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Extending tensors on polar manifolds

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Abstract Given a polar action on a Riemannian manifold, we prove surjectivity of restriction to the section for general invariant tensors, and a sharper surjectivity result in the special case of metrics. These are related to the Chevalley Restriction Theorem and Michor's Basic Forms Theorem. The proofs rely on results in the Invariant Theory of finite reflection groups and symmetric pairs, some of which may be of independent interest.

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1 Introduction

Let (M, g) be a Riemannian manifold and G a Lie group acting on M properly by isometries. Recall that, by definition (see [13,25]), this action is called *polar* if there exists an immersed sub-manifold $\Sigma \to M$ meeting all G-orbits orthogonally. Such a submanifold Σ is called a *section*, and comes with a natural action by a discrete group of isometries $W = W(\Sigma)$, called its *generalized Weyl group*. Sections are always totally geodesic, and the immersion $\Sigma \to M$ induces an isometry $\Sigma/W \to M/G$, so in particular M/G is a Riemannian orbifold.

Denote by $C^\infty(T^{k,l}M)^G$, respectively $C^\infty(T^{k,l}\Sigma)^{W(\Sigma)}$, the sets of smooth (k,l)-tensors on M, respectively Σ, which are invariant under G, respectively W. Our main result states that the natural restriction map $C^\infty(T^{k,l}M)^G \to C^\infty(T^{k,l}\Sigma)^{W(\Sigma)}$ is surjective:

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Theorem 1 *Let M be a polar G-manifold with immersed section $i : \Sigma \to M$, and $W(\Sigma)$ the generalized Weyl group associated to Σ. Define the pull-back (restriction) map*

$$\psi = i^* : C^\infty(T^{k,l}M)^G \to C^\infty(T^{k,l}\Sigma)^{W(\Sigma)}$$

by

$$[\psi(\beta)](x)(v_1, \ldots, v_l) = P^{\otimes k}[\beta(i(x)((di)_x v_1, \ldots, (di)_x v_l)]$$

where $P : T_{i(x)}M \to T_x\Sigma$ is orthogonal projection. Then ψ is surjective.

In the case of functions, that is, $(k, l) = (0, 0)$, the map ψ above is an isomorphism. This is known as the Chevalley Restriction Theorem—see [25].

Note that Theorem 1 applies to $(0, l)$-tensors with symmetry properties, such as symmetric l-tensors, exterior l-forms, etc. This can be phrased naturally in terms of Weyl's construction (see [11, Lecture 6]). Recall that Weyl's construction associates to each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $l \in \mathbb{N}$ a functor $\mathbb{S}_\lambda$ of vector spaces called its Schur functor. One recovers Λ^l and Sym^l as the Schur functors associated to $\lambda = (l)$ and $\lambda = (1, 1, \ldots, 1)$, respectively.

Corollary 1 *Let M be a Riemannian manifold with an isometric polar action by G. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $l \in \mathbb{N}$, and consider the associated Schur functor $\mathbb{S}_\lambda$. Then the (surjective) restriction map $\psi : C^\infty(T^{0,l}M)^G \to C^\infty(T^{0,l}\Sigma)^W$ induces a surjective map*

$$\psi_\lambda : C^\infty(\mathbb{S}_\lambda(T^*M))^G \to C^\infty(\mathbb{S}_\lambda(T^*\Sigma))^W$$

For context, consider a special case of Corollary 1: exterior l-forms. Then the conclusion of Corollary 1 is implied by Michor's Basic Forms Theorem—see [23,24]. In fact, Michor's Theorem gives more precise information: it states that for a polar G-manifold M with section Σ, every smooth $W(\Sigma)$-invariant l-form on Σ can be extended *uniquely* to a smooth G-invariant l-form on M which is *basic*, that is, vanishes when contracted with vectors tangent to the G-orbits.

Now consider Riemannian metrics:

Theorem 2 *Let G act polarly on the Riemannian manifold M with section Σ and generalized Weyl group W. Assume this polar action is of* classical type. *Consider the restriction map (which is surjective by Corollary 1):*

$$\psi = |_\Sigma : C^\infty(\mathrm{Sym}^2 M)^G \to C^\infty(\mathrm{Sym}^2\Sigma)^W$$

For any Riemannian metric $\sigma \in C^\infty(\mathrm{Sym}^2\Sigma)^W$, there is a Riemannian metric $\tilde{\sigma} \in C^\infty(\mathrm{Sym}^2 M)^G$ such that $\psi(\tilde{\sigma}) = \sigma$, and with respect to which the G-action is polar with the same section Σ.

See page 10 for the precise definition of *classical type*. This assumption can be removed if one is willing to accept a proof relying on calculations performed by a computer—see the Appendix. (We label statements with computer-assisted proofs "Observations".)

Observation 1 Theorem 2 is valid without the classical type assumption.

For Theorem 2, Observation 1, and Michor's Basic Forms Theorem, the proof relies on polarization results in the Invariant Theory of finite reflection groups—see Sect. 4. On the other hand, the main ingredient in the proof of Theorem 1 is a multi-variable version of the Chevalley Restriction Theorem due to Tevelev—see Sect. 2.

An application of Theorem 2 (for classical type, and Observation 1 in general) is to give a partial answer to a natural question by K. Grove: given a proper isometric action of G on a Riemannian manifold (M, g), describe the set of all metrics on M/G which are induced by smooth G-invariant metrics g_0 on M. Theorem 2 answers this question under the additional hypothesis that M is a polar G-manifold. Namely, that set of metrics on $M/G = \Sigma/W$ coincides with the set of smooth orbifold metrics.

Another application is an important step in the main reconstruction result in [13]. This was in fact our main motivation for this work.

The present paper is organized as follows.

In Sect. 2 we state Tevelev's multi-variable version of the Chevalley Restriction Theorem for isotropy representations of symmetric spaces (Theorem 3), and generalize it to the class of polar representations (Corollary 2).

Section 3 is concerned with the proofs of Theorem 1 and Corollary 1.

In Sect. 4 we show how the algebraic results behind Michor's Basic Forms Theorem [23,24], Theorem 2, and Observation 1 (namely Solomon's Theorem [29], Theorem 4, and Observation 2) are in fact results about polarizations in the Invariant Theory of finite reflection groups. We then show in detail how Theorem 2 (respectively Observation 1) follows from Theorem 4 (respectively Observation 2).

The Appendix provides proofs of Theorem 4 and Observation 2. The latter is computer-assisted.

2 Multi-variable Chevalley restriction theorem

Let (G, K) be a symmetric pair, and consider the isotropy representation of K on $V = T_K G/K$, also called an s-representation. This is polar, and any maximal abelian sub-algebra $\Sigma \subset V$ is a section. Its generalized Weyl group W is also called the "baby Weyl group". The classic Chevalley Restriction Theorem says that

$$|_\Sigma : \mathbb{R}[V]^K \to \mathbb{R}[\Sigma]^W$$

is an isomorphism (see [33, page 143]).

Now consider the diagonal action of K on V^m (respectively W on Σ^m), and the corresponding algebras of invariant (m-variable) polynomials $\mathbb{R}[V^m]^K$ (respectively $\mathbb{R}[\Sigma^m]^W$). In contrast with the single-variable case, the restriction map $|_\Sigma$ is not injective. On the other hand, surjectivity is due to Tevelev:

Theorem 3 [31] *In the notation above, the restriction map* $|_\Sigma : \mathbb{R}[V^m]^K \to \mathbb{R}[\Sigma^m]^W$ *is surjective.*

Remarks The proof of Theorem 3 relies on the Kumar–Mathieu Theorem, previously known as the PRV conjecture, see [19,20]. Joseph [17] previously proved the theorem above in the special case of the adjoint action, using similar techniques. In [31] the Theorem above is stated only for $m = 2$ factors. But on page 324 it is remarked that "Actually, this (and Josephs's) Theorem also holds for any number of summands [...] ".

We observe that Theorem 3 generalizes to the class of *polar* representations (see [5] for a treatment of polar representations).

Corollary 2 *Let* $K \subset O(V)$ *be a polar representation, with section* Σ *and generalized Weyl group* $W \subset O(\Sigma)$. *Then the* m*-variable restriction is surjective*:

$$|_\Sigma : \mathbb{R}[V^m]^K \to \mathbb{R}[\Sigma^m]^W$$

Proof Let K_0 be the connected component of K which contains the identity. It is polar with the same section Σ. Let W_0 be its generalized Weyl group, so that $W_0 \subset W$. From the classification of irreducible polar representations in [5], it follows that the maximal subgroup $\tilde{K} \subset O(V)$, containing K_0, that is orbit-equivalent to K_0, defines an s-representation. (This fact has been given a classification-free proof in [7].) Note that K_0 and $\tilde{K}$ have the same sections and generalized Weyl groups.

Theorem 3 states that

$$|_\Sigma : \mathbb{R}[V^m]^{\tilde{K}} \to \mathbb{R}[\Sigma^m]^{W_0}$$

is surjective. But since $\tilde{K} \supset K_0$, we have $\mathbb{R}[V^m]^{\tilde{K}} \subset \mathbb{R}[V^m]^{K_0}$, and so

$$|_\Sigma : \mathbb{R}[V^m]^{K_0} \to \mathbb{R}[\Sigma^m]^{W_0}$$

is again surjective.

Finally, to show $|_\Sigma : \mathbb{R}[V^m]^K \to \mathbb{R}[\Sigma^m]^W$ is surjective, let $\beta \in \mathbb{R}[\Sigma^m]^W$. Then there is $\tilde{\beta}_0 \in \mathbb{R}[V^m]^{K_0}$ which restricts to β. Define

$$\tilde{\beta} = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} h\tilde{\beta}_0$$

Since $\tilde{\beta}$ equals the average of $\tilde{\beta}_0$ over K, it is K-invariant. To show that $\tilde{\beta}|_\Sigma = \beta$, we note that each coset $hK_o \in K/K_0$ can be represented by some $h \in N(\Sigma)$. Indeed, for an arbitrary $h \in K$, $h\Sigma$ is a section for K, hence also for K_0. Since K_0 acts transitively on the sections, there is $h_0 \in K_0$ such that $hh_0^{-1} \in N(\Sigma)$. Therefore

$$\tilde{\beta}|_\Sigma = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} (h\tilde{\beta}_0)|_\Sigma = \frac{1}{|K/K_0|} \sum \beta = \beta$$

because β is W-invariant.

Note that the algebra of multi-variable polynomials $\mathbb{R}[V^m]$ is graded by m-tuples of natural numbers $(d_1, \dots, d_m)$, and similarly for $\mathbb{R}[\Sigma^m]$. Consider the subspace generated by the polynomials of degree $(*, 1, \dots, 1)$. These can be identified with those tensor fields of type $(0, m-1)$ which have polynomial coefficients, that is, members of $\mathbb{R}[V, (V^*)^{m-1}]$, respectively $\mathbb{R}[\Sigma, (\Sigma^*)^{m-1}]$.

Since this grading is preserved by the restriction map $|_\Sigma$, Corollary 2 implies:

Corollary 3 *Let $K \subset O(V)$ be a polar representation, with section Σ and generalized Weyl group $W \subset O(\Sigma)$. Then the restriction map for polynomial-coefficient invariant $(0, l-1)$-tensors*

$$|_\Sigma : \mathbb{R}[V, (V^*)^{l-1}]^K \to \mathbb{R}[\Sigma, (\Sigma^*)^{l-1}]^W$$

is surjective.

3 Extending tensors

The goal of this section is to provide proofs of Theorem 1 and Corollary 1. We start with two Lemmas that will be used in proving Theorem 1.

Lemma 1 *Let V be a polar K-representation with section Σ and generalized Weyl group W. Then restriction to Σ is a surjective map*

$$|_\Sigma : C^\infty(T^{0,l} V)^K \to C^\infty(T^{0,l} \Sigma)^W$$

Proof The space of polynomial-coefficient $(0, l)$-tensors $\mathbb{R}[V, (V^*)^l]^K \subset C^\infty(T^{0,l} V)^K$ is generated, as an $\mathbb{R}[V]^K$-module, by finitely many (homogeneous) $\sigma_1, \dots, \sigma_r$ (see [30, Proposition 2.4.14]).

Since $\mathbb{R}[V]^K = \mathbb{R}[\Sigma]^W$, Corollary 3 implies that the restrictions $\sigma_1|_\Sigma, \dots, \sigma_r|_\Sigma$ generate $\mathbb{R}[\Sigma, (\Sigma^*)^l]^W$ as an $\mathbb{R}[\Sigma]^W$-module.

Then, by an argument involving the Malgrange Division Theorem and the fact that $\mathbb{R}[\Sigma, (\Sigma^*)^l]^W$ is dense in $C^\infty(T^{0,l} \Sigma)^W$ (see [8, Lemma 3.1]), we conclude that $\sigma_1|_\Sigma, \dots, \sigma_r|_\Sigma$ generate $C^\infty(\Sigma, (\Sigma^*)^l)^W = C^\infty(T^{0,l} \Sigma)^W$ as a $C^\infty(\Sigma)^W$-module. This implies that $|_\Sigma : C^\infty(T^{0,l} V)^K \to C^\infty(T^{0,l} \Sigma)^W$ is surjective.

The next lemma describes the smooth G-invariant tensors on a tube $\mathcal{U} = G \times_K V$ in terms of smooth K-invariant tensors on the slice V.

Lemma 2 *Let $K \subset G$ be Lie groups with K compact, and V be a K-representation. Define $\mathcal{U} = G \times_K V$ to be the quotient of $G \times V$ by the free action of K given by $k \cdot (g, v) = (gk^{-1}, kv)$, and identify V with the subset of $\mathcal{U}$ which is the image of $\{1\} \times V \subset G \times V$ under the natural quotient projection $G \times V \to \mathcal{U}$.*

Then there is a K-representation H and an isomorphism

$$C^\infty(T^{0,l} V)^K \times C^\infty(V, H)^K \to C^\infty(T^{0,l}\mathcal{U})^G$$

Under this identification the restriction map

$$|_V : C^\infty(T^{0,l}\mathcal{U})^G \to C^\infty(T^{0,l}V)^K$$

corresponds to projection onto the first factor. In particular $|_V$ *is onto.*

Proof To describe H, let $p \in \mathcal{U}$ be the image of $(1,0) \in G \times V$ in $\mathcal{U}$. Then $(V^*)^{\otimes l}$ is a K-invariant subspace of $(T_p^*\mathcal{U})^{\otimes l}$, and we define H to be its K-invariant complement, so that

$$(T_p^*\mathcal{U})^{\otimes l} = (V^*)^{\otimes l} \oplus H$$

as K-representations.

We define $\Psi : C^\infty(T^{0,l}V)^K \times C^\infty(V, H)^K \to C^\infty(T^{0,l}\mathcal{U})^G$ in the following way: Given $(\beta_1, \beta_2) \in C^\infty(T^{0,l}V)^K \times C^\infty(V, H)^K$, let $\tilde{\beta} : G \times V \to T^{0,l}\mathcal{U}$ be given by

$$\tilde{\beta}(g, v) = g \cdot (\beta_1(v) + \beta_2(v))$$

Since $\tilde{\beta}$ is K-invariant, it descends to $\beta = \Psi(\beta_1, \beta_2) : \mathcal{U} \to T^{0,l}\mathcal{U}$.

The map β is smooth because $\tilde{\beta}$ is smooth and the action of K on $G \times V$ is free. Moreover β is clearly a G-invariant cross-section of the bundle $T^{0,l}\mathcal{U} \to \mathcal{U}$, and $\beta|_V = \beta_1$.

Now the proof of Theorem 1 essentially follows from Lemmas 1 and 2, together with the Slice Theorem (see [2]) and partitions of unity:

Proof of Theorem 1 First note that it is enough to consider $(0,l)$ tensors. Indeed, ψ for (k,l) tensors equals the composition of ψ for $(0, k+l)$-tensors with raising and lowering indices (using the Riemannian metric on M) to transform between (k,l)-tensors and $(0, k+l)$-tensors.

It is enough to prove surjectivity of ψ locally around each orbit in M, because of the existence of G-invariant partitions of unity subject to any cover by G-invariant open sets in M.

So let $p \in M$ be an arbitrary point, with orbit Gp, isotropy $K = G_p$, and slice $V = (T_pGp)^\perp$. The Slice Theorem (see [2]) then says that for an open G-invariant tubular neighborhood $\mathcal{U}$ of the orbit Gp there is a G-equivariant diffeomorphism

$$E : G \times_K V \to \mathcal{U}$$

From now on we will identify $\mathcal{U}$ with $G \times_K V$ through E.

The slice representation of K on V is polar (see [25]). If $\Sigma \subset V$ is a section with generalized Weyl group $W(\Sigma)$, the quotients $\mathcal{U}/G$, V/K and Σ/W are isometric.

Since the inclusion $\Sigma \to \mathcal{U}$ factors as $\Sigma \to V \to \mathcal{U}$, the restriction map ψ factors as $\psi = |_\Sigma^V \circ |_V^{\mathcal{U}}$, where

$$|_\Sigma^V : C^\infty(T^{0,l}V)^K \to C^\infty(T^{0,l}\Sigma)^W \qquad |_V^{\mathcal{U}} : C^\infty(T^{0,l}\mathcal{U})^G \to C^\infty(T^{0,l}V)^K$$

Both these maps are surjective, by Lemmas 1 and 2. Therefore ψ is surjective.

Now we turn to Corollary 1, about $(0, l)$-tensors with symmetry properties, such as exterior forms and symmetric tensors.

Proof of Corollary 1 The Schur functor $\mathbb{S}_\lambda$ is defined in terms of a certain element $c_\lambda \in \mathbb{Z}S_l$ in the group ring $\mathbb{Z}S_l$, called the Young symmetrizer associated to λ—see [11, Lecture 6]. Indeed, given a vector space V, the group S_l acts on $V^{\otimes l}$, and so c_λ determines a linear map $V^{\otimes l} \to V^{\otimes l}$. The image of this map is defined to be $\mathbb{S}_\lambda(V)$.

Thus $C^\infty(\mathbb{S}_\lambda(T^*M))$ is simply the image of the natural map

$$c_\lambda : C^\infty(T^{0,l}M) \to C^\infty(T^{0,l}M)$$

and similarly for $C^\infty(\mathbb{S}_\lambda(T^*M))^G$ (because the actions of G and S_l commute), and $C^\infty(\mathbb{S}_\lambda(T^*\Sigma))^W$.

Since the restriction map ψ is S_l-equivariant and surjective, it takes the image of

$$c_\lambda : C^\infty(T^{0,l}M)^G \to C^\infty(T^{0,l}M)^G$$

onto the image of

$$c_\lambda : C^\infty(T^{0,l}\Sigma)^W \to C^\infty(T^{0,l}\Sigma)^W$$

completing the proof.

4 Polarizations and finite reflection groups

An alternative way of proving special cases of Theorem 3 is given by the polarization technique. This has the advantage of providing explicit lifts, which we exploit to prove Theorem 2 and Observation 1.

We start by recalling the definition of polarizations (see [27] for a reference). Let U be an Euclidean vector space, and $H \to O(U)$ be a representation of the group H. Consider the diagonal action of H on m copies of U, and the corresponding algebra of invariant (m-variable) polynomials $\mathbb{R}[U^m]^H$. Identify $\mathbb{R}[U]^H$ with the elements of $\mathbb{R}[U^m]^H$ which depend only on the first variable.

The method of polarizations consists of generating multi-variable invariants from single-variable invariants. Indeed, assuming $f \in \mathbb{R}[U]^H$ is homogeneous of degree d, let $t_1, \dots, t_m$ be formal variables, and formally expand

$$f(t_1 v_1 + \cdots + t_m v_m) = \sum_{r_1+\cdots+r_m=d} t_1^{r_1} \cdots t_m^{r_m} f_{r_1,\dots,r_m}(v_1, \dots, v_m)$$

Then each $f_{r_1,\dots,r_m}$ belongs to $\mathbb{R}[U^m]^H$, and is called a polarization of f.

An alternative but equivalent definition of polarizations is given in terms of *polarization operators*—see [32]. These are differential operators D_{ij} (for $1 \le i, j \le m$) on $\mathbb{R}[U^m]^H$ defined by

$$(D_{ij}f)(u_1,\ldots,u_m)=\frac{d}{dt}\bigg|_{t=0} f(u_1,\ldots,u_j+tu_i,\ldots,u_m)$$

Then one defines the subalgebra $\mathcal{P}^m\subset\mathbb{R}[U^m]^H$ of polarizations to be the smallest subalgebra of $\mathbb{R}[U^m]^H$ containing $\mathbb{R}[U]^H$ and stable under the operators D_{ij}.

For example, if $f\in\mathbb{R}[U]^H$, then the tensors $df=D_{2,1}f\in\mathbb{R}[U^2]^H$ and Hess $f=D_{2,1}(D_{3,1}f)\in\mathbb{R}[U^3]^H$ are polarizations. Similarly, if $f_1,\ldots,f_p\in\mathbb{R}[U]^H$, then $df_1\otimes df_2\otimes\cdots\otimes df_p=(D_{2,1}f_1)\cdots(D_{p+1,1}f_p)$ is a polarization, and so is $df_1\wedge\cdots\wedge df_p$. (Here we are identifying tensor fields with multi-variable functions as in Sect. 2.)

Now consider the special case where $H=W$ is a finite group generated by reflections on $U=\Sigma$. Recall that W is the product of a finite number of irreducible reflection groups, and that irreducible finite reflection groups are classified into types: Dihedral, A_n, B_n, D_n (called "classical"), and six exceptional groups H_3, H_4, F_4, E_6, E_7, and E_8. We say a reducible W is of *classical type* if each of its factors is of classical type.

If W is irreducible of type A, B, or dihedral, then $\mathcal{P}^m=\mathbb{R}[\Sigma^m]^W$ by [15,34].

It was noted by Wallach [32] that $\mathbb{R}[\Sigma^m]^W$ is *not* generated by polarizations for W of type D_n for $n>3$ and $m>1$. He proposed a definition of generalized polarizations, and showed that these do generate all multi-variable invariants for type D. Unfortunately Wallach's generalized polarizations fail to generate all multi-variable invariants for W of type F_4 (see [15]).

For W of general type, even though $\mathcal{P}^m\neq\mathbb{R}[\Sigma^m]^W$, one can still identify geometrically interesting subspaces of $\mathbb{R}[\Sigma^m]^W$ which are contained in $\mathcal{P}^m$. For example, Solomon's Theorem [29] states that the subspace $\mathbb{R}[\Sigma,\Lambda^{m-1}\Sigma^*]^W\subset\mathbb{R}[\Sigma^m]^W$ of exterior $(m-1)$-forms is contained in $\mathcal{P}^m$. Another example is the space of symmetric 2-tensors:

Theorem 4 *Let $W\subset O(\Sigma)$ be a finite group generated by reflections. Assume W is of classical type. Then every W-invariant symmetric 2-tensor field on Σ is a sum of terms of the form $a\mathrm{Hess}(b)$, for $a,b\in\mathbb{R}[\Sigma]^W$.*

Observation 2 Theorem 4 is valid without the classical type assumption.

We provide proofs of Theorem 4 and Observation 2 above in the Appendix. The latter is computer-assisted.

Now assume $K\subset O(V)$ is a polar representation of the compact group K with section Σ, and generalized Weyl group W. Recall that the connected component of the identity K_0 is polar with the same section Σ, and denote by W_0 its generalized Weyl group. By [5], W_0 is a finite group generated by reflections. Since the operators D_{ij} commute with the restriction map $|_{\Sigma^m}:\mathbb{R}[V^m]^{K_0}\to\mathbb{R}[\Sigma^m]^{W_0}$, and the single-variable invariants coincide by the Chevalley Restriction Theorem, the image of $|_{\Sigma^m}$ must contain $\mathcal{P}^m$. In particular, this gives an alternative proof of Theorem 3 in the special case that W_0 is of classical type—see [15].

Similarly, Theorem 4 implies surjectivity of the restriction map for symmetric 2-tensors when W_0 is of classical type. In fact, we have the sharper statement:

Lemma 3 *Let $K\subset O(V)$ be a polar representation of the compact group K, with section $\Sigma\subset V$ and generalized Weyl group W. Let K_0 be the connected component* *of K containing the identity. Assume the generalized Weyl group W_0 associated to K_0 is of classical type. Consider the restriction map for symmetric 2-tensor fields $|_\Sigma : C^\infty(\mathrm{Sym}^2 V)^K \to C^\infty(\mathrm{Sym}^2 \Sigma)^W$.*

This map is surjective. Moreover, given $\beta \in C^\infty(\mathrm{Sym}^2 \Sigma)^W$ there is $\tilde{\beta} \in C^\infty(\mathrm{Sym}^2 V)^K$ such the $\tilde{\beta}|_\Sigma = \beta$ and satisfying the following property:

For all $q \in V$, and $X, Y \in T_q V$ such that X is vertical (that is, tangent to the K-orbit through q) and Y is horizontal (that is, normal to the K-orbit through q), we have $\tilde{\beta}(X, Y) = 0$.

Proof Let $\beta \in C^\infty(\mathrm{Sym}^2 \Sigma)^W$. By Theorem 4 together with [8, Lemma 3.1], β is of the form $\beta = \sum_i a_i \mathrm{Hess}(b_i)$, where $a_i, b_i \in C^\infty(\Sigma)^{W_0}$. By the Chevalley Restriction Theorem, a_i, b_i extend uniquely to $\tilde{a}_i, \tilde{b}_i \in C^\infty(V)^{K_0}$.

Define $\tilde{\beta}_0 = \sum_i \tilde{a}_i \mathrm{Hess}(\tilde{b}_i)$ and

$$\tilde{\beta} = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} h\tilde{\beta}_0$$

Then $\tilde{\beta}|_\Sigma = \beta$ by the same argument as in Corollary 2.

To show that $\tilde{\beta}$ satisfies the additional property in the statement of the Lemma, it is enough to do so for each $\mathrm{Hess}(\tilde{\beta}_i)$. Changing the section Σ if necessary, we may assume that $q, Y \in \Sigma$. Extend the given $X, Y \in T_q V$ to parallel vector fields (in the Euclidean metric), also denoted by X, Y. Let $f = d\tilde{\beta}_i(X)$.

We claim that $f|_\Sigma$ is identically zero. Indeed, since $X(q)$ is vertical, it is orthogonal to Σ, and so $X(p)$ is orthogonal to Σ for every $p \in \Sigma$. Thus, for regular $p \in \Sigma$, $X(p)$ is vertical. Since $\tilde{\beta}_i$ is constant on orbits, $f(p) = 0$ for every regular $p \in \Sigma$, and hence on all of Σ by continuity.

Therefore $\mathrm{Hess}(\tilde{\beta}_i)(X, Y) = df(Y) = 0$, because $Y \in \Sigma$.

Replacing in the proof above "Theorem 4" with "Observation 2" yields:

Observation 3 Lemma 3 is valid without the classical type assumption.

The following lemma is needed in the proofs of Theorem 2 and Observation 1.

Lemma 4 *Let V be a polar K-representation with section $\Sigma \subset V$ and generalized Weyl group W. Let $\tilde{\sigma} \in C^\infty(\mathrm{Sym}^2 V)^K$, and $\sigma = \tilde{\sigma}|_\Sigma$. Then $\sigma(0)$ is positive definite if and only if $\tilde{\sigma}(0)$ is positive definite.*

Proof Denote by K_0 the connected subgroup of K containing the identity. Recall that the action of K_0 is polar with the same section Σ. Denote by W_0 its generalized Weyl group. Consider a decomposition of V into K_0-invariant subspaces

$$V = \mathbb{R}^m \oplus V_1 \oplus \cdots \oplus V_k$$

where K_0 acts trivially on $\mathbb{R}^m$, and each V_i is irreducible and non-trivial.

By Theorem 4 in [5], each V_i is a polar K_0-representation, with section $\Sigma_i = \Sigma \cap V_i$, and we have the decomposition into W_0-invariant subspaces

$$\Sigma = \mathbb{R}^m \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_k$$

Moreover W_0 splits as a product $W_1 \times \cdots \times W_k$ (see [14, section 2.2]), where W_i is the generalized Weyl group associated to the section $\Sigma_i \subset V_i$, so that Σ_i are pairwise inequivalent as W_0-representations. This implies that V_i are pairwise inequivalent as K_0-representations.

Since the quotients V_i/K_0 and Σ_i/W_0 are isometric, irreducibility of V_i as a K_0-representation implies irreducibility of Σ_i as a W_0-representation. (Indeed, a general representation of a compact group H on Euclidean space $\mathbb{R}^n$ is irreducible if and only if the quotient S^{n-1}/H has diameter less than $\pi/2$.)

By Schur's Lemma together with the assumption $\tilde{\sigma}|_{\Sigma} = \sigma$,

$$\sigma(0) = A \oplus \lambda_1 \mathrm{Id}_{\Sigma_1} \oplus \cdots \oplus \lambda_k \mathrm{Id}_{\Sigma_k}$$
$$\tilde{\sigma}(0) = A \oplus \lambda_1 \mathrm{Id}_{V_1} \oplus \cdots \oplus \lambda_k \mathrm{Id}_{V_k}$$

where A is a symmetric $m \times m$ matrix, and $\lambda_i \in \mathbb{R}$.

Therefore $\sigma(0) > 0$ if and only if $\tilde{\sigma}(0) > 0$.

Let M be a polar G-manifold. We say M is of *classical type* if, for every $p \in M$, the slice representation of $(G_p)_0$ has generalized Weyl group of classical type. Now we are ready to prove Theorem 2:

Proof of Theorem 2 As in the proof of Theorem 1, we use partitions of unity and the Slice Theorem to reduce to the case where M is a tube $\mathcal{U} = G \times_K V$, and V is a polar representation. Let $\Sigma \subset V$ be a section, with generalized Weyl group W, so that $M/G = V/K = \Sigma/W$.

Note that it suffices to extend the given Riemannian metric $\sigma \in C^\infty(\mathrm{Sym}^2\Sigma)^W$ to a G-invariant Riemannian metric on a possibly smaller tube $G \times_K V^\epsilon$ around the orbit G/K, for some $\epsilon > 0$.

By Lemma 3, σ extends to $\beta_1 \in C^\infty(\mathrm{Sym}^2 V)^K$. By Lemma 4, $\beta_1(0)$ is positive-definite, and so by continuity, $\beta_1 > 0$ on V^ϵ for some small $\epsilon > 0$.

Choose any smooth, K-invariant and positive-definite $\beta_2 : V \to \mathrm{Sym}^2(T_K G/K)$. Then, by Lemma 2, the pair (β_1, β_2) defines $\tilde{\sigma} \in C^\infty(\mathrm{Sym}^2 M)^G$, which is positive-definite on $G \times_K V^\epsilon$ and extends the given σ. By construction, Σ is $\tilde{\sigma}$-orthogonal to G-orbits.

Finally, using Observation 3 instead of Lemma 3 gives a proof of Observation 1.

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Appendix: Hessian Theorem for finite reflection groups

In this section we provide proofs of Theorem 4 and Observation 2. The latter relies on Calculation 1, which can be checked with a computer. See [22] for the source code for a script written in GAP [26] using the package CHEVIE [12], which performs these calculations.

Note that as far as the proofs of Theorem 2 and Observation 1 are concerned, one only needs to consider crystallographic reflection groups (see [14] for a definition). Our proofs include the non-crystallographic cases for the sake of completeness.

Recall some facts about finite reflection groups: First, the algebra of invariants, $A=\mathbb{R}[\Sigma]^W$, is a free polynomial algebra with n generators, where $n=\dim\Sigma$. This is known as Chevalley's Theorem—see [1, Chapter V]. Such a set $\{\rho_i\}$ of homogeneous generators is called a set of *basic invariants*, and $d_i=\deg\rho_i$ are called the *degrees* of W.

Second, $\mathbb{R}[\Sigma]$ is a free $A=\mathbb{R}[\Sigma]^W$-module, more precisely $\mathbb{R}[\Sigma]=A\otimes H$, where H is isomorphic to the regular representation of W (see Theorem B in [3]). In particular, $\mathcal{M}=\mathbb{R}[\Sigma,\mathrm{Sym}^2(\Sigma^*)]^W$ is a free A-module of rank $(n^2+n)/2$.

Third, Σ is reducible as a W-representation if and only if $\Sigma=\Sigma_1\times\Sigma_2$ and $W=W_1\times W_2$ for two reflection groups $W_k\subset O(\Sigma_k)$—see section 2.2 in [14]. Thus the following proposition reduces the proofs of Theorem 4 and Observation 2 to the irreducible case.

Lemma 5 *Let $W_k\subseteq O(\Sigma_k)$, $k=1,2$ be finite reflection groups in the Euclidean vector spaces Σ_k, and let $W=W_1\times W_2\subset O(\Sigma)=O(\Sigma_1\times\Sigma_2)$. Then there are W-invariant polynomials on Σ whose Hessians generate $\mathbb{R}[\Sigma,\mathrm{Sym}^2(\Sigma^*)]^W$ if and only if the same holds for $W_k\subseteq O(\Sigma_k)$, $k=1,2$.*

Proof Assume there are $Q_j\in\mathbb{R}[\Sigma]^W$ whose Hessians generate $\mathbb{R}[\Sigma,\mathrm{Sym}^2(\Sigma^*)]^W$. Then the restrictions $Q_j|_{\Sigma_1}$ generate $\mathbb{R}[\Sigma_1,\mathrm{Sym}^2(\Sigma_1^*)]^{W_1}$ as an $\mathbb{R}[\Sigma_1]^{W_1}$-module.

Indeed, every $\sigma\in\mathbb{R}[\Sigma_1,\mathrm{Sym}^2(\Sigma_1^*)]^{W_1}$ can be naturally extended to $\tilde\sigma\in\mathbb{R}[\Sigma,\mathrm{Sym}^2(\Sigma^*)]^W$ that is constant on each copy of Σ_2. Then there are $a_j\in\mathbb{R}[\Sigma]^W$ such that $\tilde\sigma=\sum_j a_j\mathrm{Hess}(Q_j)$. Therefore

$$\sigma=\sum_j(a_j|_{\Sigma_1})\mathrm{Hess}(Q_j|_{\Sigma_1})$$

and similarly for $\mathbb{R}[\Sigma_2,\mathrm{Sym}^2(\Sigma_2^*)]^{W_2}$.

For the converse, let $\rho_j\in\mathbb{R}[\Sigma_1]^{W_1}$, $j=1,\dots,n_1$ and $\psi_j\in\mathbb{R}[\Sigma_2]^{W_2}$, $j=1,\dots,n_2$ be basic invariants on Σ_1 and Σ_2 respectively; and $Q_j\in\mathbb{R}[\Sigma_1]^{W_1}$, $j=1,\dots,(n_1^2+n_1)/2$, $R_j\in\mathbb{R}[\Sigma_2]^{W_2}$, $j=1,\dots,(n_2^2+n_2)/2$ be homogeneous invariants whose Hessians form a basis for the corresponding spaces of equivariant symmetric 2-tensors.

We claim that the Hessians of the following set of $W=W_1\times W_2$-invariant polynomials on $\Sigma=\Sigma_1\times\Sigma_2$ form a basis for the space of equivariant symmetric 2-tensors on Σ:

$$\{Q_j\}\cup\{R_j\}\cup\{\rho_i\psi_j\}$$

Table 1 Monomials with generating Hessians

Group	Set T of monomials in $y_1, \dots, y_n$
H_3	$\{y_j\}_{j=1}^3 \cup \{y_1 y_j\}_{j=1}^3$
H_4	$\{y_j\}_{j=1}^4 \cup \{y_1 y_j\}_{j=1}^4 \cup \{y_2^2, y_3^2\}$
F_4	$\{y_j\}_{j=1}^4 \cup \{y_1 y_j\}_{j=1}^4 \cup \{y_2^2, y_3^2\}$
E_6	$\{y_j\}_{j=1}^6 \cup \{y_1 y_j\}_{j=1}^6 \cup \{y_2 y_j\}_{j=2}^6 \cup \{y_3^2, y_3 y_5, y_3 y_6, y_4^2\}$
E_7	$\{y_j\}_{j=1}^7 \cup \{y_1 y_j\}_{j=1}^7 \cup \{y_2 y_j\}_{j=2}^7 \cup \{y_3 y_j\}_{j=3}^7 \cup \{y_4 y_5, y_4 y_6, y_4 y_7\}$
E_8	$\{y_j\}_{j=1}^8 \cup \{y_1 y_j\}_{j=1}^8 \cup \{y_2 y_j\}_{j=2}^8 \cup \{y_3 y_j\}_{j=3}^8 \cup$ $\{y_4^2, y_4 y_6, y_4 y_7, y_4 y_8, y_5^2, y_5 y_8, y_6^2\}$

Indeed, $\mathbb{R}[\Sigma, \mathrm{Sym}^2(\Sigma^*)]^W$ decomposes as

$$\mathbb{R}[\Sigma, \mathrm{Sym}^2(\Sigma_1^*)]^W \oplus \mathbb{R}[\Sigma, \mathrm{Sym}^2(\Sigma_2^*)]^W \oplus \mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W$$

The first two pieces are freely generated over $\mathbb{R}[\Sigma]^W$ by $\mathrm{Hess}\, Q_j$ and $\mathrm{Hess}\, R_j$. The third piece can be rewritten as $\mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W = \mathbb{R}[\Sigma_1, \Sigma_1^*]^{W_1} \otimes \mathbb{R}[\Sigma_2, \Sigma_2^*]^{W_2}$. By Solomon's Theorem [29], $\mathbb{R}[\Sigma_k, \Sigma_k^*]^{W_k}$, $k = 1, 2$, are freely generated by $d\rho_j$ and $d\psi_j$, so that $\mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W$ is freely generated by $(d\rho_j \otimes d\psi_j + d\psi_j \otimes d\rho_j)$. To finish the proof of the claim one uses the product rule

$$\mathrm{Hess}(\rho_i \psi_j) = d\rho_i \otimes d\psi_j + d\psi_j \otimes d\rho_j + \rho_i \mathrm{Hess}(\psi_j) + \psi_j \mathrm{Hess}(\rho_i)$$

Proof of Theorem 4 By Lemma 5, we may assume W is irreducible of classical type.

If W is of type A, type B, or dihedral, then all multi-variable invariants are generated by polarizations, by [15,34]. Hence it is enough to show that $\mathcal{P}^3 \cap \mathcal{M}$ is generated, as an A-module, by Hessians of invariants. This follows from the product rule

$$\mathrm{Hess}(\rho_i \psi_j) = d\rho_i \otimes d\psi_j + d\psi_j \otimes d\rho_j + \rho_i \mathrm{Hess}(\psi_j) + \psi_j \mathrm{Hess}(\rho_i)$$

If, on the other hand, W is of type D, then the multi-variable invariants are generated by *generalized* polarizations—see Theorems 3.1 and 3.4 in [15], or Proposition 2 in Appendix 2 of [32]. But degree considerations imply that $\mathcal{M}$ is in fact generated by (classical) polarizations, hence also by Hessians by the product rule.

Calculation 1 Let W of type H_3, H_4, F_4, E_6, E_7 or E_8. Then there is a choice of basic invariants $\rho_1, \dots \rho_n$ with $\deg(\rho_1) < \dots < \deg(\rho_n)$, and of a regular vector $v \in \Sigma$, such that $\{\mathrm{Hess}(\rho^* Q)(v) \mid Q \in T\}$ is linearly independent, where $\rho = (\rho_1, \dots, \rho_n) : \Sigma \to \mathbb{R}^n$, and T is the set of polynomials on $\mathbb{R}^n$ given in Table 1.

We remark that the computation above is independent of the choice of basic invariants and regular vector—see Lemmas 6 and 7. Moreover, one may construct a set of basic invariants consisting of "orbit Chern classes" from some linear functional $\lambda_0 : \Sigma \to \mathbb{R}$ and the degrees d_i. Namely, $\rho_i = \sum_{\lambda \in W\lambda_0} \lambda^{d_i}$—see [9,10,22,28]. See

Table 2 Degrees of exceptional finite reflection groups

Group	Degrees $d_1, \dots, d_n$
H_3	2, 6, 10
H_4	2, 12, 20, 30
F_4	2, 6, 8, 12
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30

also [16,21] for (other) explicit sets of basic invariants of exceptional groups. The degrees $d_1, \dots, d_n$ of the exceptional groups are listed in Table 2 for the convenience of the reader (see [4]).

The following Lemma is analogous to Proposition 3.13 in [14]. We will use the special case $U = \mathrm{Sym}^2 \Sigma^*$ in proving both Observation 2 from Calculation 1, and independence of the choice of regular vector v in Calculation 1.

Lemma 6 *Let $W \subset O(\Sigma)$ be an irreducible finite reflection group, and $\eta : W \to O(U)$ an orthogonal representation, with character χ. Choose a basis $\{e_1, \dots, e_l\}$ for U, and let $f_1, \dots, f_l \in \mathbb{R}[\Sigma, U]^W$ be homogeneous elements given by $f_i = \sum_j a_{ij} e_j$. Let $D = \det(a_{ij}) \in \mathbb{R}[\Sigma]$. Then:*

(a) *D is divisible by the following product over all reflections $r \in W$:*

$$J_\eta = \prod_r (\lambda_r)^{\frac{l-\chi(r)}{2}}$$

with λ_r a linear functional whose kernel equals the hyperplane fixed by r.

(b) *If $\{f_i\}$ is a basis of $\mathbb{R}[\Sigma, U]^W$ over $\mathbb{R}[\Sigma]^W$, then D and J_η have the same degree.*

(c) *$\{f_i\}$ forms a basis if and only if $D = cJ_\eta$ for some $c \in \mathbb{R} - \{0\}$.*

Proof (a) Let $r \in W$ be a reflection. The transformation $\eta(r) : U \to U$ is diagonalizable with eigenvalues $1, -1$. In particular the multiplicity of -1 equals $k = (l - \chi(r))/2$.

Assume, without loss of generality, that $e_1, \dots, e_k$ is a basis for the eigenspace of $\eta(r)$ associated to the eigenvalue -1. Then the first k columns of (a_{ij}) are odd with respect to r. By multi-linearity, D vanishes to order k on the hyperplane fixed by r. In other words, D is divisible by $(\lambda_r)^k$.

Since this is true for every reflection r, D is divisible by J_η.

(b) For any graded vector space $E = \oplus_i E_i$, denote its Poincaré series by $P_t(E) = \sum_i \dim(E_i) t^i$. Let

$$P(t) = \frac{P_t(\mathbb{R}[\Sigma, \mathrm{Sym}^2(\Sigma^*)]^W)}{P_t(\mathbb{R}[\Sigma]^W)}$$

Since $P(t) = \sum_{i=1}^l t^{\deg(f_i)}$, we have $\deg(D) = P'(1)$.

On the other hand, $P_t(\mathbb{R}[\Sigma]^W) = \Pi_{i=1}^n(1-t^{d_i})^{-1}$, while the numerator can be computed using Molien's formula [18, page 249]:

$$P_t(\mathbb{R}[\Sigma, \mathrm{Sym}^2(\Sigma^*)]^W) = \frac{1}{|W|}\sum_{g\in W}\frac{\chi(g)}{\det(1-tg)}$$

Note that for $g=1$, $\det(1-tg)=(1-t)^n$; for $g=r$ a reflection, $\det(1-tg) = (1-t)^{n-1}(1+t)$; and for all other g, $(1-t)^{n-1}$ does not divide $\det(1-tg)$. Therefore, when computing $P'(1)$, terms of the latter type vanish:

$$|W|P'(1) = \chi(1)\left.\frac{d}{dt}\right|_{t=1}\frac{\Pi_{i=1}^n(1-t^{d_i})}{(1-t)^n} + \sum_{r \text{ refl.}}\chi(r)\left.\frac{d}{dt}\right|_{t=1}\frac{\Pi_{i=1}^n(1-t^{d_i})}{(1-t)^{n-1}(1+t)}$$

$$P'(1) = \frac{1}{|W|}\left(\frac{lN|W|}{2} - \sum_{r \text{ refl.}}\chi(r)\frac{|W|}{2}\right) = \sum_{r \text{ reflection}}\frac{l-\chi(r)}{2}$$

where N is the number of reflections and we have used the identities $d_1\cdots d_n = |W|$ and $(d_1-1)+\cdots+(d_n-1)=N$. Thus $P'(1)$ equals the degree of J_η.

(c) Assume $\{f_i\}$ is a basis. By parts 1 and 2, $D = cJ_\eta$ for some $c\in\mathbb{R}$. Assume for a contradiction that $c=0$. This means that $\{f_i(v)\}$ is linearly dependent for every $v\in\Sigma$. Take a regular v, and let B be a small open W-invariant neighborhood of the orbit Wv. Since W acts freely on B, on can construct $\sigma\in C^\infty(\Sigma,U)^W$ with $\mathrm{supp}(\sigma)\subset B$, and $\sigma(v)\notin \mathrm{span}\{f_i(v)\}$. This contradicts the fact that $\mathbb{R}[\Sigma,U]^W$ is dense in $C^\infty(\Sigma,U)^W$.
Now assume $D = cJ_\eta$ for some $c\in\mathbb{R}-\{0\}$. Choose any homogeneous basis $\{f'_i\}$ of $\mathcal{M}$, and define D' analogously to D. Writing $f_i = \sum_j b_{ij}f'_j$, we see that $\det(b_{ij})\in\mathbb{R}-\{0\}$, because $D = D'\det(b_{ij})$. This implies that (b_{ij}) is invertible in the algebra of matrices with coefficients in A, so that $\{f_i\}$ is a basis of $\mathcal{M}$ over A, too.

Proof of Observation 2 Assume Calculation 1. We claim that

$$\{f_i\} = \{\mathrm{Hess}Q\ ,\ Q\in T\}$$

forms a basis for $\mathcal{M}$ over A. Indeed, by inspection, the sum of the degrees of f_i equals $\deg(J_\eta)$, which in this case is $N(n-1)$. Using Lemma 6, we see that $D = cJ_\eta$ for some $c\neq 0$, so that $\{f_i\}$ forms a basis.

Note that independence of the choice of regular vector follows from Lemma 6, because the zero set of J_η is contained in the singular set.

Lemma 7 *Calculation 1 is independent of the choice of basic invariants ρ_i.*

Proof Assume $\{\rho_i\}$ and $\{\psi_i\}$ are two sets of basic invariants, and that

$$s\{\mathrm{Hess}(\rho^*Q)(v)\mid Q\in T\}$$

is linearly independent at some (hence all) regular vector $v \in \Sigma$. By Lemma 6, $\{\mathrm{Hess}(\rho^* Q) \mid Q \in T\}$ forms a basis of $\mathcal{M}$ as a free A-module. We claim that $\{\mathrm{Hess}(\psi^* Q) \mid Q \in T\}$ is also a basis, so that in particular $\{\mathrm{Hess}(\psi^* Q)(v) \mid Q \in T\}$ is linearly independent for every regular vector v.

Recall the graded version of Nakayama's Lemma (see Exercise 4.6a in [6]): a set of homogeneous elements $f_i \in \mathcal{M}$ generates $\mathcal{M}$ as an A-module if and only if their images in $\mathcal{M}/I\mathcal{M}$ span it as real vector space, where I is the ideal of A generated by the elements of positive degree.

Since the degrees d_i are all distinct, we may assume without loss of generality that $\psi_i = \rho_i + R_i(\rho_1, \dots, \rho_{i-1})$. Note that the Hessian of a product of three or more (not necessarily distinct) basic invariants belongs to $I\mathcal{M}$, so that modulo $I\mathcal{M}$ we have $\mathrm{Hess}(\psi_i \psi_j) \equiv \mathrm{Hess}(\rho_i \rho_j)$, and

$$\mathrm{Hess}(\psi_i) \equiv \mathrm{Hess}(\rho_i) + \sum_{j,k} c_{j,k} \mathrm{Hess}(\rho_j \rho_k)$$

where $c_{j,k} \in \mathbb{R}$ vanishes unless $d_j + d_k = d_i$.

By inspection, $y_j y_k \in T$ whenever $d_j + d_k = d_i$ for some i (see Table 2). Therefore $\{\mathrm{Hess}(\psi^* Q)\ ,\ Q \in T\}$ is written in terms of $\{\mathrm{Hess}(\rho^* Q)\ ,\ Q \in T\}$ (modulo $I\mathcal{M}$) using a triangular matrix with 1's in the diagonal, showing that $\{\mathrm{Hess}(\psi^* Q) \mid Q \in T\}$ is a basis of $\mathcal{M}$.

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