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Quantitative estimates of strong unique continuation for wave equations

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Abstract The main results of the present paper consist in some quantitative estimates for solutions to the wave equation $\partial_t^2 u - \text{div}(A(x)\nabla_x u) = 0$. Such estimates imply the following strong unique continuation properties: (a) if u is a solution to the the wave equation and u is flat on a segment $\{x_0\} \times J$ on the t axis, then u vanishes in a neighborhood of $\{x_0\} \times J$. (b) Let u be a solution of the above wave equation in $\Omega \times J$ that vanishes on a a portion $Z \times J$ where Z is a portion of $\partial\Omega$ and u is flat on a segment $\{x_0\} \times J$, $x_0 \in Z$, then u vanishes in a neighborhood of $\{x_0\} \times J$. The property (a) has been proved by Lebeau (Commun Partial Differ Equ 24:777–783, 1999).

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1 Introduction

The strong unique continuation properties and the related quantitative estimates have been well understood for second order equations of elliptic [1,6,22,27] and parabolic type [5,15,28]. The three sphere inequalities [30], doubling inequalities [20], or two-sphere one cylinder inequality [16] are the typical form in which such quantitative estimates of unique continuation occur in the elliptic or in the parabolic context. We refer to [4,36] for a more extensive literature on these subjects. On the contrary, the strong properties of unique continuation are much less studied in the context of hyperbolic equations, [7,31,32].

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To the author knowledge there exits no result in the literature concerning quantitative estimates of unique continuation in the framework of hyperbolic equations. In this paper we study this issue for the wave equation

$$\partial_t^2 u - \text{div}\,(A(x)\nabla_x u) = 0, \tag{1.1}$$

($\text{div} := \sum_{j=1}^n \partial_{x_j}$) where $A(x)$ is a real-valued symmetric $n \times n$, $n \geq 2$, matrix whose entries are functions of Lipschitz class and satisfying uniform ellipticity condition.

The quantitative estimates of unique continuation for the Eq. (1.1) represent the quantitative counterparts of the following strong unique continuation property. Let u be a weak solution to (1.1) and assume that

$$\sup_{t\in J} \|u(\cdot,t)\|_{L^2(B_r)} = C_N r^N, \quad \forall N \in \mathbb{N}, \quad \forall r < 1,$$

where C_N is arbitrary and independent on r, $J = (-T, T)$ is an interval of $\mathbb{R}$. Then we have

$$u = 0 \quad \text{in } \mathcal{U},$$

where $\mathcal{U}$ is a neighborhood of $\{0\} \times J$. The above property was proved by Lebeau in [31]. As a consequence of such a result and using the weak unique continuation property proved in [23,34,35], see also [24], we have that, if the entries of A are function in $C^\infty(\mathbb{R}^n)$ then $u = 0$ in the domain of dependence of a cylinder $B_\delta \times J$, where B_δ is the ball of $\mathbb{R}^n$, $n \geq 2$, centered at 0 with a small radius δ. Previously the strong unique continuation property was proved by Masuda [32] whenever $J = \mathbb{R}$ and the entries of the matrix A are functions of C^2 class and by Baouendi and Zachmanoglou [7] whenever the entries of A are analytic functions. In both [7,32], the above property was proved also for first order perturbation of operator $\partial_t^2 u - \text{div}(A(x)\nabla u)$. Also, we recall here the papers [11,12,33]. In such papers unique continuation properties are proved along and across lower dimensional manifolds for the wave equation.

The quantitative estimate of strong unique continuation (in the interior) that we prove is, roughly speaking, the following one (for the precise statement see Theorem 2.1). Let u be a solution to (1.1) in the cylinder $B_1 \times J$ and let $r \in (0, 1)$. Assume that

$$\sup_{t\in J} \|u(\cdot,t)\|_{L^2(B_r)} \leq \varepsilon \quad \text{and} \quad \|u(\cdot,0)\|_{H^2(B_1)} \leq 1,$$

where $\varepsilon < 1$. Then

$$\|u(\cdot,0)\|_{L^2(B_{s_0})} \leq C \left|\log\left(\varepsilon^\theta\right)\right|^{-1/6}, \tag{1.2}$$

where $s_0 \in (0, 1)$, $C \geq 1$ are constants independent of u and r and

$$\theta = |\log r|^{-1}. \tag{1.3}$$

The estimate (1.2) are sharp estimate from two points of view:

(i) The logarithmic character of the estimate cannot be improved as it is shown by a well-known counterexample of John for the wave equation, [26];
(ii) The sharp dependence of θ by r. Indeed it is easy to check that the estimate (1.2) implies the strong unique continuation for the Eq. (1.1) (see Remark 2.2 for more details).

As a consequence of estimate (1.2) and some reflection transformation introduced in [1] we derive a quantitative estimate of unique continuation at the boundary (Theorem 2.3). Also, we extend (1.2) to a first order perturbation of the wave operator (Sect. 4).

One of the main purposes that led us to derive the above estimates is their applications in the framework of stability for inverse hyperbolic problems with time independent unknown boundaries from transient data with a finite time of observation. Some uniqueness results has been proved in [25]. In the paper [37] the most important tools that are used to prove a sharp stability estimate are precisely the strong unique continuation (at the interior and at the boundary) for the Eq. (1.1). The quantitative estimate of strong unique continuation was applied for the first time to the elliptic inverse problems with unknown boundaries in [3]. Concerning the parabolic inverse problems with unknown boundaries such estimates were applied in [9,10,14,18,36]. In both the cases, elliptic and parabolic, the stability estimates that were proved are optimal [13] and [2] (elliptic case), [14] (parabolic case).

The proof of (1.2) follows a similar strategy and ingredients as the one used in [31]. In particular, in order to perform a suitable transformation of the wave equation in a nonhomogeneous second order elliptic equation we use the Boman transformation [8], then, to the obtained elliptic equation, we use the Carleman estimate with singular weight, [6,17,22] and the stability estimates for the Cauchy problem. The main difference between our proof and the one of [31] relies in the different nature of the results; in our case the results are quantitative while in [31] the results are only qualitative. More precisely, in [31] the parameter ε has the particular form $\varepsilon = C_N r^N$ while in the present paper ε is a free parameter. An important consequence of this fact is that we need to control very accurately how much the error ε affects the growth of the solution to (1.1) in order to reach the above sharpness character (i) and (ii).

The plan of the paper is as follows. In Sect. 2 we state the main results of this paper, in Sect. 3 we prove the theorems of Sect. 2, in Sect. 4 we consider the case of the more general equation $q(x)\partial_t^2 u - \text{div}(A(x)\nabla_x u) = b(x) \cdot \nabla_x u + c(x)u$.

2 The main results

2.1 Notation and definition

Let $n \in \mathbb{N}$, $n \geq 2$. For any $x \in \mathbb{R}^n$, we will denote $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. Given $x \in \mathbb{R}^n$, $r > 0$, we will denote by B_r, B'_r $\widetilde{B}_r$ the ball of $\mathbb{R}^n$, $\mathbb{R}^{n-1}$ and $\mathbb{R}^{n+1}$ of radius r centered at 0. For any open set $\Omega \subset \mathbb{R}^n$ and any function (smooth enough) u we denote by $\nabla_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ the gradient of u. Also, for the gradient of u we use the notation $D_x u$. If $j = 0, 1, 2$ we denote by $D_x^j u$ the set of the derivatives of u of order j, so $D_x^0 u = u$, $D_x^1 u = \nabla_x u$ and $D_x^2 u$ is the hessian matrix $\{\partial_{x_i x_j} u\}_{i,j=1}^n$. Similar notation are used whenever other variables occur and Ω is an open subset of $\mathbb{R}^{n-1}$ or a subset $\mathbb{R}^{n+1}$. By $H^\ell(\Omega)$, $\ell = 0, 1, 2$ we denote the usual Sobolev spaces of order ℓ, in particular we have $H^0(\Omega) = L^2(\Omega)$.

For any interval $J \subset \mathbb{R}$ and Ω as above we denote by

$$\mathcal{W}(J; \Omega) = \{u \in C^0(J; H^2(\Omega)) : \partial_t^\ell u \in C^0(J; H^{2-\ell}(\Omega)), \ell = 1, 2\}.$$

We shall use the letters $C, C_0, C_1, \dots$ to denote constants. The value of the constants may change from line to line, but we shall specified their dependence everywhere they appear.

2.2 Statements of the main results

Let $A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ be a real-valued symmetric $n \times n$ matrix whose entries are measurable functions and they satisfy the following conditions for given constants $\rho_0 > 0$, $\lambda \in (0, 1]$ and $\Lambda > 0$,

$$\lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \text{for every } x, \xi \in \mathbb{R}^n, \tag{2.1a}$$

$$|A(x) - A(y)| \leq \frac{\Lambda}{\rho_0} |x - y|, \quad \text{for every } x, y \in \mathbb{R}^n. \tag{2.1b}$$

Let $q = q(x)$ be a a real-valued measurable function that satisfies

$$\lambda \leq q(x) \leq \lambda^{-1}, \quad \text{for every } x \in \mathbb{R}^n, \tag{2.2a}$$

$$|q(x) - q(y)| \leq \frac{\Lambda}{\rho_0} |x - y|, \quad \text{for every } x, y \in \mathbb{R}^n. \tag{2.2b}$$

Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; B_{\rho_0})$ be a weak solution to

$$q(x)\partial_t^2 u - \mathrm{div}\,(A(x)\nabla_x u) = 0, \quad \text{in } B_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0). \tag{2.3}$$

Let $r_0 \in (0, \rho_0]$ and denote by

$$\varepsilon := \sup_{t \in (-\lambda\rho_0, \lambda\rho_0)} \left(\rho_0^{-n} \int_{B_{r_0}} u^2(x, t) dx \right)^{1/2} \tag{2.4}$$

and

$$H := \left(\sum_{j=0}^{2} \rho_0^{j-n} \int_{B_{\rho_0}} \left| D_x^j u(x, 0) \right|^2 dx \right)^{1/2}. \tag{2.5}$$

Theorem 2.1 (estimate at the interior) *Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; B_{\rho_0})$ be a weak solution to* (2.3) *and let* (2.1) *and* (2.2) *be satisfied. There exist constants $s_0 \in (0, 1)$ and $C \geq 1$ depending on λ and Λ only such that for every $0 < r_0 \leq \rho \leq s_0\rho_0$ the following inequality holds true*

$$\|u(\cdot, 0)\|_{L^2(B_\rho)} \leq \frac{C\left(\rho_0\rho^{-1}\right)^C (H + e\varepsilon)}{\left(\theta \log\left(\frac{H+e\varepsilon}{\varepsilon}\right)\right)^{1/6}}, \tag{2.6}$$

where

$$\theta = \frac{\log(\rho_0/C\rho)}{\log(\rho_0/r_0)}. \tag{2.7}$$

The proof of Theorem 2.1 is given in Sect. 3.

Remark 2.2 Observe that estimate (2.6) implies the following property of strong unique continuation. Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; B_{\rho_0})$ be a weak solution to (2.3) and assume that

$$\sup_{t\in(-\lambda\rho_0,\lambda\rho_0)} \left(\rho_0^{-n} \int_{B_{r_0}} u^2(x,t)dx\right)^{1/2} = O(r_0^N), \quad \forall N \in \mathbb{N}, \quad \text{as } r_0 \to 0,$$

then

$$u(\cdot, t) = 0, \quad \text{for } |x| + \lambda^{-1}s_0|t| \leq s_0\rho_0. \tag{2.8}$$

It is enough to consider the case $t = 0$. If $\|u(\cdot, 0)\|_{L^2(B_{s_0\rho_0})} = 0$ there is nothing to proof, otherwise if

$$\|u(\cdot, 0)\|_{L^2(B_{s_0\rho_0})} > 0, \tag{2.9}$$

we argue by contradiction. By (2.9) it is not restrictive to assume that $H = \|u(\cdot, 0)\|_{H^2(B_{\rho_0})} = 1$. Now we apply inequality (2.6) with $\varepsilon_0 = C_N r_0^N$, $N \in \mathbb{N}$, and passing to the limit as $r_0 \to 0$ we have that (2.6) implies

$$\|u(\cdot, 0)\|_{L^2(B_{s_0\rho_0})} \leq C s_0^{-C} N^{-1/6}, \quad \forall N \in \mathbb{N},$$

by passing again to the limit as $N \to 0$ we get, by (2.9), $\|u(\cdot, 0)\|_{L^2(B_\rho)} = 0$ that contradicts (2.9).

In order to state Theorem 2.3 below let us introduce some notation. Let ϕ be a function belonging to $C^{1,1}(B'_{\rho_0})$ that satisfies

$$\phi(0) = |\nabla_{x'}\phi(0)| = 0 \tag{2.10}$$

and

$$\|\phi\|_{C^{1,1}\left(B'_{\rho_0}\right)} \leq E\rho_0, \tag{2.11}$$

where

$$\|\phi\|_{C^{1,1}\left(B'_{\rho_0}\right)} = \|\phi\|_{L^\infty\left(B'_{\rho_0}\right)} + \rho_0 \|\nabla_{x'}\phi\|_{L^\infty\left(B'_{\rho_0}\right)} + \rho_0^2 \left\|D^2_{x'}\phi\right\|_{L^\infty\left(B'_{\rho_0}\right)}.$$

For any $r \in (0, \rho_0]$ denote by

$$K_r := \{(x', x_n) \in B_r : x_n > \phi(x')\}$$

and

$$Z := \{(x', \phi(x')) : x' \in B'_{\rho_0}\}.$$

Let $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; K_{\rho_0})$ be a solution to

$$\partial_t^2 u - \text{div}\,(A(x)\nabla_x u) = 0, \quad \text{in } \ K_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0), \tag{2.12}$$

satisfying one of the following conditions

$$u = 0, \quad \text{on } \ Z \times (-\lambda\rho_0, \lambda\rho_0) \tag{2.13}$$

or

$$A\nabla_x u \cdot \nu = 0, \quad \text{on } \ Z \times (-\lambda\rho_0, \lambda\rho_0), \tag{2.14}$$

where ν denotes the outer unit normal to Z.

Let $r_0 \in (0, \rho_0]$ and denote by

$$\varepsilon = \sup_{t\in(-\lambda\rho_0,\lambda\rho_0)} \left(\rho_0^{-n} \int_{K_{r_0}} u^2(x,t)dx\right)^{1/2} \tag{2.15}$$

and

$$H = \left(\sum_{j=0}^{2} \rho_0^{j-n} \int_{K_{\rho_0}} \left|D_x^j u(x,0)\right|^2 dx\right)^{1/2}. \tag{2.16}$$

Theorem 2.3 (estimate at the boundary) *Let* (2.1) *be satisfied. Let* $u \in \mathcal{W}([-\lambda\rho_0, \lambda\rho_0]; K_{\rho_0})$ *be a solution to* (2.12) *satisfying* (2.15) *and* (2.16). *Assume that u satisfies either*

(2.13) *or* (2.14). *There exist constants* $\overline{s}_0 \in (0,1)$ *and* $C \geq 1$ *depending on* λ, Λ *and* E *only such that for every* $0 < r_0 \leq \rho \leq \overline{s}_0\rho_0$ *the following inequality holds true*

$$\|u(\cdot,0)\|_{L^2(K_\rho)} \leq \frac{C\left(\rho_0\rho^{-1}\right)^C (H+e\varepsilon)}{\left(\widetilde{\theta}\log\left(\frac{H+e\varepsilon}{\varepsilon}\right)\right)^{1/6}}, \tag{2.17}$$

where

$$\widetilde{\theta} = \frac{\log(\rho_0/C\rho)}{\log(\rho_0/r_0)}. \tag{2.18}$$

The proof of Theorem 2.3 is given in Sect. 3.2.

Remark 2.4 By arguing similarly to Remark 2.2 we have that estimate (2.17) implies the following property of strong unique continuation at the boundary. Let $u \in \mathcal{W}([-\lambda\rho_0,\lambda\rho_0];K_{\rho_0})$ be a solution to (2.12) satisfying either (2.13) or (2.14) and assume that

$$\sup_{t\in(-\lambda\rho_0,\lambda\rho_0)}\left(\rho_0^{-n}\int_{K_{r_0}} u^2(x,t)dx\right)^{1/2} = O(r_0^N), \quad \forall N\in\mathbb{N}, \quad \text{as } r_0\to 0,$$

then

$$u(x,t)=0, \quad \text{for } x\in K_{\rho(t)}, \quad t\in(-\lambda\rho_0,\lambda\rho_0),$$

where $\rho(t) = \overline{s}_0(\rho_0 - \lambda^{-1}|t|)$.

3 Proof of Theorems 2.1 and 2.3

3.1 Proof of Theorem 2.1

Observe that to prove Theorem 2.1 we can assume that $u(x,t)$ is even with respect to the variable t. Indeed defining

$$u_+(x,t) = \frac{u(x,t)+u(x,-t)}{2},$$

we see that u_+ satisfies all the hypotheses of Theorem 2.1 and, in particular, we have

$$u_+(x,0) = u(x,0),$$

$$\sup_{t\in(-\lambda\rho_0,\lambda\rho_0)}\left(\rho_0^{-n}\int_{B_{r_0}} u_+^2(x,t)dx\right)^{1/2} \leq \varepsilon,$$

and

$$\left(\sum_{j=0}^{2} \rho_0^{j-n} \int_{B_{\rho_0}} \left| D_x^j u_+(x,0) \right|^2 dx \right)^{1/2} = H,$$

also, notice that the function of ε at the right hand side of (2.6) is not decreasing. Hence, from now on we assume that $u(x,t)$ is even with respect to the variable t. Moreover it is not restrictive to assume $\rho_0 = 1$.

In order to prove Theorem 2.1 we proceed in the following way.

First step. After a standard extension of $u(\cdot,0)$ in $H^2(B_2) \cap H_0^1(B_2)$ we will construct, similarly to [31], a sequence of function $\{v_k(x,y)\}_{k\in\mathbb{N}}$, with the following properties:

(i) for every $k \in \mathbb{N}$ the function v_k belongs to $H^2(B_2) \cap H_0^1(B_2)$, in addition $v_k(x,y)$ is even with respect to the variable $y \in \mathbb{R}$,

(ii) the sequence $\{v_k(\cdot,0)\}_{k\in\mathbb{N}}$ approximates $u(\cdot,0)$ in $L^2(B_1)$, more precisely we have

$$\|u(\cdot,0) - v_k\|_{L^2(B_1)} \le CHk^{-1/6}.$$

Moreover, for every $k \in \mathbb{N}$ the function $v_k(x,y)$ is a solution to the elliptic problem,

$$\begin{cases} q(x)\partial_y^2 v_k + \operatorname{div}(A(x)\nabla_x v_k) = f_k(x,y), & \text{in } B_2 \times \mathbb{R}, \\ \|v_k(\cdot,0)\|_{L^2(B_{r_0})} \le \varepsilon, \end{cases}$$

where f_k satisfies

$$\|f_k(\cdot,y)\|_{L^2(B_2)} \le (C|y|)^{2k} \quad \forall k \in \mathbb{N}.$$

Second step. Here we derive some stability estimates of Cauchy problem for the above elliptic equation getting estimates v_k in the ball of $\mathbb{R}^{n+1}$ centered at 0 with radius $r_0/4$, (Proposition 3.6). Then we use Carleman estimates with singular weight (Theorem 3.7) for the elliptic equation and the above estimate of $\|u(\cdot,0) - v_k\|_{L^2(B_1)}$. Finally, we choose the parameter k and we get the estimate (2.6).

First step.

Let us start by introducing some notation and by giving an outline of the proof of Theorem 2.1. Let $\widetilde{u}_0$ an extension of the function $u_0 := u(\cdot,0)$ such that $\widetilde{u}_0 \in H^2(B_2) \cap H_0^1(B_2)$ and

$$\|\widetilde{u}_0\|_{H^2(B_2)} \le CH, \tag{3.1}$$

where C is an absolute constant.

Let us denote by λ_j, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ the eigenvalues associated to the Dirichlet problem

$$\begin{cases} \text{div}\,(A(x)\nabla_x v) + \omega q(x) v = 0, & \text{in } B_2, \\ v \in H_0^1\,(B_2)\,, \end{cases} \tag{3.2}$$

and by $e_j(\cdot)$ the corresponding eigenfunctions normalized by

$$\int_{B_2} e_j^2(x) q(x) dx = 1. \tag{3.3}$$

By (2.1a), (2.2) and Poincaré inequality we have for every $j \in \mathbb{N}$

$$\lambda_j = \int_{B_2} A(x)\nabla_x e_j(x) \cdot \nabla_x e_j(x) dx \geq c\lambda^2 \int_{B_2} e_j^2(x) q(x) dx = c\lambda^2 \tag{3.4}$$

where c is an absolute constant. Denote by

$$\alpha_j := \int_{B_2} \widetilde{u}_0(x) e_j(x) q(x) dx, \tag{3.5}$$

and let

$$\widetilde{u}(x,t) := \sum_{j=1}^{\infty} \alpha_j e_j(x) \cos\sqrt{\lambda_j} t. \tag{3.6}$$

Proposition 3.1 *We have*

$$\sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right) \alpha_j^2 \leq C H^2, \tag{3.7}$$

where C depends on λ, Λ only. Moreover, $\widetilde{u} \in \mathcal{W}(\mathbb{R}; B_2) \cap C^0(\mathbb{R}; H^2(B_2) \cap H_0^1(B_2))$ is an even function with respect to the variable t and it satisfies

$$\begin{cases} q(x)\partial_t^2 \widetilde{u} - div\,(A(x)\nabla_x \widetilde{u}) = 0, & in\ \ B_2 \times \mathbb{R}, \\ \widetilde{u}(\cdot, 0) = \widetilde{u}_0, & in\ \ B_2, \\ \partial_t \widetilde{u}(\cdot, 0) = 0, & in\ \ B_2. \end{cases} \tag{3.8}$$

Proof By (3.2) and (3.3) we have

$$\lambda_j \alpha_j = \int_{B_2} \widetilde{u}_0(x) \lambda_j q(x) e_j(x) dx = -\int_{B_2} \text{div}\,(A(x)\nabla_x \widetilde{u}_0(x))\, e_j(x) dx.$$

Hence, by (2.1), (2.2) and (3.1) we have

$$\sum_{j=1}^{\infty}\left(1+\lambda_j^2\right)\alpha_j^2 = \|\widetilde{u}_0\|^2_{L^2(B_2;qdx)} + \left\|\frac{1}{q}\mathrm{div}\,(A\nabla_x\widetilde{u}_0)\right\|^2_{L^2(B_2;qdx)} \leq CH^2,$$

where C depends on λ, Λ only and (3.7) follows. □

Notice that, since $\widetilde{u}(\cdot,0)=u_+(\cdot,0)$ and $\partial_t\widetilde{u}(\cdot,0)=0=\partial_t u_+(\cdot,0)$ in B_1, we have for the uniqueness to the Cauchy problem for Eq. (2.3), (see, for instance [19]),

$$\widetilde{u}(x,t)=u_+(x,t), \quad \text{for } |x|+\lambda^{-1}|t|<1. \tag{3.9}$$

Let us introduce the following nonnegative, even function ψ such that

$$\psi(t)=\begin{cases}\frac{1}{2}(1+\cos\pi t), & \text{for } |t|\leq 1,\\ 0, & \text{for } |t|>1.\end{cases} \tag{3.10}$$

Notice that $\psi\in C^{1,1}$, supp $\psi\subseteq[-1,1]$ and

$$\int_{\mathbb{R}}\psi(t)dt=1. \tag{3.11}$$

Let

$$\widehat{\psi}(\tau)=\int_{\mathbb{R}}\psi(t)e^{-i\tau t}dt=\int_{\mathbb{R}}\psi(t)\cos\tau t dt, \quad \tau\in\mathbb{R}. \tag{3.12}$$

Since ψ has compact support, $\widehat{\psi}$ is an entire function. By (3.11) we have

$$\left|\widehat{\psi}(\tau)\right|\leq\int_{\mathbb{R}}\psi(t)dt=1, \quad \text{for every } \tau\in\mathbb{R},$$

and

$$\left|\tau^2\widehat{\psi}(\tau)\right|=\left|-\int_{\mathbb{R}}\psi(t)\frac{d^2}{dt^2}\cos\tau t dt\right|=\left|-\int_{\mathbb{R}}\psi''(t)\cos\tau t dt\right|\leq\pi^2, \quad \text{for every } \tau\in\mathbb{R},$$

hence we have

$$\left|\widehat{\psi}(\tau)\right|\leq\min\left\{1,\pi^2\tau^{-2}\right\}, \quad \text{for every } \tau\in\mathbb{R}. \tag{3.13}$$

Let

$$\vartheta(t)=4\lambda^{-1}\psi(4\lambda^{-1}t), \quad t\in\mathbb{R}. \tag{3.14}$$

In the following proposition we collect the elementary properties of ϑ that we need.

Proposition 3.2 *The function ϑ is an even and non negative function such that $\vartheta \in C^{1,1}$, supp $\vartheta = [-\frac{\lambda}{4}, \frac{\lambda}{4}]$, $\int_{\mathbb{R}} \vartheta(t)dt = 1$, $\widehat{\vartheta}(\tau) = \widehat{\psi}(\frac{\lambda\tau}{4})$ and*

$$\int_{\mathbb{R}} \left|\vartheta'(t)\right| dt = 8\lambda^{-1}, \tag{3.15}$$

$$\left|\widehat{\vartheta}(\tau)\right| \leq \min\left\{1, 16\pi^2(\tau\lambda)^{-2}\right\}, \quad \textit{for every } \tau \in \mathbb{R}, \tag{3.16}$$

$$\left|\widehat{\vartheta}(\tau) - 1\right| \leq \left(\frac{\lambda\tau}{4}\right)^2, \quad \textit{for } \left|\frac{\lambda\tau}{4}\right| \leq \frac{\pi}{2}, \tag{3.17}$$

$$\frac{1}{2} \leq \widehat{\vartheta}(\tau), \quad \textit{for } \left|\frac{\lambda\tau}{4}\right| \leq \frac{1}{\sqrt{2}}. \tag{3.18}$$

Proof We limit ourselves to prove property (3.17) and (3.18), since the other properties are immediate consequences of (3.12), (3.13) and (3.14). We have

$$\left|\widehat{\vartheta}(\tau) - 1\right| \leq \int_{-1}^{1} \psi(s)\left(1 - \cos\left(\frac{\lambda s\tau}{4}\right)\right) ds. \tag{3.19}$$

Now, if $s \in [-1, 1]$ and $|\frac{\lambda\tau}{4}| \leq \frac{\pi}{2}$ then

$$1 - \cos\left(\frac{\lambda s\tau}{4}\right) \leq \left(\frac{\lambda\tau}{4}\right)^2.$$

Hence by (3.19) we get (3.17). Finally (3.18) is an immediate consequence of (3.17) □

As usual, if $f, g \in L^1(\mathbb{R})$, we denote by $(f*g)(t) := \int_{\mathbb{R}} f(t-s)g(s)ds$. Moreover we denote by $f^{*(k)} := f * f^{*(k-1)}$, for $k \geq 2$, where $f^{*(1)} := f$.

Let us define

$$\vartheta_k(t) := (k\vartheta(kt))^{*(k)}, \quad \text{for every } k \in \mathbb{N}. \tag{3.20}$$

Notice that $\vartheta_k \geq 0$, supp $\vartheta_k \subset [-\frac{\lambda}{4}, \frac{\lambda}{4}]$, $\int_{\mathbb{R}} \vartheta_k(t)dt = 1$, for every $k \in \mathbb{N}$ and

$$\widehat{\vartheta}_k(\tau) = \left(\widehat{\vartheta}(k^{-1}\tau)\right)^k, \quad \text{for every } k \in \mathbb{N}, \tau \in \mathbb{R}. \tag{3.21}$$

Moreover, by (3.17) we have

$$\lim_{k\to+\infty} \widehat{\vartheta}_k(\tau) = 1, \quad \text{for every } \tau \in \mathbb{R}. \tag{3.22}$$

For any number $\mu \in (0, 1]$ and any $k \in \mathbb{N}$ let us set

$$\varphi_{\mu,k} = \left(\vartheta_k * \varphi_\mu\right), \tag{3.23}$$

where

$$\varphi_\mu(t) = \mu^{-1}\vartheta\left(\mu^{-1}t\right), \quad \text{for every } t \in \mathbb{R}. \tag{3.24}$$

We have supp $\varphi_{\mu,k} \subset [-\frac{\lambda(\mu+1)}{4}, \frac{\lambda(\mu+1)}{4}]$, $\varphi_{\mu,k} \geq 0$ and $\int_{\mathbb{R}} \varphi_{\mu,k}(t)dt = 1$. Moreover $\varphi_{\mu,k}$ is an even function.

Now, let us define the following mollified form of the Boman transformation of $\widetilde{u}(x,\cdot)$ [8],

$$\widetilde{u}_{\mu,k}(x) = \int_{\mathbb{R}} \widetilde{u}(x,t)\varphi_{\mu,k}(t)dt, \quad \text{for } \ x \in B_2. \tag{3.25}$$

Proposition 3.3 *If $k \in \mathbb{N}$ and $\mu = k^{-1/6}$ then the following inequality holds true*

$$\left\|u(\cdot,0) - \widetilde{u}_{\mu,k}\right\|_{L^2(B_1)} \leq CHk^{-1/6}, \tag{3.26}$$

where C depends on λ only.

Proof Let $\mu \in (0,1]$. By applying the triangle inequality and taking into account (3.11) and (3.24) we have

$$\begin{aligned} \left\|u(\cdot,0) - \widetilde{u}_{\mu,k}(\cdot)\right\|_{L^2(B_1)} &\leq \left(\int_{B_1} dx \int_{-\lambda\mu/4}^{\lambda\mu/4} |u(x,0) - \widetilde{u}(x,t)|^2 \varphi_\mu(t)dt\right)^{1/2} \\ &+ \left(\int_{B_1} dx \int_{-\lambda(\mu+1)/4}^{\lambda(\mu+1)/4} |\widetilde{u}(x,t)|^2 dt\right)^{1/2} \left\|\varphi_\mu - \varphi_{\mu,k}\right\|_{L^2(\mathbb{R})} := I_1 + I_2. \end{aligned} \tag{3.27}$$

In order to estimate I_1 from above we observe that by the energy inequality, (3.1), and taking into account that $\partial_t\widetilde{u}(x,0) = 0$, we have

$$\begin{aligned} \int_{B_2} |\partial_t\widetilde{u}(x,t)|^2 dx &\leq \int_{B_2} \left(|\partial_t\widetilde{u}(x,t)|^2 + |\nabla_x\widetilde{u}(x,t)|^2\right) dx \\ &\leq \lambda^{-2} \int_{B_2} \left(|\partial_t\widetilde{u}(x,0)|^2 + |\nabla_x\widetilde{u}(x,0)|^2\right) dx \leq CH^2, \end{aligned}$$

where C depends on λ only. Therefore

$$I_1^2 \leq 2\int_{B_1} dx \left|\int_0^{\lambda\mu/4} \partial_\eta\widetilde{u}(x,\eta)d\eta\right|^2 \leq \frac{\lambda\mu}{2} \int_{B_1} dx \int_0^{\lambda\mu/4} \left|\partial_\eta\widetilde{u}(x,\eta)\right|^2 d\eta \leq CH^2\mu^2.$$

Hence

$$I_1 \leq CH\mu, \tag{3.28}$$

where C depends on λ only.

Concerning I_2, first we observe that by using Poincaré inequality, by energy inequality, and by (3.1) (recalling that $\mu \in (0,1]$), we have

$$\int_{-\lambda(\mu+1)/4}^{\lambda(\mu+1)/4} dt \int_{B_1} |\widetilde{u}(x,t)|^2\,dx \le \int_{-\lambda/2}^{\lambda/2} dt \int_{B_2} |\widetilde{u}(x,t)|^2\,dx \\ \le C \int_{-\lambda/2}^{\lambda/2} dt \int_{B_2} |\nabla_x \widetilde{u}(x,t)|^2\,dx \le CH^2, \tag{3.29}$$

where C depends on λ only.

In order to estimate from above $\|\varphi_\mu - \varphi_{\mu,k}\|_{L^2(\mathbb{R})}$ we recall that $\widehat{\varphi}_\mu(\tau) = \widehat{\vartheta}(\mu\tau)$ and $\widehat{\varphi}_{\mu,k}(\tau) = \widehat{\vartheta}(\mu\tau)(\widehat{\vartheta}(k^{-1}\tau))^k$, hence the Parseval identity and a change of variable give

$$2\pi \left\|\varphi_\mu - \varphi_{\mu,k}\right\|^2_{L^2(\mathbb{R})} = \frac{1}{\mu}\int_{\mathbb{R}} \left|\left(\widehat{\vartheta}((\mu k)^{-1}\tau)\right)^k - 1\right|^2 \left|\widehat{\vartheta}(\tau)\right|^2 d\tau. \tag{3.30}$$

By (3.16), (3.17) and (3.18) and by using the elementary inequalities $1 - e^{-z} \le z$, for every $z \in \mathbb{R}$, and $\log s \le s - 1$, for every $s > 0$, we have, whenever $|\frac{\lambda\tau}{4\mu k}| \le \frac{1}{\sqrt{2}}$,

$$0 \le 1 - \left(\widehat{\vartheta}((\mu k)^{-1}\tau)\right)^k = 1 - e^{k \log \widehat{\vartheta}((\mu k)^{-1}\tau)} \le \frac{\lambda^2\tau^2}{8\mu^2 k}. \tag{3.31}$$

Now let $\delta \in (0,1]$ be a number that we shall choose later and denote $\beta = \frac{4\mu k}{\sqrt{2}\lambda}\delta$. By (3.30), (3.16) and (3.31) we have

$$2\pi \left\|\varphi_\mu - \varphi_{\mu,k}\right\|^2_{L^2(\mathbb{R})} = \frac{1}{\mu}\int_{|\tau|\le\beta} \left|\left(\widehat{\vartheta}((\mu k)^{-1}\tau)\right)^k - 1\right|^2 \left|\widehat{\vartheta}(\tau)\right|^2 d\tau \\ + \frac{1}{\mu}\int_{|\tau|\ge\beta} \left|\left(\widehat{\vartheta}((\mu k)^{-1}\tau)\right)^k - 1\right|^2 \left|\widehat{\vartheta}(\tau)\right|^2 d\tau \\ \le \frac{1}{\mu}\int_{|\tau|\le\beta} \left(\frac{\lambda^2\tau^2}{8\mu^2 k}\right)^2 d\tau + \frac{1}{\mu}\int_{|\tau|>\beta} \left(\frac{32\pi^2}{\lambda^2\tau^2}\right)^2 d\tau \le C\left(k^3\delta^5 + \frac{1}{\delta^3\mu^4 k^3}\right), \tag{3.32}$$

where C depends on λ only. If $\mu^2 k^{3/5} \ge 1$, we choose $\delta = (\mu^2 k^3)^{-1/4}$ and by (3.32) we have

$$\left\|\varphi_\mu - \varphi_{\mu,k}\right\|_{L^2(\mathbb{R})} \le C\left(k^{3/5}\mu^2\right)^{-5/8}, \tag{3.33}$$

where C depends on λ only. Hence recalling (3.29) we have

$$I_2 \le CH\left(k^{3/5}\mu^2\right)^{-5/8}. \tag{3.34}$$

By (3.27), (3.28) and (3.34) we obtain

$$\|u(\cdot,0)-\widetilde{u}_{\mu,k}\|_{L^2(B_1)} \le CH(\mu+(k^{3/5}\mu^2)^{-5/8}). \tag{3.35}$$

Now, if $\mu = k^{-\frac{1}{6}}$, $k \ge 1$ then (3.35) implies (3.26). □

From now on we fix $\overline{\mu} := k^{-\frac{1}{6}}$ for $k \ge 1$ and we denote

$$\widetilde{u}_k := \widetilde{u}_{\overline{\mu},k}. \tag{3.36}$$

Let us introduce now, for every $k \in \mathbb{N}$ an even function $g_k \in C^{1,1}(\mathbb{R})$ such that if $|z| \le k$ then we have $g_k(z) = \cosh z$, if $|z| \ge 2k$ then we have $g_k(z) = \cosh 2k$ and such that it satisfies the condition

$$|g_k(z)| + \left|g_k'(z)\right| + \left|g_k''(z)\right| \le ce^{2k}, \quad \text{for every } z \in \mathbb{R}, \tag{3.37}$$

where c is an absolute constant.

The following proposition is the main result of this first step.

Proposition 3.4 *Let*

$$v_k(x,y) := \sum_{j=1}^{\infty} \alpha_j \widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right) g_k\left(y\sqrt{\lambda_j}\right) e_j(x), \quad \textit{for } (x,y) \in B_2 \times \mathbb{R}. \tag{3.38}$$

We have that $v_k(\cdot,y)$ belongs to $H^2(B_2) \cap H_0^1(B_2)$ for every $y \in \mathbb{R}$, $v_k(x,y)$ is an even function with respect to y and it satisfies

$$\begin{cases} q(x)\partial_y^2 v_k + div\,(A(x)\nabla_x v_k) = f_k(x,y), & in\ B_2 \times \mathbb{R}, \\ v_k(\cdot,0) = \widetilde{u}_k, & in\ B_2. \end{cases} \tag{3.39}$$

and

$$\|v_k(\cdot,0)\|_{L^2(B_{r_0})} \le \varepsilon. \tag{3.40}$$

where

$$f_k(x,y) = \sum_{j=1}^{\infty} \lambda_j \alpha_j \widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)\left(g_k''\left(y\sqrt{\lambda_j}\right) - g_k\left(y\sqrt{\lambda_j}\right)\right) q(x) e_j(x). \tag{3.41}$$

Moreover we have

$$\sum_{j=0}^{2} \|\partial_y^j v_k(\cdot,y)\|_{H^{2-j}(B_2)} \le CHe^{2k}, \quad \textit{for every } y \in \mathbb{R}, \tag{3.42}$$

$$\|f_k(\cdot,y)\|_{L^2(B_2)} \le CHe^{2k}\min\left\{1,\left(4\pi\lambda^{-1}|y|\right)^{2k}\right\},\quad \textit{for every } y\in\mathbb{R}, \quad (3.43)$$

where C depends on λ *and* Λ *only.*

Proof First of all observe that

$$\left|\widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)\right| \le \|\varphi_{\overline{\mu},k}\|_{L^1(\mathbb{R})} = 1. \quad (3.44)$$

For the sake of brevity, in what follows we shall omit k from v_k.

In order to prove that $v(\cdot,y)\in H^2(B_2)\cap H^1_0(B_2)$ for $y\in\mathbb{R}$, let $M,N\in\mathbb{N}$ such that $M>N$ and let us denote by

$$V_{M,N}(x,y) := \sum_{j=N+1}^{M}\alpha_j\widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)g_k\left(y\sqrt{\lambda_j}\right)e_j(x). \quad (3.45)$$

By (3.37) and (3.44) we have, for every $y\in\mathbb{R}$,

$$\begin{aligned}
\lambda\int_{B_2}\left|\nabla_x V_{M,N}(x,y)\right|^2dx &\le \int_{B_2}A(x)\nabla_x V_{M,N}(x,y)\cdot\nabla_x V_{M,N}(x,y)dx\\
&= \sum_{j=N+1}^{M}\left(\int_{B_2}A(x)\nabla_x e_j(x)\cdot\nabla_x V_{M,N}(x,y)dx\right)\widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)g_k\left(y\sqrt{\lambda_j}\right)\alpha_j\\
&= \sum_{j=N+1}^{M}\lambda_j\alpha_j^2\widehat{\varphi}^2_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)g_k^2\left(y\sqrt{\lambda_j}\right) \le ce^{4k}\sum_{j=N+1}^{M}\lambda_j\alpha_j^2.
\end{aligned}$$

Therefore, since $V_{M,N}(\cdot,y)\in H^1_0(B_2)$ we have

$$\|V_{M,N}(\cdot,y)\|^2_{H^1_0(B_2)} \le ce^{4k}\sum_{j=N+1}^{M}\lambda_j\alpha_j^2,\quad \text{for every }\ y\in\mathbb{R}. \quad (3.46)$$

The inequality above and (3.7) gives

$$\|V_{M,N}(\cdot,y)\|_{H^1_0(B_2)}\to 0,\quad \text{as }\ M,N\to\infty,\quad \text{for every }\ y\in\mathbb{R},$$

hence $v\in H^1_0(B_2)$.

In order to prove that $v\in H^2(B_2)$, first observe that by (3.37), (3.44) and (3.45) we have

$$\|\mathrm{div}\left(A\nabla_x V_{M,N}\right)\|^2_{L^2(B_2)} \leq c\lambda^{-1}e^{4k}\sum_{j=N+1}^{M}\lambda_j^2\alpha_j^2, \quad \text{for every} \quad y\in\mathbb{R},$$

then by the above inequality and standard L^2 regularity estimate [21] we obtain

$$\begin{aligned}\|D_x^2 V_{M,N}(\cdot,y)\|^2_{L^2(B_2)} &\leq C\|\mathrm{div}\left(A\nabla_x V_{M,N}\right)\|^2_{L^2(B_2)}\\ &\leq e^{4k}\sum_{j=N+1}^{M}\lambda_j^2\alpha_j^2, \quad \text{for every} \quad y\in\mathbb{R},\end{aligned} \tag{3.47}$$

where C depends on λ and Λ only. Hence $v\in H^2(B_2)$. Moreover by (3.7), (3.46) and (3.47) we have

$$\begin{aligned}&\|v(\cdot,y)\|_{L^2(B_2)}+\|\nabla_x v(\cdot,y)\|_{L^2(B_2)}+\|D_x^2 v(\cdot,y)\|_{L^2(B_2)}\\ &\quad\leq CHe^{2k}, \quad \text{for every} \quad y\in\mathbb{R},\end{aligned} \tag{3.48}$$

where C depends on λ and Λ only. Similarly we have $\partial_y v(\cdot,y)$, $\partial_y^2 v(\cdot,y)$, $\partial_y\nabla_x v(\cdot,y)\in L^2(B_2)$ and

$$\sum_{j=1}^{2}\|\partial_y^j D_x^{2-j}v(\cdot,y)\|_{L^2(B_2)}\leq CHe^{2k}, \quad \text{for every} \quad y\in\mathbb{R}, \tag{3.49}$$

where C depends on λ and Λ only.

Inequality (3.49) and (3.48), yields (3.42). By (3.38) we have immediately that the function v is an even function and it satisfies (3.39).

Concerning (3.40), we have by $\|\varphi_{\overline{\mu},k}\|_{L^1(\mathbb{R})}=1$, by Schwarz inequality, by (2.4) and by (3.25),

$$\|v_k(\cdot,0)\|^2_{L^2(B_{r_0})}=\int_{B_{r_0}}|\widetilde{u}_k(x)|^2\,dx\leq\int_{-\lambda(\overline{\mu}+1)/4}^{\lambda(\overline{\mu}+1)/4}\left(\int_{B_{r_0}}|u(x,t)|^2\,dx\right)\varphi_{\overline{\mu},k}(t)dt\leq\varepsilon^2.$$

Concerning (3.43), first observe that by the definition of g_k we have that $g_k''(y\sqrt{\lambda_j})-g_k(y\sqrt{\lambda_j})=0$, for $|y|\sqrt{\lambda_j}\leq k$ and $|g_k''(y\sqrt{\lambda_j})-g_k(y\sqrt{\lambda_j})|\leq ce^{2k}$, for $|y|\sqrt{\lambda_j}\geq k$. Hence, taking into account (3.16) and (3.21), we have, for every $y\in\mathbb{R}$ and for every $k\in\mathbb{N}$,

$$\begin{aligned}&\left|g_k''(y\sqrt{\lambda_j})-g_k(y\sqrt{\lambda_j})\right|\left|\widehat{\varphi}_{\overline{\mu},k}(\sqrt{\lambda_j})\right|\leq ce^{2k}\left|\widehat{\vartheta}(k^{-1}\sqrt{\lambda_j})\right|^k\chi_{\{y:|y|\sqrt{\lambda_j}\geq k\}}\\ &\quad\leq ce^{2k}\sup\left\{\left|\widehat{\vartheta}(k^{-1}\sqrt{\lambda_j})\right|^k:|y|\sqrt{\lambda_j}\geq k\right\}\leq ce^{2k}\min\left\{1,\left(4\pi\lambda^{-1}|y|\right)^{2k}\right\}.\end{aligned} \tag{3.50}$$

By (3.42) and (3.50) we have

$$\|f_k(\cdot,y)\|_{L^2(B_2)} \le ce^{2k} \min\left\{1, \left(4\sqrt{2}\pi\lambda^{-1}|y|\right)^{2k}\right\} \left(\sum_{j=1}^{\infty} \lambda_j^2 \alpha_j^2\right)^{1/2}, \quad \text{for every } y \in \mathbb{R}.$$

By the above inequality and by (3.7) we obtain (3.43). □

Second step.

In what follows we shall denote by $\widetilde{B}_r$ the ball of $\mathbb{R}^{n+1}$ of radius r centered at 0. In order to prove Proposition 3.6 stated below we need the following Lemma.

Lemma 3.5 *Let r be a positive number and let $w \in H^2(\widetilde{B}_r)$ be a solution to the problem*

$$\begin{cases} q(x)\partial_y^2 w(x,y) + div\,(A(x)\nabla_x w(x,y)) = 0, & in \ \widetilde{B}_r, \\ \partial_y w(\cdot,0) = 0, & in \ B_r, \end{cases} \tag{3.51}$$

where A satisfies (2.1) and q satisfies (2.2).

Then there exist $\beta \in (0,1)$ and $C \ge 1$ depending on λ and Λ only such that

$$\int_{\widetilde{B}_{r/4}} w^2 dxdy \le C \left(\int_{\widetilde{B}_r} w^2 dxdy\right)^{1-\beta} \left(r\int_{B_{r/2}} w^2(x,0)dx\right)^{\beta}. \tag{3.52}$$

Proof After scaling, we may assume $r = 1$. By [4, Theorem 1.7] we have

$$\|w\|_{L^2(\widetilde{B}_{1/4})} \le C\left(\|w\|_{L^2(\widetilde{B}_1)}\right)^{1-\widetilde{\beta}} \left(\|w\|_{H^{1/2}(B_{1/2})}\right)^{\widetilde{\beta}}, \tag{3.53}$$

where C and $\widetilde{\beta} \in (0,1)$ depend on λ and Λ only. Now, by the interpolation inequality, the trace inequality and standard regularity for elliptic equation [21] we have

$$\begin{aligned} \|w\|_{H^{1/2}(B_{1/2})} &\le C\|w\|^{2/3}_{L^2(B_{1/2})}\|w\|^{1/3}_{H^{3/2}(B_{1/2})} \le C\|w\|^{2/3}_{L^2(B_{1/2})}\|w\|^{1/3}_{H^2(\widetilde{B}_{3/4})} \\ &\le C'\|w\|^{2/3}_{L^2(B_{1/2})}\|w\|^{1/3}_{L^2(\widetilde{B}_1)}, \end{aligned} \tag{3.54}$$

where C' depends on λ and Λ only. By (3.53) and (3.54) we get (3.52) with $\beta = \frac{2\widetilde{\beta}}{3}$. □

Proposition 3.6 *Let v_k be defined in (3.38) and let $r_0 \le \frac{\lambda}{8}$. Then we have*

$$\|v_k\|_{L^2(\widetilde{B}_{r_0/4})} \le C\sqrt{r_0}\left(\varepsilon + H\,(C_0 r_0)^{2k}\right)^{\beta}\left(He^{2k} + H\,(C_0 r_0)^{2k}\right)^{1-\beta}. \tag{3.55}$$

where $\beta \in (0,1)$, C depend on λ and Λ only and $C_0 = 4\pi e\lambda^{-1}$.

Proof Let $w_k \in H^2\left(\widetilde{B}_{r_0}\right)$ be the solution to the following Dirichlet problem

$$\begin{cases} q(x)\partial_y^2 w_k + \operatorname{div}\left(A(x)\nabla_x w_k\right) = f_k, & \text{in } \widetilde{B}_{r_0}, \\ w_k = 0, & \text{on } \partial \widetilde{B}_{r_0}. \end{cases} \tag{3.56}$$

Notice that, since f_k is an even function with respect to y, by the uniqueness to the Dirichlet problem (3.56) we have that w_k is an even function with respect to y.

By standard regularity estimates we have

$$\|w_k\|_{L^2(\widetilde{B}_{r_0})} + r_0\|\nabla_{x,y} w_k\|_{L^2(\widetilde{B}_{r_0})} \leq C r_0^2 \|f_k\|_{L^2(\widetilde{B}_{r_0})}, \tag{3.57}$$

where C depends on λ only. By the above inequality and by the trace inequality we get

$$\begin{aligned} \|w_k(\cdot,0)\|_{L^2(B_{r_0/2})} &\leq C\left(r_0^{-1/2}\|w_k\|_{L^2(\widetilde{B}_{r_0})} + r_0^{1/2}\|\nabla_{x,y} w_k\|_{L^2(\widetilde{B}_{r_0})}\right) \\ &\leq C r_0^{3/2}\|f_k\|_{L^2(\widetilde{B}_{r_0})}, \end{aligned} \tag{3.58}$$

where C depends on λ only.

Now, denoting

$$z_k = v_k - w_k, \tag{3.59}$$

by (3.43), (3.40), (3.57) and (3.58) we have

$$\|z_k(\cdot,0)\|_{L^2(B_{r_0/2})} \leq \varepsilon + C r_0^2 H\left(C_0 r_0\right)^{2k}, \tag{3.60}$$

and

$$\|z_k\|_{L^2(\widetilde{B}_{r_0})} \leq C r_0^{1/2} H\left(e^{2k} + r_0^2\left(C_0 r_0\right)^{2k}\right), \tag{3.61}$$

where C depends on λ only.

Now by (3.56) we have

$$\begin{cases} q(x)\partial_y^2 z_k + \operatorname{div}\left(A(x)\nabla_x z_k\right) = 0, & \text{in } \widetilde{B}_{r_0}, \\ \partial_y z_k(\cdot,0) = 0, & \text{on } B_{r_0}, \end{cases}$$

hence by applying Lemma 3.5 to the function z_k and by using (3.42), (3.59), (3.60) and (3.61) the thesis follows. □

In order to prove Theorem 2.1 we use a Carleman estimate with singular weight, proved for the first time by [6]. In order to control the dependence of the various constants, we use here the following version of such a Carleman estimate that was proved, in the context of parabolic operator, in [17]. First we introduce some notation. Let P be the elliptic operator

$$P := q(x)\partial_y^2 + \operatorname{div}(A(x)\nabla_x). \tag{3.62}$$

Denote

$$\sigma(x,y) = \left(A^{-1}(0)x\cdot x + (q(0))^{-1}y^2\right)^{1/2}, \tag{3.63}$$

$$\widetilde{B}_r^\sigma = \left\{(x,y)\in\mathbb{R}^{n+1} : \sigma(x,y)\le r\right\}, \quad r>0, \tag{3.64}$$

Notice that

$$\widetilde{B}^\sigma_{\sqrt{\lambda}r} \subset \widetilde{B}_r \subset \widetilde{B}^\sigma_{r/\sqrt{\lambda}} \quad \text{, for every } r>0. \tag{3.65}$$

Theorem 3.7 *Let P be the operator (3.62) and assume that (2.1) and (2.2) are satisfied. There exists a constant $C_*>1$ depending on λ and Λ only such that, denoting*

$$\phi(s) = s\exp\left(\int_0^s \frac{e^{-C_*\eta}-1}{\eta}d\eta\right), \tag{3.66a}$$

$$\delta(x,y) = \phi\left(\sigma(x,y)/2\sqrt{\lambda}\right), \tag{3.66b}$$

for every $\tau\ge C_*$ and $U\in C_0^\infty\left(\widetilde{B}^\sigma_{2\sqrt{\lambda}/C_*}\setminus\{0\}\right)$ we have

$$\begin{aligned}\tau&\int_{\mathbb{R}^{n+1}}\delta^{1-2\tau}(x,y)\left|\nabla_{x,y}U\right|^2dxdy + \tau^3\int_{\mathbb{R}^{n+1}}\delta^{-1-2\tau}(x,y)U^2dxdy\\ &\le C_*\int_{\mathbb{R}^{n+1}}\delta^{2-2\tau}(x,y)\left|PU\right|^2dxdy.\end{aligned} \tag{3.67}$$

Conclusion of the proof of Theorem 2.1
Set

$$r_1 = \frac{\sqrt{\lambda}r_0}{16}$$

by (3.55) we have

$$\|v_k\|_{L^2\left(\widetilde{B}^\sigma_{4r_1}\right)} \le C\sqrt{r_1}S_k, \tag{3.68}$$

where C depends on λ and Λ only and

$$S_k = \left(\varepsilon + H(C_1r_1)^{2k}\right)^\beta\left(He^{2k} + H(C_1r_1)^{2k}\right)^{1-\beta}, \tag{3.69}$$

where $C_1 = 16C_0/\sqrt{\lambda}$, recall that C_0 has been introduced in Proposition 3.6.

Denote

$$\delta_0(r) := \phi(r/2\sqrt{\lambda}), \quad \text{for every } \ r > 0$$

and

$$R = \frac{\sqrt{\lambda}}{C_*}.$$

Let us consider a function $h \in C_0^2\left(0, \delta_0\left(2R\right)\right)$ such that $0 \leq h \leq 1$ and

$$\begin{aligned}
&h(s) = 1, \quad \text{for every } s \in \left[\delta_0\left(2r_1\right), \delta_0\left(R\right)\right], \\
&h(s) = 0, \quad \text{for every } s \in \left[0, \delta_0\left(r_1\right)\right] \cup \left[\delta_0\left(3R/2\right), \delta_0\left(2R\right)\right], \\
&r_1\left|h'(s)\right| + r_1^2\left|h''(s)\right| \leq c, \quad \text{for every } s \in \left[\delta_0\left(r_1\right), \delta_0\left(2r_1\right)\right], \\
&\left|h'(s)\right| + \left|h''(s)\right| \leq c, \quad \text{for every } s \in \left[\delta_0\left(R\right), \delta_0\left(3R/2\right)\right],
\end{aligned}$$

where c is an absolute constant.

Moreover, let us define

$$\zeta(x, y) = h\left(\delta(x, y)\right).$$

Notice that if $2r_1 \leq \sigma(x, y) \leq R$ then $\zeta(x, y) = 1$ and if $\sigma(x, y) \geq 2R$ or $\sigma(x, y) \leq r_1$ then $\zeta(x, y) = 0$.

For the sake of brevity, in what follows we shall omit k from v_k and f_k. By density, we can apply (3.67) to the function $U = \zeta v$ and we have, for every $\tau \geq C_*$,

$$\begin{aligned}
&\tau \int_{\widetilde{B}^\sigma_{2R}} \delta^{1-2\tau}(x, y)\left|\nabla_{x,y}\left(\zeta v\right)\right|^2 + \tau^3 \int_{\widetilde{B}^\sigma_{2R}} \delta^{-1-2\tau}(x, y)\left|\zeta v\right|^2 \\
&\leq C \int_{\widetilde{B}^\sigma_{2R}} \delta^{2-2\tau}(x, y)\left|f\right|^2 \zeta^2 + C \int_{\widetilde{B}^\sigma_{2R}} \delta^{2-2\tau}(x, y)\left|P\zeta\right|^2 v^2 \\
&\quad + C \int_{\widetilde{B}^\sigma_{2R}} \delta^{2-2\tau}(x, y)\left|\nabla_{x,y} v\right|^2 \left|\nabla_{x,y}\zeta\right|^2 := I_1 + I_2 + I_3,
\end{aligned} \tag{3.70}$$

where C depends λ and Λ only.

Estimate of I_1.

Notice that

$$\frac{\sqrt{|x|^2 + y^2}}{2C_2} \leq \delta(x, y) \leq \frac{C_2\sqrt{|x|^2 + y^2}}{2} \quad \text{for every } (x, y) \in \widetilde{B}_2, \tag{3.71}$$

where $C_2 > 1$ depends on λ and Λ only.

By (3.43), (3.65) and (3.71) we have

$$\int_{\widetilde{B}^{\sigma}_{2\sqrt{\lambda}/C_*}} \delta^{2-2\tau}(x,y)\,|f|^2\,\zeta^2 dxdy \leq \int_{\widetilde{B}_2} \left(2C_2|y|^{-1}\right)^{-2+2\tau} |f|^2\,dxdy$$
$$\leq \int_{-2}^{2} \left[\left(2C_2|y|^{-1}\right)^{-2+2\tau} \int_{B_2} |f(x,y)|^2\,dx\right] dy \leq CH^2 \int_{-2}^{2} \left(2C_2|y|^{-1}\right)^{-2+2\tau} (C_0|y|)^{4k}\,dy, \tag{3.72}$$

where C depends on λ and Λ only.

Now let k and τ satisfy the relation

$$\frac{\tau-1}{2} \leq k. \tag{3.73}$$

By (3.72) and (3.73) we get

$$I_1 \leq CH^2 (C_3)^{4k}, \tag{3.74}$$

where $C_3 = 2C_0C_2$.

Estimate of I_2

By (3.42) and (3.68) and (3.70) we have

$$I_2 \leq Cr_1^{-4} \int_{\widetilde{B}^{\sigma}_{2r_1}\setminus\widetilde{B}^{\sigma}_{r_1}} \delta^{2-2\tau}(x,y)v^2 dxdy + C\int_{\widetilde{B}^{\sigma}_{3R/2}\setminus\widetilde{B}^{\sigma}_{R}} \delta^{2-2\tau}(x,y)v^2 dxdy$$
$$\leq C\left(r_1^{-3}\delta_0^{2-2\tau}(r_1)S_k^2 + e^{4k}H^2\delta_0^{2-2\tau}(R)\right),$$

hence (3.71) gives

$$I_2 \leq C\left(\delta_0^{-1-2\tau}(r_1)S_k^2 + e^{4k}H^2\delta_0^{-1-2\tau}(R)\right), \tag{3.75}$$

Estimate of I_3

By (3.70) we have

$$I_3 \leq Cr_1^{-2}\delta_0^{2-2\tau}(r_1)\int_{\widetilde{B}^{\sigma}_{2r_1}\setminus\widetilde{B}^{\sigma}_{r_1}} \left|\nabla_{x,y}v\right|^2 dxdy + C\delta_0^{2-2\tau}(R)\int_{\widetilde{B}^{\sigma}_{3R/2}\setminus\widetilde{B}^{\sigma}_{R}} \left|\nabla_{x,y}v\right|^2 dxdy. \tag{3.76}$$

Now in order to estimate from above the righthand side of (3.76) we use the Caccioppoli inequality, (3.42), (3.43) and (3.68) and we get

$$I_3 \leq C\delta_0^{2-2\tau}(r_1)\left(r_1^{-4}\int_{\widetilde{B}^{\sigma}_{4r_1}\setminus\widetilde{B}^{\sigma}_{r_1/2}} v^2 dxdy + \int_{\widetilde{B}^{\sigma}_{4r_1}\setminus\widetilde{B}^{\sigma}_{r_1/2}} f^2 dxdy\right)$$
$$+ C\delta_0^{2-2\tau}(R)\int_{\widetilde{B}^{\sigma}_{3R/2}\setminus\widetilde{B}^{\sigma}_{R}} \left|\nabla_{x,y}v\right|^2 dxdy \leq C\left(S_k^2 + H^2\left(C_1 r_1\right)^{4k}\right)\delta_0^{-1-2\tau}(r_1)$$
$$+ CH^2 e^{4k}\delta_0^{1-2\tau}(R) := \widetilde{I}_3 \tag{3.77}$$

Now let $r_1 \leq \frac{R}{2}$ and let ρ be such that $\frac{2r_1}{\sqrt{\lambda}} \leq \rho \leq \frac{R}{\sqrt{\lambda}}$ and denote by $\widetilde{\rho} = \sqrt{\lambda}\rho$. By estimating from below trivially the left hand side of (3.70) and taking into account (3.77) we have

$$\delta_0^{1-2\tau}(\widetilde{\rho})\int_{\widetilde{B}^{\sigma}_{\widetilde{\rho}}\setminus\widetilde{B}^{\sigma}_{2r_1}} \left|\nabla_{x,y}v\right|^2 + \delta_0^{-1-2\tau}(\widetilde{\rho})\int_{\widetilde{B}^{\sigma}_{\widetilde{\rho}}\setminus\widetilde{B}^{\sigma}_{2r_1}} |v|^2 \leq I_1 + I_2 + \widetilde{I}_3. \tag{3.78}$$

Now let us add at both the side of (3.78) the quantity

$$\delta_0^{1-2\tau}(\widetilde{\rho})\int_{\widetilde{B}^{\sigma}_{2r_1}} \left|\nabla_{x,y}v\right|^2 + \delta_0^{-1-2\tau}(\widetilde{\rho})\int_{\widetilde{B}^{\sigma}_{2r_1}} v^2,$$

since this term can be estimated from above by $\widetilde{I}_3$, by using standard estimates for second order elliptic equations and by taking into account that $\delta_0(\widetilde{\rho}) \geq \delta_0(r_1)$, we have

$$\rho^2\int_{\widetilde{B}^{\sigma}_{\widetilde{\rho}}} \left|\nabla_{x,y}v\right|^2 + \int_{\widetilde{B}^{\sigma}_{\widetilde{\rho}}} v^2 \leq \delta_0^{1+2\tau}(\widetilde{\rho})\left(I_1 + I_2 + C\widetilde{I}_3\right), \tag{3.79}$$

where C depends on λ and Λ only.

Now by (3.71), (3.74), (3.75), (3.77) and (3.79) it is simple to derive that if (3.73) is satisfied then we have

$$\rho^2\int_{\widetilde{B}_{\lambda\rho}} \left|\nabla_{x,y}v\right|^2 + \int_{\widetilde{B}_{\lambda\rho}} v^2 \leq C\left[S_k^2\left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(r_1)}\right)^{1+2\tau} + H^2 C_4^k\left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(R)}\right)^{1+2\tau}\right], \tag{3.80}$$

where $C_4 > 1$ depends on λ and Λ only.

Now, by applying a standard trace inequality and by recalling that $v(\cdot, 0) = \widetilde{u}_k(\cdot, 0)$ in B_2 (where $\widetilde{u}_k$ is defined by (3.36)) we have

$$\int_{B_{\lambda\rho/2}} |\widetilde{u}_k(\cdot, 0)|^2 \leq C\rho^{-1}\left[S_k^2\left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(r_1)}\right)^{1+2\tau} + H^2 C_4^k\left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(R)}\right)^{1+2\tau}\right]. \tag{3.81}$$

By Proposition 3.3, by (3.69) and (3.81) we have, for $r_1 \leq \frac{R}{2}$

$$\rho \int_{B_{\lambda\rho/2}} |u(\cdot,0)|^2 \leq C\left(H_{k,\tau} + H^2 k^{-1/3}\right) + C\left[C_5^k \left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(r_1)}\right)^{1+2\tau} H^{2(1-\beta)}\varepsilon^{2\beta} + H^2 C_4^k \left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(R)}\right)^{1+2\tau}\right], \tag{3.82}$$

where

$$H_{k,\tau} := H^2 \left(\frac{\delta_0(\widetilde{\rho})}{\delta_0(r_1)}\right)^{1+2\tau} C_5^k r_1^{4\beta k}.$$

and C, C_5 depend on λ, Λ only.

Now let us choose $\tau = \frac{4\beta k - 1}{2}$. We have that (3.73) is satisfied and by (3.71), (3.82) we have that there exist constants $C_6 > 1$ and k_0 depending on λ and Λ only such that for every $k \geq k_0$ we have

$$\rho \int_{B_{\lambda\rho/2}} |u(\cdot,0)|^2 \leq C_6 H_1^2 \left[\left(C_6 \rho r_1^{-1}\right)^{4\beta k} \varepsilon_1^{2\beta} + (C_6\rho)^{4\beta k} + k^{-1/3}\right], \tag{3.83}$$

where

$$H_1 := H + e\varepsilon \quad \text{and } \varepsilon_1 := \frac{\varepsilon}{H + e\varepsilon}.$$

Now, let us denote by

$$\overline{k} := \left[\frac{\log \varepsilon_1}{2\log r_1}\right] + 1,$$

where, for any $s \in \mathbb{R}$, we set $[s] := \max\{p \in \mathbb{Z} : p \leq s\}$. If $\overline{k} \geq k_0$ we choose $k = \overline{k}$ so that by (3.83) we have, for $\rho \leq 1/C_6$,

$$\rho \int_{B_{\lambda\rho/2}} |u(\cdot,0)|^2 \leq C_2 H_1^2 \left(\varepsilon_1^{2\beta\theta_0} + \left(\frac{2\log(1/r_1)}{\log(1/\varepsilon_1)}\right)^{1/3}\right), \tag{3.84}$$

where

$$\theta_0 = \frac{\log(1/C_6\rho)}{2\log(1/r_1)}. \tag{3.85}$$

Otherwise, if $\overline{k} < k_0$ then multiplying both the side of such an inequality by $\log(1/C_6\rho)$ and by (3.85) we get $\theta_0 \log(1/\varepsilon_1) \leq k_0 \log(1/C_6\rho)$. Hence

$$(H + e\varepsilon)^{2\beta\theta_0} \leq (C_6\rho)^{-2\beta k_0} \varepsilon^{2\beta\theta_0}.$$

By this inequality and by (2.5) we have trivially

$$\int_{B_{\lambda\rho/2}} |u(\cdot,0)|^2 \leq (H+e\varepsilon)^2 = (H+e\varepsilon)^{2(1-\beta\theta_0)}(H+e\varepsilon)^{2\beta\theta_0}$$
$$\leq (H+e\varepsilon)^{2(1-\beta\theta_0)}(C_6\rho)^{-2\beta k_0}\varepsilon^{2\beta\theta_0}. \quad (3.86)$$

Finally by (3.84) and (3.86) we obtain (2.6). □

3.2 Proof of Theorem 2.3

First, let us assume $A(0) = I$ where I is the identity matrix $n \times n$. Following the arguments of [1] or [3] we have there exist $\rho_1, \rho_2 \in (0, \rho_0]$ such that $\frac{\rho_1}{\rho_0}, \frac{\rho_2}{\rho_0}$ depend on λ, Λ, E only and we can construct a function $\Phi \in C^{1,1}(\overline{B}_{\rho_2}(0), \mathbb{R}^n)$ such that

$$\Phi\left(B_{\rho_2}\right) \subset B_{\rho_1}, \quad (3.87a)$$

$$\Phi(y',0) = (y',\phi(y')), \quad \text{for every } y' \in B'_{\rho_2}, \quad (3.87b)$$

$$\Phi\left(B^+_{\rho_2}\right) \subset K_{\rho_1}, \quad (3.87c)$$

$$C_1^{-1}|y-z| \leq |\Phi(x)-\Phi(z)| \leq C_1|y-z|, \quad \text{for every } y,z \in B_{\rho_2}, \quad (3.87d)$$

$$C_2^{-1} \leq |\text{det} D\Phi(y)| \leq C_2, \quad \text{for every } y \in B_{\rho_2}, \quad (3.87e)$$

$$|\text{det} D\Phi(y) - \text{det} D\Phi(z)| \leq C_3|y-z|, \quad \text{for every } y,z \in B_{\rho_2}, \quad (3.87f)$$

where $C_1, C_2, C_3 \geq 1$ depend on λ, Λ, E only.

Denoting

$$\overline{A}(y) = |\text{det} D\Phi(y)|(D\Phi^{-1})(\Phi(y))A(\Phi(y))(D\Phi^{-1})^*(\Phi(y)),$$

$$v(y,t) = u(\Phi(y),t), \quad (3.88)$$

we have

$$\overline{A}(0) = I \quad (3.89a)$$

$$\overline{a}^{nk}(y',0) = \overline{a}^{kn}(y',0) = 0, \quad k = 1,\ldots,n-1. \quad (3.89b)$$

Moreover, we have that the ellipticity and Lipschitz constants of $\overline{A}$ depend on λ, Λ, E only. For every $y \in B_{\rho_2}(0)$, let us denote by $\tilde{A}(y) = \{\tilde{a}_{ij}(y)\}_{i,j=1}^n$ the matrix with entries given by

$$\tilde{a}^{ij}(y', |y_n|) = \overline{a}^{ij}(y', |y_n|), \quad \text{if either } i, j \in \{1, \dots, n-1\}, \quad \text{or } i = j = n,$$

$$\tilde{a}^{nj}(y', y_n) = \tilde{a}^{jn}(y', y_n) = \text{sgn}(y_n)\overline{a}^{nj}(y', |y_n|), \quad \text{if } 1 \le j \le n-1.$$

We have that $\tilde{A}$ satisfies the same ellipticity and Lipschitz continuity conditions as $\overline{A}$.

Now, if u satisfies the boundary condition (2.13) then we define

$$U(y,t) = \text{sgn}(y_n)v(y', |y_n|, t), \quad \text{for } (y,t) \in B_{\rho_2} \times (-\lambda\rho_2, \lambda\rho_2),$$

$$\widetilde{q}(y) = |\det D\Phi(y', |y_n|)|, \quad \text{for } y \in B_{\rho_2},$$

we have that $U \in \mathcal{W}\left((-\lambda\rho_2, \lambda\rho_2); B_{\rho_2}\right)$ is a solution to

$$\widetilde{q}(y)\partial_t^2 U - \text{div}\left(\widetilde{A}(y)\nabla U\right) = 0, \quad \text{in } B_{\rho_2} \times (-\lambda\rho_2, \lambda\rho_2). \tag{3.90}$$

Moreover, by (3.87d) we have that

$$K_{r/C_1} \subset \Phi\left(B_r^+\right) \subset K_{C_1 r}, \quad \text{for every } r \le \rho_2.$$

Now we can apply Theorem 2.1 to the function U and then by simple changes of variables in the integrals we obtain (2.17). In the general case $A(0) \ne I$ we can consider a linear transformation $G : \mathbb{R}^n \to \mathbb{R}^n$ such that setting $A'(Gx) = \frac{GA(x)G^*}{\det G}$ we have $A'(0) = I$. Therefore, noticing that

$$B_{\sqrt{\lambda}r} \subset G\left(B_r\right) \subset B_{\sqrt{\lambda^{-1}}r}, \quad \text{for every } r > 0,$$

it is a simple matter to get (2.17) in the general case.

If u satisfies the boundary condition (2.14) then we define

$$V(y,t) = v(y', |y_n|, t), \quad \text{for } (y,t) \in B_{\rho_2} \times (-\lambda\rho_2, \lambda\rho_2),$$

and we get that V is a solution to (2.12). Therefore, arguing as before we obtain again (2.17). □

4 Concluding remark: a first order perturbation

In this subsection we outline the proof of an extension of Theorems 2.1, 2.3 for solution to the equation

$$q(x)\partial_t^2 u - Lu = 0, \quad \text{in } B_{\rho_0} \times (-\lambda\rho_0, \lambda\rho_0). \tag{4.1}$$

where

$$Lu = \text{div}\,(A(x)\nabla_x u) + b(x) \cdot \nabla_x u + c(x)u, \tag{4.2}$$

and A, q satisfy (2.1), (2.2), $b = (b^1, \dots, b^n)$ $b^j \in C^{0,1}(\mathbb{R}^n)$, $c \in L^\infty(\mathbb{R}^n)$, $b(x)$ and $c(x)$ real valued. Moreover we assume

$$|b(x)| \le \lambda^{-1}\rho_0^{-1}, \quad \text{for every } x \in \mathbb{R}^n, \tag{4.3a}$$

$$|b(x) - b(y)| \le \frac{\Lambda}{\rho_0^2}\,|x - y|\,, \quad \text{for every } x, y \in \mathbb{R}^n. \tag{4.3b}$$

and

$$|c(x)| \le \lambda^{-1}\rho_0^{-2}, \quad \text{for every } x \in \mathbb{R}^n. \tag{4.4}$$

In what follows we assume $\rho_0 = 1$.
First of all we consider the case in which

$$b \equiv 0 \tag{4.5}$$

and we set

$$L_0 u = \text{div}\,(A(x)\nabla_x u) + c(x)u, \tag{4.6}$$

Let us denote by λ_j, with $\lambda_1 \le \cdots \le \lambda_m \le 0 < \lambda_{m+1} \le \cdots \le \lambda_j \le \cdots$ the eigenvalues associated to the problem

$$\begin{cases} L_0 v + \omega q(x) v = 0, & \text{in } B_2, \\ v \in H^1(B_2)\,, \end{cases} \tag{4.7}$$

and by $e_j(\cdot)$ the corresponding eigenfunctions normalized by

$$\int_{B_2} e_j^2(x) q(x) dx = 1. \tag{4.8}$$

In this case the main difference with respect to the case considered above is the presence of non positive eigenvalues $\lambda_1 \le \cdots \le \lambda_m$. In what follows we indicate the simple changes in the proof of Theorem 2.1 in order to get the same estimate (2.6) (with maybe different constants s_0 and C). Let ε and H be the same of (2.4) and (2.5).

Likewise the case $c \equiv 0$, the proof can be reduced to the even part u_+ with respect to t of solution u of Eq. (4.1). Moreover denoting again by

$$\widetilde{u}(x,t) := \sum_{j=1}^{\infty} \alpha_j e_j(x) \cos\sqrt{\lambda_j}\, t, \tag{4.9}$$

it is easy to check that instead of Proposition 3.1 we have

Proposition 4.1 *We have*

$$\sum_{j=1}^{\infty}\left(1+|\lambda_j|+\lambda_j^2\right)\alpha_j^2 \le CH^2, \tag{4.10}$$

where C depends on λ, Λ only. Moreover, $\widetilde{u}\in\mathcal{W}(\mathbb{R};B_2)\cap C^0\left(\mathbb{R};H^2(B_2)\cap H_0^1(B_2)\right)$ is an even function with respect to variable t and it satisfies

$$\begin{cases} q(x)\partial_t^2\widetilde{u}-L_0\widetilde{u}=0, & in\ B_2\times\mathbb{R},\\ \widetilde{u}(\cdot,0)=\widetilde{u}_0, & in\ B_2,\\ \partial_t\widetilde{u}(\cdot,0)=0, & in\ B_2. \end{cases} \tag{4.11}$$

Similarly to (3.9), the uniqueness to the Cauchy problem for the equation $q(x)\partial_t^2 u - L_0 u = 0$ implies

$$\widetilde{u}(x,t)=u_+(x,t), \quad \text{for } |x|+\lambda^{-1}|t|<1.$$

Likewise the Sect. 3 we set

$$\widetilde{u}_k := \widetilde{u}_{\overline{\mu},k},$$

where $\overline{\mu} := k^{-\frac{1}{6}}$, $k \ge 1$ and $\widetilde{u}_{\mu,k}$ is defined by (3.25). In the present case we set, instead of (3.38),

$$v_k(x,y) := v_k^{(1)}(x,y)+v_k^{(2)}(x,y), \tag{4.12}$$

where

$$v_k^{(1)}(x,y)=\sum_{j=1}^{m}\alpha_j\widehat{\varphi}_{\overline{\mu},k}\left(i\sqrt{|\lambda_j|}\right)\cos\left(\sqrt{|\lambda_j|y}\right)e_j(x), \quad \text{for } (x,y)\in B_2\times\mathbb{R} \tag{4.13a}$$

$$v_k^{(2)}(x,y)=\sum_{j=m+1}^{\infty}\alpha_j\widehat{\varphi}_{\overline{\mu},k}\left(\sqrt{\lambda_j}\right)g_k\left(y\sqrt{\lambda_j}\right)e_j(x), \quad \text{for } (x,y)\in B_2\times\mathbb{R}. \tag{4.13b}$$

and $g_k(z)$ is the same function introduced in Sect. 3, in particular it satisfies (3.37).

Instead of Proposition 3.4 we have

Proposition 4.2 *Let v_k be defined by (4.12). We have that $v_k(\cdot,y)$ belongs to $H^1(B_2)\cap H_0^1(B_2)$ for every $y\in\mathbb{R}$, $v_k(x,y)$ is an even function with respect to y and it satisfies*

$$\begin{cases} q(x)\partial_y^2 v_k + div\,(A(x)\nabla_x v_k) = f_k(x,y), & in\ B_2\times\mathbb{R}, \\ v_k(\cdot,0) = \widetilde{u}_k, & in\ \ B_2. \end{cases} \tag{4.14}$$

and

$$\|v_k(\cdot,0)\|_{L^2(B_{r_0})} \leq \varepsilon. \tag{4.15}$$

where

$$f_k(x,y) = \sum_{j=m+1}^{\infty} \lambda_j \alpha_j \widehat{\varphi_{\overline{\mu},k}}\left(\sqrt{\lambda_j}\right)\left(g_k''\left(y\sqrt{\lambda_j}\right) - g_k\left(y\sqrt{\lambda_j}\right)\right) e_j(x). \tag{4.16}$$

Moreover we have

$$\sum_{j=0}^{2} \|\partial_y^j v_k(\cdot,y)\|_{H^{2-j}(B_2)} \leq Ce^{\lambda\sqrt{|\lambda_1|}} He^{2k}, \quad for\ every\ \ y\in\mathbb{R}, \tag{4.17}$$

$$\|f_k(\cdot,y)\|_{L^2(B_2)} \leq CHe^{2k}\min\{1,(4\pi\lambda^{-1}|y|)^{2k}\}, \quad for\ every\ \ y\in\mathbb{R}, \tag{4.18}$$

where C depends on λ and Λ only.

Instead of Proposition 3.6 we have

Proposition 4.3 *Let v_k be defined in (4.12). Then there exists a constant c, $0<c<1$, depending on λ only such that if $r_0 \leq c$, we have*

$$\|v_k\|_{L^2(\widetilde{B}_{r_0/4})} \leq C\sqrt{r_0}e^{\lambda\sqrt{|\lambda_1|}}(\varepsilon + H(C_0r_0)^{2k})^{\beta}(He^{2k} + H(C_0r_0)^{2k})^{1-\beta}. \tag{4.19}$$

where $\beta\in(0,1)$, C depend on λ and Λ only and $C_0 = 4\pi e\lambda^{-1}$.

With propositions 4.1, 4.2, 4.3 at hand and by using Carleman estimate (3.67), the proofs of estimates (2.6) and (2.17) are straightforward, whenever (4.5) is satisfied.

In the more general case we use a well known trick, see for instance [29], to transform the Eq. (4.1) in a self-adjoint equation. Let z be a new variable and denote by $A_0(x,z) = \{a_0^{ij}(x,z)\}_{i,j=1}^{(n+1)}$ the real-valued symmetric $(n+1)\times(n+1)$ matrix whose entries are defined as follows. Let $\eta\in C^1(\mathbb{R})$ be a function such that $\eta(z)=z$, for $z\in(-1,1)$, and $|\eta(z)|+|\eta'(z)|\leq 2\lambda^{-1}$

$$a_0^{ij}(x,z) = a_0^{ij}(x), \quad \text{if}\ \ i,j\in\{1,\dots,n\},$$

$$a_0^{(n+1)j}(x,z) = a_0^{j(n+1)}(x,z) = \eta(z)b^j(x), \quad \text{if}\ \ 1\leq j\leq n,$$

$$a_0^{(n+1)(n+1)}(x,z) = K_0$$

where $K_0 = 8\lambda^{-3} + 1$. We have that A_0 satisfies

$$\lambda_0|\zeta|^2 \leq A_0(x,z)\zeta\cdot\zeta \leq \lambda_0^{-1}|\zeta|^2, \quad \text{for every } \; \zeta \in \mathbb{R}^{n+1}$$

and

$$|A_0(x,z) - A_0(y,w)| \leq \Lambda_0\left(|x-y| + |z-w|\right), \quad \text{for every } \; (x,z), (y,w) \in \mathbb{R}^{n+1}$$

where λ_0 depends on λ only and Λ_0 depends on λ, Λ only. Denote

$$\mathcal{L}U := \operatorname{div}_{x,z}\left(A_0(x,z)\nabla_{x,z}U\right) + c(x)U$$

It is easy to check that if $u(x,t)$ is a solution of (4.1) ($\rho_0 = 1$) then $U(x,z,t) := u(x,t)$ is solution to

$$q(x)\partial_t^2 U - \mathcal{L}U = 0, \quad \text{in } \; \widetilde{B}_1 \times (-\lambda, \lambda).$$

Therefore we are reduced to the case considered previously in this subsection and again the proofs of estimates (2.6) and (2.17) are now straightforward.

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