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The spectral determinant of the isotropic quantum harmonic oscillator in arbitrary dimensions

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Abstract We show that the spectral determinant of the isotropic quantum harmonic oscillator converges exponentially to one as the space dimension grows to infinity. We determine the precise asymptotic behaviour for large dimension and obtain estimates valid for all cases with the same asymptotic behaviour in the large. As a consequence, we provide an alternative proof of a conjecture posed by Bär and Schopka concerning the convergence of the determinant of the Dirac operator on S^n, determining the exact asymptotic behaviour for this case and thus improving the estimate on the rate of convergence given in the work of Møller (Math Ann 343:35–51, 2009).

Keywords Spectral determinant · Quantum harmonic oscillator · Dirac operator · Zeta function

1 Introduction

Within the last 50 years there has been an interest within the mathematical physics community in the calculation of the spectral (or functional) determinant of operators defined by elliptic operators. This quantity depends globally on the spectrum of the operator in question, and is normally quite difficult to compute, with typical situations

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for which this becomes feasible being those where the spectrum is known explicitly such as balls [4] and spheres [3, and the references therein], for instance. Some instances where the spectrum is not known explicitly but it is still possible to carry out this computation are polygons [1] or Sturm–Liouville operators [5,8].

An added difficulty stems from the fact that computing the determinant usually requires a regularisation procedure to make sense of what would otherwise be an infinite product of a sequence of numbers converging to infinity [7,11]. One of the standard ways to proceed is to build the spectral zeta function associated with the operator and then use the formula that would be correct in the finite dimensional case. More precisely, if λ_k is a non-decreasing sequence of positive real numbers such that λ_k approaches infinity as k goes to infinity, then we define the zeta function associated with this sequence by

$$\zeta_\lambda(s) = \sum_k \frac{1}{\lambda_k^s} \tag{1}$$

for s in a suitable subset of the complex plane. The spectral determinant is then defined by $e^{-\zeta_\lambda'(0)}$. When the operator has a zero eigenvalue, this is normally not counted, while if the eigenvalues λ_k are not all positive, there are other ways to define the determinant, as in the case of the Dirac operator on the sphere S^n which we will be referring to below [3].

The purpose of this paper is to study the determinant of the quantum isotropic harmonic oscillator in $\mathbb{R}^n$ with Hamiltonian given by

$$H_n = -\Delta + |x|^2, \quad x \in \mathbb{R}^n. \tag{2}$$

This defines a self-adjoint operator in $L^2(\mathbb{R}^n)$ with purely discrete spectrum whose (strictly positive) eigenvalues and eigenfunctions can be computed explicitly, allowing us to write the corresponding zeta function as in (1) above. From this it will be possible to evaluate the determinant for any given dimension and, in particular, we shall obtain a recursion formula which will make this feasible in a relatively straightforward way. However, as happens in some of these cases, the resulting expressions soon become quite involved as the dimension increases. We will thus follow another direction to obtain further information on the behaviour for large values of the dimension. In particular, we are interested in studying and quantify precisely the intriguing behaviour of spectral determinants of certain families of operators with the dimension, which have been identified numerically in [3], namely, their convergence to one as the dimension grows large. To do this, we shall obtain an integral representation of the corresponding zeta function which will then allow us to derive an explicit expression for the derivative of the zeta function at zero. This is done in terms of two closed loop integrals and a real integral on a half-line, for which we then manage to both obtain estimates valid for all dimensions which display the correct order in their asymptotic behaviour, and the exact asymptotic behaviour. In fact, we see that the two-term approximation for large n obtained in this way is already quite accurate even in low dimension.

As a by-product of our approach, and by relating the zeta function for the harmonic oscillator to that for the Dirac operator on spheres, we are able to derive precise asymptotics for the behaviour of the determinant of the latter as the dimension becomes large. As stated in Corollary E below, while the results in [9] yield a control of the decay for $|\log\det(D_{S^n})|$ of order $\left(\sqrt{5}/3\right)^n$, we show that this decay is of (precise) order $\left(\sqrt{2}/2\right)^n n^{-1/2}$ and determine explicitly the first two terms in the asymptotics. A similar result holds for the phase term. From a computational point of view, the precision of these approximations is quite good, as may be seen by comparing them with the values given in [3]. Except for the case of the two-sphere, where the error is of the order of 1.8% for the absolute value and 0.6% for the phase, in all other cases this error is below 0.18% and decays quite fast.

The structure of the paper is as follows. In the next section we state some of the basic properties about the operator H_n, together with the main results of the paper. We then prove the recurrence formula in Sect. 3, where we also compute some examples of both the zeta function and of the determinant to illustrate these results. In Sect. 4 we derive the integral representation for the zeta function, together with some of its properties. In Sect. 5 we begin by obtaining the explicit expression for the determinant. This is then analysed, first from the point of view of its asymptotic behaviour for large n, for which we explicitly determine the first two terms in the expansion of $-\log\det(H_n)$. We then obtain estimates for this quantity which are valid for all n and which display the same asymptotic behaviour (although with worse constants). Finally, in the last section we establish the connection with the Dirac operator on spheres and obtain the asymptotic behaviour for the determinant in this case.

2 Preliminaries and main results

The spectrum of (2) in $L^2(\mathbb{R}^n)$ is discrete and its eigenvalues and eigenfunctions are given by the equation

$$-\Delta u + |x|^2 u = \lambda u, \quad x \in \mathbb{R}^n. \tag{3}$$

These may be calculated explicitly via separation of variables and using the eigenvalues of the one-dimensional harmonic oscillator which are solutions of the problem

$$-\varphi''(y) + y^2\varphi = \sigma\varphi(y), \quad y \in \mathbb{R}.$$

The corresponding eigenvalues of this problem are given by $\sigma_k = 2k+1$ ($k = 0, 1, 2, \ldots$) and the associated eigenfunctions are of the form

$$\varphi_k(y) = e^{-y^2/2} h_k(y),$$

where the functions h_n are Hermite polynomials—see, for instance, [14].

We then obtain that, for the n-dimensional problem,

$$\lambda_k = 2|\alpha| + n, \text{ where } |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha_j \in \mathbb{N}_0.$$

From the above we see that for each value of $k = |\alpha|$ the corresponding eigenvalue of the form $2k + n$ will have multiplicity

$$m_k = \binom{n+k-1}{k}. \tag{4}$$

We may now define the zeta function for the quantum harmonic oscillator in dimension n by

$$\zeta_n^H(s) := \sum_{k=0}^{\infty} \frac{m_k}{(2k+n)^s}, \tag{5}$$

where the series is absolutely convergent for $\mathrm{Re}(s) > n$.

We first show that the family of harmonic zeta functions may be defined by a simple two-term recursion formula which, in particular, shows that all elements of this family may be written in terms of the Riemann zeta function.

Theorem A (A recursion formula) *The family of harmonic zeta functions defined by* (5) *satisfies the following recursion relation*

$$\zeta_{n+2}^H(s) = \frac{1}{4n(n+1)}\zeta_n^H(s-2) - \frac{n}{4(n+1)}\zeta_n^H(s), \tag{6}$$

for all positive integers n and complex numbers s in the domain of the functions involved. Furthermore,

$$\zeta_1^H(s) = \left(1 - 2^{-s}\right)\zeta(s) \quad \textit{and} \quad \zeta_2^H(s) = 2^{-s}\zeta(s-1).$$

Some of the properties of the functions $\zeta_n^H(s)$ may already be derived from this formula. It implies, for instance, some of the results in Theorem B below. However, in order to study the asymptotic behaviour and the estimates for the determinant, we will rely mostly on the integral representation formula derived in Sect. 4 in the proof of Theorem B.

In order to proceed, we shall thus derive an integral representation for $\zeta_n^H(s)$ and show that it is possible to extend this function to the whole complex plane, except possibly for a finite number of singularities. This will be done in a standard way as for the Riemann zeta function.

Theorem B (Analytic extension) *The harmonic zeta function $\zeta_n^H(s)$ defined by* (5) *may be extended to the whole complex plane as a meromorphic function whose only singularities are simple poles at the positive even integers less than or equal to n when n is even, and at the odd positive integers less than or equal to n when n is odd. In either case, the residue at the simple pole s = n is given by*

$$\underset{s=n}{Res}\left[\zeta_n^H s\right] = \frac{1}{2^n(n-1)!}.$$

In particular, $\zeta_n^H(s)$ is analytic at zero, allowing us to define the determinant for the harmonic oscillator H_n by

$$\det(H_n) = e^{-\left[\zeta_n^H(s)\right]'_{s=0}}.$$

The asymptotic behaviour of this quantity when the spatial dimension becomes large is now given in the following

Theorem C (Determinant: asymptotic behaviour) *The spectral determinant of the harmonic oscillator satisfies*

$$\lim_{n\to+\infty} \det(H_n) = 1.$$

More precisely, we have

$$\begin{aligned} -\log\det(H_n) &= \left[\zeta_n^H(s)\right]'_{s=0} \\ &= 2^{-n}\left[h_1 n^{-1/2} + h_2 n^{-3/2} + \mathrm{O}\left(n^{-5/2}\right)\right] \end{aligned}$$

as $n \to +\infty$, *where*

$$h_1 = \frac{\sqrt{2}}{\pi^{3/2}}\left[2\left(\gamma + \log\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{n\pi}{2}\right) - \pi\sin\left(\frac{n\pi}{2}\right)\right]$$

and

$$\begin{aligned} h_2 = \frac{1}{2\sqrt{2}\pi^{7/2}}\Bigg\{&\left[2\left(\pi^2 - 16\right)\gamma + 2\pi^2\log\left(\frac{\pi}{2}\right)\right. \\ &\left.+16\left(3 - 2\log\left(\frac{\pi}{2}\right)\right)\right]\cos\left(\frac{n\pi}{2}\right) \\ &+\pi\left(16 - \pi^2\right)\sin\left(\frac{n\pi}{2}\right)\Bigg\}. \end{aligned}$$

The next result provides estimates which are valid for all dimension. Here the emphasis was on deriving a simple closed form expression with the correct quantitative behaviour, thus bounding the values of the determinant.

Theorem D (Determinant: estimates) *The derivative of* $\zeta_n^H(s)$ *at zero satisfies*

$$\begin{aligned} \left|\left[\zeta_n^H(s)\right]'_{s=0}\right| \le 2^{-n}\Bigg\{&\frac{2}{\sqrt{\pi}}\left[\left(\gamma + \log\frac{\pi}{2}\right)^2 + \pi^2\right]^{1/2}\frac{\Gamma(n+1/2)}{\Gamma(n+1)} \\ &+\frac{2}{n\pi\coth\left(\frac{\pi}{2}\right)} \times \frac{1}{\sinh^n\left(\frac{\pi}{2}\right)}\Bigg\} \end{aligned}$$

for all integers n.

Remark 1 Note that the leading term in the asymptotic expansion of the above bound for large n is of the form

$$\frac{2^{-n}}{\sqrt{n}} \times \frac{2}{\sqrt{\pi}} \left[\left(\gamma + \log\left(\frac{\pi}{2}\right) \right)^2 + \pi^2 \right]^{1/2},$$

which is of the same order as that of the leading term given by Theorem C.

These results, together with the fact that the determinant of the Dirac operator on S^n is related to $\det(H_n)$, allow us to derive a proof of Conjecture 1 in [3]. A proof of this result had already been obtained by Møller in [9] and was based on a recurrence formula for certain coefficients which dominate the behaviour of the absolute value of the zeta function and the corresponding derivative for the operator D^2 defined on spheres. With our approach we are able to obtain the exact rate of decay of the determinant.

Corollary E (The Dirac operator on S^n) *For the Dirac operator on the standard round sphere S^n $(n \geq 2)$ defined by*

$$\det(D_{S^n}) = \exp\left(i\frac{\pi}{2}\zeta_n^{D^2}(0) \right) \exp\left(-\frac{1}{2} \left[\zeta_n^{D^2}(s) \right]'_{s=0} \right),$$

we have

$$\left[\zeta_n^{D^2}(s) \right]'_{s=0} = 2^{\lfloor n/2 \rfloor - n + 2} \left[d_1 n^{-1/2} + d_2 n^{-3/2} + \mathrm{O}\left(n^{-5/2}\right) \right],$$

as $n \to +\infty$, where

$$d_1 = \frac{\sqrt{2}}{\pi^{3/2}} \left[2\left(\gamma + \log(\pi)\right) \cos\left(\frac{n\pi}{2}\right) - \pi \sin\left(\frac{n\pi}{2}\right) \right]$$

and

$$d_2 = \frac{1}{2\sqrt{2}\pi^{7/2}} \left[2\left(\left(\pi^2 - 16\right) (\gamma + \log(\pi)) + 24 \right) \cos\left(\frac{n\pi}{2}\right) + \pi \left(16 - \pi^2\right) \sin\left(\frac{n\pi}{2}\right) \right].$$

The asymptotic behaviour of the phase as $n \to +\infty$ is given by

$$\begin{aligned} \phi_n^D &= \frac{\pi}{2}\zeta_n^{D^2}(0) \\ &= 2^{\lfloor n/2 \rfloor - n + 1} \cos\left(\frac{n\pi}{2}\right) \left[\sqrt{\frac{2}{\pi}} n^{-1/2} - \frac{16 - \pi^2}{2\sqrt{2}\pi^{5/2}} n^{-3/2} + \mathrm{O}\left(n^{-5/2}\right) \right]. \end{aligned}$$

Using the same approach as in the proof of Theorem D, it is possible to derive similar estimates for $\left[\zeta_n^{D^2}(s)\right]'_{s=0}$.

3 Proof of the recurrence formula

For all s with real part larger than n we have

$$\begin{aligned}
\zeta_{n+2}^H(s) &= \sum_{k=0}^{\infty} \binom{n+k+1}{k} \frac{1}{(2k+n+2)^s} \\
&= \sum_{k=1}^{\infty} \binom{n+k}{k-1} \frac{1}{(2k+n)^s} \\
&= \sum_{k=1}^{\infty} \frac{k(n+k)(n+k-1)!}{k!(n+1)n(n-1)!} \frac{1}{(2k+n)^s} \\
&= \sum_{k=1}^{\infty} \frac{k(n+k)}{n(n+1)} \binom{n+k-1}{k} \frac{1}{(2k+n)^s} \\
&= \frac{1}{n(n+1)} \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{k^2+kn+n^2/4}{(2k+n)^s} \\
&\quad - \frac{1}{n(n+1)} \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{n^2/4}{(2k+n)^s} \\
&= \frac{1}{4n(n+1)} \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{1}{(2k+n)^{s-2}} \\
&\qquad - \frac{n}{4(n+1)} \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{1}{(2k+n)^s} \\
&= \frac{1}{4n(n+1)} \zeta_n^H(s-2) - \frac{n}{4(n+1)} \zeta_n^H(s).
\end{aligned}$$

A direct evaluation of (5) for $n = 1, 2$ yields

$$\zeta_1^H(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = \left(1 - 2^{-s}\right) \zeta(s)$$

and

$$\zeta_2^H(s) = \sum_{k=0}^{\infty} \frac{k+1}{(2k+2)^s} = \frac{1}{2^s} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-1}} = 2^{-s} \zeta(s-1).$$

Since the functions $\zeta_1^H(s)$ and $\zeta_2^H(s)$ are meromorphic, the result then follows.

This recurrence formula may be used to compute $\zeta_n^H(s)$ for any n, although expressions do get quite involved even for small dimension. Some examples are as follows:

$$\begin{aligned}
\zeta_3^H(s) &= \frac{1}{8}\left(1-\frac{1}{2^{s-2}}\right)\zeta(s-2)-\frac{1}{8}\left(1-\frac{1}{2^s}\right)\zeta(s)\\
\zeta_4^H(s) &= \frac{1}{3\times 2^{s+1}}\left[\zeta(s-3)-\zeta(s-1)\right]\\
\zeta_5^H(s) &= \frac{1}{3\times 2^7}\left[\left(1-\frac{1}{2^{s-4}}\right)\zeta(s-4)\right.\\
&\quad \left.-10\left(1-\frac{1}{2^{s-2}}\right)\zeta(s-2)+9\left(1-\frac{1}{2^s}\right)\zeta(s)\right]\\
\zeta_6^H(s) &= \frac{1}{15\times 2^{s+3}}\left[\zeta(s-5)-5\zeta(s-3)+4\zeta(s-1)\right],
\end{aligned}$$

clearly displaying the dual behaviour for odd and even n.

An interesting feature of (6) is that since it does not involve the explicit dependence on the variable s it also provides an analogous recursion formula for the derivative of $\zeta_n^H(s)$, thus allowing for a more direct way to compute the determinant without having to compute any derivatives other than those for $\zeta_1^H(s)$ and $\zeta_2^H(s)$. For illustrative purposes, we present here the first six determinant values obtained in this way.

$$\begin{aligned}
&\det(H_1) = \sqrt{2} && \det(H_2) = 2^{-1/12}e^{-\zeta'(-1)}\\
&\det(H_3) = 2^{-1/16}e^{\frac{3}{8}\zeta'(-2)} && \det(H_4) = 2^{11/720}e^{\frac{1}{6}[\zeta'(-1)-\zeta'(-3)]}\\
&\det(H_5) = 2^{3/256}e^{\frac{5}{128}[\zeta'(-4)-2\zeta'(-2)]} && \det(H_6) = 2^{-191/60480}e^{\frac{1}{120}[-\zeta'(-5)+5\zeta'(-3)-4\zeta'(-1)]}
\end{aligned}$$

4 Analytic extension to the complex plane

The proof that each element in the family of functions $\zeta_n^H(s)$ defined by (5) may be extended meromorphically to the complex plane basically follows the steps used for the same purpose for the Riemann zeta function, and it will provide us with an integral representation which may then be used to obtain the desired estimates and asymptotic behaviour.

We begin with the following

Lemma 1 *The harmonic zeta functions $\zeta_n^H(s)$ have the following integral representation*

$$\zeta_n^H(s) = \frac{e^{-\pi i s}\Gamma(1-s)}{2^{n+1}\pi i}\int_C \frac{z^{s-1}}{\sinh^n(z)}\mathrm{d}z,$$

where the contour C starts from $+\infty$ on the real axis, travels along the positive part of the real axis up to some positive value, encircles the origin once without containing any of the values $\pm ik\pi$, $k\in\mathbb{N}$, and returns to $+\infty$ along the positive part of the real axis.

Proof As in [12,13, p. 18ff], we start from

$$\int_0^{+\infty} x^{s-1}e^{-(2k+n)x}\mathrm{d}x = \frac{\Gamma(s)}{(2k+n)^s},\quad \mathrm{Re}(s)>0.$$

Multiplying both sides by $\binom{n+k-1}{k}$ and summing over k from zero to infinity we obtain

$$\Gamma(s)\zeta_n^H(s) = \int_0^{+\infty} x^{s-1} \sum_{k=0}^{\infty} \binom{n+k-1}{k} e^{-(2k+n)x} \mathrm{d}x,$$

where the interchange between the series and the integral is justified by the absolute convergence of the former for $\mathrm{Re}(s) > n$.

We now claim that

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} e^{-(2k+n)x} = \frac{1}{2^n \sinh^n(x)}. \tag{7}$$

Indeed, differentiating

$$\sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$$

$(n-1)$ times with respect to y, we obtain

$$\frac{d^{n-1}}{dy^{n-1}} \sum_{k=0}^{\infty} y^k = (n-1)!(1-y)^{-n}.$$

The left-hand side in the above expression equals

$$\begin{aligned}\sum_{k=n-1}^{\infty} \left(\prod_{j=k-n+2}^{k} j \right) y^{k-n+1} &= \sum_{k=0}^{\infty} \left(\prod_{j=k+1}^{k+n-1} j \right) y^k \\ &= \sum_{k=0}^{\infty} \frac{(n+k-1)!}{k!} y^k \\ &= (n-1)! \sum_{k=0}^{\infty} \binom{n+k-1}{k} y^k,\end{aligned}$$

and thus

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} y^k = (1-y)^{-n}.$$

Letting $y = e^{-2x}$ and multiplying by e^{-nx} yields (7). We thus have

$$\Gamma(s)\zeta_n^H(s) = \int_0^{\infty} \frac{x^{s-1}}{2^n \sinh^n(x)} \mathrm{d}x, \quad \mathrm{Re}(s) > n.$$

Define now

$$I_n(s) = \int_C \frac{z^{s-1}}{2^n \sinh^n(z)} \mathrm{d}z, \tag{8}$$

where C is the contour described in the lemma. This may be evaluated by considering a path on the positive real axis from infinity to $\rho(0 < \rho < \pi)$, the circle $|z| = \rho$ described counter-clockwise, and again the real axis from ρ to infinity. On the circle of radius ρ and for $\rho \in (0, \pi/2]$ we have

$$\begin{aligned}|\sinh(z)|^2 &= \sinh^2(\rho\cos\theta) + \sin^2(\rho\sin\theta)\\ &\geq \rho^2\cos^2\theta + \frac{4}{\pi^2}\rho^2\sin^2\theta \geq \frac{4}{\pi^2}\rho^2.\end{aligned}$$

Hence

$$\left| \oint_{|z|=\rho} \frac{z^{s-1}}{2^n \sinh^n(z)} \mathrm{d}z \right| \leq \frac{\pi^{n+1}\rho^{\mathrm{Re}(s)-n}}{2^{2n-1}}$$

and we see that the integral over this part of the contour goes to zero as ρ goes to zero, provided $\mathrm{Re}(s) > n$. Hence

$$\begin{aligned}I_n(s) &= -\int_0^{+\infty} \frac{x^{s-1}}{2^n \sinh^n(x)} \mathrm{d}x + \int_0^{+\infty} \frac{\left(xe^{2\pi i}\right)^{s-1}}{2^n \sinh^n(x)} \mathrm{d}x\\ &= \left(e^{2\pi i s} - 1\right) \int_0^{+\infty} \frac{x^{s-1}}{2^n \sinh^n(x)} \mathrm{d}x\\ &= \left(e^{2\pi i s} - 1\right) \Gamma(s)\zeta_n^H(s)\\ &= \frac{2\pi i e^{\pi i s}}{\Gamma(1-s)} \zeta_n^H(s),\end{aligned}$$

from which the integral representation follows. □

The integrals $I_n(s)$ are well-defined and uniformly convergent in any finite region in the complex s-plane and thus define an analytic function of s. As with the Riemann zeta function, the only possible singularities of $\zeta_n^H(s)$ are the poles of $\Gamma(1-s)$, located at the positive integers. From the series definition of the functions $\zeta_n^H(s)$ we know that there are no singularities for $\mathrm{Re}(s) > n$, so it remains to inspect what happens at the integers $1, \ldots, n$.

First note that for positive integer s the two integrals over the positive part of the real axis from infinity to some positive ρ and then from ρ to plus infinity cancel each other out. We are thus left to analyse what happens to

$$I_n(p) = \oint_{|z|=\rho} \frac{z^{p-1}}{2^n \sinh^n z} \mathrm{d}z$$

for some ρ on $(0,\pi)$ and $p=1,\ldots,n$. The integrand may be written as

$$\left[\frac{z}{2\sinh(z)}\right]^n z^{p-n-1}$$

and we see that when $p-n-1$ is odd the residue is non-zero and integral does not vanish. We thus have simple poles at these values of p, which will be positive integers with the same parity as n up and to including n. On the other hand, for all values of p with a different parity from n this integral will vanish cancelling out the singularities of the Γ function.

When $s=n$ the integrand has a simple pole and we obtain

$$I_n(n)=\oint_{|z|=\rho}\frac{z^{n-1}}{2^n\sinh^n z}\mathrm{d}z=2^{1-n}\pi i,$$

yielding

$$\zeta_n^H(s)=\frac{2^{-n}}{\Gamma(n)}\frac{1}{s-n}+\mathrm{O}(1)\quad \text{as } s\to n$$

and concluding the proof of Theorem B.

5 The determinant

We shall use the integral representation formula given by Lemma 1 to obtain the necessary bounds to control the determinant. We shall begin by estimating $\zeta_n^H(0)$. From Lemma 1 we have

$$\zeta_n^H(s)=\frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i}I_n(s), \tag{9}$$

with $I_n(s)$ as in (8). For general complex s the integral over the part of the contour C encircling the origin no longer necessarily goes to zero as in the proof of Lemma 1, but we may still write

$$\begin{aligned}I_n(s)&=-\int_\rho^{+\infty}\frac{x^{s-1}}{2^n\sinh^n(x)}\mathrm{d}x\\&\quad+\int_\rho^{+\infty}\frac{\left(xe^{2\pi i}\right)^{s-1}}{2^n\sinh^n(x)}\mathrm{d}x+\oint_{|z|=\rho}\frac{z^{s-1}}{2^n\sinh^n(z)}\mathrm{d}z\\&=\left(e^{2\pi i s}-1\right)\int_\rho^{+\infty}\frac{x^{s-1}}{2^n\sinh^n(x)}\mathrm{d}x+\oint_{|z|=\rho}\frac{z^{s-1}}{2^n\sinh^n(z)}\mathrm{d}z\end{aligned} \tag{10}$$

and so (9) with I_n as above is valid for all complex numbers except at the simple poles of $\zeta_n^H(s)$. In particular, it is valid for s in a neighbourhood of zero and we may thus differentiate (9) with respect to s to obtain

$$\begin{aligned}\left[\zeta_n^H(s)\right]' &= \frac{-\pi i e^{-\pi i s}\Gamma(1-s)}{2\pi i} I_n(s) \\ &\quad - \frac{e^{-\pi i s}\Gamma'(1-s)}{2\pi i} I_n(s) + \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} I_n'(s) \\ &= -\pi i \zeta_n^H(s) - \frac{\Gamma'(1-s)}{\Gamma(1-s)}\zeta_n^H(s) + \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} I_n'(s) \\ &= -\left[\pi i + \frac{\Gamma'(1-s)}{\Gamma(1-s)}\right]\zeta_n^H(s) + \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i} I_n'(s).\end{aligned}$$

In particular, at s equal to zero we have

$$\left[\zeta_n^H(s)\right]'_{s=0} = (\gamma - \pi i)\,\zeta_n^H(0) + \frac{1}{2\pi i} I_n'(0).$$

In what follows we will take $\rho = \pi/2$. From (9) and (10) we have

$$\zeta_n^H(0) = \frac{1}{2\pi i} I_n(0) = \frac{2^{-n}}{2\pi i} \oint_{|z|=\pi/2} \frac{1}{z \sinh^n(z)} \mathrm{d}z, \tag{11}$$

while differentiating (10) with respect to s yields

$$\begin{aligned}I_n'(s) = \frac{2\pi i e^{2\pi i s}}{2^n} \int_{\pi/2}^{+\infty} \frac{x^{s-1}}{\sinh^n(x)} \mathrm{d}x + \frac{e^{2\pi i s} - 1}{2^n} \int_{\pi/2}^{+\infty} \frac{x^{s-1}\log x}{\sinh^n(x)} \mathrm{d}x \\ + \frac{1}{2^n} \oint_{|z|=\pi/2} \frac{z^{s-1}\log(z)}{\sinh^n(z)} \mathrm{d}z.\end{aligned}$$

Due to the contour used, the branch of the logarithm which appears in this last integral has its imaginary part with values on $[0, 2\pi)$. When s is zero this last expression simplifies, after division by $2\pi i$, to

$$\frac{1}{2\pi i} I_n'(0) = \frac{1}{2^n} \int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x + \frac{1}{2^n} \times \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\log(z)}{z \sinh^n(z)} \mathrm{d}z.$$

Hence

$$\begin{aligned}\left[\zeta_n^H(s)\right]'_{s=0} = 2^{-n} \Bigg[\frac{\gamma - \pi i}{2\pi i} \oint_{|z|=\pi/2} \frac{1}{z \sinh^n(z)} \mathrm{d}z + \int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x \\ + \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\log(z)}{z \sinh^n(z)} \mathrm{d}z \Bigg].\end{aligned} \tag{12}$$

5.1 Asymptotics for large dimension

We shall first determine the asymptotic behaviour of each of the three integrals above which make up $\left[\zeta_n^H(0)\right]'$, as n goes to infinity. It is not difficult to see (and we will make this rigorous below) that the integral on the real line from $\pi/2$ to $+\infty$ still decays exponentially with n, while this will no longer be the case for the integrals over the two complex loops. For these two integrals, one possible appropriate framework to obtain the correct asymptotic behaviour is by analysing the behaviour at saddle points along the path $|z| = \pi/2$. This will be a fairly standard procedure and we will follow closely the approach described in Chapter 4, Section 7 in [10].

We start with the evaluation of the first of these integrals which we write as

$$\oint_{|z|=\pi/2} \frac{1}{z\sinh^n(z)}\mathrm{d}z = \int_{C_+} \frac{1}{z\sinh^n(z)}\mathrm{d}z + \int_{C_-} \frac{1}{z\sinh^n(z)}\mathrm{d}z,$$

where C_+ and C_- denote the half-circles of radius $\pi/2$ on the upper and lower halves of the complex plane, respectively. Since the integrand takes complex conjugate values at complex conjugate points, we have

$$\oint_{|z|=\pi/2} \frac{1}{z\sinh^n(z)}\mathrm{d}z = 2i\,\mathrm{Im}\left[\int_{C_+} \frac{1}{z\sinh^n(z)}\mathrm{d}z\right] \tag{13}$$

and we may thus restrict our analysis to the integral over C_+. Following [10] we now write

$$\int_{C_+} \frac{1}{z\sinh^n(z)}\mathrm{d}z = \int_{C_+} e^{-np(z)}q(z)\mathrm{d}z$$

where $p(z) = \log(\sinh(z))$ and $q(z) = 1/z$. Since $p'(z) = \coth(z)$, the critical points of p (called *saddle points* in [10, pages 127ff]) are all simple and located at points of the form $\pm\pi i/2 + 2k\pi i$ $(k \in \mathbb{Z})$. Of relevance to us here is $\pi i/2$ and it is not difficult to see that the functions p and q satisfy the necessary conditions for Theorem 7.1 in [10, Chapter 4, Section 7, page 127] to hold. In particular,

$$\begin{aligned}
\mathrm{Re}\left[p(z) - p(\tfrac{\pi}{2}i)\right] &= \mathrm{Re}\left[\log\left(\sinh(z)\right) - \log\left(\sinh\left(\frac{\pi}{2}i\right)\right)\right] \\
&= \mathrm{Re}\left[\log\left(\sinh\left(\frac{\pi}{2}e^{ti}\right)\right) - \log(i)\right], \quad (t \in (0,\pi)) \\
&= \mathrm{Re}\left[\log\left(\sinh\left(\frac{\pi}{2}e^{ti}\right)\right) - \frac{\pi}{2}i\right] \\
&= \log\left|\sinh\left(\frac{\pi}{2}e^{ti}\right)\right| \\
&= \frac{1}{2}\log\left[\sinh^2\left(\frac{\pi}{2}\cos t\right) + \sin^2\left(\frac{\pi}{2}\sin t\right)\right]
\end{aligned}$$

$$\geq \frac{1}{2}\log\left[\frac{\pi^2}{4}\cos^2(t)+\sin^2(t)\right]$$
$$> 0,$$

except at the saddle point $\pi i/2$. We thus have

$$\int_{C_+}\frac{1}{z\sinh^n(z)}\mathrm{d}z = 2e^{-n\log\left[\sinh\left(\frac{\pi}{2}i\right)\right]}\sum_{k=0}^{\infty}\Gamma\left(k+\frac{1}{2}\right)\frac{a_{2k}}{n^{k+1/2}}$$
$$= 2e^{-n\frac{\pi}{2}i}\left[\frac{a_0\Gamma(1/2)}{n^{1/2}}+\frac{a_2\Gamma(3/2)}{n^{3/2}}+\mathrm{O}\left(n^{-5/2}\right)\right],$$

where formulas for the first two coefficients a_0 and a_2 are given in [10, equation (7.06) on page 127] and in this case evaluate to

$$a_0 = \frac{\sqrt{2}}{\pi}i \quad \text{and} \quad a_2 = -\frac{16-\pi^2}{\sqrt{2}\pi^3}i.$$

Replacing this above we obtain

$$\int_{C_+}\frac{1}{z\sinh^n(z)}\mathrm{d}z = 2ie^{-n\frac{\pi}{2}i}\left[\sqrt{\frac{2}{\pi}}n^{-1/2}-\frac{16-\pi^2}{2\sqrt{2}\pi^{5/2}}n^{-3/2}+\mathrm{O}\left(n^{-5/2}\right)\right]$$

and, for the original integral, it follows from (13) that

$$\oint_{|z|=\pi/2}\frac{1}{z\sinh^n(z)}\mathrm{d}z = 4i\cos\left(\frac{n\pi}{2}\right)\left[\sqrt{\frac{2}{\pi}}n^{-1/2}-\frac{16-\pi^2}{2\sqrt{2}\pi^{5/2}}n^{-3/2}+\mathrm{O}\left(n^{-5/2}\right)\right]$$

as n becomes large.

For the evaluation of the asymptotic form of the loop integral containing the logarithmic term we will also separate the circle into the two paths $C_\pm$ as above, but we shall now evaluate each of those integrals separately. The function p is the same as before – and hence so are the saddle points –, while now $q(z) = \log(z)/z$, with the branch cut of the logarithm being the half-line $[0, +\infty)$.

After some calculations we obtain

$$\oint_{|z|=\pi/2}\frac{\log(z)}{z\sinh^n(z)}\mathrm{d}z = \oint_{C_+}\frac{\log(z)}{z\sinh^n(z)}\mathrm{d}z + \oint_{C_-}\frac{\log(z)}{z\sinh^n(z)}\mathrm{d}z$$
$$= e^{-n\frac{\pi}{2}i}\left\{\sqrt{2\pi}\left[-1+\frac{2}{\pi}i\log\left(\frac{\pi}{2}\right)\right]n^{-1/2}\right.$$
$$+\frac{1}{\sqrt{2}\pi^{3/2}}\left[8-\frac{\pi^2}{2}+\frac{2i}{\pi}\left(12\right.\right.$$

$$
\begin{aligned}
& \left.\left. - \left(8 - \frac{\pi^2}{2}\right) \log\left(\frac{\pi}{2}\right)\right)\right] n^{-3/2} \Bigg\} \\
& + e^{n\frac{\pi}{2} i} \left\{ \sqrt{2\pi} \left[-3 + \frac{2}{\pi} i \log\left(\frac{\pi}{2}\right) \right] n^{-1/2} \right. \\
& + \frac{1}{\sqrt{2}\pi^{3/2}} \left[3 \left(8 - \frac{\pi^2}{2}\right) + \frac{2i}{\pi} \left(12 \right.\right. \\
& \left.\left. - \left(8 - \frac{\pi^2}{2}\right) \log\left(\frac{\pi}{2}\right)\right)\right] n^{-3/2} \Bigg\} \\
& + \mathrm{O}\left(n^{-5/2}\right) \\
= {} & 2\sqrt{2\pi} \left\{ 2 \left[-1 + \frac{i}{\pi} \log\left(\frac{\pi}{2}\right) \right] \cos\left(\frac{n\pi}{2}\right) \right. \\
& \left. - i \sin\left(\frac{n\pi}{2}\right) \right\} n^{-1/2} \\
& + \frac{\sqrt{2}}{\pi^{3/2}} \left\{ 2 \left[8 - \frac{\pi^2}{2} + \frac{i}{\pi} \left(12 \right.\right.\right. \\
& \left.\left. - \left(8 - \frac{\pi^2}{2}\right) \log\left(\frac{\pi}{2}\right)\right)\right] \cos\left(\frac{n\pi}{2}\right) \\
& \left. + i \left(8 - \frac{\pi^2}{2}\right) \sin\left(\frac{n\pi}{2}\right) \right\} n^{-3/2} \\
& + \mathrm{O}\left(n^{-5/2}\right)
\end{aligned}
$$

Regarding the remaining integral on the real line, we see from Lemma 3 below that it still decreases exponentially and has thus a much smaller contribution to the overall value of $\left[\zeta_n^H(s)\right]'_{s=0}$.

We now collect the terms obtained for the two loop integrals and, plugging them back into (12), obtain, after some lengthy calculations,

$$
\begin{aligned}
\left[\zeta_n^H(s)\right]'_{s=0} = 2^{-n} & \left\{ \frac{\sqrt{2}}{\pi^{3/2}} \left[2 \left(\gamma + \log\left(\frac{\pi}{2}\right)\right) \cos\left(\frac{n\pi}{2}\right) - \pi \sin\left(\frac{n\pi}{2}\right) \right] n^{-1/2} \right. \\
& + \frac{1}{2\sqrt{2}\pi^{7/2}} \left[2 \left(\left(\pi^2 - 16\right) \gamma + \pi^2 \log\left(\frac{\pi}{2}\right) \right.\right. \\
& \left.\left. + 8 \left(3 - 2 \log\left(\frac{\pi}{2}\right)\right)\right) \cos\left(\frac{n\pi}{2}\right) \\
& + \pi \left(16 - \pi^2\right) \sin\left(\frac{n\pi}{2}\right) \right] n^{-3/2} \\
& \left. + \mathrm{O}\left(n^{-5/2}\right) \right\}
\end{aligned} \tag{14}
$$

for large n, providing the asymptotic part in Theorem C. Further terms in the asymptotic expansion may be determined in a similar way.

5.2 Estimates for all n

To estimate the absolute value of the closed loop integrals in (12) it is convenient to consider the two integrals together. We thus want to estimate

$$\left| \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\gamma - \pi i + \log(z)}{z \sinh^n(z)} \mathrm{d}z \right|.$$

Letting $z = \frac{\pi}{2} e^{\theta i}$ with θ in $[0, 2\pi)$ we have

$$\begin{aligned}
\left| \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\gamma - \pi i + \log(z)}{z \sinh^n(z)} \mathrm{d}z \right| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\gamma + \log\left(\frac{\pi}{2}\right) + (\theta - \pi) i}{\sinh^n\left(\frac{\pi}{2} e^{\theta i}\right)} \mathrm{d}\theta \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\gamma + \log\left(\frac{\pi}{2}\right) + (\theta - \pi) i}{\sinh^n\left(\frac{\pi}{2} e^{\theta i}\right)} \right| \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sqrt{\left(\gamma + \log\left(\frac{\pi}{2}\right)\right)^2 + (\theta - \pi)^2}}{\sinh^n\left(\frac{\pi}{2} e^{\theta i}\right)} \right| \mathrm{d}\theta \\
&\leq \frac{\sqrt{\left(\gamma + \log\left(\frac{\pi}{2}\right)\right)^2 + \pi^2}}{2\pi} \int_0^{2\pi} \frac{1}{\left|\sinh^n\left(\frac{\pi}{2} e^{\theta i}\right)\right|} \mathrm{d}\theta.
\end{aligned}$$

We will now proceed to estimate this last integral, for which we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\left|\sinh^n\left(\frac{\pi}{2} e^{\theta i}\right)\right|} \mathrm{d}\theta &= \int_0^{2\pi} \frac{1}{\left[\sinh^2\left(\frac{\pi}{2}\cos\theta\right) + \sin^2\left(\frac{\pi}{2}\sin\theta\right)\right]^{n/2}} \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{\left[\sinh^2\left(\frac{\pi}{2}\sin\theta\right) + \sin^2\left(\frac{\pi}{2}\cos\theta\right)\right]^{n/2}} \mathrm{d}\theta \qquad (15) \\
&= 4 \int_0^{\pi/2} \frac{1}{\left[\sinh^2\left(\frac{\pi}{2}\sin\theta\right) + \sin^2\left(\frac{\pi}{2}\cos\theta\right)\right]^{n/2}} \mathrm{d}\theta.
\end{aligned}$$

In order to continue, we will need the following

Lemma 2 *For* θ *in* $[0, \pi/2]$ *we have*

$$\frac{1}{\sinh^2\left(\frac{\pi}{2}\sin\theta\right)+\sin^2\left(\frac{\pi}{2}\cos\theta\right)} \leq \cos^4\left(\frac{\theta}{2}\right).$$

Proof The denominator on the left-hand side of the above inequality satisfies

$$\begin{aligned}\sinh^2\left(\tfrac{\pi}{2}\sin\theta\right)+\sin^2\left(\tfrac{\pi}{2}\cos\theta\right) &\geq \left(\tfrac{\pi}{2}\sin\theta\right)^2+\tfrac{1}{3}\left(\tfrac{\pi}{2}\sin\theta\right)^4+\cos^2\theta\\ &= \tfrac{\pi^2}{4}\left(1-\cos^2\theta\right)+\tfrac{\pi^4}{48}\left(1-\cos^2\theta\right)^2+\cos^2\theta,\end{aligned}$$

where we used the inequalities

$$\sinh^2 t \geq t^2+\frac{1}{3}t^4 \text{ and } \sin t \geq \frac{2}{\pi}t \quad \left(0 \leq t \leq \frac{\pi}{2}\right).$$

Noting that

$$\cos^4\left(\frac{\theta}{2}\right)=\left(\frac{1+\cos\theta}{2}\right)^2,$$

and writing $y = \cos\theta$, we want to prove that

$$4 \leq (1+y)^2\left[\frac{\pi^2}{4}(1-y^2)+\frac{\pi^4}{48}(1-y^2)^2+y^2\right]$$

for all y on $[0, 1)$. We have thus reduced the problem to that of finding the zeros of the sixth order polynomial

$$g(y) = (1+y)^2\left[\frac{\pi^2}{4}(1-y^2)+\frac{\pi^4}{48}(1-y^2)^2+y^2\right]-4$$

on a bounded interval. The roots of the polynomial g are found to be all outside this interval, except for one at $y = 1$. We thus have that $g(y)$ is positive on $[0, 1)$, yielding the desired result. □

Using this inequality in (15) yields

$$\begin{aligned}\int_0^{2\pi}\frac{1}{\left|\sinh^n\left(\frac{\pi}{2}e^{\theta i}\right)\right|}\mathrm{d}\theta &\leq 4\int_0^{\pi/2}\cos^{2n}\left(\frac{\theta}{2}\right)\mathrm{d}\theta\\ &\leq 4\int_0^{\pi}\cos^{2n}\left(\frac{\theta}{2}\right)\mathrm{d}\theta\\ &= 4\sqrt{\pi}\frac{\Gamma(n+1/2)}{\Gamma(n+1)},\end{aligned}$$

where the integral with the cosine may be computed by transforming it to the beta function $B(n+1/2, 1/2)$ by means of the change of variable $\theta = 2\arccos(r)$. Note that it would also be possible to write the integral on $(0, \pi/2)$ in terms of the incomplete beta function $B_{1/2}(n + 1/2, 1/2)$ but, as we stated, our aim here was to get a simple explicit expression.

We have thus shown that

$$\left| \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\gamma - \pi i + \log(z)}{z \sinh^n(z)} \mathrm{d}z \right| \leq \frac{2\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)} \sqrt{\left(\gamma + \log\left(\frac{\pi}{2}\right)\right)^2 + \pi^2} \quad (16)$$

For the case of the real integral on $[\pi/2, +\infty)$ it is not difficult to obtain the leading term in the expansion for large n by using Laplace's method [10, Chapter 3, Section 7], for instance, and see that it behaves as

$$\int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x = \frac{2}{\pi \coth\left(\frac{\pi}{2}\right)} \frac{1}{n \sinh^n\left(\frac{\pi}{2}\right)} + \cdots \quad \text{as } n \to +\infty.$$

In fact, in this case it is possible to show that the integral is always smaller than or equal to this first term in its asymptotic expansion for large n and we have the sharp bound given by the following lemma.

Lemma 3 *For all positive integers n we have*

$$\int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x \leq \frac{2}{\pi \coth\left(\frac{\pi}{2}\right)} \frac{1}{n \sinh^n\left(\frac{\pi}{2}\right)}.$$

Proof Writing $\sinh(x) = t$ we have

$$x = \log\left(t + \sqrt{1+t^2}\right) \text{ and } \frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}}.$$

Replacing this in the integral we obtain

$$\int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x = \int_{\sinh(\pi/2)}^{+\infty} \frac{1}{\log\left(t + \sqrt{1+t^2}\right)} \times \frac{1}{\sqrt{t^2+1}} \times \frac{1}{t^n} \mathrm{d}t$$

We claim that

$$\frac{1}{\log\left(t + \sqrt{1+t^2}\right)} \times \frac{1}{\sqrt{t^2+1}} \leq \frac{2}{\pi \coth\left(\frac{\pi}{2}\right) t}. \quad (17)$$

Indeed,

$$\frac{d}{dt}\left[\frac{t}{\log\left(t+\sqrt{1+t^2}\right)}\times\frac{1}{\sqrt{1+t^2}}\right]=\frac{-t\sqrt{1+t^2}+\log\left(t+\sqrt{1+t^2}\right)}{\log^2\left(t+\sqrt{1+t^2}\right)\left(1+t^2\right)^{3/2}}$$

where the numerator in the fraction on the right-hand side takes the value zero at zero and its derivative with respect to t is again negative. This shows that this expression is decreasing in t and so, on the interval $[\sinh(\pi/2), +\infty)$, it is smaller than or equal to the value it takes at $\sinh(\pi/2)$, showing (17). We thus have

$$\begin{aligned}\int_{\pi/2}^{+\infty}\frac{1}{x\sinh^n(x)}\mathrm{d}x &\leq \frac{2}{\pi\coth\left(\frac{\pi}{2}\right)}\int_{\sinh(\pi/2)}^{+\infty}\frac{1}{t^{n+1}}\mathrm{d}t\\ &= \frac{2}{\pi\coth\left(\frac{\pi}{2}\right)}\frac{1}{n\sinh^n\left(\frac{\pi}{2}\right)}\end{aligned} \tag{18}$$

as desired. □

Combining the two estimates (16) and (18) finally yields the estimate in Theorem D.

6 The Dirac operator on spheres

Following [3] – see also the references therein – the determinant of the Dirac operator on S^n is given by

$$\det(D_{S^n})=\exp\left(i\frac{\pi}{2}\zeta_n^{D^2}(0)\right)\exp\left(-\frac{1}{2}\left[\zeta_n^{D^2}(s)\right]'_{s=0}\right),$$

where $\zeta_n^{D^2}(s)$ denotes the zeta function for the operator D^2 on S^n, defined in the usual way referred to above. The above expression involves a *phase term*, namely

$$\phi_n^D=\frac{\pi}{2}\zeta_n^{D^2}(0),$$

which will essentially behave as $\zeta_n^{D^2}(0)$ – this is due to the symmetry of the spectrum of the Dirac operator on S^n, for otherwise an extra term will be present [9]. The spectrum of the Dirac operator on S^n is given by

$$\lambda_k^{\pm}=\pm(k+n/2),$$

where each eigenvalue $\lambda_k^{\pm}$ has multiplicity $2^{\lfloor n/2\rfloor}m_k$, with m_k given by (4) [2]. We then have that the spectrum of D^2 is given by the squares of the above values (with twice the multiplicity), yielding the following zeta function

$$\zeta_n^{D^2}(s) = 2^{\lfloor n/2 \rfloor + 1} \sum_{k=0}^{\infty} \frac{m_k}{\left(\frac{n}{2} + k\right)^{2s}}.$$

This is easily seen to equal $2^{\lfloor n/2 \rfloor + 2s + 1} \zeta_n^H(2s)$, from which it follows that

$$\zeta_n^{D^2}(0) = 2^{\lfloor n/2 \rfloor + 1} \zeta_n^H(0)$$

and

$$\left[\zeta_n^{D^2}(s)\right]'_{s=0} = 2^{\lfloor n/2 \rfloor + 2} \left[\log(2)\, \zeta_n^H(0) + \left[\zeta_n^H(s)\right]'_{s=0}\right].$$

Because of this connection between the two operators, the asymptotic behaviour and the estimates obtained for $\zeta_n^H(0)$ and $\left[\zeta_n^H(s)\right]'_{s=0}$ above may be used to derive the corresponding asymptotic behaviour and estimates for the Dirac operator, as the asymptotic expression is affected only by the term $2^{\lfloor n/2 \rfloor}$.

Using (11), (12) and (14) we then have

$$\begin{aligned}
\left[\zeta_n^{D^2}(s)\right]'_{s=0} &= 2^{\lfloor n/2 \rfloor - n + 2} \left\{ \frac{\log(2) + \gamma - \pi i}{2\pi i} \oint_{|z|=\pi/2} \frac{1}{z \sinh^n(z)} \mathrm{d}z \right. \\
&\quad \left. + \int_{\pi/2}^{+\infty} \frac{1}{x \sinh^n(x)} \mathrm{d}x + \frac{1}{2\pi i} \oint_{|z|=\pi/2} \frac{\log(z)}{z \sinh^n(z)} \mathrm{d}z \right\} \\
&= 2^{\lfloor n/2 \rfloor - n + 2} \left\{ \frac{\sqrt{2}}{\pi^{3/2}} \left[2\left(\gamma + \log(\pi)\right) \cos\left(\frac{n\pi}{2}\right) - \pi \sin\left(\frac{n\pi}{2}\right)\right] n^{-1/2} \right. \\
&\quad + \frac{1}{2\sqrt{2}\pi^{7/2}} \left[2\left(\left(\pi^2 - 16\right)\left(\gamma + \log(\pi)\right) + 24\right) \cos\left(\frac{n\pi}{2}\right) \right. \\
&\quad \left. \left. + \pi\left(16 - \pi^2\right) \sin\left(\frac{n\pi}{2}\right)\right] n^{-3/2} + \mathrm{O}\left(n^{-5/2}\right) \right\},
\end{aligned}$$

yielding the expression in Corollary E.

Likewise, for the phase we have

$$\begin{aligned}
\phi_n^D &= \frac{\pi}{2} \zeta_n^{D^2}(0) = \frac{\pi}{2} 2^{\lfloor n/2 \rfloor + 1} \zeta_n^H(0) = \frac{1}{2i} 2^{\lfloor n/2 \rfloor - n} \oint_{|z|=\pi/2} \frac{1}{z \sinh^n(z)} \mathrm{d}z \\
&= 2^{\lfloor n/2 \rfloor - n + 1} \cos\left(\frac{n\pi}{2}\right) \left[\sqrt{\frac{2}{\pi}} n^{-1/2} - \frac{16 - \pi^2}{2\sqrt{2}\pi^{5/2}} n^{-3/2} + \mathrm{O}\left(n^{-5/2}\right)\right].
\end{aligned}$$

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