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Nonlinear commutators for the fractional p-Laplacian and applications

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Abstract We prove a nonlinear commutator estimate concerning the transfer of derivatives onto testfunctions for the fractional p-Laplacian. This implies that solutions to certain degenerate nonlocal equations are higher differentiable. Also, weakly fractional p-harmonic functions which a priori are less regular than variational solutions are in fact classical. As an application we show that sequences of uniformly bounded $\frac{n}{s}$-harmonic maps converge strongly outside at most finitely many points.

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1 Introduction

An important tool for obtaining higher differentiability of solutions to elliptic partial differential equations is the method of "differentiating the equation": As an example, take $u \in W^{1,2}$ a distributional solution to

$$\Delta u = g \in L^2_{loc}(\Omega). \tag{1.1}$$

It is easy to obtain such a solution $u \in W^{1,2}$, e.g., by the direct method of the calculus of variations. Actually, any such solution belongs to $W^{2,2}_{loc}$. To see this, we differentiate the equation:

$$\Delta \partial_i u = \partial_i g \in \left(W^{1,2}_{loc}(\Omega)\right)^*. \tag{1.2}$$

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Now $\partial_i u$ solves an elliptic equation with right-hand side in $(W^{1,2})^*$. Consequently, $\partial_i u$ belongs to $W^{1,2}_{loc}$ and it is shown that $u \in W^{2,2}_{loc}$. Let us have a closer look at this "differentiating the equation"-argument. The distributional Laplacian $(-\Delta)u$ is defined on testfunctions $\varphi \in C_c^\infty(\Omega)$,

$$(-\Delta)u[\varphi] := \int \nabla u \, \nabla \varphi \overset{(1.1)}{=} -\int g\varphi \quad \forall \varphi \in C_c^\infty.$$

Since Δ is a linear operator with constant coefficients,

$$(-\Delta)(\partial_i u)[\varphi] - (-\Delta)u[-\partial_i \varphi] = 0. \tag{1.3}$$

"Differentiating the equation" (1.2) in distributional sense becomes

$$(-\Delta)(\partial_i u)[\varphi] = -\int \nabla u \nabla(\partial_i \varphi) = -\int g \; \partial_i \varphi. \tag{1.4}$$

Higher differentiability $u \in W^{2,2}_{loc}$ then follows by duality: Take the supremum over φ with $\|\nabla \varphi\|_{L^2} \leq 1$ on both sides of (1.4), and obtain an estimate for $\partial_i \nabla u$ in terms of $g \in L^2$.

The above reasoning relies crucially on (1.3). Of course, we can replace $(-\Delta)$ and ∂_i with more general differential operators of arbitrary order: The s-Laplacian is defined as

$$(-\Delta)^s f = \mathcal{F}^{-1}(c\, |\xi|^{2s} \mathcal{F} f),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. As a distribution

$$(-\Delta)^s f[\varphi] = \int_{\mathbb{R}^n} (-\Delta)^s f \varphi.$$

Similarly to (1.3), just via integration by parts,

$$(-\Delta)^{s+\varepsilon} u[\varphi] - c(-\Delta)^s u[(-\Delta)^\varepsilon \varphi] = 0, \tag{1.5}$$

where c is a constant coming from the choice of the Fourier transform coefficients and the definition of the s-Laplacian. With (1.5) in mind one can prove a finer scale of higher differentiability results. For example,

$$u \in W^{1,2}(\Omega) \quad \text{and} \quad \Delta u \in (W^{1-\varepsilon,2}(\Omega))^* \quad \Rightarrow \quad u \in W^{1+\varepsilon,2}_{loc}(\Omega).$$

However, a statement of the form (1.5) is false for some nonlinear operators, in particular it fails for the p-Laplace

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

And indeed, even p-harmonic functions, i.e. solutions to $\Delta_p u = 0$, may not be smooth.

In this paper, we investigate to what extent the "differentiating the equation"-argument can be saved in the case of a nonlocal, nonlinear differential operator which is related to the p-Laplacian: The *fractional* p-Laplacian.

The fractional p-Laplacian of order $s \in (0, 1)$ on a domain $\Omega \subset \mathbb{R}^n$, $(-\Delta)^s_{p,\Omega} u$ is a distribution acting on testfunctions $\varphi \in C_c^\infty(\Omega)$ given by

$$(-\Delta)^s_{p,\Omega} u[\varphi] := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\,(\varphi(x) - \varphi(y))}{|x - y|^{n+sp}}\, dx\, dy.$$

It appears as the first variation of the $\dot{W}^{s,p}$-Sobolev norm

$$[u]^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy.$$

In this sense it is related to the classical p-Laplacian which appears as first variation of the $\dot{W}^{1,p}$-Sobolev norm $\|\nabla u\|_p^p$.

If $p = 2$ the fractional p-Laplacian on $\mathbb{R}^n$ becomes the usual fractional Laplace operator $(-\Delta)^s$. For an overview on the fractional Laplacian and fractional Sobolev spaces we refer to, e.g., [6,13].

The fractional p-Laplacian has recently received quite some interest, for example we refer to [2,11,12,15,18–20,23,25]. Higher regularity is one interesting and very challenging question where only very partial results are known, e.g. in [2] they obtain for $s \approx 1$ estimates in $C^{1,\alpha}$. We also refer to [5] where they show higher Sobolev regularity when the right-hand side belongs to a Sobolev space.

Since the fractional p-Laplacian is nonlinear, one cannot expect a direct analogue of (1.5). Our first result is a nonlinear commutator estimate which can play the role of (1.5). It measures how and at what price one can "transfer" derivatives to the testfunction. It implies that while an expression such as in (1.5) may not be zero, it is small—on small differential scales. For simplicity we restrict our attention to the case $p \geq 2$.

Theorem 1.1 *Let $s \in (0, 1)$, $p \in [2, \infty)$, and $\varepsilon \in [0, 1 - s)$. Take $B \subset \mathbb{R}^n$ a ball or all of $\mathbb{R}^n$. Let $u \in W^{s,p}(B)$ and $\varphi \in C_c^\infty(B)$. For a certain constant c depending on s, ε, p denote the nonlinear commutator*

$$R_\varepsilon(u, \varphi) := (-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] - c(-\Delta)^s_{p,B} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi].$$

Then we have the estimate

$$|R_\varepsilon(u, \varphi)| \leq C\, \varepsilon\, [u]^{p-1}_{W^{s+\varepsilon,p}(B)} [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)}.$$

The fact that the ε appears in the estimate of $R_\varepsilon(u, \varphi)$ is the main point in Theorem 1.1. For the proof we Taylor expand $R_\varepsilon(\cdot, \cdot)$ in ε. When computing $\frac{d}{d\delta} R_\delta$ we find a logarithmic potential operator, which we estimate in the following way:

Lemma 1.2 *For $p \in (1,\infty)$ we consider the following semi-norm expression for $\varphi \in C_c^\infty(\mathbb{R}^n)$*

$$A(\varphi) := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} k_\alpha(x,y,z)\,(-\Delta)^{\frac{\beta}{2}}\varphi(z)\,dz\right|^p \frac{dx\,dy}{|x-y|^{n+\gamma p}}\right)^{\frac{1}{p}}.$$

Here, $\alpha, \beta \in (0,n)$, $\gamma \in (0,1)$ so that $s := \gamma + \beta - \alpha \in (0,1)$, and

$$k_\alpha(x,y,z) = \left(|x-z|^{\alpha-n}\log\frac{|x-z|}{|x-y|} - |y-z|^{\alpha-n}\log\frac{|y-z|}{|x-y|}\right).$$

Then

$$A(\varphi) \leq C[\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

Having Theorem 1.1 serve as a replacement for (1.5), for small enough ε we obtain estimates "close to the differential order s" for the fractional p-Laplacian.

Theorem 1.3 *Let $s \in (0,1)$, $p \in [2,\infty)$, $\Omega \subset \mathbb{R}^n$ open. Take $u \in W^{s,p}(\Omega)$ a solution to*

$$(-\Delta)^s_{p,\Omega} u = f.$$

Then there is an $\varepsilon_0 > 0$ only depending on s, p, and Ω, so that for $\varepsilon \in (0,\varepsilon_0)$ the following holds: If $f \in (W^{s-\varepsilon(p-1),p}(\Omega))^*$ then $u \in W^{s+\varepsilon,p}_{loc}(\Omega)$.

More precisely, for any $\Omega_1 \Subset \Omega$ there is a constant $C = C(\Omega_1,\Omega,s,p)$ so that

$$[u]_{W^{s+\varepsilon,p}(\Omega_1)} \leq C\,\|f\|_{(W_0^{s-\varepsilon(p-1),p}(\Omega))^*} + C[u]_{W^{s,p}(\Omega)}.$$

Also, by Sobolev embedding, the higher differentiability $W^{s+\varepsilon,p}_{loc}$ implies higher integrability i.e. $W^{s,p+\frac{pn}{n-\varepsilon p}}_{loc}$-estimates.

A higher differentiability result similar to Theorem 1.3 was proven by Kuusi et al. [18,20]. There it is stated only for the case $p = 2$, but the proof goes through for $p \in (1,\infty)$ with only minor modifications. Their method is a generalization of Gehring's Lemma and dual pairs. Our argument is quite different and allows for a shorter proof. Both techniques are quite robust and can be easily extended to more general nonlinearities:

Theorem 1.4 *Let $s \in (0,1)$, $p \in [2,\infty)$, and a domain $\Omega \subset \mathbb{R}^n$. Let $\phi : \mathbb{R} \to \mathbb{R}$ and $K(x,y)$ be a measurable kernel so that for some $C > 1$,*

$$|\phi(t)| \leq C|t|^{p-1}, \quad \phi(t)t \geq |t|^p \quad \forall t \in \mathbb{R},$$

and

$$C^{-1}|x-y|^{-n-sp} \leq K(x,y) \leq C|x-y|^{-n-sp}.$$

We consider for $u \in W^{s,p}(\Omega)$, the distribution $\mathcal{L}_{\phi,K,\Omega}(u)$

$$\mathcal{L}_{\phi,K,\Omega}(u)[\varphi] := \int_\Omega \int_\Omega K(x,y)\, \phi(u(x)-u(y))\, (\varphi(x)-\varphi(y))\, dx\, dy.$$

Then the conclusions of Theorem 1.3 still hold if the fractional p-Laplace $(-\Delta)^s_{p,\Omega}$ is replaced with $\mathcal{L}_{\phi,K,\Omega}$.

Remark 1.5 (Limiting case as $s \to 1$) The classical p-Laplacian can be seen as a (rescaled) limit of the fractional p-Laplacian $(-\Delta)^s_{p,\Omega}$ as $s \to 1$, see [4]. Nevertheless, it seems unlikely that as $s \to 1$ there is a limit *differentiability* version of Theorem 1.1, and consequently a replacement for Theorems 1.3 and 1.4 if $p > 2$.

There is, however, a nonlinear commutator estimate due to Iwaniec [16] reminiscent of Theorem 1.1. But it concerns *integrability* instead of *differentiability*. For any u with $\operatorname{supp} u \subset \Omega$ and any $\varepsilon \in (-1,1)$ there are maps v, R so that we have the Hodge decomposition

$$|\nabla u|^\varepsilon \nabla u = \nabla v + R.$$

Moreover, $\|\nabla v\|_{\frac{q}{1+\varepsilon},\Omega} \precsim \|\nabla u\|^{1+\varepsilon}_{q,\Omega}$ for all q and, most importantly, by Iwaniec' nonlinear commutator estimate if ε is small then R is small:

$$\|R\|_{\frac{p+\varepsilon}{1+\varepsilon},\Omega} \precsim |\varepsilon| \|\nabla u\|^{1+\varepsilon}_{p+\varepsilon,\Omega}.$$

The additional ε in the last estimate allows for estimates "close to the *integrability* order p". Indeed

$$\|\nabla u\|^{p+\varepsilon}_{p+\varepsilon,\Omega} = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |\nabla u|^{p-2} \nabla u R,$$

and thus,

$$\|\nabla u\|^{p+\varepsilon}_{p+\varepsilon,\Omega} \precsim |\Delta_p u[v]| + \varepsilon \|\nabla u\|^{p-1}_{p+\varepsilon,\Omega}\, \|\nabla u\|^{1+\varepsilon}_{p+\varepsilon,\Omega}.$$

In particular, if ε is small enough and $\Delta_p u$ is in $(W_0^{1,\frac{p+\varepsilon}{1+\varepsilon}}(\Omega))^*$, then $u \in W^{1,p+\varepsilon}(\Omega)$.

The commutator estimate in Theorem 1.1 also allows to estimate very weak solutions—i.e. solutions whose initial regularity assumptions are below the variationally natural regularity:

In the local regime, the distributional p-Laplacian $\Delta_p u[\varphi]$ is well defined for $\varphi \in C_c^\infty(\Omega)$ whenever $u \in W^{1,p-1}_{loc}(\Omega)$. The variationally natural regularity assumption is however $W^{1,p}$, since Δ_p appears as first variation of $\|\nabla u\|^p_{p,\Omega}$. For the p-Laplacian, Iwaniec and Sbordone [17] showed that some very weak p-harmonic functions are in fact classical variational solutions:

Theorem 1.6 (Iwaniec–Sbordone) *For any $p \in (1, \infty)$, $\Omega \subset \mathbb{R}^n$, there are exponents $1 < r_1 < p < r_2 < \infty$ so that every (weakly) p-harmonic function,*

$$\Delta_p u = 0,$$

satisfying $u \in W^{1,r_1}_{loc}(\Omega)$ indeed belongs to $W^{1,r_2}_{loc}(\Omega)$.

Again, while the p-Laplace improves its solution's *integrability*, the fractional p-Laplace improves its solution's *differentiability*. The distributional fractional p-Laplace $(-\Delta)^s_{p,\Omega} u[\varphi]$ is well defined for $\varphi \in C_c^\infty(\Omega)$ whenever $u \in W^{q,p-1}(\Omega)$ for any $q > 0$ with $q \geq (\frac{sp-1}{p-1})_+$. We have

Theorem 1.7 *For any $s \in (0, 1)$, $p \in (2, \infty)$, $\Omega \subset \mathbb{R}^n$, there are exponents $1 < r_1 < p < r_2 < \infty$ and $t_1 < s < t_2$ so that every (weakly) s-p-harmonic map,*

$$(-\Delta)^s_{p,\Omega} u = 0,$$

satisfying $u \in W^{t_1,r_1}(\Omega)$ indeed belongs to $W^{t_2,r_2}_{loc}(\Omega)$.

The arguments for Theorem 1.7 are quite similar to the ones in Theorem 1.3, and we shall skip them.

Let us state an important application of Theorem 1.3: It is concerning fractional harmonic maps into spheres $\mathbb{S}^N \subset \mathbb{R}^{N+1}$: In [23] we proved that for $s \in (0, 1)$ critical points of the energy

$$\mathcal{E}_s(u) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x - y|^{n+s\frac{n}{s}}}\, dx\, dy, \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{S}^N$$

are Hölder continuous. Indeed, together with Theorem 1.3 the estimates in [23] imply a sharper result.

Theorem 1.8 (ε-regularity for fractional harmonic maps) *For any open set $\Omega \subset \mathbb{R}^n$ there is a $\delta > 0$ so that for any $\Lambda > 0$ there exists $\varepsilon > 0$ and the following holds: Let $u \in W^{s,\frac{n}{s}}(\Omega, \mathbb{S}^N)$ with*

$$[u]_{W^{s,\frac{n}{s}}(\Omega)} \leq \Lambda \tag{1.6}$$

be a critical point of $\mathcal{E}_s(u)$, i.e.

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_s\left(\frac{u + t\varphi}{|u + t\varphi|}\right) = 0 \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{1.7}$$

If on a ball $2B \subset \Omega$ we have

$$[u]_{W^{s,\frac{n}{s}}(2B)} \leq \varepsilon, \tag{1.8}$$

then on the ball B (the ball concentric to $2B$ with half the radius),

$$[u]_{W^{s+\delta,\frac{n}{s}}(B)} \leq C_{\Lambda,B}.$$

This kind of ε-regularity estimate is crucial for compactness and bubble analysis for fractional harmonic maps. Da Lio obtained quantization results [8] in the $p = 2$ regime for $n = 1$ and $s = \frac{1}{2}$. With the help of Theorem 1.8 one can extend her *compactness* estimates to all $s \in (0, 1)$, $n \in \mathbb{N}$. More precisely, we have the following result extending the first part of [8, Theorem 1.1].

Theorem 1.9 *Let $u_k \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^{N-1})$ be a sequence of $(s, \frac{n}{s})$-harmonic maps in the sense of (1.7) such that*

$$[u_k]_{W^{s,\frac{n}{s}}(\mathbb{R}^n,\mathbb{S}^{N-1})} \leq C.$$

Then there is $u_\infty \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^{N-1})$ and a possibly empty set $\{\alpha_1, \ldots, \alpha_l\}$ such that up to a subsequence we have strong convergence away from $\{\alpha_1, \ldots, \alpha_l\}$, that is

$$u_k \xrightarrow{k\to\infty} u_\infty \quad \text{in } W^{s,\frac{n}{s}}_{loc}(\mathbb{R}^n \backslash \{\alpha_1, \ldots, \alpha_l\}).$$

A more precise analysis of compactness and the formation of bubbles will be part of a future work.

2 Outline and notation

In Sect. 3 we will prove the commutator estimate, Theorem 1.1. Roughly speaking, we compute the kernel $\kappa_\varepsilon(x, y, z)$ of the commutator and show that its derivative *in* ε (which gives a logarithmic potential) induces a bounded operator. The latter estimate is contained in Lemma 1.2 which we shall prove via Littlewood–Paley theory in Sect. 4.

We try to keep the notation as simple as possible. For a ball B, λB denotes the concentric ball with λ-times the radius. With

$$(u)_B := |B|^{-1} \int_B u$$

we denote the mean value.

The dual norm of the p-Laplacian is denoted as

$$\|(-\Delta)^s_{p,\Omega} u\|_{(W^{t,p}_0(\Omega))^*} \equiv \sup_\varphi |(-\Delta)^s_{p,\Omega} u[\varphi]|$$

where the supremum is taken over $\varphi \in C^\infty_c(\Omega)$ with $[\varphi]_{W^{t,p}(\mathbb{R}^n)} \leq 1$.

We already defined the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$. Its inverse I^s is the Riesz potential, which for some constant $c \in \mathbb{R}$ can be written as

$$I^s g(x) = c \int_{\mathbb{R}^n} |x - z|^{s-n} g(z)\, dz. \tag{2.1}$$

In the estimates, the constants can change from line to line. Whenever we deem the constant unimportant to the argument, we will drop it, writing $A \precsim B$ if $A \leq C \cdot B$

for some constant $C > 0$. Similarly we will use $A \approx B$ whenever A and B are comparable.

3 The commutator estimate: proof of Theorem 1.1

Proof Recall that for $t \in (0, n)$ there is a constant $c \in \mathbb{R}$ so that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$c \int_{\mathbb{R}^n} |x - z|^{t-n} (-\Delta)^{\frac{t}{2}} \varphi(z)\, dz = I^t (-\Delta)^{\frac{t}{2}} \varphi(x) = \varphi(x). \tag{3.1}$$

We write

$$\begin{aligned}
(-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] &= \int_B \int_B \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left(\frac{\varphi(x)-\varphi(y)}{|x-y|^{\varepsilon p}}\right)}{|x - y|^{n+sp}}\, dx\, dy \\
&\overset{(3.1)}{=} \int_B \int_B \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left(\frac{|x-z|^{t+\varepsilon p-n} - |y-z|^{t+\varepsilon p-n}}{|x-y|^{\varepsilon p}}\right)}{|x - y|^{n+sp}} \\
&\quad \times (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz\, dx\, dy \\
&= \int_B \int_B \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) \left(|x - z|^{t-n} - |x - y|^{t-n}\right)}{|x-y|^{n+sp}} \\
&\quad \times (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz\, dx\, dy \\
&\quad + \int_B \int_B \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \kappa_\varepsilon(x, y, z)}{|x - y|^{n+sp}} \\
&\quad \times (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz\, dx\, dy
\end{aligned}$$

with

$$\kappa_\varepsilon(x, y, z) := \left(\frac{|x - z|^{t+\varepsilon p-n} - |y - z|^{t+\varepsilon p-n}}{|x - y|^{\varepsilon p}}\right) - (|x - z|^{t-n} - |x - y|^{t-n}).$$

Using again (3.1), this reads as

$$\begin{aligned}
R(u, \varphi) &:= (-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] - c(-\Delta)^s_{p,B} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi] \\
&= \int_B \int_B \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) \kappa_\varepsilon(x, y, z)}{|x-y|^{n+sp}} (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) dz dx dy.
\end{aligned}$$

Since $\kappa_0(x, y, z) = 0$ for almost all $x, y, z \in \mathbb{R}^n$,

$$\kappa_\varepsilon(x, y, z) = \int_0^\varepsilon \frac{d}{d\delta} \kappa_\delta(x, y, z)\, d\delta.$$

We denote

$$
\begin{aligned}
k_\delta(x,y,z) &:= |x-y|^{\delta p}\frac{d}{d\delta}\kappa_\delta(x,y,z)\\
&= \left(|x-z|^{t+\delta p-n}\log\frac{|x-z|}{|x-y|} - |y-z|^{t+\delta p-n}\log\frac{|y-z|}{|x-y|}\right).
\end{aligned}
$$

Thus, $R(u,\varphi)$ is equal to

$$
\int\limits_0^\varepsilon\int\limits_B\int\limits_B \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(s+\varepsilon)(p-1)}}\left(\int\limits_{\mathbb{R}^n}\frac{k_\delta(x,y,z)\,(-\Delta)^{\frac{t+\varepsilon p}{2}}\varphi(z)dz}{|x-y|^{s+\varepsilon-(\varepsilon-\delta)p}}\right)\frac{dx\,dy\,d\delta}{|x-y|^n}.
$$

With Hölder inequality we get the upper bound for $|R(u,\varphi)|$

$$
\varepsilon[u]^{p-1}_{W^{s+\varepsilon,p}(B)}\sup_{\delta\in(0,\varepsilon)}\left(\int\limits_B\int\limits_B\left|\int\limits_{\mathbb{R}^n}\frac{k_\delta(x,y,z)\,(-\Delta)^{\frac{t+\varepsilon p}{2}}\varphi(z)dz}{|x-y|^{s+\varepsilon-(\varepsilon-\delta)p}}\right|^p\frac{dx\,dy}{|x-y|^n}\right)^{\frac{1}{p}}.
$$

This falls into the realm of Lemma 1.2, for

$$
\alpha := t+\delta p,\quad \beta := t+\varepsilon p,\quad \gamma := s+\varepsilon-(\varepsilon-\delta)p,\quad \gamma+\beta-\alpha = s+\varepsilon.
$$

This concludes the proof. □

4 Logarithmic potential estimate: proof of Lemma 1.2

For the proof of Lemma 1.2 we will use the Littlewood–Paley decomposition: We refer to the Triebel monographs, e.g. [24], and [14] for a complete picture of this theory. We will only need few properties:

For a tempered distribution f we define f_j to be the Littlewood–Paley projections $f_j := P_j f$, where

$$
P_j f(x) := \int\limits_{\mathbb{R}^n} 2^{jn}p(2^j(x-z))f(z)\,dz.
$$

Here, p is a Schwartz function, and it can be chosen in a way such that

$$
\sum_{j\in\mathbb{Z}} f_j = f. \tag{4.1}
$$

For any $j \in \mathbb{Z}$ we have the estimate for Riesz potentials and derivatives (cf. (2.1))

$$\|I^s|(-\Delta)^{\frac{t}{2}} f_j|\|_p \precsim \sum_{i=j-1}^{j+1} 2^{j(t-s)} \|f_i\|_p. \tag{4.2}$$

The homogeneous semi-norm for the Triebel space $\dot{F}^s_{p,p} = \dot{B}^s_{p,p}$ is

$$\|f\|_{\dot{F}^s_{p,p}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsp} \|f_j\|_p^p \right)^{\frac{1}{p}}. \tag{4.3}$$

Crucially to us, the Triebel spaces are equivalent to Sobolev spaces: For $s \in (0, 1)$ we have the identification

$$\|f\|_{\dot{F}^s_{p,p}} \approx [f]_{W^{s,p}(\mathbb{R}^n)}. \tag{4.4}$$

Proof of Lemma 1.2 We denote

$$T\varphi(x, y) := \int_{\mathbb{R}^n} k(x, y, z)\, (-\Delta)^{\frac{\beta}{2}} \varphi(z)\, dz.$$

In order to obtain the claimed estimate, we will use two decompositions simultaneously. Firstly, we decompose into slices where $|x - y| \approx 2^{-k}$. For this denote

$$\chi_{|y| \approx 2^{-k}} := \chi_{B_{2^{-k}}(0) \backslash B_{2^{-k-1}}(0)}(y).$$

Secondly, we use the Littlewood–Paley decomposition (4.1). Then

$$A(\varphi)^p \precsim \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} I_{j,k},$$

where

$$I_{j,k} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y| \approx 2^{-k}}\, |T\varphi(x, y)|^{p-1}\ \ |T\varphi_j(x, y)|\, \frac{dx\, dy}{|x - y|^{n+\gamma p}}.$$

Set

$$a_k := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y| \approx 2^{-k}}\, |T\varphi(x, y)|^p\, \frac{dx\, dy}{|x - y|^{n+\gamma p}} \right)^{\frac{1}{p}}$$

and

$$b_j := 2^{j(\gamma+\beta-\alpha)} \|\varphi_j\|_p.$$

Note that with (4.3) and (4.4)

$$\left(\sum_{k\in\mathbb{Z}} a_k^p\right)^{\frac{1}{p}} \approx A(\varphi) \quad \text{and} \quad \left(\sum_{j\in\mathbb{Z}} b_j^p\right)^{\frac{1}{p}} \approx \|\varphi\|_{\dot{F}^s_{p,p}} \approx [\varphi]_{W^{s,p}(\mathbb{R}^n)}. \tag{4.5}$$

With Hölder inequality,

$$\begin{aligned} I_{j,k} &\precsim a_k^{p-1} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{|x-y|\approx 2^{-k}} \left|T\varphi_j(x,y)\right|^p \frac{dx\,dy}{|x-y|^{n+\gamma p}}\right)^{\frac{1}{p}} \\ &=: a_k^{p-1}\, \tilde{I}_{j,k}. \end{aligned}$$

We have to possibilities of estimating $\tilde{I}_{j,k}$, and we are going to interpolate between them:

Firstly, for any small $\sigma \in (0,\alpha)$ we can employ the estimate $|\log \frac{|x-z|}{|x-y|}| \precsim \frac{|x-y|^\sigma}{|x-z|^\sigma} + \frac{|x-z|^\sigma}{|x-y|^\sigma}$. If we recall the Riesz potentials (2.1), we see that

$$\begin{aligned} &\int_{\mathbb{R}^n} |x-z|^{\alpha-n} \log \frac{|x-z|}{|x-y|} |(-\Delta)^{\frac{\beta}{2}}\varphi_j(z)|\,dz \\ &\quad \precsim |x-y|^{-\sigma} I^{\alpha+\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|(x) + |x-y|^{\sigma} I^{\alpha-\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|(x). \end{aligned}$$

Having in mind (4.2) we obtain the estimate

$$\begin{aligned} \tilde{I}_{j,k} &\precsim 2^{k(\frac{n+\gamma p}{p})}\, 2^{k\sigma} 2^{-k\frac{n}{p}} \|I^{\alpha+\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|\|_p + 2^{k(\frac{n+\gamma p}{p})}\, 2^{-k\sigma} 2^{-k\frac{n}{p}} \\ &\quad \times \|I^{\alpha-\sigma}|(-\Delta)^{\frac{\beta}{2}}\varphi_j|\|_p \\ &\precsim 2^{(k-j)(\gamma+\sigma)}(b_{j-1}+b_j+b_{j+1}) + 2^{(k-j)(\gamma-\sigma)}(b_{j-1}+b_j+b_{j+1}). \end{aligned}$$

This is our first estimate:

$$\tilde{I}_{j,k} \precsim 2^{(k-j)(\gamma-\sigma)}\, (2^{2\sigma(k-j)}+1)\, (b_{j-1}+b_j+b_{j+1}). \tag{4.6}$$

Secondly, by a substitution we can write

$$T\varphi_j(x,y) = \int_{\mathbb{R}^n} |z|^{\alpha-n} \log \frac{|z|}{|x-y|} \left((-\Delta)^{\frac{\beta}{2}}\varphi_j(z+x) - (-\Delta)^{\frac{\beta}{2}}\varphi_j(z+y)\right) dz.$$

We use now $|f(x) - f(y)| \precsim |x-y|(\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y))$, where $\mathcal{M}$ is the Hardy–Littlewood maximal function. Then, again for any $\sigma > 0$,

$$\begin{aligned}
&|T\varphi_j(x,y)| \\
&\precsim |x-y| \int_{\mathbb{R}^n} |z|^{\alpha-n} \left|\log \frac{|z|}{|x-y|}\right| \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(z+x)\, dz \\
&\quad + |x-y| \int_{\mathbb{R}^n} |z|^{\alpha-n} \left|\log \frac{|z|}{|x-y|}\right| |\mathcal{M}(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(z+x)\, dz \\
&\precsim |x-y|^{1-\sigma} I^{\alpha+\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(x) \\
&\quad + |x-y|^{1-\sigma} I^{\alpha+\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(y) \\
&\quad + |x-y|^{1+\sigma} I^{\alpha-\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(x) \\
&\quad + |x-y|^{1+\sigma} I^{\alpha-\sigma} \mathcal{M}|(-\Delta)^{\frac{\beta}{2}} \nabla\varphi_j|(y).
\end{aligned}$$

Consequently, our second estimate is

$$\begin{aligned}
\tilde{I}_{j,k} &\precsim 2^{k(\gamma-1+\sigma)} \|I^{\alpha+\sigma}\mathcal{M}|(-\Delta)^{\frac{\beta}{2}}\nabla\varphi_j\|_p + 2^{k(\gamma-1-\sigma)} \|I^{\alpha-\sigma}\mathcal{M}|(-\Delta)^{\frac{\beta}{2}}\nabla\varphi_j\|_p \\
&\precsim 2^{k(\gamma-1+\sigma)}\, 2^{j(-\alpha-\sigma+\beta+1)} \|\varphi_j\|_p + 2^{k(\gamma-1-\sigma)}\, 2^{j(-\alpha+\sigma+\beta+1)} \|\varphi_j\|_p.
\end{aligned}$$

Together with (4.6) we thus have

$$\begin{aligned}
\tilde{I}_{k,j} &\precsim \min\{2^{(k-j)(\gamma-\sigma)}\, (2^{2\sigma(k-j)}+1), 2^{(j-k)(1-\gamma-\sigma)}\, (1+2^{(j-k)(2\sigma)})\} \\
&\quad \times (b_{j-1}+b_j+b_{j+1}).
\end{aligned}$$

In particular, since $\gamma \in (0,1)$ pick any $0 < \sigma < \min\{\gamma, 1-\gamma\}$—which, as we shall see in a moment, makes the following sums convergent:

$$\begin{aligned}
A(\varphi)^p &\precsim \sum_{j\in\mathbb{Z}} \sum_{k=j+1}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\, (b_{j-1}+b_j+b_{j+1})\, a_j^{p-1} \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)}\, (b_{j-1}+b_j+b_{j+1})\, a_j^{p-1} \\
&\quad + \sum_{j\in\mathbb{Z}} (b_{j-1}+b_j+b_{j+1})\, a_j^{p-1} \\
&=: I + II + III.
\end{aligned}$$

With Hölder inequality and (4.5),

$$III \precsim \left(\sum_{j\in\mathbb{Z}} b_j^p\right)^{\frac{1}{p}} \left(\sum_{j\in\mathbb{Z}} a_j^p\right)^{\frac{p-1}{p}} = A(\varphi)^{p-1}\, [\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

As for I, for any $\varepsilon > 0$,

$$
\begin{aligned}
I &= \sum_{j\in\mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\, b_j\, a_k^{p-1} \\
&\precsim \sum_{j\in\mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)}\, (\varepsilon^p b_j^p + \varepsilon^{-p'} a_k^p) \\
&= C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p + \varepsilon^{-p'} \sum_{j\in\mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} a_k^p \\
&= C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p + \varepsilon^{-p'} \sum_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} 2^{(j-k)(1-\gamma-\sigma)} a_k^p \\
&= C_{1-\gamma-\sigma}\varepsilon^p \sum_{j\in\mathbb{Z}} b_j^p + \varepsilon^{-p'} C_{1-\gamma-\sigma} \sum_{k\in\mathbb{Z}} a_k^p \\
&\approx \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)} + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p.
\end{aligned}
$$

The same works for II:

$$
\begin{aligned}
II &= \sum_{j\in\mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)}\, b_j\, a_k^{p-1} \\
&\precsim \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)} + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p.
\end{aligned}
$$

Together,

$$
I + II \precsim \varepsilon^p [\varphi]^p_{W^{s,p}(\mathbb{R}^n)} + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p,
$$

which holds for any $\varepsilon > 0$. Pick

$$
\varepsilon := [\varphi]_{W^{s,p}(\mathbb{R}^n)}^{-\frac{1}{p'}}\, A(\varphi)^{\frac{1}{p'}}.
$$

Then

$$
A(\varphi)^p \leq I + II + III \precsim A(\varphi)^{p-1}\, [\varphi]_{W^{s,p}(\mathbb{R}^n)}.
$$

Lemma 1.2 is proven if we divide both sides by $A(\varphi)^{p-1}$. □

5 Higher differentiability: proof of Theorem 1.3

In view of Lemma 8.1 we can assume w.l.o.g. that Ω is a bounded open set, and that the support of u is strictly contained in some open set $\Omega_1 \Subset \Omega$. Then Theorem 1.3 follows from

Lemma 5.1 *Let $\Omega_1 \Subset \Omega$ two open, bounded sets, $s \in (0,1)$, $p \in [2,\infty)$. Then there exists an $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$[u]^{p-1}_{W^{s+\varepsilon,p}(\Omega)} \precsim [u]^{p-1}_{W^{s,p}(\Omega)} + \|(-\Delta)^s_{p,\Omega} u\|_{(W_0^{s-\varepsilon(p-1),p}(\Omega))^*}.$$

Proof We can find finitely many balls $(B_k)_{k=1}^K \subset \Omega$ so that $\bigcup_{k=1}^K B_k \supset \Omega_1$. We denote with $10B_k$ the concentric balls with ten times the radius, and may assume $\bigcup_{k=1}^N 10B_k \subset \Omega$.

Denote

$$\Gamma_s := [u]^p_{W^{s,p}(\Omega)}, \quad \Gamma_{s+\varepsilon} := [u]^p_{W^{s+\varepsilon,p}(\Omega)}.$$

We then have

$$\Gamma_{s+\varepsilon} \precsim \sum_{k=1}^K [u]^p_{W^{s+\varepsilon,p}(2B_k)} + \sum_{k=1}^K \int_{\Omega\backslash 2B_k} \int_{B_k} \frac{|u(x)-u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}}\, dx\, dy.$$

As for the second term, because of the disjoint support of the integrals we find

$$\int_{\Omega\backslash 2B_k} \int_{B_k} \frac{|u(x)-u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}}\, dx\, dy \precsim (\operatorname{diam} B_k)^{-\varepsilon p}\, \Gamma_s.$$

That is

$$\Gamma_{s+\varepsilon} \precsim \sum_{k=1}^K [u]^p_{W^{s+\varepsilon,p}(2B_k)} + \Gamma_s.$$

With Lemma 8.2 and Poincaré inequality, Proposition 8.3, for any $\delta > 0$,

$$\Gamma_{s+\varepsilon} \precsim \delta^p \Gamma_{s+\varepsilon} + C_\delta \Gamma_s + \sum_{k=1}^K \delta^{-p'} \left(\sup_\varphi (-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi] \right)^{\frac{p}{p-1}}$$

where the supremum is over all $\varphi \in C_c^\infty(4B_k)$ and $[\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1$. Here we also used that $\bigcup_{k=1}^K 8B_k$ covers no more than Ω. Choosing δ sufficiently small, we can estimate $\Gamma_{s+\varepsilon}$ by

$$\Gamma_s + \sum_{k=1}^K \left(\sup \left\{ |(-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi]| : \varphi \in C_c^\infty(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

With Theorem 1.1 this can be estimated by

$$\Gamma_s + \varepsilon^{\frac{p}{p-1}} \Gamma_{s+\varepsilon} + \sum_{k=1}^{K} \left(\sup \left\{ \left| (-\Delta)^s_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi] \right| : \varphi \in C_c^\infty(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

If $\varepsilon \in [0, \varepsilon_0)$ for ε_0 small enough, we can again absorb $\Gamma_{s+\varepsilon}$. The estimate for $\Gamma_{s+\varepsilon}$ becomes

$$\Gamma_s + \sum_{k=1}^{K} \left(\sup \left\{ \left| (-\Delta)^s_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi] \right| : \varphi \in C_c^\infty(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

Next, we need to transform $(-\Delta)^{\frac{\varepsilon p}{2}} \varphi$ into a feasible testfunction, and denoting the usual cutoff function with $\eta_{6B_k} \in C_c^\infty(6B_k)$, $\eta_{6B_k} \equiv 1$ in $5B_k$

$$(-\Delta)^{\frac{\varepsilon p}{2}} \varphi =: \psi + (1 - \eta_{6B_k})(-\Delta)^{\frac{\varepsilon p}{2}} \varphi$$

Then $\psi \in C_c^\infty(6B_k)$

$$[\psi]_{W^{s-\varepsilon(p-1),p}(\Omega)} \precsim C_k [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq C_k.$$

Moreover, the disjoint support of $(1 - \eta_{6B_k})$ and φ implies (see, e.g., [3, Lemma A.1])

$$\left[(1 - \eta_{6B_k})(-\Delta)^{\frac{\varepsilon p}{2}} \varphi \right]_{\mathrm{Lip}} \leq C_k \, [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)}.$$

Consequently,

$$|(-\Delta)^s_{p,8B_k} u[(-\Delta)^{\frac{\varepsilon p}{2}} \varphi - \psi]| \precsim [u]^{p-1}_{W^{s,p}(\Omega)}.$$

Hence, our estimate for $\Gamma_{s+\varepsilon}$ now looks like

$$\Gamma_s + \sum_{k=1}^{K} \left(\sup \left\{ \left| (-\Delta)^s_{p,8B_k} u[\psi] \right| : \psi \in C_c^\infty(6B_k), [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

Finally, we need to transform the support of $(-\Delta)^{\frac{s}{2}}_p$ from $8B_k$ to Ω. Since $\operatorname{supp} \psi \subset 6B_k$, the disjoint support of the integrals gives

$$\begin{aligned}
&|(-\Delta)^s_{p,8B_k} u[\psi] - (-\Delta)^s_{p,\Omega} u[\psi]| \\
&\precsim \int_{\Omega \setminus 8B_k} \int_{7B_k} \frac{|u(x) - u(y)|^{p-1} \, |\psi(x) - \psi(y)|}{|x - y|^{n+sp}} \, dx \, dy \\
&\leq C_k [u]^{p-1}_{W^{s,p}(\Omega)} [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)}.
\end{aligned}$$

This implies the final estimate of $\Gamma_{s+\varepsilon}$ by

$$\Gamma_s + \left(\sup\left\{\left|(-\Delta)^s_{p,\Omega} u[\psi]\right| :\ \psi \in C_c^\infty(\Omega),\, [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)} \leq 1\right\}\right)^{\frac{p}{p-1}}.$$

□

6 Differentiability of *p*-harmonic maps: proof of Theorem 1.8

For $B \subset \mathbb{R}^n$, $t \in (0,1)$, we set

$$T_{t,B} u(z) = \int_B \int_B \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(|x-z|^{t-n} - |y-z|^{t-n})}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy.$$

$T_{t,B} u$ was introduced in [23] because of the following relation

$$c\int_{\mathbb{R}^n} T_{t,B} u(z)\, \varphi(z)\, dz = \int_B \int_B \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(I^t\varphi(x) - I^t\varphi(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy. \tag{6.1}$$

From [23, in particular (3.1), Lemma 3.3, 3.4, 3.5] we have the following

Theorem 6.1 *Let u satisfy (1.6) and (1.7) in an open set Ω. Assume that on the ball $2B$ for a small enough $\varepsilon > 0$ (depending on Λ) (1.8) holds. Then there is $t_0 < s$, $\sigma > 0$, so that for some $\gamma_2 > \gamma_1 \gg 1$ for any ball $B_{\gamma_2 \rho} \subset B$*

$$[u]_{W^{s,\frac{n}{s}}(B_\rho)} \precsim C_\Lambda \rho^\sigma, \tag{6.2}$$

and

$$\|T_{t_0,B_{\gamma_1\rho}} u\|_{\frac{n}{n-t_0},B_\rho} \leq C_\Lambda \rho^\sigma. \tag{6.3}$$

Estimate (6.3) looks almost as if $T_{t_0,B_{\gamma_1\rho}}$ belongs locally to a Morrey space. But the domain dependence on $B_{\gamma_1\rho}$ prevents us from exploiting this immediately. The following proposition removes the domain dependence.

Proposition 6.2 *Under the assumptions of Theorem 6.1 there exists $\gamma > 1$, $\sigma > 0$ so that*

$$\|T_{t_0,B} u\|_{\frac{n}{n-t_0},B_\rho} \leq C_{B,\Lambda} \rho^\sigma$$

for any ball so that $B_{\gamma\rho} \subset B$.

Proof Set $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq 1$ to be chosen later. Take $\gamma := 2\gamma_1$ with γ_1 from (6.3). We will always assume $\rho < 1$.

For some $\varphi \in C_c^\infty(B_{\rho^{\kappa_1}})$, $\|\varphi\|_{\frac{n}{t_0}} \leq 1$ we have

$$\begin{aligned}
&\|T_{t_0,B} u\|_{\frac{n}{n-t_0}, B_{\rho^{\kappa_1}}}\\
&\precsim \int_{\mathbb{R}^n} T_{t_0,B} u\ \varphi\\
&\overset{(6.1)}{\approx} \int_B \int_B \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\ (I^{t_0}\varphi(x) - I^{t_0}\varphi(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy.
\end{aligned}$$

We will now use several cutoffs to slice φ into the right form. This kind of arguments and the consequent (tedious) estimates have been used several times in work related to fractional harmonic maps, cf. e.g. [3,7,9,10,21–23], and we will not repeat them in detail. We will also assume that $\kappa_1 > \kappa_2 > \kappa_3$. If they are equal, to keep the "disjoint support estimates" working one needs to use cutoff functions on twice, four times etc. of the Balls.

For a cutoff function $\eta_{B_{\rho^{\kappa_2}}} \in C_c^\infty(B_{2\rho^{\kappa_2}})$, $\eta_{B_{\rho^{\kappa_2}}} \equiv 1$ on $B_{\rho^{\kappa_2}}$, we have

$$I^{t_0}\varphi := \psi + (1-\eta_{B_{\rho^{\kappa_2}}}) I^{t_0}\varphi.$$

Note that $\psi \in C_c^\infty(B_{2\rho^{\kappa_2}})$ and[1]

$$\|(-\Delta)^{\frac{t_0}{2}}\psi\|_{\frac{n}{t_0}} + [\psi]_{W^{t_0,\frac{n}{t_0}}(\mathbb{R}^n)} \precsim \|\varphi\|_{\frac{n}{t_0}}. \tag{6.4}$$

The disjoint support of $(1-\eta)$ and φ ensures (see [3, Lemma A.1])

$$[I^{t_0}\varphi - \psi]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \precsim \rho^{(\kappa_1-\kappa_2)(n-t_0)}\ \|\varphi\|_{\frac{n}{t_0}}. \tag{6.5}$$

We furthermore decompose

$$(-\Delta)^{\frac{t_0}{2}}\psi =: \phi + (1-\eta_{B_{\rho^{\kappa_3}}})(-\Delta)^{\frac{t_0}{2}}\psi.$$

Then $\phi \in C_c^\infty(B_{2\rho^{\kappa_3}})$ and

$$\|\phi\|_{\frac{n}{t_0}} \precsim \|\varphi\|_{\frac{n}{t_0}}, \tag{6.6}$$

$$\|\nabla(\psi - I^{t_0}\phi)\|_\infty \precsim \rho^{-\kappa_3+(\kappa_2-\kappa_3)n}\ \|\varphi\|_{\frac{n}{t_0}}. \tag{6.7}$$

Again with (6.1), we then have

$$\|T_{t_0,B} u\|_{\frac{n}{n-t_0}, B_\rho} \precsim |I| + |II| + |III| + |IV|$$

[1] This is true if $\frac{n}{t_0} \geq 2$, since then $[f]_{W^{t_0,\frac{n}{t_0}}} \leq \|(-\Delta)^{\frac{t_0}{2}} f\|_{\frac{n}{t_0}}$. If $\frac{n}{t_0} < 2$ one has to adapt the estimate, but the results remains true.

where

$$I := \int T_{t_0, B_{\gamma\rho}} u\, \phi,$$

$$II := \int_{B_{\gamma\rho}} \int_{B_{\gamma\rho}} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\, ((\psi - I^{t_0}\phi)(x) - (\psi - I^{t_0}\phi)(y))}{|x-y|^{n+s\frac{n}{s}}}\, dx\, dy,$$

$$III := \int_{B \backslash B_{\gamma\rho}} \int_{B_{2\rho^{\kappa_2}}} \frac{|u(x) - u(y)|^{\frac{n}{s}-2}(u(x) - u(y))\, (\psi(x) - \psi(y))}{|x - y|^{n+s\frac{n}{s}}}\, dx\, dy,$$

and

$$IV := \int_B \int_B \frac{|u(x) - u(y)|^{\frac{n}{s}-2}(u(x) - u(y))\, ((I^{t_0}\varphi - \psi)(x) - (I^{t_0}\varphi - \psi)(y))}{|x - y|^{n+s\frac{n}{s}}}\, dx\, dy.$$

With (6.6), $\operatorname{supp}\phi \subset B_{2\rho^{\kappa_3}} \subset B_{2\rho}$, and (6.3),

$$|I| \precsim \rho^{\sigma}.$$

With (6.2), (6.7) (for ρ small enough),

$$|II| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B_{\gamma\rho})} [\psi - I^{t_0}\phi]_{W^{s,\frac{n}{s}}(B_{\gamma\rho})} \precsim \rho^{\sigma(\frac{n}{s}-1)}\, \rho^{-(\kappa_3-1)} \rho^{(\kappa_2-\kappa_3)n}.$$

With the disjoint support of the integrals, Hölder inequality ($\frac{n}{t_0} > \frac{n}{s}$), and (6.4),

$$|III| \precsim [u]^{p-1}_{W^{s,\frac{n}{s}}(B)}\, \rho^{t_0-s}\, \rho^{\kappa_2(s-t_0)}\, [\psi]_{W^{t_0,\frac{n}{t_0}}(B)} \precsim \rho^{(\kappa_2-1)(s-t_0)}.$$

Lastly, with (6.5)

$$|IV| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B)} [I^{t_0}\varphi - \psi]_{W^{s,\frac{n}{s}}(B)} \precsim \rho^{(\kappa_1-\kappa_2)(n-t_0)}.$$

If we choose $\kappa_1 = \kappa_2 = \kappa_3 = 1$, we obtain

$$\|T_{t_0,B} u\|_{\frac{n}{n-t_0}, B_\rho} \precsim 1,$$

whenever $B_{2\gamma\rho} \subset B$, In particular

$$\|T_{t_0,B} u\|_{\frac{n}{n-t_0}, \frac{1}{2\gamma}B} \precsim 1. \tag{6.8}$$

On the other hand, we may take

$$\kappa_1 > \kappa_2 > \kappa_3 = 1.$$

Then we have shown that

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_{\rho^{\kappa_1}}} \precsim \rho^{\tilde{\sigma}},$$

which holds whenever $B_{\gamma\rho} \subset B$. Equivalently, for an even smaller $\tilde{\sigma}$,

$$\|T_{t_0,B}u\|_{\frac{n}{n-t_0},B_\rho} \precsim \rho^{\tilde{\sigma}},$$

which holds whenever $B_{\gamma\rho^{\frac{1}{\kappa_1}}} \subset B$. With (6.8) this estimate also holds whenever $B_{2\gamma\rho} \subset B$, with a constant depending on the radius of B. ◻

In [23] it is shown that for $t_1 > t_0$, $T_{t_1,B}u = I^{t_1-t_0}T_{t_0,B}u$. Since according to Proposition 6.2 $T_{t_0,B}u$ belongs to a Morrey space, we can apply Adams estimates on Riesz potential acting on Morrey spaces [1, Theorem 3.1 and Corollary after Proposition 3.4] and obtain an increased integrability estimate for $T_{t_1,B}u$.

Proposition 6.3 *Under the assumptions of Theorem 6.1 there are $\gamma > 1$, $t_0 < t_1 < s$, and $p_1 > \frac{n}{n-t_1}$ so that*

$$\|T_{t_1,B}u\|_{p_1,B_\rho} \leq C_\Lambda \rho^{\sigma}$$

for any ball so that $B_{\gamma\rho} \subset B$.

Now we exploit (6.1): For any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(-\Delta)^s_{\frac{n}{s},B}u[\varphi] = \int_{\mathbb{R}^n} T_{t_1,B}u \; (-\Delta)^{\frac{t_1}{2}}\varphi.$$

Let $\varphi \in C_c^\infty(B_{\frac{1}{4}\rho})$ for $B_{\gamma\rho} \subset B$. With the usual cutoff-function $\eta \in C_c^\infty(B_\rho)$, $\eta \equiv 1$ on $B_{\frac{1}{2}\rho}$

$$|(-\Delta)^s_{\frac{n}{s},B}u[\varphi]| \precsim \|T_{t_1,B}u\|_{p_1,B_\rho}\|(-\Delta)^{\frac{t}{2}}\varphi\|_{p_1',B_\rho} + \|T_{t_1,B}u\|_{\frac{n}{n-t},B_\rho}\|(-\Delta)^{\frac{t_1}{2}}\varphi\|_{\frac{n}{t},\mathbb{R}^n\backslash B_{\frac{1}{2}\rho}}.$$

By the Sobolev inequality for Gagliardo–Norms [23, Theorem 1.6], and the disjoint support [3, Lemma A.1], this implies

$$|(-\Delta)^s_{\frac{n}{s},B}u[\varphi]| \precsim C_\Lambda[\varphi]_{W^{s+t_1-\frac{n}{p_1'},\frac{n}{s}}(\mathbb{R}^n)}.$$

Since $p_1 > \frac{n}{n-t_1}$, we have $s + t_1 - \frac{n}{p_1'} < s$, and the claim of Theorem 1.8 follows from Theorem 1.3 by a covering argument. ◻

7 Compactness for $\frac{n}{s}$-harmonic maps: proof of Theorem 1.9

From the arguments in [8, Proof of Lemma 2.3.] one has the following:

Proposition 7.1 *For $s \in (0,1)$, $p \in (1,\infty)$ let $(u_k)_{k=1}^\infty \in W^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1})$, $\Lambda := \sup_{k\in\mathbb{N}}[u_k]_{W^{s,p}(\mathbb{R}^n)} < \infty$ and $\varepsilon_0 > 0$ given. Then up to a subsequence there is $u_\infty \in \dot{W}^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1})$ and a finite set of points $J = \{a_1, \ldots, a_l\}$ such that*

$$u_k \rightharpoonup u_\infty \quad \text{in } W^{s,p}(\mathbb{R}^n, \mathbb{S}^{N-1}) \text{ as } k \to \infty,$$

and for all $x \notin J$ there is $r = r_x > 0$ so that

$$\limsup_{k\to\infty}[u_k]_{W^{s,p}(B_r(x))} < \varepsilon_0.$$

This, Theorem 1.8 and the compactness of the embedding $W^{s+\delta,\frac{n}{s}}(B_r(x)) \hookrightarrow W^{s,\frac{n}{s}}(B_r(x))$ immediately implies that

$$u_k \xrightarrow{k\to\infty} u_\infty \quad \text{in } W^{s,\frac{n}{s}}_{loc}(\mathbb{R}^n \backslash J).$$

Appendix A: Useful tools

The following Lemma is used to restrict the fractional p-Laplacian to smaller sets.

Lemma 8.1 (Localization lemma) *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \Omega \subset \mathbb{R}^n$ be open sets so that* $\operatorname{dist}(\Omega_1, \Omega_2^c)$, $\operatorname{dist}(\Omega_2, \Omega_3^c)$, $\operatorname{dist}(\Omega_3, \Omega^c) > 0$. *Let $s \in (0,1)$, $p \in [2,\infty)$.*

For any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ so that

(1) *$\tilde{u} - u \equiv const$ in Ω_1*
(2) $\operatorname{supp} \tilde{u} \subset \Omega_2$
(3) $[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(\Omega)}$
(4) *For any $t \in (2s-1, s)$,*

$$\|(-\Delta)^s_{p,\Omega_3}\tilde{u}\|_{(W_0^{t,p}(\Omega_3))^*} \precsim \|(-\Delta)^s_{p,\Omega}u\|_{(W_0^{t,p}(\Omega))^*} + [u]^{p-1}_{W^{s,p}(\Omega)}.$$

The constants are uniform in u and depend only on s, t, p and the sets Ω_1, Ω_2, Ω_3, and Ω.

Proof Let $\Omega_1 \Subset \Omega$, let $\eta \equiv \eta_{\Omega_1} \in C_c^\infty(\Omega_2)$, $\eta_{\Omega_1} \equiv 1$ on Ω_1. We set

$$\tilde{u} := \eta_{\Omega_1}(u - (u)_{\Omega_1}).$$

Clearly $\tilde{u}$ satisfies property (1) and (2). We have property (3), too:

$$[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(\Omega)}.$$

We write

$$\tilde{u}(x) - \tilde{u}(y) = \underbrace{\eta(x)(u(x) - u(y))}_{a(x,y)} + \underbrace{(\eta(x) - \eta(y))(u(y) - (u)_{\Omega_1})}_{b(x,y)}.$$

Setting

$$T(a) := |a|^{p-2}a,$$

observe that

$$|T(a+b) - T(a)| \precsim |b| \left(|a|^{p-2} + |b|^{p-2}\right).$$

Also note that

$$T(a(x,y)) = \eta^{p-1}(x)|u(x) - u(y)|^{p-2}(u(x) - u(y))$$

We thus have for any $\varphi \in C_c^\infty(\Omega_3)$,

$$\begin{aligned}
&(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi] \\
&= \int_\Omega \int_\Omega \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}(\tilde{u}(x) - \tilde{u}(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\,dx\,dy \\
&= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\,\eta^{p-1}(x)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega \int_\Omega \frac{(T(a+b) - T(a))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\,dx\,dy \\
&= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\,(\eta^{p-1}(x)\varphi(x) - \eta^{p-1}(y)\varphi(y))}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad - \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\,(\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega \int_\Omega \frac{(T(a+b) - T(a))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\,dx\,dy \\
&= (-\Delta)^s_{p,\Omega}u[\eta^{p-1}\,\varphi] \\
&\quad - \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\,(\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega \int_\Omega \frac{(T(a+b) - T(a))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+sp}}\,dx\,dy.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| \\
&\precsim \|(-\Delta)^s_{p,\Omega}u\|_{(W^{t,p}_0(\Omega))^*}\ [\eta^{p-1}\varphi]_{W^{t,p}(\Omega)} \\
&\quad + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-1}\ |\eta^{p-1}(x)-\eta^{p-1}(y)|\ |\varphi(y)|}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega\int_\Omega \frac{|\eta(x)-\eta(y)|\ |u(y)-(u)_{\Omega_1}|\ \eta(x)^{p-2}\ |u(x)-u(y)|^{p-2}\ |\varphi(x)-\varphi(y)|}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega\int_\Omega \frac{|\eta(x)-\eta(y)|^{p-1}\ |u(y)-(u)_{\Omega_1}|^{p-1}\ |\varphi(x)-\varphi(y)|}{|x-y|^{n+sp}}\,dx\,dy.
\end{aligned}
$$

That is for any $t < s$

$$
\begin{aligned}
&|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| \\
&\precsim \|(-\Delta)^s_{p,\Omega}u\|_{(W^{t,p}_0(\Omega))^*}\ [\eta^{p-1}\varphi]_{W^{t,p}(\Omega)} \\
&\quad +[u]^{p-1}_{W^{s,p}(\Omega)}\ \left(\int_\Omega\int_\Omega \frac{|\eta^{p-1}(x)-\eta^{p-1}(y)|^p\ |\varphi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{\frac{1}{p}} \\
&\quad +[\varphi]_{W^{t,p}(\Omega)}\ [u]^{p-2}_{W^{s,p}(\Omega)}\left(\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\,dx\,dy\right)^{\frac{1}{p}} \\
&\quad +[\varphi]_{W^{t,p}(\Omega)}\left(\int_\Omega\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\ |u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\,dx\,dy\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Since η is bounded and Lipschitz, $\operatorname{supp}\eta \subset \Omega_2$, and $\varphi \in C^\infty_c(\Omega_3)$ we have that

$$
[\eta^{p-1}\varphi]_{W^{t,p}(\Omega)} \precsim [\varphi]_{W^{t,p}(\mathbb{R}^n)}.
$$

Also, choosing some bounded $\Omega_4 \Subset \Omega$ so that $\Omega_3 \Subset \Omega_4$,

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{|\eta^{p-1}(x)-\eta^{p-1}(y)|^p\ |\varphi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\precsim \int_{\Omega_3}\int_{\Omega_4} |x-y|^{(1-s)p-n}\ \,dx\ |\varphi(y)|^p dy \\
&\quad + \int_{\Omega_3}\int_{\mathbb{R}^n\setminus\Omega_4} |x-y|^{-n-sp}\ \,dx\ |\varphi(y)|^p dy \\
&\precsim \|\varphi\|_p^p \precsim [\varphi]^p_{W^{t,p}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, using Lipschitz continuity of η and that $2s-1<t<s$

$$\begin{aligned}
&\int_{\Omega}\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\,|u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\,dx\,dy\\
&\precsim \int_{\Omega_3} |u(y)-(u)_{\Omega_1}|^p \int_{\Omega_2} |x-y|^{-n+(t+1-2s)p}\,dx\,dy\\
&\quad+\int_{\Omega\setminus\Omega_3} |u(y)-(u)_{\Omega_1}|^p \int_{\Omega_2} \frac{1}{|x-y|^{n+sp}}\,dx\,dy\\
&\precsim \int_{\Omega_1}\int_{\Omega_3} |u(y)-u(z)|^p\,dy\,dz\\
&\quad+\int_{\Omega_1}\int_{\Omega\setminus\Omega_3} |u(y)-u(z)|^p \int_{\Omega_2} \frac{1}{|x-y|^{n+sp}}\,dx\,dy\,dz
\end{aligned}$$

Note that for $x,z\in\Omega_2$ and $y\in\Omega_3^c$ we have that $|x-y|\approx|y-z|$, and since $\Omega_1,\Omega_2,\Omega_3$ are bounded we then have

$$\int_{\Omega}\int_{\Omega_2} \frac{|\eta(x)-\eta(y)|^p\,|u(y)-(u)_{\Omega_1}|^p}{|x-y|^{n+(2s-t)p}}\,dx\,dy \precsim [u]_{W^{s,p}(\Omega)}.$$

Thus we have shown that for any $\varphi\in C_c^\infty(\Omega_3)$,

$$|(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| \precsim \left(\|(-\Delta)^s_{p,\Omega}u\|_{(W_0^{t,p}(\Omega))^*}+[u]^{p-1}_{W^{s,p}(\Omega)}\right)\,[\varphi]_{W^{t,p}(\mathbb{R}^n)}.$$

Since moreover, $\operatorname{supp}\tilde{u}\subset\Omega_2$, for any $\varphi\in C_c^\infty(\Omega_3)$,

$$|(-\Delta)^s_{p,\Omega_3}\tilde{u}[\varphi]| \precsim |(-\Delta)^s_{p,\Omega}\tilde{u}[\varphi]| + [u]^{p-1}_{W^{s,p}(\Omega)}\,[\varphi]_{W^{t,p}(\mathbb{R}^n)},$$

we get the claim. □

The next Lemma estimates the $W^{s,p}$-norm in terms of the fractional p-Laplacian.

Lemma 8.2 *Let $B\subset\mathbb{R}^n$ be a ball and $4B$ the concentric ball with four times the radius. Then for any $\delta>0$, $[u]^p_{W^{s,p}(B)}$ can be estimated by*

$$\begin{aligned}
&\delta^p[u]^p_{W^{s,p}(4B)}\\
&+\frac{C}{\delta^{p'}}\left(\sup_{\varphi}\int_{4B}\int_{4B}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dx\,dy\right)^{\frac{p}{p-1}}\\
&+\frac{C}{\delta^{p'}}\operatorname{diam}(B)^{-sp}\int_{4B}|u(x)-(u)_B|^p\,dx
\end{aligned}$$

where the supremum is over all $\varphi\in C_c^\infty(2B)$ and $[\varphi]_{W^{s,p}(\mathbb{R}^n)}\le 1$.

Proof Let $\eta \in C_c^\infty(2B)$, $\eta \equiv 1$ in B be the usual cutoff function in $2B$.

$$\psi(x) := \eta(x)(u(x) - (u)_B), \quad \text{and} \quad \varphi(x) := \eta^2(x)(u(x) - (u)_B).$$

Then,

$$[\psi]_{W^{s,p}(\mathbb{R}^n)} + [\varphi]_{W^{s,p}(\mathbb{R}^n)} \precsim [u]_{W^{s,p}(2B)}. \tag{8.1}$$

We have

$$[u]^p_{W^{s,p}(B)} \leq \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}(\psi(x)-\psi(y))\,(\psi(x)-\psi(y))}{|x-y|^{n+sp}} dx\,dy.$$

Now we observe

$$\begin{aligned}(\psi(x)-\psi(y))^2 = &(\psi(x)-\psi(y))(\eta(x)-\eta(y))(u(x)-(u)_B)\\ &+\psi(x)(\eta(y)-\eta(x))\,(u(x)-u(y))\\ &+(\varphi(x)-\varphi(y))(u(x)-u(y)).\end{aligned}$$

That is,

$$[u]^p_{W^{s,p}(B)} \precsim I + II + III,$$

with

$$I := \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dx\,dy,$$

$$II := \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}|\eta(x)-\eta(y)|\,|\psi(x)-\psi(y)|}{|x-y|^{n+sp}}\,|u(x)-(u)_B|\,dx\,dy,$$

$$III := \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-1}|\eta(x)-\eta(y)|}{|x-y|^{n+sp}}\,|\psi(x)|dx\,dy.$$

With (8.1),

$$I \leq [u]_{W^{s,p}(4B)} \sup_{[\varphi]_{W^{s,p}(\mathbb{R}^n)}\leq 1} \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dx\,dy.$$

As for II,

$$II \precsim \|\nabla\eta\|_\infty \int_{4B}\int_{4B} \frac{|u(x)-u(y)|^{p-2}|\psi(x)-\psi(y)|\,|u(x)-(u)_B|}{|x-y|^{n+sp-1}}\,dx\,dy.$$

For any $t_2 > 0$ so that $t_2 = 1 - s$, we have with Hölder's inequality

$$II \precsim \|\nabla\eta\|_\infty \int\limits_{4B}\int\limits_{4B} \frac{|u(x)-u(y)|^{p-2}|\psi(x)-\psi(y)|\,|u(x)-(u)_B|}{|x-y|^{n+s(p-2)+s-t_2}}\,dx\,dy$$

$$\precsim \text{diam}\,(B)^{-1}\,[u]^{p-2}_{W^{s,p}(4B)}\,[\psi]_{W^{s,p}(4B)}\left(\int\limits_{4B}\int\limits_{4B}\frac{|u(x)-(u)_B|^p}{|x-y|^{n-t_2p}}\,dx\,dy\right)^{\frac{1}{p}}.$$

Since $t_2 > 0$,

$$\int\limits_{4B}\int\limits_{4B}\frac{|u(x)-(u)_B|^p}{|x-y|^{n-t_2p}}\,dx\,dy \precsim (\text{diam}\,B)^{t_2p}\int\limits_{4B}|u(x)-(u)_B|^p\,dx.$$

So using again (8.1), we arrive at

$$II \precsim \text{diam}\,(B)^{-s}\,[u]^{p-1}_{W^{s,p}(4B)}\left(\int\limits_{4B}|u(x)-(u)_B|^p\,dx\right)^{\frac{1}{p}}.$$

III can be estimated the same way as II, and we have the following estimate for $[u]^p_{W^{s,p}(B)}$

$$[u]_{W^{s,p}(4B)}\,\sup_{\varphi}\int\limits_{4B}\int\limits_{4B}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dx\,dy$$

$$+[u]^{p-1}_{W^{s,p}(4B)}\,\text{diam}\,(B)^{-s}\left(\int\limits_{4B}|u(x)-(u)_B|^p\,dx\right)^{\frac{1}{p}}.$$

We conclude with Young's inequality. □

The next Proposition follows immediately from Jensen's inequality and the definition of $[u]^p_{W^{t,p}(\lambda B)}$.

Proposition 8.3 (A Poincaré type inequality) *Let B be a ball and for $\lambda \geq 1$ let λB be the concentric ball with λ times the radius. Then for any $t \in (0,1)$, $p \in (1,\infty)$,*

$$\int\limits_{\lambda B}|u(x)-(u)_B|^p\,dx \precsim \lambda^{n+tp}\text{diam}\,(B)^{tp}\,[u]^p_{W^{t,p}(\lambda B)}.$$

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