



# Enumerating submodules invariant under an endomorphism

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**Abstract** We study zeta functions enumerating submodules invariant under a given endomorphism of a finitely generated module over the ring of ( $S$ -)integers of a number field. In particular, we compute explicit formulae involving Dedekind zeta functions and establish meromorphic continuation of these zeta functions to the complex plane. As an application, we show that ideal zeta functions associated with nilpotent Lie algebras of maximal class have abscissa of convergence 2.

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## 1 Introduction

### 1.1 Zeta functions derived from endomorphisms

Throughout, rings are assumed to be commutative and unital. We say that a ring  $R$  has *polynomial submodule growth* if the following holds for every finitely generated  $R$ -module  $M$ : for each  $m \geq 1$ , the number of submodules of additive index  $m$  of  $M$  is finite and polynomially bounded as a function of  $m$ . Recall that  $R$  is *semi-local* if it contains only finitely many maximal ideals.

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**Theorem 1.1** ([21, Thm 1]) *Let  $R$  be a ring which is finitely generated over  $\mathbf{Z}$  or semi-local with finite residue fields. Then  $R$  has polynomial submodule growth if and only if it has Krull dimension at most 2.*

Let  $R$  be a ring with polynomial submodule growth, let  $M$  be a finitely generated  $R$ -module, and let  $A \in \text{End}_R(M)$ . For  $m \geq 1$ , let  $a_m(A, R)$  denote the number of  $A$ -invariant  $R$ -submodules  $U \leq M$  with  $|M : U| = m$ . We define a zeta function

$$\zeta_{A,R}(s) := \sum_{m=1}^{\infty} a_m(A, R)m^{-s}$$

and we let  $\alpha_{A,R} < \infty$  denote its abscissa of convergence; it is well-known that  $\alpha_{A,R}$  is precisely the degree of polynomial growth of the partial sums  $a_1(A, R) + \cdots + a_m(A, R)$  as a function of  $m$ .

The zeta functions  $\zeta_{A,R}(s)$  belong to the larger theory of subobject zeta functions; for a recent survey of the area, see [26]. Indeed, using the terminology from [17],  $\zeta_{A,R}(s)$  is the submodule zeta function  $\zeta_{R[A] \curvearrowright M}(s)$  of the enveloping algebra  $R[A] := \sum_{i=0}^{\infty} R \cdot A^i \subset \text{End}_R(M)$  of  $A$  acting on  $M$ .

The main results of this article, Theorems A–D, constitute a rather exhaustive analysis of the zeta functions  $\zeta_{A,R}(s)$  in the cases that  $R$  is the ring of ( $S$ -)integers of a number field or a (generic) completion of such a ring. In particular, our findings provide further evidence in support of the author’s general conjectures on submodule zeta functions stated in [17, §8].

## 1.2 Related work: invariant subspaces

The study of subspaces invariant under an endomorphism has a long history. For a finite-dimensional vector space  $V$  over the real or complex numbers and  $A \in \text{End}(V)$ , Shayman [22] investigated topological properties of the compact analytic space  $S_A$  of  $A$ -invariant subspaces of  $V$ . In particular, if  $A$  is nilpotent, then he found the subspace  $S_A(d) \subset S_A$  of  $d$ -dimensional  $A$ -invariant subspaces of  $V$  to be connected but usually singular.

For an arbitrary ground field  $F$  and a fixed number  $n$ , Ringel and Schmidmeier [16] studied the category of triples  $(V, U, T)$ , where  $V$  is a finite-dimensional vector space over  $F$ ,  $T \in \text{End}_F(V)$  satisfies  $T^n = 0$ , and  $U \leq V$  is  $F$ -invariant. While their point of view is rather different from ours, we would like to point out that they found the case of exponent  $n \geq 7$  to involve instances of so-called “wild” representation type.

## 1.3 Ideal zeta functions

In our study of the zeta functions  $\zeta_{A,R}(s)$ , we will frequently encounter another special case of submodule zeta functions, namely ideal zeta functions. Let  $R$  be a ring with polynomial submodule growth and let  $\mathbf{A}$  be a possibly non-associative  $R$ -algebra whose underlying  $R$ -module is finitely generated. We write  $\mathbf{I} \triangleleft_R \mathbf{A}$  to indicate that  $\mathbf{I}$

is a two-sided ideal of  $\mathbf{A}$  which is also an  $R$ -submodule. The *ideal zeta function* (cf. [11]) of  $\mathbf{A}$  is

$$\zeta_{\mathbf{A}}(s) := \sum_{\substack{I \triangleleft_R \mathbf{A} \\ |\mathbf{A}:I| < \infty}} |\mathbf{A} : I|^{-s}.$$

For example, the ideal zeta function of the ring of integers of a number field  $k$  is precisely the Dedekind zeta function of  $k$ . In particular, the ideal zeta function of  $\mathbf{Z}$  is the Riemann zeta function  $\zeta(s)$ . As explained in [17, Rem. 2.2 (ii)], ideal zeta functions are in fact a special case of the submodule zeta functions discussed below.

### 1.4 Global setup, Euler products, and growth rates

For the remainder of this article, let  $k$  be a number field with ring of integers  $\mathfrak{o}$ .

Let  $\mathcal{V}_k$  denote the set of non-Archimedean places of  $k$ . For  $v \in \mathcal{V}_k$ , let  $k_v$  be the  $v$ -adic completion of  $k$  and let  $\mathfrak{o}_v$  be its valuation ring. For  $S \subset \mathcal{V}_k$ , let

$$\mathfrak{o}_S = \bigcap_{v \in \mathcal{V}_k \setminus S} \mathfrak{o}_v \cap k$$

be the usual ring of  $S$ -integers of  $k$ .

In the following, we investigate  $\zeta_{A,R}(s)$ , where  $A \in \text{End}_R(M)$  and  $R = \mathfrak{o}_v$  or  $R = \mathfrak{o}_S$  for  $v \in \mathcal{V}_k$  or a finite set  $S \subset \mathcal{V}_k$ , respectively. The techniques that we use are predominantly local and valid for almost all places of  $k$  (i.e. for all but finitely many places); the exclusion of a finite number of exceptional places is common and frequently unavoidable in the theory of subobject zeta functions.

If  $M$  is a finitely generated  $\mathfrak{o}_S$ -module, then  $M \otimes_{\mathfrak{o}_S} \mathfrak{o}_v$  is a free  $\mathfrak{o}_v$ -module for almost all  $v \in \mathcal{V}_k \setminus S$ . We thus lose little by henceforth assuming that  $M = \mathfrak{o}_S^n$  and  $A \in M_n(\mathfrak{o}_S)$ , where  $M_n(R)$  denotes the algebra of  $n \times n$  matrices over a ring  $R$ . Note that if  $A \in M_n(k)$ , then  $A \in M_n(\mathfrak{o}_v)$  for almost all  $v \in \mathcal{V}_k$ . In order to exclude trivialities, unless otherwise stated, we always assume that  $n > 0$ . Being instances of submodule zeta functions, the zeta functions  $\zeta_{A,\mathfrak{o}_S}(s)$  admit natural Euler product factorisations.

**Proposition** (Cf. [17, Lemma 2.3]) *Let  $A \in M_n(\mathfrak{o}_S)$  for finite  $S \subset \mathcal{V}_k$ . Then*

$$\zeta_{A,\mathfrak{o}_S}(s) = \prod_{v \in \mathcal{V}_k \setminus S} \zeta_{A,\mathfrak{o}_v}(s).$$

The following is a consequence of deep results of du Sautoy and Grunewald on subobject zeta functions expressible in terms of what they call ‘‘cone integrals’’.

**Theorem 1.2** (Cf. [7, §4]) *Let  $A \in M_n(\mathfrak{o}_S)$  for finite  $S \subset \mathcal{V}_k$ . Then:*

- (i) The abscissa of convergence  $\alpha_{A, \mathfrak{o}_S}$  of  $\zeta_{A, \mathfrak{o}_S}(s)$  is a rational number.
- (ii)  $\zeta_{A, \mathfrak{o}_S}(s)$  admits meromorphic continuation to  $\{s \in \mathbf{C} : \operatorname{Re}(s) > \alpha_{A, \mathfrak{o}_S} - \delta\}$  for some  $\delta > 0$ . This continued function is regular on the line  $\operatorname{Re}(s) = \alpha_{A, \mathfrak{o}_S}$  except for a pole at  $s = \alpha_{A, \mathfrak{o}_S}$ .
- (iii) Let  $\beta_{A, \mathfrak{o}_S}$  denote the multiplicity of the pole of (the meromorphic continuation of)  $\zeta_{A, \mathfrak{o}_S}(s)$  at  $\alpha_{A, \mathfrak{o}_S}$ . Then there exists a real constant  $c_{A, \mathfrak{o}_S} > 0$  such that

$$a_1(A, \mathfrak{o}_S) + \dots + a_m(A, \mathfrak{o}_S) \sim c_{A, \mathfrak{o}_S} \cdot m^{\alpha_{A, \mathfrak{o}_S}} (\log m)^{\beta_{A, \mathfrak{o}_S} - 1}.$$

where  $f(m) \sim g(m)$  signifies that  $f(m)/g(m) \rightarrow 1$  as  $m \rightarrow \infty$ .

### 1.5 Matrices, polynomials, and partitions

Prior to stating our main results, we need to establish some notation and recall some terminology. By a *partition* of an integer  $n \geq 0$ , we mean a non-increasing sequence  $\lambda = (\lambda_1, \dots, \lambda_r)$  of positive integers with  $n = \lambda_1 + \dots + \lambda_r$ ; for background, we refer to [12]. We write  $|\lambda| := n$ ,  $\operatorname{len}(\lambda) := r$ , and  $\lambda_{-1} := \lambda_r$ . We write  $\lambda \vdash n$  to signify that  $\lambda$  is a partition of  $n$ . For  $i \geq 0$ , define  $\sigma_i(\lambda) := \lambda_1 + \dots + \lambda_i$ . For  $1 \leq j \leq |\lambda|$ , let  $\lambda^{-1}(j)$  be the unique number  $i \in \{1, \dots, \operatorname{len}(\lambda)\}$  with  $\sigma_{i-1}(\lambda) < j \leq \sigma_i(\lambda)$ ; equivalently,  $\lambda^{-1}(j) = \min\{i \in \{1, \dots, \operatorname{len}(\lambda)\} : j \leq \sigma_i(\lambda)\}$ . The *dual partition* of  $\lambda$  is denoted by  $\lambda^*$ . Thus, if  $|\lambda| > 0$ , then  $\lambda^* = (\mu_1, \dots, \mu_t)$ , where  $t = \lambda_1$  and  $\mu_i = \#\{i \in \{1, \dots, \operatorname{len}(\lambda)\} : \lambda_i \geq i\}$ .

For a monic polynomial  $f = X^m + a_{m-1}X^{m-1} + \dots + a_0$ , let

$$C(f) = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 & 1 \\ -a_0 & \dots & -a_{m-2} & -a_{m-1} & & \end{bmatrix}$$

be its companion matrix. Let  $A \in M_n(k)$ . It is well-known that there are monic irreducible polynomials  $f_1, \dots, f_e \in k[X]$  and partitions  $\lambda_1, \dots, \lambda_e$  of positive integers  $n_1, \dots, n_e$  such that  $n = \deg(f_1)n_1 + \dots + \deg(f_e)n_e$  and  $A$  is similar to its (primary) rational canonical form

$$\operatorname{diag}\left(C\left(f_1^{\lambda_{1,1}}\right), \dots, C\left(f_1^{\lambda_{1, \operatorname{len}(\lambda_1)}}\right), \dots, C\left(f_e^{\lambda_{e,1}}\right), \dots, C\left(f_e^{\lambda_{e, \operatorname{len}(\lambda_e)}}\right)\right)$$

over  $k$ . We call  $((f_1, \lambda_1), \dots, (f_e, \lambda_e))$  an *elementary divisor vector* of  $A$  over  $k$ ; any two elementary divisor vectors of  $A$  coincide up to reordering.

### 1.6 Main results

Recall that  $k$  is a number field with ring of integers  $\mathfrak{o}$ . Throughout,  $\mathfrak{p}_v \in \operatorname{Spec}(\mathfrak{o})$  denotes the prime ideal corresponding to a place  $v \in \mathcal{V}_k$  and  $q_v = |\mathfrak{o}/\mathfrak{p}_v|$  denotes the residue field size of  $k_v$ . Our global main result is the following.

**Theorem A** *Let  $S \subset \mathcal{V}_k$  be finite and  $A \in M_n(\mathfrak{o}_S)$ . Let  $((f_1, \lambda_1), \dots, (f_e, \lambda_e))$  be an elementary divisor vector of  $A$  over  $k$ . Write  $k_i = k[X]/(f_i)$ . Let  $\mathfrak{o}_i$  denote the ring of integers of  $k_i$ . Let  $S_i = \{w \in \mathcal{V}_{k_i} : \exists v \in S.w \mid v\}$  and write  $\mathfrak{o}_{i,S_i} := (\mathfrak{o}_i)_{S_i}$ . Then the following hold:*

- (i) *There are finitely many places  $w_1, \dots, w_\ell \in \mathcal{V}_k \setminus S$  and associated rational functions  $W_1, \dots, W_\ell \in \mathbf{Q}(X)$  such that*

$$\zeta_{A, \mathfrak{o}_S}(s) = \prod_{u=1}^{\ell} W_u(q_{w_u}^{-s}) \times \prod_{i=1}^e \prod_{j=1}^{|\lambda_i|} \zeta_{\mathfrak{o}_{i,S_i}}((\lambda_i^*)^{-1}(j) \cdot s - j + 1). \tag{1.1}$$

*In particular,  $\zeta_{A, \mathfrak{o}_S}(s)$  admits meromorphic continuation to the complex plane.*

- (ii) *The abscissa of convergence  $\alpha_{A, \mathfrak{o}_S}$  of  $\zeta_{A, \mathfrak{o}_S}(s)$  satisfies  $\alpha_{A, \mathfrak{o}_S} = \max_{1 \leq i \leq e} \text{len}(\lambda_i) \in \mathbf{N}$ .*
- (iii) *Let  $I := \{i \in \{1, \dots, e\} : \text{len}(\lambda_i) = \alpha_{A, \mathfrak{o}_S}\}$ . Then the multiplicity  $\beta_{A, \mathfrak{o}_S}$  of the pole of  $\zeta_{A, \mathfrak{o}_S}(s)$  at  $\alpha_{A, \mathfrak{o}_S}$  satisfies  $\beta_{A, \mathfrak{o}_S} = \sum_{i \in I} \lambda_{i,-1}$ .*

As we will see, part (i) is in fact a consequence of a similar formula (5.1) which is valid for almost all local zeta functions  $\zeta_{A, \mathfrak{o}_v}(s)$ . The exceptional factors  $W_u(q_{w_u}^{-s})$  in (1.1) cannot, in general, be omitted, see Example 5.5 below.

We note that the special case  $A = 0_n$  in Theorem A is consistent with the well-known formula  $\zeta_{\mathfrak{o}_S}(s)\zeta_{\mathfrak{o}_S}(s-1) \cdots \zeta_{\mathfrak{o}_S}(s-(n-1))$  for the zeta function enumerating all finite-index submodules of  $\mathfrak{o}_S^n$ . We further note that the shape of the right-hand side of (1.1) is rather similar to that of Solomon’s formula [23, Thm 1] for the zeta function enumerating submodules of finite index of a  $\mathbf{Z}G$ -lattice for a finite group  $G$ .

Local functional equations under “inversion of the residue field size” are a common, but not universal, phenomenon in the theory of subobject zeta functions; see [24, 25]. For an extension of number fields  $k'/k$  and  $v \in \mathcal{V}_k$ , let  $g_v(k')$  denote the number of places of  $k'$  which divide  $v$ .

**Theorem B** *Let  $A \in M_n(k)$  and let  $((f_1, \lambda_1), \dots, (f_e, \lambda_e))$  be an elementary divisor vector of  $A$  over  $k$ . Write  $\mu_i := \lambda_i^*$ . Then, for almost all  $v \in \mathcal{V}_k$ ,*

$$\zeta_{A, \mathfrak{o}_v}(s) \Big|_{q_v \rightarrow q_v^{-1}} = (-1)^{\sum_{i=1}^e |\lambda_i| \cdot g_v(k[X]/(f_i))} \cdot q_v^{\sum_{i=1}^e \deg(f_i) \binom{|\lambda_i|}{2} - \left(\sum_{i=1}^e \deg(f_i) \sum_{j=1}^{\lambda_{i1}} j \mu_{ij}\right) s} \cdot \zeta_{A, \mathfrak{o}_v}(s). \tag{1.2}$$

We now give a description of the operation “ $q_v \rightarrow q_v^{-1}$ ” in Theorem B. Let  $k'/k$  be an extension of number fields, let  $\mathfrak{o}'$  be the ring of integers of  $k'$ , let  $v \in \mathcal{V}_k$ , and let  $w \in \mathcal{V}_{k'}$  divide  $v$ . It is well-known that  $\zeta_{\mathfrak{o}'_w}(s) = 1/(1 - q_w^{-s}) = 1/(1 - q_v^{-f(w/v)s})$  for some  $f(w/v) \geq 1$ . After excluding finitely many places of  $k$ , the local version (5.1) of (1.1) expresses  $\zeta_{A, \mathfrak{o}_v}(s)$  as a product of factors of the form  $\zeta_{\mathfrak{o}'_w}(as - b)$ . The operation “ $q_v \rightarrow q_v^{-1}$ ” is then applied in the evident way to each of these factors.

We note that in the special case that  $(A - a1_n)^n = 0$  for some  $a \in k$ , the functional equation (1.2) follows from [25, Thm 1.2] (see [25, Rem. 1.5]).

It is natural to ask what properties of  $A$  can be inferred from its associated zeta functions. We will make frequent use of the following elementary observation.

**Lemma** *Let  $A, B \in M_n(k)$ . Suppose that  $k[A]$  and  $k[B]$  are similar (i.e.  $\text{GL}_n(k)$ -conjugate). Then for almost all  $v \in \mathcal{V}_k$ ,  $\zeta_{A, \mathfrak{o}_v}(s) = \zeta_{B, \mathfrak{o}_v}(s)$ .*

The following is another consequence of our explicit formulae.

**Theorem C** *Let  $A \in M_n(k)$  and  $B \in M_m(k)$  be nilpotent. The following are equivalent:*

- (i)  $n = m$  and  $A$  and  $B$  are similar.
- (ii) For almost all  $v \in \mathcal{V}_k$ ,  $\zeta_{A, \mathfrak{o}_v}(s) = \zeta_{B, \mathfrak{o}_v}(s)$ .
- (iii) There exists a finite  $S \subset \mathcal{V}_k$  such that  $A$  and  $B$  both have entries in  $\mathfrak{o}_S$  and such that  $\zeta_{A, \mathfrak{o}_S}(s) = \zeta_{B, \mathfrak{o}_S}(s)$ .

The nilpotency condition in Theorem C cannot, in general, be omitted, see Remark 5.8.

The author previously conjectured [17, §8.3] that generic local submodule zeta functions associated with nilpotent matrix algebras have a simple pole at zero. In the present case, our explicit formulae allow us to deduce the following.

**Theorem D** *Let  $A \in M_n(k)$ . Then for almost all  $v \in \mathcal{V}_k$ ,  $\zeta_{A, \mathfrak{o}_v}(s)$  has a pole at zero. Moreover, the following are equivalent:*

- (i) For almost all  $v \in \mathcal{V}_k$ ,  $\zeta_{A, \mathfrak{o}_v}(s)$  has a simple pole at zero.
- (ii) There exists  $a \in k$  with  $(A - a1_n)^n = 0$ .

### 1.7 Behaviour at zero in general—a conjecture

We use this opportunity to state a generalisation of our conjecture on the behaviour at zero of local submodule zeta functions (see [17, Conj. IV and §8.3]); this generalisation disposes of the mysterious nilpotency assumption found in its precursor.

For a ring  $R$  with polynomial submodule growth, a finitely generated  $R$ -module  $M$ , and  $\Omega \subset \text{End}_R(M)$ , the submodule zeta function  $\zeta_{\Omega \curvearrowright M}(s)$  is the Dirichlet series enumerating  $\Omega$ -invariant  $R$ -submodules of finite index of  $M$  (cf. [17, Def. 2.1 (ii)]).

Let  $V$  be a finite-dimensional vector space over  $k$  and let  $\mathcal{A} \subset \text{End}_k(V)$  be an associative, unital subalgebra. Let  $\text{rad}(\mathcal{A})$  denote the (nil)radical of  $\mathcal{A}$ . By the Wedderburn-Malcev Theorem [6, Thm 72.19], there exists a subalgebra  $\mathcal{S} \subset \mathcal{A}$  such that  $\mathcal{A} = \text{rad}(\mathcal{A}) \oplus \mathcal{S}$  as vector spaces (whence  $\mathcal{S} \approx_k \mathcal{A}/\text{rad}(\mathcal{A})$  is semisimple); moreover,  $\mathcal{S}$  is unique up to conjugacy under  $(1 + \text{rad}(\mathcal{A})) \leq \mathcal{A}^\times$ . Choose  $\mathfrak{o}$ -forms  $V \subset V$ ,  $\mathcal{A} \subset \text{End}_{\mathfrak{o}}(V)$  and  $\mathcal{S} \subset \text{End}_{\mathfrak{o}}(V)$  of  $V$ ,  $\mathcal{A}$ , and  $\mathcal{S}$ , respectively. We write  $X_v := X \otimes_{\mathfrak{o}} \mathfrak{o}_v$  in the following.

**Conjecture E** *For almost all  $v \in \mathcal{V}_k$ ,*

$$\left. \frac{\zeta_{\mathcal{A}_v \curvearrowright X_v}(s)}{\zeta_{\mathcal{S}_v \curvearrowright X_v}(s)} \right|_{s=0} = 1.$$

This conjecture reduces to the behaviour predicted in [17, §8.3] in the “nilpotent case”  $\mathcal{A} = \text{rad}(\mathcal{A}) \oplus k1_V$ . In order to make Conjecture E more explicit, we recall Solomon’s formula for  $\zeta_{\mathcal{S}_v \curvearrowright \mathcal{V}_v}(s)$ . Let  $\mathcal{S} = \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_r$  be the Wedderburn decomposition of the semisimple algebra  $\mathcal{S}$  (so that each  $\mathcal{S}_i$  is simple). Let  $W_i$  be a simple  $\mathcal{S}_i$ -module and decompose  $V = V_1 \oplus \dots \oplus V_r$ , where  $V_i$  is isomorphic to  $W_i^{m_i}$  and  $\mathcal{S}$  acts diagonally on  $V$ . Let  $k_i$  be the centre of  $\mathcal{S}_i$  and let  $\sigma_i$  be the ring of integers of  $k_i$ . Finally, let  $e_i$  be the Schur index of the central simple  $k_i$ -algebra  $\mathcal{S}_i$  and define  $n_i$  by  $\dim_{k_i}(\mathcal{A}_i) = n_i^2$ .

**Theorem 1.3** ([23, §4]) *For almost all  $v \in \mathcal{V}_k$ ,*

$$\zeta_{\mathcal{S}_v \curvearrowright \mathcal{V}_v}(s) = \prod_{i=1}^r \prod_{j=1}^{m_i e_i} \prod_{\substack{w \in \mathcal{V}_{k_i} \\ w|v}} \zeta_{\sigma_i, w}(n_i s - j + 1). \tag{1.3}$$

The special case  $\mathcal{A} = k[\alpha]$  ( $\alpha \in \text{End}_k(V)$ ) of Conjecture E follows from Theorem 1.3 and Theorem 5.1 below.

For a more abstract interpretation of Conjecture E, note that we may identify  $\mathcal{S}$  acting on  $V$  with  $\mathcal{A}/\text{rad}(\mathcal{A})$  acting (faithfully) on the semi-simplification of  $V$  as an  $\mathcal{A}$ -module (i.e. the direct sum of the composition factors of  $V$  as an  $\mathcal{A}$ -module).

### 1.8 Overview

In order to derive Theorems A–D, we proceed as follows. In Sect. 2, we reduce the computation of  $\zeta_{A, \sigma_S}(s)$  to the case that the minimal polynomial of  $A$  over  $k$  is a power of an irreducible polynomial. In Sect. 3, we then further reduce to the case that  $A$  is nilpotent. The heart of this article, Sect. 4, is then devoted to the explicit determination of  $\zeta_{A, \sigma_v}(s)$  for nilpotent  $A$  and almost all  $v \in \mathcal{V}_k$ ; as a by-product, in Theorem 4.4, we compute the ideal zeta function of the 2-dimensional ring  $\mathbb{Z}[[X]]$ . We then combine our findings and derive Theorems A–D in Sect. 5. Finally, as an application, in Sect. 6, we use Theorem A to compute the abscissae of convergence of some (largely unknown) submodule and ideal zeta functions.

### Notation

Throughout,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\delta_{ij}$  denotes the Kronecker symbol. The symbol “ $\subset$ ” indicates not necessarily proper inclusion. We use  $\approx_R$  to denote both the similarity of matrices over  $R$  and the existence of an  $R$ -isomorphism. Matrices act by right-multiplication on row vectors. Matrix sizes are indicated by single subscripts for square matrices and double subscripts in general; in particular,  $1_n$  and  $0_{m,n}$  denote the  $n \times n$  identity and  $m \times n$  zero matrix, respectively.

We say that a property depending on  $S$  holds for sufficiently large finite  $S \subset \mathcal{V}_k$ , if there exists a finite  $S_0 \subset \mathcal{V}_k$  such that the property holds for all finite  $S \subset \mathcal{V}_k$  with  $S \supset S_0$ . Given  $v \in \mathcal{V}_k$ , we write  $|\cdot|_v$  for the  $v$ -adic absolute value on  $k_v$  with  $|\pi|_v = q_v^{-1}$  for  $\pi \in \mathfrak{p}_v \setminus \mathfrak{p}_v^2$ .

By a  $p$ -adic field, we mean a finite extension  $K$  of the  $p$ -adic numbers  $\mathbf{Q}_p$  for some prime  $p$ . We let  $\mathfrak{O}_K$  denote the valuation ring of  $K$  and write  $q_K$  for the residue field size of  $K$ . Furthermore,  $v_K$  and  $|\cdot|_K$  denote the additive valuation and absolute value on  $K$ , respectively, normalised such that any uniformiser  $\pi$  satisfies  $v_K(\pi) = 1$  and  $|\pi|_K = q_K^{-1}$ . When the reference to  $K$  is clear, we occasionally omit the subscript “ $K$ ”.

### 2 Reduction to the case of a primary minimal polynomial

By the following, up to enlarging  $S$ , we may reduce the computation of  $\zeta_{A, \sigma_S}(s)$  to the case where the minimal polynomial of  $A$  over  $k$  is primary (i.e. a power of an irreducible polynomial).

**Proposition 2.1** *Let  $A \in M_n(k)$ . Let  $f = f_1 \cdots f_e$  be a factorisation of the minimal polynomial  $f$  of  $A$  over  $k$  into a product of pairwise coprime monic polynomials  $f_i \in k[X]$ . Let  $A_i \in M_{n_i}(k)$  denote the matrix of  $A$  acting on  $\text{Ker}(f_i(A))$  with respect to an arbitrary  $k$ -basis. Then for almost all  $v \in \mathcal{V}_k$ ,*

$$\zeta_{A, \sigma_v}(s) = \prod_{i=1}^e \zeta_{A_i, \sigma_v}(s).$$

*Proof* It is well-known that  $k^n = \text{Ker}(f_1(A)) \oplus \cdots \oplus \text{Ker}(f_e(A))$  is an  $A$ -invariant decomposition into subspaces of dimensions  $n_1, \dots, n_e$ , say, and  $f_i$  is the minimal polynomial of  $A_i$ . We may thus assume that  $A = \text{diag}(A_1, \dots, A_e)$ . By the Chinese remainder theorem, for each  $i = 1, \dots, e$ , there exists  $g_i \in k[X]$  with  $g_i \equiv \delta_{ij} \pmod{f_j}$  for  $j = 1, \dots, e$ . Hence,  $g_i(A) = \text{diag}(\delta_{i1}1_{n_1}, \dots, \delta_{ie}1_{n_e}) \in k[A]$ . Choose a finite set  $S \subset \mathcal{V}_k$  with  $A_i \in M_{n_i}(\sigma_S)$  and  $g_i \in \sigma_S[X]$  for  $i = 1, \dots, e$ .

Let  $v \in \mathcal{V}_k \setminus S$ . Write  $V := \sigma_v^n$ . The block diagonal shape of  $A$  yields an  $A$ -invariant decomposition  $V = V_1 \oplus \cdots \oplus V_e$  into free  $\sigma_v$ -modules of ranks  $n_1, \dots, n_e$ . Note that  $A$  acts as  $A_i$  on each  $V_i$  and that each  $g_i(A)$  acts as the natural map  $V \rightarrow V_i \hookrightarrow V$ . Let  $U \leq V$  be an  $\sigma_v$ -submodule. If  $U$  is  $A$ -invariant, then it decomposes as  $U = U_1 \oplus \cdots \oplus U_e$  for  $A_i$ -invariant submodules  $U_i \leq V_i$ . We conclude that  $(U_1, \dots, U_e) \mapsto U_1 \oplus \cdots \oplus U_e$  defines a bijection from

$$\left\{ (U_1, \dots, U_e) : U_i \leq_{\sigma_v} V_i \text{ and } U_i A_i \leq U_i \text{ for } i = 1, \dots, e \right\}$$

onto the set of  $A$ -invariant submodules of  $V$  whence  $\zeta_{A, \sigma_v}(s) = \zeta_{A_1, \sigma_v}(s) \cdots \zeta_{A_e, \sigma_v}(s)$ . □

### 3 Reduction to the case of a nilpotent matrix

Recall that  $\mathbf{C}(f)$  denotes the companion matrix of a polynomial  $f$ . Given a partition  $\lambda = (\lambda_1, \dots, \lambda_r)$ , let

$$\mathbf{N}(\lambda) := \text{diag}(\mathbf{C}(X^{\lambda_1}), \dots, \mathbf{C}(X^{\lambda_r})).$$



Suppose that the minimal polynomial of  $A \in M_n(k)$  is a power of an irreducible polynomial  $f$ ; we then say that  $A$  is  $(f\text{-})$ primary. The elementary divisors of  $A$  are  $f^{\lambda_1}, \dots, f^{\lambda_r}$  for a unique partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  of  $n/\deg(f)$ . We call  $\lambda$  the *type* of  $A$ .

For an extension  $k'/k$  of number fields and  $S \subset \mathcal{V}_k$ , define

$$\mathcal{D}_{k'/k}(S) = \{w \in \mathcal{V}_{k'} : \exists v \in S. w \mid v\}.$$

Hence, using the notation from Theorem B,  $\#\mathcal{D}_{k'/k}(S) = \sum_{v \in S} g_v(k')$ .

In this section, we prove the following.

**Theorem 3.1** *Let  $f \in k[X]$  be monic and irreducible. Let  $A \in M_n(k)$  be an  $f$ -primary matrix of type  $\lambda$ . Let  $k' = k[X]/(f)$ , and let  $\mathfrak{o}'$  be the ring of integers of  $k'$ . Then for almost all  $v \in \mathcal{V}_k$ ,*

$$\zeta_{A, \mathfrak{o}_v}(s) = \prod_{\substack{w \in \mathcal{V}_{k'} \\ w \mid v}} \zeta_{N(\lambda), \mathfrak{o}'_w}(s).$$

Hence, for all sufficiently large finite  $S \subset \mathcal{V}_k$ , setting  $S' = \mathcal{D}_{k'/k}(S)$ .

$$\zeta_{A, \mathfrak{o}_S}(s) = \zeta_{N(\lambda), \mathfrak{o}'_{S'}}(s).$$

*Remark 3.2* In [22, §3], the study of the variety of subspaces invariant under an endomorphism of a finite-dimensional real or complex vector space is reduced to the case of a nilpotent endomorphism. Shayman proceeds by first reducing to the case of a primary endomorphism ([22, Thm 2]) and our Proposition 2.1 proceeded along the same lines. In his setting, the minimal polynomial of a primary endomorphism is a power of a linear or quadratic irreducible and he considers these cases separately. His reasoning is similar to arguments employed in our proof of Theorem 3.1 below. We may regard the factorisation of  $\zeta_{A, \mathfrak{o}_v}(s)$  obtained by combining Proposition 2.1 and Theorem 3.1 as an arithmetic analogue of the factorisation of the space of  $A$ -invariant subspaces in [22, Thm 3]. In [22, §4], Shayman then proceeds to study invariant subspaces of nilpotent matrices in Jordan normal form. For our purposes, a slightly different normal form, introduced in Sect. 4.1, will prove advantageous.

Our proof of Theorem 3.1 requires some preparation.

### 3.1 A generalised Jordan normal form for primary matrices

Let  $\otimes$  denote the usual Kronecker product  $[a_{ij}] \otimes B = [a_{ij}B]$  of matrices. The following result is a special case of the “separable Jordan normal form” in [14, §6.2]; it can also be obtained by restriction of scalars from the usual Jordan normal form of an  $f$ -primary matrix over a minimal splitting field of  $f$  over  $k$ .

**Proposition 3.3** *Let  $f \in k[X]$  be monic and irreducible of degree  $d$ . Let  $A \in M_n(k)$  be  $f$ -primary of type  $\lambda$ . Write  $m := n/d$ . Then  $A \approx_k 1_m \otimes \mathbf{C}(f) + \mathbf{N}(\lambda) \otimes 1_d$ .*

**Lemma 3.4** *Let  $f \in k[X]$  be monic and irreducible of degree  $d$ ,  $\lambda \vdash m > 0$ , and  $A = 1_m \otimes \mathbf{C}(f) + \mathbf{N}(\lambda) \otimes 1_d$ . Then  $1_m \otimes \mathbf{C}(f) = \text{diag}(\mathbf{C}(f), \dots, \mathbf{C}(f)) \in k[A]$ .*

*Proof* Write  $\gamma := \mathbf{C}(f)$  and  $e := \lambda_1$ ; note that  $X^e$  is the minimal polynomial of  $\mathbf{N}(\lambda)$  over every field. We may naturally regard  $A$  as an  $m \times m$  matrix over the field  $k' := k[\gamma]$ . Moreover, we may identify  $k' = k[1_m \otimes \mathbf{C}(f)]$  as  $k$ -algebras. Thus,  $k[A, 1_m \otimes \mathbf{C}(f)] = k'[\gamma 1_m + \mathbf{N}(\lambda)] = k'[\mathbf{N}(\lambda)]$  whence the  $k$ -dimension of  $k[A, 1_m \otimes \mathbf{C}(f)]$  is  $|k' : k|e = de$ . As  $f^e$  is the minimal polynomial of  $A$  over  $k$ , the number  $de$  is also the  $k$ -dimension of  $k[A]$  whence the claim follows.  $\square$

Regarding the transition from the number field  $k$  to the local ring  $\mathfrak{o}_v$ , we note that the enveloping algebras of companion matrices take the expected forms over UFDs.

**Lemma 3.5** *Let  $R$  be a UFD and let  $f \in R[X]$  be monic. Then evaluation at  $\mathbf{C}(f)$  induces an isomorphism  $R[X]/(f) \approx_R R[\mathbf{C}(f)]$ .*

*Proof* Let  $K$  denote the field of fractions of  $R$ . The kernel of the natural map  $R[X] \rightarrow R[\mathbf{C}(f)]$  is  $I := R[X] \cap fK[X]$  and, clearly,  $fR[X] \subset I$ . Let  $h \in I$  so that  $h = fg$  for some  $g \in K[X]$ . By [4, Thm 7.7.2], there exists  $a \in K^\times$  with  $af, a^{-1}g \in R[X]$ . As  $f$  is monic (hence primitive),  $a \in A$  whence  $g = a(a^{-1}g) \in R[X]$  and  $h \in fR[X]$ .  $\square$

### 3.2 Properties of $S$ -integers and their completions

**Lemma 3.6** *Let  $k'/k$  be an extension of number fields. Let  $\mathfrak{o}'$  be the ring of integers of  $k'$ . Let  $S \subset \mathcal{V}_k$  be finite and  $S' = \mathcal{D}_{k'/k}(S)$ . Then  $\mathfrak{o}' \otimes_{\mathfrak{o}} \mathfrak{o}_S \approx_{\mathfrak{o}} \mathfrak{o}'_{S'}$ .*

*Proof* The following argument is taken from [5]: if  $h$  is the class number of  $k$  and  $a \in \mathfrak{o}$  generates the principal ideal  $\prod_{v \in S} \mathfrak{p}_v^h$ , then  $\mathfrak{o}_S = \mathfrak{o}[1/a]$ . We conclude that  $\mathfrak{o}' \otimes_{\mathfrak{o}} \mathfrak{o}_S = \mathfrak{o}'[1/a] = \mathfrak{o}'_{S'}$ .  $\square$

**Lemma 3.7** *Let  $f \in k[X]$  be monic and irreducible. Let  $k' = k[X]/(f)$  with ring of integers  $\mathfrak{o}'$ . Then the following holds for all sufficiently large finite  $S \subset \mathcal{V}_k$ :*

- (i)  $\mathfrak{o}_S[X]/(f) \approx_{\mathfrak{o}_S} \mathfrak{o}'_{S'}$ , where  $S' = \mathcal{D}_{k'/k}(S)$ .
- (ii)  $\mathfrak{o}_v[X]/(f) \approx_{\mathfrak{o}_v} \prod_{\substack{w \in \mathcal{V}_{k'} \\ w|v}} \mathfrak{o}'_w$  for  $v \in \mathcal{V}_k \setminus S$ .

*Proof* We freely use the exactness of localisation and completion; see [10, Prop. 2.5, Thm 7.2]. Let  $S_0 \subset \mathcal{V}_k$  be finite with  $f \in \mathfrak{o}_{S_0}[X]$ . If  $S \supset S_0$ , then  $\mathfrak{o}_{S_0}[X]/(f) \otimes_{\mathfrak{o}_{S_0}} \mathfrak{o}_S \approx_{\mathfrak{o}_S} \mathfrak{o}_S[X]/(f)$ . As  $\mathfrak{o}_{S_0}[X]/(f)$  and  $\mathfrak{o}'$  both become isomorphic to  $k'$  after base change to  $k$ , for sufficiently large finite  $S \supset S_0$ ,  $\mathfrak{o}_S[X]/(f) \approx_{\mathfrak{o}_S} \mathfrak{o}'_{S'}$  by Lemma 3.6. This proves the first part. For the second part, first note that, using (i) and Lemma 3.6,

$$\mathfrak{o}_v[X]/(f) \approx_{\mathfrak{o}_v} \mathfrak{o}_S[X]/(f) \otimes_{\mathfrak{o}_S} \mathfrak{o}_v \approx_{\mathfrak{o}_v} \mathfrak{o}'_{S'} \otimes_{\mathfrak{o}_S} \mathfrak{o}_v \approx_{\mathfrak{o}_v} \mathfrak{o}' \otimes_{\mathfrak{o}} \mathfrak{o}_v. \tag{3.1}$$

Write  $\mathfrak{o}_{(v)} := \mathfrak{o}_v \cap k$  for the  $v$ -adic valuation ring of  $k$ . It is easy to see that we may naturally identify  $\mathfrak{o}' \otimes_{\mathfrak{o}} \mathfrak{o}_{(v)}$  with the integral closure of  $\mathfrak{o}_{(v)}$  in  $k'$ . The key observation here is that if  $a \in k'$  is a root of a monic polynomial  $f(X) \in \mathfrak{o}_{(v)}[X]$ , then there exists  $m \in \mathfrak{o}$  with  $v(m) = 0$  and  $ma \in \mathfrak{o}'$ . Indeed, as in the proof of Lemma 3.6, we find  $m \in \mathfrak{o}$  such that for all  $w \in \mathcal{V}_k$ ,  $w(m) > 0$  if and only if some coefficient  $c$  of  $f(X)$  satisfies  $w(c) < 0$ . By replacing  $m$  by a suitable power, we can ensure that all coefficients of  $mf(X)$  belong to  $\mathfrak{o}$  whence  $ma$  is integral over  $\mathfrak{o}$  and thus belongs to  $\mathfrak{o}'$ .

We conclude (see [13, Ch. II, §8, Exerc. 4]) that the canonical isomorphism  $k' \otimes_k k_v \approx_{k_v} \prod_{w|v} k'_w$  ([13, Ch. II, Prop. 8.3]) induces an isomorphism  $\mathfrak{o}' \otimes_{\mathfrak{o}} \mathfrak{o}_v \approx_{\mathfrak{o}_v} \prod_{w|v} \mathfrak{o}'_w$ . Part (ii) thus follows from the latter isomorphism and (3.1).  $\square$

### 3.3 Proof of Theorem 3.1

Recall that  $a_m(A, R)$  denotes the number of  $A$ -invariant  $R$ -submodules of  $R^n$  of index  $m$ , where  $A \in M_n(R)$ .

**Proposition 3.8** *Let  $R_1, \dots, R_r$  be rings with polynomial submodule growth.*

- (i)  $R := R_1 \times \dots \times R_r$  has polynomial submodule growth.
- (ii) (Cf. [23, Lem. 1].) *Let  $A \in M_n(R)$  and let  $A_i$  denote the image of  $A$  under the map  $M_n(R) \rightarrow M_n(R_i)$  induced by the projection  $R \rightarrow R_i$ . Then  $a_m(A, R) = a_m(A_1, R_1) \cdots a_m(A_r, R_r)$  for each  $m \in \mathbf{N}$ . Thus,  $\zeta_{A,R}(s) = \zeta_{A_1,R_1}(s) \cdots \zeta_{A_r,R_r}(s)$ .*

*Proof* Decompose  $R^n = R_1^n \times \dots \times R_r^n$  with  $R$  acting diagonally on  $R^n$ . Multiplication by  $e_i = (\delta_{1i}, \dots, \delta_{ni}) \in R$  acts as the natural map  $R^n \rightarrow R_i^n \rightarrow R^n$ . Given an  $R_i$ -submodule  $U_i \leq R_i^n$  for  $i = 1, \dots, r$ , we obtain an  $R$ -submodule  $U = U_1 \times \dots \times U_r$  of  $R^n$  and it is easy to see that every  $R$ -submodule of  $R^n$  is of this form in a unique way. Evidently,  $U$  has finite index in  $R^n$  if and only if each  $U_i$  has finite index in  $R_i^n$ . Part (i) is immediate and (ii) follows since  $A$  acts as  $A_i$  on  $R_i^n$ .  $\square$

*Proof of Theorem 3.1* Assuming that the finite set  $S \subset \mathcal{V}_k$  is sufficiently large, we can make the following assumptions for all  $v \in \mathcal{V}_k \setminus S$ :

- (NOR)  $A = 1_m \otimes \mathbf{C}(f) + \mathbf{N}(\lambda) \otimes 1_d \in M_n(\mathfrak{o}_v)$  for  $d = \deg(f)$  and  $\lambda \vdash m$  (Proposition 3.3).
- (DIA)  $1_m \otimes \mathbf{C}(f) \in \mathfrak{o}_v[A]$  (Lemma 3.4).
- (INT)  $\mathfrak{o}_v[X]/(f) \approx_{\mathfrak{o}_v} \prod_{w|v} \mathfrak{o}'_w$  (Lemma 3.7).

Let  $v \in \mathcal{V}_k \setminus S$ . First note that as an  $\mathfrak{o}_v$ -module,  $\mathfrak{o}_v[\mathbf{C}(f)]$  is freely generated by  $(1_d, \mathbf{C}(f), \dots, \mathbf{C}(f)^{d-1})$ . It follows easily that  $\mathfrak{o}_v^n$  is free of rank  $m$  as an  $\mathfrak{o}_v[\mathbf{C}(f)]$ -module, where  $\mathbf{C}(f)$  acts as  $1_m \otimes \mathbf{C}(f)$ .

Using Lemma 3.5, (INT) allows us to identify  $\mathfrak{o}_v[\mathbf{C}(f)] = \mathfrak{o}_v[X]/(f) = \prod_{w|v} \mathfrak{o}'_w =: R_v$ . Thanks to (DIA), we may then regard  $A$  as an  $m \times m$  matrix over  $R_v$ . Moreover, the  $A$ -invariant  $\mathfrak{o}_v$ -submodules of  $\mathfrak{o}_v^n$  coincide with the  $A$ -invariant  $R_v$ -submodules of  $R_v^m$ . By (NOR), the latter  $R_v$ -submodules are precisely those invariant

under  $A - C(f) \cdot 1_m = N(\lambda)$ . Therefore,  $\zeta_{A, \sigma_v}(s) = \zeta_{N(\lambda), R_v}(s)$ . Noticing that the  $(0, 1)$ -matrix  $N(\lambda)$  is preserved by each projection  $R_v \rightarrow \sigma'_w$ , Proposition 3.8 shows that  $\zeta_{N(\lambda), R_v}(s) = \prod_{w|v} \zeta_{N(\lambda), \sigma'_w}(s)$  which concludes the proof.  $\square$

### 4 The case of a nilpotent matrix

Let  $\lambda \vdash n$ . Recall the definitions of  $\lambda^{-1}(j)$  from the introduction and of  $N(\lambda)$  from Sect. 3.

**Definition**  $W_\lambda(X, Y) = 1 / \prod_{j=1}^n (1 - X^{j-1} Y^{\lambda^{-1}(j)}) \in \mathbf{Q}(X, Y)$ .

Equivalently,  $W_\lambda(X, Y) = 1 / \prod_{i=1}^{\text{len}(\lambda)} \prod_{j=1}^{\lambda_i} (1 - X^{\sigma_{i-1}(\lambda) + j - 1} Y^i)$ . This section is devoted to proving the following.

**Theorem 4.1** *Let  $\lambda \vdash n$  and let  $K$  be a  $p$ -adic field. Then*

$$\zeta_{N(\lambda^*), \mathfrak{O}_K}(s) = W_\lambda(q_K, q_K^{-s}).$$

Prior to giving a proof of Theorem 4.1, we record a few consequences.

**Corollary 4.2** *Let  $A \in M_n(k)$  be nilpotent of type  $\lambda$  (see Sect. 3). Then for all sufficiently large finite sets  $S \subset \mathcal{V}_k$ ,*

$$\zeta_{A, \sigma_S}(s) = \prod_{j=1}^n \zeta_{\sigma_S} \left( (\lambda^*)^{-1}(j) \cdot s - j + 1 \right).$$

*If  $A \in M_n(\mathfrak{o})$  and  $A \approx_{\mathfrak{o}} N(\lambda)$ , then we may take  $S = \emptyset$ .*

As an application, we can determine the ideal zeta function of  $\mathbf{Z}[X]/(X^n)$ . Recall that  $\zeta(s)$  denotes the Riemann zeta function.

**Corollary 4.3** *For every prime  $p$ ,*

$$\zeta_{\mathbf{Z}_p[X]/(X^n)}(s) = 1 / \prod_{j=1}^n (1 - p^{j-1-j s}).$$

*In particular,*

$$\zeta_{\mathbf{Z}[X]/(X^n)}(s) = \prod_{j=1}^n \zeta(j s - j + 1).$$

*Proof* The matrix of multiplication by  $X$  acting on  $\mathbf{Z}[X]/(X^n)$  with respect to the basis  $(1, X, \dots, X^{n-1})$ , i.e. the companion matrix of  $X^n$ , is precisely  $N((n))$ .  $\square$

*Remark* The subalgebra zeta functions of  $\mathbf{Z}_p[X]/(X^n)$  are known only for  $n \leq 4$  and sufficiently large primes  $p$ . Moreover, the author’s computation of these zeta functions for  $n = 4$  relied on fairly involved machine calculations; see [18, §9.2]. (The formula for  $\zeta_{\mathbf{Z}_p[X]/(X^4)}(s)$  in [18] takes up about a page in total.)

Subobject zeta functions over rings other than  $\sigma_S$  or  $\sigma_V$  have received little attention so far. We obtain the following.

**Theorem 4.4** (i)  $\mathbf{Z}[[X]]$  has polynomial submodule growth.

(ii)  $\zeta_{\mathbf{Z}[[X]]}(s) = \prod_{j=1}^{\infty} \zeta(js - j + 1)$  for  $\text{Re}(s) > 1$ .

*Proof* It is well-known that the maximal ideals of  $\mathbf{Z}[[X]]$  are precisely of the form  $(X, p)$  for a rational prime  $p$ . It follows that  $X$  acts nilpotently on every  $\mathbf{Z}[[X]]$ -module of finite length. Hence, if  $U \leq_{\mathbf{Z}[[X]]} \mathbf{Z}[[X]]^d$  has finite index, then  $U$  contains  $X^n \mathbf{Z}[[X]]^d$  for some  $n \geq 1$ . As  $\mathbf{Z}[[X]]$  is Noetherian,  $U$  thus corresponds to a  $\mathbf{Z}[X]$ -submodule of  $\mathbf{Z}[X]^d / X^n \mathbf{Z}[X]^d$ . In particular, (i) follows since  $\mathbf{Z}[X]$  has polynomial submodule growth by Theorem 1.1. Moreover, Corollary 4.3 implies the identity in (ii) on the level of formal Dirichlet series.

In order to establish (absolute) convergence, let  $s > 1$  be real. By well-known facts on infinite products,  $\prod_{j=1}^{\infty} \zeta(js - j + 1)$  converges (absolutely) if and only if the same is true of  $F(s) := \sum_{j=1}^{\infty} (\zeta(js - j + 1) - 1)$ . Using the non-negativity of the coefficients of each Dirichlet series  $\zeta(js - j + 1)$ , we obtain

$$F(s) = \sum_{j=1}^{\infty} \sum_{n=2}^{\infty} n^{j-1} (n^j)^{-s} = \sum_{n=2}^{\infty} g_n n^{-s},$$

where

$$g_n := n \cdot \sum_{\substack{m \geq 2, j \geq 1 \\ n = m^j}} \frac{1}{m}.$$

We see that for  $N \geq 2$ ,

$$\sum_{n=2}^N g_n \leq N \sum_{\substack{m \geq 2, j \geq 1 \\ m^j \leq N}} \frac{1}{m} \leq N \sum_{m=2}^N \frac{2 \log N}{m} = \mathcal{O}(N(\log N)^2) = \mathcal{O}(N^{1+\varepsilon})$$

for every  $\varepsilon > 0$ . In particular,  $F(s)$  and  $\zeta_{\mathbf{Z}[[X]]}(s)$  both converge for  $\text{Re}(s) > 1$ . □

*Remark 4.5* (i) Note, in particular, that  $\zeta_{\mathbf{Z}[[X]]}(s)$  has an essential singularity at  $s = 1$  and therefore does not admit meromorphic continuation beyond its abscissa of convergence. This illustrates that Theorem 1.2 (ii) does not carry over to general ground rings with polynomial submodule growth.

(ii) In view of Theorem 1.1, it is natural to investigate the ideal growth of univariate polynomial rings over suitable 1-dimensional ground rings. Segal [20]

showed that if  $R$  is a Dedekind domain which is not a field and which has only finitely many ideals of a given finite index, then  $\zeta_{R[X]}(s) = \prod_{j=1}^{\infty} \zeta_R(j^s - j)$  is an identity of formal Dirichlet series. We note that despite the close similarity between Segal’s formula and Theorem 4.4 (ii), our approach is quite different from his.

In order to prove Theorem 4.1, we employ the  $p$ -adic integration machinery from [11]. For a ring  $R$ , let  $\text{Tr}_n(R)$  denote the  $R$ -algebra of upper triangular  $n \times n$ -matrices over  $R$ . Recall that an element of a ring is *regular* if it is not a zero divisor. Write  $\text{Tr}_n^{\text{reg}}(R) = \{\mathbf{x} \in \text{Tr}_n(R) : \det(\mathbf{x}) \in R \text{ is regular}\}$ . For a  $p$ -adic field  $K$ , let  $\mu_K$  denote the Haar measure on  $K^n$  with  $\mu_K(\mathfrak{O}_K^n) = 1$ .

**Proposition 4.6** ([11, §3]) *Let  $K$  be a  $p$ -adic field and  $A \in M_n(\mathfrak{O}_K)$ . Define  $V_K(A) := \{\mathbf{x} \in \text{Tr}_n^{\text{reg}}(\mathfrak{O}_K) : \mathfrak{O}_K^n \mathbf{x} A \subset \mathfrak{O}_K^n \mathbf{x}\}$  to be the set of upper-triangular  $n \times n$  matrices over  $\mathfrak{O}_K$  whose rows span an  $A$ -invariant  $\mathfrak{O}_K$ -submodule of finite index of  $\mathfrak{O}_K^n$ . Then*

$$\zeta_{A, \mathfrak{O}_K}(s) = (1 - q_K^{-1})^{-n} \int_{V_K(A)} |x_{11}|_K^{s-1} |x_{22}|_K^{s-2} \cdots |x_{nn}|_K^{s-n} d\mu_K(\mathbf{x}). \tag{4.1}$$

**Strategy.** In order to prove Theorem 4.1, we proceed as follows. First, in Sect. 4.1, we define a matrix  $\mathbf{A}(\lambda)$  which is similar (over  $\mathbf{Z}$ ) to  $\mathbf{N}(\lambda^*)$  so that  $\zeta_{\mathbf{N}(\lambda^*), \mathfrak{O}_K}(s) = \zeta_{\mathbf{A}(\lambda), \mathfrak{O}_K}(s)$ . As we will see in Sect. 4.2, the advantage of  $\mathbf{A}(\lambda)$  over  $\mathbf{N}(\lambda^*)$  is that the sets  $V_K(\mathbf{A}(\lambda))$  in Proposition 4.6 exhibit a natural, recursive structure. Specifically, we will define  $\mathbf{d}\lambda := (\lambda_2, \dots, \lambda_{\text{len}(\lambda)})$  and find that  $V_K(\mathbf{A}(\lambda))$  can be described in terms of  $V_K(\mathbf{A}(\mathbf{d}\lambda))$  and membership conditions for generic vectors in generic sublattices. In Sect. 4.3, the geometry of such membership conditions is elucidated by means of suitable (birational) changes of coordinates. Finally, in Sect. 4.4, we combine all these ingredients and prove Theorem 4.1.

### 4.1 A dual normal form for nilpotent matrices

**Definition** Let  $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n \geq 0$ . Define  $\mathbf{d}\lambda := (\lambda_2, \dots, \lambda_r)$ . We recursively define  $\mathbf{A}(\lambda) \in M_n(\mathbf{Z})$  as follows:

- (i) If  $r \leq 1$ , define  $\mathbf{A}(\lambda) = 0_n$ .
- (ii) If  $r > 1$ , define

$$\mathbf{A}(\lambda) = \left[ \begin{array}{c|cc} 0_{\lambda_1} & 1_{\lambda_2} & 0_{\lambda_1, \lambda_3 + \dots + \lambda_r} \\ \hline & \mathbf{A}(\mathbf{d}\lambda) & \end{array} \right]. \tag{4.2}$$

In other words,

$$\mathbf{A}(\boldsymbol{\lambda}) = \begin{bmatrix} 0_{\lambda_1} & \begin{array}{|c|} \hline 1_{\lambda_2} \\ \hline 0_{\lambda_1-\lambda_2, \lambda_2} \\ \hline \end{array} & & & \\ & 0_{\lambda_2} & \begin{array}{|c|} \hline 1_{\lambda_3} \\ \hline 0_{\lambda_2-\lambda_3, \lambda_3} \\ \hline \end{array} & & \\ & & \ddots & \ddots & \\ & & & & \begin{array}{|c|} \hline 1_{\lambda_r} \\ \hline 0_{\lambda_{r-1}-\lambda_r, \lambda_r} \\ \hline 0_{\lambda_r} \\ \hline \end{array} \end{bmatrix}. \tag{4.3}$$

By the following, the  $\mathbf{A}(\boldsymbol{\lambda})$  parameterise similarity classes of nilpotent matrices.

**Proposition 4.7**  $\mathbf{A}(\boldsymbol{\lambda}^*)$  and  $\mathbf{N}(\boldsymbol{\lambda})$  are conjugate by permutation matrices.

*Proof* Let  $T(\boldsymbol{\lambda})$  be the Young diagram of  $\boldsymbol{\lambda}$  and let  $V(\boldsymbol{\lambda})$  be the  $\mathbf{Z}$ -module freely generated by the cells of  $T$ . We use “English notation” for Young diagrams—that is,  $T(\boldsymbol{\lambda})$  consists of precisely  $\text{len}(\boldsymbol{\lambda})$  left-justified rows, indexed as  $1, \dots, \text{len}(\boldsymbol{\lambda})$  from top to bottom, such that the  $i$ th row contains precisely  $\lambda_i$  cells. (See below for an example.)

Define  $\Theta(\boldsymbol{\lambda})$  to be the endomorphism of  $V(\boldsymbol{\lambda})$  (acting on the right) which sends each cell to its right neighbour if it exists and to zero otherwise. We consider two orderings on the cells of  $T(\boldsymbol{\lambda})$  and describe the associated matrices representing  $\Theta(\boldsymbol{\lambda})$ . The *horizontal order* is defined by traversing the cells of  $T(\boldsymbol{\lambda})$  from left to right within each row, proceeding from top to bottom. For example, by labelling the cells of  $T((2, 2, 1))$  as  $1, \dots, 5$  according to the horizontal order, we obtain



Clearly,  $\mathbf{N}(\boldsymbol{\lambda})$  is the matrix of  $\Theta(\boldsymbol{\lambda})$  with respect to the horizontal order.

The *vertical order* is obtained by traversing the cells of  $T(\boldsymbol{\lambda})$  from top to bottom within each column, proceeding from left to right. In the case of  $T((2, 2, 1))$  from above, the vertical order is thus given by



Write  $\boldsymbol{\mu} := \boldsymbol{\lambda}^*$ , say  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_\ell)$ . We now show by induction on  $\ell$  that the matrix of  $\Theta(\boldsymbol{\lambda})$  with respect to the vertical order is  $\mathbf{A}(\boldsymbol{\mu})$ —it then follows, in particular, that  $\mathbf{A}(\boldsymbol{\mu})$  and  $\mathbf{N}(\boldsymbol{\lambda})$  are conjugate as claimed.

If  $\ell \leq 1$ , then  $\Theta(\boldsymbol{\lambda}) = 0$  and  $\mathbf{A}(\boldsymbol{\mu}) = 0$  so let  $\ell > 1$ . Let  $t_1, \dots, t_n$  be the cells of  $T(\boldsymbol{\lambda})$  according to the vertical order. Then  $t_i \Theta(\boldsymbol{\lambda}) = t_{\mu_1+i}$  for  $1 \leq i \leq \mu_2$  and

$t_i \Theta(\lambda) = 0$  for  $\mu_2 < i \leq \mu_1$ . Let  $\tilde{\lambda} := (\mathbf{d}\mu)^*$  and  $\tilde{V} := \mathbf{Z}t_{\mu_1+1} \oplus \cdots \oplus \mathbf{Z}t_n$ . We may naturally identify the endomorphism of  $\tilde{V}$  induced by  $\Theta(\lambda)$  with  $\Theta(\tilde{\lambda})$  acting on  $V(\tilde{\lambda})$ ; the defining basis of  $\tilde{V}$  is then ordered vertically. By induction, the matrix of  $\Theta(\lambda)$  acting on  $\tilde{V}$  with respect to the basis  $(t_{\mu_1+1}, \dots, t_n)$  is therefore  $\mathbf{A}(\mathbf{d}\mu)$  whence the claim follows from the recursive description of  $\mathbf{A}(\mu)$  in (4.2).  $\square$

For  $|\lambda| > 0$ , let  $\mathbf{B}(\lambda) \in M_{|\lambda|, |\mathbf{d}\lambda|}(\mathbf{Z})$  denote the matrix obtained by deleting the first  $\lambda_1$  columns of  $\mathbf{A}(\lambda)$ . The following consequence of (4.3) will be useful below.

**Lemma 4.8**  *$\mathbf{B}(\lambda)$  contains precisely  $\lambda_1$  zero rows and by deleting these, the  $|\mathbf{d}\lambda| \times |\mathbf{d}\lambda|$  identity matrix is obtained.*

### 4.2 Recursion

In this subsection, we give a recursive description of  $V_K(\mathbf{A}(\lambda))$  (see Proposition 4.6).

**Lemma 4.9** *Let  $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$  and let  $X$  be the generic upper triangular  $n \times n$  matrix. Partition  $X$  in the form*

$$X = \left[ \begin{array}{ccc|c} X^I_{\lambda_2} & * & & \bar{X}_{\lambda_1, |\mathbf{d}\lambda|} \\ 0_{\lambda_1 - \lambda_2, \lambda_2} & X^{\text{II}}_{\lambda_1 - \lambda_2} & & \\ \hline & 0 & & X'_{|\mathbf{d}\lambda|} \end{array} \right],$$

where subscripts are added to denote block sizes. Then

$$XA(\lambda) = \left[ \begin{array}{ccc|c} 0_{\lambda_1} & X^I & \bar{X}\mathbf{B}(\mathbf{d}\lambda) & \\ \hline & 0 & X'\mathbf{A}(\mathbf{d}\lambda) & \end{array} \right].$$

*Proof* This follows easily from (4.2).  $\square$

By Lemmas 4.8–4.9, the  $\lambda_1 \times |\mathbf{d}\lambda|$  submatrix obtained by considering the first  $\lambda_1$  rows of  $XA(\lambda)$  and then deleting the first  $\lambda_1$  columns is of the form

$$X^\lambda := \left[ \begin{array}{cccccc} x_{1,1} & \cdots & x_{1,\lambda_2} & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & x_{\lambda_2,\lambda_2} & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & & * & \cdots & * \end{array} \right], \tag{4.4}$$

where the entries marked “\*” indicate unspecified but *distinct* variables taken from  $\bar{X}$ .



**Corollary 4.10** *Let  $\lambda \vdash n$  and let  $K$  be a  $p$ -adic field. For  $\mathbf{x} \in \text{Tr}_n(K)$ , define  $\mathbf{x}'$  and  $\mathbf{x}^\lambda$  by specialising  $X'$  and  $X^\lambda$  from Lemma 4.9 and (4.4), respectively, at  $\mathbf{x}$ . Then*

$$V_K(\mathbf{A}(\lambda)) = \left\{ \mathbf{x} \in \text{Tr}_n^{\text{reg}}(\mathfrak{O}_K) : \begin{array}{l} \text{(i) each row of } \mathbf{x}^\lambda \text{ belongs to } \mathfrak{O}_K^{|\text{d}\lambda|} \mathbf{x}' \text{ and} \\ \text{(ii) } \mathbf{x}' \in V_K(\mathbf{A}(\text{d}\lambda)) \end{array} \right\}. \tag{4.5}$$

*Proof* Let  $\mathbf{x} \in \text{Tr}_n^{\text{reg}}(\mathfrak{O}_K)$ . Clearly,  $\mathbf{x} \in V_K(\mathbf{A}(\lambda))$  if and only if every row of  $\mathbf{x}\mathbf{A}(\lambda)$  is contained in the  $\mathfrak{O}_K$ -span of the rows of  $\mathbf{x}$ . By Lemma 4.9 and since  $\det(\mathbf{x}) \neq 0$ , the first  $\lambda_1$  rows of  $\mathbf{x}\mathbf{A}(\lambda)$  satisfy this condition if and only if every row of  $\mathbf{x}^\lambda$  is contained in the  $\mathfrak{O}_K$ -span of the rows of  $\mathbf{x}'$ . Similarly, the rows numbered  $\lambda_1 + 1, \dots, n$  of  $\mathbf{x}\mathbf{A}(\lambda)$  are contained in the  $\mathfrak{O}_K$ -span of  $\mathbf{x}$  if and only if each row of  $\mathbf{x}'\mathbf{A}(\text{d}\lambda)$  is contained in the  $\mathfrak{O}_K$ -span of  $\mathbf{x}'$  or, equivalently, if  $\mathbf{x}' \in V_K(\mathbf{A}(\text{d}\lambda))$ .  $\square$

### 4.3 Characterising submodule membership

Condition (i) in (4.5) leads us to investigate pairs  $(\mathbf{x}, \mathbf{y}) \in R^n \times \text{Tr}_n(R)$  (where  $R$  is a ring) such that  $\mathbf{x}$  is contained in the row span of  $\mathbf{y}$  over  $R$ . In this subsection, we study the set of all such pairs  $(\mathbf{x}, \mathbf{y})$  in the case that  $R = \mathfrak{O}_K$  for a  $p$ -adic field  $K$ .

We write  $\mathbf{A}^n = \text{Spec}(\mathbf{Z}[X_1, \dots, X_n])$  and  $\text{Tr}_n = \text{Spec}(\mathbf{Z}[Y_{ij} : 1 \leq i \leq j \leq n])$ . Let

$$E_n(R) := \{(\mathbf{x}, \mathbf{y}) \in R^n \times \text{Tr}_n(R) : \mathbf{x} \in R^n \mathbf{y}\}. \tag{4.6}$$

We identify  $\mathbf{A}^n \times \text{Tr}_n = \text{Spec}(\mathbf{Z}[X_1, \dots, X_n, Y_{11}, \dots, Y_{1n}, Y_{22}, \dots, Y_{nn}])$ . Define

$$\mathcal{C}_n := \{(\alpha, \omega) \in \mathbf{R}_{\geq 0}^n \times \text{Tr}_n(\mathbf{R}_{\geq 0}) : \omega_{ii} \leq \alpha_i \text{ for } 1 \leq i \leq n\}. \tag{4.7}$$

For a  $p$ -adic field  $K$ , we extend  $v_K$  to families of elements of  $K$  via  $v_K(a_1, \dots, a_m) = (v_K(a_1), \dots, v_K(a_m))$  and write

$$\mathcal{C}_n(K) := \left\{ (\mathbf{x}, \mathbf{y}) \in K^n \times \text{Tr}_n(K) : (v_K(\mathbf{x}), v_K(\mathbf{y})) \in \mathcal{C}_n \right\} \subset \mathfrak{O}_K^n \times \text{Tr}_n^{\text{reg}}(\mathfrak{O}_K).$$

The following lemma will play a key role in our proof of Theorem 4.1. It shows that away from sets of measure zero, a suitable  $\mathbf{Z}$ -defined change of coordinates (defined independently of  $K$ ) transforms  $E_n(\mathfrak{O}_K)$  into  $\mathcal{C}_n(K)$ .

**Lemma 4.11** *There exist*

- *closed subschemes  $V_n, V'_n \subset \mathbf{A}^n \times \text{Tr}_n$  of the form  $f_n = 0$  and  $f'_n = 0$ , respectively, where  $f_n, f'_n \in \mathbf{Z}[X, Y]$  are non-zero non-units, and*
- *an isomorphism  $\varphi_n : (\mathbf{A}^n \times \text{Tr}_n) \setminus V_n \rightarrow (\mathbf{A}^n \times \text{Tr}_n) \setminus V'_n$*

*such that the following conditions are satisfied:*

- (i) *For each  $p$ -adic field  $K$ ,  $\varphi_n^K(E_n(\mathfrak{O}_K) \setminus V_n(\mathfrak{O}_K)) = \mathcal{C}_n(K) \setminus V'_n(\mathfrak{O}_K)$ , where  $\varphi_n^K$  denotes the map induced by  $\varphi_n$  on  $K$ -points.*
- (ii) *The Jacobian determinant of  $\varphi_n$  is identically 1.*

(iii)  $\varphi_n$  commutes with (the appropriate restrictions of) the projection of  $\mathbf{A}^n \times \text{Tr}_n$  onto  $\text{Tr}_n$  and (the restrictions of) the projection onto the first coordinate of  $\mathbf{A}^n$ .

*Example* ( $n = 2$ ) Let  $K$  be a  $p$ -adic field; we drop the subscripts “ $K$ ” in the following. Let  $x, y, a, b, c \in \mathfrak{D}$  and suppose that  $x(ay - bx)abc \neq 0$ . Define  $y' := y - \frac{x}{a}b \in K$  and note that  $y' \neq 0$ . Then  $(x, y) \in \mathfrak{D}^2 \cdot \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$  if and only if  $v(a) \leq v(x)$  and  $(x, y) - \frac{x}{a}(a, b) = (0, y') \in \mathfrak{D}(0, c)$ ; the latter condition is equivalent to  $v(c) \leq v(y')$  and implies that  $y' \in \mathfrak{D}$ . We see that the map  $((x, y), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) \mapsto ((x, y'), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix})$  has the properties of  $\varphi_2$  stated in Lemma 4.11.

*Proof of Lemma 4.11* We proceed by induction. For  $n = 1$ , we let  $f_1 = f'_1 = X_1 Y_{11}$  and define  $\varphi_1$  to be the identity. Clearly, (i)–(iii) are satisfied.

Let  $n > 1$  and suppose that  $\varphi_{n-1}$  with the stated properties has been defined. Let  $K$  be a  $p$ -adic field and let  $(\mathbf{x}, \mathbf{y}) \in K^n \times \text{Tr}_n(K)$  with  $x_1 y_{11} \neq 0$ . We again drop the subscripts “ $K$ ”. Gaussian elimination shows that  $(\mathbf{x}, \mathbf{y}) \in E_n(\mathfrak{D})$  if and only if the following conditions are satisfied:

- (a)  $x_i, y_{ij} \in \mathfrak{D}$  for  $1 \leq i \leq j \leq n$ ,
- (b)  $\frac{x_1}{y_{11}} \in \mathfrak{D}$ , and
- (c)  $(x_2 - \frac{x_1}{y_{11}}y_{12}, \dots, x_n - \frac{x_1}{y_{11}}y_{1n}) \in \mathfrak{D}^{n-1} \cdot \begin{bmatrix} y_{ij} \end{bmatrix}_{2 \leq i \leq j \leq n}$ .

We will now simplify (c) using a change of coordinates. For  $2 \leq j \leq n$ , let  $x'_j := x_j - \frac{x_1}{y_{11}}y_{1j}$ . Write  $x'_1 := x_1$  and  $\mathbf{x}' := (x'_1, \dots, x'_n)$ . Note that  $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y})$  is an automorphism of the complement of  $Y_{11} = 0$  in  $\mathbf{A}^n \times \text{Tr}_n$  and that the Jacobian determinant of this map is identically 1.

Assuming that  $y_{ij} \in \mathfrak{D}$  for  $1 \leq i \leq j \leq n$  and  $\frac{x_1}{y_{11}} \in \mathfrak{D}$ , we see that  $x_j \in \mathfrak{D}$  if and only if  $x'_j \in \mathfrak{D}$ . Hence,  $(\mathbf{x}, \mathbf{y}) \in E_n(\mathfrak{D})$  if and only if (b) and the following two conditions are satisfied:

- (a')  $x'_i, y_{ij} \in \mathfrak{D}$  for  $1 \leq i \leq j \leq n$ ,
- (c')  $(x'_2, \dots, x'_n) \in \mathfrak{D}^{n-1} \cdot \begin{bmatrix} y_{ij} \end{bmatrix}_{2 \leq i \leq j \leq n}$ .

After excluding suitable hypersurfaces, our inductive hypothesis allows us to perform another change of coordinates, replacing  $x'_2, \dots, x'_n$  by  $x''_2, \dots, x''_n$ , say, such that  $(\mathbf{x}, \mathbf{y}) \in E_n(K)$  if and only if the following conditions are satisfied:

- (a'')  $x''_i, y_{ij} \in \mathfrak{D}$  for  $1 \leq i \leq j \leq n$  (where  $x''_1 := x'_1 = x_1$ ) and
- (c'')  $v(y_{ii}) \leq v(x''_i)$  for  $1 \leq i \leq n$ ;

note that (b) is implied by the case  $i = 1$  of (c'').

For (i), assuming that the product of all  $x''_i$  and  $y_{ij}$  is non-zero, conditions (a'') and (c'') are both satisfied if and only if  $(\mathbf{x}'', \mathbf{y}) \in \mathcal{C}_n(K)$ , where  $\mathbf{x}'' := (x''_1, \dots, x''_n)$ . The change of coordinates  $\mathbf{x} \mapsto \mathbf{x}''$  is defined over  $\mathbf{Z}$ , does not depend on  $K$ , and, does not modify the  $x_1$ - or  $y$ -coordinate, as required for (iii); part (ii) follows since  $\varphi_n$  is defined as a composite of maps, the Jacobian determinant of each of which is identically 1. □

*Remark 4.12* It follows from Lemma 4.11(ii) that the change of variables afforded by  $\varphi_n$  does not affect  $p$ -adic measures. Moreover, it is well-known that if  $0 \neq f \in$

$\mathfrak{O}_K[X_1, \dots, X_n]$ , then the zero locus of  $f$  in  $\mathfrak{O}_K^n$  has measure zero. We conclude that  $V_n$  and  $V'_n$  in Lemma 4.11 are without relevance for the computation of the integral in Proposition 4.6.

### 4.4 Final steps towards Theorem 4.1

By combining Corollary 4.10 and Lemma 4.11, we may reduce the computation of the integral in Proposition 4.6 for  $A = \mathbf{A}(\lambda)$  to a purely combinatorial problem.

**Proposition 4.13** *Let  $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$  and let  $K$  be a  $p$ -adic field. Then*

$$\zeta_{\mathbf{A}(\lambda), \mathfrak{O}_K}(s) = (1 - q_K^{-1})^{-n} \int_{V_\lambda(\mathfrak{O}_K)} \prod_{i=1}^n |x_i|_K^{s-i} d\mu_K(\mathbf{x}), \tag{4.8}$$

where  $V_\lambda(\mathfrak{O}_K)$  consists of those  $\mathbf{x} \in \mathfrak{O}_K^{n(n+1)/2}$  satisfying the following divisibility conditions, where the  $y_{i,j,\ell}$  below denote distinct variables among the  $x_{n+1}, \dots, x_{n(n+1)/2}$ :

- For  $2 \leq i \leq r$  and  $1 \leq j \leq \lambda_i$ ,

$$x_{\sigma_{i-1}(\lambda)+j} \mid x_{\sigma_{i-2}(\lambda)+j}, y_{i,j,1}, \dots, y_{i,j,j-1}.$$

- For  $3 \leq i \leq r$  and  $\sigma_{i-1}(\lambda) < j \leq n$ ,

$$x_j \mid y_{i,j,n+1}, \dots, y_{i,j,n+\lambda_{i-2}}.$$

*Remark* Since the  $y_{i,j,\ell}$  do not appear in the integrand in the right-hand side of (4.8), it is of no consequence precisely which of the  $x_{n+1}, \dots, x_{n(n+1)/2}$  each  $y_{i,j,\ell}$  refers to provided that distinct triples  $(i, j, \ell)$  yield different  $y_{i,j,\ell}$ .

*Proof of Proposition 4.13* If  $r \leq 1$ , the claim is trivially true so let  $r \geq 2$ .

As our first step, we combine Corollary 4.10 and Lemma 4.11 in order to transform the membership condition (i) in (4.5). Recall that the non-zero entries of  $X^\lambda$  in (4.4) are distinct variables from  $X^l$  or  $\bar{X}$ ; in particular, none of the variables in  $X^\lambda$  occurs in  $X^l$ . We may therefore use Lemma 4.11 to transform the membership condition for any fixed row of  $\mathbf{x}^\lambda$  to be contained in  $\mathfrak{O}_K^{|\mathbf{d}\lambda|} \mathbf{x}'$ . Condition (iii) in Lemma 4.11 now ensures that the coordinates corresponding to the variables in  $X^l$  remain unchanged by each such transformation. By applying Lemma 4.11 to each row of  $\mathbf{x}^\lambda$  in turn, we thus obtain the given divisibility conditions for  $i = 2$  and  $i = 3$ , respectively; here,  $x_1, \dots, x_n$  correspond to the diagonal entries  $x_{11}, \dots, x_{nn}$  in Proposition 4.6. Condition (iii) in Lemma 4.11 further ensures that the coordinates corresponding to the diagonal entries of  $X$  remain unchanged. Condition (ii) in Lemma 4.11 thus implies that the preceding transformations do not affect the integrand in (4.1).

Having transformed condition (i) in (4.5), subsequent steps then recursively apply the same procedure in order to express the condition  $\mathbf{x}' \in V_K(\mathbf{A}(\mathbf{d}\lambda))$  in Corollary 4.10

in terms of the stated divisibility conditions, taking into account the evident shifts of variable indices. Crucially, in doing so, none of the diagonal coordinates will ever be modified, again thanks to condition (iii) in Lemma 4.11. Therefore, the divisibility conditions obtained during earlier steps will never be altered by subsequent ones. The claim thus follows by induction.  $\square$

*Proof of Theorem 4.1* We once again omit subscripts “ $K$ ” in the following. Moreover, we will make repeated use of the identity

$$\int_{\{(x,y) \in \mathcal{D}^2 : x|y\}} |x|^r |y|^s d\mu(x, y) = \int_{\mathcal{D}^2} |x|^{r+s+1} |y|^s d\mu(x, y) \tag{4.9}$$

which follows by performing a change of variables  $y = xy'$  on the left-hand side. We will furthermore use the well-known identity  $\int_{\mathcal{D}} |x|^s d\mu(x) = (1 - q^{-1}) / (1 - q^{-s-1})$ .

By repeatedly applying (4.9), we can eliminate all the  $y_{i,j,\ell}$  variables and rewrite (4.8) as an integral over  $\mathcal{D}^n$ . In order to record the effect of this procedure on the integrand, we use  $\lambda$  to index  $x_1, \dots, x_n$  as follows. Let  $f(i, j) := \sigma_{i-1}(\lambda) + j$  and, for  $\mathbf{x} = (x_1, \dots, x_n)$ , write  $x_{ij} := x_{f(i,j)}$ . Define

$$U_\lambda(\mathcal{D}) := \left\{ \mathbf{x} \in \mathcal{D}^n : x_{i,j} \mid x_{i-1,j} \text{ for } 2 \leq i \leq r \text{ and } 1 \leq j \leq \lambda_i \right\}.$$

Proposition 4.7 and repeated applications of (4.9) to (4.8) show that

$$\zeta_{N(\lambda^*), \mathcal{D}}(s) = \zeta_{A(\lambda), \mathcal{D}}(s) = (1 - q^{-1})^{-n} \int_{U_\lambda(\mathcal{D})} F_\lambda(\mathbf{x}) d\mu(\mathbf{x}),$$

where

$$\begin{aligned} F_\lambda(\mathbf{x}) &= \prod_{i=1}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{s-f(i,j)} \times \prod_{i=2}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{j-1} \times \prod_{a=3}^r \prod_{i=a}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{\lambda_{a-2}} \\ &= \prod_{j=1}^{\lambda_1} |x_{1j}|^{s-j} \times \prod_{i=2}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{s-(\lambda_{i-1}+1)}; \end{aligned}$$

the second equality follows since  $s - f(i, j) + j - 1 + \sum_{a=3}^i \lambda_{a-2} = s - (\lambda_{i-1} + 1)$  for  $2 \leq i \leq r$  and  $1 \leq j \leq \lambda_i$ . Another sequence of applications of (4.9) can be used to remove the divisibility conditions in  $U_\lambda(\mathcal{D})$ , yielding

$$\begin{aligned} (1 - q^{-1})^n \zeta_{A(\lambda), \mathcal{D}}(s) &= \int_{\mathcal{D}^n} \prod_{j=1}^{\lambda_1} |x_{1j}|^{s-j} \times \prod_{i=2}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{s-j+i-1 + \sum_{a=1}^{i-1} (s-(\lambda_a+1))} d\mu(\mathbf{x}) \\ &= \int_{\mathcal{D}^n} \prod_{i=1}^r \prod_{j=1}^{\lambda_i} |x_{ij}|^{is - (\sigma_{i-1}(\lambda) + j)} d\mu(\mathbf{x}) \end{aligned}$$

$$\begin{aligned}
 &= (1 - q^{-1})^n \cdot \prod_{i=1}^r \prod_{j=1}^{\lambda_i} (1 - q^{-is + \sigma_{i-1}(\lambda) + j - 1})^{-1} \\
 &= (1 - q^{-1})^n \cdot W_\lambda(q, q^{-s}).
 \end{aligned}$$

□

### 5 Proofs of Theorems A–D

At the heart of our proofs of Theorems A–D lies the following local version of Theorem A.

**Theorem 5.1** *Let  $S \subset \mathcal{V}_k$  be finite and  $A \in M_n(\mathfrak{o}_S)$ . Let  $((f_1, \lambda_1), \dots, (f_e, \lambda_e))$  be an elementary divisor vector of  $A$  over  $k$ . Write  $k_i = k[X]/(f_i)$ . Let  $\mathfrak{o}_i$  denote the ring of integers of  $k_i$ . Then for almost all  $v \in \mathcal{V}_k$ ,*

$$\zeta_{A, \mathfrak{o}_v}(s) = \prod_{i=1}^e \prod_{j=1}^{|\lambda_i|} \prod_{\substack{w \in \mathcal{V}_{k_i} \\ w|v}} \zeta_{\mathfrak{o}_{i,w}}((\lambda_i^*)^{-1}(j) \cdot s - j + 1). \tag{5.1}$$

*Proof* Combine Proposition 2.1, Theorem 3.1, and Theorem 4.1. □

The following is a consequence of Proposition 4.6 and well-known rationality results from  $p$ -adic integration.

**Proposition 5.2** *(Cf. [11, §3]) Let  $K$  be a  $p$ -adic field and let  $A \in M_n(\mathfrak{O}_K)$ . Then  $\zeta_{A, \mathfrak{O}_K}(s) \in \mathbf{Q}(q_K^{-s})$ . Hence,  $\zeta_{A, \mathfrak{O}_K}(s)$  admits meromorphic continuation to all of  $\mathbf{C}$ .*

In order to deduce parts (ii)–(iii) of Theorem A, we will use the following corollary to the detailed analysis of analytic properties of subobject zeta functions in [7].

**Lemma 5.3** *Let  $S' \subset \mathcal{V}_k$  be finite,  $S \subset S'$ , and let  $A \in M_n(\mathfrak{o}_S)$ . Then  $\alpha_{A, \mathfrak{o}_S} = \alpha_{A, \mathfrak{o}_{S'}}$  and  $\beta_{A, \mathfrak{o}_S} = \beta_{A, \mathfrak{o}_{S'}}$ .*

*Proof* We first argue that  $\alpha_{A, \mathfrak{o}_v} < \alpha_{A, \mathfrak{o}_S}$  for each  $v \in \mathcal{V}_k \setminus S$ . The zeta function  $\zeta_{A, \mathfrak{o}_S}(s + n)$  is an Euler product of cone integrals (cf. Proposition 4.6) in the sense of [7, Def. 4.2]; cf. [7, Cor. 5.6]. Using the notation from [7], by [7, Cor. 3.4] (which is correct despite a minor, fixable mistake in [7, Prop. 3.3], see [1, Rem. 4.6]), it follows that each  $\alpha_{A, \mathfrak{o}_v}$  for  $v \in \mathcal{V}_k \setminus S$  is a number of the form  $n - B_j/A_j$  for  $j = 1, \dots, q$ . Hence, by combining [7, Cor. 4.14, Lem. 4.15], for each  $v \in \mathcal{V}_k \setminus S$ ,

$$\alpha_{A, \mathfrak{o}_v} < n + \max_{k=1, \dots, q} \frac{1 - B_k}{A_k} = \alpha_{A, \mathfrak{o}_S}.$$

Clearly,  $0 < \alpha_{A, \mathfrak{o}_{S'}} \leq \alpha_{A, \mathfrak{o}_S}$ . Define  $F(s) = \prod_{v \in S' \setminus S} \zeta_{A, \mathfrak{o}_v}(s)$  so that  $\zeta_{A, \mathfrak{o}_S}(s) = F(s)\zeta_{A, \mathfrak{o}_{S'}}(s)$  for all  $s \in \mathbf{C}$  with  $\text{Re}(s) > \alpha_{A, \mathfrak{o}_S} - \delta$  and some constant  $\delta > 0$  (see Theorem 1.2). By the above, every real pole of  $F(s)$  is less than  $\alpha_{A, \mathfrak{o}_S}$ . Since

$F(s)$  is a non-zero Dirichlet series with non-negative coefficients, we conclude that  $F(\alpha_{A, \sigma_S}) > 0$ . In particular, since  $\zeta_{A, \sigma_S}(s)$  has a pole at  $\alpha_{A, \sigma_S}$ , the same is true of  $\zeta_{A, \sigma_{S'}}(s)$  whence  $\alpha_{A, \sigma_{S'}} \geq \alpha_{A, \sigma_S}$ . Moreover,  $F(\alpha_{A, \sigma_S}) > 0$  clearly also implies that  $\beta_{A, \sigma_S} = \beta_{A, \sigma_{S'}}$ .  $\square$

*Remark 5.4* (i) The corresponding statement for subalgebra and submodule zeta functions (proved in the same way) is certainly well-known to experts in the area. Unfortunately, it does not seem to have been spelled out in the literature. For a similar statement in the context of representation zeta functions, see [2, Thm 1.4].  
 (ii) While in [7] only the case  $k = \mathbf{Q}, S = \emptyset$  is discussed, their arguments carry over to the present setting in the expected way (cf. [1] and [9, §4]).

*Proof of Theorem A* Part (i) follows from Theorem 5.1 and Proposition 5.2. Let  $\mu \vdash n$ . We now determine the largest real pole,  $\alpha$  say, and its multiplicity,  $\beta$  say, of

$$Z(s) := \prod_{j=1}^n \zeta_{\sigma_S}(\mu^{-1}(j) \cdot s - j + 1).$$

Write  $r = \text{len}(\mu)$ . Since  $\zeta_{\sigma_S}(s)$  has a unique pole at 1 (with multiplicity 1) and  $\zeta_{\sigma_S}(s_0) \neq 0$  for real  $s_0 > 1$ ,

$$\alpha = \max_{1 \leq j \leq n} \frac{j}{\mu^{-1}(j)} = \max_{1 \leq i \leq r} \max_{1 \leq j \leq \lambda_i} \frac{\sigma_{i-1}(\mu) + j}{i} = \max_{1 \leq i \leq r} \frac{\sigma_i(\mu)}{i} = \mu_1 = \text{len}(\mu^*),$$

where the penultimate equality follows since  $i\mu_{i+1} \leq \sigma_i(\mu)$  and thus  $\frac{\sigma_i(\mu)}{i} \geq \frac{\sigma_{i+1}(\mu)}{i+1}$  for  $1 \leq i \leq r - 1$ . Next,  $\beta$  is precisely the number of  $i \in \{1, \dots, r\}$  with  $\mu_1 = \frac{\sigma_i(\mu)}{i}$  or, equivalently, the largest  $\ell \geq 1$  with  $\mu_1 = \dots = \mu_\ell$ . In other words,  $\beta = \mu_{-1}^*$ .

Parts (ii)–(iii) of Theorem A now follow from Lemma 5.3 and the observation that  $Z(s) > 0$  for  $s > \alpha$ .  $\square$

*Example 5.5* The presence of the exceptional factors  $W_u(q_{w_u}^{-s})$  in Theorem A is in general unavoidable. For a simple example, let  $a \in \mathfrak{o}$  be non-zero and define  $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ . Using Proposition 4.6, a simple computation reveals that for  $v \in \mathcal{V}_k$ ,

$$\zeta_{A, \sigma_v}(s) = \frac{1 - q_v^{1-2s} + q_v^{(1-s)(v(a)+1)} \cdot (q_v^{-s} - 1)}{1 - q_v^{1-s}} \cdot \zeta_{\sigma_v}(s)\zeta_{\sigma_v}(2s - 1); \tag{5.2}$$

note that  $\zeta_{A, \sigma_v}(s) = \zeta_{\sigma_v}(s)\zeta_{\sigma_v}(2s - 1)$  whenever  $v(a) = 0$ . We further note that the exceptional factor in (5.2) in fact belongs to  $\mathbf{Z}[q_v^{-s}]$  and is thus regular at  $s = 1$ ; this is consistent with the general fact that for subobject zeta functions, each local abscissa of convergence is strictly less than the associated global one (see the proof of Lemma 5.3). Finally note the failure of (1.2) for the finitely many  $v \in \mathcal{V}_k$  with  $v(a) > 0$ .

*Remark 5.6* In view of a conjecture of Solomon proved by Bushnell and Reiner [3], it is natural to ask if the  $W_u \in \mathbf{Q}(X)$  in Theorem A are in fact always elements of  $\mathbf{Z}[X]$ .

*Proof of Theorem B* The claim follows by combining Theorem 5.1 and the following simple observation. Let  $k'/k$  be an extension of number fields, let  $\sigma'$  be the ring of integers of  $k'$ , and let  $v \in \mathcal{V}_k$  be unramified in  $k'$ . If  $w \in \mathcal{V}_{k'}$  divides  $v$ , define  $f(w/v)$  by  $q_w = q_v^{f(w/v)}$ . Define

$$Z_v(s) = \prod_{\substack{w \in \mathcal{V}_{k'} \\ w|v}} \zeta_{\sigma'_w}(s) = \prod_{\substack{w \in \mathcal{V}_{k'} \\ w|v}} (1 - q_v^{-f(w/v)s})^{-1}.$$

Then, recalling the definition of  $g_v(k')$  from p.5 and using  $\sum_{w|v} f(w/v) = |k' : k|$ ,

$$Z_v(s) \Big|_{q_v \rightarrow q_v^{-1}} = (-1)^{g_v(k')} q_v^{-|k':k|s} \cdot Z_v(s).$$

□

**Lemma 5.7** *Let  $S \subset \mathcal{V}_k$  be finite. Let  $Z(s)$  and  $Z'(s)$  be two Dirichlet series with finite abscissae of convergence. Suppose that  $Z(s) = \prod_{v \in \mathcal{V}_k \setminus S} Z_v(s)$  and  $Z'(s) = \prod_{v \in \mathcal{V}_k \setminus S} Z'_v(s)$ , where each  $Z_v(s)$  and  $Z'_v(s)$  is a series in  $q_v^{-s}$  with non-negative real coefficients. Suppose that  $Z(s) = Z'(s)$  and that  $W(X, Y), W'(X, Y) \in \mathbf{Q}(X, Y)$  satisfy  $Z_v(s) = W(q_v, q_v^{-s})$  and  $Z'_v(s) = W'(q_v, q_v^{-s})$  for almost all  $v \in \mathcal{V}_k \setminus S$ . Then  $W(X, Y) = W'(X, Y)$ .*

*Proof* Let  $S_0$  be the set of rational primes which are divisible by at least one element of  $S$ . For a rational prime  $p \notin S_0$ , define  $Z_p(s) = \prod_{v \in \mathcal{V}_k, v|p} Z_v(s)$  and define  $Z'_p(s)$  in the same way. Assuming that  $Z(s) = Z'(s)$ , it is well-known that the coefficients of the Dirichlet series  $Z(s)$  and  $Z'(s)$  coincide. We conclude that  $Z_p(s) = Z'_p(s)$  for  $p \notin S_0$ . By Chebotarev’s density theorem, there exists an infinite set of rational primes  $P$  such that each  $p \in P$  splits completely in  $k$ . Writing  $d = |k : \mathbf{Q}|$ , for almost all  $p \in P$ , we thus have  $W(p, p^{-s})^d = Z_p(s) = Z'_p(s) = W'(p, p^{-s})^d$  which easily implies  $W(X, Y)^d = W'(X, Y)^d$ . Thus,  $W(X, Y)/W'(X, Y)$  is a  $d$ th root of unity in  $\mathbf{R}(X, Y)$  and hence in  $\mathbf{R}$ , for the latter is algebraically closed in the former (see [4, Prop. 11.3.1]). The non-negativity assumptions on the coefficients of  $Z_v(s)$  and  $Z'_v(s)$  as series in  $q_v^{-s}$  now imply  $W(X, Y) = W'(X, Y)$ . □

*Proof of Theorem C* The implications “(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)” in Theorem C are obvious. Suppose that (iii) holds. Let  $\lambda$  and  $\mu$  be the types of the matrices  $A$  and  $B$ , respectively. By Theorem 5.1 and the preceding lemma,  $W_\lambda(X, Y) = W_\mu(X, Y)$ . It is easy to see that the binomials  $1 - X^a Y^b$  for  $a \geq 0$  and  $b \geq 1$  freely generate a free abelian subgroup of  $\mathbf{Q}(X, Y)^\times$ . Hence,  $\lambda = \mu$  and  $A$  and  $B$  are similar. □

*Remark 5.8* If  $A$  is nilpotent and  $\alpha \in k^\times$ , then  $A$  and  $A + \alpha 1_n$  give rise to the same local and global zeta functions without  $A$  and  $A + \alpha 1_n$  being similar. In general, equality of local and global zeta functions associated with non-nilpotent matrices  $A$  and  $B$  does not suffice to even conclude that the algebras  $k[A]$  and  $k[B]$  are similar. We give two examples to illustrate this behaviour, the first being arithmetic and the second of combinatorial origin.

- (i) By [15], there are monic irreducible polynomials  $f, g \in \mathbf{Z}[X]$  of the same degree such that the number fields  $\mathbf{Q}[X]/(f)$  and  $\mathbf{Q}[X]/(g)$  are non-isomorphic but have the same Dedekind zeta functions; moreover, as explained in [15, §1], every rational prime has the same “splitting type” in each of these two number fields. Consequently,  $\zeta_{\mathbf{C}(f), \mathbf{Z}_p}(s) = \zeta_{\mathbf{C}(g), \mathbf{Z}_p}(s)$  for almost all primes  $p$
- (ii) Recall the definition of  $W_\lambda$  from Sect. 4. A simple calculation shows that

$$W_{(2,2,1)} \cdot W_{(3,1)} = W_{(2,2)} \cdot W_{(3,1,1)}.$$

Let  $a, b \in k^\times$  be distinct and choose  $A, B \in M_9(k)$  to have elementary divisor vectors  $((X - a, (3, 2), (X - b, (2, 1, 1)))$  and  $((X - a, (2, 2)), (X - b, (3, 1, 1)))$ , respectively. Then  $k[A]$  and  $k[B]$  are not similar but  $\zeta_{A, \mathfrak{o}_v}(s) = \zeta_{B, \mathfrak{o}_v}(s)$  for almost all  $v \in \mathcal{V}_k$ .

*Remark 5.9* We further note that even for nilpotent  $A$ , the family of associated functional equations (1.2) in Theorem B does not determine  $A$  up to similarity; an example is given by two nilpotent  $7 \times 7$ -matrices with types  $(3, 1, 1, 1, 1)$  and  $(2, 2, 2, 1)$ , respectively.

*Proof of Theorem D* By Theorem 5.1,  $\zeta_{A, \mathfrak{o}_v}(s)$  has a pole at zero for almost all  $v \in \mathcal{V}_k$ . Moreover, again for almost all  $v \in \mathcal{V}_k$ , this pole is simple if and only if  $e = 1$  and almost all places of  $k$  remain inert in  $k[X]/(f_1)$ ; the latter condition is equivalent to  $f_1$  being linear. □

## 6 Applications

### 6.1 Submodules for unipotent groups

Let  $S \subset \mathcal{V}_k$  be finite, let  $M$  be a finitely generated  $\mathfrak{o}_S$ -module, and let  $\Omega \subset \text{End}_{\mathfrak{o}_S}(M)$ . We let  $\alpha_{\Omega \curvearrowright M}$  denote the abscissa of convergence of  $\zeta_{\Omega \curvearrowright M}(s)$ . As a special case (cf. [17, Rem. 2.2 (ii)]), given a possibly non-associative  $\mathfrak{o}_S$ -algebra  $A$  whose underlying  $\mathfrak{o}_S$ -module is finitely generated, we let  $\alpha_A$  denote the abscissa of convergence of its ideal zeta function  $\zeta_A(s)$ , as defined in the introduction. We now illustrate how Theorem A can sometimes be used to determine  $\alpha_{\Omega \curvearrowright M}$  or  $\alpha_A$  without computing the corresponding zeta function. The key observation is that if  $\omega \in \Omega$ , then  $\alpha_{\Omega \curvearrowright M} \leq \alpha_{\omega, \mathfrak{o}}$ ; by Theorem A(ii), the latter number can be easily read off from an elementary divisor vector of  $\omega \otimes_{\mathfrak{o}_S} k$ .

We let  $U_n$  denote the group scheme of upper unitriangular  $n \times n$  matrices. For  $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ , we regard  $U_\lambda := U_{\lambda_1} \times \dots \times U_{\lambda_r}$  as a subgroup scheme of  $U_n$  via the natural diagonal embedding. The case  $\text{len}(\lambda) = 1$  of the following has consequences for the ideal growth of nilpotent Lie algebras, see [18, §9.4].

**Proposition 6.1** *Let  $\lambda \vdash n$ . Then  $\alpha_{U_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n} = \text{len}(\lambda)$ .*

*Proof* Using the characterisation of  $U_m(k)$  as the centraliser of a maximal flag of subspaces of  $k^m$ , we see that  $\mathfrak{o}^n$  contains a  $U_\lambda(\mathfrak{o})$ -invariant submodule  $N$  such that  $U_\lambda(\mathfrak{o})$  acts trivially on  $\mathfrak{o}^n/N$  and  $\mathfrak{o}^n/N \approx_{\mathfrak{o}} \mathfrak{o}^{\text{len}(\lambda)}$ . We conclude that  $\alpha_{U_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n} \geq$



$\text{len}(\lambda)$ . For an upper bound, note that  $(1 + N(\lambda)) \in U_\lambda(\mathfrak{o})$  whence  $\alpha_{U_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n} \leq \alpha_{N(\lambda), \mathfrak{o}} = \text{len}(\lambda)$ .  $\square$

For  $|\lambda| \leq 5$  and almost all  $v \in \mathcal{V}_k$ , explicit formulae for  $\zeta_{U_\lambda(\mathfrak{o}_v) \curvearrowright \mathfrak{o}_v^n}(s)$  have been obtained by the author (see [18, §9.4] and the database included with [19]); the only unknown case for  $\text{len}(\lambda) = 6$ , namely  $\lambda = (6)$ , seems out of reach at present. In addition to their global abscissae of convergence, the  $\zeta_{U_\lambda(\mathfrak{o}_v) \curvearrowright \mathfrak{o}_v^n}(s)$  are known to generically satisfy local functional equations under inversion of  $q_v$  by [25, §5.2].

### 6.2 Lie algebras of maximal class

Let  $\mathfrak{g}$  be a finite-dimensional Lie  $k$ -algebra. For finite  $S \subset \mathcal{V}_k$ , by an  $\mathfrak{o}_S$ -form of  $\mathfrak{g}$ , we mean a Lie  $\mathfrak{o}_S$ -algebra  $\mathfrak{g}$  whose underlying module is free and such that  $\mathfrak{g} \otimes_{\mathfrak{o}_S} k \approx_k \mathfrak{g}$ .

Let  $\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots$  be the lower central series of  $\mathfrak{g}$ . Recall that  $\mathfrak{g}$  has maximal class if  $\mathfrak{g}$  is nilpotent of class  $\dim_k(\mathfrak{g}) - 1$ . Equivalently,  $\mathfrak{g}$  has maximal class if and only if  $\dim_k(\mathfrak{g}^1/\mathfrak{g}^2) = 2$  and  $\dim_k(\mathfrak{g}^i/\mathfrak{g}^{i+1}) = 1$  for  $1 \leq i \leq \dim_k(\mathfrak{g}) - 1$ .

**Proposition 6.2** *Let  $\mathfrak{g}$  be an  $\mathfrak{o}_S$ -form of a non-abelian finite-dimensional Lie  $k$ -algebra of maximal class. Then  $\alpha_{\mathfrak{g}} = 2$ .*

A proof of Proposition 6.2 using Theorem A will be given below.

We note that Proposition 6.2 is consistent with explicit calculations carried out for specific  $\mathbf{Z}$ -forms of the Lie algebras  $M_3, M_4, M_5$ , and  $\text{Fil}_4$  of maximal class and dimension at most 5 over the rationals; see [8, Ch. 2].

**Lemma 6.3** *Let  $S \subset \mathcal{V}_k$  be finite. Let  $\mathfrak{g}$  be an  $\mathfrak{o}_S$ -form of a nilpotent Lie  $k$ -algebra of finite dimension  $n$ . Let  $\mathbf{A}$  be the enveloping unital associative algebra of  $\text{ad}(\mathfrak{g})$  within  $\text{End}_{\mathfrak{o}_S}(\mathfrak{g})$ .*

- (i) *For each  $\varphi \in \mathbf{A}$ , there exists  $c \in \mathfrak{o}_S$  with  $(\varphi - c1_{\mathfrak{g}})^n = 0$ ; thus,  $\varphi \otimes_{\mathfrak{o}_S} k$  is primary.*
- (ii) *Let  $\varphi \in \mathbf{A}$  have type  $\lambda$  over  $k$ . Then  $\alpha_{\mathfrak{g}} \leq \text{len}(\lambda)$ .*

*Proof* The first part follows from Engel’s theorem and the second part is then an immediate consequence of Theorem A(ii).  $\square$

**Lemma 6.4** *Let  $\mathfrak{g}$  be an  $(n + 2)$ -dimensional non-abelian Lie  $k$ -algebra of maximal class. Then there exists a  $k$ -basis  $(x_1, x_2, y_1, \dots, y_n)$  of  $\mathfrak{g}$  such that  $[x_1, x_2] = y_1$ ,  $[x_1, y_i] = y_{i+1}$  for  $1 \leq i \leq n - 1$ , and  $[x_1, y_n] = 0$ .*

*Proof* Consider the graded Lie algebra  $\bigoplus_{i \geq 1} \mathfrak{g}^i/\mathfrak{g}^{i+1}$  associated with  $\mathfrak{g}$ . We claim that there exists an element  $a \in \mathfrak{g}/\mathfrak{g}^2$  such that  $[a, \cdot]$  maps  $\mathfrak{g}^i/\mathfrak{g}^{i+1}$  onto  $\mathfrak{g}^{i+1}/\mathfrak{g}^{i+2}$  for each  $i \geq 1$ . To see that, first note that  $[\mathfrak{g}/\mathfrak{g}^2, \mathfrak{g}^i/\mathfrak{g}^{i+1}] = \mathfrak{g}^{i+1}/\mathfrak{g}^{i+2}$  for each  $i \geq 1$ . Let  $(u, v)$  be a  $k$ -basis of  $\mathfrak{g}/\mathfrak{g}^2$ . Then  $[u, v]$  spans  $\mathfrak{g}^2/\mathfrak{g}^3$ . Moreover, if  $w_i$  spans  $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ , then the image of at least one of  $[u, w_i]$  and  $[v, w_i]$  spans  $\mathfrak{g}^{i+1}/\mathfrak{g}^{i+2}$ . Consequently, we may take  $a = u + cv$  for almost all  $c \in k$ .

Given  $a$  as above, choose  $b \in \mathfrak{g}/\mathfrak{g}^2$  such that  $(a, b)$  is a basis of  $\mathfrak{g}/\mathfrak{g}^2$ . Let  $x_1, x_2 \in \mathfrak{g}$  be preimages of  $a$  and  $b$ , respectively. Then, if we define  $y_1 = [x_1, x_2]$  and  $y_{i+1} = [x_1, y_i]$ , we obtain a basis  $(x_1, x_2, y_1, \dots, y_n)$  of the desired form.  $\square$

*Proof of Proposition 6.2* By Lemma 5.3 and Remark 5.4(i), we are free to enlarge  $S$  as needed. In particular, we may assume that  $\mathfrak{g}/\mathfrak{g}^2 \approx_{\mathfrak{o}_S} \mathfrak{o}_S^2$  whence  $\alpha_{\mathfrak{g}} \geq 2$  follows. Moreover, we may assume that  $\mathfrak{g}$  possesses an  $\mathfrak{o}_S$ -basis  $(x_1, x_2, y_1, \dots, y_n)$  as in by Lemma 6.4. The matrix of  $[x_1, \cdot]$  with respect to the basis  $(x_2, y_1, \dots, y_n, x_1)$  is precisely  $N((n+1, 1))$  whence  $\alpha_{\mathfrak{g}} \leq 2$  follows from Theorem A.  $\square$

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