



# Random walks and random tug of war in the Heisenberg group

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## Abstract

We study mean value properties of  $\mathbf{p}$ -harmonic functions on the first Heisenberg group  $\mathbb{H}$ , in connection to the dynamic programming principles of certain stochastic processes. We implement the approach of Peres and Sheffield (Duke Math J 145(1):91–120, 2008) to provide a game-theoretical interpretation of the sub-elliptic  $\mathbf{p}$ -Laplacian; and of Manfredi et al. (Proc Am Math Soc 138(3):881–889, 2010) to characterize its viscosity solutions via asymptotic mean value expansions.

## 1 Introduction

In this paper, we are concerned with mean value properties of  $\mathbf{p}$ -harmonic functions on the Heisenberg group  $\mathbb{H}$ , in connection to the dynamic programming principles of certain stochastic processes. More precisely, we develop asymptotic mean value expansions of the type:

$$\text{Average}(v, r)(q) = v(q) + cr^2 \Delta_{\mathbb{H}, \mathbf{p}}^N v(q) + o(r^2) \quad \text{as } r \rightarrow 0+, \quad (1.1)$$

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for the normalized version  $\Delta_{\mathbb{H},\mathbf{p}}^N$  of the  $\mathbf{p}$ - $\mathbb{H}$ -Laplacian  $\Delta_{\mathbb{H},\mathbf{p}}$  in (1.4)–(1.5), for any  $1 < \mathbf{p} < \infty$ . The “Average” denotes here a suitable mean value operator, acting on a given function  $v : \mathbb{H} \rightarrow \mathbb{R}$ , on a set of radius  $r$  and centered at a point  $q \in \mathbb{H}$ . This operator may be either “stochastic”:  $f$ , or “deterministic”:  $\frac{1}{2}(\sup + \inf)$ , or it may be given through various compositions or further averages of such types. The averaging set may be one of the following: the 3-dimensional Korányi ball  $B_r(q)$ ; the 2-dimensional ellipse in the tangent plane  $T_q$  passing through  $q$ , whose horizontal projection coincides with the 2-dimensional Euclidean ball of radius  $r$ ; the 1-dimensional boundary of such ellipse; or the 3-dimensional Korányi ellipsoid that is the image of  $B_r(q)$  under a suitable linear map.

For particular expansions in (1.1), we study solutions to the boundary value problems for the related *mean value equations*, posed on a bounded domain  $\mathcal{D} \subset \mathbb{H}$ , with data  $F \in \mathcal{C}(\mathbb{H})$ :

$$\text{Average}(u^\epsilon, \epsilon) = u^\epsilon \text{ in } \mathcal{D}, \quad u^\epsilon = F \text{ on } \mathbb{H} \setminus \mathcal{D}. \quad (1.2)$$

We identify the solution  $u^\epsilon$  as the value of a process with, in general, both random and deterministic components. The purely random component is related to the “stochastic” averaging part of the operator *Average* as described above, whereas the deterministic component is related to the “deterministic” part and can be interpreted as the *Tug of War game*. Recall that the Tug of War is a zero-sum, two-players game process, in which the position of the particle in  $\mathcal{D}$  is shifted according to the deterministic *strategies* of the two players. The players take turns with equal probabilities and strive to maximize or minimize the game outcome given by the value of  $F$  at the particle’s final (stopping) position.

We then examine convergence of the family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$ . For domains with *game-regular boundary*  $\partial\mathcal{D}$ , we show the uniform convergence to the viscosity solution of the Dirichlet problem:

$$\Delta_{\mathbb{H},\mathbf{p}}u = 0 \text{ in } \mathcal{D}, \quad u = F \text{ on } \partial\mathcal{D}. \quad (1.3)$$

The definition of game-regularity is process-related, and it replaces the celebrated Wiener capacity criterion [16], in the probabilistic setting that we are pursuing. Heuristically, game-regularity is equivalent to the equicontinuity of the family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$  on  $\bar{\mathcal{D}}$ , where the only obstruction is due to the possibly high probability of the event where the particle exits a prescribed neighbourhood of a boundary point while still in  $\mathcal{D}$ . In particular, we show that this scenario cannot happen when  $\mathcal{D}$  satisfies the *exterior  $\mathbb{H}$ -corkscrew condition*; indeed such domains are automatically game-regular.

The program outlined above, familiar in the linear setting of  $\mathbf{p} = 2$ , where it reflects the well-studied correspondence between the Laplace operator and the Brownian motion [12], mimics the approach put forward in the seminal papers [31,32] by Peres, Schramm, Sheffield and Wilson. There, the authors introduced the game-theoretical interpretation of the  $\infty$ -Laplacian and the  $\mathbf{p}$ -Laplacian in the Euclidean geometry, and during the past decade many follow up works appeared in the literature [7,8,21–24,26,30]. In the present context of Heisenberg geometry—in relation to the operators *Average* in (1.1) and their game-theoretical description—a preliminary

mean value characterization of viscosity  $\mathbf{p}$ - $\mathbb{H}$ -harmonic functions appeared in [14] for  $\mathbf{p} \geq 2$ , without addressing the issue of convergence. The contribution of this paper is that we carry out the indicated program in full, including the general case of exponents  $1 < \mathbf{p} < \infty$  and proving convergence, in relation to game-theoretical interpretation. We also believe that our careful clarification of certain proofs in [32] (including the inductive techniques in Lemma 8.4 and Theorem 14.1, and their application in the proofs of Theorems 8.3, 14.2 and 14.5), albeit in the present sub-Riemannian context, will benefit the reader less familiar with probability techniques.

## 1.1 The structure and results of this paper

Our contribution is divided into three parts.

Part I consists of four sections, in which we develop different mean value expansions (1.1). In Sect. 2 we begin with three averaging operators in connection to the linear case exponent  $\mathbf{p} = 2$ . The 1-dimensional and 2-dimensional expansions are proved in Proposition 2.3; validity of the related mean value properties as in (1.2) is then automatically equivalent to  $\mathbb{H}$ -harmonicity. A similar statement for the 3-dimensional average on Korányi balls, only holds in the viscosity sense (Proposition 2.5), and can be seen as a counterpart to the Gauss–Koebe–Levi–Tonelli theorem [9], where the average is taken with respect to a non-uniform probability measure.

In Sect. 3 we treat the case of the fully nonlinear operator  $\Delta_{\mathbb{H}, \infty}$ , utilizing the “deterministic” averaging rather than the “stochastic” ones as in Sect. 2. This description is in agreement with the absolutely minimizing Lipschitz extension (AMLE) property of the  $\infty$ -harmonic functions  $u$ , which states that for every open subset  $U$ , the restriction  $u|_U$  has the smallest Lipschitz constant among all the extensions of  $u|_{\partial U}$  on  $\bar{U}$  (see [1] for the Euclidean and [13] for the Heisenberg setting). In Sect. 4 we combine the averages for  $\mathbf{p} = 2$  and  $\mathbf{p} = \infty$  and propose two mean value expansions for  $\Delta_{\mathbb{H}, \mathbf{p}}$ , via superpositions that are both modeled on the interpolation property of  $\Delta_{\mathbb{H}, \mathbf{p}}$  in (1.6). These expansions are relevant for  $\mathbf{p} \geq 2$  because only then the related coefficients can be interpreted as probabilities. Expansion (4.1) was already present in the Euclidean setting in [26]. The general case of  $1 < \mathbf{p} < \infty$  is treated in Sect. 5, where we follow the Euclidean construction of [22], superposing the “deterministic” with “stochastic averaging” on the Korányi ellipsoids whose orientations and aspects ratios vary within the “deterministic averaging” sampling sets. The same expansions hold if we replace the constant exponent  $\mathbf{p}$  by a variable exponent  $\mathbf{p}(\cdot)$ , pertaining to the so-called *strong  $\mathbf{p}(\cdot)$ - $\mathbb{H}$ -Laplacian*, as pointed out in Remark 4.2.

Part II consists of four further sections, in which we display the stochastic interpretation of the 2-dimensional mean value expansion (2.3)<sub>2</sub> from Sect. 2. In Sect. 6 we define the 3-dimensional walk in  $\mathbb{H}$ , whose increments are 2-dimensional, with the third variable slaved to the first two via the Levy area process. Our process has infinite horizon, but it almost surely accumulates on  $\partial \mathcal{D}$ , whereas its expectation yields, in the limit of shrinking sampling radii, an  $\mathbb{H}$ -harmonic function. The convergence is addressed in Sect. 7; in view of equiboundedness, it suffices to prove equicontinuity. We first observe in Lemma 7.1, that this property is equivalent to the seemingly weaker property of equicontinuity at the boundary. We then introduce the standard notion of

walk-regularity of the boundary points, which turns out to be equivalent to the aforementioned boundary equicontinuity. In Sect. 8 we show that domains satisfying the exterior  $\mathbb{H}$ -corkscrew condition are walk-regular. We prove in Sect. 9 that any limit in question must be the viscosity solution to the  $\mathbb{H}$ -harmonic equation with boundary data  $F$ . By uniqueness of such solutions [5,6], we obtain the uniform convergence in the walk-regular case.

In Part III, we follow the same outline as in Part II, but for the 3-dimensional asymptotic expansion (4.2) and the nonlinear operator  $\Delta_{\mathbb{H},\mathbf{p}}$ . In Sect. 10, we define the related Tug of War game with noise and its upper and lower values. These values turn out to be both equal, as shown in Theorem 11.3 by a classical martingale argument, to the unique, continuous solution of the mean value equation in Theorem 11.1. The equation (11.1) can be hence seen as a finite difference approximation to the  $\mathbf{p}$ - $\mathbb{H}$ -Laplace Dirichlet problem with boundary data  $F$ ; existence, uniqueness and regularity of its solutions  $u_\epsilon$  at each sampling scale  $\epsilon$ , is obtained independently via analytical techniques. In particular, each  $u_\epsilon$  is continuous up to the boundary, where it assumes the values of  $F$ . In Theorem 12.1 we show that for  $F$  that is already a restriction of some  $\mathbf{p}$ - $\mathbb{H}$ -harmonic function with non-vanishing gradient, the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  uniformly converges to  $F$  at the rate that is of first order in  $\epsilon$ . Our proof uses an analytical argument and it is based on the observation that for  $s$  sufficiently large, the mapping  $q \mapsto |q|_K^s$  yields the variation that pushes the  $\mathbf{p}$ - $\mathbb{H}$ -harmonic function  $F$  into the region of  $\mathbf{p}$ -subharmonicity.

In Sect. 13 we discuss equicontinuity (and thus convergence) of the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$ , for general  $F$ . Similarly to Lemma 7.1, this property is equivalent to equicontinuity at the boundary, which is shown in Theorem 13.1 through the analytical argument, based on translation and well-posedness of (11.1). We proceed by defining the *game-regularity* of the boundary points; Definition 13.2, Lemma 13.4 and Theorem 13.5 mimic the parallel statements in [32]. In Sect. 14 we argue that, similarly to Theorem 8.3, domains that satisfy the exterior  $\mathbb{H}$ -corkscrew condition are game-regular. The proof in Theorem 14.5 uses the concatenating strategies technique and the annulus walk estimate taken from [32]. We again carefully provide the probabilistic details omitted in [32], having in mind a reader whose training is more analytically-oriented. In Sect. 13 we finally conclude that the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  converges uniformly to the unique viscosity solution to (1.3), in the game-regular case.

We remark that identical constructions and results of Part III, can be carried out for the process and the dynamic programming principle modelled on (5.4) rather than (4.2), where the advantage is that it covers any exponent in the range  $1 < \mathbf{p} < \infty$ . We indicate the necessary modifications in Remark 10.2 and leave further details to the interested reader; in the Euclidean setting we point to the paper [22].

## 1.2 Notation and preliminaries on the Heisenberg group $\mathbb{H}$

Let  $\mathbb{H} = (\mathbb{R}^3, *)$  be the first Heisenberg group, whose points we typically denote by:

$$q = (x, y, z) \in \mathbb{H}.$$

If needed, we also use the notation  $q = (q_1, q_2, q_3) = (q_{hor}, q_3)$ . The group operation is:

$$q * q' = (x, y, z) * (x', y', z') = \left( x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right),$$

and the Korányi metric  $d$  on  $\mathbb{H}$  is given through the Korányi gauge  $|q|_K$  in:

$$d(q, q') = |q^{-1} * q'|_K, \quad |q|_K = ((x^2 + y^2)^2 + 16z^2)^{1/4}.$$

The metric  $d$  is left-invariant and one-homogeneous with respect to the anisotropic dilations:

$$\rho_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z), \quad \rho_\lambda : \mathbb{H} \rightarrow \mathbb{H}, \quad \lambda > 0.$$

By  $B_r(q) = \{q' \in \mathbb{H}; d(q, q') < r\} = q * B_1(0)$  we denote an open ball with respect to the metric  $d$ , whereas the Euclidean balls in  $n = 2, 3$  dimensions are denoted by  $B_r^n(q)$ . Both types of balls, viewed as subsets of  $\mathbb{R}^3$ , are convex sets.

The differential operators constituting a basis of the Lie algebra on  $\mathbb{H}$ , are:

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z.$$

Operators  $X, Y$  correspond to differentiating at  $q$  in the directions spanning the plane:

$$T_q = \text{span} \left( (1, 0, -\frac{y}{2}), (0, 1, \frac{x}{2}) \right) = \left( \frac{y}{2}, -\frac{x}{2}, 1 \right)^\perp.$$

The horizontal gradient and the sub-Laplacian of a function  $v : \mathbb{H} \rightarrow \mathbb{R}$  are:

$$\nabla_{\mathbb{H}}v = (Xv, Yv), \quad \Delta_{\mathbb{H}}v = (X^2 + Y^2)v.$$

We will be concerned with the so-called  $\mathbf{p}$ -sub-Laplacian of  $v$ , with exponent  $\mathbf{p} \in (1, \infty)$ :

$$\Delta_{\mathbb{H}, \mathbf{p}}v = X(|\nabla_{\mathbb{H}}v|^{\mathbf{p}-2}Xv) + Y(|\nabla_{\mathbb{H}}v|^{\mathbf{p}-2}Yv), \tag{1.4}$$

and with its *normalized* (sometimes called *game-theoretical*) version:

$$\Delta_{\mathbb{H}, \mathbf{p}}^N v = \frac{\Delta_{\mathbb{H}, \mathbf{p}}v}{|\nabla_{\mathbb{H}}v|^{\mathbf{p}-2}}, \tag{1.5}$$

defined whenever  $\nabla_{\mathbb{H}, \mathbf{p}}v \neq 0$ . Clearly,  $\Delta_{\mathbb{H}, 2} = \Delta_{\mathbb{H}, 2}^N = \Delta_{\mathbb{H}}$  and it is also easy to check that:

$$\Delta_{\mathbb{H}, \mathbf{p}}v = |\nabla_{\mathbb{H}}v|^{\mathbf{p}-2}\Delta_{\mathbb{H}, \mathbf{p}}^N v = |\nabla_{\mathbb{H}}v|^{\mathbf{p}-2}(\Delta_{\mathbb{H}}v + (\mathbf{p} - 2)\Delta_{\mathbb{H}, \infty}v), \tag{1.6}$$

where  $\Delta_{\mathbb{H},\infty}$  is the  $\infty$ -sub-Laplacian given by:

$$\Delta_{\mathbb{H},\infty} v = \langle \nabla_{\mathbb{H}}^2 v : \left( \frac{\nabla_{\mathbb{H}} v}{|\nabla_{\mathbb{H}} v|} \right)^{\otimes 2} \rangle. \tag{1.7}$$

### 1.3 A brief review of results on nonlinear elliptic problems in $\mathbb{H}$

Many techniques and results valid in the Euclidean case can be extended [36] in the above context. We now indicate some general statements on the well-posedness of the Dirichlet problem in  $\mathbb{H}$ :

$$\Delta_{\mathbb{H},\mathbf{p}} v = 0 \quad \text{in } \mathcal{D}, \quad v - F \in HW_0^{1,\mathbf{p}}(\mathcal{D}). \tag{1.8}$$

This problem has a unique weak solution  $v \in HW^{1,\mathbf{p}}(\mathcal{D})$ , for every data function  $F$  in the horizontal Sobolev space  $HW^{1,\mathbf{p}}(\mathcal{D}) = \{v \in L^{\mathbf{p}}(\mathcal{D}); \nabla_{\mathbb{H}} v \in L^{\mathbf{p}}(\mathcal{D}, \mathbb{R}^2)\}$ . We also have existence and uniqueness of solutions to the corresponding obstacle problem. The  $\mathbf{p}$ - $\mathbb{H}$ -subsolutions and supersolutions, as well as the  $\mathbf{p}$ - $\mathbb{H}$ -subharmonic and superharmonic functions are defined in the usual manner. The  $\mathbf{p}$ - $\mathbb{H}$ -subsolutions have upper semicontinuous representatives that are  $\mathbf{p}$ - $\mathbb{H}$ -subharmonic. Every bounded  $\mathbf{p}$ - $\mathbb{H}$ -subharmonic function has locally  $\mathbf{p}$ -integrable horizontal derivatives; in fact it is quasicontinuous.

It is known that the horizontal derivatives of a  $\mathbf{p}$ - $\mathbb{H}$ -harmonic function are Hölder continuous. More precisely:  $\text{osc}_{B_r(q)} \nabla_{\mathbb{H}} v \leq C \left(\frac{r}{R}\right)^\alpha \left(\int_{B_R(q)} |\nabla_{\mathbb{H}} v|^{\mathbf{p}}\right)^{1/\mathbf{p}}$  for all  $B_r(q) \subset B_{R/2}(q) \subset B_R(q)$  compactly contained in  $\mathcal{D}$ , where  $\alpha \in (0, 1)$  and  $C$  depend only on  $\mathbf{p}$ . This was proved for the range  $4 < \mathbf{p} < \infty$  in [34], for  $2 \leq \mathbf{p} < \infty$  in [10], and for  $1 < \mathbf{p} < \infty$  in [29].

The standard notion of capacity for the subelliptic setting is studied in [11]. This notion coincides with the definition of capacity based on Radon measures associated to  $\mathbf{p}$ - $\mathbb{H}$ -subharmonic functions [36] in  $\mathbb{H}$ . The Wolff potential estimate extends to the subelliptic case, and yields a Wiener-type criterion for the attainment of the boundary values for any  $F \in \mathcal{C}(\bar{\mathcal{D}})$  by Perron solutions to (1.8). We also have a version of the Kellogg-type property stating that the set of irregular boundary points, where the boundary value  $F$  is not attained, has zero capacity. Points satisfying the exterior  $\mathbb{H}$ -corkscrew condition in Definition 8.1 are regular [27] (in fact, we reprove a version of this statement in Sect. 14).

General metric spaces with a doubling measure and supporting a Poincaré inequality are considered in [20]. Perron solutions in such metric spaces are studied in [8], while [7] contains the adequate notion and discussion of the balayage theory.

We remark that for elliptic symmetric equations in non-divergence form:

$$\text{trace}(A \nabla_{\mathbb{H}}^2 v) = 0 \tag{1.9}$$

many results that are classical in the Euclidean setting, remain open in  $\mathbb{H}$ . Let  $A : \mathcal{D} \rightarrow \mathbb{R}_{sym}^{2 \times 2}$  be measurable, bounded and uniformly elliptic coefficient matrix. It is not known whether a nonnegative smooth solution  $v$  to (1.9) is locally Hölder

continuous (with exponent depending only on the ellipticity constant  $C_0$  of  $A$  and  $\|v\|_{L^\infty_{loc}}$ ). In the same context, the validity of the Krylov-Safonov-Harnack inequality:  $\sup_{B_r(q)} v \leq C_1 \inf_{B_r(q)} v$ , where  $C_1 = C(C_0, \|v\|_{L^\infty(B_{2r}(q))})$  is open. However, similarly to the Euclidean case, the latter inequality implies the Hölder continuity via a scaling argument. When the right hand side of (1.9) is replaced by  $f \in L^4(\mathcal{D})$ , the following Alexandroff-Bakelman-Pucci inequality is expected:  $\|v\|_{L^\infty(B_r(q))} \leq C_2(\int_{B_r(q)} |f(q)|^4 dq)^{1/4}$  with  $C_2 = C(C_0)$ . Positive resolution of this problem would be a step towards establishing the Krylov-Safonov-Harnack inequality in  $\mathbb{H}$ . We also mention that there are further open questions regarding the isoperimetric inequality and the uniqueness of the mean curvature flow in  $\mathbb{H}$ .

**PART I: The mean value expansions in  $\mathbb{H}$**

**2 The averaging operators  $\mathcal{A}_i$  and the mean value expansions for  $\Delta_{\mathbb{H}}$**

Given a continuous function  $v : \mathbb{H} \rightarrow \mathbb{R}$  and a radius  $r > 0$ , consider the averages at  $q \in \mathbb{H}$ :

$$\begin{aligned} \mathcal{A}_1(v, r)(q) &= \int_{\partial B_r^2(0)} v(q * (a, b, 0)) \, d\sigma \\ &= \int_0^{2\pi} v(q + r(\cos \theta, \sin \theta, \frac{1}{2}(x \sin \theta - y \cos \theta))) \, d\theta, \\ \mathcal{A}_2(v, r)(q) &= \int_{B_r^2(0)} v(q * (a, b, 0)) \, d(a, b) = \int_{B_r^2(0)} v(q + r(a, b, \frac{1}{2}(xb - ya))) \, d(a, b), \\ \mathcal{A}_3(v, r)(q) &= \int_{B_r(q)} v(p) \, dp = \int_{B_1(0)} v(q * \rho_r(p)) \, dp, \\ \mathcal{A}_{3,K}(v, r)(q) &= \frac{1}{\int_{B_r(0)} \Psi(p) \, dp} \cdot \int_{B_r(q)} \Psi(q^{-1} * p)v(p) \, dp \\ &= \frac{4}{\pi} \int_{B_1(0)} \Psi(p)v(q * \rho_r(p)) \, dp. \end{aligned}$$

Above,  $\Psi$  is the density in the Gauss-Koebe-Levi-Tonelli theorem [9, Theorem 5.6.3]:

$$\Psi(q) = \frac{x^2 + y^2}{((x^2 + y^2)^2 + 16z^2)^{1/2}} = \frac{|(x, y, 0)|_K^2}{|(x, y, z)|_K^2} \quad \text{for all } q = (x, y, z) \in \mathbb{H} \setminus \{0\}.$$

Other types of 3-dimensional averages where the ball  $B_r(q)$  is replaced by its ‘‘ellipsoidal’’ image under a linear map, will be considered in Sect. 5. Recall first the fundamental relation between  $\mathcal{A}_{3,K}$  and  $\mathbb{H}$ -harmonic functions:

**Theorem 2.1** ([9])

(1) Let  $v \in C^2(\mathbb{H})$ . Then for every  $q \in \mathbb{H}$  there holds:

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \left( \mathcal{A}_{3,K}(v, r)(q) - v(q) \right) = \frac{\pi}{24} \Delta_{\mathbb{H}} v(q).$$

(2) If  $v \in C^2(\mathcal{D})$  satisfies  $\Delta_{\mathbb{H}} v = 0$  in some open set  $\mathcal{D} \subset \mathbb{H}$  then:

$$v(q) = \mathcal{A}_{3,K}(v, r)(q) \quad \text{for all } \bar{B}_r(q) \subset \mathcal{D}. \tag{2.1}$$

Conversely, if (2.1) holds for  $v \in C(\mathcal{D})$ , then  $v \in C^\infty(\mathcal{D})$  and  $\Delta_{\mathbb{H}} v = 0$  in  $\mathcal{D}$ .

We now want to develop similar properties of the operators  $\mathcal{A}_i, i \in \{1, 2, 3\}$ . Note first that  $\mathcal{A}_2$  averages the values of  $v$  on a 2-dimensional ellipse in the plane  $q + T_q$ , whose horizontal projection (i.e. projection along the normal direction  $e_3$  in  $\mathbb{R}^3$ ) equals  $B_r^2(x, y)$ . This ellipse coincides with the intersection of  $q + T_q$  and  $B_r(q)$ . The operator  $\mathcal{A}_1$  averages  $v$  on the boundary of the aforementioned ellipse and it is also easy to observe that:

$$\mathcal{A}_2(v, r)(q) = \frac{2}{r^2} \int_0^r s \mathcal{A}_1(v, s)(q) \, ds. \tag{2.2}$$

**Remark 2.2** Functions  $q \mapsto \mathcal{A}_i(v, r)(q)$  are continuous for  $v$  continuous. On the other hand, taking  $v = \mathbb{1}_{\{z>0\}}$  we get  $\mathcal{A}_1(v, \epsilon)(0, 0, \cdot) = \mathcal{A}_2(v, \epsilon)(0, 0, \cdot) = \mathbb{1}_{\{z>0\}}$ , so in general  $\mathcal{A}_1$  and  $\mathcal{A}_2$  do not return a continuous function for  $v$  discontinuous. Nevertheless, by a classical application of the monotone class theorem, it follows that for any locally bounded Borel  $v$ , the functions  $q \mapsto \mathcal{A}_1(v, r)(q)$  and  $q \mapsto \mathcal{A}_2(v, r)(q)$  are well defined and locally bounded Borel. Finally, since the operators  $\mathcal{A}_3$  and  $\mathcal{A}_{3,K}$  average on the solid 3-dimensional Korányi ball  $B_r(q)$ , they both return a continuous function for every  $v \in L^1_{loc}(\mathbb{H})$ .

Our first observation is:

**Proposition 2.3** Let  $v \in C^2(\mathbb{H})$  and  $q \in \mathbb{H}$ . We have the following expansions, as  $r \rightarrow 0$ :

$$\begin{aligned} \mathcal{A}_1(v, r)(q) &= v(q) + \frac{r^2}{4} \Delta_{\mathbb{H}} v(q) + o(r^2), \\ \mathcal{A}_2(v, r)(q) &= v(q) + \frac{r^2}{8} \Delta_{\mathbb{H}} v(q) + o(r^2). \end{aligned} \tag{2.3}$$

In particular, validity of any of the mean value properties  $i \in \{1, 2\}$  in:

$$v(q) = \mathcal{A}_i(v, r)(q) \quad \text{for all } r \in (0, r_0), \tag{2.4}$$

implies  $\Delta_{\mathbb{H}} v(q) = 0$ . Conversely, if  $\Delta_{\mathbb{H}} v = 0$  in some  $B_{r_0}(q)$ , then (2.4) holds for  $i \in \{1, 2\}$ .



**Proof** For a fixed  $q \in \mathbb{H}$ , let  $\phi(r) = \mathcal{A}_2(v, r)(q)$ . Clearly,  $\phi \in \mathcal{C}^2(0, \infty)$  and:

$$\begin{aligned} \phi(r) &= \int_{B_1^2(0)} v(q * r(a, b, 0)) \, d(a, b), \\ \phi'(r) &= \int_{B_1^2(0)} \langle (a, b), \nabla_{\mathbb{H}} v(q * r(a, b, 0)) \rangle \, d(a, b), \\ \phi''(r) &= \int_{B_1^2(0)} \langle (a, b)^{\otimes 2} : \nabla_{\mathbb{H}}^2 v(q * r(a, b, 0)) \rangle \, d(a, b). \end{aligned} \tag{2.5}$$

Passing to the limit, we obtain:  $\lim_{r \rightarrow 0} \phi(r) = v(q)$ ,  $\lim_{r \rightarrow 0} \phi'(r) = 0$  and since  $\int_{B_1^2(0)} a^2 \, d(a, b) = \frac{1}{4}$ , it also follows that:  $\lim_{r \rightarrow 0} \phi''(r) = \frac{1}{4} \Delta_{\mathbb{H}} v(q)$ . We thus conclude (2.3)<sub>2</sub> by Taylor’s theorem at  $r = 0$ . A similar calculation applied to  $r \mapsto \mathcal{A}_1(v, r)(q)$  yields (2.3)<sub>1</sub>. Assume now that  $\Delta_{\mathbb{H}} v = 0$  in  $B_{r_0}(q)$ . By the second formula in (2.5), we get:

$$\begin{aligned} \phi'(r) &= \int_{B_1^2(0)} \langle \nabla_{\mathbb{H}} v(q * r(a, b, 0)), (a, b) \rangle \, d(a, b), \\ &= \frac{1}{\pi} \int_0^1 \int_{\partial B_s^2(0)} \langle \nabla_{\mathbb{H}} v(q * r(a, b, 0)), \frac{(a, b)}{s} \rangle_s \, d\sigma(a, b) \, ds \\ &= \frac{1}{\pi} \int_0^1 s \int_{B_s^2(0)} \operatorname{div} \nabla_{\mathbb{H}} v(q * r(a, b, 0)) \, d(a, b) \, ds \\ &= \frac{1}{\pi} \int_0^1 r s \int_{B_s^2(0)} \Delta_{\mathbb{H}} v(q * r(a, b, 0)) \, d(a, b) \, ds = 0, \end{aligned}$$

for all  $r \in (0, r_0)$ . Consequently,  $\phi$  is constant so that:  $\mathcal{A}_2(v, r)(q) = \phi(r) = \lim_{r \rightarrow 0} \phi(r) = v(q)$  as claimed in (2.4) with  $i = 2$ . Differentiating (2.2) further implies (2.4) for  $i = 1$ . □

In order to weaken the smoothness assumption in Proposition 2.3, recall the mollification procedure in  $\mathbb{H}$ . Let  $J \in \mathcal{C}_0^\infty(\mathbb{H})$  be a nonnegative test function, supported in  $B_1(0)$  and such that  $\int_{\mathbb{H}} J(p) \, dp = 1$ . For  $r > 0$ , define  $J_r = \frac{1}{r^4} J \circ \rho_{1/r}$  that is supported in  $B_r(0)$  and still satisfying  $\int_{\mathbb{H}} J_r(p) \, dp = 1$ . Given  $v \in L^1_{loc}(\mathbb{H})$  the convolution with  $J_r$  is:

$$(v \star J_r)(q) = \int_{\mathbb{H}} v(p) J_r(q * p^{-1}) \, dp = \int_{B_r(0)} v(p^{-1} * q) J_r(p) \, dp.$$

Similarly as in the Euclidean case:  $v \star J_r \in \mathcal{C}^\infty(\mathbb{H})$ . Also, the family  $v \star J_\epsilon$  converges as  $\epsilon \rightarrow 0$  to  $v$  in  $L^1_{loc}(\mathbb{H})$ . When  $v \in \mathcal{C}(\mathbb{H})$  then the convergence is locally uniform and we also note that for all  $i \in \{1, 2, 3, (3, K)\}$ :

$$\mathcal{A}_i(v \star J_\epsilon, r) = \mathcal{A}_i(v, r) \star J_\epsilon \quad \text{for all } \epsilon, r > 0. \tag{2.6}$$

**Corollary 2.4** *Let  $v \in C(\mathcal{D})$  on an open set  $\mathcal{D} \subset \mathbb{H}$ . Validity of any of the mean value properties  $i \in \{1, 2\}$  in:*

$$v(q) = \mathcal{A}_i(v, r)(q) \quad \text{for all } \bar{B}_r(q) \subset \mathcal{D}$$

*implies that  $v \in C^\infty(\mathcal{D})$  and  $\Delta_{\mathbb{H}}v = 0$  in  $\mathcal{D}$ .*

**Proof** Fix an open set  $U$ , compactly contained in  $\mathcal{D}$ . By (2.6) the smooth functions  $v_\epsilon = v \star J_\epsilon$  satisfy the mean value property (2.4) for all  $r, \epsilon$  small enough and all  $q \in U$ . By Proposition 2.3, we thus obtain  $\Delta_{\mathbb{H}}v_\epsilon = 0$  in  $U$ . Consequently, (2.1) holds on  $U$  for each  $v_\epsilon$  and passing to the uniform limit with  $\epsilon \rightarrow 0$ , the same property is valid for  $v$  as well. Applying Theorem 2.1 (2), the claim follows on  $U$  and thus also on  $\mathcal{D}$ . □

For completeness, we now state the mean value property related to the operator  $\mathcal{A}_3$ . We also observe the viscosity version of the same property, which will be used for the  $\mathcal{A}_3$ -like averaging operator developed for the  $\mathbf{p}$ -Laplacian in Sects. 4 and 5. In the Euclidean setting, viscosity solutions in the sense of means have been discussed in [19].

**Proposition 2.5** (a) *Let  $v \in C^2(\mathbb{H})$  and  $q \in \mathbb{H}$ . We have the expansion, as  $r \rightarrow 0$ :*

$$\mathcal{A}_3(v, r)(q) = v(q) + \frac{r^2}{3\pi} \Delta_{\mathbb{H}}v(q) + o(r^2). \tag{2.7}$$

*In particular, validity of  $v(q) = \mathcal{A}_3(v, r)(q)$  for  $r \in (0, r_0)$  implies  $\Delta_{\mathbb{H}}v(q) = 0$ .*

(b) *Let  $v \in C(\mathcal{D})$  on an open set  $\mathcal{D} \subset \mathbb{H}$ . Validity of the mean value property (2.8) in the viscosity sense, as defined below, at every  $q \in \mathcal{D}$ , is equivalent to:  $v \in C^\infty(\mathcal{D})$  and  $\Delta_{\mathbb{H}}v = 0$  in  $\mathcal{D}$ . Namely, we say that:*

$$\mathcal{A}_3(v, r)(q) \stackrel{visc}{=} v(q) + o(r^2) \quad \text{as } r \rightarrow 0, \tag{2.8}$$

*if and only if the following two conditions are satisfied: (1) for every  $\phi \in C^2(\mathcal{D})$  such that  $\phi(q) = v(q)$  and  $\phi < v$  in  $\mathcal{D} \setminus \{q\}$ , there holds:  $\mathcal{A}_3(\phi, r)(q) - \phi(q) \leq o(r^2)$  as  $r \rightarrow 0$ ; (2) for every  $\phi \in C^2(\mathcal{D})$  such that  $\phi(q) = v(q)$  and  $\phi > v$  in  $\mathcal{D} \setminus \{q\}$ , there holds:  $\mathcal{A}_3(\phi, r)(q) - \phi(q) \geq o(r^2)$  as  $r \rightarrow 0$ .*

**Proof** Expansion (2.7) follows in view of  $\int_{B_1(0)} a^2 d(a, b, c) = 2/(3\pi)$ , by an entirely similar calculation as in Proposition 2.3 applied to  $\psi(r) = \mathcal{A}_3(v, r)(q)$ .

The proof of (b) is quite standard, hence we only sketch it. Firstly, by the same argument as in the proof of Theorem 15.2, condition (2.8) is equivalent to  $v$  being viscosity  $\mathbb{H}$ -harmonic; this statement is also a special case of the main result in [14]. Secondly, let  $\bar{B}_r(q_0) \subset \mathcal{D}$  and consider the  $\mathbb{H}$ -harmonic extension  $u$  of  $v|_{\partial B_r(q_0)}$  on  $B_r(q_0)$ , namely the unique solution to:

$$\Delta_{\mathbb{H}}u = 0 \text{ in } B_r(q_0), \quad u = v \text{ on } \partial B_r(q_0).$$

We claim that  $v \leq u$ . Indeed, if  $\sup_{B_r(q_0)}(v - u) > 0$  then also the perturbed difference  $q \mapsto v(q) - (u(q) - \epsilon|q - p_0|_K^4)$  attains its maximum in  $B_r(q_0)$ , if only  $\epsilon > 0$  is sufficiently small. Here,  $p_0 \notin \bar{D}$  is some fixed point. Call the said maximum  $\bar{q} \in B_r(q_0)$ ; using now  $\phi(q) = u(q) - \epsilon|q - p_0|_K^4$  as a test function in the definition of the viscosity solution, we obtain:

$$0 \leq \Delta_{\mathbb{H}}\phi(\bar{q}) = -\epsilon \cdot 24|(\bar{q} - p_0)_{hor}|^2 < 0,$$

which is a contradiction, proving the claim. In a similar manner, it follows that  $v \geq u$ . Thus  $v = u$  is  $\mathbb{H}$ -harmonic in  $B_r(q_0)$  and hence in the whole  $\mathcal{D}$ . Thirdly, it is easy to check that a classical  $\mathbb{H}$ -harmonic function is viscosity  $\mathbb{H}$ -harmonic. This ends the proof. □

### 3 The mean value expansion for $\Delta_{\mathbb{H},\infty}$

In this section, we develop the expansion similar to (2.7) but for the fully nonlinear operator  $\Delta_{\mathbb{H},\infty}$  in (1.7) replacing the linear  $\Delta_{\mathbb{H}}$ . The averaging in the left hand side of (3.2) is then what we call the “deterministic averaging”  $\frac{1}{2}(\sup + \inf)$ , as it corresponds to the two players’ choices of moves, in the Tug of War game modelled on the expansion (3.2), which is then interpreted as the dynamic programming principle for the related process. This construction is conceptually similar to having the “stochastic averaging”  $\mathcal{A}_3$  correspond to the Brownian motion. The proof of Theorem 3.1 is close to the arguments in [32] valid in the Euclidean case; here the application of Lagrange multipliers yields the bound on the non-horizontal component of any minimizer/maximizer of  $v$  on  $B_r(q)$ .

Given a function  $v \in C^2(\mathbb{H})$ , it is useful to observe the following Taylor expansions. Firstly, one directly checks that  $\langle \nabla v(q), p - q \rangle = \langle (\nabla_{\mathbb{H}}, Z)v(q), q^{-1} * p \rangle$  and  $\langle \nabla^2 v(q) : (p - q)^{\otimes 2} \rangle = \langle (\nabla_{\mathbb{H}}, Z)^2 v(q) : (q^{-1} * p)^{\otimes 2} \rangle$ . Consequently, there holds as  $p \rightarrow q$ :

$$v(p) = v(q) + \langle (\nabla_{\mathbb{H}}, Z)v(q), q^{-1} * p \rangle + \frac{1}{2} \langle (\nabla_{\mathbb{H}}, Z)^2 v(q) : (q^{-1} * p)^{\otimes 2} \rangle + o(|p - q|^2).$$

However, since  $(q^{-1} * p)^{\otimes 2} e_3 = o(d(p, q)^2)$  and also  $o(|p - q|^2) \leq o(d(p, q)^2)$ , we obtain the reduced second order Taylor expansion, valid in  $\mathbb{H}$  as  $p \rightarrow q$ :

$$v(p) = v(q) + \langle (\nabla_{\mathbb{H}}, Z)v(q), q^{-1} * p \rangle + \frac{1}{2} \langle \nabla_{\mathbb{H}}^2 v(q)_{sym} : ((q - p)_{hor})^{\otimes 2} \rangle + o(|q^{-1} * p|_K^2). \tag{3.1}$$

**Theorem 3.1** *Let  $v \in C^2(\mathbb{H})$  and let  $q \in \mathbb{H}$ . If  $\nabla_{\mathbb{H}}v(q) \neq 0$ , then we have the following expansion as  $r \rightarrow 0$ :*

$$\frac{1}{2} \left( \inf_{B_r(q)} v + \sup_{B_r(q)} v \right) = v(q) + \frac{r^2}{2} \Delta_{\mathbb{H},\infty}v(q) + o(r^2). \tag{3.2}$$

**Proof 1.** Without loss of generality, we may assume that  $v(q) = 0$  and  $|\nabla_{\mathbb{H}}v(q)| = 1$ . Consider an approximation of  $v$  given by its Taylor expansion in  $\mathbb{H}$ :

$$u(p) = \langle a, q^{-1} * p \rangle + \frac{1}{2} \langle A : ((q - p)_{hor})^{\otimes 2} \rangle,$$

where:

$$a = (\nabla_{\mathbb{H}}, Z)v(q) = (\nabla_{\mathbb{H}}, Z)u(q) \in \mathbb{R}^3 \quad \text{and} \quad A = \nabla_{\mathbb{H}}^2v(q)_{sym} = \nabla_{\mathbb{H}}^2u(q)_{sym} \in \mathbb{R}^{2 \times 2}.$$

We denote  $a = (a_{hor}, a_3)$  and observe that in view of  $|a_{hor}| = 1$ :

$$\Delta_{\mathbb{H},\infty}u(q) = \Delta_{\mathbb{H},\infty}v(q) = \langle A : (a_{hor})^{\otimes 2} \rangle.$$

Then by (3.1) it follows that  $\|u - v\|_{C(B_r(q))} = o(r^2)$ , and consequently:

$$\left| \inf_{B_r(q)} u - \inf_{B_r(q)} v \right| + \left| \sup_{B_r(q)} u - \sup_{B_r(q)} v \right| = o(r^2).$$

It hence suffices to prove (3.2) for the approximant  $u$ . For each  $r > 0$  consider the rescaling:

$$u_r(p) = \frac{1}{r} u(q * \rho_r(p)) = \langle a, (p_{hor}, rz) \rangle + \frac{r}{2} \langle A : p_{hor} \otimes p_{hor} \rangle, \tag{3.3}$$

defined for all  $p = (p_{hor}, z) = (x, y, z) \in \mathbb{H}$ , and note that  $\nabla_{\mathbb{H}}u_r(p) = \nabla_{\mathbb{H}}u(q * \rho_r(p))$  and  $\nabla_{\mathbb{H}}^2u_r(p) = r \nabla_{\mathbb{H}}^2u(q * \delta_r(p))$ . We will prove that as  $r \rightarrow 0$ :

$$\frac{1}{2} \left( \inf_{B_1(0)} u_r + \sup_{B_1(0)} u_r \right) = \frac{r}{2} \langle A : (a_{hor})^{\otimes 2} \rangle + o(r), \tag{3.4}$$

which will imply (3.2) for the function  $u$ , in view of:

$$\inf_{B_r(q)} u = r \inf_{B_1(0)} u_r, \quad \sup_{B_r(q)} u = r \sup_{B_1(0)} u_r \quad \text{and} \quad \Delta_{\mathbb{H},\infty}u(q) = \frac{1}{r} \Delta_{\mathbb{H},\infty}u_r(0).$$

2. Let  $\bar{p}^r, p^r \in \bar{B}_1(0)$  be such that  $u_r(\bar{p}^r) = \inf_{B_1(0)} u_r$  and  $u_r(p^r) = \sup_{B_1(0)} u_r$ . Then for every  $r > 0$  such that  $r|A| < 1$  it follows that  $\nabla u_r(p) = (a_{hor}, ra_3) + r(Ap_{hor}, 0) \neq 0$  for  $p \in B_1(0)$ , so we actually have:

$$\bar{p}^r, p^r \in \partial B_1(0).$$

The method of Lagrange multipliers implies that the following vectors are parallel:

$$\nabla u_r(p^r) \parallel (\nabla |p|_K^4)(p^r), \quad \nabla u_r(\bar{p}^r) \parallel (\nabla |p|_K^4)(\bar{p}^r).$$

Writing  $p^r = (p_{hor}^r, z^r)$  this yields:  $(a_{hor} + rAp_{hor}^r, ra_3) \parallel (|p_{hor}^r|^2 p_{hor}^r, 8z^r)$  and further:

$$p_{hor}^r = \frac{|p_{hor}^r|}{|a_{hor} + rAp_{hor}^r|} (a_{hor} + rAp_{hor}^r) \quad \text{and} \quad z^r = \frac{1}{8}ra_3 \frac{|p_{hor}^r|^3}{|a_{hor} + rAp_{hor}^r|}. \tag{3.5}$$

Consequently, we get:

$$|a_{hor} - p_{hor}^r| \leq 4|A|r, \quad |z^r| \leq \frac{1}{4}r|a_3|,$$

which implies:

$$\begin{aligned} 0 &\leq u_r(p^r) - u_r((a_{hor}, 0)) \\ &= \langle a_{hor}, p_{hor}^r \rangle - 1 + ra_3z^r + \frac{r}{2} \langle A : (p_{hor}^r)^{\otimes 2} - (a_{hor})^{\otimes 2} \rangle \\ &\leq \langle a_{hor}, p_{hor}^r \rangle - 1 + \frac{1}{4}|a_3|^2r^2 + r|A| \cdot |p_{hor}^r - a_{hor}| \leq (4|A|^2 + \frac{1}{4}|a_3|^2)r^2. \end{aligned}$$

Likewise, for the minimizer  $\bar{p}^r$  (rather than the maximizer  $p^r$  above) we have:

$$0 \geq u_r(\bar{p}^r) - u_r((-a_{hor}, 0)) \geq (4|A|^2 - \frac{1}{4}|a_3|^2)r^2,$$

which results in (3.4) because  $u_r((a_{hor}, 0)) + u_r((-a_{hor}, 0)) = r \langle A : (a_{hor})^{\otimes 2} \rangle$ .  $\square$

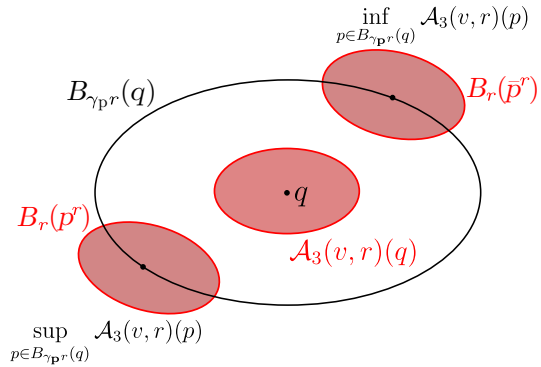
### 4 Two mean value expansions for $\Delta_{\mathbb{H}, \mathbf{p}}$ : $\mathbf{p} > 2$

Combining the mean value expansions and the averaging operators developed for  $\Delta_{\mathbb{H}}$  in Sect. 2, and for  $\Delta_{\mathbb{H}, \infty}$  in Sect. 3, we now state two mean value expansions for  $\Delta_{\mathbb{H}, \mathbf{p}}$ . Heuristically, the first formula (4.1) below, views the normalisation  $\Delta_{\mathbb{H}, \mathbf{p}}^N$  directly through the interpolation (1.6). The related averaging operator is then the superposition of:

- (1) “simple averaging” with prescribed weights  $\alpha_{\mathbf{p}}, (1 - \alpha_{\mathbf{p}})$ ,
- (2) “stochastic averaging”  $\mathcal{A}_3$ ,
- (3) “deterministic averaging”  $\frac{1}{2}(\sup + \inf)$ .

The expansion (4.1) holds for any  $\mathbf{p} > 1$ , however the “simple averaging” coefficients are feasible, in the sense that having  $\alpha_{\mathbf{p}} \in [0, 1]$  allows for their interpretation as probabilities of the stochastic versus deterministic sampling, only for  $\mathbf{p} \geq 2$ . In the

**Fig. 1** The three averaging contributions in the formula (4.2)



Euclidean setting, the parallel formula has been implemented as the dynamic programming principle for Tug of War game with noise in [26].

Our second mean value expansion (4.2) reflects the uniform “simple averaging” of: (1) “stochastic averaging” and (2) “deterministic averaging” applied to the further stochastic one. The fact that the smoothing  $\mathcal{A}_3$  is present in all three terms, results in automatic continuity of solutions to the dynamic programming principle (see Sect. 11); compare to the analysis in [26] that has been based on (4.1) and thus necessitated measurable approximations. Again, the mean value expansion (4.2) works only in the limited range of exponents  $\mathbf{p} > 2$ . In Sect. 5 we will present yet another mean value operator in the Heisenberg group, pertaining to the general case of  $\mathbf{p} \in (1, \infty)$  (Fig. 1).

**Theorem 4.1** *Let  $v \in \mathcal{C}^2(\mathbb{H})$  and let  $q \in \mathbb{H}$ . If  $\nabla_{\mathbb{H}}v(q) \neq 0$  then we have the following expansions below, valid as  $r \rightarrow 0$ :*

(1) For  $\mathbf{p} > 1$  define  $\alpha_{\mathbf{p}} = \frac{3\pi}{2(\mathbf{p} - 2) + 3\pi}$  and  $\beta_{\mathbf{p}} = \frac{2(\mathbf{p} - 2)}{2(\mathbf{p} - 2) + 3\pi}$ . Then:

$$\begin{aligned} \alpha_{\mathbf{p}}\mathcal{A}_3(v, r)(q) + \frac{\beta_{\mathbf{p}}}{2} \left( \inf_{p \in B_r(q)} v(p) + \sup_{p \in B_r(q)} v(p) \right) \\ = v(q) + \frac{r^2}{2(\mathbf{p} - 2) + 3\pi} \cdot \frac{\Delta_{\mathbb{H}, \mathbf{p}}v(q)}{|\nabla_{\mathbb{H}}v(q)|^{\mathbf{p}-2}} + o(r^2). \end{aligned} \tag{4.1}$$

In particular, for  $\mathbf{p} = 2$  we recover the expansion (2.7).

(2) For  $\mathbf{p} > 2$  define  $\gamma_{\mathbf{p}} = \left(\frac{\mathbf{p} - 2}{\pi}\right)^{1/2}$ . Then:

$$\begin{aligned} \frac{1}{3}\mathcal{A}_3(v, r)(q) + \frac{1}{3} \inf_{p \in B_{\gamma p r}(q)} \mathcal{A}_3(v, r)(p) + \frac{1}{3} \sup_{p \in B_{\gamma p r}(q)} \mathcal{A}_3(v, r)(p) \\ = v(q) + \frac{r^2}{3\pi} \cdot \frac{\Delta_{\mathbb{H}, \mathbf{p}}v(q)}{|\nabla_{\mathbb{H}}v(q)|^{\mathbf{p}-2}} + o(r^2). \end{aligned} \tag{4.2}$$

Again, the harmonic expansion (2.7) is recovered asymptotically as  $\mathbf{p} \rightarrow 2^+$ .

**Proof 1.** Summing expansions (2.7) and (3.2) weighted with coefficients  $\alpha_{\mathbf{p}}$  and  $\beta_{\mathbf{p}}$ , we get:

$$\begin{aligned} \alpha_{\mathbf{p}}\mathcal{A}_3(v, r)(q) + \frac{\beta_{\mathbf{p}}}{2} \left( \inf_{B_r(q)} v + \sup_{B_r(q)} v \right) &= v(q) + \left( \frac{\alpha_{\mathbf{p}}}{3\pi} \Delta_{\mathbb{H}} v(q) + \frac{\beta_{\mathbf{p}}}{2} \Delta_{\mathbb{H}, \infty} v(q) \right) r^2 + o(r^2) \\ &= v(q) + \frac{\alpha_{\mathbf{p}} r^2}{3\pi} \left( \Delta_{\mathbb{H}} v(q) + (\mathbf{p} - 2) \Delta_{\mathbb{H}, \infty} v(q) \right) + o(r^2) \\ &= v(q) + \frac{\alpha_{\mathbf{p}} r^2}{3\pi} |\nabla_{\mathbb{H}} v(q)|^{2-\mathbf{p}} \Delta_{\mathbb{H}, \mathbf{p}} v(q) + o(r^2), \end{aligned}$$

because  $3\pi\beta_{\mathbf{p}}/(2\alpha_{\mathbf{p}}) = \mathbf{p} - 2$  and  $\alpha_{\mathbf{p}} + \beta_{\mathbf{p}} = 1$ , proving (4.1).

2. To show (4.2), consider the function  $u(p) = \mathcal{A}_3(v, r)(p)$  and note that since  $\nabla_{\mathbb{H}} v(q) \neq 0$  we also have:  $\nabla_{\mathbb{H}} u(q) = \mathcal{A}_3(\nabla_{\mathbb{H}} v, r)(q) \neq 0$ . We may thus apply (3.2) to  $u$  and obtain:

$$\begin{aligned} &\frac{1}{2} \left( \inf_{p \in B_{\gamma_{\mathbf{p}} r}(q)} \mathcal{A}_3(v, r)(p) + \sup_{p \in B_{\gamma_{\mathbf{p}} r}(q)} \mathcal{A}_3(v, r)(p) \right) \\ &= \mathcal{A}_3(v, r)(q) + \frac{1}{2} \gamma_{\mathbf{p}}^2 r^2 \Delta_{\mathbb{H}, \infty} u(q) + o(r^2). \end{aligned}$$

Since:

$$\Delta_{\mathbb{H}, \infty} u(q) = \left\langle \mathcal{A}_3(\nabla_{\mathbb{H}}^2 v, r)(q) : \left( \frac{\mathcal{A}_3(\nabla_{\mathbb{H}} v, r)(q)}{|\mathcal{A}_3(\nabla_{\mathbb{H}} v, r)(q)|} \right)^{\otimes 2} \right\rangle = \Delta_{\mathbb{H}, \infty} v(q) + o(1),$$

it follows in view of (2.7) that:

$$\begin{aligned} &\frac{1}{3} \mathcal{A}_3(v, r)(q) + \frac{1}{3} \inf_{B_{\gamma_{\mathbf{p}} r}(q)} \mathcal{A}_3(v, r) + \frac{1}{3} \sup_{B_{\gamma_{\mathbf{p}} r}(q)} \mathcal{A}_3(v, r) \\ &= \mathcal{A}_3(v, r)(q) + \frac{\gamma_{\mathbf{p}}^2 r^2}{3} \Delta_{\mathbb{H}, \infty} v(q) + o(r^2) \\ &= v(q) + \left( \frac{1}{3\pi} \Delta_{\mathbb{H}} v(q) + \frac{\gamma_{\mathbf{p}}^2}{3} \Delta_{\mathbb{H}, \infty} v(q) \right) r^2 + o(r^2) \\ &= v(q) + \frac{r^2}{3\pi} \left( \Delta_{\mathbb{H}} v(q) + (\mathbf{p} - 2) \Delta_{\mathbb{H}, \infty} v(q) \right) + o(r^2) \\ &= v(q) + \frac{r^2}{3\pi} |\nabla_{\mathbb{H}} v(q)|^{2-\mathbf{p}} \Delta_{\mathbb{H}, \mathbf{p}} v(q) + o(r^2), \end{aligned}$$

because  $\gamma_{\mathbf{p}}^2 \pi = \mathbf{p} - 2$ . The proof is done. □

**Remark 4.2** Statement (1) of Theorem 4.1 also holds for  $\mathbf{p} = 1$ , as noted by the reviewers. The same expansion (4.2) holds if we replace the constant exponent  $\mathbf{p}$  by a variable exponent  $\mathbf{p}(\cdot) > 2$ , retaining the scaling factor  $\gamma_{\mathbf{p}} = (\mathbf{p}(\cdot) - 2)/\pi$ <sup>1/2</sup>. This formulation can be applied to the so-called *strong  $\mathbf{p}(\cdot)$ -Laplacian*:

$$\Delta_{\mathbb{H}, \mathbf{p}(\cdot)}^S v(q) = |\nabla_{\mathbb{H}} v(q)|^{\mathbf{p}(q)-2} \left( \Delta_{\mathbb{H}} v(q) + (\mathbf{p}(q) - 2) \Delta_{\mathbb{H}, \infty} v(q) \right).$$

We remark that there are different and non-equivalent ways of extending the constant exponent  $\mathbf{p}$ - $\mathbb{H}$ -Laplacian  $\Delta_{\mathbb{H},\mathbf{p}}$  to the variable exponent case [28]. The strong  $\mathbf{p}(\cdot)$ -Laplacian was introduced in the Euclidean setting in [2], and for regular functions it satisfies:

$$\Delta_{\mathbb{H},\mathbf{p}(\cdot)}^S v(q) = \Delta_{\mathbb{H},\mathbf{p}(\cdot)} v(q) - |\nabla_{\mathbb{H}} v(q)|^{\mathbf{p}(q)-2} \log(|\nabla_{\mathbb{H}} v(q)|) \langle \nabla_{\mathbb{H}} v(q), \nabla_{\mathbb{H}} \mathbf{p}(q) \rangle.$$

This connection has also been studied for weak solutions in the Euclidean case, [35]. Here,  $\Delta_{\mathbb{H},\mathbf{p}(\cdot)}$  is a particular version of the  $\mathbf{p}(\cdot)$ - $\mathbb{H}$ -Laplacian, resulting by taking the Euler-Lagrange equation of the functional  $\mathcal{E}(v) = \int_{\mathcal{D}} \frac{1}{\mathbf{p}(q)} |\nabla_{\mathbb{H}} v(q)|^{\mathbf{p}(q)} \, dq$ , namely:

$$\Delta_{\mathbb{H},\mathbf{p}(\cdot)} v(q) = \operatorname{div}_{\mathbb{H}} (|\nabla_{\mathbb{H}} v(q)|^{\mathbf{p}(q)-2} \nabla_{\mathbb{H}} v(q)).$$

A version of random Tug of War game in the context of the parabolic strong  $\mathbf{p}(x, t)$ -equation in the Euclidean setting, has been developed in [30]. There, the process is modelled on the asymptotic expansion (4.1) and results in the discontinuous approximations  $u_\epsilon$ . In our work, the game values in (11.1), modelled on the expansion (4.2), have boundary-implied regularity.

### 5 The anisotropic mean value expansion for $\Delta_{\mathbb{H},\mathbf{p}}$ : $1 < \mathbf{p} < \infty$

We now propose another mean value expansion that, unlike (4.1) and (4.2), leads to the dynamic programming principle that works for any exponent  $\mathbf{p} \in (1, \infty)$ . The key idea, developed in the Euclidean setting in [22], is to superpose:

- (1) “deterministic averaging”  $\frac{1}{2}(\sup + \inf)$  on Korányi balls, with
- (2) “stochastic averaging”  $\mathcal{A}_3$  on the “Korányi ellipsoids” defined as the images of a unit ball under suitable linear transformations.

We begin by the counterpart of Proposition 2.3 on such ellipsoids, defined as follows.

For a radius  $r > 0$ , an aspect ratio  $\alpha > 0$  and an orientation vector  $v = (v_{hor}, v_3) \in \mathbb{H}$  that we normalize to be of unit Euclidean length:  $|v_{hor}|^2 + v_3^2 = 1$ , we set the *Korányi ellipsoid* centered at a given  $q \in \mathbb{H}$  to be:

$$E(q, r; \alpha, v) = q * \rho_r \{ p + (\alpha - 1) \langle p, v \rangle v; p \in B_1(0) \}. \tag{5.1}$$

The open, bounded, smooth set  $E(q, r; \alpha, v) \subset \mathbb{H}$  is thus obtained by applying the linear map:

$$p \mapsto L(p; \alpha, v) = (p - \langle p, v \rangle v) + \alpha \langle p, v \rangle v$$

to the unit Korányi ball  $B_1(0)$ , then scaling via Heisenberg dilation  $\rho_r$ , and centering the image at  $q$  by the group operation. Given a continuous function  $v \in \mathcal{C}^2(\mathbb{H})$ , define the average:



$$\mathcal{A}_3(v, r; \alpha, \nu)(q) = \int_{E(q,r;\alpha,\nu)} v(p) \, dp = \int_{B_1(0)} v(q * \delta_r L(p; \alpha, \nu)) \, dp.$$

Observe that  $E(q, r; 1, \nu) = B_r(q)$ , so likewise:  $\mathcal{A}_3(v, r; 1, \nu) = \mathcal{A}_3(v, r)$  for all orientations  $\nu$ .

**Proposition 5.1** *Let  $v \in C^2(\mathbb{H})$  and  $q \in \mathbb{H}$ . We have the following expansion, as  $r \rightarrow 0$ :*

$$\begin{aligned} \mathcal{A}_3(v, r; \alpha, \nu)(q) &= v(q) + \frac{r^2}{3\pi} \left( \Delta_{\mathbb{H}} v(q) + (\alpha^2 - 1) \langle \nabla_{\mathbb{H}}^2 v(q) : \nu_{hor}^{\otimes 2} \rangle \right) + o(r^2). \end{aligned} \tag{5.2}$$

**Proof** As in the proof of Proposition 2.3, define the auxiliary function  $\phi(r) = \mathcal{A}_3(v, r; \alpha, \nu)(q)$ . Then  $\phi \in C^2(0, \infty)$  and it is easy to compute that:

$$\begin{aligned} \phi(r) &= \int_{B_1(0)} v \left( q * (rp_{hor} + r(\alpha - 1)\langle p, \nu \rangle \nu_{hor}, r^2 p_3 + r^2(\alpha - 1)\langle p, \nu \rangle \nu_3) \right) \, dp, \\ \phi'(r) &= \int_{B_1(0)} \left\langle p_{hor} + (\alpha - 1)\langle p, \nu \rangle \nu_{hor}, \nabla_{\mathbb{H}} \right\rangle \cdot v(q * \rho_r L(p; \alpha, \nu)) \\ &\quad + 2r(p_3 + (\alpha - 1)\langle p, \nu \rangle \nu_3) \cdot Zv(q * \rho_r L(p; \alpha, \nu)) \, dp. \end{aligned}$$

Further,  $B_1(0)$  being symmetric implies:

$$\begin{aligned} \lim_{r \rightarrow 0} \phi''(r) &= \lim_{r \rightarrow 0} \int_{B_1(0)} \left\langle (p_{hor} + (\alpha - 1)\langle p, \nu \rangle \nu_{hor})^{\otimes 2} : \nabla_{\mathbb{H}}^2 v(q * \rho_r L(p; \alpha, \nu)) \right\rangle \\ &\quad + 2(p_3 + (\alpha - 1)\langle p, \nu \rangle \nu_3) \cdot Zv(q * \rho_r L(p; \alpha, \nu)) \, dp \\ &= \left\langle \int_{B_1(0)} (p_{hor} + (\alpha - 1)\langle p, \nu \rangle \nu_{hor})^{\otimes 2} \, dp : \nabla_{\mathbb{H}}^2 v(q) \right\rangle. \end{aligned}$$

Expanding the first matrix expression in the right hand side above to:

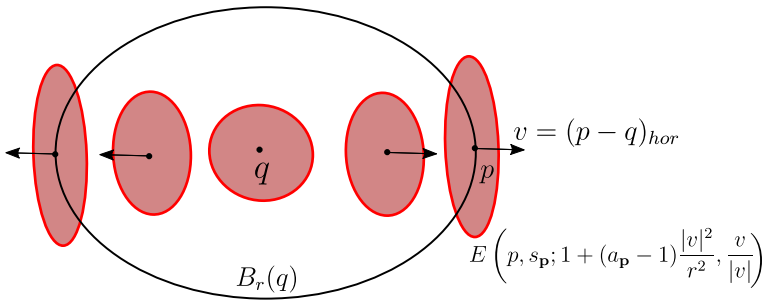
$$\int_{B_1(0)} p_{hor}^{\otimes 2} \, dp + 2(\alpha - 1) \int_{B_1(0)} \langle p, \nu \rangle (p_{hor} \otimes \nu_{hor})_{sym} \, dp + (\alpha - 1)^2 \int_{B_1(0)} \langle p, \nu \rangle^2 \, dp \cdot \nu_{hor}^{\otimes 2}$$

and using  $\int_{B_1(0)} a^2 \, d(a, b, c) = \frac{2}{3\pi}$  with  $|\nu| = 1$ , to compute  $\int_{B_1(0)} p_{hor}^{\otimes 2} \, dp = \frac{2}{3\pi} Id_2$ , and:

$$\int_{B_1(0)} \langle p, \nu \rangle (p_{hor} \otimes \nu_{hor})_{sym} \, dp = \frac{2}{3\pi} \nu_{hor}^{\otimes 2}, \quad \int_{B_1(0)} \langle p, \nu \rangle^2 \, dp = \frac{2}{3\pi},$$

we conclude that:

$$\lim_{r \rightarrow 0} \phi''(r) = \frac{2}{3\pi} \left\langle Id_2 + (\alpha^2 - 1) \nu_{hor}^{\otimes 2} : \nabla_{\mathbb{H}}^2 v(q) \right\rangle.$$



**Fig. 2** The “stochastic sampling” domains, centered at various positions  $p$  within the “deterministic sampling” domain at  $q$ , in the expansion (5.4)

As  $\lim_{r \rightarrow 0} \phi(r) = v(q)$  and  $\lim_{r \rightarrow 0} \phi'(r) = 0$ , the result follows by applying Taylor’s theorem at  $r = 0$ . □

It is clear that by choosing  $\alpha = \sqrt{\mathbf{p} - 1}$  and  $v = (\frac{\nabla_{\mathbb{H}} v(q)}{|\nabla_{\mathbb{H}} v(q)|}, 0)$ , in virtue of the interpolation (1.6) we obtain:  $\mathcal{A}_3(v, r; \alpha, v) = v(q) + \frac{r^2 |\nabla_{\mathbb{H}} v(q)|^{2-\mathbf{p}}}{3\pi} \Delta_{\mathbb{H}, \mathbf{p}} v(q) + o(r^2)$ . In order to derive a mean value expansion where the left hand side averaging does not require the knowledge of  $\nabla_{\mathbb{H}} v(q)$  and allows for the identification of a  $\mathbf{p}$ - $\mathbb{H}$ -harmonic function that is a priori only continuous, we need to additionally average over all equally probable horizontal vectors  $v_{hor}$ . Since only such horizontal orientations are relevant, we observe that the related Korányi ellipsoid in (5.1):

$$E(0, 1; \alpha, (v_{hor}, 0)) = \left\{ (p_{hor} + (\alpha - 1)\langle p_{hor}, v_{hor} \rangle v_{hor}, p_3); p \in B_1(0) \right\},$$

can be interpreted as the “Korányi lift” of the two-dimensional ellipse with radius 1:

$$\left\{ p_{hor} + (\alpha - 1)\langle p_{hor}, v_{hor} \rangle v_{hor}; p_{hor} \in B_1^2(0) \right\} \subset \mathbb{R}^2.$$

We remark that the expansion (5.4) is related to another interpolation property of  $\Delta_{\mathbb{H}, \mathbf{p}}$ :

$$\Delta_{\mathbb{H}, \mathbf{p}} v = |\nabla v|^{\mathbf{p}-2} \left( |\nabla_{\mathbb{H}} v| \Delta_{\mathbb{H}, 1} v + (\mathbf{p} - 1) \Delta_{\mathbb{H}, \infty} v \right),$$

which has first appeared, in the context of the applications of  $\Delta_{\mathbf{p}}$  to image recognition, in [18] (Fig. 2).

**Theorem 5.2** *Let  $v \in \mathcal{C}^2(\mathbb{H})$  and  $q \in \mathbb{H}$  be such that  $\nabla_{\mathbb{H}} v(q) \neq 0$ . Given  $1 < \mathbf{p} < \infty$ , let the scaling exponents  $s_p, a_p > 0$  satisfy:*

$$\frac{3\pi}{2s_p^2} + a_p^2 = \mathbf{p} - 1. \tag{5.3}$$

Then, the following expansion is valid as  $r \rightarrow 0$ :

$$\begin{aligned} & \frac{1}{2} \left( \inf_{p \in B_r(q)} + \sup_{p \in B_r(q)} \right) \mathcal{A}_3 \left( v, s_{\mathbf{p}} r; 1 + (a_{\mathbf{p}} - 1) \frac{|(p - q)_{hor}|^2}{r^2}, \frac{(p - q)_{hor}}{|(p - q)_{hor}|} \right) (p) \\ &= v(q) + \frac{r^2}{\mathbf{p} - 1} \cdot \frac{\Delta_{\mathbb{H}, \mathbf{p}} v(q)}{|\nabla_{\mathbb{H}} v(q)|^{\mathbf{p}-2}} + o(r^2). \end{aligned} \tag{5.4}$$

**Proof 1.** In the statement (5.4) and below, we often write  $v_{hor}$  instead of  $(v_{hor}, 0) \in \mathbb{H}$ , if no ambiguity arises. We define the following continuous function  $B_r(q) \ni p \mapsto f_r(p)$ :

$$f_r(p) = \mathcal{A}_3 \left( v, s_{\mathbf{p}} r; 1 + (a_{\mathbf{p}} - 1) \frac{|(p - q)_{hor}|^2}{r^2}, \frac{(p - q)_{hor}}{|(p - q)_{hor}|} \right) (p).$$

In particular, when  $(p - q)_{hor} = 0$ , the above formula still makes sense and returns:  $f_r(p) = \mathcal{A}_3(v, s_{\mathbf{p}} r)(p) = \int_{B_{s_{\mathbf{p}} r}(p)} v$ . Applying Proposition 5.1 and the Taylor expansion in (3.1), we get:

$$\begin{aligned} f_r(p) &= v(p) + \frac{r^2 s_{\mathbf{p}}^2}{3\pi} \Delta_{\mathbb{H}} v(p) \\ &+ \frac{s_{\mathbf{p}}^2 (a_{\mathbf{p}} - 1)}{3\pi} \left( 2 + (a_{\mathbf{p}} - 1) \frac{|(p - q)_{hor}|^2}{r^2} \right) (\nabla_{\mathbb{H}}^2 v(p) : (p - q)_{hor}^{\otimes 2}) + o(r^2) \\ &= v(q) + \frac{r^2 s_{\mathbf{p}}^2}{3\pi} \Delta_{\mathbb{H}} v(q) + \langle (\nabla_{\mathbb{H}}, Z)v(q), q^{-1} * p \rangle \\ &+ \left( \frac{1}{2} + \frac{s_{\mathbf{p}}^2 (a_{\mathbf{p}} - 1)}{3\pi} \left( 2 + (a_{\mathbf{p}} - 1) \frac{|(p - q)_{hor}|^2}{r^2} \right) \right) (\nabla_{\mathbb{H}}^2 v(q) : (p - q)_{hor}^{\otimes 2}) + o(r^2) \\ &= \bar{f}_r(p) + o(r^2), \end{aligned}$$

because  $o(|q^{-1} * p|_K^2)$  can be replaced by  $o(r^2)$  for  $p \in B_r(q)$ . The left hand side of (5.4) is thus:

$$\frac{1}{2} \left( \inf_{p \in B_r(q)} + \sup_{p \in B_r(q)} \right) f_r(p) = \frac{1}{2} \left( \inf_{p \in B_r(q)} + \sup_{p \in B_r(q)} \right) \bar{f}_r(p) + o(r^2). \tag{5.5}$$

Observe that Lemma 3.1 cannot be used directly to find the principal term in the expansion of the right hand side above, even though  $\nabla_{\mathbb{H}} \bar{f}_r(q) = \nabla_{\mathbb{H}} v(q) \neq 0$ , simply because the function to be minimized/maximized depends on  $r$ . We may however use the argument in the second step of proof of (3.2), as completed below.

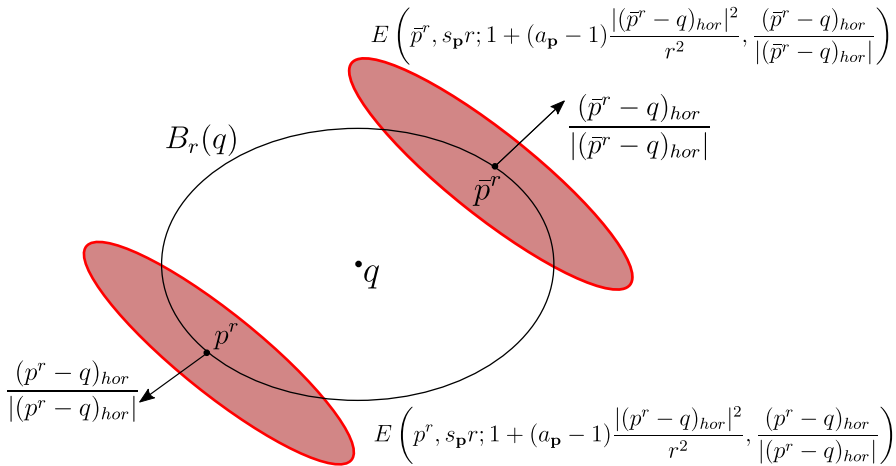


Fig. 3 The two averaging contributions in the formula (5.4)

2. We write  $\bar{f}_r(q * \rho_r(p)) = v(q) + \frac{r^2 s_p^2}{3\pi} \Delta_{\mathbb{H}} v(q) + r g_r(p)$  for  $p \in B_1(0)$ , so that:

$$g_r(p) = \langle a_{hor}, p_{hor} \rangle + r \left( a_3 p_3 + \langle A : p_{hor}^{\otimes 2} \rangle + |p_{hor}|^2 \langle B : p_{hor}^{\otimes 2} \rangle \right),$$

where:  $a = (a_{hor}, a_3) = (\nabla_{\mathbb{H}}, Z)v(q) \in \mathbb{R}^3$ ,

and:  $A = \left( \frac{1}{2} + \frac{2s_p^2(a_p - 1)}{3\pi} \right) \nabla_{\mathbb{H}}^2 v(q) \in \mathbb{R}^{2 \times 2}$ ,  $B = \frac{s_p^2(a_p - 1)^2}{3\pi} \nabla_{\mathbb{H}}^2 v(q) \in \mathbb{R}^{2 \times 2}$ .

(5.6)

Let  $\bar{p}^r, p^r \in \bar{B}_1(0)$  be, respectively, a minimizer and a maximizer of  $g_r$  on  $\bar{B}_1(0)$ . Since in view of  $a_{hor} \neq 0$  there holds  $\nabla g_r \neq 0$  in  $B_1(0)$  for all  $r$  sufficiently small, it follows that  $\bar{p}^r, p^r \in \partial B_1(0)$ . Further, the method of Lagrange multipliers yields:  $\nabla g_r(p^r) \parallel \nabla(|p|_K^4)(p^r)$  so as in (3.5):

$$p_3^r = \frac{1}{8} r a_3 \frac{|p_{hor}^r|^3}{|a_{hor} + 2r(Ap_{hor}^r + |p_{hor}^r|^2 B p_{hor}^r + \langle B : (p_{hor}^r)^{\otimes 2} \rangle p_{hor}^r)|}$$

In particular, for all  $r$  sufficiently small, we obtain (Fig. 3):

$$|p_3^r| \leq \frac{1}{4} r |a_3|. \tag{5.7}$$

We now observe that:

$$\begin{aligned} 0 &\leq g_r(p^r) - g_r\left(\frac{a_{hor}}{|a_{hor}|}\right) \\ &= \langle a_{hor}, p_{hor}^r \rangle - |a_{hor}| + r a_3 p_3^r \end{aligned}$$

$$\begin{aligned}
 &+ r \langle A : (p_{hor}^r)^{\otimes 2} - \left(\frac{a_{hor}}{|a_{hor}|}\right)^{\otimes 2} \rangle + |p_{hor}^r|^2 \langle B : (p_{hor}^r)^{\otimes 2} \rangle - \langle B : \left(\frac{a_{hor}}{|a_{hor}|}\right)^{\otimes 2} \rangle \\
 &\leq \langle a_{hor}, p_{hor}^r \rangle - |a_{hor}| + \frac{1}{4} r^2 a_3^2 + 2r(|A| + 2|B|) \left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right|,
 \end{aligned}$$

where we have used (5.7) and  $|p_{hor}^r| \left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right| \leq 2 \left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right|$ . It thus follows that:

$$\left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right|^2 \leq \frac{1}{2} r^2 \frac{a_3^2}{|a_{hor}|} + 4 \frac{r}{|a_{hor}|} (|A| + 2|B|) \left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right|,$$

resulting in:  $\left| p_{hor}^r - \frac{a_{hor}}{|a_{hor}|} \right| \leq Cr$  with  $C > 0$  depending only on  $|A|, |B|, |a_3|, \frac{1}{|a_{hor}|}$ . In conclusion:

$$0 \leq g_r(p^r) - g_r\left(\frac{a_{hor}}{|a_{hor}|}\right) \leq \frac{1}{4} r^2 a_3^2 + Cr^2(|A| + 2|B|).$$

3. Likewise, for the maximizer  $\bar{p}^r$  of  $g_r$  on  $\bar{B}_1(0)$ , we get:

$$0 \geq g_r(\bar{p}^r) - g_r\left(-\frac{a_{hor}}{|a_{hor}|}\right) \geq -\frac{1}{4} r^2 a_3^2 + Cr^2(|A| + 2|B|).$$

The two above inequalities imply:

$$\begin{aligned}
 \frac{1}{2} \left( \inf_{p \in B_1(0)} + \sup_{p \in B_1(0)} \right) g_r(p) &= \frac{1}{2} (g_r(p^r) + g_r(\bar{p}^r)) = \frac{1}{2} \left( g_r\left(\frac{a_{hor}}{|a_{hor}|}\right) \right. \\
 &+ \left. g_r\left(-\frac{a_{hor}}{|a_{hor}|}\right) \right) + O(r^2) \\
 &= \frac{r}{|a_{hor}|^2} \langle A + B : a_{hor}^{\otimes 2} \rangle + O(r^2) = r \left( \frac{1}{2} + \frac{s_{\mathbf{p}}^2(a_{\mathbf{p}}^2 - 1)}{3\pi} \right) \Delta_{\mathbb{H}, \infty} v(q) + O(r^2).
 \end{aligned}$$

Consequently, recalling the definition of  $g_r$ , we get:

$$\begin{aligned}
 \frac{1}{2} \left( \inf_{p \in B_r(q)} + \sup_{p \in B_r(q)} \right) \bar{f}_r(p) &= v(q) + \frac{r^2 s_{\mathbf{p}}^2}{3\pi} \Delta_{\mathbb{H}} v(q) \\
 &+ r^2 \left( \frac{1}{2} + \frac{s_{\mathbf{p}}^2(a_{\mathbf{p}}^2 - 1)}{3\pi} \right) \Delta_{\mathbb{H}, \infty} v(q) + o(r^2) \tag{5.8} \\
 &= v(q) + \frac{r^2 s_{\mathbf{p}}^2}{3\pi} \left[ \Delta_{\mathbb{H}} v(q) + \left( \frac{3\pi}{2s_{\mathbf{p}}^2} + a_{\mathbf{p}}^2 - 1 \right) \Delta_{\mathbb{H}, \infty} v(q) \right] + o(r^2)
 \end{aligned}$$

which directly yields (5.4) in virtue of (5.5) and (5.3). □

**PART II: The  $\mathbb{H}$ -Laplacian  $\Delta_{\mathbb{H}}$  and random walks in  $\mathbb{H}$**

**6 Horizontal  $\epsilon$ -walk in the Heisenberg group**

Let  $\mathcal{D} \subset \mathbb{H}$  be an open, bounded and connected set. In this section, we develop a probability setting related to the expansion (2.3)<sub>2</sub>. The key role is played by the 3-dimensional process  $\{Q_n\}_{n=0}^\infty$ , whose increments are 2-dimensional, with the third variable slaved to the first two via the Levy area. We then apply the classical argument and argue that  $\{Q_n\}$  accumulates a.s. on  $\partial\mathcal{D}$ , and that its expectation yields a  $\mathbb{H}$ -harmonic extension of a given boundary data  $F$ .

Let  $\Omega_1 = B_1^2(0)$  and define:

$$\Omega = (\Omega_1)^\mathbb{N} = \{\omega = \{w_i\}_{i=1}^\infty; w_i = (a_i, b_i) \in B_1^2(0) \text{ for all } i \in \mathbb{N}\}.$$

The probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is given as the countable product of  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ , where:

$$\mathbb{P}_1(D) = \frac{|D|}{|B_1^2(0)|} \text{ for all } D \in \mathcal{F}_1$$

is the normalized Lebesgue measure on the  $\sigma$ -algebra  $\mathcal{F}_1$  of Borel subsets of  $B_1^2(0)$ . For any  $n \in \mathbb{N}$ , we also define the probability space  $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$  as the product of  $n$  copies of  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ . We always identify the  $\sigma$ -algebras  $\mathcal{F}_n$  with the corresponding sub- $\sigma$ -algebras of  $\mathcal{F}$ , consisting of sets of the form  $F \times \prod_{i=n+1}^\infty \Omega_1$  for all  $F \in \mathcal{F}_n$ . Note that the sequence  $\{\mathcal{F}_n\}_{n=0}^\infty$ , where we set  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ , is a filtration of  $\mathcal{F}$ .

Given  $q_0 \in \mathcal{D}$  and a parameter  $\epsilon \in (0, 1)$ , we now recursively define the sequence of random variables  $\{Q_n^{\epsilon, q_0} : \Omega \rightarrow \mathcal{D}\}_{n=0}^\infty$ , that will converge as  $n \rightarrow \infty$  to a limiting random variable  $Q^{\epsilon, q_0}$ . We use  $\epsilon \ll 1$  as ultimately we will consider the behavior of  $Q^{\epsilon, q_0}$  as  $\epsilon \rightarrow 0$ . Also, for simplicity of notation, we often suppress the superscripts  $\epsilon$  and  $q_0$  and write  $Q_n$  or  $Q$  instead of  $Q_n^{\epsilon, q_0}$  or  $Q^{\epsilon, q_0}$ , if no ambiguity arises. Define:

$$\begin{aligned} Q_0 &\equiv q_0, \\ Q_n(w_1, \dots, w_n) &= q_{n-1} * (\epsilon \wedge d(q_{n-1}, \partial\mathcal{D}))(a_n, b_n, 0) \\ &= q_{n-1} + (\epsilon \wedge d(q_{n-1}, \partial\mathcal{D}))(a_n, b_n, \frac{1}{2}(x_{n-1}b_n - y_{n-1}a_n)), \end{aligned} \tag{6.1}$$

where  $q_{n-1} = (x_{n-1}, y_{n-1}, z_{n-1}) = Q_{n-1}(w_1, \dots, w_{n-1})$  and  $w_n = (a_n, b_n)$ .

That is, the position  $q_{n-1}$  is advanced uniformly within the 2-dimensional Korányi ellipse in  $T_{q_{n-1}}$  determined by the horizontal radius that is the minimum of  $\epsilon$  and the distance  $d(q_{n-1}, \partial\mathcal{D})$  of the current position from the boundary of  $\mathcal{D}$ .

**Lemma 6.1** *The sequence  $\{Q_n\}_{n=0}^\infty$  is a martingale relative to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$  and it converges, pointwise a.s., to some random variable  $Q : \Omega \rightarrow \partial\mathcal{D}$ .*

**Proof** The martingale property follows directly from the definition (6.1):

$$\begin{aligned} \mathbb{E}(Q_n \mid \mathcal{F}_{n-1})(w_1, \dots, w_{n-1}) &= \int_{\Omega_1} Q_n(w_1, \dots, w_n) \, d\mathbb{P}_1(w_n) \\ &= q_{n-1} + (\epsilon \wedge d(q_{n-1}, \partial\mathcal{D})) \int_{B_1^2(0)} \left( a_n, b_n, \frac{1}{2}(x_{n-1}b_n - y_{n-1}a_n) \right) \, d(a_n, b_n) \\ &= q_{n-1} \quad \text{a.s. in } \Omega_{n-1}, \end{aligned}$$

because the added linear term integrates to 0 in all components on the symmetric  $B_1^2(0)$ .

Being a bounded martingale, the sequence  $\{Q_n\}_{n=0}^\infty$  converges to some random variable  $Q : \Omega \rightarrow \bar{\mathcal{D}}$ . It remains to show that  $\mathbb{P}$ -a.s. we have:  $Q \in \partial\mathcal{D}$ . To this end, observe that:

$$\left\{ \lim_{n \rightarrow \infty} Q_n = Q \right\} \cap \{Q \in \mathcal{D}\} \subset \bigcup_{n \in \mathbb{N}, \delta \in (0, \epsilon) \cap \mathbb{Q}} A(n, \delta), \tag{6.2}$$

where  $A(n, \delta) = \left\{ d(Q_i, \partial\mathcal{D}) \geq \delta \text{ and } |Q_{i+1} - Q_i| \leq \frac{\delta}{2} \text{ for all } i \geq n \right\}$ .

Also, if  $\omega = \{w_i\}_{i=1}^\infty \in A(n, \delta)$  then for all  $i \geq n$  we have:

$$\frac{\delta}{2} \geq |Q_{i+1}(\omega) - Q_i(\omega)| \geq (\epsilon \wedge d(q_i, \partial\mathcal{D}))|w_{i+1}| \geq (\epsilon \wedge \delta)|w_{i+1}| = \delta|w_{i+1}|,$$

which implies:  $A(n, \delta) \subset \{\omega = \{w_i\}_{i=1}^\infty \in \Omega; |w_i| \leq \frac{1}{2} \text{ for all } i \geq n\}$ . We conclude that:

$$\mathbb{P}(A(n, \delta)) \leq \lim_{i \rightarrow \infty} \mathbb{P}_1(B_{1/2}^2(0))^{i-n} = 0 \quad \text{for all } n \in \mathbb{N} \text{ and all } \delta \in (0, \epsilon)$$

so that the event in the left hand side of (6.2) has probability 0 as well. □

Given a continuous function  $F : \partial\mathcal{D} \rightarrow \mathbb{R}$ , we define now:

$$u^\epsilon(q) = \mathbb{E}[F \circ Q^{\epsilon, q}] = \lim_{n \rightarrow \infty} \mathbb{E}[F \circ Q_n^{\epsilon, q}] \quad \text{for all } q \in \mathcal{D}, \tag{6.3}$$

where in the last limiting expression above we have identified  $F$  with some continuous extension of itself on  $\bar{\mathcal{D}}$ . Since for every  $n \in \mathbb{N}$  the random variable  $F \circ Q_n^{\epsilon, q}$  is jointly Borel in the variables  $q$  and  $\omega$ , it follows that  $u^\epsilon : \mathcal{D} \rightarrow \mathbb{R}$  is bounded (by  $\|F\|_{C(\partial\mathcal{D})}$ ) and Borel. It is also clear that this construction is monotone in  $F$ , in the sense that if  $F_1 \leq F_2$  on  $\partial\mathcal{D}$  then  $u_{F_1}^\epsilon \leq u_{F_2}^\epsilon$ , with obvious notation.

**Lemma 6.2** *The function  $u^\epsilon$  satisfies:*

$$u^\epsilon(q) = \mathcal{A}_2(u^\epsilon, \epsilon \wedge d(q, \partial\mathcal{D}))(q) \quad \text{for all } q \in \mathcal{D}. \tag{6.4}$$

Moreover, the sequence  $\{u^\epsilon \circ Q_n\}_{n=0}^\infty$  is a martingale relative to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$ .

**Proof** An application of Fubini’s theorem, in view of the definition in (6.1), gives directly:

$$\mathbb{E}[F \circ Q_n] = \int_{\Omega_1} \mathbb{E}[F \circ Q_{n-1}^{\epsilon, Q_1(w_1)}] \, d\mathbb{P}(w_1),$$

which implies (6.4) by passing to the limit with  $n \rightarrow \infty$  and recalling the definitions (6.3) and (6.1). To show the martingale property, we similarly check that for every  $n \in \mathbb{N}$ :

$$\begin{aligned} \mathbb{E}(u^\epsilon \circ Q_n \mid \mathcal{F}_{n-1}) &= \int_{\Omega_1} u^\epsilon \circ Q_n \, d\mathbb{P}_1(w_n) \\ &= \int_{\Omega_1} u^\epsilon(q_{n-1} * (\epsilon \wedge d(q_{n-1}, \partial\mathcal{D}))(w_n, 0)) \, d\mathbb{P}_1(w_n) = u^\epsilon(Q_{n-1}) \end{aligned}$$

is valid  $\mathbb{P}_{n-1}$ -a.s. in  $\Omega_{n-1}$ . □

**Corollary 6.3** *Assume that  $u \in \mathcal{C}^2(\mathcal{D}) \cap \mathcal{C}(\bar{\mathcal{D}})$  satisfies:*

$$\Delta_{\mathbb{H}}u = 0 \quad \text{in } \mathcal{D}, \quad u = F \quad \text{on } \partial\mathcal{D}. \tag{6.5}$$

*Then  $u^\epsilon = u$  for all  $\epsilon \in (0, 1)$ . In particular, (6.5) has at most one solution.*

**Proof** We first observe that the sequence  $\{u \circ Q_n\}_{n=0}^\infty$  is a martingale relative to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$ . This property follows by the same calculation as in the proof of Lemma 6.2, where  $u^\epsilon$  is now replaced by  $u$  and where (2.4) is used for  $u$  instead of the averaging formula (6.4). Consequently, Doob’s theorem yields:

$$u = \mathbb{E}[u \circ Q_0] = \mathbb{E}[u \circ Q_n] \quad \text{for all } n \in \mathbb{N}.$$

The right hand side above converges to  $u^\epsilon$ , as  $n \rightarrow \infty$ , which proves the first claim.

For the second claim, recall that the functions  $u^\epsilon$  depend only on the boundary values  $F = u|_{\partial\mathcal{D}}$  and not on their particular extension  $u$  on  $\bar{\mathcal{D}}$ . This yields uniqueness of the harmonic extension in (6.5). □

### 7 Convergence of $u^\epsilon$ as $\epsilon \rightarrow 0$

In this section we are concerned with the limiting properties of the family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$ . Following [12], we will introduce the process-related definition of regularity of the boundary points  $q \in \partial\mathcal{D}$ , which is the notion essentially equivalent to that of convergence to the  $\mathbb{H}$ -harmonic function with prescribed boundary data. The first observation is on transferring the estimate at the boundary of  $\mathcal{D}$  to its interior, by walk-coupling. This probabilistic technique utilizes translation invariance of solutions, similar to its analytic counterpart presented in Sect. 12 in connection with the Tug of War with noise, modelled on (4.2).



**Lemma 7.1** *In the context of definition (6.3), assume that for every  $\eta > 0$  there exist  $\delta, \epsilon_0 > 0$  such that for all  $\epsilon \in (0, \epsilon_0)$  there holds:*

$$|u^\epsilon(q') - u^\epsilon(q)| \leq \eta \quad \text{for all } q, q' \in \mathcal{D}$$

satisfying:  $d(q, \partial\mathcal{D}) < \delta, |q - q'| \leq \delta.$  (7.1)

*Then, the same uniformity property is likewise valid away from  $\partial\mathcal{D}$ . Namely, for every  $\eta > 0$  there exist  $\delta, \hat{\epsilon} > 0$  such that for all  $\epsilon \in (0, \hat{\epsilon})$  and all  $q, q' \in \mathcal{D}$  satisfying  $|q - q'| \leq \delta$ , there holds:  $|u^\epsilon(q') - u^\epsilon(q)| \leq \eta$ .*

**Proof** Fix  $\eta > 0$  and let  $\delta > \epsilon_0 > 0$  be as in (7.1). Given  $\epsilon \in (0, \epsilon_0)$  and  $q = (x, y, z), q' = (x', y', z') \in \mathcal{D}$  such that:

$$d(q, \partial\mathcal{D}), d(q', \partial\mathcal{D}) \geq \delta, \quad |q' - q| \leq \frac{\delta}{1 + \frac{1}{2}\text{diam}\mathcal{D}},$$

define the stopping time  $\tau : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$  by:

$$\tau(\omega) = \min \{n \geq 1; d(Q_n^{\epsilon, q}(\omega), \partial\mathcal{D}) < \delta \text{ or } d(Q_n^{\epsilon, q'}(\omega), \partial\mathcal{D}) < \delta\}.$$

Indeed,  $\tau$  is finite a.s. in  $\Omega$  in virtue of Lemma 6.1. For a given  $\omega \in \Omega$  with  $\tau(\omega) < \infty$ , assume without loss of generality that  $d(Q_{\tau}^{\epsilon, q}(\omega), \partial\mathcal{D}) < \delta$ . It is not hard to show (by induction on  $i = 1 \dots \tau$ ) that:

$$Q_{\tau}^{\epsilon, q'}(\omega) - Q_{\tau}^{\epsilon, q}(\omega) = q' - q + \epsilon(0, 0, \frac{1}{2}((x' - x, y' - y), \sum_{i=1}^{\tau} w_i^{\perp}))$$

and  $\epsilon \sum_{i=1}^{\tau} w_i = (x_{\tau}, y_{\tau}) - (x, y),$

where we write  $w_i^{\perp} = (a_i, b_i)^{\perp} = (b_i, -a_i)$ , and as usual:  $Q_{\tau}^{\epsilon, q} = (x_{\tau}, y_{\tau}, z_{\tau})$ . Consequently:

$$|Q_{\tau}^{\epsilon, q'}(\omega) - Q_{\tau}^{\epsilon, q}(\omega)| \leq |q' - q| + \frac{1}{2}|q' - q| \cdot |\epsilon \sum_{i=1}^{\tau} w_i| \leq (1 + \frac{1}{2}\text{diam}\mathcal{D})|q' - q| \leq \delta.$$

By Lemma 6.2, the sequence  $\{u^\epsilon \circ Q_n^{\epsilon, q'} - u^\epsilon \circ Q_n^{\epsilon, q}\}_{n=0}^{\infty}$  is a bounded martingale, so Doob's theorem yields:

$$|u^\epsilon(q') - u^\epsilon(q)| = |\mathbb{E}[u^\epsilon \circ Q_{\tau}^{\epsilon, q'} - u^\epsilon \circ Q_{\tau}^{\epsilon, q}]| \leq \int_{\Omega} |u^\epsilon \circ Q_{\tau}^{\epsilon, q'} - u^\epsilon \circ Q_{\tau}^{\epsilon, q}| d\mathbb{P} \leq \eta,$$

by (7.1). This concludes the proof, with  $\hat{\epsilon} = \epsilon_0$  and  $\hat{\delta} = \frac{\delta}{1 + \frac{1}{2}\text{diam}\mathcal{D}}.$  □

**Definition 7.2** Consider the  $\epsilon$ -walk in (6.1).

(a) We say that a boundary point  $q_0 \in \partial\mathcal{D}$  is *walk-regular* if for every  $\eta, \delta > 0$  there exists  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, 1)$  such that:

$$\mathbb{P}(Q^{\epsilon, q} \in B_\delta(q_0)) \geq 1 - \eta \quad \text{for all } \epsilon \in (0, \hat{\epsilon}) \quad \text{and } q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}.$$

(b) We say that  $\mathcal{D}$  is walk-regular if every  $q_0 \in \partial\mathcal{D}$  is walk-regular.

Observe that when  $\mathcal{D}$  is walk-regular, then (by compactness),  $\hat{\delta}$  and  $\hat{\epsilon}$  can be chosen independently of  $q_0$  (i.e. they depend only on the prescribed thresholds  $\eta, \delta$ ).

The walk-regularity is essentially equivalent to the validity of (7.1) with  $q \in \partial\mathcal{D}$ .

**Theorem 7.3** (a) *Assume that  $q_0 \in \partial\mathcal{D}$  is walk-regular. Then for every  $\eta > 0$  there exists  $\hat{\epsilon}, \hat{\delta} > 0$  such that for every  $\epsilon \in (0, \hat{\epsilon})$  there holds:*

$$|u^\epsilon(q) - F(q_0)| \leq \eta \quad \text{for all } q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}.$$

(b) *If  $q_0 \in \partial\mathcal{D}$  is not walk-regular, then there exists a continuous function  $F : \partial\mathcal{D} \rightarrow \mathbb{R}$  such that:*

$$\lim_{\hat{\epsilon}, \hat{\delta} \rightarrow 0} \sup \{ |u^\epsilon(q) - F(q_0)|; \epsilon \in (0, \hat{\epsilon}), q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D} \} > 0.$$

**Proof** To show (a), let  $\eta > 0$ , and choose  $\delta > 0$  so that  $|F(q'_0) - F(q_0)| \leq \frac{\eta}{2}$  for all  $q'_0 \in \partial\mathcal{D}$  with  $d(q'_0, q_0) < \delta$ . Further, choose  $\hat{\delta}, \hat{\epsilon}$  in Definition 7.2 (a) corresponding to  $\frac{\eta}{4\|F\|_{C(\partial\mathcal{D})} + 1}$  and  $\delta$ . Then, for all  $q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}$  and  $\epsilon \in (0, \hat{\epsilon})$  we have:

$$|u^\epsilon(q) - F(q_0)| \leq \int_{\Omega} |F \circ Q^{\epsilon, q} - F(q_0)| \, d\mathbb{P} \leq \mathbb{P}(Q^{\epsilon, q} \notin B_\delta(q_0)) \cdot 2\|F\|_{C(\partial\mathcal{D})} + \frac{\eta}{2} \leq \eta.$$

For (b), define  $F(q) = d(q, q_0)$  for all  $q \in \partial\mathcal{D}$ . By assumption, there exists  $\eta, \delta > 0$  such that for some sequences  $\epsilon_i \rightarrow 0^+$  and  $\mathcal{D} \ni q_i \rightarrow q_0$  we have:

$$\mathbb{P}(Q^{\epsilon_i, q_i} \notin B_\delta(q_0)) > \eta \quad \text{as } i \rightarrow \infty.$$

By nonnegativity of  $F$ , it follows that:

$$u^{\epsilon_i}(q_i) - F(q_0) = u^{\epsilon_i}(q_i) = \int_{\Omega} F \circ Q^{\epsilon_i, q_i} \, d\mathbb{P} \geq \int_{\{Q^{\epsilon_i, q_i} \notin B_\delta(q_0)\}} \delta \, d\mathbb{P} > \eta\delta,$$

proving the claim. □

**Theorem 7.4** *Assume that  $\mathcal{D}$  is walk-regular. Then every sequence in the family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$  has a further subsequence that converges uniformly to a continuous function  $u \in C(\bar{\mathcal{D}})$  such that  $u = F$  on  $\partial\mathcal{D}$ .*

**Proof** By Lemma 7.3 and the uniformity observation following Definition 7.2, we obtain that for every  $\eta > 0$  there exist  $\hat{\epsilon}, \hat{\delta} > 0$  such that for all  $\epsilon \in (0, \hat{\epsilon})$ :

$$|u^\epsilon(q) - F(q_0)| \leq \eta \quad \text{for all } q_0 \in \partial\mathcal{D}, \quad q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}. \tag{7.2}$$

Consequently, by uniform continuity of  $F$  on the compact metric space  $(\partial\mathcal{D}, d)$  it follows that the assumption (7.1) of Lemma 7.1 is valid. Thus the equibounded family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$  is also equi-oscillatory, so the Ascoli-Arzelà theorem implies existence of a sequence that converges uniformly to some  $u \in \mathcal{C}(\mathcal{D})$ . By (7.2) we finally conclude that  $u$  is continuous up the boundary where  $u|_{\partial\mathcal{D}} = F$ .  $\square$

We remark that the limit  $u$  above will be identified as the  $\mathbb{H}$ -harmonic function in Sect. 9.

### 8 The exterior $\mathbb{H}$ -corkscrew condition implies walk-regularity

**Definition 8.1** We say that a given boundary point  $q_0 \in \partial\mathcal{D}$  satisfies the *exterior  $\mathbb{H}$ -corkscrew condition* provided that there exists  $\mu \in (0, 1)$  such that for all sufficiently small  $r > 0$  there exists a Korányi ball  $B_{\mu r}(p_0)$  such that:

$$B_{\mu r}(p_0) \subset B_r(q_0) \setminus \mathcal{D}.$$

One can show (see [15, Theorem 1.3]) that every bounded domain  $\mathcal{D}$  of Euclidean regularity  $\mathcal{C}^{1,1}$  satisfies Definition 8.1 at each boundary point  $q_0$ . In fact, all  $\mathcal{C}^{1,1}$  domains in Carnot groups of step 2, are NTA (non-tangentially accessible), which means that they satisfy both the exterior and interior  $\mathbb{H}$ -corkscrew condition, plus a Harnack chain condition. This regularity is optimal, in the sense that  $\mathcal{C}^{1,\alpha}$  domains, for  $\alpha < 1$ , do not in general satisfy even a one-sided  $\mathbb{H}$ -corkscrew condition.

**Example 8.2** For  $\alpha \in [0, 1)$ , define  $\mathcal{D} = \{q = (x, y, z) = (q_{hor}, z) \in \mathbb{H}; |q_{hor}|^{1+\alpha} > z\}$ . Then the domain  $\mathcal{D}$  is  $\mathcal{C}^{1,\alpha}$ -regular, but the exterior  $\mathbb{H}$ -corkscrew condition does not hold at  $q_0 = 0$ . Indeed, take any  $q \notin \bar{\mathcal{D}}$  and compute:

$$\text{dist}(q, \partial\mathcal{D}) \leq d\left(q, \left(z^{\frac{1}{1+\alpha}} \frac{q_{hor}}{|q_{tan}|}, z\right)\right) = \left|z^{\frac{1}{1+\alpha}} - |q_{hor}|\right| < z^{\frac{1}{1+\alpha}}.$$

Thus, if  $q \in B_r(0)$ , we obtain:

$$\text{dist}(q, \partial\mathcal{D}) < Cr^{\frac{2}{1+\alpha}},$$

with a universal constant  $C$  depending only on  $\alpha$ . This contradicts  $\text{dist}(q, \partial\mathcal{D}) > \mu r$  for all  $\mu > 0$  as  $r \rightarrow 0$ .  $\square$

We also remark that (similarly to the Euclidean case) all bounded intrinsic Lipschitz domains are NTA, and hence satisfy the  $\mathbb{H}$ -corkscrew condition (see [15, Theorem 3]). The intrinsic Lipschitz domains, studied in [15], are domains whose boundaries are locally graphs of intrinsic Lipschitz functions acting between appropriate homogeneous subgroups of a Carnot group.

The main statement of this section is the following:

**Theorem 8.3** *If  $q_0 \in \partial\mathcal{D}$  satisfies the exterior  $\mathbb{H}$ -corkscrew condition, then  $q_0$  is walk-regular.*

Towards the proof we necessitate an inductive technique, see [32]:

**Lemma 8.4** *For a given  $q_0 \in \partial\mathcal{D}$ , assume that there exists a constant  $\theta_0 \in (0, 1)$  with the property that for all  $\delta > 0$  there exists  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, 1)$  such that:*

$$\mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_\delta(q_0)) \leq \theta_0 \quad \text{for all } \epsilon \in (0, \hat{\epsilon}), \ q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}. \quad (8.1)$$

Then  $q_0$  is walk-regular.

**Proof** Fix  $\eta, \delta > 0$  and let  $m \in \mathbb{N}$  be such that  $\theta_0^m \leq \eta$ . Define the tuples  $\{\epsilon_k\}_{k=0}^m, \{\hat{\delta}_k\}_{k=0}^{m-1}, \{\delta_k\}_{k=0}^m$  inductively, by setting:

$$\begin{aligned} \delta_m &= \frac{\delta}{2}, \quad \epsilon_m = 1, \\ \hat{\delta}_{k-1} &\in (0, \delta_k), \quad \epsilon_{k-1} \in (0, \epsilon_k) \quad \text{for all } k = 1 \dots m \\ \text{so that: } &\mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_{\delta_k}(q_0)) \leq \theta_0 \quad \text{for all } \epsilon \in (0, \epsilon_{k-1}), \ q \in B_{\hat{\delta}_{k-1}}(q_0) \cap \mathcal{D}, \\ \delta_{k-1} &\in (0, \hat{\delta}_{k-1}) \quad \text{for all } k = 2 \dots m. \end{aligned} \quad (8.2)$$

We finally set:

$$\hat{\epsilon} = \min \{ \epsilon_0, \{|\delta_k - \hat{\delta}_k|\}_{k=1}^{m-1} \} \quad \text{and} \quad \hat{\delta} = \hat{\delta}_0.$$

Fix  $q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}$  and  $\epsilon \in (0, \hat{\epsilon})$ . Then the application of Fubini’s theorem yields:

$$\mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_{\delta_k}(q_0)) \leq \theta_0 \mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_{\delta_{k-1}}(q_0)) \quad \text{for all } k = 2 \dots m.$$

Together with the inequality in (8.2) for  $k = 1$ , the above bound results in:

$$\begin{aligned} \mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_\delta(q_0)) &\leq \mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_{\delta_m}(q_0)) \\ &\leq \theta_0^{m-1} \mathbb{P}(\exists n \geq 0 \ Q_n^{\epsilon, q} \notin B_{\delta_1}(q_0)) \leq \theta_0^m \leq \eta, \end{aligned}$$

proving the validity of condition (a) in Definition 7.2. □

**Proof of Theorem 8.3** Fix  $\delta > 0$  sufficiently small and set  $\hat{\epsilon} = 1, \hat{\delta} = \delta/4$ . By assumption, there exists a subball  $B_{\mu\hat{\delta}}(p_0) \subset B_{\hat{\delta}}(q_0) \setminus \mathcal{D}$ . We will show that condition (8.1) holds, with constant  $\theta_0 = \frac{1-\mu^2/4}{1-\mu^2/9} \in (0, 1)$ , as identified below.

Fix now  $q \in B_{\hat{\delta}}(q_0) \cap \mathcal{D}$  and  $\epsilon \in (0, \hat{\epsilon})$ . Since the function  $\phi(p) = v(d(p, p_0))$  with  $v(t) = \frac{1}{t^2}$  satisfies:  $\Delta_{\mathbb{H}}\phi = 0$  in  $\mathbb{H} \setminus \{p_0\}$ , the sequence of random variables  $\{v \circ d(Q_n^{\epsilon, q}, p_0)\}_{n=0}^\infty$  is a martingale relative to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$ . Define  $\tau : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$  by:

$$\tau = \inf \{ n \geq 0; \ Q_n^{\epsilon, q} \notin B_\delta(q_0) \}.$$

For every  $n \geq 0$ , the random variable  $\tau \wedge n$  is a bounded stopping time, so:

$$v(d(q, p_0)) = \mathbb{E}[v \circ d(Q_{n \wedge \tau}^{\epsilon, q}, p_0)] \quad \text{for all } n \geq 0$$

follows by Doob’s theorem. Passing to the limit with  $n \rightarrow \infty$  we obtain:

$$v(d(q, p_0)) = \int_{\{\tau < \infty\}} v(d(Q_\tau^{\epsilon, q}, p_0)) \, d\mathbb{P} + \int_{\{\tau = \infty\}} v(d(Q^{\epsilon, q}, p_0)) \, d\mathbb{P}.$$

Since  $d(Q_\tau^{\epsilon, q}, p_0) \geq d(Q_\tau^{\epsilon, q}, q_0) - d(q_0, p_0) \geq \delta - \hat{\delta} = 3\hat{\delta}$  and  $d(Q^{\epsilon, q}, p_0) \geq \mu\hat{\delta}$ , together with  $d(q, p_0) \leq d(q, q_0) + d(q_0, p_0) \leq 2\hat{\delta}$ , the last displayed formula becomes:

$$v(2\hat{\delta}) \leq \mathbb{P}(\tau < \infty)v(3\hat{\delta}) + \mathbb{P}(\tau = \infty)v(\mu\hat{\delta}) = \mathbb{P}(\tau < \infty)(v(3\hat{\delta}) - v(\mu\hat{\delta})) + v(\mu\hat{\delta}).$$

Equivalently:

$$\mathbb{P}(\tau < \infty) \leq \frac{v(\mu\hat{\delta}) - v(2\hat{\delta})}{v(\mu\hat{\delta}) - v(3\hat{\delta})} = \frac{1/\mu^2 - 1/4}{1/\mu^2 - 1/9},$$

which ends the proof. □

### 9 Identification of the limit $u$ : a viscosity solutions proof

In this section we show that the uniform limit of the whole family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$ , defined in (6.3), coincides with the unique  $\mathbb{H}$ -harmonic extension to the given continuous data  $F$ , provided that  $\mathcal{D}$  is walk-regular. We present a viscosity solutions proof of this statement, expandable to the case of arbitrary exponent  $\mathbf{p} \in (1, \infty)$ . Indeed, in Sect. 15 we will carry out in detail the parallel construction for  $\mathbf{p} > 2$  in connection to the mean value property (4.2). The construction for  $1 < \mathbf{p} < \infty$ , feasible in the framework of (5.4), is conceptually identical and left as an exercise for an interested reader; the details pertaining to the Euclidean case can be found in [22]. We point out that another proof of Theorem 9.2 is available in connection with the discrete Levy area process.

We start with a simple general lemma about the minima of uniform approximations:

**Lemma 9.1** *Assume that a sequence of bounded functions  $\{u_n : \bar{\mathcal{D}} \rightarrow \mathbb{R}\}_{n=1}^\infty$  converges uniformly to some  $u \in \mathcal{C}(\bar{\mathcal{D}})$ , as  $n \rightarrow \infty$ . Then, for every sequence of positive numbers  $\{\delta_n\}_{n=1}^\infty$  converging to 0, every  $q_0 \in \mathcal{D}$  and every  $\phi \in \mathcal{C}^2(\bar{\mathcal{D}})$  such that:*

$$\phi(q_0) = u(q_0), \quad \phi < u \text{ in } \bar{\mathcal{D}} \setminus \{q_0\}, \tag{9.1}$$

there exists a sequence  $\{q_n \in \mathcal{D}\}_{n=1}^\infty$ , satisfying:

$$\lim_{n \rightarrow \infty} q_n = q_0 \quad \text{and} \quad u_n(q_n) - \phi(q_n) \leq \inf_{\bar{\mathcal{D}}} (u_n - \phi) + \delta_n.$$

**Proof** For every large integer  $j$  define  $\eta_j > 0$  and  $n_j \in \mathbb{N}$  such that:

$$\eta_j = \min_{\bar{\mathcal{D}} \setminus B_{1/j}^3(q_0)} (u - \phi) \quad \text{and} \quad \sup_{\bar{\mathcal{D}}} |u_n - u| \leq \frac{1}{2} \eta_j \quad \text{for all } n \geq n_j.$$

Without loss of generality, the sequence  $\{n_j\}_{j=1}^\infty$  is strictly increasing. Now, for all  $n \in [n_j, n_{j+1})$  let  $q_n \in B_{1/j}^3(q_0)$  satisfy:

$$u_n(q_n) - \phi(q_n) \leq \inf_{B_{1/j}^3(q_0)} (u_n - \phi) + \delta_n. \tag{9.2}$$

Observe that the following bound is valid for every  $q \in \bar{\mathcal{D}} \setminus B_{1/j}^3(q_0)$ :

$$\begin{aligned} u_n(q) - \phi(q) &\geq u(q) - \phi(q) - \sup_{\bar{\mathcal{D}}} |u_n - u| \geq \eta_j - \frac{1}{2} \eta_j \geq \sup_{\bar{\mathcal{D}}} |u_n - u| \\ &\geq u_n(q_0) - \phi(q_0) \geq \inf_{B_{1/j}^3(q_0)} (u_n - \phi). \end{aligned}$$

This proves the claim in view of (9.2). □

**Theorem 9.2** *The limit function  $u$  in Theorem 7.4 solves the Dirichlet problem (6.5). Automatically, the whole family  $\{u^\epsilon\}_{\epsilon \rightarrow 0}$  converges uniformly to such unique solution  $u$ , provided that  $\mathcal{D}$  is walk-regular.*

**Proof** Let  $\epsilon_i \rightarrow 0^+$  be such that  $\{u^{\epsilon_i}\}_{i=1}^\infty$  converges uniformly to  $u$  on  $\bar{\mathcal{D}}$ . Fix  $q_0 \in \mathcal{D}$  and take  $\phi \in \mathcal{C}^2(\bar{\mathcal{D}})$  satisfying (9.1). Choose a sequence  $\{q_i \in \mathcal{D}\}_{i=1}^\infty$  with the properties guaranteed by Lemma 9.1 when applied to the error sequence  $\delta_i = \epsilon_i^3$ . Recalling (6.4) we obtain:

$$\mathcal{A}_2(\phi, \epsilon_i)(q_i) \leq \mathcal{A}_2(u^{\epsilon_i}, \epsilon_i)(q_i) - (u^{\epsilon_i}(q_i) - \phi(q_i)) + \epsilon_i^3 = \phi(q_i) + \epsilon_i^3.$$

From (2.3)<sub>2</sub> we thus conclude:  $\phi(q_i) + \frac{\epsilon_i^2}{8} \Delta_{\mathbb{H}} \phi(q_i) \leq \phi(q_i) + o(\epsilon_i^2)$ , which upon passing to the limit  $i \rightarrow \infty$  implies:  $\Delta_{\mathbb{H}} \phi(q_0) \leq 0$ .

By a symmetric reasoning, we get that if  $\phi \in \mathcal{C}^2(\bar{\mathcal{D}})$  satisfies:  $\phi(q_0) = u(q_0)$  and  $\phi > u$  in  $\bar{\mathcal{D}} \setminus \{q_0\}$ , then  $\Delta_{\mathbb{H}} \phi(q_0) \geq 0$ . Finally, the same arguments as in the second part of the proof of Proposition 2.5 imply that  $u$  coincides with its own  $\mathbb{H}$ -harmonic extension in any  $\bar{B}_r(q_0) \subset \mathcal{D}$ . Therefore,  $u$  is the  $\mathbb{H}$ -harmonic extension of the continuous boundary data  $F = u|_{\partial \mathcal{D}}$  and as such it is unique, completing the proof. □

**PART III: The  $p$ - $\mathbb{H}$ -Laplacian  $\Delta_{\mathbb{H},p}$  and the random Tug of Wars in  $\mathbb{H}$**

### 10 The random Tug of War game in the Heisenberg group

In this section we develop the probability setting similar to that of Sect. 6, but related to the expansion (4.2) rather than (2.3)<sub>2</sub>. We remark that an identical construction can be carried out for the dynamic programming principle modelled on (5.4), where the advantage is that it covers any exponent  $1 < \mathbf{p} < \infty$ , see Remark 10.2. We leave the details to the interested reader; in the Euclidean setting we point to the paper [22]. Here, we always assume that  $\mathbf{p} > 2$ , whereas parallel statements for  $\mathbf{p} = 2$  follow by approximation  $\mathbf{p} \rightarrow 2^+$ .

1. Let  $\Omega_1 = B_1(0) \times \{1, 2, 3\} \times (0, 1)$  and define:

$$\Omega = (\Omega_1)^{\mathbb{N}} = \{\omega = \{(w_i, s_i, t_i)\}_{i=1}^{\infty}; \\ w_i = (a_i, b_i, c_i) \in B_1(0), s_i \in \{1, 2, 3\}, t_i \in (0, 1) \text{ for all } i \in \mathbb{N}\}.$$

The probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is given as the countable product of  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ . Here,  $\mathcal{F}_1$  is the smallest  $\sigma$ -algebra containing all products  $D \times S \times B$  where  $D \subset B_1(0) \subset \mathbb{H}$  and  $B \subset (0, 1)$  are Borel, and  $S \subset \{1, 2, 3\}$ . The probability measure  $\mathbb{P}_1$  is given as the product of: the normalized Lebesgue measure on  $B_1(0)$ , the uniform counting measure on  $\{1, 2, 3\}$  and the Lebesgue measure on  $(0, 1)$ :

$$\mathbb{P}_1(D \times S \times B) = \frac{|D|}{|B_1(0)|} \cdot \frac{|S|}{3} \cdot |B|.$$

For each  $n \in \mathbb{N}$ , we consider the probability space  $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$  that is the product of  $n$  copies of  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ . The  $\sigma$ -algebras  $\mathcal{F}_n$  is always identified with the corresponding sub- $\sigma$ -algebra of  $\mathcal{F}$ , consisting of sets of the form  $A \times \prod_{i=n+1}^{\infty} \Omega_1$  for all  $A \in \mathcal{F}_n$ . The sequence  $\{\mathcal{F}_n\}_{n=0}^{\infty}$ , where we set  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ , is a filtration of  $\mathcal{F}$ .

2. Given are two family of functions  $\sigma_I = \{\sigma_I^n\}_{n=0}^{\infty}$  and  $\sigma_{II} = \{\sigma_{II}^n\}_{n=0}^{\infty}$ , defined on the corresponding spaces of “finite histories”  $H_n = \mathbb{H} \times (\mathbb{H} \times \Omega_1)^n$ :

$$\sigma_I^n, \sigma_{II}^n : H_n \rightarrow B_1(0) \subset \mathbb{H},$$

assumed to be measurable with respect to the (target) Borel  $\sigma$ -algebra in  $B_1(0)$  and the (domain) product  $\sigma$ -algebra on  $H_n$ . For every  $q_0 \in \mathbb{H}$  and  $\epsilon \in (0, 1)$  we now recursively define the sequence of random variables:

$$\{Q_n^{\epsilon, q_0, \sigma_I, \sigma_{II}} : \Omega \rightarrow \mathbb{H}\}_{n=0}^{\infty}.$$

For simplicity of notation, we often suppress some of the superscripts  $\epsilon, q_0, \sigma_I, \sigma_{II}$  and write  $Q_n$  (or  $Q_n^{q_0}$ , or  $Q_n^{\sigma_I, \sigma_{II}}$ , etc) instead of  $Q_n^{\epsilon, q_0, \sigma_I, \sigma_{II}}$ , if no ambiguity arises. Let:

$$Q_0 \equiv q_0,$$

$$Q_n((w_1, s_1, t_1), \dots, (w_n, s_n, t_n)) = q_{n-1} * \begin{cases} \rho_{\gamma_{\mathbb{P}}^\epsilon}(\sigma_I^{n-1}(h_{n-1})) * \rho_\epsilon(w_n) & \text{for } s_n = 1 \\ \rho_{\gamma_{\mathbb{P}}^\epsilon}(\sigma_{II}^{n-1}(h_{n-1})) * \rho_\epsilon(w_n) & \text{for } s_n = 2 \\ \rho_\epsilon(w_n) & \text{for } s_n = 3 \end{cases}$$

where  $q_{n-1} = (x_{n-1}, y_{n-1}, z_{n-1}) = Q_{n-1}((w_1, s_1, t_1), \dots, (w_{n-1}, s_{n-1}, t_{n-1}))$

and  $h_{n-1} = (q_0, (q_1, w_1, s_1, t_1), \dots, (q_{n-1}, w_{n-1}, s_{n-1}, t_{n-1})) \in H_{n-1}$ .

$$(10.1)$$

In this “game”, the position  $q_{n-1}$  is first advanced (deterministically) according to the two players’ “strategies”  $\sigma_I$  and  $\sigma_{II}$  by a shift in  $B_{\gamma_{\mathbb{P}}^\epsilon}(0)$ , and then (randomly) uniformly by a further shift in the 3-dimensional Korányi ball  $B_\epsilon(0)$ . The deterministic shifts  $\rho_{\gamma_{\mathbb{P}}^\epsilon} \circ \sigma_I^{n-1}$  and  $\rho_{\gamma_{\mathbb{P}}^\epsilon} \circ \sigma_{II}^{n-1}$  are activated according to the value of the equally probable outcomes  $s_n \in \{1, 2, 3\}$ . Namely,  $s_n = 1$  results in activating  $\sigma_I$  and  $s_n = 2$  in activating  $\sigma_{II}$ , whereas  $s_n = 3$  corresponds to not activating any of these strategies.

**3.** The auxiliary variables  $t_n \in (0, 1)$  serve as a threshold for reading the eventual value from the prescribed boundary data. Let  $\mathcal{D} \subset \mathbb{H}$  be an open, bounded and connected set. Define the random variable  $\tau^{\epsilon, q_0, \sigma_I, \sigma_{II}} : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$  in:

$$\begin{aligned} \tau^{\epsilon, q_0, \sigma_I, \sigma_{II}}((w_1, s_1, t_1), (w_2, s_2, t_2), \dots) \\ = \min \{n \geq 1; t_n > d_\epsilon(q_{n-1})\} \end{aligned} \tag{10.2}$$

where:

$$d_\epsilon(q) = \frac{1}{\epsilon} \min \{ \epsilon, \text{dist}(q, \mathbb{H} \setminus \mathcal{D}) \}$$

is the scaled Korányi distance from the complement of the domain  $\mathcal{D}$ . As before, we drop the superscripts and write  $\tau$  instead of  $\tau^{\epsilon, q_0, \sigma_I, \sigma_{II}}$  if there is no ambiguity. Clearly,  $\tau$  is  $\mathcal{F}$ -measurable and, in fact, it is a stopping time relative to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$  because:

**Proposition 10.1** *In the above setting:  $\mathbb{P}(\tau < \infty) = 1$ .*

**Proof** Consider the following set of “advancing” shifts:  $D_{adv} = \{w = (a, b, c) \in B_1(0); a > \frac{1}{2}\}$ . Since  $\mathcal{D}$  is bounded, there exists  $n \geq 1$  (depending on  $\epsilon$ ) such that:

$$q_0 * \rho_\epsilon(w_1) * \dots * \rho_\epsilon(w_n) \notin \mathcal{D} \quad \text{for all } q_0 \in \mathcal{D} \quad \text{and } w_1 \in D_{adv}, i = 1 \dots n.$$

Define  $\eta = \left(\frac{|D_{adv}|}{|B_1(0)|} \cdot \frac{1}{3}\right)^n > 0$  and note that:

$$\mathbb{P}(\tau \leq n) \geq \mathbb{P}\left((D_{adv} \times \left\{\frac{1}{3}\right\} \times (0, 1))^n \times \prod_{i=n+1}^\infty \Omega_1\right) = \eta.$$

It follows by induction that:  $\mathbb{P}(\tau > kn) \leq (1 - \eta)^k$  for all  $k \in \mathbb{N}$ . The proof is concluded by observing:  $\mathbb{P}(\tau = \infty) = \mathbb{P}\left(\bigcap_{k=1}^\infty \{\tau > kn\}\right) = \lim_{k \rightarrow \infty} \mathbb{P}(\tau > kn) \leq$

$$\lim_{k \rightarrow \infty} (1 - \eta)^k = 0. \tag{□}$$



**Remark 10.2** For  $1 < p < \infty$ , one can easily construct a similar process as in (10.1), modelled on the expansion (5.4). In this case, the position of the token  $q_{n-1}$  is first advanced deterministically according to the strategies  $\sigma_I$  or  $\sigma_{II}$  by a shift  $\rho_\epsilon(y) \in B_\epsilon(0)$ , and then randomly uniformly by a shift in the Korányi ellipsoid  $\rho_\epsilon\left(E\left(0, s_p, 1 + (a_p - 1)|y_{hor}|^2, \frac{y_{hor}}{|y_{hor}|}\right)\right)$ . The deterministic shifts  $\rho_\epsilon \circ \sigma_I^{n-1}$  and  $\rho_\epsilon \circ \sigma_{II}^{n-1}$  are activated according to the value of the sequence of i.i.d. random variables  $s_n \in \{1, 2\}$ . The random stopping time  $\tau = \tau^{\epsilon, q_0, \sigma_I, \sigma_{II}}$  is defined as in (10.2) through the auxiliary variables  $t_n$ . One can check that sufficient conditions on the admissible parameters  $a_p$  and  $s_p$  in (5.3) in order to have  $\mathbb{P}(\tau < \infty) = 1$  are:

$$a_p \leq 1 \text{ and } s_p a_p > 1 \quad \text{or} \quad a_p \geq 1 \text{ and } s_p > 1.$$

The proof can be adapted from the Euclidean case (see Lemma 4.1 in [22]) by noting that the projection of a Korányi ellipsoid onto the horizontal plane reduces to a Euclidean ellipse.

4. Given the data  $F \in \mathcal{C}(\mathbb{H})$ , define the functions:

$$\begin{aligned} u_I^\epsilon(q) &= \sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}\left[F \circ (Q^{\epsilon, q, \sigma_I, \sigma_{II}})_{\tau^{\epsilon, q, \sigma_I, \sigma_{II}} - 1}\right], \\ u_{II}^\epsilon(q) &= \inf_{\sigma_{II}} \sup_{\sigma_I} \mathbb{E}\left[F \circ (Q^{\epsilon, q, \sigma_I, \sigma_{II}})_{\tau^{\epsilon, q, \sigma_I, \sigma_{II}} - 1}\right]. \end{aligned} \tag{10.3}$$

The main result in Theorem 11.3 will show that for each  $\epsilon \ll 1$  we have:  $u_I^\epsilon = u_{II}^\epsilon \in \mathcal{C}(\mathbb{H})$  coincide with the unique solution to the dynamic programming principle in Sect. 11, modelled on the expansion (4.2). It is also clear that the values of  $u_{I, II}^\epsilon$  depend only on the values of  $F$  in the  $(\gamma_p + 1)\epsilon$ -neighbourhood of  $\partial\mathcal{D}$ . In Sect. 13 we will prove that as  $\epsilon \rightarrow 0$ , the uniform limit of  $u_{I, II}^\epsilon$  that depends only on  $F|_{\partial\mathcal{D}}$ , is  $p$ - $\mathbb{H}$ -harmonic in  $\mathcal{D}$  and coincides with  $F$  on  $\partial\mathcal{D}$ .

### 11 The dynamic programming principle modelled on (4.2)

Let  $\mathcal{D} \subset \mathbb{H}$  be an open, bounded, connected domain and let  $F \in \mathcal{C}(\mathbb{H})$  be a bounded data function. We have the following:

**Theorem 11.1** *For every  $\epsilon \in (0, 1)$  there exists a unique function  $u : \mathbb{H} \rightarrow \mathbb{R}$  (denoted further by  $u_\epsilon$ ), automatically continuous and bounded, such that:*

$$\begin{aligned} u(q) &= d_\epsilon(q) \left( \frac{1}{3} \mathcal{A}_3(u, \epsilon)(q) + \frac{1}{3} \inf_{p \in B_{\gamma_p \epsilon}(q)} \mathcal{A}_3(u, \epsilon)(p) + \frac{1}{3} \sup_{p \in B_{\gamma_p \epsilon}(q)} \mathcal{A}_3(u, \epsilon)(p) \right) \\ &\quad + (1 - d_\epsilon(q)) F(q) \text{ for all } q \in \mathbb{H}. \end{aligned} \tag{11.1}$$

The solution operator to (11.1) is monotone, i.e. if  $F \leq \bar{F}$  then the corresponding solutions satisfy:  $u_\epsilon \leq \bar{u}_\epsilon$ .

**Proof 1.** We remark that by continuity of: the averaging operator  $p \mapsto \mathcal{A}_3(u, \epsilon)(p)$ , the weight function  $d_\epsilon$  and the data  $F$ , the solution of (11.1) is indeed automatically continuous. Define the operators  $T, S : \mathcal{C}(\mathbb{H}) \rightarrow \mathcal{C}(\mathbb{H})$  in:

$$(Sv)(q) = \frac{1}{3} \mathcal{A}_3(u, \epsilon)(q) + \frac{1}{3} \inf_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(u, \epsilon) + \frac{1}{3} \sup_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(u, \epsilon),$$

$$Tv = d_\epsilon Sv + (1 - d_\epsilon)F.$$

Clearly  $S$  (and likewise  $T$ ) is monotone, namely:  $Sv \leq S\bar{v}$  if  $v \leq \bar{v}$ . Observe further that:

$$\begin{aligned} |Sv(q) - S\bar{v}(q)| &\leq \frac{1}{3} \left( |\mathcal{A}_3(v - \bar{v}, \epsilon)(q)| + \left| \inf_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(v, \epsilon) - \inf_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(\bar{v}, \epsilon) \right| \right. \\ &\quad \left. + \left| \sup_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(v, \epsilon) - \sup_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(\bar{v}, \epsilon) \right| \right) \\ &\leq \frac{1}{3} \mathcal{A}_3(|v - \bar{v}|, \epsilon)(q) + \frac{2}{3} \sup_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(|v - \bar{v}|, \epsilon) \leq \sup_{B_{\gamma p^\epsilon}(q)} \mathcal{A}_3(|v - \bar{v}|, \epsilon). \end{aligned} \tag{11.2}$$

The solution  $u$  of (11.1) is obtained as the limit of iterations  $u_{n+1} = Tu_n$ , where we set  $u_0 \equiv \text{const} \leq \inf F$ . Since  $u_1 = Tu_0 \geq u_0$  on  $\mathbb{H}$  and in view of the monotonicity of  $T$ , the sequence  $\{u_n\}_{n=0}^\infty$  is non-decreasing. It is also bounded (by  $\|F\|_{\mathcal{C}(\mathbb{H})}$ ) and thus it converges pointwise to a (bounded) limit  $u : \mathbb{H} \rightarrow \mathbb{R}$ . By the calculation in (11.2),  $u$  must be a fixed point of  $T$ , hence a solution to (11.1). We also remark that the monotonicity of  $S$  yields the monotonicity of the solution operator to (11.1).

2. It remains to show uniqueness. If  $u, \bar{u}$  both solve (11.1), then define  $M = \sup_{q \in \mathbb{H}} |u(q) - \bar{u}(q)| = \sup_{q \in \mathcal{D}} |u(q) - \bar{u}(q)|$  and consider any maximizer  $q_0 \in \mathcal{D}$ , where  $|u(q_0) - \bar{u}(q_0)| = M$ . By (11.2) we obtain:

$$\begin{aligned} M &= |u(q_0) - \bar{u}(q_0)| = d_\epsilon(q_0) |Su(q_0) - S\bar{u}(q_0)| \\ &\leq \frac{d_\epsilon(q_0)}{3} \mathcal{A}_3(|u - \bar{u}|, \epsilon)(q_0) + \frac{2d_\epsilon(q_0)}{3} \sup_{B_{\gamma p^\epsilon}(q_0)} \mathcal{A}_3(|u - \bar{u}|, \epsilon) \leq d_\epsilon(q_0)M \leq M, \end{aligned}$$

yielding  $\mathcal{A}_3(|u - \bar{u}|, \epsilon)(q_0) = M$ . Consequently,  $B_\epsilon(q_0) \subset D_M = \{|u - \bar{u}| = M\}$  and hence the set  $D_M$  is open in  $\mathbb{H}$ . Since  $D_M$  is obviously closed and nonempty, there must be  $D_M = \mathbb{H}$  and since  $u - \bar{u} = 0$  on  $\mathbb{H} \setminus \mathcal{D}$ , it follows that  $M = 0$ . Thus  $u = \bar{u}$ , proving the claim.  $\square$

**Remark 11.2** It is not hard to observe that the sequence  $\{u_n\}_{n=1}^\infty$  in the proof of Theorem 11.1 converges to  $u = u_\epsilon$  uniformly. In fact, the iteration procedure  $u_{n+1} = Tu_n$  started by any bounded and continuous function  $u_0$  converges uniformly to the unique solution  $u_\epsilon$ .

**Theorem 11.3** For every  $\epsilon \in (0, 1)$ , let  $u_I^\epsilon, u_{II}^\epsilon$  be as in (10.3) and  $u_\epsilon$  as in Theorem 11.1. Then:

$$u_I^\epsilon = u_\epsilon = u_{II}^\epsilon.$$

**Proof 1.** We drop the sub/superscript  $\epsilon$  for notational convenience. To show that  $u_{II} \leq u$ , fix  $q_0 \in \mathbb{H}$  and  $\eta > 0$ . We first observe that there exists a strategy  $\sigma_{0,II}$  where  $\sigma_{0,II}^n(h_n) = \sigma_{0,II}^n(q_n)$  satisfies for every  $n \geq 0$  and  $h_n \in H_n$ :

$$\mathcal{A}_3(u, \epsilon)(q_n * \rho_{\gamma p^\epsilon}(\sigma_{0,II}^n(q_n))) \leq \inf_{B_{\gamma p^\epsilon}(q_n)} \mathcal{A}_3(u, \epsilon) + \frac{\eta}{2^{n+1}}. \tag{11.3}$$

Indeed, it suffices to show, in view of continuity of  $\mathcal{A}_3(u, \epsilon)$ , that given  $v \in \mathcal{C}(\mathbb{H})$  and  $r, \eta > 0$ , there exists an infimizing-related Borel measurable ‘‘selection’’ function  $\sigma : \mathbb{H} \rightarrow \mathbb{H}$  such that  $v(\sigma(q)) < \inf_{B_r(q)} v + \eta$  and  $\sigma(q) \in B_r(q)$  for all  $q \in \mathbb{H}$ . Using continuity of  $v$  and a localisation argument, if necessary, we note that there exists  $\delta > 0$  such that:

$$\left| \inf_{B_r(q)} v - \inf_{B_r(p)} v \right| < \frac{\eta}{2} \quad \text{for all } |p - q| < \delta.$$

Let  $\{B_\delta^3(p_i)\}_{i=1}^\infty$  be a locally finite covering of  $\mathbb{H}$ . For each  $i = 1 \dots \infty$ , choose  $q_i \in B_r(p_i)$  satisfying:  $|\inf_{B_r(p_i)} v - v(q_i)| < \frac{\eta}{2}$ . Finally, define:

$$\sigma(q) = q_i \quad \text{for } q \in B_\delta^3(p_i) \setminus \bigcup_{j=1}^{i-1} B_\delta^3(p_j).$$

Clearly, the piecewise constant function  $\sigma$  is Borel regular and infimizing-related with the prescribed parameters  $r, \eta$ .

**2.** Fix a strategy  $\sigma_I$  and consider the following sequence of random variables  $M_n : \Omega \rightarrow \mathbb{R}$ :

$$M_n = (u \circ Q_n) \mathbb{1}_{\tau > n} + (F \circ Q_{\tau-1}) \mathbb{1}_{\tau \leq n} + \frac{\eta}{2^n}.$$

We show that  $\{M_n\}_{n=0}^\infty$  is a supermartingale with respect to the filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$ . Clearly:

$$\begin{aligned} \mathbb{E}(M_n | \mathcal{F}_{n-1}) &= \mathbb{E}((u \circ Q_n) \mathbb{1}_{\tau > n} | \mathcal{F}_{n-1}) + \mathbb{E}((F \circ Q_{n-1}) \mathbb{1}_{\tau = n} | \mathcal{F}_{n-1}) \\ &\quad + \mathbb{E}((F \circ Q_{\tau-1}) \mathbb{1}_{\tau < n} | \mathcal{F}_{n-1}) + \frac{\eta}{2^n} \quad \text{a.s.} \end{aligned} \tag{11.4}$$

We readily observe that:  $\mathbb{E}((F \circ Q_{\tau-1}) \mathbb{1}_{\tau < n} | \mathcal{F}_{n-1}) = (F \circ Q_{\tau-1}) \mathbb{1}_{\tau < n}$ . Further, writing  $\mathbb{1}_{\tau = n} = \mathbb{1}_{\tau \geq n} \mathbb{1}_{t_n > d_\epsilon(q_{n-1})}$ , it follows that:

$$\begin{aligned} \mathbb{E}((F \circ Q_{n-1}) \mathbb{1}_{\tau = n} | \mathcal{F}_{n-1}) &= \mathbb{E}(\mathbb{1}_{t_n > d_\epsilon(q_{n-1})} | \mathcal{F}_{n-1}) \cdot (F \circ Q_{n-1}) \mathbb{1}_{\tau \geq n} \\ &= \mathbb{P}_1(t_n > d_\epsilon(q_{n-1})) \cdot (F \circ Q_{n-1}) \mathbb{1}_{\tau \geq n} \\ &= (1 - d_\epsilon(q_{n-1})) (F \circ Q_{n-1}) \mathbb{1}_{\tau \geq n} \quad \text{a.s.} \end{aligned}$$

Similarly, since  $\mathbb{1}_{\tau > n} = \mathbb{1}_{\tau \geq n} \mathbb{1}_{t_n \leq d_\epsilon(q_{n-1})}$ , we get in view of (11.3):

$$\begin{aligned} \mathbb{E}((u \circ Q_n) \mathbb{1}_{\tau > n} \mid \mathcal{F}_{n-1}) &= \int_{\Omega_1} u \circ Q_n \, d\mathbb{P}_1 \cdot d_\epsilon(q_{n-1}) \mathbb{1}_{\tau \geq n} \\ &= \left( \mathcal{A}_3(u, \epsilon)(q_{n-1}) + \mathcal{A}_3(u, \epsilon)(q_{n-1} * \rho_{\gamma_{\mathbf{p}}}(\sigma_I^{n-1})) \right. \\ &\quad \left. + \mathcal{A}_3(u, \epsilon)(q_{n-1} * \rho_{\gamma_{\mathbf{p}}}(\sigma_{0,II}^{n-1})) \right) \frac{d_\epsilon(q_{n-1})}{3} \mathbb{1}_{\tau \geq n} \\ &\leq ((Su) \circ Q_{n-1} + \frac{\eta}{2^n}) d_\epsilon(q_{n-1}) \mathbb{1}_{\tau \geq n} \quad \text{a.s.} \end{aligned}$$

Concluding, by (11.1) the decomposition (11.4) yields:

$$\begin{aligned} \mathbb{E}(M_n \mid \mathcal{F}_{n-1}) &\leq \left( d_\epsilon(q_{n-1})((Su) \circ Q_{n-1}) + (1 - d_\epsilon(q_{n-1}))(F \circ Q_{n-1}) \right) \mathbb{1}_{\tau \geq n} \\ &\quad + (F \circ Q_{\tau-1}) \mathbb{1}_{\tau \leq n-1} + \frac{\eta}{2^{n-1}} = M_{n-1} \quad \text{a.s.} \end{aligned}$$

3. The supermartingale property of  $\{M_n\}_{n=0}^\infty$  being established, we conclude that:

$$u(q_0) + \eta = \mathbb{E}[M_0] \geq \mathbb{E}[M_\tau] = \mathbb{E}[F \circ Q_{\tau-1}] + \frac{\eta}{2^\tau}.$$

Thus:

$$u_{II}(q_0) \leq \sup_{\sigma_I} \mathbb{E}[F \circ (Q^{\sigma_I, \sigma_{II,0}})_{\tau-1}] \leq u(q_0) + \eta.$$

As  $\eta > 0$  was arbitrary, we obtain the claimed comparison  $u_{II}(q_0) \leq u(q_0)$ . For the reverse inequality  $u(q_0) \leq u_I(q_0)$ , we use a symmetric argument, with an almost-maximizing strategy  $\sigma_{0,I}$  and the resulting submartingale  $\tilde{M}_n = (u \circ Q_n) \mathbb{1}_{\tau > n} + (F \circ Q_{\tau-1}) \mathbb{1}_{\tau \leq n} - \frac{\eta}{2^n}$ , along a given yet arbitrary strategy  $\sigma_{II}$ . The obvious estimate  $u_I(q_0) \leq u_{II}(q_0)$  concludes the proof.  $\square$

### 12 The first convergence theorem

We prove the first convergence result below, via an analytical argument, although a probabilistic one is possible as well, in view of the interpretation of  $u_\epsilon$  in Theorem 11.3. Our proof mimics the construction for the Euclidean case in [22], which is based on the observation that for  $s$  sufficiently large, the mapping  $q \mapsto |q|_K^s$  yields the variation that pushes the  $\mathbf{p}$ - $\mathbb{H}$ -harmonic function  $F$  into the region of  $\mathbf{p}$ - $\mathbb{H}$ -subharmonicity.

**Theorem 12.1** *Let  $F \in \mathcal{C}^2(\mathbb{H})$  be a bounded data function that satisfies on some open set  $U$ , compactly containing  $\mathcal{D}$ :*

$$\Delta_{\mathbb{H},\mathbf{p}} F = 0 \quad \text{and} \quad \nabla_{\mathbb{H}} F \neq 0 \quad \text{in } U. \tag{12.1}$$

Then the solutions  $u_\epsilon$  of (11.1) converge to  $F$  uniformly in  $\mathbb{H}$ , namely:

$$\|u_\epsilon - F\|_{C(\mathcal{D})} \leq C\epsilon \quad \text{as } \epsilon \rightarrow 0, \tag{12.2}$$

with a constant  $C$  depending on  $F, U, \mathcal{D}$  and  $\mathbf{p}$ , but not on  $\epsilon$ .

**Proof 1.** We first note that since  $u_\epsilon = F$  on  $\mathbb{H} \setminus \mathcal{D}$  by construction, (12.2) indeed implies the uniform convergence of  $u_\epsilon$  in  $\mathbb{H}$ . Also, by applying a left translation it not restrictive to assume that  $U$  does not intersect the interior of the cylinder  $\{q = (x, y, z) = (q_{hor}, z) \in \mathbb{H} : |q_{hor}|^2 = x^2 + y^2 \leq 1\}$ . In particular, this implies  $|q|_K \geq 1$  for all  $q \in \mathcal{D}$ .

We now show that there exists  $s \geq 4$  and  $\hat{\epsilon} > 0$  such that the following functions:

$$v_\epsilon(q) = F(q) + \epsilon|q|_K^s$$

satisfy, for every  $\epsilon \in (0, \hat{\epsilon})$ :

$$\nabla_{\mathbb{H}} v_\epsilon \neq 0 \quad \text{and} \quad \Delta_{\mathbb{H}, \mathbf{p}} v_\epsilon \geq \epsilon s \cdot |\nabla_{\mathbb{H}} v_\epsilon|^{\mathbf{p}-2} \quad \text{in } \bar{\mathcal{D}}. \tag{12.3}$$

Fix  $q \in \bar{\mathcal{D}}, \epsilon \in (0, 1)$  and denote  $a = \nabla_{\mathbb{H}} v_\epsilon(q)$  and  $b = \nabla_{\mathbb{H}} F(q)$ . By (12.1) it follows that:

$$\Delta_{\mathbb{H}, \mathbf{p}} v_\epsilon(q) = |\nabla_{\mathbb{H}} v_\epsilon(q)|^{\mathbf{p}-2} (I + II + III), \tag{12.4}$$

where:

$$\begin{aligned} I &= \epsilon \Delta_{\mathbb{H}}(|q|_K^s), \\ II &= \epsilon(\mathbf{p} - 2) \langle \nabla_{\mathbb{H}}^2(|q|_K^s) : \frac{a}{|a|} \otimes \frac{a}{|a|} \rangle, \\ III &= (\mathbf{p} - 2) \langle \nabla_{\mathbb{H}}^2 F(q) : \frac{a}{|a|} \otimes \frac{a}{|a|} - \frac{b}{|b|} \otimes \frac{b}{|b|} \rangle. \end{aligned}$$

Denoting  $\xi = |q_{hor}|^2 q_{hor} + 4z q_{hor}^\perp = (x(x^2 + y^2) - 4yz, y(x^2 + y^2) + 4xz) \in \mathbb{R}^2$ , a further computation shows that:

$$\begin{aligned} \nabla_{\mathbb{H}}(|q|_K^s) &= s|q|_K^{s-4} \xi, \\ \Delta_{\mathbb{H}}(|q|_K^s) &= s(s + 2)|q_{hor}|^2 |q|_K^{s-4}, \\ \nabla_{\mathbb{H}}^2(|q|_K^s) &= s|q|_K^{s-8} ((s - 4)\xi \otimes \xi + 3|q_{hor}|^2 |q|_K^4 I d_2). \end{aligned}$$

Consequently, we have:

$$\begin{aligned} \langle \nabla_{\mathbb{H}}^2(|q|_K^s) : \frac{a}{|a|} \otimes \frac{a}{|a|} \rangle &= s|q|_K^{s-8} \left( (s - 4) \langle \xi \otimes \xi : \frac{a}{|a|} \otimes \frac{a}{|a|} \rangle + 3|q_{hor}|^2 |q|_K^4 \right) \\ &\geq 3s|q_{hor}|^2 |q|_K^{s-4}. \end{aligned}$$

Also, since  $|\xi| = |q_{hor}| |q|_K^2 \leq |q_{hor}|^2 |q|_K^2$ , observe that:

$$\begin{aligned} \left| \langle \nabla_{\mathbb{H}}^2 F(q) : \frac{a}{|a|} \otimes \frac{a}{|a|} - \frac{b}{|b|} \otimes \frac{b}{|b|} \rangle \right| &\leq 4 |\nabla_{\mathbb{H}}^2 F(q)| \frac{|a-b|}{|b|} \leq 4\epsilon s |\xi| |q|_K^{s-4} \frac{|\nabla_{\mathbb{H}}^2 F(q)|}{|\nabla_{\mathbb{H}} F(q)|} \\ &\leq 4\epsilon s |q_{hor}|^2 \cdot |q|_K^{s-2} \frac{|\nabla_{\mathbb{H}}^2 F(q)|}{|\nabla_{\mathbb{H}} F(q)|} \end{aligned}$$

We gather the estimates above to get, in view of (12.4):

$$\begin{aligned} \Delta_{\mathbb{H}, \mathbf{p}} v_\epsilon(q) &\geq |\nabla_{\mathbb{H}} v_\epsilon(q)|^{\mathbf{p}-2} \epsilon s \cdot |q_{hor}|^2 |q|_K^{s-4} \\ &\quad \times \left( (s+2) + 3(\mathbf{p}-2) - 4(\mathbf{p}-2) |q|_K^2 \cdot \frac{|\nabla_{\mathbb{H}}^2 F(q)|}{|\nabla_{\mathbb{H}} F(q)|} \right), \end{aligned}$$

It is clear that for  $s$  large enough, the quantity in parentheses above is uniformly bounded from below by 1 on  $\bar{\mathcal{D}}$ . This justifies the second bound in (12.3), since  $|q|_K, |q_{hor}| \geq 1$  on  $\bar{\mathcal{D}}$ . Finally, choosing  $\hat{\epsilon}$  sufficiently small we ensure that  $\nabla_{\mathbb{H}} v_\epsilon \neq 0$  in  $\bar{\mathcal{D}}$  for  $\epsilon \in (0, \hat{\epsilon})$ .

2. We claim that  $s$  and  $\hat{\epsilon}$  in step 1 can further be chosen in a way that for all  $\epsilon \in (0, \hat{\epsilon})$ :

$$v_\epsilon \leq S_\epsilon v_\epsilon \quad \text{in } \bar{\mathcal{D}}. \tag{12.5}$$

Indeed, a careful analysis of the remainder terms in Taylor’s expansion (4.2) reveals that:

$$S_\epsilon v_\epsilon(q) = v_\epsilon(q) + \frac{r^2}{3\pi} \cdot \frac{\Delta_{\mathbb{H}, \mathbf{p}} v_\epsilon(q)}{|\nabla_{\mathbb{H}} v_\epsilon(q)|^{\mathbf{p}-2}} + R_2(\epsilon, q), \tag{12.6}$$

where:

$$|R_2(\epsilon, q)| \leq C_{\mathbf{p}} \epsilon^2 \text{osc}_{B_\epsilon(q)} (|\nabla_{\mathbb{H}}^2 v_\epsilon| + |Z v_\epsilon|) + C \epsilon^3.$$

Above, we denoted by  $C_{\mathbf{p}}$  a constant depending only on  $\mathbf{p}$ , whereas  $C$  is a constant depending only on the quantities  $|\nabla_{\mathbb{H}} v_\epsilon(q)|$  and  $\|\nabla^2 v_\epsilon\|_{\mathcal{C}(B_\epsilon(q))}$ , that remain uniformly bounded in  $q \in \bar{\mathcal{D}}$  for small  $\epsilon$ . Since  $v_\epsilon$  is the sum of the smooth on  $U$  function  $q \mapsto \epsilon |q|_K^s$ , and a  $\mathbf{p}$ -harmonic function  $u$  that is also smooth in virtue of its non vanishing horizontal gradient (see [33]), we obtain that:  $\text{osc}_{B_\epsilon(q)} (|\nabla_{\mathbb{H}}^2 v_\epsilon| + |T v_\epsilon|) \leq C_u \epsilon$ , with  $C_u$  depending only on the third derivatives of  $u$  (and on  $\mathcal{D}$ ). In conclusion, (12.6) and (12.3) imply (12.5) for  $s$  sufficiently large, because:

$$v_\epsilon(q) - S_\epsilon v_\epsilon(q) \leq -\frac{\epsilon^2}{3\pi} \frac{\Delta_{\mathbb{H}, \mathbf{p}} v_\epsilon(q)}{|\nabla_{\mathbb{H}} v_\epsilon(q)|^{\mathbf{p}-2}} + \epsilon^3 (C_{\mathbf{p}} C_u + C) \leq \epsilon^3 \left( -\frac{s}{3\pi} + C_{\mathbf{p}} C_u + C \right) \leq 0.$$

3. Let  $A$  be a compact set in:  $\mathcal{D} \subset A \subset U$ . Fix  $\epsilon \in (0, \hat{\epsilon})$  and for each  $q \in A$  consider:

$$\phi_\epsilon(q) = v_\epsilon(q) - u_\epsilon(q) = F(q) - u_\epsilon(q) + \epsilon|q|_K^s.$$

By (12.5) and (11.1) we get:

$$\begin{aligned} \phi_\epsilon(q) &= d_\epsilon(q)(v_\epsilon(q) - S_\epsilon u_\epsilon(q)) + (1 - d_\epsilon(q))(v_\epsilon(q) - F(q)) \\ &\leq d_\epsilon(q)(S_\epsilon v_\epsilon(q) - S_\epsilon u_\epsilon(q)) + (1 - d_\epsilon(q))(v_\epsilon(q) - F(q)) \\ &\leq d_\epsilon(q)\left(\frac{1}{3}\mathcal{A}_3(\phi_\epsilon, \epsilon)(q) + \frac{2}{3} \sup_{B_{(1+\gamma)\epsilon}(q)} \phi_\epsilon\right) + (1 - d_\epsilon(q))(v_\epsilon(q) - F(q)). \end{aligned} \tag{12.7}$$

Define:

$$M_\epsilon = \max_A \phi_\epsilon.$$

We claim that there exists  $q_0 \in A$  with  $d_\epsilon(q_0) < 1$  and such that  $\phi_\epsilon(q_0) = M_\epsilon$ . To prove the claim, define  $\mathcal{D}^\epsilon = \{q \in \mathcal{D}; \text{dist}(q, \partial\mathcal{D}) \geq \epsilon\}$ . We can assume that the closed set  $\mathcal{D}^\epsilon \cap \{\phi_\epsilon = M_\epsilon\}$  is nonempty; otherwise the claim would be obvious. Let  $\mathcal{D}_0^\epsilon$  be a nonempty connected component of  $\mathcal{D}^\epsilon$  and denote  $\mathcal{D}_M^\epsilon = \mathcal{D}_0^\epsilon \cap \{\phi_\epsilon = M_\epsilon\}$ . Clearly,  $\mathcal{D}_M^\epsilon$  is closed in  $\mathcal{D}_0^\epsilon$ ; we now show that it is also open. Let  $q \in \mathcal{D}_M^\epsilon$ . Since  $d_\epsilon(q) = 1$  from (12.7) it follows that:

$$M_\epsilon = \phi_\epsilon(q) \leq \frac{1}{3}\mathcal{A}_3(\phi_\epsilon, \epsilon)(q) + \frac{2}{3} \sup_{B_{(1+\gamma)\epsilon}(q)} \phi_\epsilon \leq M_\epsilon.$$

Consequently,  $\mathcal{A}_3(\phi_\epsilon, \epsilon)(q) = M_\epsilon$ , implying  $\phi_\epsilon \equiv M_\epsilon$  in  $B_\epsilon(q)$  and thus openness of  $\mathcal{D}_M^\epsilon$  in  $\mathcal{D}_0^\epsilon$ . In particular,  $\mathcal{D}_M^\epsilon$  contains a point  $\bar{q} \in \partial\mathcal{D}^\epsilon$ . Repeating the previous argument for  $\bar{q}$  results in  $\phi_\epsilon \equiv M_\epsilon$  in  $B_\epsilon(\bar{q})$ , proving the claim.

We now complete the proof of Theorem 12.1 by deducing a bound on  $M_\epsilon$ . If  $M_\epsilon = \phi_\epsilon(q_0)$  for some  $q_0 \in \bar{\mathcal{D}}$  with  $d_\epsilon(q_0) < 1$ , then (12.7) yields:  $M_\epsilon = \phi_\epsilon(q_0) \leq d_\epsilon(q_0)M_\epsilon + (1 - d_\epsilon(q_0))(v_\epsilon(q_0) - F(q_0))$ , which implies:

$$M_\epsilon \leq v_\epsilon(q_0) - F(q_0) = \epsilon|q_0|_K^s.$$

On the other hand, if  $M_\epsilon = \phi_\epsilon(q_0)$  for some  $q_0 \in A \setminus \mathcal{D}$ , then  $d_\epsilon(q_0) = 0$ , hence likewise:  $M_\epsilon = \phi_\epsilon(q_0) = v_\epsilon(q_0) - F(q_0) = \epsilon|q_0|_K^s$ . In either case:

$$\max_{\bar{\mathcal{D}}}(F - u_\epsilon) \leq \max_{\bar{\mathcal{D}}} \phi_\epsilon + C\epsilon \leq 2C\epsilon$$

where  $C = \max_{q \in U} |q|_K^s$  is independent of  $\epsilon$ . A symmetric argument applied to  $-u$  after noting that  $(-u)_\epsilon = -u_\epsilon$  gives:  $\min_{\bar{\mathcal{D}}}(u - u_\epsilon) \geq -2C\epsilon$ . The proof is done.  $\square$

### 13 Convergence of $u_\epsilon$ and game-regularity

Towards checking convergence of  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$ , we first prove a counterpart of Lemma 7.1 for the case of  $\mathbf{p} > 2$ . Namely, we will show that equicontinuity of  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  on  $\mathbb{H}$  is a consequence of equicontinuity “at  $\partial\mathcal{D}$ ”. This last property will be, in turn, implied by the “game-regularity” condition, which mimics the “walk-regularity” Definition 7.2 in the context of the stochastic Tug of War as in Sect. 10. The aforementioned condition, given in Definition 13.2, and the following Lemma 13.4 and Theorem 13.5 are adapted from the same statements in the seminal paper [32], where another Tug of War was proposed in the Euclidean setting, for  $\mathbf{p} \in (1, \infty)$ .

Let  $\mathcal{D} \subset \mathbb{H}$  be an open, bounded, connected domain and let  $F \in \mathcal{C}(\mathbb{H})$  be a bounded data function. We have the following:

**Theorem 13.1** *Let  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  be the family of solutions to (11.1). Assume that for every  $\eta > 0$  there exists  $\delta > 0$  and  $\epsilon \in (0, 1)$  such that for all  $\epsilon \in (0, \hat{\epsilon})$  there holds:*

$$|u_\epsilon(q'_0) - u_\epsilon(q_0)| \leq \eta \quad \text{for all } q'_0 \in \mathcal{D}, q_0 \in \partial\mathcal{D} \\ \text{satisfying } |q_0 - q'_0| \leq \delta. \tag{13.1}$$

Then the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  is equicontinuous in  $\bar{\mathcal{D}}$ .

**Proof 1.** We present an analytical proof. A probabilistic argument is available as well, based on a game translation argument as in the proof of Lemma 7.1. For every small  $\hat{\delta} > 0$  the set  $\mathcal{D}^{\hat{\delta}}$  below is open, bounded and connected, where we define:

$$\mathcal{D}^{\hat{\delta}} = \{q \in \mathcal{D}; \text{dist}(q, \mathbb{H} \setminus \mathcal{D}) > \hat{\delta}\} \quad \text{and} \quad d_\epsilon^{\hat{\delta}}(q) = \frac{1}{\epsilon} \min\{\epsilon, \text{dist}(q, \mathbb{H} \setminus \mathcal{D}^{\hat{\delta}})\}.$$

Fix  $\eta > 0$ . In view of (13.1) and since without loss of generality the data function  $F$  is constant outside of some large bounded superset of  $\mathcal{D}$  in  $\mathbb{H}$ , there exists  $\hat{\delta} > 0$  satisfying:

$$|u_\epsilon(w * q) - u_\epsilon(q)| \leq \eta \quad \text{for all } q \in \mathbb{H} \setminus \mathcal{D}^{\hat{\delta}}, |w| \leq \hat{\delta}, \epsilon \in (0, \hat{\epsilon}). \tag{13.2}$$

Further, let  $\delta \in (0, \hat{\delta})$  be such that:

$$|q'_0 * q_0^{-1}| \leq \hat{\delta} \quad \text{and} \quad |q^{-1} * q'_0 * q_0^{-1} * q|_K \leq \frac{\hat{\delta}}{2} \tag{13.3} \\ \text{for all } q \in \mathcal{D} \quad \text{and all } q_0, q'_0 \in \bar{\mathcal{D}} \text{ satisfying } |q_0 - q'_0| \leq \delta.$$

Fix  $q_0, q'_0 \in \bar{\mathcal{D}}$  as above and let  $\epsilon \in (0, \frac{\hat{\delta}}{2})$ . Consider the following function  $\tilde{u}_\epsilon \in \mathcal{C}(\mathbb{H})$ :

$$\tilde{u}_\epsilon(q) = u_\epsilon(q'_0 * q_0^{-1} * q) + \eta.$$



Then  $\mathcal{A}_3(\tilde{u}_\epsilon, \epsilon)(q) = \mathcal{A}_3(u_\epsilon, \epsilon)(q'_0 * q_0^{-1} * q) + \eta$  and  $\inf_{B_{\gamma\rho^\epsilon}(q)} \mathcal{A}_3(\tilde{u}_\epsilon, \epsilon) = \inf_{B_{\gamma\rho^\epsilon}(q'_0 * q_0^{-1} * q)} \mathcal{A}_3(u_\epsilon, \epsilon) + \eta$ , with the same identity valid for the supremum as well. Consequently:

$$\begin{aligned} (S_\epsilon \tilde{u}_\epsilon)(q) &= \frac{1}{3} \mathcal{A}_3(\tilde{u}_\epsilon, \epsilon)(q) + \frac{1}{3} \inf_{B_{\gamma\rho^\epsilon}(q)} \mathcal{A}_3(\tilde{u}_\epsilon, \epsilon) + \frac{1}{3} \sup_{B_{\gamma\rho^\epsilon}(q)} \mathcal{A}_3(\tilde{u}_\epsilon, \epsilon) \\ &= (S_\epsilon u_\epsilon)(q'_0 * q_0^{-1} * q) + \eta = u_\epsilon(q'_0 * q_0^{-1} * q) + \eta = \tilde{u}_\epsilon(q) \quad \text{for all } q \in \mathcal{D}^\delta. \end{aligned} \tag{13.4}$$

Indeed, in view of  $q \in \mathcal{D}^\delta$  and (13.3) we have:

$$d(q'_0 * q_0^{-1} * q, \mathbb{H} \setminus \mathcal{D}) \geq d(q, \mathbb{H} \setminus \mathcal{D}) - d(q'_0 * q_0^{-1} * q, q) \geq \hat{\delta} - \frac{\hat{\delta}}{2} = \frac{\hat{\delta}}{2} > \epsilon.$$

2. It follows now from (13.4) that for all  $\epsilon \in (0, \frac{\hat{\delta}}{2})$  and all  $q_0, q'_0 \in \bar{\mathcal{D}}$  satisfying  $|q_0 - q'_0| \leq \delta$ :

$$\tilde{u}_\epsilon = d_\epsilon^\delta(S_\epsilon \tilde{u}_\epsilon) + (1 - \tilde{d}_\epsilon^\delta) \tilde{u}_\epsilon \quad \text{in } q \in \mathbb{H}.$$

On the other hand,  $u_\epsilon$  itself similarly solves the same problem above, subject to its own data  $u_\epsilon$  on  $\mathbb{H} \setminus \mathcal{D}^\delta$ . Since for every  $q \in \mathbb{H} \setminus \mathcal{D}^\delta$  we have:  $\tilde{u}_\epsilon(q) - u_\epsilon(q) = u_\epsilon(q'_0 * q_0^{-1} * q) - u_\epsilon(q) + \eta \geq 0$  in view of (13.2) and (13.3), the monotonicity property in Theorem 11.1 yields:

$$u_\epsilon \leq \tilde{u}_\epsilon \quad \text{in } \mathbb{H}.$$

Thus, in particular:  $u_\epsilon(q_0) - u_\epsilon(q'_0) \leq \eta$ . Exchanging  $q_0$  with  $q'_0$  we get the opposite inequality, and hence  $|u_\epsilon(q_0) - u_\epsilon(q'_0)| \leq \eta$ , establishing the claimed equicontinuity of  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  in  $\bar{\mathcal{D}}$ . □

Following [32] we introduce the following definition. A point  $q_0 \in \partial\mathcal{D}$  will be called game-regular if, whenever the game starts near  $q_0$ , one of the “players” has a strategy for making the game terminate near the same  $q_0$ , with high probability. More precisely:

**Definition 13.2** Consider the Tug of War game with noise in (10.1) and (10.3).

- (a) We say that a point  $q_0 \in \partial\mathcal{D}$  is *game-regular* if for every  $\eta, \delta > 0$  there exist  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, 1)$  such that the following holds. Fix  $\epsilon \in (0, \hat{\epsilon})$  and  $p_0 \in B_{\hat{\delta}}(q_0)$ ; there exists then a strategy  $\sigma_{0,I}$  with the property that for every strategy  $\sigma_{II}$  we have:

$$\mathbb{P}(Q_{\tau-1} \in B_\delta(q_0)) \geq 1 - \eta. \tag{13.5}$$

- (b) We say that  $\mathcal{D}$  is game-regular if every boundary point  $q_0 \in \partial\mathcal{D}$  is game-regular.

**Remark 13.3** As in Definition 7.2 of walk-regularity, if condition (b) holds, then  $\hat{\delta}$  and  $\hat{\epsilon}$  in part (a) can be chosen independently of  $q_0$ . Also, game-regularity is symmetric with respect to  $\sigma_I$  and  $\sigma_{II}$ .

**Lemma 13.4** Assume that for every bounded data  $F \in C(\mathbb{H})$ , the family of solutions  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  of (11.1) is equicontinuous in  $\bar{\mathcal{D}}$ . Then  $\mathcal{D}$  is game-regular.

**Proof** Fix  $q_0 \in \partial\mathcal{D}$  and let  $\eta, \delta \in (0, 1)$ . Define the data function:  $F(q) = -\min\{1, d(q, q_0)\}$ . By assumption and since  $u_\epsilon(q_0) = F(q_0) = 0$ , there exists  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, 1)$  such that:

$$|u_\epsilon(p_0)| < \eta\delta \quad \text{for all } p_0 \in B_{\hat{\delta}}(q_0) \text{ and } \epsilon \in (0, \hat{\epsilon}).$$

Consequently:

$$\sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}[F \circ (Q^{\epsilon, p_0})_{\tau-1}] = u_I^\epsilon(p_0) > -\eta\delta,$$

and thus there exists  $\sigma_{0,I}$  with the property that:  $\mathbb{E}[F \circ (Q^{\epsilon, p_0, \sigma_{0,I}, \sigma_{II}})_{\tau-1}] > -\eta\delta$  for every strategy  $\sigma_{II}$ . Then:

$$\mathbb{P}(Q_{\tau-1} \notin B_\delta(q_0)) \leq -\frac{1}{\delta} \int_{\Omega} F(Q_{\tau-1}) \, d\mathbb{P} < \eta,$$

proving (13.5) and hence game-regularity of  $q_0$ . □

**Theorem 13.5** Assume that  $\mathcal{D}$  is game-regular. Then, for every bounded data  $F \in C(\mathbb{H})$ , the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  of solutions to (11.1) is equicontinuous in  $\bar{\mathcal{D}}$ .

**Proof** In virtue of Theorem 13.1 it is enough to validate the condition (13.1). To this end, fix  $\eta > 0$  and let  $\delta > 0$  be such that:

$$|F(p) - F(q_0)| \leq \frac{\eta}{3} \quad \text{for all } q_0 \in \partial\mathcal{D} \text{ and } p \in B_\delta(q_0). \tag{13.6}$$

By Remark 13.3 and Definition 13.2, we may choose  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, \delta)$  such that for every  $\epsilon \in (0, \hat{\epsilon})$ , every  $q_0 \in \partial\mathcal{D}$  and every  $p_0 \in B_{\hat{\delta}}(q_0)$ , there exists a strategy  $\sigma_{0,II}$  with the property that for every  $\sigma_I$  there holds:

$$\mathbb{P}((Q^{\epsilon, p_0, \sigma_I, \sigma_{0,II}})_{\tau-1} \in B_\delta(q_0)) \geq 1 - \frac{\eta}{6\|F\|_{C(\mathbb{H})} + 1}. \tag{13.7}$$

Let  $q_0 \in \partial\mathcal{D}$  and  $q'_0 \in \mathcal{D}$  satisfy  $|q_0 - q'_0| \leq \hat{\delta}$ . Then:

$$\begin{aligned} u_\epsilon(q'_0) - u_\epsilon(q_0) &= u_{II}^\epsilon(q'_0) - F(q_0) \leq \sup_{\sigma_I} \mathbb{E}[F \circ (Q^{\epsilon, q'_0, \sigma_I, \sigma_{0,II}})_{\tau-1} - F(q_0)] \\ &\leq \mathbb{E}[F \circ (Q^{\epsilon, q'_0, \sigma_{0,I}, \sigma_{0,II}})_{\tau-1} - F(q_0)] + \frac{\eta}{3}, \end{aligned}$$

for some almost-supremizing strategy  $\sigma_{0,I}$ . Thus, by (13.6) and (13.7):

$$\begin{aligned} u_\epsilon(q'_0) - u_\epsilon(q_0) &\leq \int_{\{Q_{\tau-1} \in B_\delta(q_0)\}} |F(Q_{\tau-1}) - F(q_0)| \, d\mathbb{P} \\ &\quad + \int_{\{Q_{\tau-1} \notin B_\delta(q_0)\}} |F(Q_{\tau-1}) - F(q_0)| \, d\mathbb{P} + \frac{\eta}{3} \\ &\leq \frac{\eta}{3} + 2\|F\|_{C(\mathbb{H})} \mathbb{P}(Q_{\tau-1} \notin B_\delta(q_0)) + \frac{\eta}{3} \leq \eta. \end{aligned}$$

The remaining inequality  $u_\epsilon(q'_0) - u_\epsilon(q_0) > -\eta$  is obtained by a reverse argument. □

### 14 Concatenating strategies, the annulus walk and the exterior $\mathbb{H}$ -corkscrew condition as sufficient for game-regularity

We start with a result on concatenating strategies, which contains a condition equivalent to the game-regularity criterion in Definition 13.2 (a). This is similar to the proof of Theorem 14.1, both derived from the construction in [32] for Euclidean setting. Let  $\mathcal{D} \subset \mathbb{H}$  be an open, bounded connected domain.

**Theorem 14.1** *For a given  $q_0 \in \partial\mathcal{D}$ , assume that there exists  $\theta_0 \in (0, 1)$  such that for every  $\delta > 0$  there exists  $\hat{\delta} \in (0, \delta)$  and  $\hat{\epsilon} \in (0, 1)$  with the following property. Fix  $\epsilon \in (0, \hat{\epsilon})$  and choose an initial position  $p_0 \in B_\delta(q_0)$ ; there exists a strategy  $\sigma_{0,II}$  such that for every  $\sigma_I$  we have:*

$$\mathbb{P}(\exists n < \tau \quad Q_n \notin B_\delta(q_0)) \leq \theta_0. \tag{14.1}$$

Then  $q_0$  is game-regular.

**Proof 1.** Under condition (14.1), construction of an optimal strategy realising the (arbitrarily small) threshold  $\eta$  in (13.5) is carried out by concatenating the  $m$  optimal strategies corresponding to the achievable threshold  $\eta_0$ , on  $m$  concentric balls, where  $(1 - \eta_0)^m = 1 - \theta_0^m \geq 1 - \eta$ .

Fix  $\eta, \delta > 0$ . We want to find  $\hat{\epsilon}$  and  $\hat{\delta}$  such that (13.5) holds. Observe first that for  $\theta_0 \leq \eta$  the claim follows directly from (14.1). In the general case, let  $m \in \{2, 3, \dots\}$  be such that:

$$\theta_0^m \leq \eta. \tag{14.2}$$

Below we inductively define the radii  $\{\delta_k\}_{k=1}^m$ , together with the quantities  $\{\hat{\delta}(\delta_k)\}_{k=1}^m$ ,  $\{\hat{\epsilon}(\delta_k)\}_{k=1}^m$  from the assumed condition (14.1). Namely, for every initial position in  $B_{\hat{\delta}(\delta_k)}(q_0)$  in the Tug of War game with step less than  $\hat{\epsilon}(\delta_k)$ , there exists a strategy  $\sigma_{0,II,k}$  guaranteeing exiting  $B_{\delta_k}(q_0)$  (before the process is stopped) with probability at most  $\theta_0$ . We set  $\delta_m = \delta$  and find  $\hat{\delta}(\delta_m) \in (0, \delta)$  and  $\hat{\epsilon}(\delta_m) \in (0, 1)$ , with the indicated choice of the strategy  $\sigma_{0,II,m}$ . Decreasing the value of  $\hat{\epsilon}(\delta_m)$  if necessary, we then set:

$$\delta_{m-1} = \hat{\delta}(\delta_m) - (1 + \gamma_{\mathbf{p}})\hat{\epsilon}(\delta_m) > 0.$$

Similarly, having constructed  $\delta_k > 0$ , we find  $\hat{\delta}(\delta_k) \in (0, \delta_k)$  and  $\hat{\epsilon}(\delta_k) \in (0, \hat{\epsilon}(\delta_{k+1}))$  and define:

$$\delta_{k-1} = \hat{\delta}(\delta_k) - (1 + \gamma_p)\hat{\epsilon}(\delta_k) > 0.$$

Eventually, we call:

$$\hat{\delta} = \hat{\delta}(\delta_1), \quad \hat{\epsilon} = \hat{\epsilon}(\delta_1).$$

To show that the condition of game-regularity at  $q_0$  is satisfied, we will concatenate the strategies  $\{\sigma_{0,II,k}\}_{k=1}^m$  by switching to  $\sigma_{0,II,k+1}$  immediately after the token exits  $B_{\delta_k}(q_0) \subset B_{\hat{\delta}(\delta_{k+1})}(q_0)$ . This construction is carried out in the next step.

2. Fix  $p_0 \in B_{\hat{\delta}}(q_0)$  and let  $\epsilon \in (0, \hat{\epsilon})$ . Define the strategy  $\sigma_{0,II}$ :

$$\sigma_{0,II}^n = \sigma_{0,II}^n(q_0, (q_1, w_1, s_1, t_1), \dots, (q_n, w_n, s_n, t_n)) \quad \text{for all } n \geq 0,$$

separately in the following two cases.

Case 1. If  $q_k \in B_{\delta_1}(q_0)$  for all  $k \leq n$ , then we set:

$$\sigma_{0,II}^n = \sigma_{0,II,1}^n(q_0, (q_1, w_1, s_1, t_1), \dots, (q_n, w_n, s_n, t_n)).$$

Case 2. Otherwise, define:

$$k \doteq k(x_0, x_1, \dots, x_n) = \max \left\{ 1 \leq k \leq m - 1; \exists 0 \leq i \leq n \ q_i \notin B_{\delta_k}(q_0) \right\}$$

$$i \doteq \min \left\{ 0 \leq i \leq n; q_i \notin B_{\delta_k}(q_0) \right\}.$$

and set:

$$\sigma_{0,II}^n = \sigma_{0,II,k+1}^{n-i}(q_i, (q_{i+1}, w_{i+1}, s_{i+1}, t_{i+1}), \dots, (q_n, w_n, s_n, t_n)).$$

It is not hard to check that each  $\sigma_{0,II}^n : H_n \rightarrow B_1(0) \subset \mathbb{H}$  is Borel measurable, as required. Let  $\sigma_I$  be now any opposing strategy. Then, a classical argument via Fubini's theorem, gives:

$$\mathbb{P}(\exists n < \tau \ Q_n \notin B_{\delta_k}(q_0)) \leq \theta_0 \mathbb{P}(\exists n < \tau \ Q_n \notin B_{\delta_{k-1}}(q_0)) \quad \text{for all } k = 2 \dots m,$$

so consequently:

$$\mathbb{P}(\exists n < \tau \ Q_n \notin B_{\delta}(q_0)) \leq \theta_0^{m-1} \mathbb{P}(\exists n \leq \tau \ Q_n \notin B_{\delta_1}(q_0)) \leq \theta_0^m.$$

This yields the result by (14.2) and completes the proof. □

The proof of game-regularity in what follows will be based on the concatenating strategies technique above and the analysis of the annulus walk below. Namely, we will

derive an estimate on the probability of exiting a given annular domain  $\tilde{\mathcal{D}}$  through the external portion of its boundary. We show that when the ratio of the annulus thickness and the distance of the initial token position  $q_0$  from the internal boundary is large enough, then this probability may be bounded by a universal constant  $\theta_0 < 1$ . When  $\mathbf{p} \geq 4$ , then  $\theta_0$  converges to 0 as the indicated ratio goes to  $\infty$ .

**Theorem 14.2** *For given radii  $0 < R_1 < R_2 < R_3$ , consider the annulus  $\tilde{\mathcal{D}} = B_{R_3}(0) \setminus \bar{B}_{R_1}(0) \subset \mathbb{H}$ . For every  $\xi > 0$ , there exists  $\hat{\epsilon} \in (0, 1)$  depending on  $R_1, R_2, R_3$  and  $\xi, \mathbf{p}$ , such that for every  $p_0 \in \tilde{\mathcal{D}} \cap B_{R_2}(0)$  and every  $\epsilon \in (0, \hat{\epsilon})$ , there exists a strategy  $\tilde{\sigma}_{0,11}$  with the property that for every strategy  $\tilde{\sigma}_I$  there holds:*

$$\mathbb{P}\left(\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0)\right) \leq \frac{v(R_2) - v(R_1)}{v(R_3) - v(R_1)} + \xi. \tag{14.3}$$

Here,  $v : (0, \infty) \rightarrow \mathbb{R}$  is given by:

$$v(t) = \begin{cases} \operatorname{sgn}(\mathbf{p} - 4) t^{\frac{\mathbf{p}-4}{\mathbf{p}-1}} & \text{for } \mathbf{p} \neq 4 \\ \log t & \text{for } \mathbf{p} = 4, \end{cases} \tag{14.4}$$

and  $\{\tilde{Q}_n = \tilde{Q}_n^{\epsilon, p_0, \tilde{\sigma}_I, \tilde{\sigma}_{0,11}}\}_{n=0}^\infty$  and  $\tilde{\tau} = \tilde{\tau}^{\epsilon, p_0, \tilde{\sigma}_I, \tilde{\sigma}_{0,11}}$  denote, as usual, the random variables corresponding to positions and stopping time in the Tug of War game (10.1) on  $\tilde{\mathcal{D}}$ .

**Proof** Consider the radial function  $u : \mathbb{H} \setminus \{0\} \rightarrow \mathbb{R}$  given by  $u(q) = v(|q|_K)$ , where  $v$  is as in (14.4). Recall that:

$$\Delta_{\mathbb{H}, \mathbf{p}} u = 0 \quad \text{and} \quad \nabla_{\mathbb{H}} u \neq 0 \quad \text{in} \quad \mathbb{H} \setminus \{0\}. \tag{14.5}$$

Let  $\tilde{u}_\epsilon$  be the family of solutions to (11.1) with the data  $F$  given by a smooth and bounded modification of  $u$  outside of the annulus  $B_{2R_3}(0) \setminus \bar{B}_{R_1/2}(0)$ . By Theorem 12.1, there exists a constant  $C > 0$ , depending only on  $\mathbf{p}, u$  and  $\tilde{\mathcal{D}}$ , such that:

$$\|\tilde{u}_\epsilon - u\|_{C(\tilde{\mathcal{D}})} \leq C\epsilon \quad \text{as} \quad \epsilon \rightarrow 0.$$

For a given  $q_0 \in \tilde{\mathcal{D}} \cap B_{R_2}(0)$ , there exists thus a strategy  $\tilde{\sigma}_{0,11}$  so that for every  $\tilde{\sigma}_I$  we have:

$$\mathbb{E}\left[u \circ (\tilde{Q}^{\epsilon, q_0, \tilde{\sigma}_I, \tilde{\sigma}_{0,11}})_{\tilde{\tau}-1}\right] - u(q_0) \leq 2C\epsilon. \tag{14.6}$$

We now estimate:

$$\begin{aligned} \mathbb{E}[u \circ \tilde{Q}_{\tilde{\tau}-1}] - u(q_0) &= \int_{\{\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0)\}} u(\tilde{Q}_{\tilde{\tau}-1}) \, d\mathbb{P} + \int_{\{\tilde{Q}_{\tilde{\tau}-1} \in B_{R_1+\epsilon}(0)\}} u(\tilde{Q}_{\tilde{\tau}-1}) \, d\mathbb{P} - u(q_0) \\ &\geq \mathbb{P}(\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0))v(R_3 - \epsilon) \\ &\quad + \left(1 - \mathbb{P}(\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0))\right)v(R_1 - \gamma_{\mathbf{p}}\epsilon) - v(R_2), \end{aligned}$$

where we used the fact that  $v$  in (14.4) is an increasing function. Recalling (14.6), this implies:

$$\mathbb{P}(\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0)) \leq \frac{v(R_2) - v(R_1 - \gamma_{\mathbf{p}}\epsilon) + 2C\epsilon}{v(R_3 - \epsilon) - v(R_1 - \gamma_{\mathbf{p}}\epsilon)}. \tag{14.7}$$

The proof of (14.3) is now complete, by continuity of the right hand side with respect to  $\epsilon$ . □

By inspecting the quotient in the right hand side of (14.3) we obtain:

**Corollary 14.3** *The function  $v$  in (14.4) satisfies, for any fixed  $0 < R_1 < R_2$ :*

$$\begin{aligned} (a) \quad \lim_{R_3 \rightarrow \infty} \frac{v(R_2) - v(R_1)}{v(R_3) - v(R_1)} &= \begin{cases} 1 - \left(\frac{R_2}{R_1}\right)^{\frac{\mathbf{p}-4}{\mathbf{p}-1}} & \text{for } 2 < \mathbf{p} < 4 \\ 0 & \text{for } \mathbf{p} \geq 4, \end{cases} \\ (b) \quad \lim_{M \rightarrow \infty} \frac{v(MR_1) - v(R_1)}{v(M^2R_1) - v(R_1)} &= \begin{cases} \frac{1}{2} & \text{for } \mathbf{p} = 4 \\ 0 & \text{for } \mathbf{p} > 4. \end{cases} \end{aligned}$$

Consequently, the estimate (14.3) can be replaced by:

$$\mathbb{P}(\tilde{Q}_{\tilde{\tau}-1} \notin \bar{B}_{R_3-\epsilon}(0)) \leq \theta_0 \tag{14.8}$$

valid for any  $\theta_0 > 1 - \left(\frac{R_2}{R_1}\right)^{\frac{\mathbf{p}-4}{\mathbf{p}-1}}$  if  $\mathbf{p} \in (2, 4)$ , and any  $\theta_0 > 0$  if  $\mathbf{p} \geq 4$ , upon choosing  $R_3$  sufficiently large with respect to  $R_1$  and  $R_2$ . Alternatively, when  $\mathbf{p} > 4$ , the same bound can be achieved by setting  $R_2 = MR_1$ ,  $R_3 = M^2R_1$  with the ratio  $M$  large enough.

The results of Theorem 14.2 and Corollary 14.3 are invariant under scaling (Fig. 4), i.e.:

**Remark 14.4** The bounds (14.3) and (14.8) remain true if we replace  $R_1, R_2, R_3$  by  $rR_1, rR_2, rR_3$ , the domain  $\tilde{D}$  by  $\rho_r\tilde{D}$  and  $\hat{\epsilon}$  by  $r\hat{\epsilon}$ , for any  $r > 0$ .

**Theorem 14.5** *Let  $q_0 \in \partial\mathcal{D}$  satisfy the exterior  $\mathbb{H}$ -corkscrew condition, as in Definition 8.1. Then  $q_0$  is game-regular.*

**Proof** With the help of Theorem 14.2, we will show that the assumption of Theorem 14.1 is satisfied, with probability  $\theta_0 < 1$  depending only on  $\mathbf{p} > 2$  and  $\mu \in (0, 1)$  in Definition 8.1. Namely, set  $R_1 = 1$ ,  $R_2 = \frac{2}{\mu}$  and  $R_3 > R_2$  according to Corollary 14.3 (a) in order to have  $\theta_0 = \theta_0(\mathbf{p}, R_1, R_2) < 1$ . Further, set  $r = \frac{\delta}{2R_3}$ . Using the corkscrew condition, we obtain:

$$B_{2rR_1}(q'_0) \subset B_{\delta/(\mu R_3)}(q_0) \setminus \tilde{D},$$

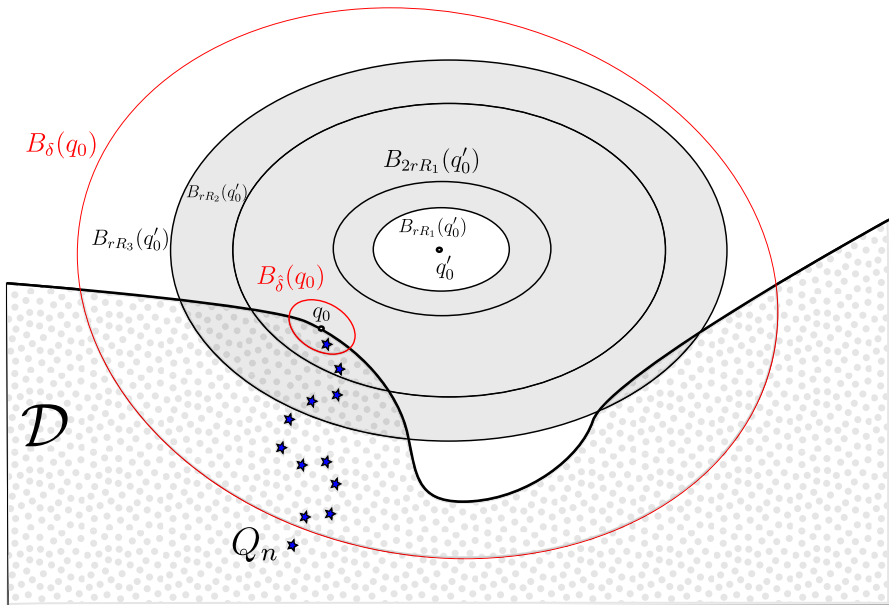


Fig. 4 The concentric balls and the annuli in the proof of Theorem 14.5

for some  $q'_0 \in \mathbb{H}$ . In particular:  $d(q_0, q'_0) < rR_2$ , so  $q_0 \in B_{rR_2}(q'_0) \setminus \bar{B}_{2rR_1}(q'_0)$ . It now easily follows that there exists  $\hat{\delta} \in (0, \delta)$  with the property that:

$$B_{\hat{\delta}}(q_0) \subset B_{rR_2}(q'_0) \setminus \bar{B}_{2rR_1}(q'_0).$$

Finally, we observe that  $B_{rR_3}(q'_0) \subset B_{\delta}(q_0)$  because  $rR_3 + rR_2 < 2rR_3 = \delta$ .

Let  $\hat{\epsilon}/r > 0$  be as in Theorem 14.2, applied to the annuli with radii  $R_1, R_2, R_3$ , in view of Remark 14.4. For a given  $p_0 \in B_{\hat{\delta}}(q_0)$  and  $\epsilon \in (0, \hat{\epsilon})$ , let  $\tilde{\sigma}_{0,II}$  be the strategy ensuring validity of the bound (14.8) in the annulus walk on  $q'_0 * \tilde{D}$ . For a given strategy  $\sigma_I$  there clearly holds:

$$\begin{aligned} & \left\{ \omega \in \Omega; \exists n < \tau^{\epsilon, p_0, \sigma_I, \sigma_{0,II}}(\omega) \quad Q_n^{\epsilon, p_0, \sigma_I, \sigma_{0,II}}(\omega) \notin B_{\hat{\delta}}(q_0) \right\} \\ & \subset \left\{ \omega \in \Omega; \tilde{Q}_{\tilde{\tau}-1}^{\epsilon, p_0, \tilde{\sigma}_I, \tilde{\sigma}_{0,II}}(\omega) \notin B_{rR_3-\epsilon}(q'_0) \right\}. \end{aligned}$$

The final claim follows by (14.8) and by applying Theorem 14.1. □

**Remark 14.6** By Corollary 14.3 (b) and adjusting the arguments in [32] to the Heisenberg group geometry, one can show that every open, bounded  $\mathcal{D} \subset \mathbb{H}$  is game-regular for  $p > 4$ . This will be the content of a separate work.

### 15 Uniqueness and identification of the limit $u$ in Theorem 13.5

Let  $F \in C(\mathbb{H})$  be a bounded data function and let  $\mathcal{D}$  be open, bounded and game-regular. In virtue of Theorem 13.5 and the Ascoli-Arzelà theorem, every sequence in the family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  of solutions to (11.1) has a further subsequence converging uniformly to some  $u \in C(\mathbb{H})$  and satisfying  $u = F$  on  $\mathbb{H} \setminus \mathcal{D}$ . We will show that such limit  $u$  is in fact unique.

Recall first the definition of the  $\mathbf{p}$ - $\mathbb{H}$ -harmonic viscosity solution, that should be compared with the definition in the statement of Proposition 2.5, valid for  $\mathbf{p} = 2$ .

**Definition 15.1** We say that  $u \in C(\bar{\mathcal{D}})$  is a *viscosity solution* to the following problem:

$$\Delta_{\mathbb{H}, \mathbf{p}} u = 0 \quad \text{in } \mathcal{D}, \quad u = F \quad \text{on } \partial \mathcal{D}, \tag{15.1}$$

if the latter boundary condition holds and if:

- (1) for every  $q_0 \in \mathcal{D}$  and every  $\phi \in C^2(\bar{\mathcal{D}})$  such that:

$$\phi(q_0) = u(q_0), \quad \phi < u \quad \text{in } \bar{\mathcal{D}} \setminus \{q_0\} \quad \text{and} \quad \nabla_{\mathbb{H}} \phi(q_0) \neq 0, \tag{15.2}$$

there holds:  $\Delta_{\mathbb{H}, \mathbf{p}} \phi(q_0) \leq 0$ ,

- (2) for every  $q_0 \in \mathcal{D}$  and every  $\phi \in C^2(\bar{\mathcal{D}})$  such that:

$$\phi(q_0) = u(q_0), \quad \phi > u \quad \text{in } \bar{\mathcal{D}} \setminus \{q_0\} \quad \text{and} \quad \nabla_{\mathbb{H}} \phi(q_0) \neq 0,$$

there holds:  $\Delta_{\mathbb{H}, \mathbf{p}} \phi(q_0) \geq 0$ .

**Theorem 15.2** Assume that the sequence  $\{u_\epsilon\}_{\epsilon \in J, \epsilon \rightarrow 0}$  of solutions to (11.1) with a bounded data function  $F \in C(\mathbb{H})$ , converges uniformly as  $\epsilon \rightarrow 0$  to some limit  $u \in C(\mathbb{H})$ . Then  $u$  must be the viscosity solution to (15.1).

**Proof** Fix  $q_0 \in \mathcal{D}$  and let  $\phi$  be a test function as in (15.2). Using the same argument as in the proof of Lemma 9.1, we observe that there exists a sequence  $\{q_\epsilon\}_{\epsilon \in J} \in \mathcal{D}$ , such that:

$$\lim_{\epsilon \rightarrow 0, \epsilon \in J} q_\epsilon = q_0 \quad \text{and} \quad u_\epsilon(q_\epsilon) - \phi(q_\epsilon) = \min_{\bar{\mathcal{D}}} (u_\epsilon - \phi). \tag{15.3}$$

Since by (15.3) we have:  $\phi(q) \leq u_\epsilon(q) + (\phi(q_\epsilon) - u_\epsilon(q_\epsilon))$  for all  $q \in \bar{\mathcal{D}}$ , it follows that:

$$\begin{aligned} & \frac{1}{3} \left( \mathcal{A}_3(\phi, \epsilon)(q_\epsilon) + \inf_{B_{\gamma \mathbf{p}^\epsilon}(q_\epsilon)} \mathcal{A}_3(\phi, \epsilon) + \sup_{B_{\gamma \mathbf{p}^\epsilon}(q_\epsilon)} \mathcal{A}_3(\phi, \epsilon) \right) - \phi(q_\epsilon) \\ & \leq \frac{1}{3} \left( \mathcal{A}_3(u_\epsilon, \epsilon)(q_\epsilon) + \inf_{B_{\gamma \mathbf{p}^\epsilon}(q_\epsilon)} \mathcal{A}_3(u_\epsilon, \epsilon) + \sup_{B_{\gamma \mathbf{p}^\epsilon}(q_\epsilon)} \mathcal{A}_3(u_\epsilon, \epsilon) \right) \\ & \quad + (\phi(q_\epsilon) - u_\epsilon(q_\epsilon)) - \phi(q_\epsilon) = 0, \end{aligned} \tag{15.4}$$



for all  $\epsilon$  small enough to guarantee that  $d_\epsilon(q_\epsilon) = 1$ . On the other hand, (4.2) yields:

$$\begin{aligned} & \frac{1}{3} \left( \mathcal{A}_3(\phi, \epsilon)(q_\epsilon) + \inf_{B_{\gamma\mathbf{p}\epsilon}(q_\epsilon)} \mathcal{A}_3(\phi, \epsilon) + \sup_{B_{\gamma\mathbf{p}\epsilon}(q_\epsilon)} \mathcal{A}_3(\phi, \epsilon) \right) - \phi(q_\epsilon) \\ &= \frac{\epsilon^2}{3\pi} \cdot \frac{\Delta_{\mathbb{H}, \mathbf{p}}\phi(q_\epsilon)}{|\nabla_{\mathbb{H}}\phi(q_\epsilon)|^{\mathbf{p}-2}} + o(\epsilon^2), \end{aligned}$$

for  $\epsilon$  small enough to get  $\nabla_{\mathbb{H}}\phi(q_\epsilon) \neq 0$ . Combining the above with (15.4) gives:

$$\Delta_{\mathbb{H}, \mathbf{p}}\phi(q_\epsilon) \leq o(1).$$

Passing to the limit with  $\epsilon \rightarrow 0$ ,  $\epsilon \in J$  establishes the desired inequality  $\Delta_{\mathbb{H}, \mathbf{p}}\phi(q_0) \leq 0$  and proves part (1) of Definition 15.1. The verification of part (2) is done along the same lines. □

Since the viscosity solutions  $u \in \mathcal{C}(\bar{\mathcal{D}})$  of (15.1) are unique [6], in view of Theorems 15.2 and 13.5 we obtain:

**Corollary 15.3** *Let  $F \in \mathcal{C}(\mathbb{H})$  be a bounded data function and let  $\mathcal{D}$  be open, bounded and game-regular. The family  $\{u_\epsilon\}_{\epsilon \rightarrow 0}$  of solutions to (11.1) converges uniformly in  $\bar{\mathcal{D}}$  to the unique viscosity solution of (15.1).*

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