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Dynamics of the Ericksen–Leslie equations with general Leslie stress I: the incompressible isotropic case

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Abstract The Ericksen–Leslie model for nematic liquid crystals in a bounded domain with general Leslie and isotropic Ericksen stress tensor is studied in the case of a non-isothermal and incompressible fluid. This system is shown to be locally strongly well-posed in the L_p-setting, and a dynamical theory is developed. The equilibria are identified and shown to be normally stable. In particular, a local solution extends to a unique, global strong solution provided the initial data are close to an equilibrium or the solution is eventually bounded in the topology of the natural state manifold. In this case, the solution converges exponentially to an equilibrium, in the topology of the state manifold. The above results are proven *without* any structural assumptions on the Leslie coefficients and in particular *without* assuming Parodi's relation.

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1 Introduction: the Ericksen–Leslie model with general Leslie stress

In their pioneering articles, Ericksen [7] and Leslie [21] developed a continuum theory for the flow of nematic liquid crystals. Their theory models nematic liquid crystal

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flow from a hydrodynamical point of view and describes the evolution of the underlying system under the influence of the velocity u of the fluid and the orientation configuration d of rod-like liquid crystals. We already observe here that the modulus $|d|_2$ of the director field d must equal 1 pointwise, as d represents a direction field. For more information see e.g. [1,3,5,35] or [15,27]. The original derivation [7,21] is based on the conservation laws for mass, linear and angular momentums as well as on constitutive relations given by Leslie in [21].

Following arguments from thermodynamics and employing the entropy principle, we proposed in [14,15] thermodynamically consistent models of Ericksen–Leslie type, even in the case of compressible fluids. Let us emphasize that these models *contain the classical Ericksen–Leslie model in its general form as a special case*.

A related class of models also dealing with the non-isothermal situation was presented by Feireisl et al. [8] as well as by Feireisl, Frémond, Rocca and Schimperna in [9]. Their models include stretching as well as rotational terms and are consistent with the fundamental laws of thermodynamics. The equation for the director d is, however, given in the penalized form, which does not seem to be physical. They show that the presence of the term $|\nabla d|_2^2$ in the internal energy as well as the stretching term $d \cdot \nabla u$ give rise, in order to respect the laws of thermodynamics, to two new non dissipative contributions in the stress tensor S and in the flux q. It is interesting to note that these two new contributions coincide with the extra terms derived by Sun and Liu [36] by different methods. It seems that these extra terms are non physical and arise there for purely mathematical reasons.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with smooth enough boundary, the general Ericksen–Leslie model in the non-isothermal situation derived as in [14,15] reads as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } \Omega, \\ \rho \mathcal{D}_t u + \nabla \pi = \operatorname{div} S & \text{in } \Omega, \\ \rho \mathcal{D}_t \epsilon + \operatorname{div} q = S : \nabla u - \pi \operatorname{div} u + \operatorname{div}(\rho \partial_{\nabla d} \psi \mathcal{D}_t d) & \text{in } \Omega, \\ \gamma \mathcal{D}_t d - \mu_V V d = P_d\big(\operatorname{div}(\rho \frac{\partial \psi}{\partial \nabla d}) - \rho \nabla_d \psi\big) + \mu_D P_d D d & \text{in } \Omega, \\ u = 0, \quad q \cdot \nu = 0 & \text{on } \partial\Omega, \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) = \theta_0, \quad d(0) = d_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here the unknown variables ρ, u denote the density and velocity of the fluid, respectively, ϵ the internal energy and d the so called director, which - we recall - must have modulus 1. Moreover, $D = \frac{1}{2}(\nabla u]^T + \nabla u)$ denotes the rate of deformation tensor, the vorticity tensor V defined by $V = \frac{1}{2}(\nabla u - [\nabla u]^T)$ is skew-symmetric, and q denotes the heat flux. By $\mathcal{D}_t = \partial_t + u \cdot \nabla$ we denote the Lagrangian derivative and P_d is defined as $P_d = I - d \otimes d$.

Note that the condition $|d|_2 = 1$ is preserved by smooth solutions, as $P_d d = 0$ as well as $(V d|d) = 0$, hence $\mathcal{D}_t |d|_2^2 = 0$.

In addition, ψ denotes the free energy, which, following Oseen [29] and Frank [12], see also Virga [38], is given by the Oseen-Frank functional $\psi^F(d, \nabla d)$ as

$$\begin{aligned} \psi^F(d, \nabla d) := k_1(\operatorname{div} d)^2 + k_2(d \cdot \operatorname{curl} d)^2 + k_3|d \times \operatorname{curl} d|^2 \\ +(k_2 + k_4)(tr(\nabla d)^2 - (\operatorname{div} d)^2), \end{aligned} \tag{1.2}$$

where $k_1, \ldots, k_4$ are the so called *Frank coefficients*, which may depend on ρ and θ.

These equations have to be supplemented by the thermodynamical laws

$$\epsilon = \psi + \theta\eta, \quad \eta = -\partial_\theta \psi, \quad \kappa = \partial_\theta \epsilon, \quad \pi = \rho^2 \partial_\rho \psi, \tag{1.3}$$

and by the constitutive laws

$$\begin{cases} S = S_N + S_E + S_L^{stretch} + S_L^{diss}, \\ S_N = 2\mu_s D + \mu_b \mathrm{div}\, u\, I, \\ S_E = -\rho \frac{\partial\psi}{\partial\nabla d}[\nabla d]^{\mathsf{T}}, \\ S_L^{stretch} = \frac{\mu_D+\mu_V}{2\gamma} \mathsf{n}\otimes d + \frac{\mu_D-\mu_V}{2\gamma} d\otimes \mathsf{n}, \quad \mathsf{n} = \mu_V V d + \mu_D P_d D d - \gamma \mathcal{D}_t d, \\ S_L^{diss} = \frac{\mu_P}{\gamma}(\mathsf{n}\otimes d + d\otimes \mathsf{n}) + \frac{\gamma\mu_L + \mu_P^2}{2\gamma}(P_d D d\otimes d + d\otimes P_d D d) + \mu_0 (Dd|d) d\otimes d, \end{cases} \tag{1.4}$$

and

$$q = -\tilde{\alpha}_0 \nabla\theta - \tilde{\alpha}_1 (d|\nabla\theta) d. \tag{1.5}$$

Here all coefficients μ_j, $\tilde{\alpha}_j$ and γ are functions of $\rho, \theta, d, \nabla d$. For thermodynamical consistency we require the conditions

$$\mu_s \geq 0, \quad 2\mu_s + n\mu_b \geq 0, \quad \tilde{\alpha}_0 \geq 0, \quad \tilde{\alpha}_0 + \tilde{\alpha}_1 \geq 0, \quad \mu_0, \mu_L \geq 0, \quad \gamma > 0. \tag{1.6}$$

We also note that the natural boundary condition at $\partial\Omega$ for d becomes

$$\nu_i \nabla_{\partial_i d} \psi = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Observe that this condition is *fully nonlinear*, in general. Physically, it means that the boundary does not interact with the director field. Otherwise one would have to model such interactions and it seems to be unclear whether this could be done in a physically consistent way by simply imposing Dirichlet boundary conditions. For this reason we employ the *Neumann condition* for d throughout this paper. Actually, we can prove local well-posedness also in the case of Dirichlet boundary conditions, but the set of equilibria becomes more complicated in this case. Our results on stability and long time behaviour are only valid for constant equilibria.

In the case of *isotropic elasticy* with constant density and temperature one has $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ and so the Oseen-Frank energy reduces to the Dirichlet energy, i.e.

$$\psi(d, \nabla d) := \psi^F(d, \nabla d) = \frac{1}{2}|\nabla d|^2,$$

and thus div $\left(\frac{\partial \psi^F}{\partial(\partial\nabla d)}\right) = \Delta d$. Then the Ericksen stress tensor simplifies to

$$S_E = -\lambda \nabla d [\nabla d]^{\mathsf{T}}, \tag{1.8}$$

where $\lambda = \rho\partial_\tau\psi$, and the natural boundary condition at $\partial\Omega$ for d becomes the Neumann condition $\partial_\nu d = 0$ on $\partial\Omega$.

It is the aim of this article to investigate the above Ericksen–Leslie system analytically, in the case of isotropic elasticity and for incompressible as well as compressible fluids. To this end, Part I of this article will concentrate on the case of incompressible fluids, whereas Part II will investigate the compressible case. Here *incompressibility* means that the density ρ is constant and *isotropy* means that the free energy ψ is a function of ϱ, θ and $\tau = |\nabla d|_2^2/2$, only.

For the convenience of the reader, we rewrite the model in the incompressible and isotropic case, which then reads as follows.

$$\begin{cases} \rho\mathcal{D}_t u + \nabla\pi = \operatorname{div} S & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \rho\mathcal{D}_t\epsilon + \operatorname{div} q = S : \nabla u + \operatorname{div}(\lambda\nabla d\mathcal{D}_t d) & \text{in } \Omega, \\ \gamma\mathcal{D}_t d - \mu_V V d - \operatorname{div}[\lambda\nabla]d = \lambda|\nabla d|^2 d + \mu_D P_d D d & \text{in } \Omega, \\ u = 0, \quad q\cdot\nu = 0, \quad \partial_\nu d = 0 & \text{on } \partial\Omega, \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) = \theta_0, \quad d(0) = d_0 & \text{in } \Omega. \end{cases} \tag{1.9}$$

These equations have to be supplemented by the thermodynamical laws for the internal energy ϵ, entropy η, heat capacity κ given in (1.3), by Ericksen's tension $\lambda = \partial\rho_\tau\psi$, and by the constitutive laws (1.4) as above, with Ericksen stress of the form (1.8), and with q satifying

$$q = -\alpha\nabla\theta. \tag{1.10}$$

Note that all coefficients μ_j, α and γ are functions of θ and τ, by the principle of equi-presence.

For further purposes, it is convenient to write the equation for the internal energy as an equation for the temperature θ. It reads as

$$\rho\kappa\mathcal{D}_t\theta + \operatorname{div} q = S : \nabla u + \operatorname{div}(\lambda\nabla)d\cdot\mathcal{D}_t d + (\theta\partial_\theta\lambda)\nabla d\nabla\mathcal{D}_t d.$$

Observe the appearance of unusual third order terms due to the presence of $\mathcal{D}_t d$ in the Leslie stress S_L as well as in the last term of the energy balance. This alludes a peculiarity of the system, which has to be overcome in the analysis.

Let us emphasize that in the case $\mu_V = \gamma$, our parameters $\mu_s, \mu_0, \mu_V, \mu_D, \mu_P, \mu_L$ are in one-to-one correspondence to the celebrated Leslie parameters $\alpha_1, \dots, \alpha_6$ given in the Leslie stress σ_L defined by

$$\sigma_L := \alpha_1(d^T Dd)d\otimes d + \alpha_2 N\otimes d + \alpha_3 d\otimes N + \alpha_4 D + \alpha_5(Dd)\otimes d + \alpha_6 d\otimes(Dd), \tag{1.11}$$

where D denotes the deformation tensor as above and

$$N := N(u,d) := \partial_t d + (u\cdot\nabla)d - Vd,$$

with V as above. This shows that our model (1.9), (1.3), (1.4) *contains the classical isothermal and isotropic Ericksen–Leslie model* given by

$$\begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla \pi = -\operatorname{div}\,(\nabla d[\nabla d]^T) + \operatorname{div}\,\sigma_L & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\, u = 0 & \text{in } (0,T)\times\Omega, \\ d_t + u\nabla d - Vd + \frac{\lambda_2}{\lambda_1} Dd = -\frac{1}{\lambda_1}(\Delta d + |\nabla d|^2 d) + \frac{\lambda_2}{\lambda_1}(d^T Dd)d & \text{in } (0,T)\times\Omega, \\ (u, \partial_\nu d) = (0,0) & \text{on } (0,T)\times\partial\Omega, \\ (u,d)_{|t=0} = (u_0, d_0) & \text{in } \Omega. \end{cases} \tag{1.12}$$

as a special case; here we have $\lambda_1 = -\gamma/\lambda$ and $\lambda_2 = \mu_D/\lambda$. Note that in the simplified model no third order terms appear, which considerably simplifies this problem.

It was shown by Parodi [30] in 1970 that, *assuming Onsager's reciprocal relation*, one is lead to the relation

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \tag{1.13}$$

where the coefficients α_j denote the Leslie coefficients introduced above.

The analysis of the Ericksen–Leslie system began by the pioneering work of Lin [22,23] and Lin and Liu [24,25], who introduced and studied the nowadays called *isothermal simplified model*. They studied the situation where the nonlinearity in the equation for d is replaced by a Ginzburg-Landau energy functional and proved the existence of global weak solutions under suitable assumptions on the intial data. Wang proved in [39] global well-posedness for the simplified system for initial data being small in $BMO^{-1} \times BMO$ in the case of a whole space $\Omega = \mathbb{R}^n$ by combining techniques of Koch and Tataru with methods from harmonic maps.

Concerning the situation of *bounded domains*, a rather complete understanding of the well-posedness as well as the dynamics of the simplified system subject to Neumann conditions for d was obtained in [13]. First results on the existence of global weak solutions to the simplified system subject to Dirichlet boundary conditions in two dimensions go back to Lin et al. [26]. Recently, considering the simplified system in three dimensions subject to Dirichlet conditions $d = d_b$ on $\partial\Omega$, Huang et al. [17] constructed examples of small initial data for which one has finite time blow up of (u, d).

It seems that Coutard and Shkoller [4] were the first who considered so called *stretching terms* analytically in the equation for d. More precisely, they replaced the equation for d in the *simplified model* by a Ginzburg-Landau type approximation *including stretching* of the form

$$\gamma(\partial_t d + u \cdot \nabla d + d \cdot \nabla u) = \Delta d - \frac{1}{\varepsilon^2}(|d|_2^2 - 1)d \quad \text{in } (0,T)\times\Omega. \tag{1.14}$$

They proved local well-posedness for (1.14) as well as a global existence result for small data. Note, however, that in this case the presence of the stretching term $d \cdot \nabla u$ causes loss of total energy balance and, moreover, the condition $|d|_2 = 1$ in $(0,T)\times\Omega$, is not preserved anymore.

For recent results on the general Ericksen–Leslie model with vanishing Leslie and general Ericksen stress we refer to the articles [16,28]. For results including stretching terms for d, we refer to the articles [14,15,18,26,40,41], which contain well-posedness criteria for the general system under various assumptions on the Leslie coefficients.

The main new idea in investigating the general Ericksen–Leslie equations analytically in the strong sense is, similiarly to the situation of the simplified and isothermal system treated in [13], to regard them as a quasilinear parabolic evolution equation. Restricting ourselves in Part I to the case of *incompressible fluids* and to the case of isotropic elasticity, we present in the following a rather complete dynamic theory for the underlying equations. It seems to be the first well-posedness result for the *Ericksen–Leslie equations dealing with general Leslie stress* S_L *without assuming additional conditions on the Leslie coefficients*.

The results given in the three main theorems below answer all questions concerning well-posedness, stability and longtime behaviour for the *general* Ericksen–Leslie system subject to Neumann boundary conditions for d in a very satisfactory way. It is proved by means of techniques involving maximal L_p-regularity and quasilinear parabolic evolution equations. For these methods, we refer to the booklet by Denk, Hieber, and Prüss [6], the articles [19,20,31,34] and to the monograph by Prüss and Simonett [33]. For the convenience of the reader we have summarized the relevant results from these papers in Sect. 3.

Let us also emphasize that for obtaining our well-posedeness results in the strong sense *no structural conditions* on the Leslie coefficients are imposed and that in particular *Parodi's relation* (1.13) on the Leslie coefficients is *not* being assumed.

Moreover, the equilibria of the system have been identified in our recent paper [14]—which are zero velocities and constant density, temperature and director—and there it also has been proved that these are thermodynamically stable. The negative total entropy has been shown to be a strict Lyapunov functional; in particular, the model is thermodynamically consistent.

We further mention at this point the series of articles [8–11], in which it is shown that their particular systems admit a global weak solution for a natural class of initial data.

For more information on modeling and analysis of the Ericksen–Leslie system, we refer e.g. to [3,5] and to the survey articles [15,27].

2 Thermodynamical stability and consistency

In this short section we recall from [14,15] that the above model (1.9), (1.3), (1.4), (1.10) has the following thermodynamical properties. To this end, we introduce the following

Assumption (P):

$$\mu_s > 0, \quad \alpha > 0, \quad \mu_0, \mu_L \geq 0, \quad \kappa, \gamma > 0, \quad \lambda, \lambda + 2\tau\partial_\tau\lambda > 0. \tag{2.1}$$

Theorem 2.1 ([14], Theorem 1) *Assume that condition (P) holds. Then the incompressible and isotropic model* (1.9)*,* (1.3)*,* (1.4)*,* (1.10) *has the following properties:*

(i) Along smooth solutions the total energy

$$\mathsf{E} := \int_\Omega \Big[\frac{\rho}{2}|u|_2^2 + \rho\epsilon\Big]\,dx$$

is preserved.

(ii) Along smooth solutions the total entropy

$$\mathsf{N} := \int_\Omega \rho\eta\,dx$$

is non-decreasing.

(iii) The negative total entropy $-\mathsf{N}$ is a strict Lyapunov functional.

(iv) The condition $|d|_2 = 1$ is preserved along smooth solutions.

(v) The equilibria of the system are given by the set of constants

$$\mathcal{E} = \{(0, \theta_*, d_*) : \ \theta_* \in (0, \infty),\ d_* \in \mathbb{R}^n,\ |d_*|_2 = 1\}. \tag{2.2}$$

Here θ_* is uniquely determined by the identity

$$\epsilon(\theta_*, 0) = \mathsf{E}_0/\rho|\Omega|.$$

(vi) The equilibria are the critical points of the total entropy with prescribed energy.

(vii) The second variation of N with prescribed energy at an equilibrium is negative semi-definite.

3 Background: quasilinear parabolic evolution equations

In this section we briefly recall some results on abstract quasilinear parabolic problems

$$\dot{v} + A(v)v = F(v), \quad t > 0, \quad v(0) = v_0, \tag{3.1}$$

which are employed in the proofs of our main theorems. These results are due to Prüss [31], Prüss and Simonett [32], Köhne et al. [19], and Prüss et al. [34]; a convenient reference for this theory is the monograph by Prüss and Simonett [33], Chapter 5.

Assume that $(A, F) : V_\mu \to \mathcal{L}(X_1, X_0) \times X_0$ and $v_0 \in V_\mu$. Here the spaces X_1, X_0 are Banach spaces such that $X_1 \hookrightarrow X_0$ with dense embedding and V_μ is an open subset of the real interpolation space

$$X_{\gamma,\mu} := (X_0, X_1)_{\mu-1/p,p}, \quad \mu \in (1/p, 1].$$

We are mainly interested in solutions v of (3.1) having *maximal L_p-regularity*, i.e.

$$v \in H^1_p(J; X_0) \cap L_p(J; X_1) =: \mathbb{E}_1(J), \quad \text{where } J = (0, T).$$

The trace space of this class of functions is given by $X_\gamma := X_{\gamma,1}$. However, to see and exploit the effect of parabolic regularization in the L_p-framework it is also useful to consider solutions in the class of weighted spaces

$$v \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) =: \mathbb{E}_{1,\mu}(J), \quad \text{which means } \; t^{1-\mu} v \in \mathbb{E}_1(J).$$

The trace space for this class of weighted spaces is given by $X_{\gamma,\mu}$. In our approach it is crucial to know that the operators $A(v)$ have the property of *maximal L_p-regularity*. Recall that an operator A_0 in X_0 with domain X_1 has maximal L_p-regularity, if the linear problem

$$\dot{v} + A_0 v = f, \quad t \in J, \; v(0) = 0,$$

admits a unique solution $v \in \mathbb{E}_1(J)$, for any given $f \in L_p(J; X_0) =: \mathbb{E}_0(J)$. It has been proved in [32] that in this case maximal regularity also holds in the weighted spaces.

Proposition 3.1 *Let $p \in (1, \infty)$, $v_0 \in V_\mu$ be given and suppose that (A, F) satisfies*

$$(A, F) \in C^1(V_\mu; \mathcal{L}(X_1, X_0) \times X_0), \tag{3.2}$$

for some $\mu \in (1/p, 1]$. Assume in addition that $A(v_0)$ has maximal L_p-regularity.

Then there exist $a = a(v_0) > 0$ and $r = r(v_0) > 0$ with $\bar{B}_{X_{\gamma,\mu}}(v_0, r) \subset V_\mu$ such that problem (3.1) has a unique solution

$$v = v(\cdot, v_1) \in \mathbb{E}_{1,\mu}(0, a) \cap C([0, T]; V_\mu),$$

on $[0, a]$, for any initial value $v_1 \in \bar{B}_{X_{\gamma,\mu}}(v_0, r)$. In addition,

$$t \partial_t v \in \mathbb{E}_{1,\mu}(0, a),$$

in particular, for each $\delta \in (0, a)$ we have

$$v \in H^2_p((\delta, a); X_0) \cap H^1_p((\delta, a); X_1) \hookrightarrow C^1([\delta, a]; X_\gamma) \cap C^{1-1/p}([\delta, a]; X_1),$$

i.e. the solution regularizes instantly.

The next result provides information about the continuation of local solutions.

Corollary 3.2 *Let the assumptions of Theorem 3.1 be satisfied and assume that $A(v)$ has maximal L_p-regularity for all $v \in V_\mu$. Then the solution v of (3.1) has a maximal interval of existence $J(v_0) = [0, t_+(v_0))$, which is characterized by the following alternatives:*

(i) *Global existence: $t_+(v_0) = \infty$;*
(ii) $\liminf_{t \to t_+(v_0)} \operatorname{dist}_{X_{\gamma,\mu}}(v(t), \partial V_\mu) = 0$;
(iii) $\lim_{t \to t_+(v_0)} v(t)$ *does not exist in $X_{\gamma,\mu}$.*

Next we assume that there is an open set $V \subset X_\gamma$ such that

$$(A, F) \in C^1(V, \mathcal{L}(X_1, X_0) \times X_0). \tag{3.3}$$

Let $\mathcal{E} \subset V \cap X_1$ denote the set of equilibrium solutions of (3.1), which means that

$$v \in \mathcal{E} \quad \text{if and only if} \quad v \in V \cap X_1 \text{ and } A(v)v = F(v).$$

Given an element $v_* \in \mathcal{E}$, we assume that v_* is contained in an m-dimensional *manifold of equilibria*. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a C^1-function $\Psi : U \to X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = v_*$,
- the rank of $\Psi'(0)$ equals m, and (3.4)
- $A(\Psi(\zeta))\Psi(\zeta) = F(\Psi(\zeta)), \quad \zeta \in U.$

We suppose that the operator $A(v_*)$ has the property of maximal L_p-regularity, and define the full linearization of (3.1) at v_* by

$$A_0 w = A(v_*)w + (A'(u_*)w)v_* - F'(v_*)w \quad \text{for } w \in X_1. \tag{3.5}$$

After these preparations we can state the following result on convergence of solutions starting near v_* which is called the *generalized principle of linearized stability*.

Proposition 3.3 *Let* $1 < p < \infty$. *Suppose* $v_* \in V \cap X_1$ *is an equilibrium of* (3.1), *and suppose that the functions* (A, F) *satisfy* (3.3). *Suppose further that* $A(v_*)$ *has the property of maximal* L_p*-regularity and let* A_0 *be defined in* (3.5). *Suppose that* v_* *is* normally stable, *which means*

i) near v_* *the set of equilibria* $\mathcal{E}$ *is a* C^1*-manifold in* X_1 *of dimension* $m \in \mathbb{N}$,
ii) the tangent space for $\mathcal{E}$ *at* v_* *is isomorphic to* $\mathsf{N}(A_0)$,
iii) 0 *is a* semi-simple eigenvalue *of* A_0, *i.e.* $\mathsf{N}(A_0) \oplus \mathsf{R}(A_0) = X_0$,
iv) $\sigma(A_0)\backslash\{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Then v_* *is stable in* X_γ, *and there exists* $\delta > 0$ *such that the unique solution* v *of* (3.1) *with initial value* $v_0 \in X_\gamma$ *satisfying* $|v_0 - v_*|_\gamma \le \delta$ *exists on* $\mathbb{R}_+$ *and converges at an exponential rate in* X_γ *to some* $v_\infty \in \mathcal{E}$ *as* $t \to \infty$.

The next result contains information on bounded solutions in the presence of compact embeddings and of a strict Lyapunov functional.

Proposition 3.4 *Let* $p \in (1, \infty)$, $\mu \in (1/p, 1)$, $\bar{\mu} \in (\mu, 1]$, *with* $V_\mu \subset X_{\gamma,\mu}$ *open. Assume that* $(A, F) \in C^1(V_\mu; \mathcal{L}(X_1, X_0) \times X_0)$, *and that the embedding* $X_{\gamma,\bar{\mu}} \hookrightarrow X_{\gamma,\mu}$ *is compact. Suppose furthermore that* v *is a maximal solution which is bounded in* $X_{\gamma,\bar{\mu}}$ *and satisfies*

$$\operatorname{dist}_{X_{\gamma,\mu}}(v(t), \partial V_\mu) \ge \eta > 0, \quad \textit{for all } t \ge 0. \tag{3.6}$$

Suppose that $\Phi \in C(V_\mu \cap X_\gamma; \mathbb{R})$ *is a strict Lyapunov functional for* (3.1)*, which means that* Φ *is strictly decreasing along non-constant solutions.*

Then $t_+(v_0) = \infty$*, i.e.* v *is a global solution of* (3.1)*. Its* ω*-limit set* $\omega_+(v_0) \subset \mathcal{E}$ *in* X_γ *is nonempty, compact and connected. If, in addition, there exists* $v_* \in \omega_+(v_0)$ *which is normally stable, then* $\lim_{t\to\infty} v(t) = v_*$ *in* X_γ*.*

4 Main results

In order to formulate the main well-posedness result for the Ericksen–Leslie system in the incompressible and isotropic case and to have access to the tools presented in Sect. 3, we introduce a functional analytic setting as follows. Denote the principal variable by $v = (u, \theta, d)$ and let us rewrite system (1.9), (1.3), (1.4), (1.10) as a quasi-linear evolution equation of the form

$$\dot{v} + A(v)v = F(v), \quad t > 0,\ v(0) = v_0, \tag{4.1}$$

replacing $\mathcal{D}_t d$ appearing in the the equations for u and θ by the equation for d. We also apply the *Helmholtz projection* $\mathbb{P}$ to the equation for u. Then v belongs to the base space X_0 defined by

$$X_0 := L_{q,\sigma}(\Omega) \times L_q(\Omega; \mathbb{R}) \times H^1_q(\Omega; \mathbb{R}^n),$$

where $1 < p, q < \infty$ and σ indicates solenoidal vector fields. The regularity space will be

$$X_1 := \{u \in H^2_q(\Omega; \mathbb{R}^n) \cap L_{q,\sigma}(\Omega) : u = 0 \text{ on } \partial\Omega\} \times Y_1,$$

with

$$Y_1 := \{(\theta, d) \in H^2_q(\Omega) \times H^3_q(\Omega; \mathbb{R}^n) : \partial_\nu \theta = \partial_\nu d = 0 \text{ on } \partial\Omega\}.$$

We consider solutions v within the class

$$v \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) = \mathbb{E}_{1,\mu}(J),$$

where $J = (0, a)$ with $0 < a \le \infty$ is an interval and $\mu \in (1/p, 1]$. The time trace space of this class is given by

$$X_{\gamma,\mu} := \{u \in B^{2(\mu-1/p)}_{qp}(\Omega)^n \cap L_{q,\sigma}(\Omega) : u = 0 \text{ on } \partial\Omega\} \times Y_{\gamma,\mu}, \tag{4.2}$$

where

$$Y_{\gamma,\mu} = \{(\theta, d) \in B^{2(\mu-1/p)}_{qp}(\Omega) \times B^{1+2(\mu-1/p)}_{qp}(\Omega; \mathbb{R}^n) : \partial_\nu \theta = \partial_\nu d = 0 \text{ on } \partial\Omega\},$$

whenever the boundary traces exist. Note that

$$X_{\gamma,\mu} \hookrightarrow B_{qp}^{2(\mu-1/p)}(\Omega)^{n+1} \times B_{qp}^{1+2(\mu-1/p)}(\Omega)^n \hookrightarrow C(\overline{\Omega})^{n+1} \times C^1(\overline{\Omega})^n,$$

provided

$$\frac{1}{p} + \frac{n}{2q} < \mu \leq 1. \tag{4.3}$$

For brevity we set $X_\gamma := X_{\gamma,1}$. Finally, the state manifold of the problem is defined by

$$\mathcal{SM} = \{v \in X_\gamma : \theta(x) > 0,\ |d(x)|_2 = 1 \text{ in } \Omega\}.$$

We assume the following regularity on the parameter functions:

Regularity assumption (R)
The parameter functions are assumed to satisfy

$$\begin{aligned} &\mu_j, \alpha, \gamma \in C^2((0,\infty) \times [0,\infty)) \text{ for } j = S, V, D, P, L, 0, \text{ and} \\ &\psi \in C^4((0,\infty) \times [0,\infty)). \end{aligned} \tag{4.4}$$

The fundamental well-posedness results regarding the general isotropic incompressible Ericksen–Leslie system reads as follows.

Theorem 4.1 (Local Well-Posedness) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^{3-} and assume conditions (R) and (P). Assume that $J = (0, a)$, $1 < p, q, < \infty$ and $\mu \in (1/p, 1]$ are subject to (4.3) and $v_0 \in X_{\gamma,\mu}$. Then for some $a = a(v_0) > 0$, there is a unique solution*

$$v \in H^1_{p,\mu}(J, X_0) \cap L_{p,\mu}(J; X_1),$$

of (4.1), i.e. of (1.9), (1.3), (1.4), (1.10) on J. Moreover,

$$v \in C([0, a]; X_{\gamma,\mu}) \cap C((0, a]; X_\gamma),$$

i.e. the solution regularizes instantly in time. It depends continuously on v_0 and exists on a maximal time interval $J(v_0) = [0, t^+(v_0))$. Moreover,

$$t\partial_t v \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1), \quad a < t^+(v_0).$$

Furthermore,

$$|d(\cdot,\cdot)|_2 \equiv 1, \quad \mathsf{E}(t) \equiv \mathsf{E}_0, \quad t \in J,$$

and $-N$ is a strict Lyapunov functional. In addition, problem (4.1) generates a local semi-flow in its natural state manifold $\mathcal{SM}$.

Below, we denote by $\bar{\mathcal{E}}$ the set

$$\bar{\mathcal{E}} = \{(0, \theta_*, d_*) : \theta_* > 0,\ d_* \in \mathbb{R}^n\}$$

of constant equilibria of the system when ignoring the constraints $|d|_2 = 1$ and $\mathsf{E} = \mathsf{E}_0$. The next result concerns stability of equilibria.

Theorem 4.2 (Stability of Equilibria) *Assume conditions (R) and (P). Then any equilibrium $v_* \in \overline{\mathcal{E}}$ of* (4.1) *is stable in X_γ. Moreover, for each $v_* \in \overline{\mathcal{E}}$ there is $\delta > 0$ such that if $|v_0 - v_*|_{X_{\gamma,\mu}} \leq \delta$, then the solution v of* (4.1) *with initial value v_0 exists globally in time and converges at an exponential rate in X_γ to some $v_\infty \in \overline{\mathcal{E}}$.*

The third result concerns global existence and convergence of solutions to equilibria in the topology of the state manifold $\mathcal{SM}$.

Theorem 4.3 (Long-Time Behaviour) *Assume conditions (R) and (P) and let v be the solution of equation* (4.1) *with $v_0 \in \mathcal{SM}$. Then the following assertions hold.*

(a) Suppose that for some $\bar{\mu} \in (1/p + n/2q + 1/2, 1]$ we have

$$\sup_{t \in (0, t^+(v_0))} [|v(t)|_{X_{\gamma,\overline{\mu}}} + |1/\theta(t)|_{L_\infty}] < \infty. \tag{4.5}$$

Then $t^+(v_0) = \infty$ and v is a global solution.
(b) If v is a global solution, bounded in $X_{\gamma,\bar{\mu}}$ and with $1/\theta$ bounded, then v converges exponentially in $\mathcal{SM}$ to an equilibrium $v_\infty \in \mathcal{E}$ of (4.1) *as $t \to \infty$.*
(c) If v is global solution of (4.1) *which converges to an equilibrium in $\mathcal{SM}$, then* (4.5) *valid.*

Remark 4.4 Let us emphasize that the above theorem holds true *without any structural assumptions* besides condition (P) on the Leslie coefficients. In particular, the above well-posedness results hold true *without* assuming Parodi's relation (1.13).

Remarks 4.5 (a) Wu, Xu and Liu considered in [41] the *isothermal penalized* Ericksen–Leslie model and gave a formal physical derivation of the Ericksen–Leslie model based on an energy variational approach assuming Parodi's relation. Then they prove that, under certain assumptions on the data and the Leslie coefficients, the isothermal penalized Ericksen–Leslie system admits a unique, global solution provided the viscosity is large enough and study as well its longtime behaviour. Moreover, *assuming Parodi's relation*, but not largeness of the viscosity, they show global well-posedness and Lypunov stability for the *penalized* Ericksen–Leslie system near local energy minimizers.

(b) Wang et al. [40] proved local well-posedness of the *isothermal* general Ericksen–Leslie system as well as global well-posedness for small initial data under various conditions on the Leslie coefficients, which ensure that the energy of the system is dissipated.

(c) It is interesting to compare our above results with a recent result due to Huang et al. [17], where they considered the simplified system subject to Dirichlet boundary conditions $d = d_b$ on $\partial\Omega$ and where they constructed examples of small initial data for which one has finite time blow up of (u, d) for the solution of the simplified system.

5 Maximal L^p-regularity of the linearization

The main task to apply the results in Sect. 3 is to establish maximal L_p-regularity of the linearized problem. To prove this, we linearize Eq. (1.9) at an initial value $v_0 = (u_0, \theta_0, d_0)$ and drop all terms of lower order. This yields the principal linearization

$$\begin{cases} \mathcal{L}_\pi(\partial_t, \nabla)v_\pi = f & \text{in } J \times \Omega, \\ u = \partial_\nu\theta = \partial_\nu d = 0 & \text{on } J \times \partial\Omega, \\ u = \theta = d = 0 & \text{on } \{0\} \times \Omega. \end{cases} \tag{5.1}$$

Here $J = (0, a)$, $v_\pi = (u, \pi, \theta, d)$ is the unknown, and $f = (f_u, f_\pi, f_\theta, f_d)$ are the given data. Denote the spatial co-variable by ξ and by z that in time. Assume that $z \in \Sigma_\phi := \{z \in \mathbb{C}\backslash\{0\} : 0 \le |argz| < \pi\}$. Then the differential operator $L = \mathcal{L}_\pi(\partial_t, \nabla)$ is defined via its symbol $\mathcal{L}_\pi(z, i\xi)$, which is

$$\mathcal{L}_\pi(z, i\xi) := \begin{bmatrix} M_u(z,\xi) & i\xi & 0 & izR_1(\xi)^{\mathsf{T}} \\ i\xi^{\mathsf{T}} & 0 & 0 & 0 \\ 0 & 0 & m_\theta(z,\xi) & -iz\theta_0 ba(\xi) \\ -iR_0(\xi) & 0 & -iba(\xi) & M_d(z,\xi) \end{bmatrix}, \quad \xi \in \mathbb{R}^n, z \in \Sigma_\phi, \tag{5.2}$$

with $b = \partial_\theta\lambda$, and $\lambda_1 = \partial_\tau\lambda$. We also introduce the parabolic part of this symbol by dropping pressure gradient and divergence, i.e.

$$\mathcal{L}(z, i\xi) = \begin{bmatrix} M_u(z,\xi) & 0 & izR_1(\xi)^{\mathsf{T}} \\ 0 & m_\theta(z,\xi) & iz\theta_0 ba(\xi) \\ -iR_0(\xi) & iba(\xi) & M_d(z,\xi) \end{bmatrix}, \quad \xi \in \mathbb{R}^n, z \in \Sigma_\phi. \tag{5.3}$$

The entries of these matrices are given by

$$\begin{aligned} m_\theta(z,\xi) &:= \rho\kappa z + \alpha|\xi|^2, \\ a(\xi) &:= \xi \cdot \nabla d_0, \\ M_d(z,\xi) &:= \gamma z + \lambda|\xi|^2 + \lambda_1 a(\xi) \otimes a(\xi) = m_d(z,\xi) + \lambda_1 a(\xi) \otimes a(\xi), \\ R_0(\xi) &:= \frac{\mu_D + \mu_V}{2} P_0\xi \otimes d_0 + \frac{\mu_D - \mu_V}{2}(\xi|d_0)P_0, \\ R_1(\xi) &:= \left(\frac{\mu_D + \mu_V}{2} + \mu_P\right) P_0\xi \otimes d_0 + \left(\frac{\mu_D - \mu_V}{2} + \mu_p\right)(\xi|d_0)P_0, \\ M_u(z,\xi) &:= \rho z + \mu_s|\xi|^2 + \mu_0(\xi|d_0)^2 d_0 \otimes d_0 + a_1(\xi|d_0)P_0\xi \otimes d_0 \\ &\quad + a_2(\xi|d_0)^2 P_0 + a_3|P_0\xi|^2 d_0 \otimes d_0 + a_4(\xi|d_0)d_0 \otimes P_0\xi. \end{aligned}$$

Here $P_0 = P_{d_0} = I - d_0 \otimes d_0$, and a_j are certain coefficients. Note that the above coefficients depend on x through the dependence of the parameter functions on the initial value $v_0(x)$. The maximal regularity result for (5.1) employed below reads as follows.

Theorem 5.1 *Let $J = (0, a)$, $1 < p, q < \infty$, and assume condition (R) and (P). Then Eq. (5.1) admits a unique solution $v_\pi = (u, \pi, \theta, d)$ satisfying*

$$\begin{aligned}(u, \theta) &\in {}_0H^1_p(J; L_q(\Omega))^{n+1} \cap L_p(J; H^2_q(\Omega))^{n+1},\\ \pi &\in L_p(J; \dot{H}^1_q(\Omega)),\\ d &\in {}_0H^1_p(J; H^1_q(\Omega))^n \cap L_p(J; H^3_q(\Omega))^n,\end{aligned}$$

if and only if

$$\begin{aligned}(f_u, f_\theta) &\in L_p(J; L_q(\Omega))^{n+1},\\ f_d &\in L_p(J; H^1_q(\Omega))^n,\\ f_\pi &\in {}_0H^1_p(J; H^{-1}_q(\Omega)) \cap L_p(J; H^1_q(\Omega)).\end{aligned}$$

Further, the solution map $f \mapsto v_\pi$ is continuous between the corresponding spaces.

Let us remark that if we replace ∂_t by $\partial_t + \omega$, where $\omega > 0$ is a sufficiently large constant, then the assertion of Theorem 5.1 holds true also for $J = (0, \infty)$.

Proof We subdivide the proof into 5 steps.

Step 1: The Principal Symbol with Constant Coefficients in $\Omega = \mathbb{R}^n$.
To extract the structure of $\mathcal{L}$, we introduce the symbols

$$\begin{aligned}&R(\xi) := (\xi|d_0)P_0 + P_0\xi \otimes d_0, \quad R_\mu(\xi) := \mu_-(\xi|d_0)P_0 + \mu_+ P_0\xi \otimes d_0,\\ &\mu_\pm := \mu_D \pm \mu_V + \mu_P.\end{aligned}$$

Then M_u simplifies to

$$M_u = m_u + \mu_0(\xi|d_0)^2 d_0 \otimes d_0 + \frac{\mu_L}{4} R^{\mathsf{T}} R + \frac{1}{4\gamma} R^{\mathsf{T}}_\mu R_\mu + \frac{\mu_P \mu_V}{2\gamma}(\xi|d_0)(R - R^{\mathsf{T}}),$$

and we also have

$$R_1 = R_\mu - R_0 \quad \text{and} \quad m_u(z, \xi) = \rho z + \mu_s|\xi|^2.$$

Next, we let $v = (u, w)$, $v_\pi = (v, \pi, w)$ and $w = (\theta, d)$. Then, setting

$$J = \operatorname{diag}(I, 1/\theta_0, zI),$$

and multiplying the second of $\mathcal{L}$ by $1/\theta_0$ as well as the last line with $\bar{z}$, we obtain the estimate

$$\begin{aligned}\mathrm{Re}(\mathcal{L}v|Jv) &= \mathrm{Re}\, m_u|u|_2^2 + \mathrm{Re}\, m_\theta|\theta|^2 + \mathrm{Re}\, z(\lambda_0|\xi|^2|d|_2^2 + \lambda_1|(a(\xi)|d)|^2)\\ &\quad + \frac{\mu_L}{4}|Ru|_2^2 + \frac{1}{4\gamma}|R_\mu u|_2^2 + \mathrm{Re}[iz(d|R_\mu u) + \gamma|z|^2|d|_2^2\\ &\geq c[\mathrm{Re}\, z(|u|_2^2 + |\theta|^2 + |\xi|^2|d|_2^2) + |\xi|^2(|u|_2^2 + |\theta|^2)\\ &\quad + (2\gamma|z||d|_2 - |R_\mu u|_2)^2],\end{aligned}$$

provided

$$\rho, \mu_s, \kappa, \gamma, \alpha, \lambda, \lambda + 2\tau\partial_\tau\lambda > 0 \text{ and } \mu_0, \mu_L \geq 0.$$

One could even relax the assumptions on μ_0 and μ_L to $2\mu_s + \mu_0 > 0$ and $2\mu_s + \mu_L > 0$, but we will not do this here. This means that the symbol $\bar{J}\mathcal{L}$ is accretive for $\mathrm{Re}\, z > 0$, i.e. it is strongly elliptic.

Let us emphasize that we do not need any structural conditions on the coefficients $\mu_D, \mu_V, \mu_P, \partial_\theta\lambda$.

Step 2: Schur Reductions.
In this step we perform a Schur reduction to reduce the above symbol to a symbol only for u. To this end, we consider the subsystem for w, i.e. the equation

$$\begin{bmatrix} m_\theta(z,\xi) & -iz\theta_0 b a(\xi)^{\mathsf{T}} \\ -iba(\xi) & m_d(z,\xi) + \lambda_1 a(\xi)\otimes a(\xi)\end{bmatrix}\begin{bmatrix}\theta\\ d\end{bmatrix} = \begin{bmatrix} f_\theta \\ f_d + iR_0(\xi)u\end{bmatrix}.$$

To solve this system, we follow the strategy developed in [14] and introduce the new variable $\delta = (a(\xi)|d)$. Then, multiplying the second equation with $a(\xi)$ we obtain the system

$$\begin{bmatrix} m_\theta(z,\xi) & -iz\theta_0 b \\ -ib|a(\xi)|^2 & m_d(z,\xi) + \lambda_1|a(\xi)|^2\end{bmatrix}\begin{bmatrix}\theta\\ \delta\end{bmatrix} = \begin{bmatrix} f_\theta \\ (f_d|a(\xi)) + i(R_0(\xi)u|a(\xi))\end{bmatrix}.$$

This system is easily solved to the result

$$\begin{bmatrix}\theta\\ \delta\end{bmatrix} = \frac{1}{det(z,\xi)}\begin{bmatrix} m_d(z,\xi) + \lambda_1|a(\xi)|^2 & iz\theta_0 b \\ ib|a(\xi)|^2 & m_\theta(z,\xi)\end{bmatrix}\begin{bmatrix} f_\theta \\ (f_d + iR_0(\xi)u|a(\xi))\end{bmatrix},$$

where

$$det(z,\xi) = m_\theta(z,\xi)(m_d(z,\xi) + \lambda_1|a(\xi)|^2) + z\theta_0 b^2|a(\xi)|^2.$$

Note that this symbol behaves like $(z + |\xi|^2)^2$ as soon as $\rho, \kappa, \lambda, \lambda + 2\tau\partial_\tau\lambda > 0$. Knowing $\delta = (a(\xi)|d)$ and θ, we are now able to determine d. As a result we obtain

$$d = m_d^{-1}[f_d + iR_0 u + iba(\xi)\theta - \lambda_1 a(\xi)\delta].$$

Following the arguments given in [14], we see that θ and d belong to the right regularity classes, whenever f_θ, f_d and u are so.

In order to extract the Schur complement for u, we set $f_\theta = f_d = 0$ and compute d. This yields

$$d = i\left[\frac{1}{m_d}(I - a_0 \otimes a_0) + \frac{m_\theta}{det} a_0 \otimes a_0\right] R_0 u = i\left[\frac{1}{m_d} P_{a_0} + \frac{m_\theta}{det} Q_{a_0}\right] R_0 u. \tag{5.4}$$

with $a_0(\xi) = a(\xi)/|a(\xi)|$ if $a(\xi) \neq 0$ and $a_0(\xi) = 0$ otherwise. This is the representation of d needed for the Schur complement of u.

Step 3: The Generalized Stokes Symbol.
We insert (5.4) into the equation for u to obtain the generalized Stokes symbol for (u, π) and obtain

$$M(z, i\xi) = M_u(z, \xi) - zR_1^{\mathsf{T}}(\xi)[\frac{1}{m_d(z,\xi)} P_{a_0}(\xi) + \frac{m_\theta(z,\xi)}{det(z,\xi)} Q_a(\xi)] R_0(\xi) \tag{5.5}$$

As the Schur reduction preserves accretivity, even with the same accretivity constant, we obtain

$$\mathrm{Re}(M(z, i\xi)u|u) \geq \mathrm{Re}\, m_u(z, \xi)|u|^2 = \big(\rho \mathrm{Re}\, z + \mu_s|\xi|^2\big)|u|^2.$$

This shows that M is strongly elliptic. For this reason we may now apply the method developed by Bothe and Prüss [2] or Prüss and Simonett [33], Section 7.1, to prove maximal L_p-regularity of the resulting generalized Stokes problem. In these references we need to replace λ by z and the symbol $z + \mathcal{A}(\xi)$ by $M(z, i\xi)$. We will not do this here in detail and refer the reader to Section 7.1 of [33] for this analysis.

Step 4: The Lopatinskii-Shapiro Condition.
In order to guarantee the solvability of the above problem in a half-space, we need to replace the co-variable ξ by the one-dimensional differential operator $\xi - i\nu\partial_y$, where $(\xi|\nu) = 0$. The Lopatinskii-Shapiro condition then means that the problem

$$\begin{aligned} \mathcal{L}(z, i\xi + \nu\partial_y)v = 0, \quad y > 0,\\ u(0) = \partial_y\theta = \partial_y d = 0, \end{aligned} \tag{5.6}$$

admits only the zero solution in $L_2(\mathbb{R}_+)^{2n+1}$, for all $(z, \xi) \neq (0, 0)$.

In order to prove this condition, suppose that $v(y)$ is a solution of the ODE system (5.6), which belongs to $L_2(\mathbb{R}_+)^{2n+1}$. Taking the inner product with v in $L_2(\mathbb{R}_+)$, taking real parts, integrating by parts with respect to y and employing the boundary conditions, we obtain the estimate

$$\begin{aligned} c\mathrm{Re}(\mathcal{L}(z, i\xi + \nu\partial_y)v|v)_{L_2} \geq \mathrm{Re}\, z[|u|_{L_2}^2 + |\theta|_{L_2}^2 + |\xi|^2|d|_{L_2}^2 + |\partial_y d|_{L_2}^2] + |z|^2|d|_{L_2}^2 \\ + |\xi|^2(|u|_{L_2}^2 + |\theta|_{L_2}^2) + |\partial_y u|_{L_2}^2 + |\partial_y\theta|_{L_2}^2. \end{aligned}$$

This shows that the Lopatinskii-Shapiro is valid. Hence, to prove maximal L_p-regularity in the half space case, we may proceed in the following way. First we perform the same Schur reductions as in Step 2 and as in [14]. This yields the unique existence of θ and d in the right regularity class. We then employ the half-space theory for the generalized Stokes symbol M by the methods in Bothe and Prüss [2] or Prüss and Simonett [33], Section 7.2, to obtain maximal L_p-regularity for the half-space case.

Step 5: General Domains and Variable Coefficients.
The results of Step 3 and Step 4 extend by a perturbation argument to a bent half-space, and to the case of variable coefficients with small deviation from constant ones. We then may apply a localization procedure to cover the case of general domains with smooth boundaries and variable coefficients. For details we refer at this point e.g. to Sections 6.3 and 7.3. of the monograph [33] by Prüss and Simonett. This completes the proof of Theorem 5.1. □

6 Proofs of the main results

In this section we present the proofs of the above three main results. They are based on the theory of quasilinear parabolic evolution equations, see Sect. 3.

Proof of Theorem 4.1 As already discussed above, we rewrite the system (1.3), (1.4), (1.9), (1.10) as a quasi-linear evolution equation of the form

$$\dot{v} + A(v)v = F(v), \quad t > 0, \; v(0) = v_0, \tag{6.1}$$

replacing $\mathcal{D}_t d$ appearing in the the equations for u and θ by the equation for d. Here $v = (u, \theta, d)$. We further apply the Helmholtz projection $\mathbb{P}$ to the equation for u and recall the base space

$$X_0 = L_{q,\sigma}(\Omega) \times Y_0,$$

with $Y_0 = L_q(\Omega) \times H^1_q(\Omega; \mathbb{R}^n)$ as well as the regularity space X_1 as above, i.e.

$$X_1 = \{u \in H^2_q(\Omega; \mathbb{R}^n) \cap L_{q,\sigma}(\Omega) : \; u = 0 \text{ on } \partial\Omega\} \times Y_1,$$

with

$$Y_1 = \{(\theta, d) \in H^2_q(\Omega) \times H^3_q(\Omega; \mathbb{R}^n) : \; \partial_\nu \theta = \partial_\nu d = 0 \text{ on } \partial\Omega\}.$$

In order to prove local well-posedness of the system (1.3), (1.4), (1.9), (1.10) we may now resort to the abstract theory presented in Sect. 3.

Note first that by Theorem 5.1 the quasi-linear part $A(v)$ has maximal L_p-regularity. A result by Prüss and Simonett [32], Theorem 2.4, implies that $A(v)$ also admits maximal regularity in $L_{p,\mu}(J; X_0)$, hence also in the situation of time weights. Recalling the solution space $\mathbb{E}_{1,\mu}(J) = H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$, we see that the time-trace space $X_{\gamma,\mu}$ of $\mathbb{E}_{1,\mu}(J)$ is given as in (4.2), and the embedding

$$X_{\gamma,\mu} \hookrightarrow B^{2(\mu-1/p)}_{qp}(\Omega)^{n+1} \times B^{1+2(\mu-1/p)}_{qp}(\Omega)^n \hookrightarrow C^1(\overline{\Omega})^{n+1} \times C^2(\overline{\Omega})^n, \tag{6.2}$$

holds, provided

$$\frac{1}{p} + \frac{n}{2q} + 1/2 < \mu \le 1.$$

Here $B^s_{pq}(\Omega)$ denote as usual the Besov spaces; see e.g. Triebel [37]. Therefore, the mappings A and F satisfy the assumptions of the local existence theorem Theorem 3.1, as well as of Corollary 3.2, hence we obtain local well-posedness for (6.1) and strong solutions on a maximal time interval.

Even more, if only

$$\frac{1}{p} + \frac{n}{2q} < \mu \le 1$$

holds, also the results in LeCrone et al. [20], Theorem 2.1, apply, and we obtain local strong solutions if the initial values only satisfy

$$(u_0, \theta_0, d_0) \in B^{2(\mu-1/p)}_{qp}(\Omega)^{n+1} \times B^{1+2(\mu-1/p)}_{qp}(\Omega)^n$$

plus compatibility conditions, which means that it is enough to assume that $u_0, \theta_0, d_0, \nabla d_0$ are Hölder continuous, choosing μ close to $1/p$ which is possible if q is large enough.

Recalling that the state manifold of (6.1) is given by

$$\mathcal{SM} = \{(u, \theta, d) \in X_\gamma : \theta > 0,\ |d|_2 = 1\},$$

where $X_\gamma := X_{\gamma,1}$, we see by these results that $\mathcal{SM}$ is locally positive invariant for the semi-flow, the total energy E is preserved and the negative total entropy $-\mathsf{N}$ is a strict Lyapunov functional for the semi-flow on $\mathcal{SM}$. This completes the proof of Theorem 4.1. □

Proof of Theorems 4.2 and 4.3 The linearization of the system (1.3), (1.4), (1.9) at an equilibrium $v_* = (0, \theta_*, d_*)$ is given by the operator $A_* = A(v_*)$ defined in the base space X_0 with domain $D(A_*) = X_1$. This operator has maximal L_p-regularity. Moreover, A_* is the negative generator of a compact analytic C_0-semigroup having compact resolvent, due to the compact embedding of $X_1 = D(A_*)$ into X_0. Hence, its spectrum consists only of countably many eigenvalues of finite multiplicity.

Lemma 6.1 *Let $z \ne 0$ be an eigenvalue of A_*. Then* $\mathrm{Re}\, z < 0$.

Proof Suppose that $z \in \mathbb{C}\backslash\{0\}$ is an eigenvalue of A_* with $\mathrm{Re}\, z \ge 0$. Then

$$\begin{aligned} \mathcal{L}_\pi(z, \nabla) v_\pi &= 0 \quad \text{in } \Omega, \\ u = \partial_\nu \theta = \partial_\nu d &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

where $(v_\pi) = (u, \theta, d)$ as above. Multiplying the equation for d with $\bar{z}$ and taking the inner product of this equation with v_π in $L_2(\Omega)$ yields by integration by parts the estimate

$$0 = \mathrm{Re}(\mathcal{L}_\pi(z, \nabla)v_\pi | v_\pi)_{L_2} \geq c\Big[\mathrm{Re}\, z(|u|^2_{L_2} + |\theta|^2_{L_2} + |\nabla d|^2_{L_2}) \\ + |z|^2|d|_{L_2} + |\nabla u|^2_{L_2} + |\nabla\theta|^2_{L_2}\Big].$$

This implies $u = \theta = d = 0$. Hence A_* does not have eigenvalues in the L_2-setting with nonnegative real parts, except for $z = 0$. Due to elliptic regularity, eigenvalues are independent of p, and so the assertion follows for A_* defined in X_0. □

The above lemma states that all eigenvalues of A_* expect for 0 are stable. In addition, the eigenvalue 0 is semi-simple. Its eigenspace is given by

$$\mathsf{N}(A_*) = \{(0, \vartheta, \mathsf{d}) : \vartheta \in \mathbb{R}, \mathsf{d} \in \mathbb{R}^n\},$$

and hence coincides with the set of constant equilibria $\bar{\mathcal{E}}$ determined in Theorem 2.1 when ignoring the constraint $|d|_2 = 1$ and conservation of energy. Therefore each such equilibrium is normally stable. Hence, the assertion of Theorem 4.2 follows by means of the generalized principle of linearized stability, Proposition 3.3.

Finally, we note that the assertion of Theorem 4.3 follows from Proposition 3.4., as we have compact embeddings and $-\mathsf{N}$ serves as a strict Lyapunov functional. □

References

1. Atkin, R., Sluchin, T., Stewart, I.W.: Reflections on the life and work of Frank Matthews Leslie. J. Non-Newton. Fluid Mech. **119**, 7–23 (2004)
2. Bothe, D., Prüss, J.: L^p-theory for a class of non-Newtonian fluids. SIAM J. Math. Anal. **39**, 379–421 (2007)
3. Chandrasekhar, S.: Liquid Crystals. Cambridge University Press, Cambridge (1992)
4. Coutand, D., Shkoller, S.: Well-posedness of the full Ericksen-Leslie model of nematic liquid crystals. C. R. Acad. Sci. Paris Sér. I Math. **333**, 919–924 (2001)
5. DeGennes, P.G., Prost, J.: The Physics of Liquid Crystals. Oxford University Press, Oxford (1995)
6. Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Am. Math. Soc., **166** (2003)
7. Ericksen, J.L.: Hydrostatic theory of liquid crystals. Arch. Ratio. Mech. Anal. **9**, 371–378 (1962)
8. Feireisl, E., Rocca, E., Schimperna, G.: On a non-isothermal model for nematic liquid crystals. Nonlinearity **24**, 243–257 (2011)
9. Feireisl, E., Frémond, M., Rocca, E., Schimperna, G.: A new approach to non-isothermal models for nematic liquid crystals. Arch. Ration. Mech. Anal. **205**, 651–672 (2012)
10. Feireisl, E., Rocca, E., Schimperna, G., Zarnescu, A.: Evolution of non-isothermal Landau-de Gemmes nematic liquid crystal flows with singular potential. arXiv:1207.1643
11. Feireisl, E., Rocca, E., Schimperna, G., Zarnescu, A.: Nonisothermal nematic liquid crystal flows with Ball-Majumdar free energy. arXiv:1310.8474
12. Frank, F.C.: On the theory of liquid crystals. Discuss. Faraday Soc. **25**, 19–28 (1958)
13. Hieber, M., Nesensohn, M., Prüss, J., Schade, K.: Dynamics of nematic lquid crystals: the quasilinear approach. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 397–408 (2016)
14. Hieber, M., Prüss, J.: Thermodynamic consistent modeling and analysis of nematic liquid crystal flows. In: Springer Proc. Math. and Statistics (2016) **(to appear)**
15. Hieber, M., Prüss, J.: Modeling and analysis of the Ericksen–Leslie equations for nematic liquid crystal flows. In: Giga, Y., Novotny, A. (eds.) Handbook of mathematical analysis in mechanics of viscous fluids. Springer, New York (**to appear**)
16. Hong, M., Li, J., Xin, Z.: Blow-up criteria of strong solutions to the Ericksen-Leslie system in $\mathbb{R}^3$. Comm. Partial Differ. Equ. **39**, 1284–1328 (2014)

17. Huang, T., Lin, F., Liu, C., Wang, C.: Finite time singularities of the nematic liquid crystal flow in dimension three. Arch. Ration. Mech. Anal. **221**, 1223–1254 (2016)
18. Huang, J., Lin, F., Wang, C.: Regularity and existence of global solutions to the Ericksen-Leslie system in $\mathbb{R}^2$. Comm. Math. Phys. **331**, 805–850 (2014)
19. Köhne, M., Prüss, J., Wilke, M.: On quasilinear parabolic evolution equations in weighted L_p-spaces. J. Evol. Equ. **10**, 443–463 (2010)
20. LeCrone, J., Prüss, J., Wilke, M.: On quasilinear parabolic evolution equations in weighted L^p-spaces II. J. Evol. Equ. **14**, 509–533 (2014)
21. Leslie, F.M.: Some constitutive equations for liquid crystals. Arch. Ration. Mech. Anal. **28**, 265–283 (1968)
22. Lin, F.: Nonlinear theory of defects in nematic liquid crystals: phase transition and flow phenomena. Commun. Pure Appl. Math. **42**, 789–814 (1989)
23. Lin, F.: On nematic liquid crystals with variable degree of freedom. Commun. Pure Appl. Math. **44**, 453–468 (1991)
24. Lin, F., Liu, Ch.: Nonparabolic dissipative systems modeling the flow of liquid crystals. Commun. Pure Appl. Math. **48**, 501–537 (1995)
25. Lin, F., Liu, Ch.: Existence of solutions for the Ericksen-Leslie system. Arch. Ration. Mech. Anal. **154**, 135–156 (2000)
26. Lin, F., Lin, J., Wang, C.: Liquid crystal flows in two dimensions. Arch. Ration. Mech. Anal. **197**, 297–336 (2010)
27. Lin, F., Wang, C.: Recent developments of analysis for hydrodynamic flow of nematic liquid crystals. Philos. Trans. R. Soc. Lon. Ser. A, Math. Phys. Eng. Sci. **372**, 20130361 (2014)
28. Ma, W., Gong, H., Li, J.: Global strong solutions to incompressible Ericksen-Leslie system in $\mathbb{R}^3$. Nonlinear Anal. **109**, 230–235 (2014)
29. Oseen, C.W.: The theory of liquid crystals. Trans. Faraday Soc. **29**, 883–899 (1933)
30. Parodi, O.: Stress tensor for a nematic liquid crystal. J. Physique **31**, 581–584 (1970)
31. Prüss, J.: Maximal regularity for evolution equations in L_p-spaces. Conf. Semin. Mat. Univ. Bari (2002) **285**, 1–39 (2003)
32. Prüss, J., Simonett, G.: Maximal regularity for evolution equations in weighted L_p-spaces. Arch. Math. (Basel) **82**, 415–431 (2004)
33. Prüss, J., Simonett, G.: Moving Interfaces and Quasilinear Parabolic Evolution Equations. Monographs in Mathematics, vol. 105. Birkhäuser Basel (2016). doi:10.1007/978-3-319-27698-4
34. Prüss, J., Simonett, G., Zacher, R.: On convergence of solutions to equilibria for quasilinear parabolic problems. J. Differ. Equ. **246**, 3902–3931 (2009)
35. Stewart, I.W.: The Static and Dynamic Continuum Theory of Liquid Crystals. The Liquid Crystal Book Series. Taylor and Francis, UK (2004)
36. Sun, H., Liu, Ch.: On energetic variational approaches in modeling the nematic liquid crystal flows. Disc. Cont. Dyn. Syst. **23**, 455–475 (2009)
37. Triebel, H.: Theory of Function Spaces II. Birkhäuser, Basel (1992)
38. Virga, E.G.: Variational theories for liquid crystals. Chapman-Hall, London (1994)
39. Wang, C.: Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data. Arch. Ration. Mech. Anal. **200**, 1–19 (2011)
40. Wang, W., Zhang, P., Zhang, Z.: Well-posedness of the Ericksen-Leslie system. Arch. Ration. Mech. Anal. **210**, 837–855 (2013)
41. Wu, H., Xu, X., Liu, Ch.: On the general Ericksen-Leslie system: Parodi's relation, well-posedness and stability. Arch. Ration. Mech. Anal. **208**, 59–107 (2013)