In the proof of Lemma 2.2, we need the following Chern–Lu inequality as in the proof the parabolic Schwarz lemma [18]

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_\omega \hat{\omega}_Y \leq C \operatorname{tr}_\omega \hat{\omega}_Y,$$

where $C > 0$ depends also on the upper bound of the bisectonal curvature of (Y, ω_Y) . In turn this implies the equation below which will be used later.

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_\omega \hat{\omega}_Y \leq -\frac{|\nabla \operatorname{tr}_\omega \hat{\omega}_Y|^2}{\operatorname{tr}_\omega \hat{\omega}_Y} + C(\operatorname{tr}_\omega \hat{\omega}_Y)^2 \leq -c_0|\nabla \operatorname{tr}_\omega \hat{\omega}_Y|^2 + C, \tag{2.1}$$

where $c_0 = C^{-1} > 0$ is the reciprocal of the constant C in (i) Lemma 2.2.

2.1 Kähler metrics from the blown up

We will construct a smooth function σ_Y on Y such that $\sigma_Y = 0$ precisely on N . Choose a finite open cover $\{V_\alpha\}_{\alpha=1}^J$ of N in Y and complex coordinates $\{w_{\alpha,i}\}_{i=1}^n$ on V_α such that $N \cap V_\alpha = \{w_{\alpha,1} = \dots = w_{\alpha,k} = 0\}$. We also denote $V_0 = Y \setminus \cup_{\alpha=1}^J \frac{1}{2}\bar{V}_\alpha$ and we may also assume that $V_0 \cap N = \emptyset$. Take a partition of unity $\{\theta_\alpha\}_{\alpha=0}^J$ subordinate to the open cover $\{V_\alpha\}_{\alpha=0}^J$, and we define a smooth function

$$\sigma_Y = \theta_0 \cdot 1 + \sum_{\alpha=1}^J \theta_\alpha \cdot \sum_{j=1}^k |w_{\alpha,j}|^2 \in C^\infty(Y),$$

and it is straightforward to see from the construction that σ_Y vanishes precisely along N . Since $\{w_{\alpha,i}\}_{i=1}^k$ are defining functions of N , it follows that if $V_\alpha \cap V_\beta \neq \emptyset$, then the function

$$f_{\alpha\beta} := \frac{\sum_{j=1}^k |w_{\alpha,j}|^2}{\sum_{j=1}^k |w_{\beta,j}|^2}, \quad \text{on } V_\alpha \cap V_\beta$$

is never-vanishing and bounded from above. Since the cover is finite we have

$$0 < c \leq \inf_{\alpha, \beta} \inf_{y \in V_\alpha \cap V_\beta \neq \emptyset} f_{\alpha\beta}(y) \leq \sup_{\alpha, \beta} \sup_{y \in V_\alpha \cap V_\beta \neq \emptyset} f_{\alpha\beta}(y) \leq C < \infty. \tag{2.2}$$

We denote $\sigma_X = \pi^* \sigma_Y$ to be the pull-back of σ_Y to X .

Lemma 2.3 (see also [10]) *There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the (1, 1)-form*

$$\omega_\varepsilon := \pi^* \omega_Y + \varepsilon i \partial \bar{\partial} \log \sigma_X$$

is positive definite on $X \setminus E$ and extends to a smooth Kähler metric on X .

Proof We only need to prove the positivity of ω_ε near E , which is in fact local. So we may assume the map π is defined from an open set $U \subset X$ to V_α given by

$$w_{\alpha,1} = z_1, w_{\alpha,2} = z_1 z_2, \dots, w_{\alpha,k} = z_1 z_k, w_{\alpha,k+1} = z_{k+1}, \dots, w_{\alpha,n} = z_n,$$

where $\{z_i\}$ are the complex coordinates on U such that $E \cap U = \{z_1 = 0\}$. It loses no loss of generality to assume ω_Y on V_α is just the Euclidean metric $\omega_{\mathbb{C}^n} = \sum_j i dw_{\alpha,j} \wedge d\bar{w}_{\alpha,j}$.

We note that on V_α

$$\sigma_Y = \left(\sum_{\beta=1, V_\beta \cap V_\alpha \neq \emptyset}^J \theta_\beta f_{\beta\alpha} \right) \cdot \sum_{j=1}^k |w_{\alpha,j}|^2 =: \phi_\alpha \cdot \sum_{j=1}^k |w_{\alpha,j}|^2.$$

From (2.2), we know that ϕ_α is a smooth function with a strict positive lower bound on V_α . In particular $\omega_Y + \varepsilon i \partial \bar{\partial} \log \phi_\alpha > 0$ on V_α for any $0 < \varepsilon \leq \varepsilon_0 \ll 1$.

We calculate

$$\begin{aligned} \pi^* \omega_Y &= \left(1 + \sum_{j=2}^k |z_j|^2 \right) dz_1 \wedge d\bar{z}_1 + \sum_{j=2}^k (z_1 \bar{z}_j dz_j \wedge d\bar{z}_1 + \bar{z}_1 z_j dz_1 \wedge d\bar{z}_j) \\ &\quad + |z_1|^2 \sum_{j=2}^k dz_j \wedge d\bar{z}_j + \sum_{j=k+1}^n dz_j \wedge d\bar{z}_j, \end{aligned} \tag{2.3}$$

and note that on U

$$\log \sigma_X = \log \phi_\alpha + \log |z_1|^2 + \log \left(1 + \sum_{j=2}^k |z_j|^2 \right),$$

so on $U \setminus E$ we have

$$i \partial \bar{\partial} \log \sigma_X = i \partial \bar{\partial} \log \phi_\alpha + \frac{\sum_{i,j=2}^k ((1 + |z'|^2) \delta_{ij} - \bar{z}_i z_j) \sqrt{-1} dz_i \wedge d\bar{z}_j}{(1 + |z'|^2)^2}, \tag{2.4}$$

where $z' = (z_2, \dots, z_k)$ and the second term on RHS is nonnegative in z' -directions, which is just the Fubini-Study metric in the coordinates z' . By straightforward calculations, we see that if ε is small enough the $(1, 1)$ -form $\pi^*\omega_Y + \varepsilon i\partial\bar{\partial} \log \sigma_X$ is positive on $X \setminus E$ and extends to a Kähler metric on X . \square

Remark 2.1 Globally from the above calculations we see that

$$\omega_\varepsilon = \pi^*\omega_Y + \varepsilon i\partial\bar{\partial} \log \sigma_X - \varepsilon[E],$$

where $[E]$ denotes the current of integration along E .

We will denote $\omega_X = \pi^*\omega_Y + \varepsilon_0 i\partial\bar{\partial} \log \sigma_X - \varepsilon_0[E]$ to be a fixed Kähler metric obtained from the Lemma 2.3. The following inequality follows from the local expression of $\pi^*\omega_Y$ as in the proof of Lemma 2.3.

Lemma 2.4 *There exists a uniform constant $C > 1$ such that*

$$C^{-1}\hat{\omega}_Y \leq \omega_X \leq \frac{C}{\sigma_X}\hat{\omega}_Y, \tag{2.5}$$

where the second inequality is understood on $X \setminus E$.

3 The proof of the main theorem

Now we are ready to derive the crucial estimates on ω along the Kähler–Ricci flow (1.1).

Lemma 3.1 *There exists uniform constants $C > 0$ and $\delta \in (0, 1)$ such that along the flow (1.1) we have*

$$\omega \leq C \frac{\omega_0}{\sigma_X^{1-\delta}}, \quad \text{on } X \setminus E \times [0, T]. \tag{3.1}$$

The proof is almost the same as that of Lemma 2.5 in [20], with minor modification using Lemma 2.3. For completeness, we provide a sketched proof.

Proof Fix an $\epsilon \in (0, 1)$ and define

$$Q_\epsilon = \log \text{tr}_{\omega_0} \omega + A \log \sigma_X^{1+\epsilon} \text{tr}_{\hat{\omega}_Y} \omega - A^2\varphi,$$

where $A > 0$ is a constant to be determined later. First of all, $Q_\epsilon|_{t=0} \leq C$ for a constant C independent of $\epsilon \in (0, 1)$, which can be seen from (2.5). Observe that for each time $t_0 \in (0, T)$, $\max_X Q_\epsilon$ can only be achieved on $X \setminus E$, since $Q_\epsilon(x) \rightarrow -\infty$ as $x \rightarrow E$. Thus we assume the maximum of Q_ϵ is obtained at (x_0, t_0) for some $x_0 \in X \setminus E$. From the Chern–Lu inequality (e.g. Eq. (2.1)) the following holds on $X \setminus E$

$$\left(\frac{\partial}{\partial t} - \Delta \right) Q_\epsilon \leq C \text{tr}_\omega \omega_0 - A \text{tr}_\omega (A\hat{\omega}_t + (1 + \epsilon)i\partial\bar{\partial} \log \sigma_X) + A^2 \log \frac{\Omega}{\omega^n} + C,$$

where the constant C depends on the lower bound of the bisectional curvature of (X, ω_0) and the upper bound of bisectional curvature of (Y, ω_Y) . Since $\hat{\omega}_t \geq c_1 \hat{\omega}_Y$ for a uniform $c_1 > 0$ and any $t \in [0, T)$, by Lemma 2.3 for $A > 0$ large enough $A\hat{\omega}_t + (1 + \epsilon)i\partial\bar{\partial} \log \sigma_X \geq c_2 \omega_0$ on $X \setminus E$ for some $c_2 > 0$. If $A > 0$ is taken even larger then at (x_0, t_0) , we have

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta \right) Q_\epsilon \leq -2 \operatorname{tr}_\omega \omega_0 + A^2 \log \frac{\Omega}{\omega^n} + C \leq -\operatorname{tr}_\omega \omega_0 + C,$$

where in the last inequality we use

$$-\operatorname{tr}_\omega \omega_0 + A^2 \log \frac{\Omega}{\omega^n} \leq -\operatorname{tr}_\omega \omega_0 + nA^2 \log \operatorname{tr}_\omega \omega_0 + C \leq C,$$

as $\log x \leq \epsilon x + C(\epsilon)$ for any $x \in (0, \infty)$. So we have $\operatorname{tr}_\omega \omega_0(x_0, t_0) \leq C$. Then

$$\operatorname{tr}_{\omega_0} \omega|_{(x_0, t_0)} \leq \frac{\omega^n}{\omega_0^n} (\operatorname{tr}_\omega \omega_0)^{n-1}|_{(x_0, t_0)} \leq C.$$

Observing that from (2.5), $\sigma_X \operatorname{tr}_{\hat{\omega}_Y} \omega \leq C \operatorname{tr}_{\omega_0} \omega$ on $X \setminus E$, thus $\sup_X Q_\epsilon \leq C$ for some uniform constant $C > 0$. Letting $\epsilon \rightarrow 0$, we get

$$\log \operatorname{tr}_{\omega_0} \omega + A \log \sigma_X \operatorname{tr}_{\hat{\omega}_Y} \omega \leq C, \quad \text{on } X \setminus E \times [0, T).$$

Finally from $C \operatorname{tr}_{\hat{\omega}_Y} \omega \geq \operatorname{tr}_{\omega_0} \omega$ we see from the above that

$$\log \operatorname{tr}_{\omega_0} \omega + \log \sigma_X^A (\operatorname{tr}_{\omega_0} \omega)^A \leq C,$$

so $\operatorname{tr}_{\omega_0} \omega \leq C \sigma_X^{-A/(1+A)}$ on $X \setminus E$, and we can then take $\delta = \frac{1}{1+A} \in (0, 1)$. □

Next we will show the distance function defined by ω_t is Hölder-continuous with respect to the fixed metric (X, ω_0) .

Lemma 3.2 *There exists a uniform constant $C > 0$ such that for any $p, q \in X$, it holds that*

$$d_{\omega_t}(p, q) \leq C d_{\omega_0}(p, q)^\delta, \quad \forall t \in [0, T),$$

where $\delta \in (0, 1)$ is the constant determined in Lemma 3.1.

Proof It suffices to prove the estimate near E , say on $T(E)$, a tubular neighborhood of E , since ω_t is uniformly equivalent to ω_0 outside $T(E)$. Choose coordinates charts $\{U_\alpha\}$ covering $T(E)$ and local coordinates $\{z_{\alpha,i}\}_{i=1}^n$ such that $U_\alpha \cap E = \{z_{\alpha,1} = 0\}$. We may assume that the cover is fine enough such that any $p, q \in T(E)$ with $d_{\omega_0}(p, q) \leq \frac{1}{2}$ must lie in the same U_α . Since we have only finitely many such U_α , we will work on one of them only and omit the subscript α for simplicity. Furthermore the fixed Kähler metric ω_0 is uniformly equivalent to the Euclidean metric $\omega_{\mathbb{C}^n}$ on U , so

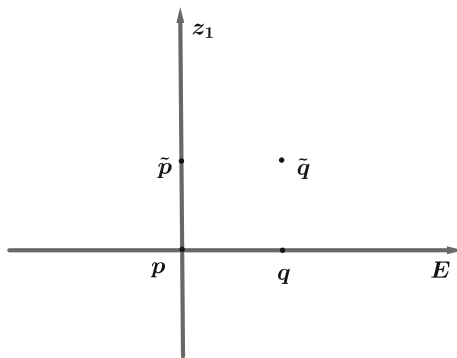
without loss of generality we assume $\omega_0 = \omega_{\mathbb{C}^n}$ on U . Recall that Lemma 3.1 implies that on $U \setminus E$ it holds that

$$\omega_t \leq C \frac{\omega_{\mathbb{C}^n}}{|z_1|^{2(1-\delta)}}, \quad \forall t \in [0, T), \tag{3.2}$$

since $\sigma_X \sim |z_1|^2$ on U .

Take any two points $p, q \in U$ with $d_{\omega_0}(p, q) = d < \frac{1}{4}$. We will consider different cases depending on the positions of p, q .

• **Case 1** $p, q \in E$. Rotating the coordinates if necessary we may assume $p = 0$ and $q = (d, 0, \dots, 0)$. We pick two points $\tilde{p} = (d, 0, \dots, 0)$ and $\tilde{q} = (d, d, 0, \dots, 0)$ as shown the picture below. From (3.2), we have



$$d_{\omega_t}(p, \tilde{p}) \leq L_{\omega_t}(\overline{p\tilde{p}}) \leq C \int_0^d \frac{1}{r^{1-\delta}} dr \leq Cd^\delta,$$

where $\overline{p\tilde{p}}$ denotes the (Euclidean) line segment connecting p and \tilde{p} . Similarly $d_{\omega_t}(q, \tilde{q}) \leq Cd^\delta$. On the other hand,

$$d_{\omega_t}(\tilde{p}, \tilde{q}) \leq L_{\omega_t}(\overline{\tilde{p}\tilde{q}}) \leq \frac{C}{d^{1-\delta}} L_{\omega_{\mathbb{C}^n}}(\overline{\tilde{p}\tilde{q}}) = Cd^\delta.$$

If we denote $\gamma = \overline{p\tilde{p}} + \overline{\tilde{p}\tilde{q}} + \overline{\tilde{q}q}$ to be the piecewise line segment connecting p and q , then we have

$$d_{\omega_t}(p, q) \leq L_{\omega_t}(\gamma) \leq Cd^\delta = Cd_{\omega_0}(p, q)^\delta.$$

We remark that $\gamma \subset X \setminus E$, except the two end points p, q .

• **Case 2** $\min(d_{\omega_0}(p, E), d_{\omega_0}(q, E)) \leq d$. The (Euclidean) projections of p, q to E , denoted by p', q' , respectively, must satisfy $d_{\omega_0}(p', q') \leq d$. From the assumption it follows that $d_{\omega_0}(p, p') \leq 2d$ and $d_{\omega_0}(q, q') \leq 2d$. By similar arguments as above using (3.2) we have

$$d_{\omega_t}(p, p') \leq Cd^\delta, \quad d_{\omega_t}(q, q') \leq Cd^\delta,$$

and by Case 1 $d_{\omega_t}(p', q') \leq Cd_{\omega_0}(p', q')^\delta \leq Cd^\delta$. By triangle inequality we get the desired estimate $d_{\omega_t}(p, q) \leq Cd^\delta$.

• **Case 3** $\min(d_{\omega_0}(p, E), d_{\omega_0}(q, E)) \geq d$. Every point in the (Euclidean) line segment \overline{pq} has norm of z_1 -coordinates no less than d , therefore

$$d_{\omega_t}(p, q) \leq L_{\omega_t}(\overline{pq}) \leq Cd^{-(1-\delta)}L_{\omega_{\mathbb{C}^n}}(\overline{p, q}) = Cd^\delta.$$

Combining the all the cases above, we finish the proof of the lemma. □

Next we will prove the Hölder continuity of φ_t with respect to (X, ω_0) . To begin with, we first prove the gradient estimate of $\Phi := (T - t)\dot{\varphi} + \varphi$ with respect to the evolving metrics (X, ω_t) (c.f. [3]).

Lemma 3.3 *There exists a uniform constant $C > 0$ such that*

$$\sup_X |\nabla_{\omega_t} \Phi|_{\omega_t} \leq C, \quad \forall t \in [0, T].$$

Proof Taking $\frac{\partial}{\partial t}$ on both sides of (1.2), we get

$$\frac{\partial}{\partial t} \dot{\varphi} = \Delta \dot{\varphi} + \frac{1}{T} \operatorname{tr}_\omega(\hat{\omega}_Y - \omega_0) = \Delta \dot{\varphi} + \frac{1}{T-t} \operatorname{tr}_\omega \hat{\omega}_Y + \frac{1}{T-t} \Delta \varphi,$$

where we used the equation $-\frac{1}{T} \operatorname{tr}_\omega \omega_0 = -\frac{n}{T-t} + \frac{t}{T(T-t)} \operatorname{tr}_\omega \hat{\omega}_Y + \frac{1}{T-t} \Delta \varphi$. Then we have the equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Phi = \operatorname{tr}_\omega \hat{\omega}_Y - n \geq -n. \tag{3.3}$$

By maximum principle, it follows that $\inf_X \Phi \geq -C$ for some constant depending also on T . Recall Φ is also bounded above by Lemma 2.1. And combining (3.3) with Bochner formula the following equation holds:

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla \Phi|_\omega^2 = -|\nabla \nabla \Phi|^2 - |\nabla \bar{\nabla} \Phi|^2 + 2Re \langle \nabla \Phi, \bar{\nabla} \operatorname{tr}_\omega \hat{\omega}_Y \rangle.$$

Fix a constant $B := \sup_{X \times [0, T]} |\Phi| + 2$. By direct calculations the following equation holds

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \frac{|\nabla \Phi|^2}{B - \Phi} &= \frac{\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla \Phi|^2}{B - \Phi} + \frac{|\nabla \Phi|^2 \left(\frac{\partial}{\partial t} - \Delta \right) \Phi}{(B - \Phi)^2} + 2Re \left\langle \nabla \log(B - \Phi), \bar{\nabla} \frac{|\nabla \Phi|^2}{B - \Phi} \right\rangle \\ &= \frac{-|\nabla \nabla \Phi|^2 - |\nabla \bar{\nabla} \Phi|^2 + 2Re \langle \nabla \Phi, \bar{\nabla} \operatorname{tr}_\omega \hat{\omega}_Y \rangle}{B - \Phi} + \frac{|\nabla \Phi|^2 (\operatorname{tr}_\omega \hat{\omega}_Y - n)}{(B - \Phi)^2} \\ &\quad + 2Re \left\langle \nabla \log(B - \Phi), \bar{\nabla} \frac{|\nabla \Phi|^2}{B - \Phi} \right\rangle. \end{aligned} \tag{3.4}$$

From the Eq. (2.1), we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) \frac{\text{tr}_\omega \hat{\omega}_Y}{B - \Phi} \\ & \leq \frac{-c_0 |\nabla \text{tr}_\omega \hat{\omega}_Y|^2 + C}{B - \Phi} + \frac{\text{tr}_\omega \hat{\omega}_Y (\text{tr}_\omega \hat{\omega}_Y - n)}{(B - \Phi)^2} + 2Re \left\langle \nabla \log(B - \Phi), \bar{\nabla} \frac{\text{tr}_\omega \hat{\omega}_Y}{B - \Phi} \right\rangle. \end{aligned} \tag{3.5}$$

Denote

$$G = \frac{|\nabla \Phi|^2}{B - \Phi} + A \frac{\text{tr}_\omega \hat{\omega}_Y}{B - \Phi}, \quad \text{where } A = 10c_0^{-1}.$$

By (3.4), (3.5) and Cauchy–Schwarz inequality we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) G \\ & \leq \frac{-|\nabla \nabla \Phi|^2 - |\nabla \bar{\nabla} \Phi|^2 - 9|\nabla \text{tr}_\omega \hat{\omega}_Y|^2}{B - \Phi} + CG + C + 2Re \langle \nabla \log(B - \Phi), \bar{\nabla} G \rangle. \end{aligned}$$

Assuming the maximum of G is attained at (x_0, t_0) , we may assume at this point $|\nabla \Phi| \geq A$, otherwise we are done yet. Then at this point $\left(\frac{\partial}{\partial t} - \Delta\right)G \geq 0$ and $\nabla G = 0$, hence we have $2|\nabla \Phi| \cdot \nabla |\nabla \Phi| = -G\nabla \Phi - A\nabla \text{tr}_\omega \hat{\omega}_Y$. Taking norm on both side we get at (x_0, t_0)

$$\frac{|G\nabla \Phi + A\nabla \text{tr}_\omega \hat{\omega}_Y|^2}{2|\nabla \Phi|^2} = 2|\nabla |\nabla \Phi||^2 \leq |\nabla \nabla \Phi|^2 + |\nabla \bar{\nabla} \Phi|^2, \tag{3.6}$$

where we used the Kato’s inequality in the last inequality. Therefore at (x_0, t_0) , we have

$$\begin{aligned} 0 & \leq (B - \Phi)^{-1} \left(-\frac{1}{2}G^2 + AG \frac{|\nabla \text{tr}_\omega \hat{\omega}_Y|}{|\nabla \Phi|} + \frac{A^2}{2|\nabla \Phi|^2} |\nabla \text{tr}_\omega \hat{\omega}_Y|^2 - 9|\nabla \text{tr}_\omega \hat{\omega}_Y|^2 \right) + CG + C \\ & \leq -\frac{G^2}{4(B - \Phi)} + CG + C, \end{aligned}$$

so at (x_0, t_0) , $G \leq C$. From this we get the desired bound on $|\nabla \Phi|$. □

An immediate consequence of the gradient bound is the uniform Hölder continuity of φ_t on (X, ω_0) .

Corollary 3.1 *There exists a uniform constant $C > 0$ such that*

$$|\varphi_t(p) - \varphi_t(q)| \leq Cd_{\omega_0}(p, q)^\delta, \quad \forall p, q \in X, \text{ and } \forall t \in [0, T],$$

where $\delta > 0$ is the same constant as in Lemma 3.1.

Proof Recall the definition of Φ , that

$$\Phi_t = (T - t)\dot{\varphi} + \varphi = (T - t)^2 \frac{\partial}{\partial t} \left(\frac{\varphi_t}{T - t} \right).$$

By the gradient bound in Lemma 3.3 and distance estimate in Lemma 3.2, for any fixed points $p, q \in X$, we have

$$|\Phi_t(p) - \Phi_t(q)| \leq Cd_{\omega_t}(p, q) \leq Cd_{\omega_0}(p, q)^\delta.$$

So

$$\left| \frac{\partial}{\partial t} \left(\frac{\varphi_t}{T - t} \right)(p) - \frac{\partial}{\partial t} \left(\frac{\varphi_t}{T - t} \right)(q) \right| \leq \frac{Cd_{\omega_0}(p, q)^\delta}{(T - t)^2}, \tag{3.7}$$

integrating (3.7) over $t \in [0, t_1]$ we get by noting that $\varphi_0 \equiv 0$ that

$$\left| \frac{\varphi_{t_1}(p)}{T - t_1} - \frac{\varphi_{t_1}(q)}{T - t_1} \right| \leq Cd_{\omega_0}(p, q)^\delta \int_0^{t_1} \frac{1}{(T - t)^2} dt = Cd_{\omega_0}(p, q)^\delta \frac{t_1}{T(T - t_1)},$$

cancelling the common factor $\frac{1}{T - t_1}$ on both sides we get the desired estimate since $t_1 \in (0, T)$ is arbitrarily chosen. \square

Remark 3.1 By an argument in [8], the Hölder continuity of φ_t implies that the distance functions satisfy the estimate in Lemma 3.2.

Recall that the exceptional divisor E is a $\mathbb{C}\mathbb{P}^{k-1}$ -bundle over N and we identify N with the zero section of this bundle. Denote the bundle map by $\hat{\pi} : E \rightarrow N$ which is the restriction of $\pi : X \rightarrow Y$ to E . From Corollary 3.1, we see that the limit $\varphi_T \in PSH(X, \hat{\omega}_Y)$ is also Hölder continuous in (X, ω_0) . Since $\hat{\omega}_Y|_{\hat{\pi}^{-1}(y)} = 0$ for any $y \in N$, we know that $\varphi_T|_{\hat{\pi}^{-1}(y)} = \text{const}$ for each $y \in N$ since $\hat{\pi}^{-1}(y)$ is connected. Thus φ_T descends to a bounded function in $PSH(Y, \omega_Y)$, which we will still denote by φ_T . We shall show φ_T is also Hölder continuous in (Y, ω_Y) with a possible different Hölder component.

Lemma 3.4 *There exists a uniform constant $C > 0$ such that*

$$|\varphi_T(p) - \varphi_T(q)| \leq Cd_{\omega_Y}(p, q)^{\delta_Y}, \quad \forall p, q \in Y, \tag{3.8}$$

where $\delta_Y = \min\{\delta(1 - \delta), \delta^2\} \in (0, 1)$.

Proof We denote the zero section of the $\mathbb{C}\mathbb{P}^{k-1}$ -bundle $\hat{\pi} : E \rightarrow N$ by \hat{N} , and it is well-known that $\hat{N} \cong N$. It suffices to show (3.8) for p, q in a fixed tubular neighborhood $T(N)$ of N , since on $Y \setminus T(N)$ the metric $\pi^*\omega_Y = \hat{\omega}_Y$ is equivalent to ω_0 , and the estimate follows from Corollary 3.1.

Choose coordinates charts $(V_\alpha, w_{\alpha,j})$ covering $T(N)$ such that $V_\alpha \cap N = \{w_{\alpha,1} = \dots = w_{\alpha,k} = 0\}$. We also assume that any $p, q \in T(N)$ with $d_{\omega_Y}(p, q) \leq 1$ lie in

the same V_α , if the charts are chosen sufficiently fine. We will work in a fixed chart (V, w_i) , and omit the subscript α . On this open set ω_Y is equivalent to the Euclidean metric $\omega_{\mathbb{C}^n}$ in (\mathbb{C}^n, w_i) , so without loss of generality, we may assume $\omega_Y = \omega_{\mathbb{C}^n}$ on V . The map $\pi : U \rightarrow V$ can be locally expressed as

$$w_1 = z_1, w_2 = z_1 z_2, \dots, w_k = z_1 z_k, w_{k+1} = z_{k+1}, \dots, w_n = z_n, \tag{3.9}$$

where (U, z_i) is an open chart on X . The zero section \hat{N} can be locally expressed as $\hat{N} \cap U = \{z_1 = \dots = z_k = 0\}$.

We consider different cases depending on the positions of p, q in V . Denote $0 < d = d_{\omega_Y}(p, q) \leq 1/4$.

• **Case 1** we assume $p, q \in N$. Take the unique pre-images under $\hat{\pi}$ of p, q in \hat{N} , \hat{p}, \hat{q} , respectively. We know that $\varphi_T(p) = \underline{\varphi_T}(\hat{p})$ and $\varphi_T(q) = \varphi_T(\hat{q})$. The line segment $\overline{p\hat{q}}$ is contained in N and similarly $\overline{\hat{p}\hat{q}}$ is contained in \hat{N} as well. From the local expressions (2.3) and (2.4) of $\omega_X := \pi^*\omega_Y + \varepsilon_{0i} \partial \bar{\partial} \log \sigma_X$, we conclude that $d_{\omega_Y}(p, q) = L_{\omega_Y}(\overline{p\hat{q}})$ is comparable to $L_{\omega_X}(\overline{\hat{p}\hat{q}})$, which is no less than $c_1 d_{\omega_0}(\hat{p}, \hat{q})$, for some uniform $c_1 > 0$. Therefore

$$|\varphi_T(p) - \varphi_T(q)| = |\varphi_T(\hat{p}) - \varphi_T(\hat{q})| \leq C d_{\omega_0}(\hat{p}, \hat{q})^\delta \leq C d_{\omega_Y}(p, q)^\delta,$$

as desired.

• **Case 2** if $0 < \min\{d_{\omega_Y}(p, N), d_{\omega_Y}(q, N)\} \leq 2d^{1-\delta}$. Take the orthogonal projections of p and q to N , p', q' respectively. In other words, p' (q' resp.) has the same (w_{k+1}, \dots, w_n) -coordinates as p (q resp.) but the first k -coordinates are zero. From the assumption we know that $d_{\omega_Y}(p, p') = L_{\omega_Y}(\overline{pp'}) \leq 3d^{1-\delta}$ and $d_{\omega_Y}(q, q') = L_{\omega_Y}(\overline{qq'}) \leq 3d^{1-\delta}$. The pull-back of the line segment $\overline{pp'}$ under π is also a line segment $\overline{\pi^{-1}(p)\hat{p}'}$ in (U, z_i) connecting $\pi^{-1}(p)$ and a unique point $\hat{p}' \in \hat{\pi}^{-1}(p') \subset E$, and $\hat{p}' = (0, \frac{w_2}{w_1}, \dots, \frac{w_k}{w_1}, w_{k+1}, \dots, w_n)$, where w_j denotes the w_j -coordinate at p . It holds that $\varphi_T(p') = \varphi_T(\hat{p}')$ since \hat{p}' lies at the fiber over p' . Again from the local expressions (2.3) and (2.4) of ω_X , it follows that $L_{\omega_X}(\overline{\pi^{-1}(p)\hat{p}'})$ is comparable to the length of $\overline{pp'}$ under ω_Y , therefore

$$d_{\omega_0}(\pi^{-1}(q), \hat{p}') \leq C L_{\omega_X}(\overline{\pi^{-1}(p)\hat{p}'}) \leq C L_{\omega_Y}(\overline{pp'}) \leq C d^{1-\delta},$$

from which we derive that

$$|\varphi_T(p) - \varphi_T(p')| = |\varphi_T(\pi^{-1}(p)) - \varphi_T(\hat{p}')| \leq C d_{\omega_0}(\pi^{-1}(p), \hat{p}')^\delta \leq C d^{\delta_Y}.$$

Similar estimate also holds for $|\varphi_T(\pi^{-1}(q)) - \varphi_T(q')|$. Since $p', q' \in E$ and $d_{\omega_Y}(p', q') \leq d$, by Case 1 we also have $|\varphi_T(p') - \varphi_T(q')| \leq C d^\delta$. The desired estimate (3.8) in this case then follows from triangle inequality.

• **Case 3** $\min\{d_{\omega_Y}(p, N), d_{\omega_Y}(q, N)\} > 2d^{1-\delta}$. The line segment $\gamma(s) = \overline{pq}$ is strictly away from N , in fact, $\sigma_Y(\gamma(s)) \geq d^{2(1-\delta)}$. Therefore the pull-back $\hat{\gamma}(s) = \pi^{-1}(\gamma(s))$

joins $\pi^{-1}(p)$ to $\pi^{-1}(q)$ and $\sigma_X(\hat{\gamma}(s)) \geq d^{2(1-\delta)}$. From (2.5) that $\omega_X \leq C \frac{\pi^* \omega_Y}{\sigma_X}$ on $X \setminus E$ we have

$$d_{\omega_0}(\pi^{-1}(p), \pi^{-1}(q)) \leq CL_{\omega_X}(\hat{\gamma}) \leq \frac{C}{d^{1-\delta}} L_{\omega_Y}(\gamma) \leq Cd^\delta.$$

Therefore

$$|\varphi_T(p) - \varphi_T(q)| = |\varphi_T(\pi^{-1}(p)) - \varphi_T(\pi^{-1}(q))| \leq Cd_{\omega_0}(\pi^{-1}(p), \pi^{-1}(q))^\delta \leq Cd^{\delta^2} \leq Cd^{\delta_Y},$$

as desired.

Combining the cases discussed above, we finish the proof of the lemma. □

The positive $(1, 1)$ -form $\omega_T = \omega_Y + i\partial\bar{\partial}\varphi_T$ defines a Kähler metric g_T on $Y \setminus N$, with the associated function $\tilde{d}_T : Y \setminus N \times Y \setminus N \rightarrow [0, \infty)$ defined by

$$\tilde{d}_T(p, q) := \inf \left\{ \int_{\gamma \setminus N} \sqrt{g_T(\gamma', \gamma')} \mid \gamma \subset Y \text{ and } \gamma \text{ joins } p \text{ to } q \right\}$$

for any $p, q \in Y \setminus N$ and γ is taken over all piecewise smooth curves in Y with only finitely many intersections with N . With this distance function, $(Y \setminus N, \tilde{d}_T)$ becomes a metric space, which may not be complete. We want to extend the distance function to the whole Y . To begin with, we need a trick from [8].

Lemma 3.5 *There exists a uniform constant $C > 0$ such that for any $p \in Y \setminus N$ and $r_p = d_{\omega_Y}(p, N) > 0$*

$$\tilde{d}_T(p, q) \leq Cd_{\omega_Y}(p, q)^{\delta_Y/2}, \quad \forall q \in B_{\omega_Y}(p, r_p/2).$$

Proof The ball $B := B_{\omega_Y}(p, r_p/2)$ is strictly away from N so ω_T is smooth on B . The function $d_p(x) = \tilde{d}_T(p, x)$ is Lipschitz continuous and satisfies $|\nabla d_p|_{\omega_T} \leq 1$ a.e.. For any $r \leq \frac{r_p}{2}$, we have

$$\begin{aligned} \int_{B_{\omega_Y}(p,r)} |\nabla d_p|_{\omega_Y}^2 \omega_Y^n &\leq \int_{B_{\omega_Y}(p,r)} |\nabla d_p|_{\omega_T}^2 (\text{tr}_{\omega_Y} \omega_T) \omega_Y^n \\ &\leq \int_{B_{\omega_Y}(p,r)} (n + \Delta_{\omega_Y} \varphi_T) \omega_Y^n \\ &\leq Cr^{2n} + \int_{B_{\omega_Y}(p,1.5r)} |\varphi_T(x) - \varphi_T(p)| |\Delta_{\omega_Y} \eta| \omega_Y^n \\ &\leq Cr^{2n} + Cr^{\delta_Y+2n-2} \leq Cr^{2n-2+\delta_Y}, \end{aligned}$$

where η is a standard cut-off function supported in $B_{\omega_Y}(p, 1.5r)$ and identically equal to 1 on $B_{\omega_Y}(p, r)$, and it satisfies $|\Delta_{\omega_Y} \eta| \leq Cr^{-2}$. Then by Poincare inequality and Campanato’s lemma (see Theorem 3.1 in [7]) we get

$$\tilde{d}_T(p, q) = d_p(q) = |d_p(q) - d_p(p)| \leq C d_{\omega_Y}(p, q)^{\delta_Y/2},$$

for any $q \in B_{\omega_Y}(p, r_p/2)$. □

Lemma 3.6 *There exist constants $C > 0$ and $\delta_0 \in (0, 1)$ such that*

$$\tilde{d}_T(p, q) \leq C d_{\omega_Y}(p, q)^{\delta_0}, \quad \forall p, q \in Y \setminus N. \tag{3.10}$$

Proof We use the same notation as in the proof of Lemma 3.4. It suffices to show (3.10) for $p, q \in V$ where V is a fixed coordinate chart in Y and recall locally the map $\pi : U \rightarrow V$ is given by (3.9). Let $d = d_{\omega_Y}(p, q) < 1/4$.

In case $\min\{d_{\omega_Y}(p, N), d_{\omega_Y}(q, N)\} > 2d$, then $q \in B_{\omega_Y}(p, \frac{1}{2}d_{\omega_Y}(p, N))$. By Lemma 3.5, it follows that $\tilde{d}_T(p, q) \leq C d^{\delta_Y/2}$. So it only remains to consider the case when the minimum above is $\leq 2d$. Let $p', q' \in N \cap V$ be the orthogonal projection (assuming $\omega_Y = \omega_{\mathbb{C}^n}$) of p, q to N , respectively. Then $\max\{d_{\omega_Y}(p, p'), d_{\omega_Y}(q, q')\} \leq 3d$ and $d_{\omega_Y}(p', q') \leq d$. Choose the unique pre-images $\hat{p}, \hat{q} \in \hat{N} \subset E$ in the zero section \hat{N} of the bundle $\hat{\pi} : E \rightarrow N$, of p', q' , i.e. $\hat{\pi}(\hat{p}) = p'$ and $\hat{\pi}(\hat{q}) = q'$. From the local expressions (2.3) and (2.4) of $\omega_X = \pi^*\omega_Y + \varepsilon_0 i \partial \bar{\partial} \log \sigma_X$, it can be shown that $d_{\omega_X}(\hat{p}, \hat{q}) \leq C d_{\omega_Y}(p', q') \leq Cd$. As in the proof of Case 1 in Lemma 3.2 with p, q in that lemma replaced by \hat{p}, \hat{q} here. Recall that the piecewise line segment γ which connects \hat{p} and \hat{q} lies outside E , except the two end points. Furthermore γ is chosen independent of $t \in [0, T)$ and we have

$$\int_{\gamma} \sqrt{g_t(\gamma', \gamma')} = L_{\omega_t}(\gamma) \leq C d_{\omega_X}(\hat{p}, \hat{q})^{\delta} \leq C d^{\delta}, \tag{3.11}$$

since $g_t \rightarrow g_T$ (locally) smoothly on $\gamma \setminus \{\hat{p}, \hat{q}\}$, letting $t \rightarrow T^-$ and applying Fatou's lemma to (3.11), we get

$$\int_{\gamma} \sqrt{g_T(\gamma', \gamma')} \leq C d^{\delta}.$$

Denote the image curve $\gamma_0 = \pi(\gamma) \subset Y$ which joins p' to q' and is contained in $Y \setminus N$ except the end points. It follows then that $L_{\omega_T}(\gamma_0) \leq C d^{\delta}$. The line segment $\gamma_1(s) = \overline{pp'}$ is given by

$$\gamma_1(s) = (s w_1(p), \dots, s w_k(p), w_{k+1}(p), \dots, w_n(p)), \quad s \in [0, 1]$$

and its pull-back to X , $\hat{\gamma}_1(s) = \pi^{-1}(\gamma_1(s))$ is locally given by

$$\hat{\gamma}_1(s) = (s w_1(p), \frac{w_2(p)}{w_1(p)}, \dots, \frac{w_k(p)}{w_1(p)}, w_{k+1}(p), \dots, w_n(p)), \quad s \in [0, 1].$$

By the estimate in Lemma 3.1, it follows that

$$\int_{\hat{\gamma}_1} \sqrt{g_t(\hat{\gamma}'_1, \hat{\gamma}'_1)} \leq C \int_{\hat{\gamma}_1} \sqrt{\frac{g_X(\hat{\gamma}'_1, \hat{\gamma}'_1)}{s^{2(1-\delta)} |\mathbf{w}(p)|^{2(1-\delta)}}} \leq C \int_{\hat{\gamma}_1} \frac{\sqrt{\pi^* \omega_Y(\hat{\gamma}'_1, \hat{\gamma}'_1)}}{s^{1-\delta} |\mathbf{w}(p)|^{1-\delta}} \leq C |\mathbf{w}(p)|^{\delta} \leq C d^{\delta},$$

where $\mathbf{w}(p) = (w_1(p), \dots, w_k(p))$. By Fatou’s lemma and letting $t \rightarrow T^-$, we get

$$L_{\omega_T}(\gamma_1) = \int_{\hat{\gamma}_1} \sqrt{g_T(\hat{\gamma}'_1, \hat{\gamma}'_1)} \leq Cd^\delta.$$

Similarly the line segment $\gamma_2 = \overline{qq'}$ also have $L_{\omega_T}(\gamma_2) \leq Cd^\delta$. Now we define a piecewise smooth curve

$$\tilde{\gamma} = \gamma_1 + \gamma_0 + \gamma_2,$$

which joins p to q and lies entirely outside N , except the two points p' and q' . And combining the estimates above we get

$$L_{\omega_T}(\tilde{\gamma}) = L_{\omega_T}(\gamma_1) + L_{\omega_T}(\gamma_0) + L_{\omega_T}(\gamma_2) \leq Cd^\delta.$$

Then by definition

$$\tilde{d}_T(p, q) \leq L_{\omega_T}(\tilde{\gamma}) \leq Cd^\delta = Cd_{\omega_Y}(p, q)^\delta.$$

From the discussions above, (3.10) follows for $\delta_0 = \min(\delta_Y/2, \delta)$. □

We now extend the distance function \tilde{d}_T to Y , for any $p \in Y \setminus N$ and $q \in N$, we define the distance

$$d_T(p, q) := \lim_{i \rightarrow \infty} \tilde{d}_T(p, q_i), \tag{3.12}$$

where $\{q_i\} \subset Y \setminus N$ is a sequence of points such that $d_{\omega_Y}(q, q_i) \rightarrow 0$. We need to justify d_T is well-defined, i.e. the limit exists and is independent of the choice of the sequence $\{q_i\}$.

Lemma 3.7 *The limit in (3.12) exists and for any other sequence $\{q'_i\} \subset Y \setminus N$ converging to q in (Y, ω_Y) , the following holds*

$$\lim_{i \rightarrow \infty} \tilde{d}_T(p, q_i) = \lim_{i \rightarrow \infty} \tilde{d}_T(p, q'_i).$$

Proof This is in fact an immediate consequence of Lemma 3.6. Observe that

$$|\tilde{d}_T(p, q_i) - \tilde{d}_T(p, q_j)| \leq \tilde{d}_T(q_i, q_j) \leq Cd_{\omega_Y}(q_i, q_j)^{\delta_0} \rightarrow 0, \text{ as } i, j \rightarrow \infty.$$

Thus $\{\tilde{d}_T(p, q_i)\}_{i=1}^\infty$ is a Cauchy sequence hence it converges. On the other hand, similarly we have

$$|\tilde{d}_T(p, q_i) - \tilde{d}_T(p, q'_i)| \leq Cd_{\omega_Y}(q_i, q'_i)^\delta \rightarrow 0, \text{ as } i \rightarrow \infty,$$

and it then follows that the limit is independent of the choice of $\{q_i\}$ converging to q . □

We then define the distance between points in N as follows: for any $p, q \in N$

$$d_T(p, q) := \lim_{i \rightarrow \infty} \tilde{d}_T(p_i, q_i),$$

for two sequences $Y \setminus N \supset \{p_i\} \rightarrow p$ and $Y \setminus N \supset \{q_i\} \rightarrow q$ under d_{ω_Y} . It can be checked similar as Lemma 3.7 that the limit exists and is independent of the choice of sequences converging to p or q . Thus (Y, d_T) defines a compact metric space, since $d_T(p, q) \leq Cd_{\omega_Y}(p, q)^{\delta_0}$ for any $p, q \in Y$, which follows from Lemma 3.6.

We now turn to the Gromov–Hausdorff (GH) convergence of the flow. The proof is motivated by [15] (see also [3,5,29]).

Lemma 3.8 *For any $t_i \rightarrow T^-$, there exists a subsequence which we still denote by $\{t_i\}$ such that as compact metric spaces*

$$(X, \omega_{t_i}) \xrightarrow{d_{GH}} (Z, d_Z)$$

for some compact metric space (Z, d_Z) .

Proof For any $\epsilon > 0$, we choose an ϵ -net $\{x_j\}_{j=1}^{N_{i,\epsilon}} \subset (X, \omega_{t_i})$, in the sense that $d_{\omega_{t_i}}(x_j, x_{j'}) > \epsilon$ and the open balls $\{B_{\omega_{t_i}}(x_j, 2\epsilon)\}_j$ cover (X, ω_{t_i}) . From Lemma 3.2, we have

$$\epsilon < d_{\omega_{t_i}}(x_j, x_{j'}) \leq Cd_{\omega_0}(x_j, x_{j'})^\delta,$$

thus under the fixed metric d_{ω_0} , each pair of points $(x_j, x_{j'})$ from the ϵ -net has distance at least $C^{-1/\delta}\epsilon^{1/\delta}$, thus the balls $\{B_{\omega_0}(x_j, C^{-1/\delta}\epsilon^{1/\delta}/2)\}_j$ are disjoint, so for some $c > 0$

$$N_{i,\epsilon}c\epsilon^{1/\delta^{2n}} = \sum_{j=1}^{N_{i,\epsilon}} c\epsilon^{1/\delta^{2n}} \leq \int_{\cup_j B_{\omega_0}(x_j, C^{-1/\delta}\epsilon^{1/\delta}/2)} \omega_0^n \leq \int_X \omega_0^n,$$

from which we derive an upper bound of $N_{i,\epsilon} \leq N_\epsilon$, which is independent of i . Then by Gromov’s precompactness theorem [4], there exists a compact metric space (Z, d_Z) , such that up to a subsequence $(X, \omega_{t_i}) \xrightarrow{d_{GH}} (Z, d_Z)$. □

Lemma 3.9 *There exists an open and dense subset $Z^\circ \subset Z$ such that (Z°, d_Z) and $(Y \setminus N, d_T)$ are homeomorphic and locally isometric.*

Proof For notational convenience we denote $Y^\circ = Y \setminus N$. The maps $\pi_i = \pi : (X, \omega_{t_i}) \rightarrow (Y, \omega_Y)$ are Lipschitz by the estimate $\pi^*\omega_Y \leq C\omega_{t_i}$ as in (ii) of Lemma 2.2. The target space (Y, ω_Y) is compact, so by Arzela–Ascoli theorem up to a subsequence of $\{t_i\}$, along the GH convergence $(X, \omega_{t_i}) \xrightarrow{d_{GH}} (Z, d_Z)$, the maps $\pi_i \xrightarrow{GH} \pi_Z$, for some map $\pi_Z : (Z, d_Z) \rightarrow (Y, \omega_Y)$, in the sense that for any $(X, \omega_{t_i}) \ni x_i \xrightarrow{d_{GH}} z \in Z, \pi_i(x_i) \xrightarrow{d_{\omega_Y}} \pi_Z(z)$ in Y . π_Z is also Lipschitz from (Z, d_Z)

to (Y, ω_Y) , i.e. $d_{\omega_Y}(\pi_Z(z_1), \pi_Z(z_2)) \leq Cd_Z(z_1, z_2)$ for any $z_1, z_2 \in Z$. We denote $Z^\circ = \pi_Z^{-1}(Y^\circ)$, and we will show that $\pi_Z|_{Z^\circ} : (Z^\circ, d_Z) \rightarrow (Y^\circ, d_T)$ is homeomorphic and locally isometric, and $Z^\circ \subset Z$ is open and dense. The openness of $Z^\circ \subset Z$ follows from the continuity of the map $\pi_Z : (Z, d_Z) \rightarrow (Y, d_{\omega_Y})$ and the fact that $Y^\circ \subset Y$ is open.

• **$\pi_Z|_{Z^\circ}$ is injective:** suppose $z_1, z_2 \in Z^\circ = \pi_Z^{-1}(Y^\circ)$ are mapped to the same point $y \in Y^\circ$, $\pi_Z(z_1) = \pi_Z(z_2) = y$. Since (Y°, ω_T) is an incomplete smooth Riemannian manifold and locally in Y° , d_T is induced from the Riemannian metric, we can find a small $r = r_y > 0$ such that the metric ball $(B_{\omega_T}(y, 2r), \omega_T)$ is geodesically convex. Choose two sequence of points $z_{1,i}, z_{2,i} \in (X, \omega_i)$ converging in GH sense to $z_1, z_2 \in Z$, respectively. From the convergence of $\pi_i \xrightarrow{GH} \pi_Z$, we obtain $d_{\omega_Y}(\pi_i(z_{1,i}), \pi_Z(z_1)) \xrightarrow{i \rightarrow \infty} 0$ and $d_{\omega_Y}(\pi_i(z_{2,i}), \pi_Z(z_2)) \xrightarrow{i \rightarrow \infty} 0$. By Lemma 3.6, the same limits hold with d_{ω_Y} replaced by d_T . In particular this implies that $d_T(\pi_i(z_{1,i}), \pi_i(z_{2,i})) \xrightarrow{i \rightarrow \infty} 0$ and both $\pi_i(z_{1,i})$ and $\pi_i(z_{2,i})$ lie inside $B_{\omega_T}(y, r/2)$ when i is large enough. We can find ω_T -geodesics $\gamma_i \subset B_{\omega_T}(y, r)$ connecting $\pi_i(z_{1,i})$ and $\pi_i(z_{2,i})$, and by the uniform and smooth convergence of $\omega_i \rightarrow \omega_T$ on $\pi^{-1}(B_{\omega_T}(y, 2r))$, it follows that

$$0 \leq d_{\omega_i}(z_{1,i}, z_{2,i}) \leq L_{\omega_i}(\hat{\gamma}_i) \leq L_{\omega_T}(\gamma_i) + \epsilon_i = d_T(\pi(z_{1,i}), \pi(z_{2,i})) + \epsilon_i \xrightarrow{i \rightarrow \infty} 0,$$

where $\hat{\gamma}_i = \pi^{-1}(\gamma_i)$ is a curve joining $z_{1,i}$ to $z_{2,i}$ and $\{\epsilon_i\}$ is a sequence tending to zero. From the definition of GH convergence we see that

$$d_Z(z_1, z_2) = \lim_{i \rightarrow \infty} d_{\omega_i}(z_{1,i}, z_{2,i}) = 0.$$

Hence $z_1 = z_2$ and $\pi_Z|_{Z^\circ}$ is injective.

• **$\pi_Z|_{Z^\circ} : (Z^\circ, d_Z) \rightarrow (Y^\circ, d_T)$ is a local isometry.** We first explain what the local isometry means. It says that for any $z \in Z^\circ$ and $y = \pi_Z(z) \in Y^\circ$, we can find open sets $z \in U \subset Z^\circ$ and $y \in V \subset Y^\circ$ such that $\pi_Z|_U : (U, d_Z) \rightarrow (V, d_T)$ is an isometry.

There exists a small $r = r_y > 0$ such that the metric ball $(B_{\omega_T}(y, 3r), \omega_T) \subset Y^\circ$ and is geodesically convex. Take $U = (\pi_Z|_{Z^\circ})^{-1}(B_{\omega_T}(y, r))$. Since $B_{\omega_T}(y, r)$ is also open in (Y, ω_Y) , it can be seen that U is open in Z° and is a neighborhood of $z \in Z^\circ$. We will show $\pi_Z|_U : (U, d_Z) \rightarrow (B_{\omega_T}(y, r), \omega_T)$ is an isometry, i.e. for any $z_1, z_2 \in U$, and $y_1 = \pi_Z(z_1), y_2 = \pi_Z(z_2)$, we have $d_Z(z_1, z_2) = d_T(y_1, y_2)$.

We choose sequences of points $z_{1,i}, z_{2,i} \in (X, \omega_i)$ converging in GH sense to z_1, z_2 , respectively, as before. It then follows from $\pi_i \xrightarrow{GH} \pi_Z$ and Lemma 3.6 that $d_T(\pi_i(z_{a,i}), y_a) \rightarrow 0$ as $i \rightarrow \infty$, for each $a = 1, 2$. In particular when i is large enough, $\pi_i(z_{a,i}) \in B_{\omega_T}(y, 1.1r)$. Choose a minimal ω_i -geodesic $\hat{\gamma}_i$ joining $z_{1,i}$ to $z_{2,i}$, and we have

$$d_{\omega_i}(z_{1,i}, z_{2,i}) = L_{\omega_i}(\hat{\gamma}_i) \xrightarrow{i \rightarrow \infty} d_Z(z_1, z_2).$$

Denote the image $\gamma_i = \pi_i(\hat{\gamma}_i)$ which is a continuous curve joining $\pi_i(z_{1,i})$ to $\pi_i(z_{2,i})$. If $\gamma_i \subset B_{\omega_T}(y, 3r)$ (for a subsequence of i), since ω_{t_i} converge smoothly and uniformly to ω_T on the compact subset $\pi^{-1}(B_{\omega_T}(y, 3r))$, it follows

$$d_T(\pi_i(z_{1,i}), \pi_i(z_{2,i})) \leq L_{\omega_T}(\gamma_i) \leq L_{\omega_{t_i}}(\hat{\gamma}_i) + \epsilon_i \xrightarrow{i \rightarrow \infty} d_Z(z_1, z_2).$$

In case $\gamma_i \not\subset B_{\omega_T}(y, 3r)$ for i large enough, we have

$$\begin{aligned} d_T(\pi_i(z_{1,i}), \pi_i(z_{2,i})) &\leq 2.5r \leq L_{\omega_T}(\gamma_i \cap B_{\omega_T}(y, 3r)) \\ &\leq L_{\omega_{t_i}}(\gamma_i) + \epsilon_i \xrightarrow{i \rightarrow \infty} d_Z(z_1, z_2). \end{aligned}$$

Observe that $d_T(\pi_i(z_{1,i}), \pi_i(z_{2,i})) \xrightarrow{i \rightarrow \infty} d_T(\pi_Z(z_1), \pi_Z(z_2)) = d_T(y_1, y_2)$. So by the discussion in both cases, it follows that $d_T(y_1, y_2) \leq d_Z(z_1, z_2)$. To see the reverse inequality, by the geodesic convexity of $(B_{\omega_T}(y, 3r), \omega_T)$, we can find minimal ω_T -geodesics $\sigma_i \subset B_{\omega_T}(y, 3r)$ connecting $\pi_i(z_{1,i})$ and $\pi_i(z_{2,i})$ for i large enough. The pull-back $\hat{\sigma}_i = \pi^{-1}(\sigma_i) \subset B_{\omega_T}(y, 3r)$ joins $z_{1,i}$ to $z_{2,i}$, again by the local smooth convergence of ω_{t_i} to ω_T , we have

$$\begin{aligned} d_{\omega_{t_i}}(z_{1,i}, z_{2,i}) &\leq L_{\omega_{t_i}}(\hat{\sigma}_i) \leq L_{\omega_T}(\sigma_i) + \epsilon_i \\ &= d_T(\pi_i(z_{1,i}), \pi_i(z_{2,i})) + \epsilon_i \xrightarrow{i \rightarrow \infty} d_T(y_1, y_2), \end{aligned}$$

letting $i \rightarrow \infty$ we get $d_Z(z_1, z_2) \leq d_T(y_1, y_2)$. Thus we show that $d_Z(z_1, z_2) = d_T(y_1, y_2)$, as desired.

• **$\pi_Z|_{Z^\circ}$ is surjective.** This follows from the definition. Indeed, for any $y \in Y^\circ$, take $z = z_i = \pi^{-1}(y) \in (X, \omega_{t_i})$, up to a subsequence $z_i \xrightarrow{d_{GH}} z_0 \in Z$. Since $\pi_i \xrightarrow{GH} \pi_Z$, we get $d_{\omega_Y}(y, \pi_Z(z_0)) = d_{\omega_Y}(\pi_i(z_i), \pi_Z(z_0)) \rightarrow 0$ as $i \rightarrow \infty$. So $\pi_Z(z_0) = y$ and $z_0 \in Z^\circ$ is the pre-image of y under $\pi_Z|_{Z^\circ}$.

Combining the discussions above, we see that $\pi_Z|_{Z^\circ} : (Z^\circ, d_Z) \rightarrow (Y^\circ, d_T)$ is a bijection and thus a homeomorphism (noting that the continuity of the maps $\pi_Z|_{Z^\circ}$ and $(\pi_Z|_{Z^\circ})^{-1}$ follow from the local isometry property).

It only remains to show $Z^\circ \subset Z$ is dense. Suppose not, there exists a point $z_0 \in Z$ such that $B_{d_Z}(z_0, \bar{\epsilon}) \subset Z \setminus Z^\circ$ for some $\bar{\epsilon} > 0$. Choose a sequence of points $x_i \in (X, \omega_{t_i})$ such that $x_i \xrightarrow{d_{GH}} z_0$. We claim that $d_{\omega_{t_i}}(x_i, E) \rightarrow 0$ as $i \rightarrow \infty$, where E is the exceptional divisor of the blown-down map $\pi : X \rightarrow Y$. If not, then $d_{\omega_{t_i}}(x_i, E) \geq a_0 > 0$ for a sequence of large i 's, by Lemma 3.2, under the fixed metric w_0 , $d_{\omega_0}(x_i, E) \geq C^{-1/\delta} a_0^{1/\delta} > 0$, thus $\{x_i\} \subset K$, for some compact subset $K \Subset X \setminus E$. It then follows that $\pi_i(x_i) \in \pi(K) \Subset Y^\circ$, and this contradicts the fact that $d_{\omega_Y}(\pi_i(x_i), \pi_Z(z_0)) \rightarrow 0$ and $\pi_Z(z_0) \notin Y^\circ$. Therefore, we may assume without loss of generality that $x_i \in E$ for all i . Moreover, from Lemma 3.10 below, we may replace $x_i \in E$ by the point in the same fiber as x_i of the $\mathbb{C}\mathbb{P}^{k-1}$ -bundle $\hat{\pi} : E \rightarrow N$ and the zero section \hat{N} . So we can assume in addition that $x_i \in \hat{N}$. Denote the points

$y_i = \pi_i(x_i) \in N$ and $y_0 = \pi_Z(z_0) \in N$. From $\pi_i \xrightarrow{GH} \pi_Z$ and $x_i \xrightarrow{d_{GH}} z_0$, we have $d_{\omega_Y}(y_i, y_0) \rightarrow 0$ as $i \rightarrow \infty$.

We may choose a coordinates chart (V, w_j) as before, which is centered at y_0 and contains all but finitely many y_i , and $N \cap V = \{w_1 = \dots = w_k = 0\}$. We take an open set (U, z_j) over (V, w_j) , such that the map $\pi : U \rightarrow V$ is expressed as in (3.9). We fix a point $p \in V \setminus N$ whose w -coordinate is $w(p) = (r, 0, \dots, 0)$ for some $r > 0$ to be determined. Take $\hat{p} = \pi^{-1}(p)$ and its z -coordinate is $z(\hat{p}) = (r, 0, \dots, 0)$. The point(s) $\hat{p}_i = \hat{p} \in (X, \omega_{t_i})$ converge (up to a subsequence) in GH sense to some point $p_Z \in Z$, and as above, we have $d_{\omega_Y}(p, \pi_Z(p_Z)) = d_{\omega_Y}(\pi_i(\hat{p}_i), \pi_Z(p_Z)) \rightarrow 0$ as $i \rightarrow \infty$, so $p = \pi(p_Z) \in Y^\circ$ and $p_Z \in Z^\circ$. From the assumption we have $d_Z(z_0, p_Z) \geq \bar{\varepsilon} > 0$. On the other hand, by the local expressions (2.3) and (2.4) of $\omega_X = \pi^*\omega_Y + \varepsilon_0 i \partial \bar{\partial} \log \sigma_X$, we find that line segments $\hat{p}\hat{z}_0 + \hat{z}_0 x_i$ in (U, z_j) have ω_X -length $\leq Cr + \epsilon_i$ for some sequence $\epsilon_i \rightarrow 0$, where we denote $\hat{z}_0 = \pi^{-1}(p_0) \cap \hat{N}$, i.e. \hat{z}_0 is the origin in (U, z_j) . So $d_{\omega_0}(\hat{p}_i, x_i) \leq C(r + \epsilon_i)$ and by Lemma 3.2, $d_{\omega_{t_i}}(\hat{p}_i, x_i) \leq C(r + \epsilon_i)^\delta$. Letting $i \rightarrow \infty$ we get $d_Z(p_Z, z_0) \leq Cr^\delta$. If we choose r small such that $Cr^\delta = \bar{\varepsilon}/2$, we would get a contradiction. Therefore $Z^\circ \subset Z$ is dense. \square

By exactly the same proof of Lemma 3.2 in [20], we have

Lemma 3.10 *There is a uniform constant $C > 0$ such that*

$$\text{diam}(\hat{\pi}^{-1}(y), \omega_t) \leq C(T - t)^{1/3}, \quad \forall t \in [0, T), \text{ and } \forall y \in N.$$

That is to say, the diameters of the fibers of $\hat{\pi} : E \rightarrow N$ degenerate at a uniform rate as $O((T - t)^{1/3})$.

Lemma 3.11 *The map $\pi_Z : (Z, d_Z) \rightarrow (Y, d_T)$ is a homeomorphism.*

Note that the target space is equipped with the metric d_T , not the metric d_{ω_Y} .

Proof From Lemma 3.6, we get for any $z_1, z_2 \in Z$

$$d_T(\pi_Z(z_1), \pi_Z(z_2)) \leq C d_{\omega_Y}(\pi_Z(z_1), \pi_Z(z_2))^{\delta_0} \leq C d_Z(z_1, z_2)^{\delta_0},$$

so the map $\pi_Z : (Z, d_Z) \rightarrow (Y, d_T)$ is continuous.

• **π_Z is injective.** Suppose $z_1, z_2 \in Z$ satisfies $\pi_Z(z_1) = \pi_Z(z_2) = y \in Y$. If $y \in Y^\circ$, then $z_1, z_2 \in Z^\circ$, $z_1 = z_2$ by the injectivity of $\pi_Z|_{Z^\circ}$. So we only need to consider the case $y \in Y \setminus Y^\circ = N$ and thus $z_1, z_2 \in Z \setminus Z^\circ$. Pick sequences of points $x_{1,i}, x_{2,i} \in (X, \omega_{t_i})$ converging in GH sense to z_1, z_2 , respectively. By similar arguments as in the proof of Lemma 2.3, without loss of generality we can assume $x_{1,i}, x_{2,i} \in \hat{N} \subset E$. Denote $y_{1,i} = \pi(x_{1,i})$ and $y_{2,i} = \pi(x_{2,i})$. We then have

$$d_{\omega_Y}(y_{1,i}, y) = d_{\omega_Y}(\pi_i(x_{1,i}), \pi_Z(z_1)) \xrightarrow{i \rightarrow \infty} 0,$$

and similarly $d_{\omega_Y}(y_{2,i}, y) \rightarrow 0$ as well, and this implies that $d_{\omega_Y}(y_{1,i}, y_{2,i}) \rightarrow 0$. Since $x_{1,i}$ and $x_{2,i}$ are both in the zero section \hat{N} , from the local expressions (2.3) and

(2.4) of the metric $\omega_X = \pi^* \omega_Y + \varepsilon_0 i \partial \bar{\partial} \log \sigma_X$, we see that $d_{\omega_X}(x_{1,i}, x_{2,i}) \rightarrow 0$ as $i \rightarrow \infty$. Then by Lemma 3.2 again, we get $d_{\omega_i}(x_{1,i}, x_{2,i}) \leq C d_{\omega_X}(x_{1,i}, x_{2,i})^\delta \rightarrow 0$. Letting $i \rightarrow \infty$ we get $d_Z(z_1, z_2) = 0$, thus $z_1 = z_2$. This proves the injectivity of π_Z .

• π_Z is surjective. This follows from the definition. In fact, we only need to show any $p \in Y \setminus Y^\circ = N$ lies in the image of π_Z . We fix the point $\hat{p} \in \hat{N}$ with $\hat{\pi}(\hat{p}) = p$. $\hat{p}_i = \hat{p} \in (X, \omega_i)$ converge up to subsequence in GH sense to a point $p_Z \in Z$. Then $d_{\omega_Y}(p, \pi_Z(p_Z)) = d_{\omega_Y}(\pi_i(\hat{p}_i), \pi_Z(p_Z)) \rightarrow 0$ by definition of $\pi_i \xrightarrow{GH} \pi_Z$. It then follows that $\pi_Z(p_Z) = p$.

Thus, $\pi_Z : (Z, d_Z) \rightarrow (Y, d_Y)$ is bijective and continuous. It is also a homeomorphism since (Z, d_Z) is compact. \square

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