Fuglede's conjecture holds in $\mathbb{Q}_p$

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Abstract

We prove Fuglede's conjecture in $\mathbb{Q}_p$ which states that a Borel set of positive and finite Haar measure in $\mathbb{Q}_p$ is a spectral set if and only if it tiles $\mathbb{Q}_p$ by translations.

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1 Introduction

Let G be a locally compact abelian group and $\widehat{G}$ its dual group. Denote by $\mathfrak{m}$ (sometimes by dx) the Haar measure on G. Consider a Borel measurable subset Ω in G with $0 < \mathfrak{m}(\Omega) < \infty$. We say that Ω is a *tile* of G by translations if there exists a set $T \subset G$ of translates such that $\sum_{t \in T} 1_\Omega(x - t) = 1$ for almost all $x \in G$, where 1_A denotes the indicator function of a set A. The set T is then called a *tiling complement* of Ω and (Ω, T) is called a *tiling pair*.

In the case of $G = \mathbb{R}$ considered as an additive group, compact sets of positive measure that tile $\mathbb{R}$ by translations were extensively studied [1,29,33]. The simplest case concerns compact sets consisting of finite number of unit intervals all of whose endpoints are integers. This tiling problem can be reformulated in terms of finite subsets of $\mathbb{Z}$ which tile the group $\mathbb{Z}$. See [37] for references on the study of tiling problem in the group $\mathbb{Z}$. There are also investigations on the existence of tiles having infinitely many connected components. A large class of such tiles arises from self-similar constructions by Bandt [4], Gröchenig and Haas [14], Kenyon [27], Lagarias and Wang [32] et al. A structure theorem for bounded tiles in $\mathbb{R}$ is obtained by Lagarias

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and Wang in [33] where it is proved that all tilings of $\mathbb{R}$ by a bounded region (compact set with zero boundary measure) must be periodic, and that the corresponding tiling complements are rational up to affine transformations.

As we shall see, the tiles in the field $\mathbb{Q}_p$ of p-adic numbers have relatively simple structures. By an *almost compact open set* we mean a Borel set $\Omega \subset \mathbb{Q}_p$ such that there exists a compact open set Ω' satisfying

$$\mathfrak{m}(\Omega \backslash \Omega') = \mathfrak{m}(\Omega' \backslash \Omega) = 0.$$

One of our main results, stated below, concerns with the structure of tiles in $\mathbb{Q}_p$.

Theorem 1.1 *Assume that $\Omega \subset \mathbb{Q}_p$ is a Borel set of positive and finite Haar measure. If Ω tiles $\mathbb{Q}_p$ by translations, then it is an almost compact open set.*

In [8], it is proved that any *bounded* tile of $\mathbb{Q}_p$ is an almost compact open set. The proof in [8] is different from the one in the present paper. Without using the theory of distributions, that proof is more direct and easily understandable. But the boundedness is assumed as an extra condition.

We say that a Borel set $\Omega \subset G$ of positive and finite Haar measure is a *spectral set* if there exists a set $\Lambda \subset \widehat{G}$ which is an orthonormal basis of the Hilbert space $L^2(\Omega)$. Such a set Λ is called a *spectrum* of Ω and (Ω, Λ) is called a *spectral pair*.

When $G = \mathbb{R}^n$, Fuglede [12] formulated the following conjecture: *A Borel set $\Omega \subset \mathbb{R}^n$ of positive and finite Lebesgue measure is a spectral set if and only if it is a tile.* Fuglede [12] proved the conjecture in $\mathbb{R}^d$ under the extra assumption that the spectrum or the tiling complement is a lattice of $\mathbb{R}^d$. There are many positive results under different extra assumptions before the work [39] where Tao gave a counterexample: there exists a spectral subset of $\mathbb{R}^d$ with $d \geq 5$ which is not a tile. After that, Matolcsi [28], Matolcsi and Kolountzakis [22,23], Farkas and Révész [10], Farkas, Matolcsi and Móra [11] gave a series of counterexamples which show that both directions of Fuglede's conjecture fail in $\mathbb{R}^d (d \geq 3)$. However, the conjecture is still open in low dimensions $d = 1, 2$. There has been an effort to prove or disprove Fuglede's conjecture in $\mathbb{R}$ and $\mathbb{R}^2$, or for special classes of subsets of $\mathbb{R}^n$ like unions of intervals [5,6,30,31] and convex bodies [15,26].

The Fuglede's conjecture can be generalized to locally compact abelian groups: *A Borel set $\Omega \subset G$ of positive and finite Haar measure is a spectral set if and only if it is a tile.* In its generality, this generalized conjecture is not true. We could rather ask for which groups it holds. The question arises even for finite groups. The counterexamples in $\mathbb{R}^d$, $d \geq 3$ are actually constructed, based on counterexamples in finite groups. Some substantial works have been done for some finite groups [2,13,16,24,25,35].

Let $p \geq 2$ be prime and $\mathbb{Q}_p$ be the field of p-adic numbers. In the present paper, we will prove that Fuglede's conjecture in $\mathbb{Q}_p$ holds.

Theorem 1.2 *Assume that $\Omega \subset \mathbb{Q}_p$ is a Borel set of positive and finite Haar measure. Then Ω is a spectral set if and only if it is a tile of $\mathbb{Q}_p$.*

In [9], the assertion in Theorem 1.2 was proved under the additional assumption that Ω is a compact open set in $\mathbb{Q}_p$, and furthermore, the compact open spectral sets

were characterized by their p-homogeneity (see the definition of p-homogeneity in Section 1 of [9]).

Theorem 1.3 [9, Theorem 1.1] *Let Ω be a compact open set in $\mathbb{Q}_p$. The following statements are equivalent:*

(1) Ω *is a spectral set;*
(2) Ω *tiles $\mathbb{Q}_p$ by translations;*
(3) Ω *is p-homogenous.*

Theorems 1.1 and 1.3 are two main steps towards the proof of Fuglede's conjecture in $\mathbb{Q}_p$. Actually, by Theorems 1.1 and 1.3, to prove Theorem 1.2, it suffices to prove "Ω is a spectral set $\Longrightarrow$ Ω is a tile".

As pointed out in [7, Proposition 3.1], which follows from [17], that (Ω, Λ) is a spectral pair in $\mathbb{Q}_p$ is equivalent to

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \sum_{\lambda \in \Lambda} |\widehat{1_\Omega}|^2(\xi - \lambda) = \mathfrak{m}(\Omega)^2, \tag{1.1}$$

where $\widehat{1_\Omega}$ is the Fourier transform of 1_Ω. By definition, (Ω, T) is a tiling pair of $\mathbb{Q}_p$ means that

$$\sum_{t \in T} 1_\Omega(x - t) = 1, \quad \mathfrak{m}\text{-a.e. } x \in \mathbb{Q}_p. \tag{1.2}$$

A subset E of $\mathbb{Q}_p$ is said to be *uniformly discrete* if E is countable and $\inf_{x,y \in E} |x - y|_p > 0$, where $|\cdot|_p$ denotes the p-adic absolute value on $\mathbb{Q}_p$. Remark that if E is uniformly discrete, then $\text{Card}(E \cap K) < \infty$ for any compact subset K of $\mathbb{Q}_p$ so that

$$\mu_E = \sum_{\lambda \in E} \delta_\lambda \tag{1.3}$$

defines a discrete Radon measure. Observe that both (1.1) and (1.2) are of the form

$$\mu_E * f = w, \tag{1.4}$$

where f is a non-negative integrable function and $w > 0$ is a positive number. Actually, both spectrum Λ and tiling complement T in $\mathbb{Q}_p$ are uniformly discrete (see Proposition 2.3). The above convolution equation (1.4) will be our main concern.

Our proofs of Theorems 1.1 and 1.2 will be based on the analysis of the set of zeros of the Fourier transforms $\widehat{\mu_\Lambda}$ and $\widehat{\mu_T}$, where both μ_Λ and μ_T are considered as Bruhat–Schwartz distributions and their Fourier transforms $\widehat{\mu_\Lambda}$ and $\widehat{\mu_T}$ are also Bruhat–Schwartz distributions.

The article is organized as follows. In Sect. 2, we present preliminaries on the field $\mathbb{Q}_p$ of p-adic numbers and on the $\mathbb{Z}$-module generated by the p^n-th roots of unity. Some useful facts from the theory of Bruhat–Schwartz distributions are also presented. It is proved that a particular structure is shared by the set of zeros of $\widehat{\mu_E}$ for all uniform

discrete sets E. In Sect. 3, we study the functional equation $\mu_E * f = 1$ under the assumption that E is uniformly discrete and f is a non null and non-negative integrable function. Such a relation has strong constraints on f and on E. For example, we prove that $\widehat{f}$ has compact support. Section 4 is devoted to the proof that tiles in $\mathbb{Q}_p$ are all almost compact open and spectral sets in $\mathbb{Q}_p$ are all almost bounded. It is then proved that tiles in $\mathbb{Q}_p$ are spectral sets. The proof that spectral sets in $\mathbb{Q}_p$ are tiles is given in Sect. 5. In the last section, we prove the dual Fuglede's conjecture in $\mathbb{Q}_p$ and give some remarks on the case of higher dimensions.

2 Preliminaries

Our study on the Fuglede's conjecture on $\mathbb{Q}_p$ is strongly related to the following functional equation

$$\mu_E * f = 1 \tag{2.1}$$

where $E \subset \mathbb{Q}_p$ is a uniformly discrete set and $f \in L^1(\mathbb{Q}_p)$ is a non-negative integrable function such that $\int_{\mathbb{Q}_p} f \, d\mathfrak{m} > 0$.

The convolution in (2.1) is understood as a convolution of Bruhat–Schwartz distributions and even as a convolution in the Colombeau algebra of generalized functions. One of reasons is that the Fourier transform of the infinite Radon measure μ_E is not defined for the measure μ_E but for the distribution μ_E. Both spectral sets and tiles are characterized by special cases of the equation (2.1). In fact, (1.1) means that a spectral pair (Ω, Λ) is characterized by (2.1) with $f = \mathfrak{m}(\Omega)^{-2}|\widehat{1_\Omega}|^2$ and $E = \Lambda$; (1.2) means that a tiling pair (Ω, T) is characterized by (2.1) with $f = 1_\Omega$ and $E = T$.

In this section, after having presented some basic facts like the Lebesgue density theorem, the uniform discreteness of spectrum and of tiling complement etc, we will recall some result from [36] on the $\mathbb{Z}$-module generated by p^n-th roots of unity, which is a key for our study. Then we will present some useful facts from the theory of Bruhat–Schwartz distributions and from the theory of the Colombeau algebra of generalized functions. At the end, we will investigate the set of zeros of the Fourier transform $\widehat{\mu_E}$.

2.1 The field $\mathbb{Q}_p$ of *p*-adic numbers

We start with a quick recall of p-adic numbers. Consider the field $\mathbb{Q}$ of rational numbers and a prime $p \geq 2$. Any nonzero number $r \in \mathbb{Q}$ can be written as $r = p^v \frac{a}{b}$ where $v, a, b \in \mathbb{Z}$ and $(p, a) = 1$ and $(p, b) = 1$ [here (x, y) denotes the greatest common divisor of the two integers x and y]. We define $|r|_p = p^{-v_p(r)}$ for $r \neq 0$ and $|0|_p = 0$. Then $|\cdot|_p$ is a non-Archimedean absolute value. That means

(i) $|r|_p \geq 0$ with equality only when $r = 0$;
(ii) $|rs|_p = |r|_p |s|_p$;
(iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ under $|\cdot|_p$. The ring $\mathbb{Z}_p$ of p-adic integers is the set of p-adic numbers with absolute value ≤ 1. A typical element x of $\mathbb{Q}_p$ is of the form

$$x = \sum_{n=v}^{\infty} a_n p^n \quad (v \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \text{ and } a_v \neq 0). \tag{2.2}$$

Here, $v_p(x) := v$ is called the *p-valuation* of x.

A non-trivial additive character on $\mathbb{Q}_p$ is defined by

$$\chi(x) = e^{2\pi i \{x\}}$$

where $\{x\} = \sum_{n=-v_p(x)}^{-1} a_n p^n$ is the fractional part of x in (2.2). From this character we can get all characters χ_ξ of $\mathbb{Q}_p$, by defining $\chi_\xi(x) = \chi(\xi x)$. We remark that

$$\chi(x) = e^{2\pi i k / p^n}, \quad \text{if } x \in \frac{k}{p^n} + \mathbb{Z}_p \ (k, n \in \mathbb{Z}), \tag{2.3}$$

and

$$\int_{p^{-n}\mathbb{Z}_p} \chi(x) dx = 0 \quad \text{for all } n \geq 1. \tag{2.4}$$

The map $\xi \mapsto \chi_\xi$ from $\mathbb{Q}_p$ to $\widehat{\mathbb{Q}}_p$ is an isomorphism. We write $\widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p$ and identify a point $\xi \in \mathbb{Q}_p$ with the point $\chi_\xi \in \widehat{\mathbb{Q}}_p$. For more information on $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}}_p$, the reader is referred to the book [40].

The following notation will be used in the whole paper.

Notation:

- $\mathbb{Z}_p^\times := \mathbb{Z}_p \backslash p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$, the group of units of $\mathbb{Z}_p$.
- $B(0, p^n) := p^{-n}\mathbb{Z}_p$, the (closed) ball centered at 0 of radius p^n.
- $B(x, p^n) := x + B(0, p^n)$.
- $S(x, p^n) := B(x, p^n) \backslash B(x, p^{n-1})$, a "sphere".
- $\mathbb{L} := \{\{x\} : x \in \mathbb{Q}_p\}$, a complete set of representatives of the cosets of the additive subgroup $\mathbb{Z}_p$.
- $\mathbb{L}_n := p^{-n}\mathbb{L}$.

The Lebesgue density theorem holds in $\mathbb{Q}_p$ for the Haar measure.

Proposition 2.1 [34] *Let $\Omega \subset \mathbb{Q}_p$ be a bounded Borel set such that $\mathfrak{m}(\Omega) > 0$. Then*

$$\lim_{n \to \infty} \frac{\mathfrak{m}(B(x, p^{-n}) \cap \Omega)}{\mathfrak{m}(B(x, p^{-n}))} = 1, \quad \mathfrak{m}\text{-a.e. } x \in \Omega.$$

2.2 Fourier Transform

The Fourier transform of $f \in L^1(\mathbb{Q}_p)$ is defined to be

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p} f(x) \overline{\chi_\xi(x)} dx \quad (\forall \xi \in \widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p).$$

A complex function f defined on $\mathbb{Q}_p$ is called *uniformly locally constant* if there exists $n \in \mathbb{Z}$ such that

$$f(x + u) = f(x) \quad \forall x \in \mathbb{Q}_p, \forall u \in B(0, p^n).$$

The following proposition shows that for an integrable function f, having compact support and being uniformly locally constant are dual properties for f and its Fourier transform.

Proposition 2.2 *Let $f \in L^1(\mathbb{Q}_p)$ be a complex-value integrable function.*

(1) *If f has compact support, then $\widehat{f}$ is uniformly locally constant.*
(2) *If f is uniformly locally constant, then $\widehat{f}$ has compact support.*

Proof (1) Suppose that f is supported in $B(0, p^n)$. For any $x \in \mathbb{Q}_p$ and any $u \in B(0, p^{-n})$, we have

$$\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int_{B(0, p^n)} f(y)\overline{\chi(\xi y)}(\overline{\chi(uy) - 1})dy.$$

Notice that if $y \in B(0, p^n)$, we have $|uy|_p \leq 1$. So by (2.3),

$$\chi(uy) - 1 = 0.$$

Therefore, $\widehat{f}(\xi + u) - \widehat{f}(\xi) = 0$ for all $u \in B(0, p^{-n})$. Thus $\widehat{f}$ is uniformly locally constant.

(2) Suppose that $f(x + u) = f(x)$ for any $x \in \mathbb{Q}_p$ and any $u \in B(0, p^n)$. Observing

$$\mathbb{Q}_p = \mathbb{L}_n + B(0, p^n) = \bigcup_{z \in \mathbb{L}_n} B(z, p^n),$$

we deduce

$$\widehat{f}(\xi) = \sum_{z \in \mathbb{L}_n} \int_{B(z, p^n)} f(y)\overline{\chi(\xi y)}dy = \sum_{z \in \mathbb{L}_n} f(z) \int_{B(z, p^n)} \overline{\chi(\xi y)}dy.$$

By (2.4), we have $\int_{B(z, p^n)} \chi(\xi y)dy = 0$ for $|\xi|_p > p^{-n}$. Therefore, $\widehat{f}(\xi) = 0$ for $|\xi|_p > p^{-n}$.   □

The above proposition is the key which will allow us to prove that a tile Ω in $\mathbb{Q}_p$ is an almost compact open set [see Theorem 4.1 (2)], by showing that the support of $\widehat{1_\Omega}$ is compact (see Proposition 3.3). Consequently, a tile is a spectral set by Theorem 1.3.

Now we prove that the spectra and tiling complements are always uniformly discrete.

Proposition 2.3 *Let $\Omega \subset \mathbb{Q}_p$ be a Borel set of positive and finite Haar measure.*

(1) *If (Ω, Λ) is a spectral pair, then Λ is uniformly discrete.*
(2) *If (Ω, T) is a tiling pair, then T is uniformly discrete.*

Proof (1) By the fact $\widehat{1_\Omega}(0) = \mathfrak{m}(\Omega) > 0$ and the continuity of the function $\widehat{1_\Omega}$, there exists an integer n_0 such that $\widehat{1_\Omega}(x) \neq 0$ for all $x \in B(0, p^{n_0})$. This, together with the orthogonality

$$\widehat{1_\Omega}(\lambda - \lambda') = 0 \quad \forall \, \lambda, \lambda' \in \Lambda \text{ distinct,}$$

implies that $|\lambda - \lambda'|_p > p^{n_0}$ for different $\lambda, \lambda' \in \Lambda$.

(2) Consider the continuous function on $\mathbb{Q}_p$ defined by

$$f(x) := 1_\Omega * 1_{-\Omega}(x) = \int_{\mathbb{Q}_p} 1_\Omega(y) 1_\Omega(y - x) dy = \mathfrak{m}(\Omega \cap (\Omega + x)).$$

The fact $f(0) = \mathfrak{m}(\Omega) > 0$ and the continuity of f imply that there exists an integer n_0 such that

$$\mathfrak{m}(\Omega \cap (\Omega + t)) > 0, \quad \text{for } t \in B(0, p^{n_0}). \tag{2.5}$$

For different $t \in T$ and $t' \in T$, the tiling property implies $\mathfrak{m}((\Omega + t) \cap (\Omega + t')) = 0$. So

$$\mathfrak{m}(\Omega \cap (\Omega + t - t')) = \mathfrak{m}((\Omega + t) \cap (\Omega + t')) = 0$$

by the translation invariance of $\mathfrak{m}$. Thus we must have $|t - t'|_p > p^{n_0}$, by (2.5). □

2.3 $\mathbb{Z}$-module generated by p^n-th roots of unity

Let $m \geq 2$ be an integer and let $\omega_m = e^{2\pi i/m}$, which is a primitive m-th root of unity. Denote by $\mathcal{M}_m$ the set of integral points $(a_0, a_1, \ldots, a_{m-1}) \in \mathbb{Z}^m$ such that

$$\sum_{j=0}^{m-1} a_j \omega_m^j = 0.$$

The set $\mathcal{M}_m$ is clearly a $\mathbb{Z}$-module. In the following we assume that $m = p^n$ is a power of a prime number.

Lemma 2.4 [36, Theorem 1] *If $(a_0, a_1, \ldots, a_{p^n-1}) \in \mathcal{M}_{p^n}$, then for any integer $0 \leq i \leq p^{n-1} - 1$ we have $a_i = a_{i+jp^{n-1}}$ for all $j = 0, 1, \ldots, p - 1$.*

Lemma 2.4 has the following two special forms. The first one is an immediate consequence.

Lemma 2.5 *Let* $(b_0, b_1, \ldots, b_{p-1}) \in \mathbb{Z}^p$. *If* $\sum_{j=0}^{p-1} e^{2\pi i b_j / p^n} = 0$, *then subject to a permutation of* $(b_0, \ldots, b_{p-1})$, *there exists* $0 \leq r \leq p^{n-1} - 1$ *such that*

$$b_j \equiv r + jp^{n-1} \pmod{p^n}$$

for all $j = 0, 1, \ldots, p - 1$.

Lemma 2.6 [9, Lemma 2.5] *Let* C *be a finite subset of* $\mathbb{Z}$. *If* $\sum_{c \in C} e^{2\pi i c / p^n} = 0$, *then* $p \mid \mathrm{Card}(C)$ *and* C *is decomposed into* $\mathrm{Card}(C)/p$ *disjoint subsets* $C_1, C_2, \ldots,$ $C_{\mathrm{Card}(C)/p}$, *such that each* C_j *consists of* p *points and*

$$\sum_{c \in C_j} e^{2\pi i c / p^n} = 0.$$

Now applying Lemmas 2.5 and 2.6, we have the following lemma which will be useful in the paper.

Lemma 2.7 *Let* $C \subset \mathbb{Q}_p$ *be a finite set.*

(1) *If* $\sum_{c \in C} \chi(c) = 0$, *then* $p \mid \mathrm{Card}(C)$ *and* $\sum_{c \in C} \chi(xc) = 0$ *for any* $x \in \mathbb{Z}_p^\times$ *(i.e.* $|x|_p = 1$*).*
(2) *If there exists* $\xi \in \mathbb{Q}_p$ *such that* $\sum_{c \in C} \chi(\xi c) = 0$, *then for any* $c \in C$, *there exists* $c' \in C$ *such that* $|c - c'|_p = p/|\xi|_p$.
(3) *If there exists a finite set* $\mathbb{I} \subset \mathbb{Z}$ *such that*

$$\sum_{c \in C} \chi(p^i c) = 0 \quad \text{for all } i \in \mathbb{I}, \tag{2.6}$$

then $p^{\mathrm{Card}(\mathbb{I})} \mid \mathrm{Card}(C)$.

Proof (1) It is a direct consequence of Lemmas 2.5 and 2.6. See [9, Lemma 2.6] for details.
(2) Let $C = \{c_1, c_2, \ldots, c_m\}$. Recall that $\chi(\xi c) = e^{2\pi i \{\xi c\}}$. There exist an integer n and a subset $\{n_1, n_2, \ldots, n_m\}$ of $\mathbb{Z}$ such that

$$\chi(\xi c_k) = e^{2\pi i n_k / p^n}, \quad k = 1, 2, \ldots, m.$$

By Lemma 2.6, $p \mid \mathrm{Card}(C)$ and C is decomposed into $\mathrm{Card}(C)/p$ disjoint subsets $C_1, \ldots, C_{\mathrm{Card}(C)/p}$, such that each C_j consists of p points and

$$\sum_{c \in C_j} \chi(\xi c) = 0.$$

Without loss of generality, assume that $c \in C_1$. By Lemma 2.5, we have

$$|c - c'|_p = \frac{p}{|\xi|_p}, \quad \text{if } c' \in C_1 \backslash \{c\}.$$

(3) Assume that $\mathbb{I} = \{i_1, i_2, \ldots, i_n\}$ with $i_1 < i_2 < \cdots < i_n$. By Lemma 2.6 and the equality (2.6) with $i = i_1$, we have $p \mid \mathrm{Card}(C)$ and C can be decomposed into $\mathrm{Card}(C)/p$ subsets $C_1, C_2, \ldots, C_{\mathrm{Card}(C)/p}$ such that each C_j consists of p points and

$$\sum_{c \in C_j} \chi(p^{i_1}c) = 0.$$

By Lemma 2.5, if c and c' lie in the same C_j, we have

$$|c - c'|_p = p^{1+i_1}. \tag{2.7}$$

Now we consider the equality (2.6) when $i = i_2$. Since $i_1 < i_2$, (2.7) and (2.3) imply that the function

$$c \mapsto \chi(p^{i_2}c)$$

is constant on each C_j. From each C_j, take one element $\widetilde{c}_j$. Let $\widetilde{C}$ be the set consisting of these $\widetilde{c}_j$.

Since each C_j contains p elements, the equality (2.6) with $i = i_2$ is equivalent to

$$\sum_{\widetilde{c} \in \widetilde{C}} \chi(p^{i_2}c) = 0.$$

By Lemma 2.6, $p \mid \mathrm{Card}(\widetilde{C})$, which implies $p^2 \mid \mathrm{Card}(C)$. By induction, we get $p^n \mid \mathrm{Card}(C)$. $\qquad\square$

2.4 Bruhat–Schwartz distributions in $\mathbb{Q}_p$

Here we give a brief description of the theory of Bruhat–Schwartz distributions following [3,38,40]. Let $\mathcal{E}$ denote the space of the uniformly locally constant functions. The space $\mathcal{D}$ of *Bruhat–Schwartz test functions* is, by definition, constituted of uniformly locally constant functions of compact support. Such a test function $f \in \mathcal{D}$ is a finite linear combination of indicator functions of the form $1_{B(x,p^k)}(\cdot)$, where $k \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$. The largest of such numbers k is denoted by $\ell := \ell(f)$ and is called the *parameter of constancy* of f. Since $f \in \mathcal{D}$ has compact support, the minimal number $\ell' := \ell'(f)$ such that the support of f is contained in $B(0, p^{\ell'})$ exists and will be called the *parameter of compactness* of f.

Clearly, $\mathcal{D} \subset \mathcal{E}$. The space $\mathcal{D}$ is provided with a topology of topological vector space as follows: a sequence $\{\phi_n\} \subset \mathcal{D}$ is called a *null sequence* if there is a fixed pair of $l, l' \in \mathbb{Z}$ such that each ϕ_n is constant on every ball of radius p^l and is supported by the ball $B(0, p^{l'})$ and the sequence ϕ_n tends uniformly to zero.

A *Bruhat–Schwartz distribution* f on $\mathbb{Q}_p$ is by definition a continuous linear functional on $\mathcal{D}$. The value of f at $\phi \in \mathcal{D}$ will be denoted by $\langle f, \phi \rangle$. Note that linear functionals on $\mathcal{D}$ are automatically continuous. This property allows us to easily construct distributions. Denote by $\mathcal{D}'$ the space of Bruhat–Schwartz distributions. The space $\mathcal{D}'$ is equipped with the weak topology induced by $\mathcal{D}$.

A locally integrable function f is considered as a distribution: for any $\phi \in \mathcal{D}$,

$$\langle f, \phi \rangle = \int_{\mathbb{Q}_p} f \phi \, d\mathfrak{m}.$$

The discrete measure μ_E defined by (1.3) is also a distribution: for any $\phi \in \mathcal{D}$,

$$\langle \mu_E, \phi \rangle = \sum_{\lambda \in E} \phi(\lambda).$$

Here for each ϕ, the sum is finite because E is uniformly discrete and each ball contains at most a finite number of points in E. Since the test functions in $\mathcal{D}$ are uniformly locally constant and have compact support, the following proposition is a direct consequence of Proposition 2.2 or of the fact (see also [7, Lemma 4]) that

$$\widehat{1_{B(c, p^k)}}(\xi) = \chi(-c\xi) p^k 1_{B(0, p^{-k})}(\xi). \tag{2.8}$$

Proposition 2.8 [38, Chapter II 3] *The Fourier transform $f \mapsto \widehat{f}$ is a homeomorphism from $\mathcal{D}$ onto $\mathcal{D}$.*

The *Fourier transform of a distribution* $f \in \mathcal{D}'$ is a new distribution $\widehat{f} \in \mathcal{D}'$ defined by the duality

$$\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle, \quad \forall \, \phi \in \mathcal{D}.$$

The Fourier transform $f \mapsto \widehat{f}$ is a homeomorphism of $\mathcal{D}'$ onto $\mathcal{D}'$ under the weak topology [38, Chapter II 3].

2.5 Zeros of the Fourier transform of a discrete measure

Let $f \in \mathcal{D}'$ be a distribution in $\mathbb{Q}_p$. A point $x \in \mathbb{Q}_p$ is called a *zero* of f if there exists an integer n_0 such that

$$\langle f, 1_{B(y, p^n)} \rangle = 0, \quad \text{for all } y \in B(x, p^{n_0}) \text{ and all integers } n \le n_0.$$

Denote by $\mathcal{Z}_f$ the set of all zeros of f. Remark that $\mathcal{Z}_f$ is the maximal open set O on which f vanishes, i.e. $\langle f, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$ such that the support of ϕ is contained in O.

The *support* of a distribution f is defined as the complementary set of $\mathcal{Z}_f$ and is denoted by $\mathrm{supp}(f)$.

Let E be a uniformly discrete set in $\mathbb{Q}_p$. The following proposition characterizes the structure of $\mathcal{Z}_{\widehat{\mu_E}}$, the set of zeros of the Fourier transform of the discrete measure μ_E. It is bounded and is a union of spheres centered at 0.

Proposition 2.9 *Let E be a uniformly discrete set in $\mathbb{Q}_p$.*

(1) *If* $\xi \in \mathcal{Z}_{\widehat{\mu_E}}$, *then* $S(0, |\xi|_p) \subset \mathcal{Z}_{\widehat{\mu_E}}$.
(2) *The set* $\mathcal{Z}_{\widehat{\mu_E}}$ *is bounded.*

Proof First remark that by using (2.8) we get

$$\langle \widehat{\mu_E}, 1_{B(\xi, p^{-n})} \rangle = \langle \mu_E, \widehat{1_{B(\xi, p^{-n})}} \rangle = p^{-n} \sum_{\lambda \in E \cap B(0, p^n)} \overline{\chi(\xi\lambda)}. \tag{2.9}$$

This expression will be used several times.

(1) By definition, $\xi \in \mathcal{Z}_{\widehat{\mu_E}}$ implies that there exists an integer n_0 such that

$$\langle \widehat{\mu_E}, 1_{B(\xi, p^{-n})} \rangle = 0, \quad \forall\, n \geq n_0.$$

By (2.9), this is equivalent to

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi\lambda) = 0, \quad \forall\, n \geq n_0. \tag{2.10}$$

For any $\xi' \in S(0, |\xi|_p)$, we have $\xi' = u\xi$ for some $u \in \mathbb{Z}_p^\times$. By Lemma 2.7 (1) and the equality (2.10), we obtain

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi'\lambda) = \sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi\lambda) = 0, \quad \forall\, n \geq n_0.$$

Thus, again by (2.9), $\langle \widehat{\mu_E}, 1_{B(\xi', p^{-n})} \rangle = 0$ for $n \geq n_0$. We have thus proved $S(0, |\xi|_p) \subset \mathcal{Z}_{\widehat{\mu_E}}$.
(2) Fix $\lambda_0 \in E$. By the discreteness of E, there exists an integer n_0 such that

$$\forall \lambda \in E \backslash \{\lambda_0\}, \quad |\lambda - \lambda_0|_p \geq p^{-n_0}.$$

We are going to show that $\mathcal{Z}_{\widehat{\mu_E}} \subset B(0, p^{n_0+1})$, which is equivalent to that $\xi \notin \mathcal{Z}_{\widehat{\mu_E}}$ when $|\xi|_p \geq p^{n_0+2}$. To this end, we will prove that for all integer n large enough such that $\lambda_0 \in B(0, p^n)$ and all ξ such that $|\xi|_p \geq p^{n_0+2}$, we have

$$\langle \widehat{\mu_E}, 1_{B(\xi, p^{-n})} \rangle \neq 0.$$

In fact, if this is not the case, then by (2.9),

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi\lambda) = 0.$$

Thus, by Lemma 2.7 (2), for the given λ_0, we can find $\lambda \in E \cap B(0, p^n)$, such that $|\lambda - \lambda_0|_p = p/|\xi|_p < p^{-n_0}$, a contradiction.

$\square$

Remark that n_0 in the above proof depends only on the structure of E. Define

$$n_E := \max_{\substack{\lambda, \lambda' \in E \\ \lambda \neq \lambda'}} v_p(\lambda - \lambda'). \tag{2.11}$$

According to the above proof of the second assertion of Proposition 2.9, we immediately get

$$\mathcal{Z}_{\widehat{\mu_E}} \subset B(0, p^{n_E+1}). \tag{2.12}$$

2.6 Convolution and multiplication of distributions

Denote

$$\Delta_k := 1_{B(0, p^k)}, \quad \theta_k := \widehat{\Delta}_k = p^k \cdot 1_{B(0, p^{-k})}.$$

Let $f, g \in \mathcal{D}'$ be two distributions. We define the *convolution* of f and g by

$$\langle f * g, \phi \rangle = \lim_{k \to \infty} \langle f(x), \langle g(\cdot), \Delta_k(x)\phi(x + \cdot) \rangle \rangle,$$

if the limit exists for all $\phi \in \mathcal{D}$.

Proposition 2.10 [3, Proposition 4.7.3] *If $f \in \mathcal{D}'$, then $f * \theta_k \in \mathcal{E}$ with the parameter of constancy at least $-k$.*

We define the *multiplication* of f and g by

$$\langle f \cdot g, \phi \rangle = \lim_{k \to \infty} \langle g, (f * \theta_k)\phi \rangle,$$

if the limit exists for all $\phi \in \mathcal{D}$. The above definition of convolution is compatible with the usual convolution of two integrable functions and the definition of multiplication is compatible with the usual multiplication of two locally integrable functions.

The following proposition shows that both the convolution and the multiplication are commutative when they are well defined and the convolution of two distributions is well defined if and only if the multiplication of their Fourier transforms is well defined.

Proposition 2.11 [40, Sections 7.1 and 7.5] *Let $f, g \in \mathcal{D}'$ be two distributions. Then*

(1) *If $f * g$ is well defined, so is $g * f$ and $f * g = g * f$.*
(2) *If $f \cdot g$ is well defined, so is $g \cdot f$ and $f \cdot g = g \cdot f$.*
(3) *$f * g$ is well defined if and only if $\widehat{f} \cdot \widehat{g}$ is well defined. In this case, we have $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ and $\widehat{f \cdot g} = \widehat{f} * \widehat{g}$.*

The following proposition justifies an intuition.

Proposition 2.12 *Let $f, g \in \mathcal{D}'$ be two distributions. If $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, then $f \cdot g$ is well defined and $f \cdot g = 0$.*

Proof Let $\phi \in \mathcal{D}$ with parameter of constancy ℓ and parameter of compactness ℓ'.

By the assumption $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, the distance between the two compact sets $\operatorname{supp}(f) \cap B(0, p^{\ell'})$ and $\operatorname{supp}(g) \cap B(0, p^{\ell'})$ is strictly positive. So there exists an integer $n \leq \ell'$ such that for any $x \in \operatorname{supp}(f) \cap B(0, p^{\ell'})$, the ball $B(x, p^n)$ is contained in $\mathcal{Z}_g \cap B(0, p^{\ell'})$.

Therefore, the compact open set

$$X := \bigcup_{x \in \operatorname{supp}(f) \cap B(0, p^{\ell'})} B(x, p^n)$$

satisfies $\operatorname{supp}(f) \cap B(0, p^{\ell'}) \subset X \subset \mathcal{Z}_g \cap B(0, p^{\ell'})$.

Define $\phi_1(x) := \phi(x) \cdot 1_X(x)$.

It follows that

$$\operatorname{supp}(\phi - \phi_1) \subset B(0, p^{\ell'}) \backslash X \subset \mathcal{Z}_f \cap B(0, p^{\ell'}).$$

Thus, we have

$$
\begin{aligned}
\langle f \cdot g, \phi \rangle = \langle f \cdot g, (\phi - \phi_1 + \phi_1) \rangle &= \langle f \cdot g, \phi - \phi_1 \rangle + \langle f \cdot g, \phi_1 \rangle \\
&= \lim_{k \to \infty} \langle f, (g * \theta_k)(\phi - \phi_1) \rangle + \lim_{k \to \infty} \langle g, (f * \theta_k)\phi_1 \rangle \\
&= 0,
\end{aligned}
$$

where the existence of the last two limits is due to

$$\operatorname{supp}((g * \theta_k)(\phi - \phi_1)) \subset \operatorname{supp}(\phi - \phi_1) \subset \mathcal{Z}_f,$$

and

$$\operatorname{supp}((f * \theta_k)\phi_1) \subset \operatorname{supp}\phi_1 \subset \mathcal{Z}_g.$$

$\square$

The multiplication of some special distributions has a simple form. That is the case for the multiplication of a uniformly locally constant function and a distribution.

Proposition 2.13 [40, Section 7.5, Example 2] *Let $f \in \mathcal{E}$ and let $G \in \mathcal{D}'$. Then for any $\phi \in \mathcal{D}$, we have $\langle f \cdot G, \phi \rangle = \langle G, f\phi \rangle$.*

For a distribution $f \in \mathcal{D}'$, we define its *regularization* by the sequence of test functions [3, Proposition 4.7.4]

$$\Delta_k \cdot (f * \theta_k) \in \mathcal{D}.$$

The regularization of a distribution converges to the distribution with respect to the weak topology.

Proposition 2.14 [3, Lemma 14.3.1] *Let f be a distribution in $\mathcal{D}'$. Then $\Delta_k \cdot (f * \theta_k) \to f$ in $\mathcal{D}'$ as $k \to \infty$. Moreover, for any test function $\phi \in \mathcal{D}$ we have*

$$\langle \Delta_k \cdot (f * \theta_k), \phi \rangle = \langle f, \phi \rangle, \quad \forall k \geq \max\{-\ell, \ell'\},$$

where ℓ and ℓ' are the parameter of constancy and the parameter of compactness of the function ϕ defined in Sect. 2.4.

This approximation of distribution by test functions allows us to construct a space which is bigger than the space of distributions. This larger space is the Colombeau algebra, which will be presented below. Recall that in the space of Bruhat–Schwartz distributions, the convolution and the multiplication are not well defined for all couples of distributions. But in the Colombeau algebra, the convolution and the multiplication are well defined and these two operations are associative.

2.7 Colombeau algebra of generalized functions

Consider the set $\mathcal{P} := \mathcal{D}^{\mathbb{N}}$ of all sequences $\{f_k\}_{k \in \mathbb{N}}$ of test functions. We introduce an algebra structure on $\mathcal{P}$, defining the operations componentwise

$$\{f_k\} + \{g_k\} = \{f_k + g_k\},$$
$$\{f_k\} \cdot \{g_k\} = \{f_k \cdot g_k\},$$

where $\{f_k\}, \{g_k\} \in \mathcal{P}$.

Let $\mathcal{N}$ be the sub-algebra of elements $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{P}$ such that for any compact set $K \subset \mathbb{Q}_p$ there exists $N \in \mathbb{N}$ such that $f_k(x) = 0$ for all $k \geq N$, $x \in K$. Clearly, $\mathcal{N}$ is an ideal in the algebra $\mathcal{P}$.

Then we introduce the *Colombeau-type algebra*

$$\mathcal{G} = \mathcal{P}/\mathcal{N}.$$

The equivalence class of sequences which defines an element in $\mathcal{G}$ will be denoted by $\mathbf{f} = [f_k]$, called a generalized function.

For any $\mathbf{f} = [f_k], \mathbf{g} = [g_k] \in \mathcal{G}$, the addition and multiplication are defined as

$$\mathbf{f} + \mathbf{g} = [f_k + g_k], \quad \mathbf{f} \cdot \mathbf{g} = [f_k \cdot g_k].$$

Obviously, $(\mathcal{G}, +, \cdot)$ is an associative and commutative algebra.

Theorem 2.15 [3, Theorem 14.3.3] *The map $f \mapsto \mathbf{f} = [\Delta_k(f * \theta_k)]$ from $\mathcal{D}'$ to $\mathcal{G}$ is a linear embedding.*

Each distribution $f \in \mathcal{D}'$ is embedded into $\mathcal{G}$ by the mapping which associates f with the generalized function determined by the regularization of f. Thus we obtain that the multiplication defined on $\mathcal{D}'$ is associative in the following sense.

Proposition 2.16 *Let $f, g, h \in \mathcal{D}'$. If $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are well defined as multiplications of distributions, we have*

$$(f \cdot g) \cdot h = f \cdot (g \cdot h).$$

3 Study on $\mu_E * f = 1$

In this section, we will study the following equation

$$\mu_E * f = 1 \tag{3.1}$$

where $0 \leq f \in L^1(\mathbb{Q}_p)$ with $\int_{\mathbb{Q}_p} f \, d\mathfrak{m} > 0$, and E is a uniformly discrete subset of $\mathbb{Q}_p$. Suppose that (E, f) is a solution of (3.1). We will investigate the density of E and even the distribution of E, and the supports of the Fourier transforms $\widehat{\mu_E}$ and $\widehat{f}$.

We say that a uniformly discrete set E has a *bounded density* if the following limit exists for some $x_0 \in \mathbb{Q}_p$

$$D(E) := \lim_{k \to \infty} \frac{\mathrm{Card}(B(x_0, p^k) \cap E)}{\mathfrak{m}(B(x_0, p^k))},$$

which is called *the density* of E. Actually, if the limit exists for some $x_0 \in \mathbb{Q}_p$, then it exists for all $x \in \mathbb{Q}_p$ and the limit is independent of x. In fact, for any $x_0, x_1 \in \mathbb{Q}_p$, when k is large enough such that $|x_0 - x_1|_p < p^k$, we have $B(x_0, p^k) = B(x_1, p^k)$.

For a function $g : \mathbb{Q}_p \to \mathbb{R}$, denote

$$\mathcal{N}_g := \{x \in \mathbb{Q}_p : g(x) = 0\}.$$

If $g \in C(\mathbb{Q}_p)$ is a continuous function, then $\mathcal{N}_g$ is a closed set and $\mathcal{Z}_g$ is the set of interior points of $\mathcal{N}_g$.

But, the support of g as a continuous function is equal to the support of g as a distribution.

The following theorem gets together some properties of the solution (E, f) of the equation (3.1), which will be proved in this section.

Theorem 3.1 *Let $0 \leq f \in L^1(\mathbb{Q}_p)$ with $\int_{\mathbb{Q}_p} f \, d\mathfrak{m} > 0$, and E be a uniformly discrete subset of $\mathbb{Q}_p$. Suppose that the Eq. (3.1) is satisfied by f and E. Then the following statements hold.*

(1) *The support of $\widehat{f}$ is compact.*
(2) *The set $\mathcal{Z}_{\widehat{\mu_E}}$ is bounded and it is the union of the punctured ball $B(0, p^{-n_f}) \backslash \{0\}$ and some spheres.*
(3) *The density $D(E)$ exists and equals to $1 / \int_{\mathbb{Q}_p} f \, d\mathfrak{m}$. Furthermore, there exists an integer $n_f \in \mathbb{Z}$ such that for all integers $n \geq n_f$ we have*

$$\forall \xi \in \mathbb{Q}_p, \quad \mathrm{Card}(E \cap B(\xi, p^n)) = p^n D(E).$$

Theorem 3.1 (1) and (2) will be proved in § 3.1, the distribution of E will be discussed in Sect. 3.2 and the equality $D(E) = 1/\int_{\mathbb{Q}_p} f \, d\mathfrak{m}$ will be proved in Sect. 3.3.

3.1 Compactness of supp$(\widehat{f})$ and structure of $\mathcal{Z}_{\widehat{\mu_E}}$

Our discussion is based on the functional equation $\widehat{f} \cdot \widehat{\mu_E} = \delta_0$, which is implied by $f * \mu_E = 1$ (see Proposition 2.11).

Proposition 3.2 *Let $g \in C(\mathbb{Q}_p)$ be a continuous function and let $G \in \mathcal{D}'$ be a distribution. Suppose that the product $H = g \cdot G$ is well defined. Then*

$$\mathcal{Z}_H \subset \mathcal{N}_g \cup \mathcal{Z}_G.$$

Consequently, if $f * \mu_E = 1$ with $f \in L^1(\mathbb{Q}_p)$, then $\mathbb{Q}_p \backslash \{0\} \subset \mathcal{N}_{\widehat{f}} \cup \mathcal{Z}_{\widehat{\mu_E}}$, which is equivalent to

$$\{\xi \in \mathbb{Q}_p : \widehat{f}(\xi) \neq 0\} \backslash \{0\} \subset \mathcal{Z}_{\widehat{\mu_E}}. \tag{3.2}$$

Proof The second assertion follows directly from the first assertion because $\widehat{f} \cdot \widehat{\mu_E} = \delta_0$ and $\mathcal{Z}_{\delta_0} = \mathbb{Q}_p \backslash \{0\}$.

We now prove the first assertion. It suffices to prove $\mathcal{Z}_H \backslash \mathcal{N}_g \subset \mathcal{Z}_G$.

Take an arbitrary test function $\phi \in \mathcal{D}$ such that

$$\text{supp}(\phi) \subset \mathcal{Z}_H \backslash \mathcal{N}_g.$$

Since supp$(\phi) \subset \{x : g(x) \neq 0\}$, we can define the function

$$h(x) = \begin{cases} \frac{1}{g(x)}, & \text{for } x \in \text{supp}\,\phi, \\ 0, & \text{elsewhere.} \end{cases}$$

Since g is continuous, it is bounded away from 0 on supp ϕ. So, the function h is bounded and compactly supported. Hence it belongs to $L^1(\mathbb{Q}_p)$. On the other hand, we have $h(x)g(x) = 1_{\text{supp}(\phi)}(x) \in \mathcal{D}$. Thus

$$\langle G, \phi \rangle = \langle G, 1_{\text{supp}(\phi)} \cdot \phi \rangle = \langle 1_{\text{supp}(\phi)} \cdot G, \phi \rangle = \langle (h \cdot g) \cdot G, \phi \rangle$$

where we have used Proposition 2.13 for the second equality. Notice that supp$(h) \subset \mathcal{Z}_H$. By Proposition 2.12, $h \cdot H$ is well defined and $h \cdot H = 0$. By the associativity of the multiplication (see Proposition 2.16), we get

$$\langle G, \phi \rangle = \langle h \cdot H, \phi \rangle = 0.$$

Thus we have proved $\mathcal{Z}_H \backslash \mathcal{N}_g \subset \mathcal{Z}_G$. $\qquad \square$

Notice that $\widehat{f}(0) = \int_{\mathbb{Q}_p} f \, d\mathfrak{m} > 0$ and that $\widehat{f}$ is a continuous function. It follows that there exists a small ball where $\widehat{f}$ is nonvanishing. Let

$$n_f := \min\{n \in \mathbb{Z} : \widehat{f}(x) \neq 0, \text{ if } x \in B(0, p^{-n})\}. \tag{3.3}$$

Proposition 3.3 *The Fourier transform $\widehat{f}$ has compact support. The set $\mathcal{Z}_{\widehat{\mu_E}}$ is bounded and*

$$B(0, p^{-n_f}) \backslash \{0\} \subset \mathcal{Z}_{\widehat{\mu_E}}. \tag{3.4}$$

Proof The boundedness of $\mathcal{Z}_{\widehat{\mu_E}}$ is already proved. See Proposition 2.9 (2). This together with (3.2) implies the compactness of $\text{supp}(\widehat{f})$. We get (3.4) immediately from (3.2). $\qquad\square$

3.2 Distribution of E

The uniformly discrete set E involved in Eq. (3.1) shares the following uniform distribution property.

Proposition 3.4 *The cardinality of $E \cap B(\xi, p^{n_f})$ is independent of $\xi \in \mathbb{Q}_p$. Consequently, the set E admits a bounded density $D(E)$. Moreover, for all integers $n \geq n_f$, we have*

$$\forall \xi \in \mathbb{Q}_p, \quad \text{Card}(E \cap B(\xi, p^n)) = p^n D(E). \tag{3.5}$$

Proof For simplicity, we denote

$$E_n^\xi := E \cap B(\xi, p^n),$$

and write $E_n := E \cap B(0, p^n)$ when $\xi = 0$. It suffices to prove

$$\forall \xi \in \mathbb{Q}_p, \quad \text{Card}(E_{n_f}) = \text{Card}(E_{n_f}^\xi).$$

For any given $\xi \in \mathbb{Q}_p$, let $k = -v_p(\xi)$. If $k \leq n_f$, then $E_{n_f} = E_{n_f}^\xi$. So obviously $\text{Card}(E_{n_f}) = \text{Card}(E_{n_f}^\xi)$.

Now we suppose $k > n_f$. Then, consider any η satisfying

$$B(\eta, p^{-k}) \subset B(0, p^{-n_f}) \backslash \{0\}.$$

By (3.4) in Proposition 3.3, we have $\langle \widehat{\mu_E}, 1_{B(\eta, p^{-k})} \rangle = 0$, which by (2.9), is equivalent to

$$\sum_{\lambda \in E_k} \chi(\eta\lambda) = 0. \tag{3.6}$$

Taking $\eta = p^{k-1}$ in (3.6), we have $\sum_{\lambda \in E_k} \chi(p^{k-1}\lambda) = 0$. Observe that

$$B(0, p^k) = \bigsqcup_{i=0}^{p-1} B(ip^{-k}, p^{k-1})$$

and that the function $\chi(p^{k-1}\cdot)$ is constant on each ball of radius p^{k-1}. So we have

$$0 = \sum_{\lambda \in E_k} \chi(p^{k-1}\lambda) = \sum_{i=0}^{p-1} \chi\left(\frac{i}{p}\right) \mathrm{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right).$$

Applying Lemma 2.4, we obtain

$$\mathrm{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right) = \mathrm{Card}\left(E_{k-1}^{\frac{j}{p^k}}\right), \quad \forall\, 0 \le i, j \le p-1. \tag{3.7}$$

Similarly, taking $\eta = p^{k-2}$ in (3.6), we have

$$0 = \sum_{0 \le i, j \le p-1} \chi\left(\frac{i}{p^2} + \frac{j}{p}\right) \mathrm{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right).$$

Again, Lemma 2.4 implies

$$\mathrm{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \mathrm{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{m}{p^{k-1}}}\right) \quad \forall\, 0 \le i, j, m \le p-1.$$

Since

$$\sum_{j=0}^{p-1} \mathrm{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \mathrm{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right),$$

by (3.7), we get

$$\mathrm{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \mathrm{Card}\left(E_{k-2}^{\frac{l}{p^k} + \frac{m}{p^{k-1}}}\right), \quad \forall\, 0 \le i, j, l, m \le p-1.$$

We continue these arguments for all $\eta = p^{k-1}, \ldots, p^{n_f}$. By induction, we have

$$\mathrm{Card}(E_{n_f}^{\xi_1}) = \mathrm{Card}(E_{n_f}^{\xi_2}), \quad \forall\, \xi_1, \xi_2 \in \mathbb{L}_{n_f} \cap B(0, p^k). \tag{3.8}$$

Since $|\xi|_p = p^k$, there exists $\xi' \in \mathbb{L}_{n_f} \cap B(0, p^k)$, such that $E_{n_f}^{\xi} = E_{n_f}^{\xi'}$. Thus by (3.8),

$$\mathrm{Card}(E_{n_f}^{\xi}) = \mathrm{Card}(E_{n_f}^{\xi'}) = \mathrm{Card}(E_{n_f}).$$

The formula (3.5) follows immediately because each ball of radius p^n with $n \ge n_f$ is a disjoint union of p^{n-n_f} balls of radius p^{n_f} so that

$$\mathrm{Card}(E_n) = p^{n-n_f} \mathrm{Card}(E_{n_f}).$$

$\square$

3.3 Equality $D(E) = 1/\int_{\mathbb{Q}_p} f d\mathfrak{m}$

Proposition 3.5 *The density $D(E)$ of E satisfies*

$$D(E) = \frac{1}{\int_{\mathbb{Q}_p} f(x) dx}.$$

Proof By the integrability of f, the quantity

$$\epsilon_n := \int_{\mathbb{Q}_p \setminus B(0, p^n)} f(x) dx$$

tends to zero as the integer $n \to \infty$. Integrating the equality (3.1) over the ball $B(0, p^n)$, we have

$$\mathfrak{m}(B(0, p^n)) = \sum_{\lambda \in E} \int_{B(0, p^n)} f(x - \lambda) dx. \tag{3.9}$$

Now we split the sum in (3.9) into two parts, according to $\lambda \in E \cap B(0, p^n)$ or $\lambda \in E \setminus B(0, p^n)$. Denote $I := \int_{\mathbb{Q}_p} f d\mathfrak{m}$. For $\lambda \in E \cap B(0, p^n)$, we have

$$\int_{B(0, p^n)} f(x - \lambda) dx = \int_{B(0, p^n)} f(x) dx = I - \epsilon_n.$$

It follows that

$$\sum_{\lambda \in E \cap B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx = \text{Card}(E_n) \cdot (I - \epsilon_n). \tag{3.10}$$

Notice that

$$\int_{B(0, p^n)} f(x - \lambda) dx = \int_{B(-\lambda, p^n)} f(x) dx. \tag{3.11}$$

For $\lambda \in E \setminus B(0, p^n)$, the ball $B(-\lambda, p^n)$ is contained in $\mathbb{Q}_p \setminus B(0, p^n)$. We partition the uniformly discrete set $E \setminus B(0, p^n)$ into P_j's such that each P_j is contained in a ball of radius p^n in $\mathbb{Q}_p \setminus B(0, p^n)$. Thus the integrals in (3.11) for the λ's in the same P_j are equal. Let λ_j be a representative of P_j. Then we have

$$\sum_{\lambda \in E \setminus B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx = \sum_j \text{Card}(P_j) \cdot \int_{B(-\lambda_j, p^n)} f(x) dx.$$

However, by (3.5) in Proposition 3.4, $\text{Card}(P_j) = D(E)\mathfrak{m}(B(0, p^n))$ if $n \geq n_f$. Thus, for each integer $n \geq n_f$,

$$\sum_{e \in E \setminus B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx = D(E)\mathfrak{m}(B(0, p^n)) \sum_j \int_{B(-\lambda_j, p^n)} f(x) dx$$

$$= D(E)\mathfrak{m}(B(0, p^n)) \int_{\mathbb{Q}_p \setminus B(0, p^n)} f(x)dx$$

$$= D(E)\mathfrak{m}(B(0, p^n))\epsilon_n. \tag{3.12}$$

Thus from (3.9), (3.10) and (3.12), we finally get

$$\left| \mathfrak{m}(B(0, p^n)) - \text{Card}(E \cap B(0, p^n)) \cdot I \right| \leq 2D(E)\mathfrak{m}(B(0, p^n))\epsilon_n.$$

We conclude by dividing $\mathfrak{m}(B(0, p^n))$ and then letting $n \to \infty$. $\square$

4 Tiles are spectral sets

The key for the proof of the following theorem is the fact that the equation $\mu_E * f = 1$ imposes that the support of the Fourier transform $\widehat{f}$ is compact (Proposition 3.3). We also use the fact that the Fourier transform of an integrable function with compact support is uniformly locally constant (Proposition 2.2). The Lebesgue density theorem (Proposition 2.1) is also used. We recall that a point x in a Borel set Ω satisfying the equality in Proposition 2.1 is called a *density point* of Ω.

Theorem 4.1 *Without distinguishing sets which are equal modulo a set of zero Haar measure, we have the following assertions:*

(1) *If Ω is a spectral set in $\mathbb{Q}_p$, then it is bounded.*
(2) *If Ω is a tile in $\mathbb{Q}_p$, then it is compact and open.*
(3) *Tiles in $\mathbb{Q}_p$ are spectral sets.*

Proof Since we do not distinguish sets which are equal modulo a set of zero Haar measure, we can assume that all the points in Ω are density points of Ω, according to the Lebesgue density theorem (Proposition 2.1).

(1) Assume that (Ω, Λ) is a spectral pair in $\mathbb{Q}_p$, which is equivalent to

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \mu_\Lambda * |\widehat{1_\Omega}|^2(\xi) = \mathfrak{m}(\Omega)^2.$$

By Proposition 3.3, $\widehat{|\widehat{1_\Omega}|^2}$ has compact support. Observe that

$$\widehat{|\widehat{1_\Omega}|^2}(\xi) = 1_\Omega * 1_{-\Omega}(\xi) = \int_{\mathbb{Q}_p} 1_\Omega(x)1_\Omega(x - \xi)dx = \mathfrak{m}(\Omega \cap (\Omega + \xi)). \tag{4.1}$$

We claim

$$\text{supp}(\widehat{|\widehat{1_\Omega}|^2}) = \overline{\Omega - \Omega}.$$

Since the inclusion $\text{supp}(\widehat{|\widehat{1_\Omega}|^2}) \subset \overline{\Omega - \Omega}$ is obvious, we need only to show

$$\overline{\Omega - \Omega} \subset \text{supp}(\widehat{|\widehat{1_\Omega}|^2}). \tag{4.2}$$

In fact, let $\xi \in \Omega - \Omega$. Write $\xi = z_1 - z_2$ with $z_1, z_2 \in \Omega$. By (4.1), we have

$$\widehat{|\widehat{1_\Omega}|^2}(\xi) = \mathfrak{m}(\Omega \cap (\Omega + z_1 - z_2)) = \mathfrak{m}((\Omega - z_1) \cap (\Omega - z_2)).$$

Since z_1, z_2 are density points of Ω, 0 is a density point of both $\Omega - z_1$ and $\Omega - z_2$. This fact implies

$$\mathfrak{m}((\Omega - z_1) \cap (\Omega - z_2)) > 0.$$

Thus we have proved $\Omega - \Omega \subset \text{supp}(\widehat{|\widehat{1_\Omega}|^2})$. Then (4.2) follows. Since $\text{supp}(\widehat{|\widehat{1_\Omega}|^2})$ is compact, the set Ω is bounded.

(2) The main argument is the same as in (1). Assume that (Ω, T) is a tiling pair in $\mathbb{Q}_p$, which means

$$\mu_T * 1_\Omega(x) = 1, \quad \mathfrak{m}\text{-a.e. } x \in \mathbb{Q}_p.$$

By Proposition 3.3, $\widehat{1_\Omega}$ has compact support. Then by the first assertion of Proposition 2.2, 1_Ω is almost uniformly locally constant, i.e. Ω is, up to a zero measure set, a union of balls with the same radius. Since Ω is of finite measure, the number of these balls is finite. So, Ω is almost compact open.

(3) It is an immediate consequence of (2) and Theorem 1.3.

$\square$

The above proof of the fact "tiles are spectral sets" is partially based on Theorem 1.3 and partially on "tiles are compact and open" [Theorem 4.1 (2)]. The proof of "spectral sets are tiles" will be not based on Theorem 1.3. That means, we are not going to show that a spectral set is a compact open set up to a set of zero measure. But the boundedness of a spectral set [Theorem 4.1 (1)] will be used.

5 Spectral sets are tiles

In this section, we prove that a spectral set in $\mathbb{Q}_p$ is a tile. Assume that (Ω, Λ) is a spectral pair. The proof will be based on our knowledge on the set $\mathscr{Z}_{\widehat{\mu}_\Lambda}$. As we will see in the proof, a tiling complement can be easily constructed.

Theorem 5.1 *If Ω is a spectral set in $\mathbb{Q}_p$, then it is a tile.*

Proof Suppose that (Ω, Λ) is a spectral pair. That means (see 1.1)

$$\sum_{\lambda \in \Lambda} |\widehat{1_\Omega}|^2(x - \lambda) = \mathfrak{m}(\Omega)^2, \quad \mathfrak{m}\text{-a.e. } x \in \mathbb{Q}_p.$$

In other words, $\mu_\Lambda * f = 1$ where

$$f = |\widehat{1_\Omega}|^2 / \mathfrak{m}(\Omega)^2.$$

Let n_f be the integer defined by (3.3). Recall that $B(0, p^{-n_f})$ is the biggest ball centered at 0 over which $\hat{f}$ does not vanish. We first remark that

$$\forall n \geq n_f, \quad \text{Card}(\Lambda_n) = p^n \mathfrak{m}(\Omega) \tag{5.1}$$

where

$$\Lambda_n := \Lambda \cap B(0, p^n).$$

In fact, Plancherel Theorem implies $\int_{\mathbb{Q}_p} f \, d\mathfrak{m} = 1/\mathfrak{m}(\Omega)$.

So, by Proposition 3.5, the set Λ is of bounded density with density

$$D(\Lambda) = \frac{1}{\int_{\mathbb{Q}_p} f \, d\mathfrak{m}} = \mathfrak{m}(\Omega).$$

Thus (5.1) follows from (3.5) in Proposition 3.4.

By Theorem 4.1, the spectral set Ω is bounded, up to a Haar null measure set. Without loss of generality, we assume that $\Omega \subset \mathbb{Z}_p$. This is because we can scale Ω by a factor, say, p^n and the spectrum by p^{-n}.

Recall that $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ has the following properties: every sphere $S(0, p^{-n})$ either is contained in $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ or does not intersect $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ (Lemma 2.9); all spheres $S(0, p^{-n})$ with $n \geq n_f$ are contained in $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ (see 3.4). Note that $n_f \geq 0$ because of $\text{supp}|\widehat{1_\Omega}|^2 = \overline{\Omega - \Omega} \subset \mathbb{Z}_p$. Let

$$\mathbb{I} := \left\{ 0 \leq n < n_f : S(0, p^{-n}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}} \right\},$$
$$\mathbb{J} := \{ 0 \leq n < n_f : S(0, p^{-n}) \cap \mathcal{Z}_{\widehat{\mu_\Lambda}} = \emptyset \}.$$

Remark that $\mathbb{I}$ can be empty as well as $\mathbb{J}$. For $\mathcal{N}_{\widehat{f}}$, we claim

$$\bigcup_{j \in \mathbb{J}} S(0, p^{-j}) \subset \mathcal{N}_{\widehat{f}}. \tag{5.2}$$

In fact, otherwise for some $\xi \in S(0, p^{-j})$ with $j \in \mathbb{J}$ we have $\widehat{f}(\xi) \neq 0$. By (3.2) and Proposition 2.9 (1), $S(0, p^{-j}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}}$. This contradicts the fact $j \in \mathbb{J}$.

Observe that $\widehat{f}(\xi) = \mathfrak{m}(\Omega \cap (\Omega + \xi))/\mathfrak{m}(\Omega)^2$ (see 4.1) and that

$$(U - U) \setminus \{0\} \subset \bigcup_{j \in \mathbb{J}} S(0, p^{-j})$$

where

$$U := \left\{ \sum_{j \in \mathbb{J}} \alpha_j p^j, \alpha_j \in \{0, 1, \ldots, p-1\} \right\}.$$

Then from (5.2), we get

$$(U - U)\backslash\{0\} \subset \{\xi \in \mathbb{Q}_p : \mathfrak{m}(\Omega \cap (\Omega + \xi)) = 0\}. \tag{5.3}$$

We claim that Ω is a tile of $\mathbb{Z}_p$ with tiling complement U, so that Ω is a tile of $\mathbb{Q}_p$ with tiling complement $U + \mathbb{L}$.

In fact, the disjointness (up to a set of zero measure) of $\Omega + \xi_1$ and $\Omega + \xi_2$ for any two distinct $\xi_1, \xi_2 \in U$ follows directly from (5.3) and the invariance of the Haar measure.

So to show that Ω is a tile of $\mathbb{Z}_p$, it suffices to prove that the total measure of all $\Omega + \xi$ with $\xi \in U$ is equal to

$$\mathfrak{m}(\Omega + U) = \text{Card}(U) \cdot \mathfrak{m}(\Omega) = 1. \tag{5.4}$$

For $i \in \mathbb{I}$, we have $B(p^i, p^{-n_f}) \subset S(0, p^{-i})$ so that $B(p^i, p^{-n_f}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}}$, which implies

$$0 = \langle \widehat{\mu_\Lambda}, 1_{B(p^i, p^{-n_f})} \rangle = \langle \mu_\Lambda, \widehat{1_{B(p^i, p^{-n_f})}} \rangle = p^{-n_f} \sum_{\lambda \in \Lambda_{n_f}} \overline{\chi(p^i \lambda)}.$$

By Lemma 2.7 (3),

$$p^{\text{Card}(\mathbb{I})} \leq \text{Card}(\Lambda_{n_f}).$$

Then by the fact $\text{Card}(U) = p^{\text{Card}(\mathbb{J})}$ and (5.1), we have

$$\text{Card}(U) \cdot \mathfrak{m}(\Omega) = p^{\text{Card}(\mathbb{J})} \cdot \frac{\text{Card}(\Lambda_{n_f})}{p^{n_f}} \geq p^{\text{Card}(\mathbb{J})} \cdot \frac{p^{\text{Card}(\mathbb{I})}}{p^{n_f}} = 1.$$

We can now conclude (5.4) because $\Omega + U$ is contained in $\mathbb{Z}_p$ so that $\mathfrak{m}(\Omega + U) \leq 1$. □

6 Some remarks

6.1 Dual Fuglede's conjecture

Theorem 1.2 asserts that all spectral sets (equivalently tiles) in $\mathbb{Q}_p$ are almost compact open. What is the topological structure of the corresponding spectra and tiling complements? This question has been answered by Theorem 1.2 of [9] for the compact open spectral sets in $\mathbb{Q}_p$. Recall that the spectra and tiling complements of compact open spectral sets are infinite p-homogeneous discrete sets Λ (see [9, Section 2.8]) such that there exists an integer N such that

$$(\infty, N] \cap \mathbb{Z} \subset I_\Lambda, \tag{6.1}$$

where $I_\Lambda := \{v_p(x - y) : x, y \in \Lambda \text{ distinct}\}$, called the set of admissible p-orders of Ω. On the other hand, for any given infinite p-homogeneous discrete sets Λ satisfying (6.1), we can construct two compact open sets Ω_1 and Ω_2 such that (Ω_1, Λ) is a spectral pair and (Ω_2, Λ) is a tiling pair. For details, we refer the readers to [9, Theorems 1.2 and 5.1]. So the dual Fuglede's conjecture holds in $\mathbb{Q}_p$ (see [18] for the dual Fuglede's conjecture in $\mathbb{R}^d$): *a subset of $\mathbb{Q}_p$ is a spectrum if and only if it is a tiling complement.*

6.2 The density of E satisfying (2.1) in $\mathbb{Q}_p^d$

Assume that E is a discrete subset in $\mathbb{Q}_p^d$ such that $\mathrm{Card}(E \cap K) < \infty$ for any compact subset K of $\mathbb{Q}_p^d$. Then $\mu_E = \sum_{\lambda \in E} \delta_\lambda$ defines a discrete Radon measure in $\mathbb{Q}_p^d$. Remark that E is not necessarily assumed uniformly discrete. Suppose that E satisfies the Eq. (2.1) with $f \in L^1(\mathbb{Q}_p^d)$ which is a non-negative integrable function such that $\int_{\mathbb{Q}_p^d} f \, d\mathfrak{m} > 0$. Then for each $x_0 \in \mathbb{Q}_p^d$, the limit

$$D(E) = \lim_{k \to \infty} \frac{\mathrm{Card}(B(x_0, p^k) \cap E)}{\mathfrak{m}(B(x_0, p^k))}$$

exists and

$$D(E) = \frac{1}{\int_{\mathbb{Q}_p^d} f \, d\mathfrak{m}}.$$

This is similar to the real case, see [20] and [21, Lemma 2].

6.3 Tiles and spectral sets in $\mathbb{Q}_p^2$ which are not compact open.

For a uniformly discrete set E in the higher dimensional space $\mathbb{Q}_p^d$ with $d \geq 2$, the zero set of the Fourier transform of the measure μ_E is not necessarily bounded. In other words, Proposition 2.9 does not hold in $\mathbb{Q}_p^d$ with $d \geq 2$. For example, let $E = \{(0, 0), (0, 1), \ldots, (0, p-1)\}$ which is a finite subset of $\mathbb{Q}_p^2$. One can check that

$$\mathcal{Z}_{\widehat{\mu_E}} = \mathbb{Q}_p \times p^{-1} \mathbb{Z}_p^\times.$$

Bounded tiles of $\mathbb{Q}_p^2$ are not necessarily almost compact open. Let us construct such a bounded tile of $\mathbb{Q}_p^2$. We partition $\mathbb{Z}_p$ into p Borel sets of same Haar measure, denoted by A_i ($i = 0, 1, \ldots, p-1$). We assume that one of A_i is not almost compact open. For example, set $S = \bigcup_{n=1}^\infty B(p^n, p^{-n-1})$, which is a union of countable disjoint balls. Let

$$A_0 = S \cup (B(1, p^{-1}) \backslash (1 + S)),$$
$$A_1 = (B(0, p^{-1}) \cup B(1, p^{-1})) \backslash A_0,$$

$$A_i = B(i, p^{-1}) \text{ for } 2 \le i \le p - 1.$$

Then define

$$\Omega := \bigcup_{i=0}^{p-1} A_i \times B(i, p^{-1}) \subset \mathbb{Z}_p \times \mathbb{Z}_p.$$

The set Ω is not almost compact open, because any small ball centered at $(0, 0)$ meets both Ω and $(\mathbb{Z}_p \times \mathbb{Z}_p \backslash \Omega)$ with positive measure. But it is a tile of $\mathbb{Q}_p^2$ with tiling complement

$$T = \mathbb{L} \times \mathbb{L}_{-1}.$$

Just like on $\mathbb{R}^d$ (see [12]), it can be proved that $(\Omega, \mathbb{L} \times \mathbb{L}_{-1})$ is a tiling pair if and only if $(\Omega, \mathbb{L} \times \mathbb{L}_1)$ is a spectral pair (see [19]).

6.4 Higher dimensional cases

However, the situation changes in higher dimension. In fact, it is proved in [2] that there exist spectral sets which are not tiles in $\mathbb{F}_p^5$ for all odd primes p and in $\mathbb{F}_p^4$ for all odd primes p such that $p \equiv 3 \mod 4$, where $\mathbb{F}_p$ is the prime field with p elements. This implies that there exist compact open spectral sets which are not tiles in $\mathbb{Q}_p^5$ for all odd primes p and in $\mathbb{Q}_p^4$ for all odd primes p such that $p \equiv 3 \mod 4$. Recently, Ferguson and Sothanaphan [13] proved that for all odd primes p there are spectral sets of $\mathbb{F}_p^4$ that do not tile. So there is no need to distinguish the case $p \equiv 3 \mod 4$.

For $d = 3$, there exists a spectral set in the finite group $(\mathbb{Z}/8\mathbb{Z})^3$ which is not a tile [23]. This implies that there exists a compact open spectral set which is not a tile in $\mathbb{Q}_2^3$.

Fuglede's conjecture in $\mathbb{Q}_p^2$ is open, even under the compact open assumption.

References

1. Adler, A., Holroyd, F.C.: Some results on one-dimensional tilings. Geom. Dedic. **10**, 49–58 (1981)
2. Aten, C., Ayachi, B., Bau, E., FitzPatrick, D., Iosevich, A., Liu, H., Lott, A., MacKinnon, I., Maimon, S., Nan, S., Pakianathan, J., Petridis, G., Rojas Mena, C., Sheikh, A., Tribone, T., Weill, J., Yu, C.: Tiling sets and spectral sets over finite fields. J. Funct. Anal. **273**(8), 2547–2577 (2017)
3. Albeverio, S., Khrennikov, A., Shelkovich, V.: Theory of p-Adic Distributions: Linear and Nonlinear Models. Oxford University Press, Oxford (2010)
4. Bandt, C.: Self-similar sets 5: integer matrices and fractal tilings of $\mathbb{R}^n$. Proc. Am. Math. Soc. **112**, 549–562 (1991)
5. Bose, D., Madan, S.: "Spectral implies tiling" for three intervals revisited. Forum Math. **26**(4), 1247–1260 (2014)
6. Bose, D., Kumar, C.P.A., Krishnan, R., Madan, S.: On Fuglede's conjecture for three intervals. Online J. Anal. Comb. No. **5**, 24 (2010)
7. Fan, A.H.: Spectral measures on local fields. In: Bohner, M. (ed.) Difference Equations, Discrete Dynamical Systems and Applications, Springer Proceedings in Mathematics & Statistics 150, pp. 15–35. Springer, Geneva (2015). arXiv:1505.06230

8. Fan, A.H., Fan, S.L.: Bounded Tiles in $\mathbb{Q}_p$ are Compact Open Sets. arXiv:1511.06404
9. Fan, A.H., Fan, S.L., Shi, R.X.: Compact open spectral sets in $\mathbb{Q}_p$. J. Funct. Anal. **271**(12), 3628–3661 (2016)
10. Farkas, B., Révész, S.G.: Tiles with no spectra in dimension 4. Math. Scand. **98**, 44–52 (2006)
11. Farkas, B., Matolcsi, M., Móra, P.: On Fuglede's conjecture and the existence of universal spectra. J. Fourier Anal. Appl. **12**(5), 483–494 (2006)
12. Fuglede, B.: Commuting self-adjoint partial differential operators and a group theoretic problem. J. Funct. Anal. **16**, 101–121 (1974)
13. Ferguson, S.J., Sothanaphan, N.: Fuglede's conjecture fails in 4 dimensions over odd prime fields. Discret. Math. (2019). https://doi.org/10.1016/j.disc.2019.04.026. (published on line)
14. Gröchenig, K.H., Haas, A.: Self-similar lattice tilings. J. Fourier Anal. **1**, 131–170 (1994)
15. Iosevich, A., Katz, N., Tao, T.: The Fuglede spectral conjecture holds for convex planar domains. Math. Res. Lett. **10**(5–6), 559–569 (2003)
16. Iosevich, A., Mayeli, A., Pakianathan, J.: The Fuglede conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$. Anal. PDE **10**(4), 757–764 (2017)
17. Jorgensen, P., Pedersen, S.: Dense analytic subspaces in fractal L^2-spaces. J. Anal. Math. **75**, 185–228 (1998)
18. Jorgensen, P., Pedersen, S.: Spectral pairs in Cartesian coordinates. J. Fourier Anal. Appl. **5**, 285–302 (1999)
19. Kadir, M.: Spectral sets and tiles on vector space over local fields. J. Math. Anal. Appl. **440**(1), 240–249 (2016)
20. Kolountzakis, M.N., Lagarias, J.C.: Structure of tilings of the line by a function. Duke Math. J **82**(3), 653–678 (1996)
21. Kolountzakis, M.N.: The study of translational tiling with Fourier analysis. In: Proceedings of the Milano Conference on Fourier Analysis and Convexity, pp. 131–187 (2001)
22. Kolountzakis, M.N., Matolcsi, M.: Tiles with no spectra. Forum Math. **18**, 519–528 (2006)
23. Kolountzakis, M.N., Matolcsi, M.: Complex Hadamard matrices and the spectral set conjecture. Collect. Math. **57**, 281–291 (2006)
24. Kiss, G., Malikiosis, R.D., Somlai, G., Vizer, M.: On the Discrete Fuglede and Pompeiu Problems. arXiv:1807.02844
25. Malikiosis, R.D., Kolountzakis, M.N.: Fuglede's conjecture on cyclic groups of order $p^n q$. Discret. Anal. **12**, 16 (2017)
26. Lev, N., Matolcsi, M.: The Fuglede conjecture for convex domains is true in all dimensions. arXiv:1904.12262
27. Kenyon, R.: Self-Similar Tilings. Ph.D. thesis, Princeton University (1990)
28. Matolcsi, M.: Fuglede conjecture fails in dimension 4. Proc. Am. Math. Soc. **133**, 3021–3026 (2005)
29. Lau, K.S., Rao, H.: On one-dimensional self-similar tilings and the pq-tilings. Trans. Am. Math. Soc. **355**, 1401–1414 (2003)
30. Łaba, I.: Fuglede's conjecture for a union of two intervals. Proc. Am. Math. Soc. **129**(10), 2965–2972 (2001)
31. Łaba, I.: The spectral set conjecture and multiplicative properties of roots of polynomials. J. Lond. Math. Soc. **2**(65), 661–671 (2002)
32. Lagarias, J.C., Wang, Y.: Integral self-affine tiles in $\mathbb{R}^n$ I. Standard and non-standard digits sets. J. Lond. Math. Soc. (2) **54**, 161–179 (1996)
33. Lagarias, J.C., Wang, Y.: Tiling the line with translates of one tile. Invent. Math. **124**, 341–365 (1996)
34. Popken, J., Turkstra, H.: A P-adic analogue of a theorem of Lebesgue in the theory of measure. Nederl. Akad. Wetensch. Proc. **49**, 802–814 (1946). (Indagationes Math. 8, 505–517)
35. Shi, R.X.: Fuglede's Conjecture Holds on Cyclic Groups $\mathbb{Z}_{pqr}$. arXiv:1805.11261
36. Schoenberg, I.J.: A note on the cyclotomic polynomial. Mathematika **11**(02), 131–136 (1964)
37. Tijdeman, R.: Decomposition of the integers as a direct sum of two subsets. In: David, S. (ed.) Number Theory Seminar Paris 1993, pp. 261–276. Cambridge University Press, Cambridge (1995)
38. Taibleson, M.H.: Fourier Analysis on Local Fields. Princeton University Press, Princeton (1975)
39. Tao, T.: Fuglede's conjecture is false in 5 and higher dimensions. Math. Res. Lett. **11**, 251–258 (2004)
40. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: P-adic Analysis and Mathematical Physics. World Scientific Publishing, Singapore (1994)

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