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Limited range multilinear extrapolation with applications to the bilinear Hilbert transform

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Abstract We prove a limited range, off-diagonal extrapolation theorem that generalizes a number of results in the theory of Rubio de Francia extrapolation, and use this to prove a limited range, multilinear extrapolation theorem. We give two applications of this result to the bilinear Hilbert transform. First, we give sufficient conditions on a pair of weights w_1, w_2 for the bilinear Hilbert transform to satisfy weighted norm

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inequalities of the form

$$BH : L^{p_1}\left(w_1^{p_1}\right) \times L^{p_2}\left(w_2^{p_2}\right) \longrightarrow L^p(w^p),$$

where $w = w_1 w_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$. This improves the recent results of Culiuc *et al.* by increasing the families of weights for which this inequality holds and by pushing the lower bound on p from 1 down to $\frac{2}{3}$, the critical index from the unweighted theory of the bilinear Hilbert transform. Second, as an easy consequence of our method we obtain that the bilinear Hilbert transform satisfies some vector-valued inequalities with Muckenhoupt weights. This reproves and generalizes some of the vector-valued estimates obtained by Benea and Muscalu in the unweighted case. We also generalize recent results of Carando, *et al.* on Marcinkiewicz-Zygmund estimates for multilinear Calderón-Zygmund operators.

Mathematics Subject Classification 42B25 · 42B30 · 42B35

1 Introduction

The Rubio de Francia theory of extrapolation is a powerful tool in harmonic analysis. In its most basic form, it shows that if, for a fixed value p_0, $1 < p_0 < \infty$, an operator T satisfies a weighted norm inequality of the form

$$\|Tf\|_{L^{p_0}(w)} \leq C\|f\|_{L^{p_0}(w)} \tag{1.1}$$

for every weight w in the Muckenhoupt class A_{p_0}, then for every p, $1 < p < \infty$,

$$\|Tf\|_{L^{p}(w)} \leq C\|f\|_{L^{p}(w)} \tag{1.2}$$

whenever $w \in A_p$. Since its discovery in the early 1980s, extrapolation has been generalized in a variety of ways, yielding weak-type inequalities, vector-valued inequalities, and inequalities in other scales of Banach function spaces. We refer the reader to [10] for the development of extrapolation; for more recent results we refer the reader to [8,13,18].

Extrapolation has been also extended to the multilinear setting. In [20] it was shown that if a given operator T satisfies

$$\|T(f_1,\dots,f_m)\|_{L^p((w_1\cdots w_m)^p)} \leq C\prod_{j=1}^m \|f_j\|_{L^{p_j}\left(w_j^{p_j}\right)}$$

for fixed exponents $1 < p_1,\dots,p_m < \infty$, $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, and all weights $w_j^p \in A_{p_j}$, then the same estimate holds for all possible values of p_j. An extension to the scale of variable Lebesgue spaces was given in [11].

In this paper we develop a theory of limited range, multilinear extrapolation. In the linear case, limited range extrapolation was developed in [2] by Auscher and the second author. They proved that if inequality (1.1) holds for a given $0 < p_- < p_0 < p_+ < \infty$ and for all $w \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$, then for all $p_- < p < p_+$ and $w \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$, (1.2) holds. Conditions like this arise naturally in the study of the Riesz transforms and other operators associated to elliptic differential operators.

Our first theorem extends limited range extrapolation to the multilinear setting. To state our results we use the abstract formalism of extrapolation families. Given $m \geq 1$, hereafter $\mathcal{F}$ will denote a family of $(m+1)$-tuples $(f, f_1, \ldots, f_m)$ of non-negative measurable functions. This approach to extrapolation has the advantage that, for instance, vector-valued inequalities are an immediate consequence of our extrapolation results. We will discuss applying this formalism to prove norm inequalities for specific operators below. For complete discussion of this approach to extrapolation in the linear setting, see [10].

Theorem 1.3 *Given $m \geq 1$, let $\mathcal{F}$ be a family of extrapolation $(m+1)$-tuples. For each j, $1 \leq j \leq m$, suppose we have parameters r_j^- and r_j^+, and an exponent $p_j \in (0, \infty)$, $0 \leq r_j^- \leq p_j \leq r_j^+ \leq \infty$, such that given any collection of weights $w_1, \ldots, w_m$ with $w_j^{p_j} \in A_{\frac{p_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{p_j}\right)'}$ and $w = w_1 \cdots w_m$, we have the inequality*

$$\|f\|_{L^p(w^p)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}\left(w_j^{p_j}\right)} \tag{1.4}$$

for all $(f, f_1, \ldots, f_m) \in \mathcal{F}$ such that $\|f\|_{L^p(w^p)} < \infty$, where $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ and C depends on n, p_j, $[w_j]_{A_{\frac{p_j}{r_j^-}}}$, $[w_j]_{RH_{\left(\frac{r_j^+}{p_j}\right)'}}$. Then for all exponents q_j, $r_j^- < q_j < r_j^+$, all weights $w_j^{q_j} \in A_{\frac{q_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{q_j}\right)'}$ and $w = w_1 \cdots w_m$,

$$\|f\|_{L^q(w^q)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{q_j}\left(w_j^{q_j}\right)}, \tag{1.5}$$

for all $(f, f_1, \ldots, f_m) \in \mathcal{F}$ such that $\|f\|_{L^q(w^q)} < \infty$, where $\frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}$ and C depends on n, p_j, q_j, $[w_j]_{A_{\frac{q_j}{r_j^-}}}$, $[w_j]_{RH_{\left(\frac{r_j^+}{q_j}\right)'}}$. Moreover, for the same family of exponents and weights, and for all exponents s_j, $r_j^- < s_j < r_j^+$,

$$\left\| \left(\sum_k (f^k)^s \right)^{\frac{1}{s}} \right\|_{L^q(w^q)} \leq C \prod_{j=1}^{m} \left\| \left(\sum_k (f_j^k)^{s_j} \right)^{\frac{1}{s_j}} \right\|_{L^{q_j}\left(w_j^{q_j}\right)}, \tag{1.6}$$

for all $\{(f^k, f_1^k, \dots, f_m^k)\}_k \subset \mathcal{F}$ *such that the left-hand side is finite and where* $\frac{1}{s} = \sum_{j=1}^m \frac{1}{s_j}$ *and* C *depends on* n, p_j, q_j, s_j, $[w_j]_{A_{\frac{q_j}{r_j^-}}}$, $[w_j]_{RH_{\left(\frac{r_j^+}{q_j}\right)'}}$.

Remark 1.7 When $r_j^- = 1$ and $r_j^+ = \infty$ in Theorem 1.3 we get a version of the multilinear extrapolation theorem from [20] for extrapolation families. The original result was given in terms of operators.

Theorem 1.3 is a consequence of a linear, restricted range, off-diagonal extrapolation theorem, which we believe is of interest in its own right. It generalizes the classical Rubio de Francia extrapolation, the off-diagonal extrapolation theory of Harboure, Macías and Segovia [21], and the limited range extrapolation theorem proved by Auscher and the second author [2].

Theorem 1.8 *Given* $0 \le p_- < p_+ \le \infty$ *and a family of extrapolation pairs* $\mathcal{F}$, *suppose that for some* $p_0, q_0 \in (0,\infty)$ *such that* $p_- \le p_0 \le p_+$, $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \ge 0$, *and all weights* w *such that* $w^{p_0} \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$,

$$\left(\int_{\mathbb{R}^n} f^{q_0} w^{q_0}\,dx\right)^{\frac{1}{q_0}} \le C \left(\int_{\mathbb{R}^n} g^{p_0} w^{p_0}\,dx\right)^{\frac{1}{p_0}} \tag{1.9}$$

for all $(f,g) \in \mathcal{F}$ *such that* $\|f\|_{L^{q_0}(w^{q_0})} < \infty$, *and the constant* C *depends on* n, p_0, q_0, $[w^{p_0}]_{A_{\frac{p_0}{p_-}}}$, $[w^{p_0}]_{RH_{\left(\frac{p_+}{p_0}\right)'}}$. *Then for every* p, q *such that* $p_- < p < p_+$, $0 < q < \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, *and every weight* w *such that* $w^p \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$,

$$\left(\int_{\mathbb{R}^n} f^{q} w^{q}\,dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} g^{p} w^{p}\,dx\right)^{\frac{1}{p}} \tag{1.10}$$

for all $(f,g) \in \mathcal{F}$ *such that* $\|f\|_{L^{q}(w^{q})} < \infty$, *and* C *depends on* n, p, q, $[w^p]_{A_{\frac{p}{p_-}}}$, $[w^p]_{RH_{\left(\frac{p_+}{p}\right)'}}$.

In Theorems 1.3 and 1.8 we make the *a priori* assumption that the left-hand sides of both our hypothesis and conclusion are finite, and this plays a role in the proof. In certain applications this assumption is reasonable: for instance, when proving Coifman-Fefferman type inequalities (cf. [10]). However, when using extrapolation to prove norm inequalities for operators we would like to remove this assumption, as the point is to conclude that the left-hand side is finite. But in fact, we can do this by an easy approximation argument. This immediately yields the following corollaries.

Corollary 1.11 *Under the same hypotheses as Theorem* 1.3, *if we assume that* (1.4) *holds for all* $(f, f_1, \dots, f_m) \in \mathcal{F}$ *(whether or not the left-hand side is finite) then the conclusion* (1.5) *holds for all* $(f, f_1, \dots, f_m) \in \mathcal{F}$ *(whether or not the left-hand side is finite). Analogously, the vector-valued inequality* (1.6) *holds for all families* $\{(f^k, f_1^k, \dots, f_m^k)\}_k \subset \mathcal{F}$ *(whether or not the left-hand side is finite).*

Corollary 1.12 *Under the same hypotheses as Theorem 1.8, if we assume that (1.9) holds for all $(f, g) \in \mathcal{F}$ (whether or not the left-hand side is finite) then the conclusion (1.10) holds for all $(f, g) \in \mathcal{F}$ (whether or not the left-hand side is finite).*

In the statement of Theorem 1.8 there are some restrictions on the allowable exponents p and q. We make these explicit here; these restrictions will play a role in the proof below.

Remark 1.13 Define $q_\pm$ by

$$\frac{1}{q_\pm} - \frac{1}{p_\pm} = \frac{1}{q_0} - \frac{1}{p_0}. \tag{1.14}$$

Because of our assumptions that $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \geq 0$ and $0 \leq p_- \leq p_0 \leq p_+ \leq \infty$ it follows that $0 \leq q_- \leq q_0 \leq q_+ \leq \infty$. Moreover, the fact that $p_- < p < p_+$ yields that $q_- < q < q_+$. Note that if we were to allow that $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} < 0$, we could choose p very close to p_+ and the associated q would be negative, which would not make sense.

Moreover, we have that the following hold:

(i) If $q_0 = p_0$, then $q_\pm = p_\pm$ and $q = p$.
(ii) If $p_0 > q_0$, then $0 \leq q_- < p_-$, $q_+ < p_+ \leq \infty$ and $q < p$.
(iii) If $p_0 < q_0$, then $0 \leq p_- < q_-$, $p_+ < q_+ \leq \infty$ and $p < q$.

Remark 1.15 When $p_0 \geq q_0$ we automatically have that $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \geq 0$. Further, this implies that all of the weights which appear in both our hypothesis and conclusion (i.e, $w^{p_0}, w^{q_0}, w^p, w^q$) are in A_∞. Consequently, they are locally integrable, and so all the Lebesgue spaces that appear in the statement contain the characteristic functions of compact sets. In fact, since $w^{p_0} \in A_\infty$, $w^{q_0} \in A_\infty$ (see Lemma 2.1 below). The same is true for w^p and w^q, since by Remark 1.13, $p \geq q$.

When $p_0 < q_0$, the condition $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \geq 0$ imposes an upper bound for q_0: $q_0 \leq p_0(p_+/p_0)'$. A similar bound holds for q. Thus (by Lemma 2.1) $w^{q_0},\ w^q \in A_\infty$ and so again all the weights involved are in A_∞ and thus locally integrable.

Theorem 1.8 and Corollary 1.12 generalize several known extrapolation results.

(i) The classical Rubio de Francia extrapolation theorem (see e.g. [10, Theorems 1.4 and 3.9] for the precise formulation) corresponds to the case $p_- = 1$, $p_+ = \infty$, $q_0 = p_0$.
(ii) The A_∞ extrapolation theorem in [9] (see also [10, Corollary 3.15]) corresponds to the case $p_- = 0$, $p_+ = \infty$, and $q_0 = p_0$.
(iii) The extrapolation theorem for weights in the reverse Hölder classes [29, Lemma 3.3, (b)] corresponds to the case $p_- = 0$, $p_+ = 1$, and $q_0 = p_0$.
(iv) The limited range extrapolation theorem in [2, Theorem 4.9] (see also [10, Theorems 3.31]), corresponds to the case $0 < p_- < p_+ \leq \infty$, $q_0 = p_0$.
(v) The off-diagonal extrapolation theorem in [21] (see also [10, Theorem 3.23]) corresponds to the case $p_- = 1$, $p_0 < q_0$, $p_+ = \big(\frac{1}{p_0} - \frac{1}{q_0}\big)^{-1}$. To see this, we

recall the well-known fact that $w \in A_{p_0,q_0}$, that is,

$$\sup_Q \left(\fint_Q w^{q_0}\, dx \right)^{\frac{1}{q_0}} \left(\fint_Q w^{-p_0'}\, dx \right)^{\frac{1}{p_0'}} < \infty,$$

if and only if $w^{p_0} \in A_{p_0} \cap RH_{\frac{q_0}{p_0}} = A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$. Note that in this case $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} = 0$.

Our generalization of off-diagonal extrapolation involves weighted norm inequalities that have already appeared in the literature in the context of fractional powers of second divergence form elliptic operators with complex bounded measurable coefficients. More precisely, in [3] it was shown that for a certain operator T_α, there exist $1 \le r_- < 2 < r_+ \le \infty$ such that $T_\alpha : L^r(w^r) \to L^s(w^s)$ for every $r_- < r < s < r_+$ and for every $w \in A_{1+\frac{1}{r_-}-\frac{1}{r}} \cap RH_{s\left(\frac{r_+}{s}\right)'}$. By applying Theorem 1.8 we could prove the same result via extrapolation if we could show that there exist $r_- < r_0 < s_0 < r_+$ such that $T_\alpha : L^{r_0}(w^{r_0}) \to L^{s_0}(w^{s_0})$ for every $w \in A_{1+\frac{1}{r_-}-\frac{1}{r_0}} \cap RH_{s_0(\frac{r_+}{s_0})'}$. Note that the latter condition can be written as $w^{r_0} \in A_{\frac{r_0}{p_-}} \cap RH_{(\frac{p_+}{r_0})'}$ with $p_- = r_-$ and $\frac{1}{p_+} = \frac{1}{r_0} - \frac{1}{s_0} + \frac{1}{r_+}$, and in this case $\frac{1}{s_0} - \frac{1}{r_0} + \frac{1}{p_+} = \frac{1}{r_+} \ge 0$, so the hypotheses of Theorem 1.8 hold.

A restricted range, off-diagonal extrapolation theorem has previously appeared in the literature. Duoandikoetxea [18, Theorem 5.1] proved that if for some $1 \le p_0 < \infty$ and $0 < q_0,\ r_0 < \infty$, and all weights $w \in A_{p_0,r_0}$ (note that unlike in the classical definition of this class he does not require $p_0 \le q_0$), if (1.9) holds, then for all $1 < p < \infty$ and $0 < q,\ r < \infty$ such that $\frac{1}{p_0} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{q} = \frac{1}{r_0} - \frac{1}{r}$, and all weights $w \in A_{p,r}$, (1.10) holds.

This result is contained in Theorem 1.8 in the particular case when $r_0 \ge \max\{p_0, q_0\}$ if we take $p_- = 1$ and $p_+ = \left(\frac{1}{p_0} - \frac{1}{r_0}\right)^{-1}$. In this case, (because $r_0 \ge p_0$) $w \in A_{p_0,r_0}$ if and only if $w^{p_0} \in A_{p_0} \cap RH_{\frac{r_0}{p_0}} = A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$. Moreover, in this scenario $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \ge 0$ since $r_0 \ge q_0$.

Despite this overlap, our results are different. We eliminate the restriction $p_0,\ p > 1$ as we can take $0 \le p_- < 1$. On the other hand, we cannot recapture his result for values of $r_0 < \max\{p_0, q_0\}$.

Finally, in light of Remark 1.15, we note that [18, Theorem 5.1] allows for weights w^{q_0} or w^{p_0} that may not be locally integrable unless one assumes $r_0 \ge \max\{p_0, q_0\}$. For example, if we fix $0 < r_0 < \max\{p_0, q_0\}$ and let $w(x) = |x|^{-\frac{n}{\max\{p_0,q_0\}}}$, then it is easy to see that $w^{r_0} \in A_1$ and so $w \in A_{p_0,r_0}$, but either w^{p_0} or w^{q_0} is not locally integrable (and so the characteristic function of the unit ball centered at 0 does not belong to $L^{p_0}(w^{p_0})$ or to $L^{q_0}(w^{q_0})$). In light of this, we believe the condition $r_0 \ge \min\{p_0, q_0\}$ is not unduly restrictive.

Remark added in Proof: After this paper was submitted, we discovered that Theorem 1.8 is indeed equivalent to [18, Theorem 5.1] in the case $r_0 \ge \{p_0, q_0\}$. This can be shown using some complicated rescaling argument along the lines of the one in the forthcoming paper [24]. We would like to emphasize that the formulations and proofs are however different, as in ours the goal is to obtain ranges where the estimates hold.

1.1 Applications

To demonstrate the power of our multilinear extrapolation theorem, we use Theorem 1.3 to prove results for the bilinear Hilbert transform and for multilinear Calderón-Zygmund operators. We first consider the bilinear Hilbert transform, which is defined by

$$BH(f_1, f_2)(x) = \text{p.v.} \int_{\mathbb{R}} f_1(x-t)g(x+t)\frac{dt}{t}.$$

The problem of finding bilinear L^p estimates for this operator was first raised by Calderón in connection with the Cauchy integral problem (though it was apparently not published until [23]). Lacey and Thiele [26,27] showed that for $1 < p_1, \ p_2 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$,

$$\|BH(f_1, f_2)\|_{L^p} \leq C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

The problem of weighted norm inequalities for the bilinear Hilbert transform has been raised by a number of authors: see [15,16,20,30]. The first such results were recently obtained by Culiuc, di Plinio and Ou [14].

Theorem 1.16 *Given $1 < p_1, \ p_2 < \infty$, define p by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and assume that $p > 1$. For $i = 1, 2$, let w_i be such that $w_i^{2p_i} \in A_{p_i}$, and define $w = w_1 w_2$. Then*

$$\|BH(f_1, f_2)\|_{L^p(w^p)} \leq C\|f_1\|_{L^{p_1}\left(w_1^{p_1}\right)}\|f_2\|_{L^{p_2}\left(w_2^{p_2}\right)}, \tag{1.17}$$

where $C = C(p_i, [w_i^{2p_i}]_{A_{p_i}})$.

If we apply Theorem 1.3, we can extend Theorem 1.16 to a larger collection of weights and exponents. In particular, we can remove the restriction that $p > 1$, replacing it with $p > \frac{2}{3}$, the same threshold that appears in the unweighted theory.

Theorem 1.18 *Given arbitrary $1 < p_1, \ p_2 < \infty$, define $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and assume that $p > 1$. For every $i = 1, 2$, let $r_i^- = \frac{2p_i}{1+p_i} < q_i < 2p_i = r_i^+$. Then, for all $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$ —or, equivalently, $w_i^{2r_i} \in A_{r_i}$ for $r_i = \left(\frac{2}{q_i} - \frac{1}{p_i}\right)^{-1}$— if we write $w = w_1 w_2$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have that*

$$\|BH(f_1, f_2)\|_{L^q(w^q)} \leq C\|f_1\|_{L^{q_1}\left(w_1^{q_1}\right)}\|f_2\|_{L^{q_2}\left(w_2^{q_2}\right)}. \tag{1.19}$$

In particular, given arbitrary $1 < q_1, \ q_2 < \infty$ so that $q > \frac{2}{3}$ where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, there exist values $1 < p_1, \ p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, in such a way *that if we set* $r_i^- = \frac{2p_i}{1+p_i}$, $r_i^+ = 2p_i$ *then* $r_i^- < q_i < r_i^+$, *and for all weights* w_i *with* $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$ *(or, equivalently,* $w_i^{2r_i} \in A_{r_i}$ *for* $r_i = \left(\frac{2}{q_i} - \frac{1}{p_i}\right)^{-1}$*) and* $w = w_1 w_2$,

$$\|BH(f_1, f_2)\|_{L^q(w^q)} \le C\|f_1\|_{L^{q_1}\left(w_1^{q_1}\right)}\|f_2\|_{L^{q_2}\left(w_2^{q_2}\right)}. \tag{1.20}$$

Remark 1.21 We can state Theorem 1.18 in a different but equivalent form. For instance, in the second part of that result, if we let $v_i = w_i^{q_i}$, then our hypothesis becomes $v_i \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$, and the conclusion is that

$$BH : L^{q_1}(v_1) \times L^{q_2}(v_2) \longrightarrow L^q\left(v_1^{\frac{q}{q_1}} v_w^{\frac{q}{q_2}}\right).$$

In [14], for instance, Theorem 1.16 is stated in this form. We chose the form that we did because it seems more natural when working with off-diagonal inequalities.

Remark 1.22 In [14] the authors actually proved Theorem 1.16 for a more general family of bilinear multiplier operators introduced by Muscalu, Tao and Thiele [31]. Theorem 1.18 immediately extends to these operators. We refer the interested reader to these papers for precise definitions. This extension actually shows that that the bound $p > 1$ in Theorem 1.16 and the bound $p > \frac{2}{3}$ in Theorem 1.18 are natural and in some sense the best possible. In [25, Theorem 2.14], Lacey gave an example of an operator which does not satisfy a bilinear estimate when $p < 2/3$; in [14, Remark 1.2] the authors show that Theorem 1.16 applies to this operator. Hence, if Theorem 1.16 could be extended to include the case $p < 1$, we would get weighted estimates for this operator. But by extrapolation, these would yield inequalities below the threshold $q = \frac{2}{3}$. Indeed, we could apply the first part of Theorem 1.18 with those fixed exponents $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} > 1$ and $w_1 = w_2 \equiv 1$ to obtain that this operator maps $L^{q_1} \times L^{q_2}$ into L^q for every $r_i^- = \frac{2p_i}{1+p_i} < q_i < 2p_i = r_i^+$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. If we fix $0 < \epsilon < \min\{\frac{1}{2}(\frac{1}{p} - 1), \frac{1}{p_i'}\}$ and let $\frac{1}{q_i} := \frac{1}{2}(\frac{1}{p_i} + 1 - \epsilon)$, we would have that $r_i^- < q_i < r_i^+$ and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2p} + 1 - \epsilon > \frac{3}{2}.$$

Given q_1, q_2, as part of the proof of Theorem 1.18 we construct the parameters r_i^-, r_i^+ needed to define the weight classes. Thus, while we show that such weights exist, it is not clear from the statement of the theorem what weights are possible. To illustrate the different kinds of weight conditions we get, we give some special classes of weights, and in particular we give a family of power weights.

Corollary 1.23 *Given* $1 < q_1, q_2 < \infty$, *define* q *by* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, *and assume further that* $q > \frac{2}{3}$. *Then,*

$$\|BH(f_1, f_2)\|_{L^q(w^q)} \le C\|f_1\|_{L^{q_1}\left(w_1^{q_1}\right)}\|f_2\|_{L^{q_2}\left(w_2^{q_2}\right)} \tag{1.24}$$

holds for all $w_i^{q_i} \in A_{\max\{1,\frac{q_i}{2}\}} \cap RH_{\max\{1,\frac{2}{q_i}\}}$ *and* $w = w_1 w_2$. *In particular,*

$$BH : L^{q_1}(|x|^{-a}) \times L^{q_2}(|x|^{-a}) \longrightarrow L^q(|x|^{-a}), \tag{1.25}$$

if $a = 0$ *or if*

$$1 - \min\left\{\max\left\{1, \frac{q_1}{2}\right\}, \max\left\{1, \frac{q_2}{2}\right\}\right\} < a < \min\left\{1, \frac{q_1}{2}, \frac{q_2}{2}\right\}. \tag{1.26}$$

As a result, (1.25) *holds for all* $0 \le a < \frac{1}{2}$.

Remark 1.27 By Corollary 1.23 we get weighted estimates for the bilinear Hilbert transform in exactly the same range where the unweighted estimates are known to hold. (Note that when $a = 0$ we recover the unweighted case.) Rather than taking equal weights in (1.25), we can also give this inequality for more general power weights of the form $w_i = |x|^{-a_i/q_i}$; details are left to the interested reader.

Remark 1.28 As a consequence of Corollary 1.23 we see that even in the range of exponents covered by Theorem 1.16 from [14], we get a larger class of weights. Fix $1 < q_1, q_2 < \infty$ and assume that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < 1$. First, it is easy to show (see Lemma 2.1 below) that $w_i^{2q_i} \in A_{q_i}$ if an only if $w_i^{q_i} \in A_{\frac{1+q_i}{2}} \cap RH_2$. Hence, if we further assume that $w_i^{q_i} \in A_1$ this condition becomes $w_i^{q_i} \in A_1 \cap RH_2$ or, equivalently, (see Lemma 2.1 below) $w_i^{2q_i} \in A_1$. Hence, as a corollary of Theorem 1.16 we get that $BH : L^{q_1}(w_1^{q_2}) \times L^{q_2}(w_2^{q_2}) \longrightarrow L^q(w^q)$ for all $w_i^{2q_i} \in A_1$. But by Corollary 1.23, again assuming that $w_i^{q_i} \in A_1$, we can allow $w^{q_i} \in A_1 \cap RH_{\max\{1,\frac{2}{q_i}\}}$, or equivalently, $w_i^{\max\{2,q_i\}} \in A_1$ which is weaker than $w_i^{2q_i} \in A_1$ since $\max\{2, q_i\} < 2q_i$.

Further, when $1 < q_i \le 2$, Corollary 1.23 gives the class of weights $w_i^{q_i} \in A_1 \cap RH_{\frac{2}{q_i}}$. To compare this with Theorem 1.16 from [14] note that their condition is, as explained above, $w_i^{q_i} \in A_{\frac{1+q_i}{2}} \cap RH_2$ and hence we can weaken $w^{q_i} \in RH_2$ to $w^{q_i} \in RH_{\frac{2}{q_i}}$ at the cost of assuming that $w^{q_i} \in A_1$. Alternatively, if $q_i \ge 2$, our condition becomes $w^{q_i} \in A_{\frac{q_i}{2}}$, which removes any reverse Hölder condition for w^{q_i} at the cost of assuming that $w^{q_i} \in A_{\frac{q_i}{2}} \subset A_{\frac{1+q_i}{2}}$.

We can also prove vector-valued inequalities for the bilinear Hilbert transform for the same weighted Lebesgue spaces as in the scalar inequality. Even in the unweighted case, vector-valued inequalities were an open question until recently. Benea and Muscalu [4,5] (see also [22,32] for earlier results) proved that given $1 < s_1, s_2 \le \infty$ and s such that $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$ and $s > \frac{2}{3}$, then there exist q_1, q_2, q such that

$$\left\|\left(\sum_k |BH(f_k, g_k)|^s\right)^{\frac{1}{s}}\right\|_q \le C\left\|\left(\sum_k |f_k|^{s_1}\right)^{\frac{1}{s_1}}\right\|_{q_1}\left\|\left(\sum_k |g_k|^{s_2}\right)^{\frac{1}{s_2}}\right\|_{q_2}$$

where $1 < q_1,\ q_2 \le \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and, depending on the values of the s_i, there are additional restrictions on the possible values of the q_i. (See [5, Theorem 5] for a precise statement or (5.4) below.) An alternative proof of these estimates when $s > 1$ is given in [14].

By using the formalism of extrapolation pairs, vector-valued inequalities are an immediate consequence of extrapolation. Hence, as a consequence of Theorem 1.18 we get the following generalization of the results in [4,5,14]. We note that for some triples $s_1,\ s_2,\ s$ our method does not let us recover the full range of spaces gotten in [4,5] but we do get weighted estimates in our range.

Theorem 1.29 *Given arbitrary* $1 < p_1,\ p_2 < \infty$*, define* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and assume that* $p > 1$*. For every* $i = 1, 2$*, let* $r_i^- = \frac{2p_i}{1+p_i} < q_i, s_i < 2p_i = r_i^+$*. Then, for all* $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$ *—or, equivalently,* $w_i^{2r_i} \in A_{r_i}$ *for* $r_i = \left(\frac{2}{q_i} - \frac{1}{p_i}\right)^{-1}$*— if we write* $w = w_1 w_2$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ *and* $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$*, there holds*

$$\left\| \left(\sum_k |BH(f_k, g_k)|^s \right)^{\frac{1}{s}} \right\|_{L^q(w^q)} \le C \left\| \left(\sum_k |f_k|^{s_1} \right)^{\frac{1}{s_1}} \right\|_{L^{q_1}\left(w_1^{q_1}\right)} \left\| \left(\sum_k |g_k|^{s_2} \right)^{\frac{1}{s_2}} \right\|_{L^{q_2}\left(w_2^{q_2}\right)}. \tag{1.30}$$

In particular, for every $1 < s_1,\ s_2 < \infty$ *such that* $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} < \frac{3}{2}$*, and for every* $1 < q_1, q_2 < \infty$ *such that* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$*, if*

$$\left| \frac{1}{s_1} - \frac{1}{q_1} \right| < \frac{1}{2}, \qquad \left| \frac{1}{s_2} - \frac{1}{q_2} \right| < \frac{1}{2}, \qquad \textit{and} \qquad \sum_{i=1}^{2} \max \left\{ \frac{1}{q_i}, \frac{1}{s_i} \right\} < \frac{3}{2}, \tag{1.31}$$

there are values $1 < p_1,\ p_2 < \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$*, in such a way that if we set* $r_i^- = \frac{2p_i}{1+p_i}$, $r_i^+ = 2p_i$ *then* $r_i^- < q_i, s_i < r_i^+$*, and hence* (1.30) *holds for all weights* w_i *with* $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$ *(or, equivalently,* $w_i^{2r_i} \in A_{r_i}$ *for* $r_i = \left(\frac{2}{q_i} - \frac{1}{p_i}\right)^{-1}$*) and* $w = w_1 w_2$.

Remark 1.32 Theorem 1.29 contains the vector-valued inequalities that follow immediately from our extrapolation result applied to the weighted norm inequalities obtained in [14] (cf. Therorem 1.16). However, more general weighted estimates for the bilinear Hilbert transform are implicit in the arguments of [14]. These in turn produce vector-valued inequalities in a wider range of exponents. We shall elaborate on this in Sect. 5 below.

Remark 1.33 In [4, Proposition 10] the authors also prove iterated vector-valued inequalities of the form

$$\Big\| \Big(\sum_j \Big(\sum_k |BH(f_{jk}, g_{jk})|^s \Big)^{\frac{t}{s}} \Big)^{\frac{1}{t}} \Big\|_p$$
$$\leq \Big\| \Big(\sum_j \Big(\sum_k |f_{jk}|^{s_1} \Big)^{\frac{t_1}{s_1}} \Big)^{\frac{1}{t_1}} \Big\|_{p_1} \Big\| \Big(\sum_j \Big(\sum_k |g_{jk}|^{s_2} \Big)^{\frac{t_2}{s_2}} \Big)^{\frac{1}{t_2}} \Big\|_{p_2},$$

again with restrictions on the possible values of the p_i depending on the s_i and t_i. We can easily prove some of these inequalities by extrapolation; moreover, we can also prove prove weighted versions. After the proof of Theorem 1.29 we sketch how this is done. Here we note in passing that iterated vector-valued inequalities have recently appeared in another setting: see [1].

As we did with the scalar inequalities we give some specific examples of classes of weights for which the bilinear Hilbert transform satisfies weighted vector-valued inequalities.

Corollary 1.34 *Given* $1 < s_1, s_2 < \infty$ *such that* $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} < \frac{3}{2}$, *and* $1 < q_1, q_2 < \infty$ *such that* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$, *if*

$$\left| \frac{1}{s_1} - \frac{1}{q_1} \right| < \frac{1}{2}, \qquad \left| \frac{1}{s_2} - \frac{1}{q_2} \right| < \frac{1}{2}, \qquad \text{and} \qquad \sum_{i=1}^{2} \max\left\{ \frac{1}{q_i}, \frac{1}{s_i} \right\} < \frac{3}{2}, \tag{1.35}$$

then (1.30) *holds for all* $w_i^{q_i} \in A_{\max\{1, \frac{q_i}{2}, \frac{q_i}{s_i}\}} \cap RH_{\max\{1, \frac{2}{q_i}, [1-q_i(\frac{1}{s_i}-\frac{1}{2})]^{-1}\}}$. *In particular,*

$$\Big\| \Big(\sum_k |BH(f_k, g_k)|^s \Big)^{\frac{1}{s}} \Big\|_{L^q(|x|^{-a})}$$
$$\leq C \Big\| \Big(\sum_k |f_k|^{s_1} \Big)^{\frac{1}{s_1}} \Big\|_{L^{q_1}(|x|^{-a})} \Big\| \Big(\sum_k |g_k|^{s_2} \Big)^{\frac{1}{s_2}} \Big\|_{L^{q_2}(|x|^{-a})}. \tag{1.36}$$

holds if $a \in \{0\} \cup (a_-, a_+)$ *where*

$$a_- = 1 - \min\left\{ \max\left\{1, \frac{q_1}{2}, \frac{q_1}{s_1}\right\}, \max\left\{1, \frac{q_2}{2}, \frac{q_2}{s_2}\right\} \right\}$$
$$a_+ = \min\left\{ 1, \frac{q_1}{2}, \frac{q_2}{2}, 1 - q_1\left(\frac{1}{s_1} - \frac{1}{2}\right), 1 - q_2\left(\frac{1}{s_2} - \frac{1}{2}\right) \right\} \tag{1.37}$$

Remark 1.38 The conditions in (1.35) guarantee that $a_- \leq 0 < a_+$, hence the set $\{0\} \cup (a_-, a_+)$ defines a non-empty interval. On the other hand, this interval can be arbitrarily small. For instance, take $q_1 = s_1 = 2$, $q_2 = 2$, $s_2 = t$ with $1 < t < 2$. Then (1.35) is satisfied and we have that $a_- = 0$ and $a_+ = 2(1 - \frac{1}{t})$. Thus, $\{0\} \cup (a_-, a_+) = [0, a_+)$ and $a_+ \to 0$ as $t \to 1^+$: that is, in the limit we just get the Lebesgue measure. Notice, however, that in the context of the first part of Corollary 1.34, as $t \to 1^+$, the conditions on the weights become $w_1^2 \in A_1$ and $w_2^2 \in A_2 \cap RH_\infty$. Hence, we can take $w_1(x) = |x|^{-\frac{a_1}{q_1}}$ and $w_2(x) = |x|^{-\frac{a_2}{q_2}}$ with $0 \le a_1 < 1$ and $-1 < a_2 \le 0$. (Of course if $a_1 = a_2 = a$, then $a = 0$ as observed above.)

As a final application we use extrapolation to prove Marcinkiewicz-Zygmund inequalities for multilinear Calderón-Zygmund operators. Weighted norm inequalities for these operators have been considered by several authors: we refer the reader to [20,28] for precise definitions of these operators and weighted norm inequalities for them. Very recently, Carando, Mazzitelli and Ombrosi [6] proved the following weighted Marcinkiewicz-Zygmund inequalities.

Theorem 1.39 *For $m \ge 1$, let T be an m-linear Calderón-Zygmund operator. Given $1 < q_1, \dots, q_m < \infty$, q such that $\frac{1}{q} = \sum \frac{1}{q_i}$, and weights w_i such that $w_i^{q_i} \in A_{q_i}$,*

$$\left\| \left(\sum_{k_1,\dots,k_m} \left| T\left(f_{k_1}^1, \dots, f_{k_m}^m\right)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w^q)} \le C \prod_{i=1}^m \left\| \left(\sum_{k_i} |f_{k_i}^i|^2 \right)^{\frac{1}{2}} \right\|_{L^{q_i}\left(w_i^{q_i}\right)}, \tag{1.40}$$

where $w = w_1 \dots w_m$. If $1 < r < 2$ and if we further assume $1 < q_i < r$, then again for all weights w_i such that $w_i^{q_i} \in A_{q_i}$,

$$\left\| \left(\sum_{k_1,\dots,k_m} \left| T\left(f_{k_1}^1, \dots, f_{k_m}^m\right)\right|^r \right)^{\frac{1}{r}} \right\|_{L^q(w^q)} \le C \prod_{i=1}^m \left\| \left(\sum_{k_i} |f_{k_i}^i|^r \right)^{\frac{1}{r}} \right\|_{L^{q_i}\left(w_i^{q_i}\right)}, \tag{1.41}$$

where $w = w_1 \dots w_m$.

By using extrapolation we can prove that inequality (1.41) holds for $1 < r < 2$ with the same family of exponents as in (1.40) for $r = 2$.

Theorem 1.42 *For $m \ge 1$, let T be an m-linear Calderón-Zygmund operator. Given $1 < r < 2$, $1 < q_1, \dots, q_m < \infty$, q such that $\frac{1}{q} = \sum \frac{1}{q_i}$, and weights w_i such that $w_i^{q_i} \in A_{q_i}$, then inequality (1.41) holds.*

Remark 1.43 In [6] the authors actually prove that Theorem 1.39 holds for weights in the larger class $A_{\mathbf{p}}$ introduced in [28]. However, it is not known whether multilinear extrapolation holds for these weights. We also do not know if Theorem 1.42 can be extended to this larger family of weights.

The remainder of this paper is organized as follows. In Sect. 2 we gather some definitions and basic results about weights. In Sect. 3 we prove all of our extrapolation results. In Sect. 4 we give the proofs of all of the applications. Finally, in Sect. 5 we discuss some results that are implicit in [14] and that can be used to get more general vector-valued inequalities for the bilinear Hilbert transform.

Throughout this paper n will denote the dimension of the underlying space, $\mathbb{R}^n$. A constant C may depend on the dimension n, the underlying parameters p_-, p_+, $p, \dots$, and the A_p and RH_s constants of the associated weights. It will not depend on the specific weight. The value of a constant C may change from line to line. Throughout we will use the conventions that $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$, and $1' = \infty$ and $\infty' = 1$.

2 Preliminaries

In this section we give the basic properties of weights that we will need below. For proofs and further information, see [17,19]. By a weight we mean a non-negative function v such that $0 < v(x) < \infty$ a.e. For $1 < p < \infty$, we say $v \in A_p$ if

$$[v]_{A_p} = \sup_Q \fint_Q v\,dx \left(\fint_Q v^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $\fint_Q v\,dx = |Q|^{-1}\int_Q v\,dx$. The quantity $[v]_{A_p}$ is called the A_p constant of v. Note that it follows at once from this definition that if $v \in A_p$, then $v^{1-p'} \in A_{p'}$. When $p = 1$ we say $v \in A_1$ if

$$[v]_{A_1} = \sup_Q \fint_Q v(y)\,dy \operatorname*{ess\,sup}_{x\in Q} v(x)^{-1} < \infty.$$

The A_p classes are properly nested: for $1 < p < q$, $A_1 \subsetneq A_p \subsetneq A_q$. We denote the union of all the A_p classes, $1 \le p < \infty$, by A_∞.

Given $1 < s < \infty$, we say that a weight v satisfies the reverse Hölder inequality with exponent s, denoted $w \in RH_s$ if

$$[v]_{RH_s} = \sup_Q \left(\fint_Q v^s\,dx\right)^{\frac{1}{s}} \left(\fint_Q v\,dx\right)^{-1} < \infty.$$

When $s = \infty$ we say $v \in RH_\infty$ if

$$[v]_{RH_\infty} = \sup_Q \operatorname*{ess\,sup}_{x\in Q} v(x) \left(\fint_Q v\,dx\right)^{-1} < \infty.$$

The reverse Hölder classes are also properly nested: if $s < t < \infty$, then $RH_\infty \subsetneq RH_t \subsetneq RH_s$. Define RH_1 to be the union of all the RH_s classes, $1 < s \le \infty$. We have that $RH_1 = A_\infty$. A given v is in RH_s for some $s > 1$ if and only if there exists $p > 1$ such that $v \in A_p$. Equivalently, if $v \in A_\infty$, there exists $1 \le p < \infty$ and $1 < s \le \infty$ such that $v \in A_p \cap RH_s$.

The A_p and RH_s classes satisfy openness properties: given $v \in A_p$, $1 < p < \infty$, then there exists $\epsilon > 0$ depending only on $[v]_{A_p}$, p and n, such that $v \in A_{p-\epsilon}$; also given $v \in RH_s$, $1 < s < \infty$, then there exists $\epsilon > 0$ depending only on $[v]_{RH_s}$, s, and n, such that $v \in RH_{s+\epsilon}$.

The condition $v \in A_p \cap RH_s$ can be restated using the following result. The first part is from [12, Theorem 2.2]; the second is just gotten by the duality of A_p weights.

Lemma 2.1 *Given $1 \le p < \infty$, $1 \le s < \infty$, the weight $v \in A_p \cap RH_s$ if and only if $v^s \in A_q$, where $q = s(p-1)+1$, that is,*

$$\sup_Q \left(\fint_Q v^s\,dx \right)^{\frac{1}{s}} \left(\fint_Q v^{1-p'}\,dx \right)^{p-1} < \infty. \tag{2.2}$$

In this case also have that $v^{1-p'} \in A_{q'}$.

We can also easily construct weights $v \in A_p \cap RH_s$. The next result can be proved directly from the definitions of the weight classes; essentially the same argument is used to prove the easier half of the Jones factorization theorem. See [12, Theorem 5.1] or [7, Theorem 4.4].

Lemma 2.3 *Given weights $v_1,\ v_2 \in A_1$, then for all $1 \le p < \infty$, $1 < s \le \infty$,*

$$v = v_1^{\frac{1}{s}} v_2^{1-p} \in A_p \cap RH_s.$$

3 Proofs of extrapolation results

Our proof is similar in spirit to the proofs of off-diagonal and limited range extrapolation in [10, Theorems 3.23 and 3.31]. To better understand the heuristic argument that underlies our proof, we refer the reader to the discussion in [13, Section 4]. We have split the proof split into four cases.

3.1 Proof of Theorem 1.8. Case I: $p_- > 0$ and $p_- < p_0 < p_+$

Fix $p_- < p < p_+$ and w such that $w^p \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$. Fix an extrapolation pair $(f, g) \in \mathcal{F}$; we may assume that $0 < \|f\|_{L^q(w^q)},\ \|g\|_{L^p(w^p)} < \infty$. For if $\|f\|_{L^q(w^q)} = 0$ or if $\|g\|_{L^p(w^p)} = \infty$, then (1.10) is trivially true. And if $\|g\|_{L^p(w^p)} = 0$, then (1.9) implies that $\|f\|_{L^{q_0}(w^{q_0})} = 0$, and so $f = 0$ a.e. and thus $\|f\|_{L^q(w^q)} = 0$, which again gives us (1.10).

We now fix some exponents based on our weight w. By Lemma 2.1 we have that $w^{p\left(\frac{p_+}{p}\right)'} \in A_\tau$, where

$$\tau = \left(\frac{p_+}{p}\right)'\left(\frac{p}{p_-} - 1\right) + 1 = \frac{\frac{1}{p_-} - \frac{1}{p}}{\frac{1}{p} - \frac{1}{p_+}} + 1 = \frac{\frac{1}{p_-} - \frac{1}{p_+}}{\frac{1}{p} - \frac{1}{p_+}}. \tag{3.1}$$

For future reference we note that

$$\tau' = \frac{\frac{1}{p_-} - \frac{1}{p_+}}{\frac{1}{p_-} - \frac{1}{p}}. \tag{3.2}$$

From Remark 1.13 we have that

$$\frac{1}{q_+} - \frac{1}{p_+} = \frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p}. \tag{3.3}$$

Define the number s by

$$s = q_0 - \frac{q_0}{p_0}\frac{q}{\tau}\left(\frac{p_0}{p_-} - 1\right) = q_0 q\left(\frac{1}{q} - \frac{1}{\tau}\left(\frac{1}{p_-} - \frac{1}{p_0}\right)\right); \tag{3.4}$$

we will explain our choice of s below. For later use, we prove that $0 < s < \min(q, q_0)$. First, we have that $s > 0$: by (3.1), the fact that $p_0 < p_+$ and (3.3) we obtain

$$\frac{1}{q} - \frac{1}{\tau}\left(\frac{1}{p_-} - \frac{1}{p_0}\right) = \frac{1}{q} - \frac{\frac{1}{p} - \frac{1}{p_+}}{\frac{1}{p_-} - \frac{1}{p_+}}\left(\frac{1}{p_-} - \frac{1}{p_0}\right) > \frac{1}{q} - \frac{1}{p} + \frac{1}{p_+} = \frac{1}{q_+} \geq 0.$$

To show that $s < \min(q, q_0)$, we claim

$$s = q - \frac{q q_0}{p_0}\frac{1}{\tau'}\frac{1}{\left(\frac{p_+}{p_0}\right)'} = q_0 q\left(\frac{1}{q_0} - \left(1 - \frac{1}{\tau}\right)\left(\frac{1}{p_0} - \frac{1}{p_+}\right)\right). \tag{3.5}$$

To see that this holds, we use the fact that $\frac{1}{q} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{p_0}$:

$$\begin{aligned}
\frac{1}{q} - \frac{1}{\tau}\left(\frac{1}{p_-} - \frac{1}{p_0}\right) &= \frac{1}{q} - \frac{1}{\tau}\left(\frac{1}{p_-} - \frac{1}{p_+}\right) - \frac{1}{\tau}\left(\frac{1}{p_+} - \frac{1}{p_0}\right) \\
&= \frac{1}{q} - \frac{1}{p} + \frac{1}{p_+} - \frac{1}{\tau}\left(\frac{1}{p_+} - \frac{1}{p_0}\right) \\
&= \frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} - \frac{1}{\tau}\left(\frac{1}{p_+} - \frac{1}{p_0}\right) \\
&= \frac{1}{q_0} - \left(1 - \frac{1}{\tau}\right)\left(\frac{1}{p_0} - \frac{1}{p_+}\right).
\end{aligned}$$

It follows at once from (3.4) and (3.5) that $s < \min(q, q_0)$.

We now prove our main estimate. By rescaling and duality, we have that

$$\|f\|^s_{L^q(w^q)} = \|f^s\|_{L^{\frac{q}{s}}(w^q)} = \int_{\mathbb{R}^n} f^s h_2 w^q \, dx,$$

where h_2 is a non-negative function in $L^{(\frac{q}{s})'}(w^q)$ with $\|h_2\|_{L^{(\frac{q}{s})'}(w^q)} = 1$. Now let H_1 and H_2 be non-negative functions such that $0 < H_1 < \infty$ a.e., and $h_2 \le H_2$; we will determine their exact values below. Fix $\alpha = \frac{s}{\left(\frac{q_0}{s}\right)'}$. Then by Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n} f^s h_2 w^q\,dx &\le \int_{\mathbb{R}^n} f^s H_1^{-\alpha} H_1^{\alpha} H_2 w^q\,dx \\
&\le \left(\int_{\mathbb{R}^n} f^{q_0} H_1^{-\alpha\frac{q_0}{s}} H_2 w^q\,dx\right)^{\frac{s}{q_0}} \left(\int_{\mathbb{R}^n} H_1^{\alpha\left(\frac{q_0}{s}\right)'} H_2 w^q\,dx\right)^{1/\left(\frac{q_0}{s}\right)'} \\
&= I_1^{\frac{s}{q_0}} \times I_2^{1/\left(\frac{q_0}{s}\right)'}.
\end{aligned} \tag{3.6}$$

We first estimate I_2. Assume that $H_1 \in L^q(w^q)$ with $\|H_1\|_{L^q(w^q)} \le C_1 < \infty$, and that $H_2 \in L^{\left(\frac{q}{s}\right)'}(w^q)$ with $\|H_2\|_{L^{\left(\frac{q}{s}\right)'}(w^q)} \le C_2 < \infty$. Then again by Hölder's inequality,

$$\begin{aligned}
I_2 &\le \left(\int_{\mathbb{R}^n} H_1^{\alpha\left(\frac{q_0}{s}\right)'\frac{q}{s}} w^q\,dx\right)^{\frac{s}{q}} \left(\int_{\mathbb{R}^n} H_2^{\left(\frac{q}{s}\right)'} w^q\,dx\right)^{1/\left(\frac{q}{s}\right)'} \\
&\le C_2 \left(\int_{\mathbb{R}^n} H_1^q w^q\,dx\right)^{\frac{s}{q}} \le C_1^s C_2.
\end{aligned}$$

To estimate I_1 we want to apply (1.9); to do so we need to show that $I_1 < \infty$. Assume that $f \le H_1 \|f\|_{L^q(w^q)}$; then we have that

$$\begin{aligned}
I_1 &\le \|f\|_{L^q(w^q)}^{q_0} \int_{\mathbb{R}^n} H_1^{q_0} H_1^{-\alpha\frac{q_0}{s}} H_2 w^q\,dx \\
&= \|f\|_{L^q(w^q)}^{q_0} \int_{\mathbb{R}^n} H_1^{s} H_2 w^q\,dx = \|f\|_{L^q(w^q)}^{q_0} \times I_2 < \infty.
\end{aligned}$$

Define $\varphi = \left(\frac{q}{s}\right)' \frac{q_0}{p_0}$. Then $\varphi > 1$: by (3.5) we have that

$$\frac{s}{q}\frac{p_0}{q_0} = \frac{p_0}{q_0} - \frac{1}{\tau'}\frac{1}{\left(\frac{p_+}{p_0}\right)'},$$

and so

$$\frac{1}{\varphi} = \frac{p_0}{q_0}\frac{1}{\left(\frac{q}{s}\right)'} = \frac{p_0}{q_0}\left(1 - \frac{s}{q}\right) = \frac{1}{\tau'}\frac{1}{\left(\frac{p_+}{p_0}\right)'} < 1.$$

Now let $W^{q_0} = H_1^{-\alpha\frac{q_0}{s}} H_2 w^q$ and assume that $W^{p_0} \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$. Since I_1 is finite, $f \in L^{q_0}(W^{q_0})$. Thus, by (1.9) and Hölder's inequality,

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} f^{q_0} W^{q_0}\,dx \\
&\le C\left(\int_{\mathbb{R}^n} g^{p_0} W^{p_0}\,dx\right)^{\frac{q_0}{p_0}} \\
&= C\left(\int_{\mathbb{R}^n} g^{p_0} H_1^{-\alpha\frac{p_0}{s}} H_2^{\frac{p_0}{q_0}} w^{q\frac{p_0}{q_0}} w^{-q} w^q\,dx\right)^{\frac{q_0}{p_0}} \\
&\le \left(\int_{\mathbb{R}^n} g^{p_0\varphi'} H_1^{-\alpha\frac{p_0}{s}\varphi'} w^{q\left(\frac{p_0}{q_0}-1\right)\varphi'} w^q\,dx\right)^{\frac{q_0}{\varphi' p_0}} \left(\int_{\mathbb{R}^n} H_2^{\left(\frac{q}{s}\right)'} w^q\,dx\right)^{\frac{q_0}{\varphi p_0}}.
\end{aligned}$$

The second integral on the last line is bounded by $C_2^{\frac{q_0}{\varphi p_0}\left(\frac{q}{s}\right)'} = C_2$, so it remains to show that the first integral is bounded by $\|g\|_{L^p(w^p)}^{q_0}$. If we have that

$$g^{p_0\varphi'} H_1^{-\alpha\frac{p_0}{s}\varphi'} w^{q\left(\frac{p_0}{q_0}-1\right)\varphi'} \le H_1^q \|g\|_{L^p(w^p)}^{p_0\varphi'},$$

then the first integral would be bounded by $\|H_1\|_{L^q(w^q)}^{\frac{q_0 q}{\varphi' p_0}} \|g\|_{L^p(w^p)}^{q_0} \le C_1^{\frac{q_0 q}{\varphi' p_0}} \|g\|_{L^p(w^p)}^{q_0}$. This, combined with inequality (3.6) would yield inequality (1.10) and the proof would be complete.

Therefore, to complete the proof we need to show that we can construct non-negative functions H_1 and H_2 such that

$$\|H_1\|_{L^q(w^q)} \le C_1, \tag{3.7}$$

$$g^{p_0\varphi'} H_1^{-\alpha\frac{p_0}{s}\varphi'} w^{q\left(\frac{p_0}{q_0}-1\right)\varphi'} \le H_1^q \|g\|_{L^p(w^p)}^{p_0\varphi'}, \tag{3.8}$$

$$0 < H_1 < \infty, \quad f \le H_1 \|f\|_{L^q(w^q)}, \tag{3.9}$$

$$\|H_2\|_{L^{\left(\frac{q}{s}\right)'}(w^q)} \le C_2, \tag{3.10}$$

$$h_2 \le H_2; \tag{3.11}$$

and such that the weight $W = H_1^{-\frac{\alpha}{s}} H_2^{\frac{1}{q_0}} w^{\frac{q}{q_0}}$ satisfies

$$W^{p_0} = H_1^{-\frac{\alpha p_0}{s}} H_2^{\frac{p_0}{q_0}} w^{\frac{q p_0}{q_0}} \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}. \tag{3.12}$$

We will first prove that (3.7), (3.8) and (3.9) hold. Since $\alpha\frac{p_0}{s} = \frac{p_0}{q_0}(q_0 - s)$, one can see that (3.8) is equivalent to

$$g^{p_0} w^{q\left(\frac{p_0}{q_0}-1\right)} \le H_1^{\frac{q}{\varphi'} + \frac{p_0}{q_0}(q_0-s)} \|g\|_{L^p(w^p)}^{p_0}. \tag{3.13}$$

Using the fact that $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, we have that

$$\begin{aligned}\frac{q}{\varphi'} + \frac{p_0}{q_0}(q_0 - s) &= q - \frac{p_0}{q_0}\frac{q}{\left(\frac{q}{s}\right)'} + \frac{p_0}{q_0}(q_0 - s) = q - \frac{p_0}{q_0}(q - s) + \frac{p_0}{q_0}(q_0 - s)\\ &= q - \frac{p_0}{q_0}(q - q_0) = q\left(1 - p_0\left(\frac{1}{q_0} - \frac{1}{q}\right)\right)\\ &= q\left(1 - p_0\left(\frac{1}{p_0} - \frac{1}{p}\right)\right) = q\frac{p_0}{p}.\end{aligned}$$

Similarly, we have that

$$q\left(\frac{p_0}{q_0} - 1\right) = qp_0\left(\frac{1}{q_0} - \frac{1}{p_0}\right) = qp_0\left(\frac{1}{q} - \frac{1}{p}\right) = p_0\left(1 - \frac{q}{p}\right).$$

Therefore, (3.13) (and hence (3.8)) is equivalent to

$$g^{\frac{p}{q}} w^{\frac{p}{q}-1} \leq H_1 \|g\|_{L^p(w^p)}^{\frac{p}{q}}. \tag{3.14}$$

To construct a function H_1 that satisfies (3.7), (3.9), and (3.14), we use the Rubio de Francia iteration algorithm. As we noted above, $w^{p\left(\frac{p_+}{p}\right)'} \in A_\tau$, so the maximal operator is bounded on $L^\tau(w^{p\left(\frac{p_+}{p}\right)'})$. Hence, for non-negative $G \in L^\tau(w^{p(\frac{p_+}{p})'})$ we can define the iteration algorithm

$$\mathcal{R}_1 G = \sum_{k=0}^{\infty} \frac{M^k G}{2^k \|M\|^k_{L^\tau\left(w^{p\left(\frac{p_+}{p}\right)'}\right)}}.$$

Then we have that that $G \leq \mathcal{R}_1 G$, $\mathcal{R}_1 G \in A_1$, and $\|\mathcal{R}_1 G\|_{L^\tau(w^{p(\frac{p_+}{p})'})} \leq 2\|G\|_{L^\tau(w^{p(\frac{p_+}{p})'})}$ (cf. [10, Proof of Theorem 3.9]). Now define δ and ϵ by

$$\delta\tau = q, \qquad \epsilon\tau = q - p\left(\frac{p_+}{p}\right)',$$

and let

$$H_1 = \mathcal{R}_1(h_1^\delta w^\epsilon)^{\frac{1}{\delta}} w^{-\frac{\epsilon}{\delta}}, \qquad h_1 = \frac{f}{\|f\|_{L^q(w^q)}} + \frac{g^{\frac{p}{q}} w^{\frac{p}{q}-1}}{\|g\|^{\frac{p}{q}}_{L^p(w^p)}}.$$

Then

$$\max\left(\frac{f}{\|f\|_{L^q(w^q)}}, \frac{g^{\frac{p}{q}} w^{\frac{p}{q}-1}}{\|g\|_{L^p(w^p)}^{\frac{p}{q}}}\right) \leq h_1 \leq H_1,$$

and so both (3.9) and (3.14) hold. Moreover,

$$\|h_1\|_{L^q(w^q)} \leq 2^{1-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \frac{f^q w^q}{\|f\|_{L^q(w^q)}^q} + \frac{g^p w^p}{\|g\|_{L^p(w^p)}^p} \, dx\right)^{\frac{1}{q}} = 2,$$

and so

$$\begin{aligned}\|H_1\|_{L^q(w^q)} &= \|\mathcal{R}_1 \left(h_1^\delta w^\epsilon\right)\|_{L^\tau\left(w^{p\left(\frac{p_+}{p}\right)'}\right)}^{\frac{1}{\delta}} \\ &\leq 2^{\frac{1}{\delta}} \|h_1^\delta w^\epsilon\|_{L^\tau\left(w^{p\left(\frac{p_+}{p}\right)'}\right)}^{\frac{1}{\delta}} = 2^{\frac{1}{\delta}} \|h_1\|_{L^q(w^q)} \leq 2^{1+\frac{1}{\delta}} = C_1.\end{aligned}$$

This gives us (3.7).

The construction of H_2 and the proof of (3.10) and (3.11) are similar to the argument for H_1. By Lemma 2.1, if we set

$$\sigma = p\left(\left(\frac{p}{p_-}\right)' - 1\right),$$

then $w^{-\sigma} \in A_{\tau'}$ and so the maximal operator is bounded on $L^{\tau'}(w^{-\sigma})$. Hence, if we define the Rubio de Francia iteration algorithm for non-negative $F \in L^{\tau'}(w^{-\sigma})$ by

$$\mathcal{R}_2 F = \sum_{k=0}^{\infty} \frac{M^k F}{2^k \|M\|_{L^{\tau'}(w^{-\sigma})}^k},$$

then we have that $F \leq \mathcal{R}_2 F$, $\mathcal{R}_2 F \in A_1$, and $\|\mathcal{R}_2 F\|_{L^{\tau'}(w^{-\sigma})} \leq 2\|F\|_{L^{\tau'}(w^{-\sigma})}$. Define β and γ by

$$\beta\tau' = \left(\frac{q}{s}\right)', \qquad \gamma\tau' = \sigma + q.$$

If we now let

$$H_2 = \mathcal{R}_2 \left(h_2^\beta w^\gamma\right)^{\frac{1}{\beta}} w^{-\frac{\gamma}{\beta}},$$

then we immediately get (3.11). Moreover, we have that

$$\|H_2\|_{L^{(\frac{q}{s})'}(w^q)} = \left\|\mathcal{R}_2\left(h_2^\beta w^\gamma\right)\right\|_{L^{\tau'}(w^{-\sigma})}^{\frac{1}{\beta}} \leq 2^{\frac{1}{\beta}} \left\|h_2^\beta w^\gamma\right\|_{L^{\tau'}(w^{-\sigma})}^{\frac{1}{\beta}} = 2^{\frac{1}{\beta}} \|h_2\|_{L^{(\frac{q}{s})'}(w^q)} = 2^{\frac{1}{\beta}} = C_2.$$

This gives us (3.10).

Finally, we will show that (3.12) holds. By Lemma 2.3, (3.12) holds if there exist $\mu_1,\ \mu_2 \in A_1$ such that

$$H_1^{-\frac{\alpha p_0}{s}} H_2^{\frac{p_0}{q_0}} w^{\frac{q p_0}{q_0}} = W^{p_0} = \mu_2^{\frac{1}{\left(\frac{p_+}{p_0}\right)'}} \mu_1^{1-\frac{p_0}{p_-}}.$$

By the A_1 property of the Rubio de Francia iteration algorithms, we have that

$$\mu_1 = H_1^{\frac{q}{\tau}} w^{\frac{q}{\tau} - \frac{p}{\tau}\left(\frac{p_+}{p}\right)'} = \mathcal{R}_1(h_1^\delta w^\epsilon) \in A_1,$$
$$\mu_2 = H_2^{\frac{1}{\tau'}\left(\frac{q}{s}\right)'} w^{\frac{\sigma}{\tau'} + \frac{q}{\tau'}} = \mathcal{R}_2(h^\beta w^\gamma) \in A_1.$$

If we substitute these expressions into the above formula and equate exponents, we see that equality holds if

$$\frac{\alpha p_0}{s} = \frac{q}{\tau}\left(\frac{p_0}{p_-} - 1\right), \tag{3.15}$$

$$\frac{p_0}{q_0} = \frac{1}{\tau'}\left(\frac{q}{s}\right)' \frac{1}{\left(\frac{p_+}{p_0}\right)'}, \tag{3.16}$$

$$\frac{q p_0}{q_0} = \left(\frac{\sigma}{\tau'} + \frac{q}{\tau'}\right) \frac{1}{\left(\frac{p_+}{p_0}\right)'} + \left(\frac{q}{\tau} - \frac{p}{\tau}\left(\frac{p_+}{p}\right)'\right)\left(1 - \frac{p_0}{p_-}\right). \tag{3.17}$$

If we use our choice of α on the left-hand side of (3.15) and (3.4) on the right-hand side, it is straightforward to see that (3.15) holds. Additionally, if we use (3.5) on the right-hand side of (3.16), we see that the latter also holds. (It was the necessity of these two identities for the proof that is the reason for our original choice of s.) To show that (3.17) holds, note that by (3.2) and our choice of σ we have that

$$\frac{\sigma}{\tau'} = \frac{p}{\frac{p}{p_-} - 1} \frac{\frac{1}{p_-} - \frac{1}{p}}{\frac{1}{p_-} - \frac{1}{p_+}} = \frac{1}{\frac{1}{p_-} - \frac{1}{p_+}}.$$

Given this we can expand the right-hand side of (3.17):

$$\left(\frac{\sigma}{\tau'} + \frac{q}{\tau'}\right) \frac{1}{\left(\frac{p_+}{p_0}\right)'} + \left(\frac{q}{\tau} - \frac{p}{\tau}\left(\frac{p_+}{p}\right)'\right)\left(1 - \frac{p_0}{p_-}\right)$$

$$
\begin{aligned}
&= \left(\frac{1}{\frac{1}{p_-} - \frac{1}{p_+}} + q \frac{\frac{1}{p_-} - \frac{1}{p}}{\frac{1}{p_-} - \frac{1}{p_+}} \right) p_0 \left(\frac{1}{p_0} - \frac{1}{p_+} \right) \\
&\quad + \left(q \frac{\frac{1}{p} - \frac{1}{p_+}}{\frac{1}{p_-} - \frac{1}{p_+}} - \frac{1}{\frac{1}{p_-} - \frac{1}{p_+}} \right) p_0 \left(\frac{1}{p_0} - \frac{1}{p_-} \right) \\
&= \frac{p_0}{\frac{1}{p_-} - \frac{1}{p_+}} \left[\frac{1}{p_-} - \frac{1}{p_+} + \frac{q}{p} \left(\frac{1}{p_+} - \frac{1}{p_-} \right) + \frac{q}{p_0} \left(\frac{1}{p_-} - \frac{1}{p_+} \right) \right] \\
&= q p_0 \left[\frac{1}{q} - \frac{1}{p} + \frac{1}{p_0} \right] \\
&= \frac{q p_0}{q_0}.
\end{aligned}
$$

This completes the proof of Case I. □

3.2 Proof of Theorem 1.8. Case II: $p_0 = p_-$

Fix $p_- < p < p_+$ and w such that $w^p \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$ and note that in this case $p_- = p_0 > 0$ and $q_- = q_0$ by (1.14). The proof is similar to the proof of Case I and we indicate the main changes. First, in this case (3.4) gives $s = q_0 > 0$. Thus, $s = q_0 = q_- < q$ by (1.14) and the fact that $p_- < p$. Furthermore (3.5) holds in this case.

We now argue as before, but in this case we do not need to introduce H_1. Since $s < q$, by rescaling and duality we have that

$$
\|f\|_{L^q(w^q)}^s = \|f^s\|_{L^{\frac{q}{s}}(w^q)} = \int_{\mathbb{R}^n} f^s h_2 w^q \, dx \leq \int_{\mathbb{R}^n} f^s H_2 w^q \, dx,
$$

where h_2 is a non-negative function in $L^{(\frac{q}{s})'}(w^q)$ with $\|h_2\|_{L^{(\frac{q}{s})'}(w^q)} = 1$ and H_2 is such that $h_2 \leq H_2$; we will determine the exact value below. If we assume further that $\|H_2\|_{L^{\left(\frac{q}{s}\right)'}(w^q)} \leq C_2 < \infty$, it follows by assumption that

$$
\int_{\mathbb{R}^n} f^s H_2 w^q \, dx \leq \|f^s\|_{L^{\frac{q}{s}}(w^q)} \|H_2\|_{L^{\left(\frac{q}{s}\right)'}(w^q)} \leq C_2 \|f\|_{L^q(w^q)}^s < \infty.
$$

Define $\varphi = \left(\frac{q}{s}\right)' \frac{q_0}{p_0} = \left(\frac{q}{q_0}\right)' \frac{q_0}{p_0}$; then we have that

$$
\frac{1}{\varphi} = \frac{p_0}{q_0} \frac{1}{\left(\frac{q_0}{s}\right)'} = \frac{p_0}{q_0} \left(1 - \frac{q_0}{q}\right) = p_0 \left(\frac{1}{q_0} - \frac{1}{q} \right) = p_0 \left(\frac{1}{p_0} - \frac{1}{p} \right) = 1 - \frac{p_0}{p} < 1,
$$

which implies that $\varphi' = \frac{p}{p_0}$. Now let $W^{q_0} = H_2 w^q$ and assume that $W^{p_0} \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'} = A_1 \cap RH_{\left(\frac{p_+}{p_-}\right)'}$, or equivalently (by Lemma 2.1), $W^{p_0\left(\frac{p_+}{p_-}\right)'} \in A_1$. Then by our hypothesis (1.9) we get

$$\begin{aligned}
\|f\|^s_{L^q(w^q)} &= \int_{\mathbb{R}^n} f^{q_0} W^{q_0}\,dx \\
&\le C\left(\int_{\mathbb{R}^n} g^{p_0} W^{p_0}\,dx\right)^{\frac{q_0}{p_0}} \\
&= C\left(\int_{\mathbb{R}^n} g^{p_0} H_2^{\frac{p_0}{q_0}} w^{q\frac{p_0}{q_0}} w^{-q} w^q\,dx\right)^{\frac{q_0}{p_0}} \\
&\le \left(\int_{\mathbb{R}^n} g^{p_0\varphi'} w^{q\left(\frac{p_0}{q_0}-1\right)\varphi'} w^q\,dx\right)^{\frac{q_0}{\varphi' p_0}} \left(\int_{\mathbb{R}^n} H_2^{\left(\frac{q}{s}\right)'} w^q\,dx\right)^{\frac{q_0}{\varphi p_0}}. \\
&\le C_2\left(\int_{\mathbb{R}^n} g^{p_0\varphi'} w^{q\left(\frac{p_0}{q_0}-1\right)\varphi'} w^q\,dx\right)^{\frac{q_0}{\varphi' p_0}} \\
&= C_2\left(\int_{\mathbb{R}^n} g^p w^p\,dx\right)^{\frac{q_0}{p_0}},
\end{aligned}$$

where in the last equality we have used that

$$\begin{aligned}
q\Big(\frac{p_0}{q_0}-1\Big)\varphi' + q &= q\Big(\frac{p_0}{q_0}-1\Big)\frac{p}{p_0} + q = qp\Big(\frac{1}{q_0}-\frac{1}{p_0}\Big) + q \\
&= qp\Big(\frac{1}{q}-\frac{1}{p}\Big) + q = p.
\end{aligned}$$

Therefore, to complete the proof we need to show that we can construct a non-negative function H_2 such that

$$\|H_2\|_{L^{\left(\frac{q}{s}\right)'}(w^q)} \le C_2, \tag{3.18}$$

$$h_2 \le H_2; \tag{3.19}$$

and such that the weight $W = H_2^{\frac{1}{q_0}} w^{\frac{q}{q_0}}$ satisfies

$$W^{p_0\left(\frac{p_+}{p_-}\right)'} = H_2^{\frac{p_0}{q_0}\left(\frac{p_+}{p_-}\right)'} w^{\frac{qp_0}{q_0}\left(\frac{p_+}{p_-}\right)'} \in A_1. \tag{3.20}$$

We construct H_2 exactly as in the proof of Case I, and as before we have (3.18) and (3.19). It remains to show (3.20). By (3.5),

$$\frac{1}{\beta}\frac{p_0}{q_0}\Big(\frac{p_+}{p_-}\Big)' = \frac{\tau'}{\left(\frac{q}{s}\right)'}\frac{p_0}{q_0}\Big(\frac{p_+}{p_0}\Big)' = \frac{1}{\left(\frac{q}{s}\right)'}\frac{q-s}{s} = 1.$$

On the other hand, recalling that $p_0 = p_-$ and $s = q_0$ we obtain

$$q - \frac{\gamma}{\beta} = q - \frac{\sigma + q}{\left(\frac{q}{s}\right)'} = q_0 - \frac{p}{\left(\frac{q}{s}\right)'\left(\frac{p}{p_-} - 1\right)} = q_0 - q_0 \frac{\frac{1}{q_0} - \frac{1}{q}}{\frac{1}{p_0} - \frac{1}{p}} = 0.$$

Thus,

$$\begin{aligned} W^{p_0\left(\frac{p_+}{p_-}\right)'} &= H_2^{\frac{p_0}{q_0}\left(\frac{p_+}{p_-}\right)'} w^{\frac{qp_0}{q_0}\left(\frac{p_+}{p_-}\right)'} \\ &= \mathcal{R}_2(h_2^\beta w^\gamma)^{\frac{1}{\beta}\frac{p_0}{q_0}\left(\frac{p_+}{p_-}\right)'} w^{\frac{p_0}{q_0}\left(\frac{p_+}{p_-}\right)'\left(q - \frac{\gamma}{\beta}\right)} = \mathcal{R}_2(h_2^\beta w^\gamma) \in A_1, \end{aligned}$$

which concludes the proof of Case II. □

3.3 Proof of Theorem 1.8. Case III: $p_0 = p_+$ and $p_- > 0$

Fix $p_- < p < p_+$ and w such that $w^p \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$ and note that in this case $p_+ = p_0 < \infty$ and $q_+ = q_0$ by (1.14). We again follow the proof of Case I and we indicate the main changes. First, if we define s as in (3.4) and since (3.5) is also valid in this context, then $0 < s = q < q_+ = q_0$ by (1.14) and the fact that $p < p_+$.

We now argue as before, but in this case we do not need to use duality or introduce H_2. Since $s = q$, if we fix $\alpha = \frac{s}{\left(\frac{q_0}{s}\right)'}$, then by Hölder's inequality,

$$\begin{aligned} \|f\|_{L^q(w^q)}^{q_0} &= \left(\int_{\mathbb{R}^n} f^s H_1^{-\alpha} H_1^\alpha w^q \, dx\right)^{\frac{q_0}{q}} \\ &\le \left(\int_{\mathbb{R}^n} f^{q_0} H_1^{-\alpha\frac{q_0}{s}} w^q \, dx\right)\left(\int_{\mathbb{R}^n} H_1^{\alpha\left(\frac{q_0}{s}\right)'} w^q \, dx\right)^{\frac{q_0}{q\left(\frac{q_0}{s}\right)'}} \\ &\le C_1^{q_0/\left(\frac{q_0}{s}\right)'} \int_{\mathbb{R}^n} f^{q_0} H_1^{-\alpha\frac{q_0}{s}} w^q \, dx, \end{aligned}$$

where $0 < H_1 < \infty$ is in $L^q(w^q)$ with $\|H_1\|_{L^q(w^q)} \le C_1 < \infty$. We will determine the exact value below. If we also assume that $f \le H_1\|f\|_{L^q(w^q)}$, then

$$\begin{aligned} \int_{\mathbb{R}^n} f^{q_0} H_1^{-\alpha\frac{q_0}{s}} w^q \, dx &\le \|f\|_{L^q(w^q)}^{q_0} \int_{\mathbb{R}^n} H_1^{q_0} H_1^{-\alpha\frac{q_0}{s}} w^q \, dx \\ &= \|f\|_{L^q(w^q)}^{q_0} \int_{\mathbb{R}^n} H_1^s w^q \, dx \le C_1^q \|f\|_{L^q(w^q)}^{q_0} < \infty. \end{aligned}$$

Thus, we can apply (1.9) if we let $W^{q_0} = H_1^{-\alpha \frac{q_0}{s}} w^q$ and assume that $W^{p_0} \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'} = A_{\frac{p_+}{p_-}} \cap RH_\infty$:

$$\begin{aligned}
\|f\|_{L^q(w^q)}^{q_0} &\le C_1^{q_0/\left(\frac{q_0}{s}\right)'} \int_{\mathbb{R}^n} f^{q_0} W^{q_0}\,dx \\
&\le C\left(\int_{\mathbb{R}^n} g^{p_0} W^{p_0}\,dx\right)^{\frac{q_0}{p_0}} \\
&= C\left(\int_{\mathbb{R}^n} g^{p_0} H_1^{-\alpha\frac{p_0}{s}} w^{q\frac{p_0}{q_0}} w^{-q} w^q\,dx\right)^{\frac{q_0}{p_0}} \\
&\le C\|g\|_{L^p(w^p)}^{q_0}\left(\int_{\mathbb{R}^n} H_1^q w^q\,dx\right)^{\frac{q_0}{p_0}} \le C C_1^{\frac{q_0 q}{p_0}}\|g\|_{L^p(w^p)}^{q_0},
\end{aligned}$$

provided H_1 satisfies

$$g^{p_0} H_1^{-\alpha\frac{p_0}{s}} w^{q\left(\frac{p_0}{q_0}-1\right)} \le H_1^q \|g\|_{L^p(w^p)}^{p_0}.$$

To complete the proof we need to show that we can construct H_1 such that

$$\|H_1\|_{L^q(w^q)} \le C_1, \tag{3.21}$$

$$g^{p_0} H_1^{-\alpha\frac{p_0}{s}} w^{q\left(\frac{p_0}{q_0}-1\right)} \le H_1^q \|g\|_{L^p(w^p)}^{p_0}, \tag{3.22}$$

$$0 < H_1 < \infty, \quad f \le H_1 \|f\|_{L^q(w^q)}, \tag{3.23}$$

and such that the weight $W = H_1^{-\frac{\alpha}{s}} w^{\frac{q}{q_0}}$ satisfies

$$W^{p_0} = H_1^{-\frac{\alpha p_0}{s}} w^{\frac{q p_0}{q_0}} \in A_{\frac{p_0}{p_-}} \cap RH_\infty. \tag{3.24}$$

Since $\alpha\frac{p_0}{s} = \frac{p_0}{q_0}(q_0 - s)$, (3.22) is equivalent to

$$g^{p_0} w^{q\left(\frac{p_0}{q_0}-1\right)} \le H_1^{q+\frac{p_0}{q_0}(q_0-s)} \|g\|_{L^p(w^p)}^{p_0}. \tag{3.25}$$

Using the fact that $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and that $s = q$ we have that

$$q + \frac{p_0}{q_0}(q_0 - s) = q + p_0 q\left(\frac{1}{q} - \frac{1}{q_0}\right) = q + p_0 q\left(\frac{1}{p} - \frac{1}{p_0}\right) = q\frac{p_0}{p}.$$

Similarly, we have that

$$q\left(\frac{p_0}{q_0} - 1\right) = q p_0\left(\frac{1}{q_0} - \frac{1}{p_0}\right) = q p_0\left(\frac{1}{q} - \frac{1}{p}\right) = p_0\left(1 - \frac{q}{p}\right).$$

Therefore, (3.25) (and hence (3.22)) is equivalent to

$$g^{\frac{p}{q}} w^{\frac{p}{q}-1} \leq H_1 \|g\|_{L^p(w^p)}^{\frac{p}{q}}. \tag{3.26}$$

We now construct H_1 exactly as in the proof of Case I, and we obtain as before (3.23), (3.26), and (3.21). It remains to show (3.24). By (3.4)

$$\frac{\alpha p_0}{\delta s} = \frac{\tau p_0}{q\left(\frac{q_0}{s}\right)'} = \frac{1}{\left(\frac{q_0}{s}\right)'} \frac{\frac{p_0}{p_-} - 1}{\frac{q_0 - s}{q_0}} = \frac{p_0}{p_-} - 1,$$

and also, since $p_0 = p_+$,

$$\begin{aligned}
\frac{\epsilon \alpha p_0}{\delta s} &= \left(\frac{p_0}{p_-} - 1\right)\epsilon = \left(\frac{p_0}{p_-} - 1\right)\left(\frac{q}{\tau} - \frac{p\left(\frac{p_+}{p}\right)'}{\tau}\right) \\
&= p_0\left(\frac{1}{p_-} - \frac{1}{p_0}\right)\left(q\frac{\frac{1}{p} - \frac{1}{p_+}}{\frac{1}{p_-} - \frac{1}{p_+}} - \frac{1}{\frac{1}{p_-} - \frac{1}{p_+}}\right) = qp_0\left(\frac{1}{p} - \frac{1}{p_0} - \frac{1}{q}\right) \\
&= -\frac{qp_0}{q_0}.
\end{aligned}$$

Together, these imply that

$$W^{p_0} = H_1^{-\frac{\alpha p_0}{s}} w^{\frac{qp_0}{q_0}} = \mathcal{R}_1(h_1^\delta w^\epsilon)^{1 - \frac{p_0}{p_-}} \in A_{\frac{p_0}{p_-}} \cap RH_\infty;$$

the inclusion follows from Lemma 2.3 and the fact that $\mathcal{R}_1(h_1^\delta w^\epsilon) \in A_1$. This completes the proof of Case III. □

3.4 Proof of Theorem 1.8. Case IV: $p_- = 0$ and $p_- < p_0 \leq p_+$

In this case we adapt ideas from [29, Section 3.1]. Fix p, q such that $0 = p_- < p < p_+$, $0 < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and let v be such that $v^p \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'} = RH_{\left(\frac{p_+}{p}\right)'}$. Since $RH_1 = A_\infty$, there exists $0 < \epsilon < \min\{p_0, p\}$ such that $v^p \in A_{\frac{p}{\epsilon}}$. Set $\widetilde{p}_- = \epsilon > 0$; then $\widetilde{p}_- < p_0 \leq p_+$ and (1.9) holds for all $w^{p_0} \in A_{\frac{p_0}{\widetilde{p}_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'} \subset RH_{\left(\frac{p_+}{p_0}\right)'} = A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$. Thus, we can use Cases I and III with $\widetilde{p}_- > 0$ in place of p_- to conclude that (1.10) holds for every $\widetilde{p}, \widetilde{q}$ such that $\widetilde{p}_- < \widetilde{p} < p_+$, $0 < \widetilde{q} < \infty$ and $\frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}} = \frac{1}{p_0} - \frac{1}{q_0}$, and every weight w such that $w^{\widetilde{p}} \in A_{\frac{\widetilde{p}}{\widetilde{p}_-}} \cap RH_{\left(\frac{p_+}{\widetilde{p}}\right)'}$. If we take $\widetilde{p} = p$, $\widetilde{q} = q$ and $w = v$, our choice of ϵ guarantees that $\widetilde{p}_- = \epsilon < p < p_+$, $0 < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$. Moreover, $v^p \in A_{\frac{p}{\epsilon}} \cap RH_{\left(\frac{p_+}{p}\right)'} = A_{\frac{\widetilde{p}}{\widetilde{p}_-}} \cap RH_{\left(\frac{p_+}{\widetilde{p}}\right)'}$. Thus, (1.10) holds and the proof of Case IV is complete. □

3.5 Proof of Theorem 1.3

Our proof of Theorem 1.3 is a modification of the proof of multilinear extrapolation in [18, Theorem 6.1]. We include the details so that we can explain the use of families of extrapolation pairs. The essential idea is to reduce the problem to a linear one by acting on one function at a time.

For $2 \le j \le m$, fix weights w_j such that $w_j^{p_j} \in A_{\frac{p_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{p_j}\right)'}$. Fix functions f_j, $2 \le j \le m$, such that there exists functions f and g with $(f, g, f_2, \dots, f_m) \in \mathcal{F}$. Assume that for each j, $0 < \|f_j\|_{L^{p_j}(w_j^{p_j})} < \infty$. (We will remove this restriction below.) Define the new family of extrapolation pairs

$$\mathcal{F}_1 = \left\{ (F, g) = \left(f \prod_{j=2}^{m} w_j \|f_j\|^{-1}_{L^{p_j}\left(w_j^{p_j}\right)}, g \right) : (f, g, f_2, \dots, f_m) \in \mathcal{F} \right\}.$$

If $f \in L^p(w^p)$, then $F \in L^p(w_1^p)$, so by our hypothesis (1.4),

$$\|F\|_{L^p(w_1^p)} \le C\|g\|_{L^{p_1}\left(w_1^{p_1}\right)} \tag{3.27}$$

for all $w_1^{p_1} \in A_{\frac{p_1}{r_1^-}} \cap RH_{\left(\frac{r_1^+}{p_1}\right)'}$. Note that $p < p_1$ and so $\frac{1}{p} - \frac{1}{p_1} + \frac{1}{r_1^+} > 0$. Therefore, by Theorem 1.8, for all pairs $(F, g) \in \mathcal{F}_1$ with $\|F\|_{L^q(w_1^q)} < \infty$, and for all $r_1^- < q_1 < r_1^+$ and all $w_1^{q_1} \in A_{\frac{q_1}{r_1^-}} \cap RH_{\left(\frac{r_1^+}{q_1}\right)'}$,

$$\|F\|_{L^q\left(w_1^q\right)} \le C\|g\|_{L^{q_1}\left(w_1^{q_1}\right)},$$

where $\frac{1}{q} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{p_1}$ and so $\frac{1}{q} = \frac{1}{q_1} + \sum_{j=2}^{m} \frac{1}{p_j}$. Therefore, by our definition of F, $\|f\|_{L^q(w^q)} < \infty$ and we can rewrite this as

$$\|f\|_{L^q(w^q)} \le C\|g\|_{L^{q_1}\left(w_1^{q_1}\right)} \prod_{j=2}^{m} \|f_j\|_{L^{p_j}\left(w_1^{p_j}\right)}.$$

This inequality still holds even if we remove the restriction $0 < \|f_j\|_{L^{p_j}(w_j^{p_j})} < \infty$. If for some j, $\|f_j\|_{L^{p_j}(w_j^{p_j})} = \infty$, this inequality clearly holds; if $\|f_j\|_{L^{p_j}(w_j^{p_j})} = 0$, then (1.4) implies that $f = F = 0$, and this inequality again holds.

We can repeat this argument for any such collection of f_j, $2 \le j \le m$. Therefore, we have shown that for all $(f_1, \dots, f_m) \in \mathcal{F}$ with $f \in L^q(w^q)$,

$$\|f\|_{L^q(w^q)} \le C\|f_1\|_{L^{q_1}\left(w_1^{q_1}\right)} \prod_{j=2}^{m} \|f_j\|_{L^{p_j}\left(w_1^{p_j}\right)}.$$

To complete the proof, fix $f_1, f_3, \dots f_m$, and repeat the above argument in the second coordinate, etc. Then by induction we get the desired conclusion.

We now prove the vector-valued inequalities (1.6). The extension of scalar inequalities to vector-valued inequalities via extrapolation is well-known in the linear case: see [10, Corollary 3.12]. The argument is nearly the same in the multilinear setting. Fix $s_j, r_j^- < s_j < r_j^+$, for $1 \le j \le m$ and set $\frac{1}{s} = \sum_{j=1}^m \frac{1}{s_j}$. Define a new family

$$\widetilde{\mathcal{F}} = \Bigg\{(F, F_1, \dots, F_m) = \Bigg(\Big(\sum_k (f^k)^s\Big)^{\frac{1}{s}}, \Big(\sum_k (f_1^k)^{s_1}\Big)^{\frac{1}{s_1}}, \dots, \\ \Big(\sum_k (f_m^k)^{s_m}\Big)^{\frac{1}{s_m}}\Bigg) : \{(f^k, f_1^k, \dots, f_m^k)\}_k \subset \mathcal{F}\Bigg\}.$$

Without loss of generality we may assume that all of the sums in the definition of $\widetilde{\mathcal{F}}$ are finite; the conclusion for infinite sums follows by the monotone convergence theorem. Then, given any collection of weights $w_1, \dots, w_m$ with $w_j^{s_j} \in A_{\frac{s_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{s_j}\right)'}$ and $w = w_1 \cdots w_m$, if $\|F\|_{L^s(w^s)} < \infty$, then by (1.5) we have that

$$\begin{aligned} \|F\|_{L^s(w^s)} &= \Big(\sum_k \|f^k\|^s_{L^s(w^s)}\Big)^{\frac{1}{s}} \le C \Big(\sum_k \prod_{j=1}^m \|f_j^k\|^s_{L^{s_j}(w_j^{s_j})}\Big)^{\frac{1}{s}} \\ &\le C \prod_{j=1}^m \Big(\sum_k \|f_j^k\|^{s_j}_{L^{s_j}(w_j^{s_j})}\Big)^{\frac{1}{s_j}} = C \prod_{j=1}^m \|F_j\|_{L^{s_j}(w_j^{s_j})}, \end{aligned} \tag{3.28}$$

where in the second estimate we used Hölder's inequality with respect to sums. We can now apply the first part of Theorem 1.3 to $\widetilde{F}$, where we use (3.28) for the initial estimate in place of (1.4). We thus get

$$\|F\|_{L^q(w^q)} \le C \prod_{j=1}^m \|F_j\|_{L^{q_j}\left(w_j^{q_j}\right)}. \tag{3.29}$$

for all exponents $q_j, r_j^- < q_j < r_j^+$, all weights $w_j^{q_j} \in A_{\frac{q_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{q_j}\right)'}$, $w = w_1 \cdots w_m$, and $\frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}$. Inequality (3.29) holds for all $(F, F_1, \dots, F_m) \in \widetilde{\mathcal{F}}$ for which $\|F\|_{L^q(w^q)} < \infty$. But this is exactly (1.6) and the proof is complete. □

3.6 Proof of Corollaries 1.11 and 1.12

We will prove Corollary 1.12; the proof of Corollary 1.11 is identical. The proof follows as in [29, Section 3.1]. Given a family of extrapolation pairs $\mathcal{F}$ as in the statement and any $N > 0$, define the new family

$$\mathcal{F}_N := \big\{(f_N, g) : (f, g) \in \mathcal{F},\ f_N := f\chi_{\{x\in B(0,N): f(x)\le N\}}\big\}.$$

Note that for all $0 < r < \infty$ and $w^r \in A_\infty$,

$$\|f_N\|^r_{L^r(w^r)} \le N^r w^r(B(0, N)) < \infty. \tag{3.30}$$

Since $f_N \le f$, by our hypothesis we get that (1.9) holds for every pair in $\mathcal{F}_N$ (with a constant independent of N) with a left-hand side that is always finite by (3.30) and Remark 1.15. Therefore, we can apply Theorem 1.8 to $\mathcal{F}_N$ to conclude that (1.10) holds for every pair $(f_N, g) \in \mathcal{F}_N$ (with a constant that is again independent of N), since again the left-hand side is always finite. The desired inequality follows at once if we let $N \to \infty$ and apply the monotone convergence theorem. □

4 Proofs of the applications

We now prove Theorems 1.18, 1.29, and 1.39, and Corollary 1.23. We also sketch the ideas needed to prove the result in Remark 1.33.

4.1 Proof of Theorem 1.18

We start with the first part of the theorem. Let $p_1, p_2 \in (1,\infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, fix $w_1^{2p_1} \in A_{p_1}$, $w_2^{2p_2} \in A_{p_2}$, and let $w = w_1w_2$. Then by Theorem 1.16, $BH : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^p(w^p)$. By Lemma 2.1, $w_i^{2p_i} \in A_{p_i}$ if and only if $w^{p_i} \in A_{\frac{p_i+1}{2}} \cap RH_2$. Thus, if we set $r_i^- = \frac{2p_i}{p_i+1}$ and $r_i^+ = 2p_i$, then $1 < r_i^- < p_i < r_i^+ < \infty$ and $w_i^{p_i} \in A_{\frac{p_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{p_i}\right)'}$. We can then apply Corollary 1.11 to the family

$$\mathcal{F} = \big\{(|BH(f, g)|, |f|, |g|) : f, g \in L_c^\infty\big\}$$

to conclude that for all $r_i^- < q_i < r_i^+$ and $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$, the bilinear Hilbert transform BH is bounded from $L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2})$ into $L^q(w^q)$ where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $w = w_1w_2$. (Here we use the fact that L_c^∞ is dense any space $L^r(w^r)$ if w^r is locally integrable, and the fact that BH is bilinear to extend the inequality on triples in $\mathcal{F}$ that we get from Theorem 1.3 to all of $L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2})$.) Again by Lemma 2.1, the conditions on the weights are equivalent to $w_i^{2r_i} \in A_{r_i}$, where $r_i = \left(\frac{2}{q_i} - \frac{1}{p_i}\right)^{-1}$. Note that $1 < r_i < \infty$ since $r_i^- < q_i < r_i^+$. This completes the proof of the first part of Theorem 1.18.

To prove the second part of the theorem, fix $1 < q_1, q_2 < \infty$ such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$. We want to use the previous argument: therefore, we need to find

$p_1, p_2 \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$ and $r_i^- < q_i < r_i^+$, where

$$\frac{1}{r_i^+} = \frac{1}{2p_i} < \frac{1}{q_i} < \frac{1}{r_i^-} = \frac{1}{2p_i} + \frac{1}{2}. \tag{4.1}$$

Since $1 < p_1, p_2 < \infty$, this can be rewritten as

$$0 \le 2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} - \frac{1}{2}\right) < \frac{1}{p_i} < 2\min\left\{\frac{1}{2}, \frac{1}{q_i}\right\} \le 1. \tag{4.2}$$

Before choosing p_1, p_2 we claim that

$$\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} < \frac{3}{2}. \tag{4.3}$$

To see that this holds, note that

$$\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} = \begin{cases} 1 & \text{if } \max\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\} \le \frac{1}{2}, \\ \frac{1}{2} + \max\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\} & \text{if } \min\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\} \le \frac{1}{2} \le \max\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\}, \\ \frac{1}{q_1} + \frac{1}{q_2} & \text{if } \min\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\} > \frac{1}{2}. \end{cases}$$

and in every case this is strictly smaller than $\frac{3}{2}$ since $q_1, q_2 > 1$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$.

Now define

$$\frac{1}{p_i} := 2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} - \frac{1}{2} + \eta_i\right), \qquad i = 1, 2, \tag{4.4}$$

where we fix $\eta_1, \eta_2 > 0$ so that

$$\eta_1 + \eta_2 < \frac{3}{2} - \sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} \quad \text{and} \quad 0 < \eta_i < \min\left\{\frac{1}{q_i}, \frac{1}{q_i'}\right\}, \quad i = 1, 2. \tag{4.5}$$

That we can find such η_1, η_2 follows from (4.3). (As will be clear from the proof, we can choose η_i as close to 0 as we want; we will use this fact in the proof of Corollary 1.23 below.)

With this choice we claim that (4.2) holds and also that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$. We first prove the latter inequality: by the first condition in (4.5),

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = -2 + 2\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}\right\} + 2\sum_{i=1}^{2} \eta_i < 1.$$

To prove (4.2) we first observe that since $\eta_i > 0$,

$$2\left(\max\left\{\frac{1}{2},\frac{1}{q_i}\right\}-\frac{1}{2}\right) < 2\left(\max\left\{\frac{1}{2},\frac{1}{q_i}\right\}-\frac{1}{2}+\eta_i\right) = \frac{1}{p_i}.$$

To obtain the other half of (4.2) we consider two cases. If $\max\{\frac{1}{2},\frac{1}{q_i}\} = \frac{1}{2}$, then

$$\frac{1}{p_i} = 2\eta_i < \frac{2}{q_i} = 2\min\left\{\frac{1}{2},\frac{1}{q_i}\right\}.$$

On the other hand, if $\max\{\frac{1}{2},\frac{1}{q_i}\} = \frac{1}{q_i}$, then

$$\frac{1}{p_i} = \frac{2}{q_i} - 1 + 2\eta_i < \frac{2}{q_i} - 1 + \frac{2}{q_i'} = 1 = 2\min\left\{\frac{1}{2},\frac{1}{q_i}\right\}.$$

This completes the proof of (4.2) and hence the proof of Theorem 1.18. □

4.2 Proof of Corollary 1.23

This result follows by considering more carefully the proof of Theorem 1.18. Fix $1 < q_1, q_2 < \infty$ such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$ and $w_i^{q_i} \in A_{\max\{1,\frac{q_i}{2}\}} \cap RH_{\max\{1,\frac{2}{q_i}\}}$. We now choose p_i as in (4.4) and (4.5), though below we will take η_i much smaller. As we showed above, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, and (4.2) holds. Hence, (4.1) holds and so by the first part of Theorem 1.18, we get that the bilinear Hilbert transform is bounded from $L^{q_1}(u_1^{q_1}) \times L^{q_2}(u_2^{q_2})$ into $L^q(u^q)$ where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $u = u_1u_2$, for all $u_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{(\frac{r_i^+}{q_i})'}$, with

$$\frac{q_i}{r_i^-} = q_i\left(\frac{1}{2p_i}+\frac{1}{2}\right) = q_i\left(\max\left\{\frac{1}{2},\frac{1}{q_i}\right\}+\eta_i\right) = \max\left\{1,\frac{q_i}{2}\right\} + q_i\eta_i,$$

and

$$\begin{aligned}\frac{1}{(\frac{r_i^+}{q_i})'} &= 1 - \frac{q_i}{r_i^+} = 1 - \frac{q_i}{2p_i} = 1 - q_i\left(\max\left\{\frac{1}{2},\frac{1}{q_i}\right\}-\frac{1}{2}+\eta_i\right)\\ &= \min\left\{1,\frac{q_i}{2}\right\} - q_i\eta_i.\end{aligned}$$

Note that $w_i^{q_i} \in A_{\max\{1,\frac{q_i}{2}\}}$ immediately implies that $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}}$. On the other hand, since $w_i^{q_i} \in RH_{\max\{1,\frac{2}{q_i}\}}$, by the openness of the reverse Hölder classes we can find $0 < \theta < 1$ close to 1 such that $w_i^{q_i} \in RH_{\frac{1}{\theta}\max\{1,\frac{2}{q_i}\}}$. Therefore, in choosing the η_i we assume that (4.5) holds and that $0 < \eta_i < (1-\theta)\min\left\{1, \frac{q_i}{2}\right\}$. But then

$$\frac{1}{\left(\frac{r_i^+}{q_i}\right)'} = \min\left\{1, \frac{q_i}{2}\right\} - q_i\eta_i > \min\left\{1, \frac{q_i}{2}\right\} - q_i(1-\theta)\min\left\{\frac{1}{2}, \frac{1}{q_i}\right\}$$
$$= \theta\min\left\{1, \frac{q_i}{2}\right\}.$$

Hence $\left(\frac{r_i^+}{q_i}\right)' < \frac{1}{\theta}\max\{1, \frac{2}{q_i}\}$ which gives that $w_i^{q_i} \in RH_{\left(\frac{r_i^+}{q_i}\right)'}$. We have thus shown that $w^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$, which implies that the bilinear Hilbert transform is bounded from $L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2})$ into $L^q(w^q)$. This completes the proof of (1.24).

Finally, let $w_i(x) = |x|^{-\frac{a}{q_i}}$ so that $w(x) = w_1(x)w_2(x) = |x|^{-\frac{a}{q}}$. Then, using the well known properties of power weights, we have that $w_i^{q_i} \in A_{\max\{1, \frac{q_i}{2}\}} \cap RH_{\max\{1, \frac{2}{q_i}\}}$ if and only if

$$1 - \max\left\{1, \frac{q_i}{2}\right\} < a < 1 \qquad \text{and} \qquad -\infty < a < \frac{1}{\max\left\{1, \frac{2}{q_i}\right\}} = \min\left\{1, \frac{q_i}{2}\right\},$$

and when $\max\{1, \frac{q_i}{2}\} = 1$ we can also allow $a = 0$ in the first condition. From all these we easily see that (1.25) holds provided either $a = 0$ or a satisfies (1.26). This completes the proof. □

4.3 Proof of Theorem 1.29

The proof of the first part of Theorem 1.29 is now straightforward given Theorem 1.3 and Corollary 1.11. Indeed, Theorem 1.18 provides the initial weighted norm inequalities for the family

$$\mathcal{F} = \{(|BH(f, g)|, |f|, |g|) : f, g \in L_c^\infty.\}$$

(see the proof of Theorem 1.18). Thus, Corollary 1.11 applies and (1.6) yields (1.30) for functions $f_k, g_k \in L_c^\infty$. By a standard approximation argument we get the desired inequality for $f_k \in L^{q_1}(w_1^{q_1})$ and $g_k \in L^{q_2}(w_2^{q_2})$.

To prove the second part of Theorem 1.29 we modify the argument in the second part of the proof of Theorem 1.18. Fix q_i, s_i as in the statement; then by the first part of Theorem 1.29 we need to find $1 < p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$ and $r_i^- < q_i < r_i^+$ with

$$\frac{1}{r_i^+} = \frac{1}{2p_i} < \frac{1}{q_i}, \frac{1}{s_i} < \frac{1}{r_i^-} = \frac{1}{2p_i} + \frac{1}{2}. \tag{4.6}$$

Since $1 < p_1, p_2 < \infty$, (4.6) can be rewritten as

$$0 \le 2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} - \frac{1}{2}\right) < \frac{1}{p_i} < 2\min\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} \le 1. \tag{4.7}$$

Before choosing p_1, p_2, we first claim that

$$\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} < \frac{3}{2}. \tag{4.8}$$

To show this we argue as we did to prove (4.3): if at least one of the maxima is $\frac{1}{2}$, then since the other maxima is strictly smaller than 1 we get the desired estimate. If none of the maxima is $\frac{1}{2}$, then by the last condition in (1.31),

$$\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} = \sum_{i=1}^{2} \max\left\{\frac{1}{q_i}, \frac{1}{s_i}\right\} < \frac{3}{2}.$$

We now choose p_i: fix $\eta_i > 0$ and let

$$\frac{1}{p_i} := 2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} - \frac{1}{2} + \eta_i\right), \qquad i = 1, 2, \tag{4.9}$$

where we choose the η_i sufficiently small so that

$$\eta_1 + \eta_2 < \frac{3}{2} - \sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} \tag{4.10}$$

and

$$0 < \eta_i < \min\left\{\frac{1}{q_i}, \frac{1}{q_i'}, \frac{1}{s_i}, \frac{1}{s_i'}, \frac{1}{2} - \left|\frac{1}{s_i} - \frac{1}{q_i}\right|\right\}. \tag{4.11}$$

Such a choice of η_1, η_2 is possible by (4.8) and (1.31). By (4.10) we have that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = -2 + 2\sum_{i=1}^{2} \max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} + 2\sum_{i=1}^{2} \eta_i < 1.$$

To prove (4.7) we first observe that since $\eta_i > 0$,

$$2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} - \frac{1}{2}\right) < 2\left(\max\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\} - \frac{1}{2} + \eta_i\right) = \frac{1}{p_i}.$$

To get the second estimate in (4.7) we consider two cases. If $\max\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\} = \frac{1}{2}$, then

$$\frac{1}{p_i} = 2\eta_i < 2\min\left\{\frac{1}{q_i}, \frac{1}{s_i}\right\} = 2\min\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\}.$$

On the other hand, if $\max\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\} = \max\{\frac{1}{q_i}, \frac{1}{s_i}\}$ and we write $\frac{1}{\alpha_i} = \max\{\frac{1}{q_i}, \frac{1}{s_i}\}$ and $\frac{1}{\beta_i} = \max\{\frac{1}{q_i}, \frac{1}{s_i}\}$, we obtain

$$\begin{aligned}
\frac{1}{p_i} &= 2\max\left\{\frac{1}{q_i}, \frac{1}{s_i}\right\} - 1 + 2\eta_i \\
&< 2\max\left\{\frac{1}{q_i}, \frac{1}{s_i}\right\} - 1 + 2\min\left\{\frac{1}{q_i'}, \frac{1}{s_i'}, \frac{1}{2} - \left|\frac{1}{s_i} - \frac{1}{q_i}\right|\right\} \\
&= \frac{2}{\alpha_i} - 1 + 2\min\left\{\frac{1}{\alpha_i'}, \frac{1}{2} - \left(\frac{1}{\alpha_i} - \frac{1}{\beta_i}\right)\right\} \\
&= 2\min\left\{\frac{1}{2}, \frac{1}{\beta_i}\right\} \\
&= 2\min\left\{\frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}\right\}.
\end{aligned}$$

This completes the proof of (4.7) and hence that of Theorem 1.29. □

4.4 Proof of Remark 1.33

To prove the iterated vector-valued inequality in Remark 1.33, we simply repeat the argument used to prove the first part of Theorem 1.29. For our starting estimate we form the new family

$$\mathcal{F}' = \Bigg\{(h, f, g) = \left(\left(\sum_k |BH(f_k, g_k)|^s\right)^{\frac{1}{s}}, \left(\sum_k |f_k|^{s_1}\right)^{\frac{1}{s_1}}, \left(\sum_k |g_k|^{s_2}\right)^{\frac{1}{s_2}}\right) : f_k, g_k \in L_c^\infty\Bigg\};$$

then (1.30) gives us the starting estimate

$$\|h\|_{L^q(w^q)} \leq C\|f\|_{L^{q_1}\left(w_1^{q_1}\right)}\|g\|_{L^{q_2}\left(w_2^{q_2}\right)}.$$

We then again apply vector-valued extrapolation using the family

$$\mathcal{F}'' = \Bigg\{(H, F, G)$$

$$= \left\{ \left(\Big(\sum_j h_j^t\Big)^{\frac{1}{t}}, \Big(\sum_j f_j^{t_1}\Big)^{\frac{1}{t_1}}, \Big(\sum_j g_j^{t_2}\Big)^{\frac{1}{t_2}} \right) : (h_j, f_j, g_j) \in \mathcal{F}' \right\}$$

to get iterated vector-valued inequalities. Details are left to the interested reader. □

4.5 Proof of Corollary 1.34

Similar to our approach in the proof of Corollary 1.23, here we take a closer look at the proof of Theorem 1.29. Fix $1 < q_1, q_2, s_1, s_2 < \infty$ and $w_i^{q_i}$ as in the statement. We choose p_i as in (4.9), (4.10) and (4.11), but again we will choose η_i much smaller. Then as we proved above, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, and (4.7) holds. Note that the latter implies (4.6) and hence, by the first part of Theorem 1.29, we obtain that the bilinear Hilbert transform satisfies (1.30), provided we show that $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{(\frac{r_i^+}{q_i})'}$, where

$$\frac{q_i}{r_i^-} = q_i \left(\frac{1}{2p_i} + \frac{1}{2} \right) = q_i \left(\max\left\{ \frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i} \right\} + \eta_i \right) = \max\left\{ 1, \frac{q_i}{2}, \frac{q_i}{s_i} \right\} + q_i \eta_i$$

and

$$\begin{aligned} \frac{1}{(\frac{r_i^+}{q_i})'} &= 1 - \frac{q_i}{r_i^+} = 1 - \frac{q_i}{2p_i} = 1 - q_i \left(\max\left\{ \frac{1}{2}, \frac{1}{q_i}, \frac{1}{s_i}, \right\} - \frac{1}{2} + \eta_i \right) \\ &= \min\left\{ 1, \frac{q_i}{2}, 1 - q_i \left(\frac{1}{s_1} - \frac{1}{2} \right) \right\} - q_i \eta_i . \end{aligned}$$

Note that $w_i^{q_i} \in A_{\max\{1, \frac{q_i}{2}, \frac{q_i}{s_i}\}}$ immediately gives us that $w_i^{q_i} \in A_{\frac{q_i}{r_i^-}}$. On the other hand, since $w_i^{q_i} \in RH_{\max\{1, \frac{2}{q_i}, [1 - q_i(\frac{1}{s_i} - \frac{1}{2})]^{-1}\}}$, by the openness of the reverse Hölder classes we can find $0 < \theta < 1$ close to 1 such that $w_i^{q_i} \in RH_{\frac{1}{\theta}\max\{1, \frac{2}{q_i}, [1 - q_i(\frac{1}{s_i} - \frac{1}{2})]^{-1}\}}$. We therefore assume, in addition to (4.10), (4.11), that $0 < \eta_i < (1 - \theta) \min\left\{ \frac{1}{2}, \frac{1}{q_i}, \frac{1}{q_i} - \frac{1}{s_i} + \frac{1}{2} \right\}$; this choice is possible because of the first two conditions in (1.35). But then

$$\begin{aligned} \frac{1}{(\frac{r_i^+}{q_i})'} &= \min\left\{ 1, \frac{q_i}{2}, 1 - q_i \left(\frac{1}{s_1} - \frac{1}{2} \right) \right\} - q_i \eta_i \\ &> \min\left\{ 1, \frac{q_i}{2}, 1 - q_i \left(\frac{1}{s_1} - \frac{1}{2} \right) \right\} - q_i (1 - \theta) \min\left\{ \frac{1}{2}, \frac{1}{q_i}, \frac{1}{q_i} - \frac{1}{s_i} + \frac{1}{2} \right\} \\ &= \theta \min\left\{ 1, \frac{q_i}{2}, 1 - q_i \left(\frac{1}{s_i} - \frac{1}{2} \right) \right\}. \end{aligned}$$

Hence $\left(\frac{r_i^+}{q_i}\right)' < \frac{1}{\theta}\max\{1, \frac{2}{q_i}, [1-q_i(\frac{1}{s_i}-\frac{1}{2})]^{-1}\}$, so $w_i^{q_i} \in RH_{\left(\frac{r_i^+}{q_i}\right)'}$. We have thus shown that $w^{q_i} \in A_{\frac{q_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{q_i}\right)'}$ which yields (1.30).

To complete our proof we need to establish (1.36). Let $w_i(x) = |x|^{-\frac{a}{q_i}}$ so that $w(x) = w_1(x)w_2(x) = |x|^{-\frac{a}{q}}$. Then, using the well known properties of power weights, we have that $w_i^{q_i} \in A_{\max\{1,\frac{q_i}{2},\frac{q_i}{s_i}\}} \cap RH_{\max\{1,\frac{2}{q_i},[1-q_i(\frac{1}{s_i}-\frac{1}{2})]^{-1}\}}$ if and only if

$$1 - \max\left\{1, \frac{q_i}{2}, \frac{q_i}{s_i}\right\} < a < 1;$$

when $\max\{1, \frac{q_i}{2}\} = 1$ we can also allow $a = 0$, and

$$-\infty < a < \frac{1}{\max\left\{1, \frac{2}{q_i}, \left[1-q_i\left(\frac{1}{s_i}-\frac{1}{2}\right)\right]^{-1}\right\}} = \min\left\{1, \frac{q_i}{2}, 1-q_i\left(\frac{1}{s_i}-\frac{1}{2}\right)\right\}.$$

From all these estimates we see that (1.36) holds provided $a \in \{0\} \cup (a_-, a_+)$ with $a_\pm$ defined in (1.37). This completes the proof. □

4.6 Proof of Theorem 1.42

The desired result follows directly from extrapolation. Fix $1 < r < 2$ and define the family of $(m+1)$-tuples

$$\begin{aligned}\mathcal{F} = \Bigg\{&(F, F_1, \dots, F_m)\\ &= \left(\left(\sum_{k_1,\dots,k_m}\left|T\left(f_{k_1}^1,\dots,f_{k_m}^m\right)\right|^r\right)^{\frac{1}{r}}, \left(\sum_{k_1}\left|f_{k_1}^1\right|^r\right)^{\frac{1}{r}}, \dots,\right.\\ &\left.\left(\sum_{k_m}\left|f_{k_m}^m\right|^r\right)^{\frac{1}{r}}\right) : f_{k_j}^j \in L_c^\infty\Bigg\}.\end{aligned}$$

Now fix $1 < q_1, \dots, q_m < r < 2$ and let $\frac{1}{q} = \sum \frac{1}{q_j}$. Then by Theorem 1.39, for all weights w_j such that weights $w_j^{q_i} \in A_{q_j}$, and $(F, F_1, \dots, F_m) \in \mathcal{F}$,

$$\|F\|_{L^q(w^q)} \le C \prod_{j=1}^m \|F_j\|_{L^{q_j}\left(w_j^{q_j}\right)}. \tag{4.12}$$

Therefore, by Corollary 1.11 applied with $r_j^- = 1$, $r_j^+ = \infty$, $1 \le j \le m$, we immediately conclude that for any $1 < q_1, \dots, q_m < \infty$ and weights $w_j^{q_i} \in A_{q_j}$, inequality (4.12) holds, which yields (1.41) for functions in L^∞_c. The desired inequality then follows for $f^j_{k_j} \in L^{q_j}(w_j^{q_j})$ by a standard approximation argument. □

5 More general vector-valued inequalities

In this section we explain how to obtain, via extrapolation, vector-valued inequalities in a larger range than we proved in Theorem 1.29. The starting point is implicit in the proof of [14, Corollary 4]: from it one can show that (1.17) holds provided

$$w_i^{p_i} \in A_{1+(1-\theta_i)(p_i-1)} \cap RH_{\frac{1}{1-\theta_3}}, \tag{5.1}$$

where $1 < p_1, p_2, p < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and where $\theta_1, \theta_2, \theta_3 \in (0, 1)$ are arbitrary parameters satisfying

$$\frac{\theta_1}{p_1'} \le \frac{1}{2}, \qquad \frac{\theta_2}{p_2'} \le \frac{1}{2}, \qquad \frac{\theta_3}{p} \le \frac{1}{2}, \qquad \frac{\theta_1}{p_1'} + \frac{\theta_2}{p_2'} + \frac{\theta_3}{p} = 1. \tag{5.2}$$

In [14] the authors chose $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}$, which then gives Theorem 1.16.

If we now fix the parameters $\theta_1, \theta_2, \theta_3 \in (0, 1)$, we can rewrite (5.1) as

$$w_i^{p_i} \in A_{\frac{p_i}{r_i^-}} \cap RH_{\left(\frac{r_i^+}{p_i}\right)'}, \qquad \text{where} \quad \frac{1}{r_i^-} = 1 - \frac{\theta_i}{p_i'} \quad \text{and} \quad \frac{1}{r_i^+} = \frac{\theta_3}{p_i}. \tag{5.3}$$

Given this, we can apply our extrapolation result to obtain vector-valued inequalities by varying p_1, p_2, p and $\theta_1, \theta_2, \theta_3$. We claim that, as a result, (1.30) holds (taking $w_1 = w_2 \equiv 1$ for simplicity, but of course some natural weighted norm inequalities are also possible) whenever $1 < s_1, s_2, q_1, q_2 < \infty$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} < \frac{3}{2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}$ and if there exist $0 \le \gamma_1, \gamma_2, \gamma_3 < 1$ with $\gamma_1 + \gamma_2 + \gamma_3 = 1$ such that

$$\max\left\{\frac{1}{s_1}, \frac{1}{q_1}\right\} < \frac{1+\gamma_1}{2}, \quad \max\left\{\frac{1}{s_2}, \frac{1}{q_2}\right\} < \frac{1+\gamma_2}{2}, \quad \max\left\{\frac{1}{s'}, \frac{1}{p'}\right\} < \frac{1+\gamma_3}{2}, \tag{5.4}$$

and, additionally,

$$\min\left\{\frac{1}{s_1}, \frac{1}{q_1}\right\} + \min\left\{\frac{1}{s_2}, \frac{1}{q_2}\right\} > \frac{1-\gamma_3}{2}. \tag{5.5}$$

Note that in (5.4) it could be that $p \le 1$ (or analogously $s \le 1$), in which case $\frac{1}{p'} = 1 - \frac{1}{p} \le 0$. If we compare our conditions with those in [5, Theorem 5] (see also [14, Appendix A]), we see that ours impose the extra restrictions (5.5) and $s_i, q_i < \infty$. Also, note that the last condition in (5.4) is implied by (5.5); nevertheless we make it explicit in order to compare our conditions with those of [5, Theorem 5].

We now sketch how to prove our claim. Define

$$m_1 = \min\left\{\frac{1}{s_1}, \frac{1}{q_1}\right\}, \quad m_2 = \min\left\{\frac{1}{s_2}, \frac{1}{q_2}\right\}, \quad \widetilde{m}_1 = \frac{2}{1-\gamma_3} m_1, \quad \widetilde{m}_2 = \frac{2}{1-\gamma_3} m_2.$$

With this notation, (5.5) becomes $\widetilde{m}_1 + \widetilde{m}_2 > 1$. The first step is to show that there exist $0 < \eta_1, \eta_2 < 1$ such that

$$\eta_1 + \eta_2 = 1, \qquad \eta_1 < \widetilde{m}_1, \qquad \eta_2 < \widetilde{m}_2. \tag{5.6}$$

To prove this we consider two cases. If $|\widetilde{m}_1 - \widetilde{m}_2| < 1$, we just need to pick $\eta_1 := \frac{1}{2} + \frac{\widetilde{m}_1 - \widetilde{m}_2}{2}$, $\eta_2 := \frac{1}{2} + \frac{\widetilde{m}_2 - \widetilde{m}_1}{2}$. On the other hand, if $|\widetilde{m}_1 - \widetilde{m}_2| \geq 1$ then either $\widetilde{m}_1 \geq 1$ or $\widetilde{m}_2 \geq 1$. If $\widetilde{m}_1 \geq 1$, let $\eta_1 = 1 - \epsilon$, $\eta_2 = \epsilon$ with $0 < \epsilon \ll 1$; if $\widetilde{m}_2 \geq 1$, let $\eta_1 = \epsilon$, $\eta_2 = 1 - \epsilon$ with $0 < \epsilon \ll 1$.

Once η_1, η_2 are chosen we consider two cases. When $0 < \eta_1 \leq \eta_2 < 1$, we take

$$\frac{2\eta_2}{\eta_1} < p_1 < \frac{2}{(1-\gamma_3)\eta_1}, \qquad p_2 = p_1 \frac{\eta_1}{\eta_2}, \qquad p = p_1\eta_1 = p_2\eta_2.$$

Then we have that $p_1, p_2 > 2$ and

$$1 = \eta_1 + \eta_2 \leq 2\eta_2 < p_1\eta_1 = p < \frac{2}{1-\gamma_3}. \tag{5.7}$$

When $0 < \eta_2 < \eta_1 < 1$, we choose

$$\frac{2\eta_1}{\eta_2} < p_2 < \frac{2}{(1-\gamma_3)\eta_2}, \qquad p_1 = p_2 \frac{\eta_2}{\eta_1}, \qquad p = p_1\eta_1 = p_2\eta_2.$$

Again we have $p_1, p_2 > 2$ and

$$1 = \eta_1 + \eta_2 < 2\eta_1 < p_2\eta_2 = p < \frac{2}{1-\gamma_3}. \tag{5.8}$$

In both cases we have $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}(\eta_1 + \eta_2) = \frac{1}{p} < 1$. Now let

$$\theta_1 = p_1' \frac{1-\gamma_1}{2}, \qquad \theta_2 = p_2' \frac{1-\gamma_2}{2}, \qquad \theta_3 = p \frac{1-\gamma_3}{2}.$$

Then $\theta_1, \theta_2, \theta_3 > 0$ since $\gamma_1, \gamma_2, \gamma_3 < 1$. From (5.7) or (5.8) we have that $\theta_3 < 1$, and, since $p_1, p_2 > 2$, it follows that $\theta_i < 1 - \gamma_i \leq 1$ for $i = 1, 2$. We also have that $\frac{\theta_1}{p_1'} \leq \frac{1}{2}$, $\frac{\theta_2}{p_2'} \leq \frac{1}{2}$, and $\frac{\theta_3}{p} \leq \frac{1}{2}$ since $\gamma_1, \gamma_2, \gamma_3 \geq 0$. Finally, since we assumed that $\gamma_1 + \gamma_2 + \gamma_3 = 1$, we get

$$\frac{\theta_1}{p_1'} + \frac{\theta_2}{p_2'} + \frac{\theta_3}{p} = \frac{1-\gamma_1}{2} + \frac{1-\gamma_2}{2} + \frac{1-\gamma_3}{2} = 1.$$

Therefore, (5.2) holds and, as observed above, it follows that (5.1) yields (1.17).

By extrapolation (arguing as we did in the proof of Theorem 1.29) and using (5.3), we get that (1.30) holds provided

$$\frac{\theta_3}{p_i} = \frac{1}{r_i^+}. < \frac{1}{s_i}, \frac{1}{q_i} < \frac{1}{r_i^-} = 1 - \frac{\theta_i}{p_i'}, \qquad i = 1, 2. \tag{5.9}$$

Hence, we need to show that (5.4) and (5.5) imply that (5.9) holds. First, from (5.4) we get that

$$\max\left\{\frac{1}{s_i}, \frac{1}{q_i}\right\} < \frac{1+\gamma_i}{2} = 1 - \frac{\theta_i}{p_i'} = \frac{1}{r_i^-}, \qquad i = 1, 2.$$

Second, (5.6) yields

$$\frac{1}{r_i^+} = \frac{\theta_3}{p_i} = \frac{p}{p_i}\frac{1-\gamma_3}{2} = \eta_i < \widetilde{m}_i \frac{1-\gamma_3}{2} = m_i = \min\left\{\frac{1}{s_i}, \frac{1}{q_i}\right\}.$$

Hence, (5.9) holds, and this completes our sketch of the proof.

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