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Singular fibers and Kodaira dimensions

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Abstract Let $f : X \to \mathbb{P}^1$ be a non-isotrivial semi-stable family of varieties of dimension m over $\mathbb{P}^1$ with s singular fibers. Assume that the smooth fibers F are minimal, i.e., their canonical line bundles are semiample. Then $\kappa(X) \leq \kappa(F) + 1$. If $\kappa(X) = \kappa(F) + 1$, then $s > \frac{4}{m} + 2$. If $\kappa(X) \geq 0$, then $s \geq \frac{4}{m} + 2$. In particular, if $m = 1$, $s = 6$ and $\kappa(X) = 0$, then the family f is Teichmüller.

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1 Introduction

We always work over the complex number field $\mathbb{C}$. Let $f : S \to \mathbb{P}^1$ be a nontrivial fibration of semi-stable curves of genus $g \geq 1$. It is a classical problem to determine the lower bound for the number s of singular fibers in the fibration f, see [1,6,7,9,15,17–

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19,23]. In [1], Beauville first proved that $s \geq 4$ and conjectured that $s \geq 5$ when $g \geq 2$. In [15], the second author confirmed Beauville's conjecture. Later, Tu, Zamora and the second author proved in [17] that $s \geq 6$ if S has non-negative Kodaira dimension. It is conjectured that $s \geq 7$ if S is of general type. The first purpose of this note is to confirm this conjecture.

Theorem 1 *Let $f : S \to \mathbb{P}^1$ be a nontrivial semi-stable fibration of curves of genus $g \geq 2$ over $\mathbb{P}^1$ with s singular fibers. If S is of general type, then $s \geq 7$.*

This conjecture has been verified for $g \leq 5$ ([17,23] or $g \geq 58$ ([16], unpublished) by using the *strict canonical class inequality* established by the second author [15]. Recently, the authors in [10] have also proved this conjecture under the condition that the family is birationally equivalent to a pencil of curves with only simple base points on the minimal model of S.

We can find in [17] the examples of surfaces of general type admitting a semi-stable fibration over $\mathbb{P}^1$ with 7 singular fibers.

It is an interesting phenomenon that when the number of singular fibers is minimal, the family is of very interesting arithmetic and geometric properties. When $g = 1$ and $s = 4$, Beauville [2] proved that the family of curves must be *modular*, and there are exactly 6 such families. In [14], the authors prove that for a non-isotrivial family of semi-stable K3 surfaces $f : X \to \mathbb{P}^1$ on a Calabi-Yau manifold X, we have $s \geq 4$ and if $s = 4$, the family is *modular*.

Theorem 2 *As in Theorem 1, if the Kodaira dimension of S is zero and $s = 6$, then the family must be Teichmüller and $\omega^2_{S/\mathbb{P}^1} = 6g - 6$.*

Here a family of curves is said to be Teichmüller, if up to a suitable finite étale cover of S, it comes from a Teichmüller curve.

For each type of surfaces of Kodaira dimension zero, Tu [18] has constructed an example with a semi-stable family of curves over $\mathbb{P}^1$ admitting exactly 6 singular fibers.

When the Kodaira dimension of the surface is 1, the minimal number s should be 6 or 7. We have not found examples with $s = 6$, and we tend to believe that there are no such examples. On the other hand, we give more precise description of such surfaces.

Theorem 3 *With the notation as in Theorem 1. Suppose the Kodaira dimension of S is 1 and $s = 6$. Then S is simply connected, $p_g(S) = q(S) = 0$, the canonical elliptic fibration on S admits exactly two multiple fibers, one of the multiplicities is 2, and the second one is $n = 3$ or 5.*

(1) *If $n = 3$, then $6g - 5 \leq \omega^2_{S/\mathbb{P}^1} \leq 6g - 3$.*
(2) *If $n = 5$, then $\omega^2_{S/\mathbb{P}^1} = 6g - 3$.*

If the stability assumption is dropped, then it is only known that $s \geq 3$ for any non-isotrivial fibration of curves over $\mathbb{P}^1$, even if we require that two of the singular fibers be semi-stable [1,6].

Our method works also for the higher dimensional cases.

Theorem 4 *Let $f : X \to \mathbb{P}^1$ be a non-isotrivial semi-stable family of varieties of dimension m over $\mathbb{P}^1$ with s singular fibers. Assume that the smooth fibers F are minimal, i.e., their canonical line bundles are semiample. Then $\kappa(X) \leq \kappa(F) + 1$.*

(1) *If $\kappa(X) \geq 0$, then $s \geq \frac{4}{m} + 2$. In particular, $s \geq 6$ when $m = 1$, and $s \geq 4$ when $m = 2$ or 3.*
(2) *If $\kappa(X) = \kappa(F) + 1$, then $s > \frac{4}{m} + 2$. In particular, $s \geq 7$ when $m = 1$, $s \geq 5$ when $m = 2$, and $s \geq 4$ when $m = 3$ or 4.*

Note that when X is of general type, we know that F must be also of general type and the equality $\kappa(X) = \kappa(F) + 1$ holds. Hence the lower bound $s > \frac{4}{m} + 2$ holds in this case.

This note is organized as follows. Theorems 1–3 are proved in Sect. 2, and Theorem 4 is proved in Sect. 3.

2 Variations of the Hodge structures

In this section, we would like to prove Theorems 1–3. The main technique is based on the variation of the Hodge structures attached to a semi-stable family of curves, especially to a Teichmüller family.

2.1 Preliminaries

In this subsection, we give a brief recall about the Teichmüller curve and the associated variation of the Hodge structures, and derive some inequalities. For more details, we refer to [3, 12, 13].

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of pairs (F, ω), where $\omega \neq 0$ is a holomorphic one-form on $F \in \mathcal{M}_g$. Here and in the following, we consider the moduli problems in the sense of stacks (or one should take a suitable level structure). Let $\Omega\mathcal{M}_g(m_1, \ldots, m_k) \subseteq \Omega\mathcal{M}_g \to \mathcal{M}_g$ be the stratum of pairs (F, ω) such that ω admits exactly k distinct zeros of order $m_1, \ldots, m_k$ respectively. There is a natural action of $SL_2(\mathbb{R})$ on each stratum $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$. Each orbit projects to a complex geodesics in $\mathcal{M}_g$. When the projection of such an orbit is closed, it gives a so-called *Teichmüller curve*. After a suitable unramified cover and compactification of a given Teichmüller curve, one gets a universal family $f : S \to B$, which is a semi-stable family of curves of genus g. Moreover, there exist disjoint sections $D_1, \ldots D_k$ of f such that the restriction $\left(\sum_{i=1}^k m_i D_i\right)\big|_F$ to each fiber F is just the zero locus of ω.

Denote by s the number of singular fibers contained in f. According to the classical Arakelov inequality (see for instance the proof in [21, Proposition 1.2] and Theorem 6 below for the generalization), for any line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$, the following upper bound on the slope of $\mathcal{L}$ holds.

$$\deg(\mathcal{L}) = \mu(\mathcal{L}) \leq \frac{2g(B) - 2 + s}{2}.$$

In the case when f comes from a Teichmüller curve as above, the Hodge bundle $f_*\omega_{S/B}$ contains a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ with maximal slope (see for instance [12, Proposition 2.4] or [13, Theorem 5.5]):

$$2\deg(\mathcal{L}) = 2g(B) - 2 + s. \tag{1}$$

Consider the logarithmic Higgs bundle $(f_*\omega_{S/B} \oplus R^1 f_*\mathcal{O}_S,\, \theta)$ associated to the fibration f, which corresponds to the weight-one local system $R^1 f_*\mathbb{Q}_{S^0}$; here $f : S^0 \to B^0$ is the smooth part of f. The Higgs field θ is simply the edge morphism

$$f_*\omega_{S/B} \cong f_*\Omega^1_{S/B}(\log \Upsilon) \longrightarrow R^1 f_*\mathcal{O}_S \otimes \Omega^1_B(\log \Delta)$$

of the tautological sequence

$$0 \longrightarrow f^*\Omega^1_B(\log \Delta) \longrightarrow \Omega^1_S(\log \Upsilon) \longrightarrow \Omega^1_{S/B}(\log \Upsilon) \longrightarrow 0,$$

where $\Upsilon \to \Delta$ is denoted to be the singular locus of f. By Viehweg and Zuo [20], the existence of a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ with maximal slope is equivalent to the existence of a rank two Higgs subbundle $(\mathcal{L} \oplus \mathcal{L}^{-1},\, \theta)$ with maximal Higgs field contained in the logarithmic Higgs bundle $(f_*\omega_{S/B} \oplus R^1 f_*\mathcal{O}_S,\, \theta)$ associated to the fibration f.

Conversely, one has the following theorem, which is due to Möller [12].

Theorem 5 *Let $f : S \to B$ be a semi-stable fibration of curves of genus $g \geq 2$ over a smooth projective curve with s singular fibers. Suppose that there exists a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ satisfying the equality (1) above. Then the family f comes from a Teichmüller curve; that is, the induced map $B^0 \to \mathcal{M}_g$ is a finite unramified cover of a Teichmüller curve. Here $f : S^0 \to B^0$ is the smooth part of f.*

Since the relative canonical sheaf of a fibration of curves over a Teichmüller curve has a very special form [see (3) below], we can derive the following upper bound on $\omega^2_{S/B}$.

Proposition 1 *Let $f : S \to B$ be a semi-stable fibration of curves as in Theorem 5, and assume also that there exists a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ with the equality (1). Then*

$$\omega^2_{S/B} \leq \frac{3}{2}(g-1)\big(2g(B) - 2 + s\big). \tag{2}$$

Proof By Theorem 5, the induced map $B^0 \to \mathcal{M}_g$ is finite unramified covering of a Teichmüller curve. Hence after a suitable unramified base change, there exist disjoint sections $D_1, \ldots D_k$ of f such that the relative canonical sheaf $\omega_{S/B}$ has the form (cf. [3])

$$\omega_{S/B} \cong f^*\mathcal{L} \otimes \mathcal{O}_S\left(\sum_{i=1}^{k} m_i D_i\right), \tag{3}$$

where $\mathcal{L} \subseteq f_*\omega_{S/B}$ is the line subbundle satisfying the equality (1). Note that the inequality (2) is invariant under any finite unramified base change. Thus we may assume that $\omega_{S/B}$ already has the form as above.

As D_i's are disjoint sections, it follows that $D_i \cdot D_j = 0$ for $i \neq j$, and that

$$(\omega_{S/B} + D_i) \cdot D_i = 0, \quad \forall\, 1 \leq i \leq k.$$

Combining these with (3), one gets that $D_i^2 = -\frac{1}{m_i+1} \cdot \deg \mathcal{L}$. Note also that $\sum_{i=1}^k m_i = \deg \omega_F = 2g-2$, where ω_F is the canonical sheaf on a general fiber of f. Hence by (3) again, we obtain that

$$\begin{aligned}\omega_{S/B}^2 &= 4(g-1) \cdot \deg \mathcal{L} + \sum_{i=1}^k m_i^2 D_i^2 \\ &= \left(4(g-1) - \sum_{i=1}^k \frac{m_i^2}{m_i+1}\right) \deg \mathcal{L}.\end{aligned}$$

As $\sum_{i=1}^k m_i = 2g-2$, one gets easily that

$$\sum_{i=1}^k \frac{m_i^2}{m_i+1} \geq \sum_{i=1}^k \frac{m_i}{2} = g-1.$$

Therefore,

$$\omega_{S/B}^2 \leq 3(g-1) \cdot \deg \mathcal{L} = \frac{3}{2}(g-1)\big(2g(B) - 2 + s\big).$$

This completes the proof. □

In the case when $f : S \to \mathbb{P}^1$ is a semi-stable fibration of curves of genus $g \geq 2$ over $\mathbb{P}^1$ with $s = 6$ singular fibers, we have the following easy criterion when f comes from a Teichmüller curve.

Lemma 1 *Let $f : S \to \mathbb{P}^1$ be a semi-stable fibration of curves of genus $g \geq 2$ over $\mathbb{P}^1$ with $s = 6$ singular fibers. If the geometric genus $p_g(S) := \dim H^0(S, \omega_S) > 0$, then there exists a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ satisfying the equality (1), and hence f comes from a Teichmüller curve.*

Proof As a locally free sheaf on $\mathbb{P}^1$, the direct image sheaf $f_*\omega_{S/\mathbb{P}^1}$ is isomorphic to a direct sum of invertible sheaves:

$$f_*\omega_{S/\mathbb{P}^1} \cong \bigoplus_{i=1}^g \mathcal{O}_{\mathbb{P}^1}(d_i).$$

Note that $d_i \geq 0$ due to the semi-positivity of the direct image sheaf $f_*\omega_{S/\mathbb{P}^1}$ (cf. [4]), and that $d_i \leq \frac{1}{2}\big(2g(B) - 2 + s\big) = 2$ due to the Arakelov type inequality (cf. [20]).

Without loss of generality, we assume that $0 \le d_1 \le \cdots \le d_g \le 2$. By Fujita [4, Theorem 3.1], we obtain that

$$d_1 = \cdots = d_{q(S)} = 0, \quad \text{and} \quad d_i > 0 \quad \forall\, i \ge q(S) + 1,$$

where $q(S) := \dim H^1(S, \omega_S)$ is the irregularity of S. Hence

$$\deg f_*\omega_{S/\mathbb{P}^1} = \sum_{i=q(S)+1}^{g} d_i.$$

On the other hand, it is well-known that

$$\deg f_*\omega_{S/\mathbb{P}^1} = \chi(\omega_S) - (g-1)\big(g(\mathbb{P}^1) - 1\big) = g + p_g(S) - q(S).$$

Therefore, $d_g = 2$ once $p_g(S) > 0$. In other word, the line subbundle $\mathcal{O}_{\mathbb{P}^1}(d_g) \subseteq f_*\omega_{S/\mathbb{P}^1}$ satisfies the equality (1). □

Corollary 1 *Let $f : S \to \mathbb{P}^1$ be a semi-stable fibration of curves of genus $g \ge 2$ over $\mathbb{P}^1$ with $s = 6$ singular fibers. If the geometric genus $p_g(S) > 0$, then f comes from a Teichmüller curve and*

$$\omega^2_{S/\mathbb{P}^1} \le 6(g-1). \tag{4}$$

Proof This is a combination of Lemma 1 and Proposition 1. □

2.2 Proof of Theorem 1

By Tan et al. [17, Theorem 0.1], $s \ge 6$ if S is of general type (actually, the inequality $s \ge 6$ holds once the Kodaira dimension of S is non-negative). To complete the proof, it suffices to deduce a contradiction if $s = 6$.

Since S is of general type, we may assume that $g \ge 5$ by Tan et al. [17, Theorem 0.1(2)], and according to Tan et al. [17, Theorem 0.2] one has

$$\omega^2_{S/\mathbb{P}^1} \ge 6g - 6 + \frac{1}{2}\left(\omega_X^2 + \sqrt{\omega_X^2}\sqrt{\omega_X^2 + 8g - 8}\right), \tag{5}$$

where X is the minimal model of S. Hence we may assume that $p_g(S) = 0$ by Corollary 1. It then follows that

$$\deg f_*\omega_{S/\mathbb{P}^1} = \chi(\omega_S) - (g-1)\big(g(\mathbb{P}^1) - 1\big) = g - q(S).$$

Let δ_f be the number of nodes contained in the fibers of f. Then by Noether's formula, one has

$$\delta_f = 12 \deg f_*\omega_{S/\mathbb{P}^1} - \omega^2_{S/\mathbb{P}^1} = 12\big(g - q(S)\big) - \omega^2_{S/\mathbb{P}^1}.$$

According to Tan [15], for any integer $e \geq 2$, we have the following inequality

$$\begin{aligned} \omega_{S/\mathbb{P}^1}^2 &\leq (2g-2)\Big(2g\big(\mathbb{P}^1\big) - 2 + \frac{(e-1)s}{e}\Big) + \frac{3\delta_f}{e^2}, \\ &= (2g-2)\Big(4 - \frac{6}{e}\Big) + \frac{3\left(12\big(g - q(S)\big) - \omega_{S/\mathbb{P}^1}^2\right)}{e^2}. \end{aligned}$$

Hence

$$\omega_{S/\mathbb{P}^1}^2 \leq \frac{e^2}{e^2+3}(2g-2)\Big(4 - \frac{6}{e}\Big) + \frac{36\big(g - q(S)\big)}{e^2+3}.$$

Taking $e = 3$, one obtains

$$\omega_{S/\mathbb{P}^1}^2 \leq 6g - 3 - 3q(S) \leq 6g - 3. \tag{6}$$

Combining this with (5), one obtains that

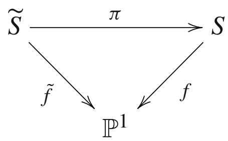
$$\begin{aligned} &\sqrt{\omega_X^2}\,\sqrt{\omega_X^2 + 8g - 8} \leq 6 - \omega_X^2, \\ &\Longrightarrow \omega_X^2\left(\omega_X^2 + 8g - 8\right) \leq (6 - \omega_X^2)^2, \\ &\Longrightarrow \omega_X^2 \leq \frac{9}{2g+1} < 1, \quad \text{since } g \geq 5. \end{aligned}$$

This gives a contradiction. □

2.3 Proof of Theorem 2

Let X be the minimal model of S. If X is either an abelian surface or a K3 surface, then $p_g(S) > 0$, and hence the conclusion follows directly from Corollary 1 and [17, Theorem 0.2].

In the remaining cases, X must be either an Enriques surface or a bielliptic surface according to the classification of surfaces with Kodaira dimension equal to zero. Let K_X be the canonical divisor of X. Then there exists an $n > 1$ such that $nK_X \equiv 0$. Hence one can construct a finite étale cover $\pi : \widetilde{S} \to S$ such that $p_g(\widetilde{S}) > 0$ and that the Kodaira dimension of $\widetilde{S}$ is still zero. Moreover $\tilde{f} := f \circ \pi : \widetilde{S} \to \mathbb{P}^1$ is still a semi-stable fibration with 6 singular fibers by Beauville [1, Lemma 3].



Since π is finite étale,

$$\omega^2_{\tilde{S}/\mathbb{P}^1} = \deg(\pi)\cdot\omega^2_{S/\mathbb{P}^1}, \quad \tilde{g}-1 = \deg(\pi)\cdot(g-1), \tag{7}$$

where $\tilde{g}$ is the genus of a general fiber of $\tilde{f}$. By construction, $\widetilde{S}$ is either an abelian surface or a K3 surface, so $p_g(\tilde{S}) = 1$. Hence the family $\tilde{f}$ comes from a Teichmüller curve and $\omega^2_{\tilde{S}/\mathbb{P}^1} = 6(\tilde{g}-1)$ by the above argument. Therefore, the family f is Teichmüller. Moreover, Together with (7), we obtain $\omega^2_{S/\mathbb{P}^1} = 6g-6$ as required. □

2.4 Proof of Theorem 3

Because the Kodaira dimension of S is 1, by Tan et al. [17, Theorem 0.2] we have

$$\omega^2_{S/\mathbb{P}^1} \geq 6g-5. \tag{8}$$

Hence $p_g(S) = 0$ by Corollary 1. Similar to the proof of Theorem 1, the inequality (6) holds. Thus $q(S) = 0$ by (6) and (8).

As the Kodaira dimension of S is 1, the minimal model X of S admits an elliptic fibration

$$h : X \longrightarrow C.$$

Since $q(X) = q(S) = 0$, it follows that $C \cong \mathbb{P}^1$. Let $\{n_1\Gamma_1, \ldots, n_r\Gamma_r\}$ be the set of multiple fibers of h with $2 \leq n_1 \leq \cdots \leq n_r$. Then the canonical sheaf of X is given by Griffiths and Harris (cf. [5, § IV-5])

$$\omega_X = h^*\Big(\mathcal{O}_{\mathbb{P}^1}(-1)\Big) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r}(n_i-1)\Gamma_i\right). \tag{9}$$

We claim first that $r = 2$. Indeed, it is clear that $r \geq 2$ by (9) since $\kappa(X) = 1$, and that $r < 3$, since otherwise by an unramified cover one can construct a new surface $\widetilde{S}$ with $p_g(\widetilde{S}) > 0$. Moreover, similar to the proof of Theorem 2, one shows that $\widetilde{S}$ is still a semi-stably fibred over $\mathbb{P}^1$ with 6 singular fibers. This is a contradiction by the above argument.

We claim also that $n_1 \nmid n_2$. Suppose n_1 divides n_2, one can construct an unramified cover S'' over S, which is still semi-stably fibred over $\mathbb{P}^1$ with 6 singular fibers. Moreover, the minimal model of S'' admits an elliptic fibration with only one multiple fiber. This is again a contradiction by the above argument.

Let F be a general fiber of f and F_0 its image in X. Let Γ be a general fiber of h and $d = \gcd(n_1, n_2)$. Then there exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1n_1 + m_2n_2 = d$. Let $\Gamma_0 = m_2\Gamma_1 + m_1\Gamma_2$. Then numerically,

$$\Gamma_0 = \frac{m_2}{n_1}\cdot n_1\Gamma_1 + \frac{m_1}{n_2}\cdot n_2\Gamma_2 \sim_{\text{num}} \left(\frac{m_2}{n_1}+\frac{m_1}{n_2}\right)\Gamma = \frac{d}{n_1n_2}\cdot\Gamma.$$

Moreover, by (9), one has the following numerical equivalence:

$$\omega_X \sim_{\text{num}} \left(1 - \frac{1}{n_1} - \frac{1}{n_2}\right) \Gamma \sim_{\text{num}} \left(\frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d}\right) \Gamma_0.$$

According to the proof of [17, Theorem 2.1], one has

$$\begin{aligned} \omega_{S/\mathbb{P}^1}^2 &\geq 6g - 6 + \omega_X \cdot F_0 \\ &= 6g - 6 + \left(\frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d}\right) \Gamma_0 \cdot F_0 \\ &\geq 6g - 6 + \left(\frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d}\right). \end{aligned}$$

From $n_1 \nmid n_2$ and (6), we see that there are only two possibilities as stated in Theorem 3.

It remains to show that S is simply connected. Since $\chi(\mathcal{O}_X) = 1 > 0$, it follows from Noether's formula that the elliptic fibration h admits at least one singular fiber. Moreover, we have shown that h has exactly two multiply fibers whose multiplicities are coprime. From [11, § II.2-Theorem 10], it follows that X, and hence also S, are both simply connected. This completes the proof. □

3 Arakelov type inequality

In this section, we generalize our results to the high dimension cases, i.e., we prove Theorem 4. The technique uses the Arakelov type inequality, which is deduced from the variation of the Hodge structures attached to such families.

The Arakelov type inequality for the direct image of the relative pluri-canonical sheaves goes back to Viehweg and the last author [22,24]. This kind of inequality is generalized in the recent work [8]. The following form can be found in [24, Theorem 4.4] and [8, Prop 3.1 & Remark 3.2], which is the key to our proof.

Theorem 6 *Let $f : X \to B$ be a semi-stable family of varieties of relative dimension $m \geq 1$ over a smooth projective curve of genus $g(B)$ with s singular fibers. Assume that the smooth fibers F are minimal, i.e., their canonical line bundles are semiample. Let $\omega_{X/B}$ be the relative canonical sheaf, and $\mathcal{E} \subseteq f_*\big(\omega_{X/B}^{\otimes k}\big)$ be any non-zero subsheaf. Then the slope $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$ satisfies that*

$$\mu(\mathcal{E}) \leq \frac{mk\big(2g(B) - 2 + s\big)}{2}.$$

The main idea of proving Theorem 4 is to compute the plurigenera by applying Riemann-Roch theorem for the direct image sheaves $f_*\big(\omega_X^{\otimes k}\big)$ on the base curve. Combining with the asymptotic behavior of the plurigenera, we complete the proof.

Proof (Proof of Theorem 4) Since the base is a rational curve $\mathbb{P}^1$, it follows that

$$\omega_X = \omega_{X/\mathbb{P}^1} \otimes f^*\omega_{\mathbb{P}^1} = \omega_{X/\mathbb{P}^1} \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-2).$$

Hence for any $k \geq 1$, one has

$$f_*\big(\omega_X^{\otimes k}\big) = f_*\big(\omega_{X/\mathbb{P}^1}^{\otimes k}\big) \otimes \mathcal{O}_{\mathbb{P}^1}(-2k). \tag{10}$$

Let $\mathcal{E} \subseteq f_*\big(\omega_X^{\otimes k}\big)$ be any subsheaf. Then by (10), $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2k)$ is a subsheaf of $f_*\big(\omega_{X/\mathbb{P}^1}^{\otimes k}\big)$. Thus by Theorem 6, one obtains

$$\mu(\mathcal{E}) + 2k = \mu\big(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2k)\big) \leq \frac{mk}{2} \cdot (s-2);$$

equivalently, we have

$$\mu(\mathcal{E}) \leq \frac{k}{2}\big(m(s-2)-4\big). \tag{11}$$

As a locally free sheaf on $\mathbb{P}^1$, the direct image sheaf $f_*\big(\omega_X^{\otimes k}\big)$ is isomorphic to a direct sum of invertible sheaves,

$$f_*\big(\omega_X^{\otimes k}\big) \cong \bigoplus_{i=1}^{r_k} \mathcal{O}_{\mathbb{P}^1}(d_i), \quad r_k = \operatorname{rank} f_*\big(\omega_X^{\otimes k}\big).$$

By (11), we have

$$d_i \leq \frac{k}{2}\big(m(s-2)-4\big), \quad i = 1, 2, \ldots, r_k.$$

Hence

$$\begin{aligned} \dim H^0(X, \omega_X^{\otimes k}) &= \dim H^0\big(\mathbb{P}^1, f^*(\omega_X^{\otimes k})\big) = \sum_{d_i \geq 0} (d_i + 1) \\ &\leq \max\left\{0,\ \left[\frac{k}{2}\big(m(s-2)-4\big)\right] + 1\right\} \cdot \operatorname{rank} f_*\big(\omega_X^{\otimes k}\big). \end{aligned}$$

Here '[•]' stands for the integral part.

According to the definition of the Kodaira dimension of a variety, when k is sufficiently large, one has

$$\begin{cases} \operatorname{rank} f_*\big(\omega_X^{\otimes k}\big) = \dim H^0(F, \omega_F^{\otimes k}) \sim k^{\kappa(F)}; \\ \dim H^0(X, \omega_X^{\otimes k}) \sim k^{\kappa(X)}. \end{cases}$$

Hence $\kappa(X) \leq \kappa(F) + 1$. Moreover, if $\kappa(X) \geq 0$, then

$$\frac{1}{2}\big(m(s-2)-4\big) \geq 0, \quad \text{i.e.,} \quad s \geq \frac{4}{m} + 2;$$

and if $\kappa(X) = \kappa(F) + 1$, then

$$\frac{1}{2}\big(m(s-2)-4\big) > 0, \quad \text{i.e.,} \quad s > \frac{4}{m} + 2.$$

This completes the proof. □

Remark 1 Recall that the volume of a projective variety X is defined to be

$$\mathrm{Vol}(X) = \limsup_{k} \frac{(\dim X)! \cdot \dim H^0(X, \omega_X^{\otimes k})}{k^{\dim X}}.$$

The above proof shows also that for a variety of general type semi-stably fibred over $\mathbb{P}^1$ with s singular fibers, one has

$$\mathrm{Vol}(X) \le \frac{(m+1)\big(m(s-2)-4\big)}{2} \mathrm{Vol}(F),$$

where F is a general fiber of f. In particular, when X is of general type and $f : X \to \mathbb{P}^1$ is a semi-stable fibration of curves of genus $g \ge 2$ with s singular fibers, one computes that

$$\mathrm{Vol}(X) = \omega_{X_0}^2, \quad \mathrm{Vol}(F) = 2g - 2,$$

where X_0 is the minimal model of X. Hence the above proof shows that in this case,

$$\omega_{X_0}^2 \le 2(s-6)(g-1).$$

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