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Heat kernel on smooth metric measure spaces and applications

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Abstract We derive a Harnack inequality for positive solutions of the f-heat equation and Gaussian upper and lower bound estimates for the f-heat kernel on complete smooth metric measure spaces with Bakry–Émery Ricci curvature bounded below. Both upper and lower bound estimates are sharp when the Bakry–Émery Ricci curvature is nonnegative. The main argument is the De Giorgi–Nash–Moser theory. As applications, we prove an L^1_f-Liouville theorem for f-subharmonic functions and an L^1_f-uniqueness theorem for f-heat equations when f has at most linear growth. We also obtain eigenvalues estimates and f-Green's function estimates for the f-Laplace operator.

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1 Introduction

This is a sequel to our earlier work [49], we investigate heat kernel estimates on smooth metric measure spaces. For Riemannian manifolds, there are two classical methods for heat kernel estimates. One is the gradient estimate technique developed by Li and Yau [27], which they used to derive two-sided Gaussian bounds for the heat kernel on Riemannian manifolds with Ricci curvature bounded below. The other is the

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Moser iteration technique invented by Moser [32]. Grigor'yan [18] and Saloff-Coste [41–43] developed this technique and derived heat kernel estimates on Riemannian manifolds satisfying the volume doubling property and the Poincaré inequality. There has been lots of work on improving heat kernel estimates on Riemannian manifolds and generalizing heat kernel estimates to general spaces' see the excellent surveys [19,20,43] and references therein.

In this paper we will investigate heat kernel estimates on smooth metric measure spaces and various applications. Let (M, g) be an n-dimensional complete Riemannian manifold, and let f be a smooth function on M. The triple $(M, g, e^{-f}dv)$ is called a complete smooth metric measure space, where dv is the volume element of g, and $e^{-f}dv$ (for short, $d\mu$) is called the weighted volume element or the weighted measure. On a smooth metric measure space, the m-Bakry–Émery Ricci curvature [2,29,40] is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,$$

where Ric is the Ricci curvature of (M, g), ∇^2 is the Hessian with respect to g, and $m \in \mathbb{R} \cup \{\pm\infty\}$ (when $m = 0$ we require f to be a constant). m-Bakry–Émery Ricci curvature is a natural generalization of Ricci curvature on Riemannian manifolds, see [2,3,29,30,46] and references therein. In particular, a smooth metric measure space satisfying

$$\mathrm{Ric}_f^m = \lambda g$$

for some $\lambda \in \mathbb{R}$ is called an m-quasi-Einstein manifold (see [8]), which can be considered as a natural generalization of an Einstein manifold. When $0 < m < \infty$, $(M^n \times F^m, g_M + e^{-2\frac{f}{m}} g_F)$, with (F^m, g_F) an Einstein manifold, is a warped product Einstein manifold. When $m = 2 - n$, (M^n, g) is a conformally Einstein manifold; in fact $\bar{g} = e^{\frac{f}{(n-2)}} g$ is the Einstein metric. When $m = 1$, (M^n, g) is the so-called static manifold in general relativity. When $m = \infty$, we write

$$\mathrm{Ric}_f = \mathrm{Ric}_f^\infty,$$

and the quasi-Einstein equation reduces to a gradient Ricci soliton. The gradient Ricci soliton is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Ricci solitons play an important role in the Ricci flow and Perelman's resolution of the Poincaré conjecture and the geometrization conjecture; see [6,22] and references therein for nice surveys.

On a smooth metric measure space $(M, g, e^{-f}dv)$, the f-Laplacian Δ_f is defined as

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to $e^{-f}dv$. The f-heat equation is defined as

$$(\partial_t - \Delta_f)u = 0.$$

We denote the f-heat kernel by $H(x, y, t)$, that is, for each $y \in M$, $H(x, y, t) = u(x, t)$ is the minimal positive solution of the f-heat equation satisfying the initial condition $\lim_{t\to 0} u(x, t) = \delta_{f,y}(x)$, where $\delta_{f,y}(x)$ is the f-delta function defined by

$$\int_M \phi(x)\delta_{f,y}(x)e^{-f}dv = \phi(y)$$

for any $\phi \in C_0^\infty(M)$. Similarly a function u is said to be f-harmonic if $\Delta_f u = 0$, and f-subharmonic (f-superharmonic) if $\Delta_f u \ge 0$ ($\Delta_f u \le 0$). It is easy to see that the absolute value of an f-harmonic function is a nonnegative f-subharmonic function. The weighted L^p-norm (or L^p_f-norm) is defined as

$$\|u\|_p = \left(\int_M |u|^p e^{-f}dv\right)^{1/p}$$

for any $0 < p < \infty$. We say that u is L^p_f-integrable, i.e. $u \in L^p_f$, if $\|u\|_p < \infty$.

Recall that for Riemannian manifolds, using the classical Bochner formula, Li and Yau [27] derived the gradient estimate and heat kernel estimate. For smooth metric measure spaces with $m < \infty$, there is an analogue of the Bochner formula for Ric_f^m,

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla u|^2 &= |\nabla^2 u|^2 + \langle\nabla\Delta_f u, \nabla u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m}|\langle\nabla f, \nabla u\rangle|^2\\ &\ge \frac{(\Delta_f u)^2}{m+n} + \langle\nabla\Delta_f u, \nabla u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u).\end{aligned} \tag{1.1}$$

Therefore when $m < \infty$, the Bochner formula for Ric_f^m can be considered as the Bochner formula for the Ricci tensor of an $(n+m)$-dimensional manifold, and for smooth metric measure spaces with Ric_f^m bounded below, one has nice f-mean curvature comparison and f-volume comparison theorems which are similar to classical ones for Riemannian manifolds (see [3,45]); in particular the comparison theorems do not depend on f. Li [27] derived an analogue of the Li–Yau estimate, which he used to f-heat kernel estimates and several Liouville theorems. Charalambous and Lu [9] obtained f-heat kernel estimates and essential spectrum by analyzing a family of warped product manifolds.

Unfortunately when $m = \infty$, due to the lack of the extra term $\frac{1}{m}|\langle\nabla f, \nabla u\rangle|^2$ in the Bochner formula (1.1), one can derive only local f-mean curvature comparison and local f-volume comparison (see [46]), which highly rely on the potential function f, and this makes it much more difficult to investigate smooth metric measure spaces with Ric_f bounded below. According to [35,36], there seem to be essential obstacles to deriving Li–Yau gradient estimate directly using the Bochner formua (1.1), even with strong growth assumptions on f. It is interesting to point out that for f-harmonic functions, Munteanu and Wang [35,36] obtained Yau's gradient estimate using both Yau's idea and the De Giorgi–Nash–Moser theory, under appropriate assumptions on f.

In this paper, without any assumption on f, we derive a Harnack inequality for positive solutions of the f-heat equation, and local Gaussian bounds for the f-heat kernel on smooth metric measure spaces using the De Giorgi–Nash–Moser theory.

Moreover, similar to [35,36], in each step one needs to figure out the accurate coefficients, which play key roles in the applications. As applications, we prove a Liouville theorem for f-subharmonic functions, eigenvalues estimates for the f-Laplacian, and f-Green's functions estimates.

Let us first state the local f-heat kernel estimates,

Theorem 1.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$. For any point $o \in M$ and $R > 0$, denote*

$$A(R) = \sup_{x \in B_o(3R)} |f(x)|, \quad A'(R) = \sup_{x \in B_o(3R)} |\nabla f(x)|.$$

Then for any $\epsilon > 0$, there exist constants $c_1(n, \epsilon)$, $c_i(n)$, $2 \leq i \leq 6$ such that

$$\begin{aligned} &\frac{c_1\, e^{c_2 A + c_3(1+A)\sqrt{Kt}}}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \exp\left(-\frac{d^2(x,y)}{(4+\epsilon)t}\right) \\ &\quad \geq H(x,y,t) \geq \frac{c_4 e^{-c_5(A'^2+K)t}}{V_f(B_x(\sqrt{t}))} \exp\left(-\frac{d^2(x,y)}{c_6 t}\right) \end{aligned} \tag{1.2}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$. $\lim_{\epsilon \to 0} c_1(n, \epsilon) = \infty$.

When f is bounded, the first named author [48] obtained f-heat kernel upper and lower bounds estimates. When $\mathrm{Ric}_f \geq 0$, the authors [49] obtained f-heat kernel upper bound estimates without assumptions on f.

It is interesting to point out that when $\mathrm{Ric}_f \geq 0$, both lower bound and upper bound estimates are sharp. For the lower bound, let $(\mathbb{R}, g_0, e^{-f}dx)$ be a 1-dimensional steady Gaussian soliton, where g_0 is the Euclidean metric and $f(x) = \pm x$. The f-heat kernel is given by (see [49])

$$H(x,y,t) = \frac{e^{\pm\frac{x+y}{2}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Obviously, the lower bound estimate is achieved by the above f-heat kernel for steady Gaussian soliton as long as t is large enough. For the upper bound estimate, when $\mathrm{Ric}_f \geq 0$, the authors [49] proved a sharp L^1_f-Liouville theorem for f-subharmonic functions using the f-heat kernel upper bound. If one improved the upper bound, then we would improve the (sharp) Liouville theorem.

Remark 1.2 The factor A' in the lower bound estimate comes from the Harnack inequality in Theorem 1.3. It will be more interesting to derive a sharp lower bound in terms of A instead of A', if possible.

The proof of upper bound on the f-heat kernel uses a weighted mean value inequality and Davies's integral estimate [15]. The proof of lower bound follows from a Harnack inequality and a chaining argument, while the proof of the Harnack inequality follows from the arguments in [42,43].

To state the Harnack inequality, let us first introduce some notations. For any point $x \in M$ and $r > 0$, $s \in \mathbb{R}$, and $0 < \varepsilon < \eta < \delta < 1$, we denote $B = B_x(r)$, $\delta B = B_x(\delta r)$ and

$$Q = B \times (s - r^2, s), \quad Q_- = \delta B \times (s - \delta r^2, s - \eta r^2), \quad Q_+ = \delta B \times (s - \varepsilon r^2, s).$$

Theorem 1.3 *Let $(M, g, e^{-f} dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$. Let u be a positive solution to the f-heat equation in Q, there exist constants c_1 and c_2 depending on n, ε, η and δ, such that*

$$\sup_{Q_-} u \leq c_1 e^{c_2 (A'^2 + K) r^2} \inf_{Q_+} u,$$

where $A'(r) = \sup_{y \in B_x(3r)} |\nabla f(y)|$.

By a different volume comparison, we get another form of the Harnack inequality and lower bound on the f-heat kernel.

Theorem 1.4 *Under the assumptions of Theorems 1.1 and 1.3, respectively, we have*

$$\sup_{Q_-}\{u\} \leq \exp\{c_1 e^{c_2 A} [(1 + A^2) K r^2 + 1]\} \cdot \inf_{Q_+}\{u\},$$

where $A = A(r) = \sup_{y \in B_x(3r)} |f(y)|$, and

$$H(x, y, t) \geq \frac{c_4}{V_f(B_x(\sqrt{t}))} \times \exp\left[-c_5 e^{c_6 A} \left((1 + A^2) K t + 1 + \frac{d^2(x, y)}{t}\right)\right], \tag{1.3}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$, where $A = A(R) = \sup_{x \in B_o(3R)} |f(x)|$. In particular, when f is bounded, we get

$$H(x, y, t) \geq \frac{c_1 e^{-c_2 K t}}{V_f(B_x(\sqrt{t}))} \times \exp\left(-\frac{d^2(x, y)}{c_3 t}\right). \tag{1.4}$$

Next we derive several applications of the f-heat kernel estimates. First we prove a Liouville theorem for f-subharmonic functions. Recall that Pigola et al. [39] proved that any nonnegative L^1_f-integrable f-superharmonic function must be constant if Ric_f is bounded below, without any assumption on f. However, as the authors proved in [49], for f-subharmonic functions, the condition on f is necessary. In fact we provided [49] explicit counterexamples illustrating that f cannot grow faster than quadratically when $\mathrm{Ric}_f \geq 0$ (see also [10]). Now we show that the L^1_f-Liouville theorem also holds for f-subharmonic functions when $\mathrm{Ric}_f \geq -(n-1)K$ and f has at most linear growth.

Theorem 1.5 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K > 0$. Assume there exist nonnegative constants a and b such that*

$$|f|(x) \leq ar(x) + b,$$

where $r(x)$ is the distance function to a fixed point $o \in M$. Then any nonnegative L^1_f-integrable f-subharmonic function must be identically constant. In particular, any L^1_f-integrable f-harmonic function must be identically constant.

There have been various Liouville type theorems for f-subharmonic and f-harmonic functions on smooth metric measure spaces and gradient Ricci solitons under different conditions; see Brighton [4], Cao and Zhou [7], Munteanu and Sesum [34], Munteanu and Wang [35,36], Petersen and Wylie [38], and Wei and Wylie [46] for details.

By a similar argument to [24] (see also [49]), we also prove an L^1_f-uniqueness theorem for solutions of the f-heat equation, see Theorem 5.3 in Sect. 5.

Second we derive lower bounds for eigenvalues of the f-Laplace operator on compact smooth metric measure spaces, by adapting the classical argument of Li and Yau [27],

Theorem 1.6 *Let $(M, g, e^{-f}dv)$ be an n-dimensional compact smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of the f-Laplacian Δ_f. Then there exists a constant C depending only on n and $A = \max_{x \in M} f(x)$, such that*

$$\lambda_k \geq \frac{C(k+1)^{2/n}}{d^2}, \quad K = 0,$$
$$\lambda_k \geq \frac{C}{d^2}\left(\frac{k+1}{\exp(C\sqrt{K}d)}\right)^{\frac{2}{n+4A}}, \quad K > 0,$$

for all $k \geq 1$, where d is the diameter of M.

Upper bounds were proved by Hassannezhad [23], and Colbois et al. [13], they depend on norms of the potential function and the conformal class of the metric. For the first eigenvalue, there have been more interesting results. When M is compact and $\mathrm{Ric}_f \geq \frac{a}{2} > 0$, Andrews and Ni [1], and Futaki et al. [17] derived lower bounds on the first eigenvalue, which depend on the diameter of the manifold. When M is complete noncompact, Munteanu and Wang [35–37], and Wu [47] obtained first eigenvalue estimates under appropriate assumptions on f. Cheng and Zhou [12] proved an interesting Obata type theorem.

Finally we discuss f-Green's functions estimates. We first get upper and lower estimates for f-Green's functions when f is bounded, similar to the classical estimates of Li and Yau [27] for Riemannian manifolds.Recall that the f-Green's function on $(M, g, e^{-f}dv)$ is defined as

$$G(x, y) = \int_0^\infty H(x, y, t)dt$$

if the integral on the right hand side converges. It is easy to check that G is positive and satisfies

$$\Delta_f G = -\delta_{f,y}(x).$$

Theorem 1.7 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq 0$ and f bounded. If $G(x, y)$ exists, then there exist constants c_1 and c_2 depending only on n and $\sup f$, such that*

$$c_1 \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t}))dt \leq G(x, y) \leq c_2 \int_{r^2}^\infty V_f^{-1}(B_x(\sqrt{t}))dt, \tag{1.5}$$

where $r = r(x, y)$.

Recently Dai et al. [14] observed that every gradient steady Ricci soliton admits a positive f-Green's function, hence it is f-nonparabolic. We provide an alternative proof using a criterion of Li and Tam [25,26], and the f-heat kernel for steady Gaussian Ricci solitons,

Theorem 1.8 *Let (M^n, g, f) be a complete gradient steady soliton. Then there exists a positive smooth f-Green function, and therefore the gradient steady soliton is f-nonparabolic.*

In [44], Song et al. investigated several properties of f-Green's functions on smooth metric measure spaces. Pigola et al. [39] proved that gradient shrinking Ricci solitons are f-parabolic.

The paper is organized as follows. In Sect. 2, we recall comparison theorems for the Bakry–Émery Ricci curvature bounded below, which we use to derive a local f-volume doubling property, a local f-Neumann Poincaré inequality, a local Sobolev inequality and mean value inequalities for the f-heat equation. In Sect. 3, we prove a Moser's Harnack inequality of f-heat equation following the arguments of Saloff-Coste [42,43]. In Sect. 4, we prove local Gaussian upper and lower bounds on the f-heat kernel. In Sect. 5, following the arguments of [49], we establish a new L^1_f-Liouville theorem for an f-harmonic function and a new L^1_f-uniqueness property for nonnegative solutions of the f-heat equation. In Sect. 6, we apply upper bounds of the f-heat kernel to get the eigenvalue estimates of the f-Laplacian on compact smooth metric measure spaces. In Sect. 7, we derive Green function estimates for smooth metric measure spaces with $\mathrm{Ric}_f \geq 0$ and f bounded, and for gradient steady Ricci solitons.

2 Poincaré, Sobolev and mean value inequalities

Recall that for any point $p \in M$ and $R > 0$, we denote

$$A(R) = A(p, R) = \sup_{x \in B_p(3R)} |f(x)|, \quad A'(R) = A'(p, R) = \sup_{x \in B_p(3R)} |\nabla f(x)|.$$

When there is no confusion we write A, A' for short. We start from a relative f-volume comparison theorem of Wei and Wylie [46].

Lemma 2.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. If $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$, then*

$$\frac{V_f(B_x(R_1, R_2))}{V_f(B_x(r_1, r_2))} \leq \frac{V_K^{n+4A}(B_x(R_1, R_2))}{V_K^{n+4A}(B_x(r_1, r_2))} \tag{2.1}$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \leq R_1$, $r_2 \leq R_2$, where $B_x(R_1, R_2) = B_x(R_2)\backslash B_x(R_1)$, and $A = A(x, \frac{1}{3}R_2)$. Here $V_K^{n+4A}(B_x(r))$ denotes the volume of the ball in the model space M_K^{n+4A}, i.e., the simply connected space form with constant sectional curvature $-K$ and dimension $n + 4A$. Similarly we have

$$\frac{V_f(B_x(R_1, R_2))}{V_f(B_x(r_1, r_2))} \leq \frac{V_K^{n+4A'R_2}(B_x(R_1, R_2))}{V_K^{n+4A'R_2}(B_x(r_1, r_2))}, \tag{2.2}$$

where $A' = A'(x, \frac{1}{3}R_2)$.

Remark 2.2 Following the proofs, $A(R)$ in all following lemmas, propositions, theorems and corollaries can be replaced by $RA'(R)$. We will apply the first volume comparison (2.1) to derive heat kernel upper bound, and the second volume comparison (2.2) to derive Harnack inequality and heat kernel lower bound.

Proof of Lemma 2.1 Applying the weighted Bochner formula (1.1) and an ODE argument, Wei and Wylie (see (3.19) in [46]) proved the following f-mean curvature comparison theorem. Recall that the weighted mean curvature $m_f(r)$ is defined as

$$m_f(r) = m(r) - \nabla f \cdot \nabla r = \Delta_f\, r.$$

If $Ric_f \geq -(n-1)K$, then

$$\begin{aligned} m_f(r) &\leq (n-1)\sqrt{K}\coth(\sqrt{K}\,r) + \frac{2K}{\sinh^2(\sqrt{K}\,r)}\int_0^r (f(t) - f(r))\cosh(2\sqrt{K}\,t)dt \\ &\leq (n-1+4A)\sqrt{K}\cdot\coth(\sqrt{K}\,r) \end{aligned} \tag{2.3}$$

along any minimal geodesic segment from x. In geodesic polar coordinates, the volume element is written as

$$dv = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1},$$

where $d\theta_{n-1}$ is the standard volume element of the unit sphere S^{n-1}. Let

$$\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta).$$

By the first variation of the area,

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = (\ln(\mathcal{A}(r,\theta)))' = m(r,\theta).$$

Therefore

$$\frac{\mathcal{A}'{}_f}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta).$$

So for $r < R$,

$$\frac{\mathcal{A}_f(R,\theta)}{\mathcal{A}_f(r,\theta)} \le \frac{\mathcal{A}_K^{n+4A}(R)}{\mathcal{A}_K^{n+4A}(r)}.$$

That is $\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_K^{n+4A}(r)}$ is nonincreasing in r, where $\mathcal{A}_K^{n+4A}(r)$ is the volume element in the simply connected hyperbolic space of constant sectional curvature $-K$ and dimension $n+4A$. Applying Lemma 3.2 in [51], we get

$$\frac{\int_{R_1}^{R_2} \mathcal{A}_f(R,\theta)dt}{\int_{r_1}^{r_2} \mathcal{A}_f(r,\theta)dt} \le \frac{\int_{R_1}^{R_2} \mathcal{A}_K^{n+4A}(R,\theta)dt}{\int_{r_1}^{r_2} \mathcal{A}_K^{n+4A}(r,\theta)dt}$$

for $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \le R_1$ and $r_2 \le R_2$. Integrating along the sphere direction proves (2.1).

The second volume comparison (2.2) follows from an observation for the weighted mean curvature,

$$\begin{aligned} m_f(r) &\le (n-1)\sqrt{K}\coth(\sqrt{K}\,r) + \frac{2K}{\sinh^2(\sqrt{K}\,r)}\int_0^r (f(t)-f(r))\cosh(2\sqrt{K}\,t)dt \\ &\le (n-1+4A'r)\sqrt{K}\cdot\coth(\sqrt{K}\,r). \end{aligned} \tag{2.4}$$

□

Let $V_K^{n+4A}(B_x(r))$ be the volume of the ball of radius r in the simply connected hyperbolic space of constant sectional curvature $-K$ and dimension $n+4A$. If $K > 0$, the model space is the hyperbolic space. If $K = 0$, the model space is the Euclidean space. In any case, we have the estimate

$$\omega_{n+4A}\cdot r^{n+4A} \le V_K(B_x(r)) \le \omega_{n+4A}\cdot r^{n+4A}e^{(n-1+4A)\sqrt{K}\,r} \tag{2.5}$$

where ω_{n+4A} is the volume of the unit ball in $(n+4A)$-dimensional Euclidean space.

Similar to [49], Lemma 2.1 implies a local f-volume doubling property. Indeed, in (2.1), letting $r_1 = R_1 = 0$, $r_2 = r$ and $R_2 = 2r$, from (2.5) we get

$$V_f(B_x(2r)) \le 2^{n+4A}e^{2(n-1+4A)\sqrt{K}\,r}\cdot V_f(B_x(r)). \tag{2.6}$$

This local f-volume doubling property is crucial in our proof of Poincaré inequality, Sobolev inequality, mean-value inequality, and Harnack inequality.

From Lemma 2.1, we also have the following,

Lemma 2.3 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space. If* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant* $K > 0$*, then*

$$V_f(B_x(r)) \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+r)}}{r^{n+4A}} V_f(B_y(r)),$$

where $A = A(y, d(x, y) + r)$.

Proof We let $r_1 = 0$, $r_2 = r$, $R_1 = d(x, y) - r$ and $R_2 = d(x, y) + r$ in Lemma 2.1. Then using (2.5) we have

$$\frac{V_f(B_y(d(x, y) + r)) - V_f(B_y(d(x, y) - r))}{V_f(B_y(r))} \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+r)}}{r^{n+4A}}.$$

Therefore we get

$$\begin{aligned} V_f(B_x(r)) &\leq V_f(B_y(d(x, y) + r)) - V_f(B_y(d(x, y) - r)) \\ &\leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+r)}}{r^{n+4A}} V_f(B_y(r)). \end{aligned}$$

□

Following the argument of [5] (see also [43] or [35]), applying Lemma 2.1 we get a local Neumann Poincaré inequality on complete smooth metric measure spaces.

Lemma 2.4 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space with* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant* $K \geq 0$*. Then,*

$$\int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 d\mu \leq c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r} \cdot r^2 \int_{B_x(r)} |\nabla\varphi|^2 d\mu \tag{2.7}$$

for any $\varphi \in C^\infty(B_x(r))$*, where* $\varphi_{B_x(r)} = \int_{B_x(r)} \varphi d\mu / \int_{B_x(r)} d\mu$.

Remark 2.5 By Remark 2.2, the coefficient $c_2 A + c_3(1 + A)\sqrt{K}r$ in Lemma 2.4 and all following lemmas, propositions, theorems, and corollaries, can be replaced by $c_2(A' + \sqrt{K})r + c_3 A'\sqrt{K}r^2$.

Combining Lemmas 2.1 and 2.4 and the argument of [21] (see also [49]), we obtain a local Sobolev inequality on smooth metric measure spaces.

Lemma 2.6 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space with* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant* $K \geq 0$*. Then there exists* $\nu > 2$*, such that*

$$\left(\int_{B_x(r)} |\varphi|^{\frac{2\nu}{\nu-2}} d\mu\right)^{\frac{\nu-2}{\nu}} \leq \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r} \cdot r^2}{V_f(B_x(r))^{\frac{2}{\nu}}} \int_{B_x(r)} (|\nabla\varphi|^2 + r^{-2}|\varphi|^2) d\mu \tag{2.8}$$

for any $\varphi \in C^\infty(B_x(r))$.

Applying Lemma 2.6 we obtain a mean value inequality for solutions to the f-heat equation, which is similar to Theorem 5.2.9 in [43] (see also [49]).

Proposition 2.7 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. Assume (2.8) holds. Fix $0 < p < \infty$. There exist constants $c_1(n, p, \nu)$, $c_2(n, p, \nu)$ and $c_3(n, p, \nu)$ such that for any $s \in \mathbb{R}$ and $0 < \delta < 1$, any smooth positive subsolution u of the f-heat equation in the cylinder $Q = B_x(r) \times (s - r^2, s)$ satisfies*

$$\sup_{Q_\delta}\{u^p\} \le \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r}}{(1-\delta)^{2+\nu}\, r^2\, V_f(B_x(r))} \cdot \int_Q u^p\, d\mu\, dt,$$

where $Q_\delta = B_x(\delta r) \times (s - \delta r^2, s)$.

Similar to Proposition 2.7, we have

Proposition 2.8 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. Assume (2.8) holds. Fix $0 < p_0 < 1 + \nu/2$. There exist constants $c_1(n, p_0, \nu)$, $c_2(n, p_0, \nu)$ and $c_3(n, p_0, \nu)$ such that for any $s \in \mathbb{R}$, $0 < \delta < 1$, and $0 < p \le p_0$, any smooth positive supersolution u of the f-heat equation in the cylinder $Q = B_x(r) \times (s - r^2, s)$ satisfies*

$$\|u\|^p{}_{p_0, Q'_\delta} \le \left\{\frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r}}{(1-\delta)^{2+\nu}\, r^2\, V_f(B_x(r))}\right\}^{1-p/p_0} \cdot \|u\|^p{}_{p,Q},$$

where $Q'_\delta := B_x(\delta r) \times (s - r^2, s - (1-\delta)r^2)$. On the other hand, for any $0 < p < \bar{p} < \infty$, there exist constants $c_4(n, \bar{p}, \nu)$, $c_5(n, \bar{p}, \nu)$ and $c_6(n, \bar{p}, \nu)$ such that

$$\sup_{Q_\delta}\{u^{-p}\} \le \frac{c_4 e^{c_5 A + c_6(1+A)\sqrt{K}r}}{(1-\delta)^{2+\nu}\, r^2\, V_f(B_x(r))} \cdot \|u^{-1}\|^p{}_{p,Q},$$

where $||u||_{p,Q} = \left(\int_Q |u(x,t)|^p d\mu dt\right)^{1/p}$.

Proof of Proposition 2.8 For any nonnegative test function $\phi \in C_0^\infty(B)$ and any supersolution of the heat equation, we have

$$\int_B (\phi\partial_t u + \nabla\phi\nabla u)d\mu \ge 0.$$

Let $\phi = \epsilon q u^{q-1}\psi^2$, $w = u^{q/2}$ for $-\infty < q \le p(1+\nu/2)^{-1} < 1$ and $q \ne 0$, where $\epsilon = 1$ if $q > 0$ and $\epsilon = -1$ if $q < 0$. We get

$$\epsilon \int_B (\psi^2\partial_t w^2 + 4(1 - 1/q)\psi^2|\nabla w|^2 + 4w\psi\langle\nabla w, \nabla\psi\rangle)d\mu \ge 0.$$

When $q > 0$. Since

$$2w\psi\langle\nabla w, \nabla\psi\rangle \geq -a^{-2}\psi^2|\nabla w|^2 - a^2w^2|\nabla\psi|^2$$

for any $a > 0$, we get

$$-\int_B \psi^2\partial_t(w^2)d\mu + c_1\int_B |\nabla(\psi w)|^2 d\mu \leq c_2\|\nabla\psi\|_\infty^2 \int_{\mathrm{supp}(\psi)} w^2 d\mu,$$

where c_1 and c_2 depend only on q. Multiplying a nonnegative smooth function $\lambda(t)$, we have

$$-\partial_t\int_B \lambda^2\psi^2w^2d\mu + c_1\lambda^2\int_B |\nabla(\psi w)|^2 d\mu \leq c_3\lambda(\lambda\|\nabla\psi\|_\infty^2 + \|\psi\lambda'\|_\infty)\int_B w^2 d\mu.$$

Choose ψ and λ such that

$$\begin{aligned}
&0 \leq \psi \leq 1, \quad \mathrm{supp}\psi \subset \sigma B, \quad \psi = 1 \ \text{ in } \sigma' B, \quad |\nabla\psi| \leq (\kappa r)^{-1},\\
&0 \leq \lambda \leq 1, \quad \lambda = 1 \ \text{ in } (-\infty, s - \sigma r^2], \quad \lambda = 0 \ \text{ in } [s - \sigma' r^2, \infty), \quad |\lambda'| \leq (\kappa r^2)^{-1},
\end{aligned}$$

where $0 < \sigma' < \sigma < 1$, $\kappa = \sigma - \sigma'$. Let $I_\sigma = [s - \sigma r^2, s]$, and integrate the above inequality on $[s - r^2, t]$ for $t \in I_{\sigma'}$. We get

$$\sup_{I_{\sigma'}}\int_{\sigma' B} w^2 d\mu + c_1\int_{Q_{\sigma'}} |\nabla w|^2 d\mu dt \leq c_4(\kappa r)^{-2}\int_{Q_\sigma} w^2 d\mu dt.$$

By Hölder inequality and Proposition 2.6, for any $\phi \in C_0^\infty(B)$, we get

$$\begin{aligned}
\int_B \phi^{2(1+2/\nu)}d\mu &\leq \left(\int_B \phi^{2\nu/(\nu-2)}d\mu\right)^{(\nu-2)/\nu}\left(\int_B \phi^2 d\mu\right)^{2/\nu}\\
&\leq C(B)\left(\int_B (|\nabla\phi|^2 + r^{-2}\phi^2)d\mu\right)\left(\int_B \phi^2 d\mu\right)^{2/\nu},
\end{aligned}$$

where $C(B) := c_1 e^{c_2 A + c_3(1+A)\sqrt{K}r}\, r^2 V_f^{-2/\nu}$. Therefore

$$\int_{Q_{\sigma'}} u^{q\theta} d\mu dt \leq c_3 C(B)\left((r\kappa)^{-2}\int_{Q_\sigma} u^q d\mu dt\right)^\theta, \tag{2.9}$$

where $\theta = 1 + 2/\nu$. Let $p_i = p_0\theta^{-i}$, notice that by Hölder inequality, for any $p_i < p = \eta p_i + (1-\eta)p_{i-1} \leq p_{i-1}$ with $0 \leq \eta < 1$,

$$\|u\|_p^p \leq \|u\|_{p_i}^{\eta p_i}\|u\|_{p_{i-1}}^{(1-\eta)p_{i-1}},$$

so it suffices to prove the estimate for all p_i.

Fix i, and let $q_j = p_i\theta^j$, $1 \le j \le i-1$, so $0 < q_j < p_0(1+\nu/2)^{-1}$. Let $\sigma_0 = 1$, $\sigma_i = \sigma_{i-1} - \kappa_i$, where $\kappa_i = (1-\delta)2^{-i}$, so $\sigma_i = 1 - (1-\delta)\sum_1^i 2^{-j} > \delta$. Plugging into inequality (2.9), we get

$$\int_{Q'_{\sigma_j}} u^{q_0\theta^j} d\mu dt \le c_4^j C(B)\left((1-\delta)^{-2} r^{-2} \int_{Q'_{\sigma_{j-1}}} u^{q_0\theta^{j-1}} d\mu dt\right)^{\theta},$$

for $1 \le j \le i$. Therefore

$$\int_{Q'_{\sigma_i}} u^{p_0} d\mu dt \le c_4^{\sum(i-j)\theta^{j+1}} C(B)^{\sum\theta^j} [(1-\delta)r]^{-2\sum\theta^{j+1}} \left(\int_Q u^{p_i} d\mu dt\right)^{\theta^i},$$

where the summation is taken from 0 to $i-1$. Therefore we obtain

$$\left(\int_{Q'_{\sigma_i}} u^{p_0} d\mu dt\right)^{p_i/p_0} \le [c_5(1-\delta)^{-2-\nu} E(B)]^{1-p_i/p_0} \left(\int_Q u^{p_i} d\mu dt\right),$$

where $E(B) = C(B)^{\nu/2} r^{-2-\nu}$.

When $q < 0$. We get

$$\int_B (\psi^2 \partial_t w^2 + 4(1-1/q)\psi^2|\nabla w|^2 + 4w\psi\langle\nabla w, \nabla\psi\rangle) d\mu \le 0.$$

Applying the mean value inequality to the last term, we get similarly

$$\int_B \psi^2 \partial_t(w^2) d\mu + c_6 \int_B |\nabla(\psi w)|^2 e^{-f} dv \le c_7 \|\nabla\psi\|_\infty^2 \int_{\mathrm{supp}(\psi)} w^2 d\mu.$$

By the above argument, we can obtain

$$\int_{Q_{\sigma'}} w^{2\theta} d\mu dt \le c_8 C(B)\left((r\kappa)^{-2} \int_{Q_\sigma} w^2 d\mu dt\right)^{\theta},$$

where $\theta = 1 + 2/\nu$. For any $\alpha > 1$, $v = u^\alpha$ satisfies

$$\partial_t v - \Delta_f v \ge -\frac{\alpha-1}{\alpha} v^{-1}|\nabla v|^2,$$

applying the above argument again, we also have

$$\int_{Q_{\sigma'}} w^{2\alpha\theta} d\mu dt \le c_9 C(B)\left((r\kappa)^{-2} \int_{Q_\sigma} w^{2\alpha} d\mu dt\right)^{\theta}.$$

Let $\kappa_i = (1-\delta)2^{-i-1}$, and $\sigma_0 = 1$, $\sigma_i = \sigma_{i-1} - \kappa_i = 1 - \sum_1^i \kappa_j$, and $\alpha_i = \theta^i$. We get

$$\left(\int_{Q_{\sigma_{i+1}}} w^{2\theta^{i+1}} d\mu dt\right)^{\theta^{-i-1}} \le C(B)\left(c_{10}^{i+1}[(1-\delta)r]^{-2}\int_{Q_{\sigma_i}} w^{2\theta^i} d\mu dt\right)^{\theta}$$
$$\le C(B)^{\sum\theta^{-j-1}} c_{10}^{\sum(j+1)\theta^{-j-1}}[(1-\delta)r]^{-2\sum\theta^{-j}}\int_Q w^2 d\mu dt,$$

where the summation is from 1 to $i+1$. Therefore when $i \to \infty$, we get

$$\sup_{Q_\delta} w^2 \le c_5 C(B)^{\nu/2}[(1-\delta)r]^{-2-\nu}\|w\|_{2,Q}^2$$

and the conclusion follows. □

3 Moser's Harnack inequality for f-heat equation

In this section we prove Moser's Harnack inequalities for the f-heat equation using Moser iteration, which will lead to the sharp lower bound estimate for the f-heat kernel in the next section. The arguments mainly follow those in [32,33,42,43], while more delicate analysis is required to get the accurate estimates, which depend on the potential function. Throughout this section, we will use the second f-volume comparison, i.e., (2.2) in Sect. 2.

Recall the notations defined in Introduction. for any point $x \in M$ and $r > 0$, $s \in \mathbb{R}$, and $0 < \varepsilon < \eta < \delta < 1$, we denote $B = B_x(r)$, $\delta B = B_x(\delta r)$ and

$$Q = B \times (s-r^2, s), \quad Q_\delta = \delta B \times (s-\delta r^2, s), \quad Q'_\delta = \delta B \times (s-r^2, s-(1-\delta)r^2),$$
$$Q_- = \delta B \times (s-\delta r^2, s-\eta r^2), \quad Q_+ = \delta B \times (s-\varepsilon r^2, s).$$

With the above notations, we have the main result in this section.

Theorem 3.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge -(n-1)K$ for some constant $K \ge 0$. For any point $x \in M$, $r > 0$, and any parameters $0 < \varepsilon < \eta < \delta < 1$, let u be a smooth solution of the f-heat equation in Q, then there exist constants c_1 and c_2 both depending on n, ε, η and δ, such that*

$$\sup_{Q_-} u \le c_1 e^{c_2(A'^2+K)r^2} \inf_{Q_+} u,$$

where $A' = A'(x, r+1)$.

Remark 3.2 The coefficient in Theorem 3.1 comes from the second volume comparison Lemma 2.1. On the other hand, the first volume comparison in Lemma 2.1 leads to another Harnack inequality,

$$\sup_{Q_-}\{u\} \le \exp\{c_1 e^{c_2 A}[(1+A^2)Kr^2+1]\} \cdot \inf_{Q_+}\{u\}.$$

Since its proof is very similar to that of Theorem 3.1, we omit the proof here.

We first modify the f-Poincaré inequality (2.7) in Sect. 2 to a weighted version, which can be derived by adapting a Whitney-type covering argument, see Sections 5.3.3–5.3.5 in [43],

Let $\xi : [0,\infty) \to [0,1]$ be a non-increasing function such that $\xi(t) = 0$ for $t > 1$, and for some positive constant β

$$\xi\left(t + \frac{1-t}{2}\right) \ge \beta\xi(t), \quad 1/2 \le t \le 1.$$

Let $\Psi_B(z) := \xi(\rho(x,z)/r)$ for $z \in B = B(x,r)$ and $\Psi_B(z) = 0$ for $z \in M\backslash B$, we write $\Psi(z)$ for short. Then

Lemma 3.3 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge -(n-1)K$ for some constant $K \ge 0$. There exist constants $c_1(n,\xi)$, $c_2(n)$ and $c_3(n)$ such that, for any $B_x(r) \subset M$, we have*

$$\int_{B_x(r)} |\varphi - \varphi_\Psi|^2 \Psi d\mu \le c_1 e^{c_2(A'+\sqrt{K})r + c_3 A'\sqrt{K}r^2} \cdot r^2 \int_{B_x(r)} |\nabla\varphi|^2 \Psi d\mu \qquad (3.1)$$

for all $\varphi \in C^\infty(B_x(r))$, where $\varphi_\Psi = \int_B \varphi\Psi d\mu / \int_B \Psi d\mu$.

Secondly, for a positive solution u to the f-heat equation, we derive an estimate for the level set of $\log u$, the proof of which depends on Lemma 3.3. This inequality is important for the iteration arguments in Lemma 3.5. In the following, we denote $d\bar{\mu} = d\mu \times dt$ by the natural product measure on $M \times \mathbb{R}$.

Lemma 3.4 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. Assume that (2.6) and (2.7) hold in $B_x(r)$. Fix $s \in \mathbb{R}$, $\delta, \tau \in (0,1)$. For any smooth positive solution u of the f-heat equation in $Q = B_x(r) \times (s - r^2, s)$, there exists a constant $c = c(u)$ depending on u such that for all $\lambda > 0$,*

$$\bar{\mu}(\{(z,t) \in R_+ | \log u < -\lambda - c\}) \le C_0 \lambda^{-1},$$
$$\bar{\mu}(\{(z,t) \in R_- | \log u > \lambda - c\}) \le C_0 \lambda^{-1},$$

where $C_0 = c_1(n,\delta,\tau)e^{c_2(A'+\sqrt{K})r + c_3 A'\sqrt{K}r^2} V_f(B)r^2$. Here $R_+ = \delta B \times (s - \tau r^2, s)$ and $R_- = \delta B \times (s - r^2, s - \tau r^2)$.

Proof By shrinking the ball B a little, we can assume that u is a positive solution in $B_x(r') \times (s - r^2, s)$ for some $r' > r$. Let $\omega = -\log u$. Then for any nonnegative function $\psi \in C_0(B_x(r'))$, we have

$$\partial_t \int \psi^2 \omega d\mu = -\int \psi^2 u^{-1} \Delta_f u d\mu = \int [-\psi^2 |\nabla\omega|^2 - 2\psi \nabla\omega \cdot \nabla\psi] d\mu.$$

By Cauchy–Schwarz inequality $2|ab| \leq 1/2a^2 + 2b^2$, we obtain

$$\partial_t \int \psi^2 \omega d\mu + 1/2 \int^2 |\nabla \omega|^2 d\mu \leq 2||\nabla \psi||_\infty^2 V_f(\mathrm{supp}(\psi)).$$

Fix $0 < \delta < 1$ and define function ξ such that $\xi = 1$ on $[0, \delta]$, $\xi(t) = \frac{1-t}{1-\delta}$ on $[\delta, 1]$ and $\xi = 0$ on $[1, \infty)$. We set $\Psi = \xi(\rho(x, \cdot)/r)$. Clearly, we can apply the above to $\psi = \Psi$. Then Lemma 3.3 can be applied with Ψ^2 as a weight function. Thus, we have

$$\int |\nabla \omega|^2 \Psi^2 d\mu \geq (c_\delta r^2 e^{c_2(A'+\sqrt{K})r + c_3 A' \sqrt{K} r^2})^{-1} \int |\omega - W|^2 \Psi^2 d\mu,$$

where $W := \int \Psi^2 \omega d\mu / \int \Psi^2 d\mu$. Noticing that $\int \Psi^2$ is comparable to V_f, so

$$\partial_t W + C_1^{-1} \int_{\delta B} |\omega - W|^2 \leq C_2,$$

where $C_1 = C(\delta, \tau) e^{c_2(A'+\sqrt{K})r + c_3 A' \sqrt{K} r^2} r^2 V_f$ and $C_2 = C(\delta, \tau) r^{-2}$. Letting $s' = s - \tau r^2$, the above inequality can be written as

$$\partial_t \overline{W} + C_1^{-1} \int_{\delta B} |\overline{\omega} - \overline{W}|^2 \leq 0,$$

where $\overline{\omega}(z, t) = \omega(z, t) - C_2(t - s')$ and $\overline{W}(z, t) = W(z, t) - C_2(t - s')$.

Now we set

$$c = W(s') = \overline{W}(s'),$$

and for $\lambda > 0$, $s - r^2 < t < s$, we define two sets

$$\Omega_t^+(\lambda) = \{z \in \delta B, \bar{\omega}(z, t) > c + \lambda\} \quad \text{and} \quad \Omega_t^-(\lambda) = \{z \in \delta B, \bar{\omega}(z, t) < c - \lambda\}.$$

Then if $t > s'$, we have

$$\overline{\omega}(z, t) - \overline{W}(t) \geq \lambda + c - \overline{W}(t) > \lambda$$

in $\Omega_t^+(\lambda)$, since $c = \overline{W}(s')$ and $\partial_t \overline{W} \leq 0$. Similarly, if $t < s'$, then we have

$$\overline{\omega}(z, t) - \overline{W}(t) \leq -\lambda + c - \overline{W}(s') < -\lambda$$

in $\Omega_t^-(\lambda)$. Hence, if $t > s'$, we obtain

$$\partial_t \overline{W}(t) + C_1^{-1} |\lambda + c - \overline{W}(t)|^2 \mu(\Omega_t^+(\lambda)) \leq 0$$

and namely,

$$-C_1\partial_t(|\lambda + c - \overline{W}(t)|^{-1}) \geq \mu(\Omega_t^+(\lambda)).$$

Integrating from s' to s,

$$\bar{\mu}(\{(z,t) \in R_+, \overline{\omega} > c + \lambda\}) \leq C_1\lambda^{-1}.$$

Recalling that $-\log u = \omega = \overline{\omega} + C_2(t - s')$, hence

$$\bar{\mu}(\{(z,t) \in R_+, \log u < -\lambda - c\}) \leq (\max\{C_1, C_2 r^4 V_f\})\lambda^{-1}.$$

This gives the first estimate of the lemma. The second estimate follows from a similar argument by working with Ω_t^- and $t < s'$. □

Thirdly, in order to finish the proof of Theorem 3.1, we need the following elementary lemma. This is in fact an iterated procedure. We let R_σ, $0 < \sigma \leq 1$ be a collection of subset for some space-time endowed with the measure $d\bar{\mu}$ such that $R_{\sigma'} \subset R_\sigma$ if $\sigma' \leq \sigma$. Indeed, R_σ will be one of the collections Q_δ or Q'_δ.

Lemma 3.5 *Let γ, C, $1/2 \leq \delta < 1$, $p_1 < p_0 \leq \infty$ be positive constants, and let φ be a positive smooth function on R_1 such that*

$$||\varphi||_{p_0,R_{\sigma'}} \leq \{C(\sigma - \sigma')^{-\gamma} V_f^{-1}(R_1)\}^{1/p-1/p_0}||\varphi||_{p,R_\sigma}$$

for all σ, σ', p satisfying $1/2 \leq \delta \leq \sigma' < \sigma \leq 1$ and $0 < p \leq p_1 < p_0$. Besides, if φ also satisfies

$$Vol_f(\{z \in R_1, \ln\varphi > \lambda\}) \leq C V_f(R_1)\lambda^{-1}$$

for all $\lambda > 0$, then we have

$$||\varphi||_{p_0,R_\delta} \leq (V_f(R_1))^{1/p_0} e^{C_1(1+C^3)},$$

where C_1 depends only on γ, δ and a positive lower bound on $1/p_1 - 1/p_0$.

Proof Without loss of generality we may assume that $Vol_f(R_1) = 1$. Let

$$\zeta = \zeta(\sigma) := \ln(||\varphi||_{p_0,R_\sigma}), \quad \delta \leq \sigma < 1.$$

We divide R_σ into two sets: $\{\ln\varphi > \zeta/2\}$ and $\{\ln\varphi \leq \zeta/2\}$. Then

$$\begin{aligned} ||\varphi||_{p,R_\sigma} &\leq ||\varphi||_{p_0,R_\sigma} \cdot V_f(\{z \in R_\sigma, \ln\varphi > \zeta/2\})^{1/p-1/p_0} + e^{\zeta/2} \\ &\leq e^\zeta \left(\frac{2C}{\zeta}\right)^{1/p-1/p_0} + e^{\zeta/2}, \end{aligned}$$

where $p < p_0$. Here in the first inequality we used the Hölder inequality, and in the second inequality we used the second assumption of lemma. In the following we want to choose p such that the last two terms in above are equal, and $0 < p \le p_1$. This is possible if

$$(1/p - 1/p_0)^{-1} = (2/\zeta) \ln \left(\frac{\zeta}{2C} \right) \le (1/p_1 - 1/p_0)^{-1}$$

and the last inequality is satisfied as long as

$$\zeta \ge C_2 C,$$

where C_2 depends only on a positive lower bound on $1/p_1 - 1/p_0$. Now we assume p and ζ have been chosen as above. Then we obtain

$$||\varphi||_{p,R_\sigma} \le 2e^{\zeta/2}.$$

Using the first assumption of the lemma and the definition of κ, we have

$$\begin{aligned} \kappa(\sigma') &\le \ln\{2(C(\sigma - \sigma')^{-\gamma})^{1/p - 1/p_0} e^{\zeta/2}\} \\ &= (1/p - 1/p_0) \ln[C(\sigma - \sigma')^{-\gamma}] + \ln 2 + \zeta/2 \end{aligned}$$

for any $\delta \le \sigma' < \sigma \le 1$. According to our choice of p above, we get

$$\kappa(\sigma') \le \frac{\zeta}{2} \left\{ \frac{\ln[C(\sigma - \sigma')^{-\gamma}]}{\ln(\zeta/C)} + \frac{2\ln 2}{\zeta} + 1 \right\}.$$

Here, on one hand, if we choose

$$\zeta \ge 16C^3(\sigma - \sigma')^{-2\gamma} + 8\ln 2,$$

then the above inequality becomes

$$\zeta(\sigma') \le \frac{3}{4}\zeta.$$

On the other hand, if the assumption of κ above in not satisfied, we can have

$$\zeta(\sigma') \le \zeta(\sigma) \le C_2 C + 16C^3(\sigma - \sigma')^{-2\gamma} + 8\ln 2.$$

Therefore, in any case

$$\zeta(\sigma') \le \frac{3}{4}\zeta(\sigma) + C_3(1 + C^3)(\sigma - \sigma')^{-2\gamma}.$$

for any $\delta \le \sigma' < \sigma \le 1$, where $C_3 = C_2 + 16 + 8\ln 2$. From this, an routine iteration (see [33], p. 733) yields

$$\zeta(\delta) \le C_4(1-\delta)^{-2\gamma}(1+C^3),$$

where C_4 depends on C_3 and γ. This completes the proof of the lemma. □

Now, applying Lemmas 3.4, 3.5 and Proposition 2.8, we get the following Harnack inequality.

Theorem 3.6 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. Assume that (2.6) and (2.7) hold in $B_x(r)$. Fix $\tau \in (0,1)$ and $0 < p_0 < 1 + \nu/2$. For any $s \in \mathbb{R}$ and $0 < \varepsilon < \eta < \delta < 1$, any smooth positive solution u of the f-heat equation in the cylinder $Q = B_x(r) \times (s - r^2, s)$ satisfies*

$$\| u \|_{p_0, Q_-} \le (r^2 V_f)^{\frac{1}{p_0}} e^{c_1 F(r)} \inf_{Q_+} u,$$

where $c_1 = c_1(n, \varepsilon, \eta, \delta, p_0)$ and $F(r) = e^{c_2(A'+\sqrt{K})r + c_3 A'\sqrt{K}r^2}$, $A' = A'(x, r)$. Hence we have

$$\sup_{Q_-} u \le e^{c_4 F(r)} \inf_{Q_+} u,$$

where $c_4 = c_4(n, \varepsilon, \eta, \delta)$.

Proof of Theorem 3.6 We let u be a positive solution to the f-heat equation in Q. Let also $\delta, \tau \in (0,1)$ be fixed. Using Proposition 2.8 and Lemma 3.4, we see that Lemma 3.5 can be applied to $e^c u$ (resp. $e^{-c}u^{-1}$), where $c = c(u)$ is defined as in Lemma 3.4, with

$$R_\sigma = \sigma\delta B \times (s - r^2, s - \tau r^2 - \sigma\tau r^2) \quad (\text{resp. } R_\sigma = \sigma\delta B \times (s - \sigma\tau r^2, s))$$

and $0 < p_1 = p_0/2 < p_0 < 1 + \nu/2$ (resp. $0 < p_1 = 1 < p_0 = \infty$). Hence for any $0 < \varepsilon < \eta < \delta < 1$ and Q_-, Q_+ as defined as above, we have

$$e^c \| u \|_{p_0, Q_-} \le (r^2 V_f)^{1/p_0} e^{c_1 F(r)}$$

and

$$e^{-c} \sup_{Q_+}\{u^{-1}\} \le e^{c_4 F(r)},$$

where $c_1 = c_1(n, \varepsilon, \eta, \delta, p_0)$, $c_4 = c_4(n, \varepsilon, \eta, \delta)$ and $F(r) = e^{c_2(A'+\sqrt{K})r + c_3 A'\sqrt{K}r^2}$. The theorem follows from this and Proposition 2.7. □

Finally, we finish the proof of Theorem 3.1 by applying the standard chain argument to Theorem 3.6.

Proof of Theorem 3.1 Let $(t_-, x_-) \in Q_-$, $(t_+, x_+) \in Q_+$, and let $\tau = t_+ - t_-$. Notice that $\tau \sim r^2$ and $d = d(x_-, x_+) < r$. Let $t_i = t_- + \frac{i\tau}{N}$ and $x_i \in \frac{1+\delta}{2}B$ for $0 \le i \le N$, such that $x_0 = x_-$, $x_N = x_+$, and $d(x_i, x_{i+1}) \le C_\delta \frac{d}{N}$. Choose N to be the smallest number such that

$$N \ge C_{\varepsilon,\eta,\delta}(A' + \sqrt{K})^2 r^2,$$

where $A' = A'(x, r+1)$, applying Theorem 3.6 with $r' = (\frac{\tau}{N})^{\frac{1}{2}}$, then we have

$$\begin{aligned} u(t_-, x_-) &\le e^{c_4 F(r')(N+1)} u(t_+, x_+) \\ &\le e^{c_4 F\left(\frac{1}{C(A'+\sqrt{K})}\right)(N+1)} u(t_+, x_+) \\ &\le \exp[c(A' + \sqrt{K})^2 r^2 + c] u(t_+, x_+), \end{aligned}$$

where c depends on n, ε, η and δ. This finishes the proof of Theorem 3.1. □

4 Gaussian upper and lower bounds of the f-heat kernel

In this section, following the arguments in [43], we derive Gaussian upper and lower bounds for the f-heat kernel on smooth metric measure spaces. The upper bound estimate follows from the f-mean value inequality in Proposition 2.7 and a weighted version of Davies integral estimate (see [49]). The lower bound estimate follows from the local Harnack inequality in Sect. 3.

Let us first state the weighted Davies integral estimate, see [49] for the proof,

Lemma 4.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Let $\lambda_1(M) \ge 0$ be the bottom of the L^2_f-spectrum of the f-Laplacian on M. Assume that B_1 and B_2 are bounded subsets of M. Then*

$$\int_{B_1}\int_{B_2} H(x, y, t)d\mu(x)d\mu(y) \le V_f^{1/2}(B_1)V_f^{1/2}(B_2)\exp\left(-\lambda_1(M)t - \frac{d^2(B_1, B_2)}{4t}\right), \tag{4.1}$$

where $d(B_1, B_2)$ denotes the distance between the sets B_1 and B_2.

Proof of upper bound estimate in Theorem 1.1 For $x \in B_o(R/2)$, denote $u(y, s) = H(x, y, s)$. Assume $t \ge r_2^2$, applying Proposition 2.7 to u, we have

$$\begin{aligned} \sup_{(y,s)\in Q_\delta} H(x, y, s) &\le \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K} r_2}}{r_2^2 V_f(B_2)} \cdot \int_{t-1/4r_2^2}^{t}\int_{B_2} H(x, \zeta, s)d\mu(\zeta)ds \\ &= \frac{c_1 e^{c_2 A + c_3(1+A)\sqrt{K} r_2}}{4V_f(B_2)} \cdot \int_{B_2} H(x, \zeta, s')d\mu(\zeta) \end{aligned} \tag{4.2}$$

for some $s' \in (t - 1/4r_2^2, t)$, where $Q_\delta = B_y(\delta r_2) \times (t - \delta r_2^2, t)$ with $0 < \delta < 1/4$, and $B_2 = B_y(r_2) \subset B_o(R)$ for $y \in B_o(R/2)$, $A = A(x, R) \le A(o, 2R)$. Applying

Proposition 2.7 and the same argument to the positive solution

$$v(x,s)=\int_{B_2}H(x,\zeta,s)d\mu(\zeta)$$

of the f-heat equation, for the variable x with $t\geq r_1^2$, we also get

$$\begin{aligned}\sup_{(x,s)\in\bar{Q}_\delta}\int_{B_2}H(x,\zeta,s)d\mu(\zeta)&\leq\frac{c_1e^{c_2A+c_3(1+A)\sqrt{K}r_1}}{r_1^2V_f(B_1)}\\&\quad\cdot\int_{t-1/4r_1^2}^{t}\int_{B_1}\int_{B_2}H(\xi,\zeta,s)d\mu(\zeta)d\mu(\xi)ds\\&=\frac{c_1e^{c_2A+c_3(1+A)\sqrt{K}r_1}}{4V_f(B_1)}\cdot\int_{B_1}\int_{B_2}H(\xi,\zeta,s'')d\mu(\zeta)d\mu(\xi)\end{aligned}\tag{4.3}$$

for some $s''\in(t-1/4r_1^2,t)$, where $\bar{Q}_\delta=B_x(\delta r_1)\times(t-\delta r_1^2,t)$ with $0<\delta<1/4$, and $B_1=B_x(r_1)\subset B_o(R)$ for $x\in B_o(R/2)$. Now letting $r_1=r_2=\sqrt{t}$ and combining (4.2) with (4.3), the smooth f-heat kernel satisfies

$$H(x,y,t)\leq\frac{c_1e^{c_2A+c_3(1+A)\sqrt{Kt}}}{V_f(B_1)V_f(B_2)}\cdot\int_{B_1}\int_{B_2}H(\xi,\zeta,s'')d\mu(\zeta)d\mu(\xi)\tag{4.4}$$

for all $x,y\in B_o(R/2)$ and $0<t<R^2/4$. Using Lemma 4.1 and noticing that $s''\in(\frac{3}{4}t,t)$, then (4.4) becomes

$$H(x,y,t)\leq\frac{c_1e^{c_2A+c_3(1+A)\sqrt{Kt}}}{V_f(B_x(\sqrt{t}))^{1/2}V_f(B_y(\sqrt{t}))^{1/2}}\times\exp\left(-\frac{3}{4}\lambda_1t-\frac{d^2(B_1,B_2)}{4t}\right)\tag{4.5}$$

for all $x,y\in B_o(R/2)$ and $0<t<R^2/4$. Notice that if $d(x,y)\leq2\sqrt{t}$, then $d(B_x(\sqrt{t}),B_y(\sqrt{t}))=0$ and hence

$$-\frac{d^2(B_x(\sqrt{t}),B_y(\sqrt{t}))}{4t}=0\leq1-\frac{d^2(x,y)}{4t},$$

and if $d(x,y)>2\sqrt{t}$, then $d(B_x(\sqrt{t}),B_y(\sqrt{t}))=d(x,y)-2\sqrt{t}$, and hence

$$-\frac{d^2(B_x(\sqrt{t}),B_y(\sqrt{t}))}{4t}=-\frac{(d(x,y)-2\sqrt{t})^2}{4t}\leq-\frac{d^2(x,y)}{4(1+\epsilon)t}+C(\epsilon)$$

for some constant $C(\epsilon)$, where $\epsilon>0$. Here if $\epsilon\to0$, then the constant $C(\epsilon)\to\infty$. Therefore in any case, Eq. (4.5) becomes

$$H(x,y,t)\leq\frac{C(\epsilon)e^{c_2A+c_3(1+A)\sqrt{Kt}}}{V_f(B_x(\sqrt{t}))^{1/2}V_f(B_y(\sqrt{t}))^{1/2}}\times\exp\left(-\frac{3}{4}\lambda_1t-\frac{d^2(x,y)}{4(1+\epsilon)t}\right)$$

for all $x,y\in B_o\left(\frac{1}{2}R\right)$ and $0<t<R^2/4$. □

Moreover, in Theorem 1.1, if $K > 0$. According to Lemma 2.3, we know that

$$V_f(B_x(\sqrt{t})) \leq \frac{e^{(n-1+4A)\sqrt{K}(d(x,y)+\sqrt{t})}}{t^{n/2+2A}} V_f(B_y(\sqrt{t}))$$

for all $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$. Substituting this into Theorem 1.1 yields the following result.

Corollary 4.2 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space with* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant* $K > 0$*. For any point* $o \in M$*,* $R > 0$*,* $\epsilon > 0$*, there exist constants* $c_1(n, \epsilon)$*,* $c_2(n)$ *and* $c_3(n)$*, such that*

$$H(x, y, t) \leq \frac{c_1\, e^{c_2 A + c_3(1+A)\sqrt{K}(d(x,y)+\sqrt{t})}}{V_f(B_x(\sqrt{t})\ t^{n/4+A}} \times \exp\left(-\frac{d^2(x, y)}{(4+\epsilon)t}\right) \tag{4.6}$$

for all $x, y \in B_o(\frac{1}{4}R)$ *and* $0 < t < R^2/4$*. Here* $\lim_{\epsilon \to 0} c_1(n, \epsilon) = \infty$*.*

When $K = 0$, see the estimate in [49].

Next we derive the lower bound estimate. First, from the Harnack inequality in Theorem 3.1 we get the following estimate,

Proposition 4.3 *Under the same assumptions of Theorem 3.1, there exists a constant* $c(n)$ *such that, for any two positive solutions* $u(x, s)$ *and* $u(y, t)$ *of the* f*-heat equation in* $B_o(R/2) \times (0, T)$*,* $0 < s < t < T$*,*

$$\ln\left(\frac{u(x, s)}{u(y, t)}\right) \leq c(n)\left[\left(A'^2 + K + \frac{1}{R^2} + \frac{1}{s}\right)(t-s) + \frac{d^2(x, y)}{t-s}\right].$$

Proof Let $u(x, s)$ and $u(y, t)$ be two positive solutions to the f-heat equation in $B_o(\delta R) \times (0, T)$, where $x, y \in B_o(\delta R)$ and $0 < s < t < T$. Let N be an integer, which will be chosen later. We set $t_i = s + i(t-s)/N$. We remark that it is possible to find a sequence of points $x_i \in \frac{1+\delta}{2}B$ such that $x_0 = x$, $x_N = y$ and $Nd(x_i, x_{i+1}) \geq C_\delta d(x, y)$. Now we choose N to be the smallest integer such that

$$\tau/N \leq s/2, \quad \tau/N \leq C_\delta^{-1} R^2, \quad \tau = t - s$$

and if $d(x, y)^2 \geq \tau$,

$$\tau/N \geq d(x, y)^2/N^2.$$

Under the above conditions, we choose

$$N = c_\delta\left(\frac{\tau}{R^2} + \frac{\tau}{s} + \frac{d(x, y)^2}{\tau}\right).$$

Now we apply Theorem 3.1 to compare $u(x_i, t_i)$ with $u(x_{i+1}, t_{i+1})$ with $r' = (\tau/N)^{1/2}$. Therefore

$$\ln\left(\frac{u(x,s)}{u(y,t)}\right) \le c_1\left[(A'^2+K)\frac{\tau}{N}+1\right]\cdot N \le c_1'\left[(A'^2+K)\tau+\frac{\tau}{R^2}+\frac{\tau}{s}+\frac{d(x,y)^2}{\tau}\right],$$

where c_1' depends on n and δ, and $\tau = t - s$. Then the conclusion follows by letting $\delta = 1/2$. □

From Corollary 4.3, we get the following lower bound for f-heat kernel,

Theorem 4.4 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge -(n-1)K$ for some constant $K > 0$. For any point $o \in M$ and $R > 0$, there exist constants $c_1(n)$, $c_2(n)$ and $c_3(n)$ such that*

$$H(x,y,t) \ge \frac{c_1 e^{-c_2(A'^2+K)t}}{V_f(B_x(\sqrt{t}))}\times \exp\left(-\frac{d^2(x,y)}{c_3 t}\right), \tag{4.7}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$.

Proof of Theorem 4.4 and the second part of Theorem 1.1 Let $u(y,t) = H(x,y,t)$ with x fixed and $s = t/2$ in Proposition 4.3 and then we get

$$H(x,y,t) \ge H(x,x,t/2)\times \exp\left[-c_1\left((A'^2+K)t+1+\frac{t}{R^2}+\frac{d^2(x,y)}{t}\right)\right] \tag{4.8}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < \infty$.

In the following we will show that Moser's Harnack inequality leads to a lower bound of the on-diagonal f-heat kernel $H(x,x,t)$. Indeed we define

$$u(y,t) = \begin{cases} P_t\phi(y) & \text{if } t > 0 \\ \phi(y) & \text{if } t \le 0, \end{cases}$$

where $P_t = e^{t\Delta_f}$ is the heat semigroup of Δ_f, and ϕ is a smooth function such that $0 \le \phi \le 1$, $\phi = 1$ on $B = B_x(\sqrt{t})$ and $\phi = 0$ on $M\backslash 2B$.

$u(y,t)$ satisfies $(\partial_t - \Delta_f)u = 0$ on $B \times (-\infty, \infty)$. Applying the local Harnack inequality, first to u, and then to the f-heat kernel $(y,s) \to H(x,y,s)$, we have

$$\begin{aligned} 1 = u(x,0) &\le \exp\{c_1[(A'^2+K)t+1]\}\,u(x,t/2) \\ &= \exp\{c_1[(A'^2+K)t+1]\}\int_{B(x,\sqrt{t})} H(x,y,t/2)\phi(y)d\mu(y) \\ &\le \exp\{c_1[(A'^2+K)t+1]\}\int_{B(x,2\sqrt{t})} H(x,y,t/2)d\mu(y) \\ &\le \exp\{2c_1[(A'^2+K)t+1]\}V_f(B_x(2\sqrt{t}))H(x,x,t). \end{aligned}$$

From this, we have

$$H(x,x,t/2) \geq V_f^{-1}(B_x(\sqrt{2t}))\exp[-c_1((A'^2+K)t+2)]$$

for $0<\sqrt{t}<R/2$. Since (2.6) implies

$$V_f(B_x(\sqrt{2t})) \leq V_f(B_x(2\sqrt{t})) \leq c_1 e^{c_2(A'+\sqrt{K})\sqrt{t}+c_3A'\sqrt{K}t} V_f(B_x(\sqrt{t})),$$

we then obtain

$$H(x,x,t/2) \geq V_f^{-1}(B_x(\sqrt{t}))c_4\exp[-c_5((A'^2+K)t+1)]$$

for $0<\sqrt{t}<R/2$. Plugging this into (4.8) yields (4.7). □

5 L_f^1-Liouville theorem

In this section, inspired by the work of Li [24], we prove a Liouville theorem for f-subharmonic functions, and a uniqueness result for solutions of f-heat equation, by applying the f-heat kernel upper bound estimates. Our results not only extend the classical L^1-Liouville theorems proved by Li [24], but also generalize the weighted versions in [28,48,49].

Firstly we prove an L_f^1-Liouville theorem for f-harmonic functions when the Bakry-Émery Ricci curvature is bounded below and f is of linear growth.

Theorem 5.1 *Let $(M,g,e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K>0$. Assume there exist nonnegative constants a and b such that*

$$|f|(x) \leq ar(x)+b \quad for\ all\ x\in M,$$

where $r(x)$ is the geodesic distance function to a fixed point $o\in M$. Then any nonnegative L_f^1-integrable f-subharmonic function must be identically constant. In particular, any L_f^1-integrable f-harmonic function must be identically constant.

Sketch proof of Theorem 5.1 We first show that the assumptions of Theorem 5.1 imply the integration by parts formula

$$\int_M \Delta_{f\,y}H(x,y,t)h(y)d\mu(y) = \int_M H(x,y,t)\Delta_f\, h(y)d\mu(y)$$

for any nonnegative L_f^1-integrable f-subharmonic function h. This can be proved by our upper bound of f-heat kernel in Theorem 1.1. Then following the arguments of [49], applying the regularity theory of f-harmonic functions, we obtain the L_f^1-Liouville result. See the proof of Theorem 1.5 in [49] for the details. □

Now we are ready to check the integration by parts formula, similar to the proof of Theorem 4.3 in [49],

Proposition 5.2 *Under the same assumptions of Theorem 5.1, for any nonnegative L^1_f-integrable f-subharmonic function h, we have*

$$\int_M \Delta_{f\,y} H(x,y,t)h(y)d\mu(y) = \int_M H(x,y,t)\Delta_f h(y)d\mu(y).$$

Proof of Proposition 5.2 By the Green formula on $B_o(R)$, we have

$$\begin{aligned}&\left|\int_{B_o(R)} \Delta_{f\,y} H(x,y,t)h(y)d\mu(y) - \int_{B_o(R)} H(x,y,t)\Delta_f h(y)d\mu(y)\right|\\ &\quad\le \int_{\partial B_o(R)} H(x,y,t)|\nabla h|(y)d\mu_{\sigma,R}(y) + \int_{\partial B_o(R)} |\nabla H|(x,y,t)h(y)d\mu_{\sigma,R}(y),\end{aligned}$$

where $d\mu_{\sigma,R}$ denotes the weighted area measure induced by $d\mu$ on $\partial B_o(R)$. In the following we will show that the above two boundary integrals vanish as $R\to\infty$.

Consider a large R and assume $x\in B_o(R/8)$. By Proposition 2.7, we have the f-mean value inequality

$$\begin{aligned}\sup_{B_o(R)} h(x) &\le c_1 e^{c_2(aR+b)+c_3(1+aR+b)\sqrt{K}R} V_f^{-1}(2R)\int_{B_o(2R)} h(y)d\mu(y)\\ &\le Ce^{\alpha(1+K)R^2} V_f^{-1}(2R)\int_{B_o(2R)} h(y)d\mu(y),\end{aligned}\tag{5.1}$$

where constants C and α depend on n, a and b. Let $\phi(y)=\phi(r(y))$ be a nonnegative cut-off function satisfying $0\le\phi\le 1$, $|\nabla\phi|\le\sqrt{3}$ and $\phi(r(y))=1$ on $B_o(R+1)\backslash B_o(R)$, $\phi(r(y))=1$ on $B_o(R-1)\cup(M\backslash B_o(R+2))$. Since h is f-subharmonic, by the integration by parts formula and Cauchy–Schwarz inequality, we have

$$\begin{aligned}0\le\int_M \phi^2 h\Delta_f h d\mu &= -2\int_M \phi h\langle\nabla\phi\nabla h\rangle d\mu - \int_M \phi^2|\nabla h|^2 d\mu\\ &\le 2\int_M |\nabla\phi|^2 h^2 d\mu - \frac{1}{2}\int_M \phi^2|\nabla h|^2 d\mu.\end{aligned}$$

Then using the definition of ϕ and (5.1), we have that

$$\begin{aligned}\int_{B_o(R+1)\backslash B_o(R)} |\nabla h|^2 d\mu \le& 4\int_M |\nabla\phi|^2 h^2 d\mu\\ \le& 12\int_{B_o(R+2)} h^2 d\mu\\ \le& 12\sup_{B_o(R+2)} h\cdot\|h\|_{L^1(\mu)}\\ \le& \frac{Ce^{\alpha(1+K)(R+2)^2}}{V_f(2R+4)}\cdot\|h\|^2_{L^1(\mu)}.\end{aligned}$$

On the other hand, the Cauchy–Schwarz inequality also implies

$$\int_{B_o(R+1)\backslash B_o(R)} |\nabla h| d\mu \le \left(\int_{B_o(R+1)\backslash B_o(R)} |\nabla h|^2 d\mu\right)^{1/2} \cdot [V_f(R+1)\backslash V_f(R)]^{1/2}.$$

Combining the above two inequalities we get

$$\int_{B_o(R+1)\backslash B_o(R)} |\nabla h| d\mu \le C_1 e^{\alpha(1+K)R^2} \cdot \|h\|_{L^1(\mu)}, \tag{5.2}$$

where $C_1 = C_1(n, a, b, K)$.

We now estimate the f-heat kernel $H(x, y, t)$. Recall that, by letting $\epsilon = 1$ in Corollary 4.2, the f-heat kernel $H(x, y, t)$ satisfies

$$\begin{aligned} H(x, y, t) &\le \frac{c_1\, e^{c_2 A + c_3(1+A)\sqrt{K}(d(x,y)+\sqrt{t})}}{V_f(B_x(\sqrt{t})\ t^{n/4+A}} \times \exp\left(-\frac{d^2(x, y)}{5t}\right) \\ &\le \frac{c_4\, e^{c_5 R}}{V_f(B_x(\sqrt{t}))t^{c_7(R+1)}} \exp\left[c_6\sqrt{K}(1+R)(d(x, y)+\sqrt{t}) - \frac{d^2(x, y)}{5t}\right] \end{aligned} \tag{5.3}$$

for any $x, y \in B_o(R/2)$ and $0 < t < R^2/4$, where c_4, c_5, c_6 and c_7 are all constants depending only on n, a and b. Together with (5.2) we get

$$\begin{aligned} J_1 &:= \int_{B_o(R+1)\backslash B_o(R)} H(x, y, t)|\nabla g|(y) d\mu(y) \\ &\le \sup_{y \in B_o(R+1)\backslash B_o(R)} H(x, y, t) \cdot \int_{B_o(R+1)\backslash B_o(R)} |\nabla g| d\mu \\ &\le \frac{C_2 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))t^{c_7(R+2)}} \\ &\quad \times \exp\left[c_5 R - \frac{(R - d(o, x))^2}{5t} + c_9\sqrt{K}(R+2)(R+1+d(o, x)+\sqrt{t})\right], \end{aligned}$$

where $C_2 = C_2(n, a, b, K)$. Notice that

$$t^{-c_7(R+2)} = e^{-c_7(R+2)\ln t} \le e^{c_7(R+2)\frac{1}{t}} \quad \text{when } t \to 0.$$

Thus, for T sufficiently small and for all $t \in (0, T)$ there exists a constant $\beta > 0$ such that

$$J_1 \le \frac{C_3 \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))} \times \exp\left(-\beta R^2 + c\frac{d^2(o, x)}{t}\right),$$

where $C_3 = C_3(n, a, b, K)$. Therefore for all $t \in (0, T)$ and all $x \in M$, $J_1 \to 0$ as $R \to \infty$.

By a similar argument, we can show that

$$\int_{B_o(R+1)\setminus B_o(R)} |\nabla H|(x,y,t)h(y)d\mu \to 0$$

as $R \to \infty$. We first estimate $\int_{B_o(R+1)\setminus B_o(R)} |\nabla H|(x,y,t)d\mu$.

$$\begin{aligned}\int_M \phi^2(y)|\nabla H|^2(x,y,t)d\mu &= -2\int_M \big\langle H(x,y,t)\nabla\phi(y), \phi(y)\nabla H(x,y,t)\big\rangle d\mu \\ &\quad - \int_M \phi^2(y)H(x,y,t)\Delta_f H(x,y,t)d\mu \\ &\le 2\int_M |\nabla\phi|^2(y)H^2(x,y,t)d\mu \\ &\quad + \frac{1}{2}\int_M \phi^2(y)|\nabla H|^2(x,y,t)d\mu \\ &\quad - \int_M \phi^2(y)H(x,y,t)\Delta_f H(x,y,t)d\mu,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{B_o(R+1)\setminus B_o(R)} |\nabla H|^2 \\ &\quad \le \int_M \phi^2(y)|\nabla H|^2(x,y,t) \\ &\quad \le 4\int_M |\nabla\phi|^2 H^2 - 2\int_M \phi^2 H\Delta_f H \\ &\quad \le 12\int_{B_o(R+2)\setminus B_o(R-1)} H^2 + 2\int_{B_o(R+2)\setminus B_o(R-1)} H|\Delta_f H| \\ &\quad \le 12\int_{B_o(R+2)\setminus B_o(R-1)} H^2 + 2\left(\int_{B_o(R+2)\setminus B_o(R-1)} H^2\right)^{\frac{1}{2}}\left(\int_M (\Delta_f H)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.4}$$

Notice that by Theorem 4.1 in [46], if $\mathrm{Ric}_f \ge -(n-1)K$, then

$$V_f(B_o(R)) \le A + B\exp\left[\frac{(n-1)K}{2}R^2\right]$$

for all $R > 1$, so we have

$$\int_1^\infty \frac{R}{\log V_f(B_o(R))}dR = \infty. \tag{5.5}$$

By Theorem 3.13 in [20], $(M, g, e^{-f}dv)$ is stochastically complete, i.e.,

$$\int_M H(x, y, t)e^{-f}dv(y) = 1. \tag{5.6}$$

Using (5.3) and (5.6), we get

$$\begin{aligned}
&\int_{B_o(R+2)\backslash B_o(R-1)} H^2(x, y, t)d\mu \\
&\quad\le \sup_{y\in B_o(R+2)\backslash B_o(R-1)} H(x, y, t) \\
&\quad\le \frac{c_4}{V_f(B_x(\sqrt{t}))t^{c_7(R+3)}} \times \exp\left[-\frac{(R-1-d(o,x))^2}{5t}\right] \\
&\qquad\times \exp[c_5(R+2) + c_6\sqrt{K}(3+R)(R+2-d(o,x)+\sqrt{t})] \\
&\quad= \frac{c_4}{V_f(B_x(\sqrt{t}))} \times \exp\left[-\frac{(R-1-d(o,x))^2}{5t} + c_7(R+3)\ln\frac{1}{t}\right] \\
&\qquad\times \exp[c_5(R+2) + c_6\sqrt{K}(3+R)(R+2-d(o,x)+\sqrt{t})].
\end{aligned} \tag{5.7}$$

From (4.7) in [49], there exists a constant $C > 0$ such that

$$\int_M (\Delta_f H)^2(x, y, t)d\mu \le \frac{C}{t^2}H(x, x, t). \tag{5.8}$$

Combining (5.4), (5.7) and (5.8), we obtain

$$\begin{aligned}
\int_{B_o(R+1)\backslash B_o(R)} |\nabla H|^2 d\mu \le{}& C_4[V_f^{-1} + t^{-1}V_f^{-\frac{1}{2}}H^{\frac{1}{2}}(x, x, t)] \\
&\times \exp\left[-\frac{(R-1-d(o,x))^2}{10t} + c_7(R+3)\ln\frac{1}{t}\right] \\
&\times \exp[c_5R + c_6\sqrt{K}(3+R)(R+2-d(o,x)+\sqrt{t})]
\end{aligned}$$

where $V_f = V_f(B_x(\sqrt{t}))$ and $C_4 = C_4(n, a, b)$. Hence we get

$$\begin{aligned}
\int_{B_o(R+1)\backslash B_o(R)} |\nabla H| d\mu &\le \left[V_f(B_o(R+1))\backslash V_f(B_o(R))\right]^{1/2} \times \left[\int_{B_o(R+1)\backslash B_o(R)} |\nabla H|^2 d\mu\right]^{1/2} \\
&\le C_4 V_f^{1/2}(B_o(R+1))[V_f^{-1} + t^{-1}V_f^{-\frac{1}{2}}H^{\frac{1}{2}}(x, x, t)]^{1/2} \\
&\quad\times \exp\left[-\frac{(R-1-d(o,x))^2}{20t} + \frac{c_7}{2}(R+3)\ln\frac{1}{t}\right] \\
&\quad\times \exp[c_5R + c_6\sqrt{K}(3+R)(R+2-d(o,x)+\sqrt{t})].
\end{aligned} \tag{5.9}$$

Therefore, by (5.1) and (5.9), we obtain

$$\begin{aligned} J_2 &= \int_{B_o(R+1)\setminus B_o(R)} |\nabla H(x,y,t)| h(y) d\mu(y) \\ &\le \sup_{y\in B_o(R+1)\setminus B_o(R)} h(y) \cdot \int_{B_o(R+1)\setminus B_o(R)} |\nabla H(x,y,t)| d\mu(y) \\ &\le \frac{C_5 \|g\|_{L^1(\mu)}}{V_f^{1/2}(B_o(2R+2))} \cdot [V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x,x,t)]^{1/2} \\ &\quad \times \exp\left[\alpha(1+K)(R+1)^2 - \frac{(R-1-d(o,x))^2}{20t} + \frac{c_7}{2}(R+3)\ln\frac{1}{t}\right] \\ &\quad \times \exp[c_5 R + c_6\sqrt{K}(3+R)(R+2-d(o,x)+\sqrt{t})], \end{aligned}$$

where $C_5 = C_5(n,a,b)$. Similar to the case of J_1, we choose T sufficiently small, then for all $t \in (0,T)$ and all $x \in M$, $J_2 \to 0$ when $R \to \infty$.

Now by the mean value theorem, for any $R > 0$ there exists $\bar{R} \in (R, R+1)$ such that

$$\begin{aligned} J &= \int_{\partial B_o(\bar{R})} [H(x,y,t)|\nabla h|(y) + |\nabla H|(x,y,t)h(y)]\, d\mu_{\sigma,\bar{R}}(y) \\ &= \int_{B_o(R+1)\setminus B_p(R)} [H(x,y,t)|\nabla h|(y) + |\nabla H|(x,y,t)h(y)]\, d\mu(y) \\ &= J_1 + J_2. \end{aligned}$$

By the above argument, we choose T sufficiently small, then for all $t \in (0,T)$ and all $x \in M$, $J \to 0$ as $\bar{R} \to \infty$. Therefore Proposition 5.2 holds for T sufficiently small. Then the semigroup property of the f-heat equation implies Proposition 5.2 holds for all time $t > 0$. □

Theorem 5.1 leads to a uniqueness property for L^1-solutions of the f-heat equation, which generalizes the classical result of Li [24]. The proof is very similar to the one in [49], so we omit it.

Theorem 5.3 *Under the same assumptions of Theorem 5.1, if $u(x,t)$ is a nonnegative function defined on $M \times [0,+\infty)$ satisfying*

$$(\partial_t - \Delta_f)u(x,t) \le 0, \quad \int_M u(x,t)e^{-f} dv < +\infty$$

for all $t > 0$, and

$$\lim_{t\to 0} \int_M u(x,t)e^{-f} dv = 0,$$

then $u(x,t) \equiv 0$ for all $x \in M$ and $t \in (0,+\infty)$. In particular, any L^1_f-solution of the f-heat equation is uniquely determined by its initial data in L^1_f.

6 Eigenvalue estimate

In this section we derive eigenvalue estimates of the f-Laplace operator compact smooth metric measure spaces, using the upper bound estimate of the f-heat kernel and an argument of Li and Yau [27].

When the Bakry–Émery Ricci curvature is nonnegative, we have

Theorem 6.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional closed smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of the f-Laplacian. Then there exists a constant C depending only on n and $\max_{x\in M} f(x)$, such that*

$$\lambda_k \geq \frac{C(k+1)^{2/n}}{d^2}$$

for all $k \geq 1$, where d is the diameter of M.

Proof Since $Ric_f \geq 0$, from Theorem 1.1, we have

$$H(x, x, t) \leq \frac{C}{V_f(B_x(\sqrt{t}))}, \tag{6.1}$$

where C is a constant depending only on n and $B = \max_{x\in M} f(x)$. Notice that the f-heat kernel can be written as

$$H(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y),$$

where φ_i is the eigenfunction of Δ_f corresponding to λ_i, $\|\phi_i\|_{L^2_f} = 1$. By the f-volume comparison theorem (see Lemma 2.1 in [49]), we get, for any $t \leq d^2$,

$$\frac{V_f(B_x(d))}{V_f(B_x(\sqrt{t}))} \leq e^{4B}\left(\frac{d}{\sqrt{t}}\right)^n.$$

Taking the weighted integral on both sides of (6.1), we conclude that

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \leq C\int_M V_f^{-1}(B_x(\sqrt{t}))d\mu \leq C\int_M p(t)d\mu,$$

where

$$p(t) = \begin{cases} e^{4B}\left(\frac{d}{\sqrt{t}}\right)^n V_f^{-1}(B_x(d)), & \text{if } \sqrt{t} \leq d \\ e^{4B}V_f^{-1}(M), & \text{if } \sqrt{t} > d. \end{cases}$$

which implies that $(k+1)e^{-\lambda_k t} \leq Cq(t)$ for any $t > 0$, that is

$$Ce^{\lambda_k t}q(t) \geq (k+1), \quad \text{for any } t > 0, \tag{6.2}$$

where

$$q(t)=\begin{cases} e^{4B}\left(\frac{d}{\sqrt{t}}\right)^n, & \text{if } \sqrt{t}\le d\\ e^{4B}, & \text{if } \sqrt{t}>d. \end{cases}$$

It is easy to see that $e^{\lambda_k t}q(t)$ takes its minimum at $t_0=\frac{n}{2\lambda_k}$. Plugging to (6.2) we get the lower bound for λ_k.

Similarly, when the Bakry–Émery Ricci curvature is bounded below, we have a similar estimate. We omit the proof since it is the same as $\mathrm{Ric}_f\ge 0$ case.

Theorem 6.2 *Let $(M,g,e^{-f}dv)$ be an n-dimensional closed smooth metric measure space with $\mathrm{Ric}_f\ge -(n-1)K$ for some constant $K>0$. Let $0=\lambda_0<\lambda_1\le\lambda_2\le\dots$ be eigenvalues of the f-Laplacian. Then there exists a constant C depending only on n and $B=\max_{x\in M} f(x)$, such that*

$$\lambda_k\ge\frac{C}{d^2}\left(\frac{k+1}{\exp(C\sqrt{K}d)}\right)^{\frac{2}{n+4B}}$$

for all $k\ge 1$, where d is the diameter of M.

7 f-Green's function estimate

In this section, we will discuss the Green's function of the f-Laplacian and f-parabolicity of smooth metric measure spaces. It was proved by Malgrange [31] that every Riemannian manifold admits a Green's function of Laplacian. Varopoulos [45] proved that a complete manifold (M,g) has a positive Green's function only if

$$\int_1^\infty \frac{t}{V_p(t)}dt<\infty, \tag{7.1}$$

where $V_p(t)$ is the volume of the geodesic ball of radius t with center at p. For Riemannian manifolds with nonnegative Ricci curvature, Varopoulos [45] and Li and Yau [27] proved (7.1) is the sufficient and necessary condition for the existence of positive Green's function.

On an n-dimensional complete smooth metric measure space $(M,g,e^{-f}dv)$, let $H(x,y,t)$ be a f-heat kernel, recall the f-Green's function

$$G(x,y)=\int_0^\infty H(x,y,t)dt$$

if the integral on the right hand side converges. From the f-heat kernel estimates, it is easy to get the following two-sided estimates for f-Green's function, which is similar to Li–Yau estimate [27] of Green's function for Riemannian manifolds with nonnegative Ricci curvauture,

Theorem 7.1 *Let $(M,g,e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f\ge 0$ and $|f|\le C$ for some nonnegative constant C.*

If $G(x, y)$ exists, then there exist constants c_1 and c_2 depending only on n and C, such that

$$c_1 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt \le G(x, y) \le c_2 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt, \tag{7.2}$$

where $r = r(x, y)$.

As a corollary, we get a necessary and sufficient condition of the existence of positive f-Green's function on smooth metric measure spaces with nonnegative Bakry–Émery Ricci curvature and bounded potential function,

Corollary 7.2 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge 0$ and $|f| \le C$ for some nonnegative constant C. There exists a positive f-Green's function $G(x, y)$ if and only if*

$$\int_1^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt < \infty.$$

Proof of Theorem 7.1 Since $Ric_f \ge 0$ and $|f| \le C$, Theorem 1.1 holds for any $0 < t < \infty$ by letting $R \to \infty$. For the lower bound estimate, we have

$$\begin{aligned} G(x, y) \ge \int_{r^2}^{\infty} H(x, y, t)dt &\ge c_3(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{c_4 t}\right) dt \\ &\ge c_5(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt. \end{aligned}$$

Hence the left hand side of (7.2) follows.

For the upper bound estimate, it suffices to show that

$$\int_0^{r^2} H(x, y, t)dt \le c_6(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt. \tag{7.3}$$

By the definition of G and Theorem 1.1,

$$\begin{aligned} G(x, y) = \int_0^{\infty} H(x, y, t)dt &= \int_0^{r^2} H(x, y, t)dt + \int_{r^2}^{\infty} H(x, y, t)dt \\ &\le \int_0^{r^2} H(x, y, t)dt + c_7(n, C) \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt \\ &\le c_8 \int_0^{r^2} V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{5t}\right) dt \\ &\quad + c_7 \int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{t}))dt, \end{aligned}$$

where c_7 and c_8 depend on n and C. Letting $s = r^4/t$, where $r^2 < s < \infty$, we get

$$\int_0^{r^2} V_f^{-1}(B_x(\sqrt{t})) \exp\left(\frac{-r^2}{5t}\right) dt = \int_{r^2}^{\infty} V_f^{-1}\left(B_x\left(\frac{r^2}{\sqrt{s}}\right)\right) \exp\left(\frac{-s}{5r^2}\right) \frac{r^4}{s^2} ds.$$

On the other hand, the f-volume comparison theorem (see Lemma 2.1 in [49]) gives

$$V_f^{-1}\left(B_x\left(\frac{r^2}{\sqrt{s}}\right)\right) \leq V_f^{-1}(B_x(\sqrt{s}))e^{4C}\left(\frac{s}{r^2}\right)^n.$$

Therefore we get

$$\int_0^{r^2} H(x,y,t)dt \leq c_9(n,C)\int_{r^2}^{\infty} V_f^{-1}(B_x(\sqrt{s}))\left(\frac{s}{r^2}\right)^{n-2}\exp\left(\frac{-s}{5r^2}\right)ds.$$

Since the function $x^{n-2}e^{-x/5}$ is bound from above, Eq. (7.3) follows. □

Next we discuss f-nonparabolicity of steady Ricci solitons using a criterion of Li and Tam [25,26], and the f-heat kernel for steady Gaussian Ricci soliton. A smooth metric measure space $(M^n, g, e^{-f}dv)$ is called f-nonparabolic if it admits a positive f-Green's function. An end, E, with respect to a compact subset $\Omega \subset M$ is an unbounded connected component of M. When we say that E is an end, it is implicitly assumed that E is an end with respect to some compact subset $\Omega \subset M$. Munteanu and Wang [35] proved that if $\mathrm{Ric}_f \geq 0$, there exists at most one f-nonparabolic end on $(M^n, g, e^{-f}dv)$.

First we observe that the criterion of Li and Tam [25,26] can be generalized to smooth metric measure spaces,

Lemma 7.3 *Let $(M^n, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. There exists an f-Green's function $G(x,y)$ which is smooth on $M\times M\backslash D$, where $D = \{(x,x)|x \in M\}$. Moreover, $G(x,y)$ can be taken to be positive if and only if there exists a positive nonconstant f-superharmonic function u on $M\backslash B_o(r)$ with the property that*

$$\liminf_{x\to\infty} u(x) < \inf_{x\in\partial B_o(r)} u(x).$$

Proof of Theorem 1.8 Let (M, g, f) be a nontrivial gradient steady soliton, we have

$$\Delta f + R = 0 \quad \text{and} \quad R + |\nabla f|^2 = a.$$

Chen [11] proved that $R \geq 0$, so $a > 0$. It was proved in [16,38] (see also [50]) that $\liminf R = 0$, and either $R > 0$ or $R \equiv 0$.

By the Bochner formula, we get

$$\Delta_f R = -2|Ric|^2 \leq 0.$$

If $R > 0$ on M, then it is a nonconstant positive f-superharmonic function, and $\liminf_{x\to\infty} R(x) = 0$. Therefore, by Lemma 7.3, we conclude $G(x,y)$ is positive.

If $R \equiv 0$, then by Proposition 4.3 in [38], (M^n, g) splits isometrically as $(N^{n-k} \times \mathbb{R}^k, g_N + g_0)$, where (N^{n-k}, g_N) is a Ricci-flat manifold, and $(\mathbb{R}^k, g_0, f)$ is a steady

Gausian Ricci soliton with $f = \langle u, x\rangle + v$ for some $u, v \in \mathbb{R}^n$. Therefore a f-Green's function on $(\mathbb{R}^k, g_0, f)$ is a f-Green's function on (M, g, f).

By [49], for one-dimensional steady Gaussian Ricci soliton, the f-heat kernel is given by

$$H_{\mathbb{R}}(x, y, t) = \frac{e^{\pm\frac{x+y}{2}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

for any $x, y \in \mathbb{R}$ and $t > 0$. Therefore for any $x, y \in \mathbb{R}$,

$$G(x, y) = \int_0^\infty H_{\mathbb{R}}(x, y, t)dt < \infty,$$

hence there exists a positive f-Green function.

For higher dimensional steady Gaussian Ricci soliton $(\mathbb{R}^k,\ g_0, f)$, define

$$H_{\mathbb{R}^k}(x, y, t) = H_{\mathbb{R}}(x_1, y_1, t) \times H_{\mathbb{R}}(x_2, y_2, t) \times \cdots \times H_{\mathbb{R}}(x_k, y_k, t),$$

where $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$, and $H_{\mathbb{R}}(x_i, y_i, t)$ is the f-heat kernel for $(\mathbb{R}, g_0, u_i x_i + v_i)$. It is easy to check that $H_{\mathbb{R}^k}(x, y, t)$ is an f-heat kernel on $(\mathbb{R}^k,\ g_0, f)$.

Then for any $x, y \in \mathbb{R}^k$,

$$G(x, y) = \int_0^\infty H_{\mathbb{R}^k}(x, y, t)dt < \infty.$$

Therefore there exists a positive f-Green function on an k-dimensional steady Gaussian soliton. □

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