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# On the generalization of Forelli's theorem

**Jae-Cheon Joo[1] · Kang-Tae Kim[2] · Gerd Schmalz[3]**

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**Abstract** The purpose of this paper is to present a solution to perhaps the final remaining case in the line of study concerning the generalization of Forelli's theorem on the complex analyticity of the functions that are: (i) $\mathcal{C}^\infty$ smooth at a point, and (ii) holomorphic along the complex integral curves generated by a contracting holomorphic vector field with an isolated zero at the same point.

**Keywords** Forelli's theorem · Complex-analyticity · Vector field

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✉ Gerd Schmalz
schmalz@une.edu.au

Jae-Cheon Joo
jcjoo91@uni-wuppertal.de

Kang-Tae Kim
kimkt@postech.ac.kr

1 Department of Mathematics, University of Wuppertal, Gaussstr. 20, 42119 Wuppertal, Germany

2 Department of Mathematics, POSTECH, Pohang 790-784, Republic of Korea

3 School of Science and Technology, University of New England, Armidale, NSW 2351, Australia

## 1 Introduction

Among the theorems concerning the complex-analyticity of functions of several complex variables, the most exploited should be the Hartogs analyticity theorem. The second may be the following theorem of Forelli:

**Theorem 1.1** (Forelli [3], Stoll [10]) *Let $F: B^n \to \mathbb{C}$ be a complex-valued function defined on the unit ball $B^n$ in $\mathbb{C}^n$. If $F$ satisfies the following two conditions:*

(i) *$F \in \mathcal{C}^\infty(0)$, i.e., for every positive integer $k$ there exists an open neighborhood $U_k$ of the origin 0 such that $F \in \mathcal{C}^k(U_k)$;*
(ii) *For every $v \in \mathbb{C}^n$ with $\|v\| = 1$, the function $\varphi_v(\zeta) := F(\zeta v)$ defined on the unit disc $B^1$ in $\mathbb{C}$ is holomorphic in the complex variable $\zeta$,*

*then $F$ is holomorphic.*

It was a surprise that the condition (i) turned out impossible to be relaxed to a finite differentiability; consider, for instance, the function

$$F(z_1, z_2) = \begin{cases} \dfrac{\bar{z}_2}{\bar{z}_1} \cdot z_1^{k+2} & \text{if } z_1 \neq 0 \\ 0 & \text{if } z_1 = 0. \end{cases}$$

This function is $\mathcal{C}^k$ everywhere, satisfies the condition (i) of the hypothesis, but is nowhere holomorphic.

On the other hand, the generalizations have occurred recently in the following two natural directions. The first direction concerns the case that the domain is the union of holomorphic discs passing through a single point of the domain. In this direction, E. M. Chirka presented the complex two dimensional case in [2] and asked whether all dimensional generalization is possible. Responding to the question , the authors, in the earlier paper [6], presented the following result:

**Theorem 1.2** (Chirka [2], Joo-Kim-Schmalz [6]) *If $\Omega$ is a domain in $\mathbb{C}^n$ with a $\mathcal{C}^1$ radial foliation by holomorphic discs (nonlinear, in general) at a point $p \in \Omega$, then any function $F: \Omega \to \mathbb{C}$ satisfying*

(i) *$F \in \mathcal{C}^\infty(p)$*
(ii) *$F$ is holomorphic along the leaves,*

*is holomorphic on $\Omega$.*

This seems to have settled the first direction. Therefore, it is natural to shift the focus onto the other direction of generalization. It starts with the re-interpretation of the condition (ii) of the original Forelli's theorem that the analyticity of the given function $F$ along the radial complex lines is equivalent to the condition that *$F$ is holomorphic along the complex integral curves of the complex Euler vector field* $E = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$. As soon as this new viewpoint is taken, the following natural question arises:

**Question** *Let $X$ be a holomorphic vector field vanishing only at the origin. Replace the condition* (2) *in the statement of Theorem* 1.1 *by the condition:*

*"$f$ is holomorphic along the complex integral curves of $X$."*
*Then, for which $X$ would the conclusion continue to hold?*
*The answer to this question, in the generic subcase where $X$ is diagonalizable, was given in* [7].

**Theorem 1.3** (Kim-Poletsky-Schmalz [7]) *Let $F\colon B^n \to \mathbb{C}$ be a function defined on the unit open ball in $\mathbb{C}^n$, and let $X = \sum_{k=1}^n \alpha_k z_k \frac{\partial}{\partial z_k}$, where $\alpha_1, \ldots, \alpha_n$ are complex numbers satisfying $\alpha_j/\alpha_\ell > 0$ for any $j, \ell \in \{1, \ldots, n\}$. If $F$ satisfies the following two conditions:*

(1) $F \in \mathcal{C}^\infty(0)$
(2) *$F$ is holomorphic along the complex integral curves of $X$,*

*then $F$ is holomorphic on a neighborhood of the origin.*

We remark that this is modified to fit to the context of this article; it was proved originally in [7] under the condtion that $F$ has a formal Taylor series at the origin, weaker than Forelli's original condition that $F \in \mathcal{C}^\infty(0)$.

It is well-known however that the diagonalizable holomorphic vector fields do not always satisfy the additional condition on its coefficients specified in the above stated theorem. But then, it is shown in [7] with explicit examples that the conclusion fails if any of the ratios $\alpha_j/\alpha_k$ should take complex non-real, or real-but-negative, values. (See also the discussion following Definition 2.1 in Sect. 2.2.)

On the other hand, in the light of the original theorem of Forelli and subsequent generalizations, the case of contracting holomorphic vector fields that are not diagonalizable should be investigated, since their complex integral curves also form a singular foliation at the origin. In the case of complex dimension two, all such vector fields, up to a change of local coordinates, take the form

$$X = \alpha \left( z\frac{\partial}{\partial z} + (mw + \beta z^m)\frac{\partial}{\partial w} \right)$$

where $m$ is a positive integer, $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in \mathbb{C}$.

Indeed, the purpose of this paper is to give the answer to this seemingly final remaining case. For the sake of clarity of the exposition, we present here the version of the main theorem of this paper in complex dimension two; the most general all-dimensional statement shall be presented in the next section as it needs further terminology concerning vector fields.

**Theorem 1.4** *Let $X = \alpha z\frac{\partial}{\partial z} + \alpha(mw + \beta z^m)\frac{\partial}{\partial w}$, where $m$ is a positive integer, $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in \mathbb{C}$. If a complex-valued function $F\colon B^2 \to \mathbb{C}$ satisfies the conditions:*

(1) $F \in \mathcal{C}^\infty(0)$
(2) *$F$ is holomorphic along every complex integral curve of $X$,*

*then $F$ is holomorphic on a neighborhood of the origin.*

We remark in passing that the nonzero complex number $\alpha$ appearing in the statement above does not play any significant role. Moreover, the domain of the function $F$ may be any open subset containing the origin in $\mathbb{C}^2$ due to the local nature of the theorem. For the global version, see the discussion in Sect. 5.1.

## 2 Contracting fields, aligned fields and main theorem

### 2.1 Contracting holomorphic vector fields

We start with a holomorphic vector field $X$ defined in an open neighborhood of the origin in $\mathbb{C}^n$. $X$ is said to be *contracting at the origin* if the flow-diffeomorphism, say $\Phi_t$, of $\operatorname{Re} X$, for some $t < 0$, is contracting at 0, i.e., the map satisfies: (1) $\Phi_t(0) = 0$, and (2) every eigenvalue of the matrix $d\Phi_t|_0$ has absolute value less than 1.

The contracting vector fields have been extensively studied. So we shall only describe small part of the theory which is directly related to the theme of this paper. In particular, the Poincaré-Dulac theorem implies that, if $X$ is a contracting holomorphic vector field then, up to a change of holomorphic local coordinate system at the origin, $X$ can be written in the following form:

$$X = \sum_{j=1}^{n} \big(\lambda_j z_j + g_j(z)\big) \frac{\partial}{\partial z_j}, \tag{2.1}$$

where:

(1) $0 < \operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_2 \leq \cdots \leq \operatorname{Re}\lambda_n$.
(2) $g_1 \equiv 0$.
(3) For every $j \in \{2, \ldots, n\}$, $g_j(z)$ is a holomorphic polynomial in the variables $z_1, \ldots, z_{j-1}$ only, vanishing at the origin. If the identity $\lambda_j = \sum_{k=1}^{j-1} m_k \lambda_k$ holds for some nonnegative integers $m_k$ with $\sum_{k=1}^{j-1} m_k \geq 1$ (called the *resonance relation for* $\lambda_j$), then the condition

$$g_j(e^{\lambda_1\zeta} z_1, \ldots, e^{\lambda_{j-1}\zeta} z_{j-1}) = e^{\lambda_j\zeta} g_j(z_1, \ldots, z_{j-1})$$

must also hold. If no resonance relation holds for $\lambda_j$, then $g_j \equiv 0$.

The natural question to ask at this stage is whether Forelli's theorem can be generalized to the case of all contracting holomorphic vector fields. The answer is negative; this was already known to be impossible even for the diagonalizable case (cf. [7]). We shall see this in further generality in the next section.

### 2.2 Aligned holomorphic vector fields

The following definition introduces the optimal condition for the generalization of Forelli's theorem.

**Definition 2.1** (*Aligned fields*) Let $X$ be a holomorphic vector field of $\mathbb{C}^n$ contracting at the origin. Take its Poincaré-Dulac normal form (cf. [1,9,11])

$$X = \sum_{j=1}^{n} \left(\lambda_j z_j + g_j(z)\right) \frac{\partial}{\partial z_j}$$

as in (2.1) above. The vector field $X$ is called *aligned*, if $\lambda_j/\lambda_k > 0$ for every $j, k \in \{1, \ldots, n\}$.

Notice that, in the Poincaré-Dulac normal form of an aligned vector field, every variable $z_j$ appears. Note also that every $\lambda_j$ can be taken to be positive.

If $X$ is *not* aligned on the contrary, then there exists a $\mathcal{C}^\infty$ function, say $f$, in a neighborhood of the origin satisfying $\overline{X} f \equiv 0$ (i.e., $f$ is holomorphic along every complex integral curve of $X$) while $f$ is nowhere holomorphic. Thus the generalization of Forelli's theorem fails with such an $X$. The two-dimensional examples given in [7] verify this. For the sake of clarity of the exposition, we describe the examples briefly:

Let the holomorphic vector field $X$ under consideration be not aligned. Then one can always extract, from its Poincaré-Dulac normal form, two distinct complex variables $z$ and $w$ (among the variables $z_1, \ldots, z_n$) such that $X$ contains a linear combination of $\alpha z \frac{\partial}{\partial z} + \beta w \frac{\partial}{\partial w}$ with the value of the ratio $\alpha/\beta$ not real-positive, and that the remaining part of $X$ includes neither $\frac{\partial}{\partial z}$ nor $\frac{\partial}{\partial w}$. (For instance, if $X$ were given as

$$\begin{aligned} X = z_1 \frac{\partial}{\partial z_1} + (2z_2 + i z_1^2) \frac{\partial}{\partial z_2} \\ + (1+i) z_3 \frac{\partial}{\partial z_3} + (1-i) z_4 \frac{\partial}{\partial z_4} + (2z_5 - z_3 z_4) \frac{\partial}{\partial z_5}, \end{aligned}$$

then $z = z_1$ and $w = z_3$ and the vector field we consider is therefore $z \frac{\partial}{\partial z} + (1+i) w \frac{\partial}{\partial w}$. Of course then $\alpha = 1$, $\beta = 1 + i$ in this case.)

We are to show, either in the case of $\alpha/\beta < 0$ or in the case of $\alpha/\beta \in \mathbb{C} \backslash \mathbb{R}$, that a non-holomorphic but $\mathcal{C}^\infty$ smooth function $f$ depending only on two complex variables $z, w$ (and thus holomorphic in any other variables) can exist satisfying $\overline{X} f = 0$, identically. If $-t = \alpha/\beta < 0$, then the function

$$f(z_1, \ldots, z_n) = \begin{cases} \exp\left(-\frac{1}{|w|^t |z|}\right) & \text{if } zw \neq 0 \\ 0 & \text{if } zw = 0 \end{cases} \tag{2.2}$$

is such an example.

If $\alpha/\beta$ is non-real then, changing the complex parameter $\zeta$ for the flow curve of $X$ by $\tau\zeta$ for an appropriate $\tau \in \mathbb{C}$ and changing also the role of $z$ and $w$, we may assume without loss of generality that $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 > 0, \alpha_2 > 0$ and $\beta = t\bar{\alpha}$ with $t > 0$. If we let $\gamma = \frac{1}{2\alpha_1} - \frac{i}{2\alpha_2}$, then there exists a constant $b > 1$ such that the function

$$f(z_1, \ldots, z_n) := \begin{cases} \exp\left[\left(\gamma \log|z| + \frac{\bar{\gamma}}{t} \log|w|\right)^b\right] & \text{if } zw \neq 0 \\ 0 & \text{if } zw = 0 \end{cases}$$

becomes such an example. For further detailed exposition, the reader is invited to read Sect. 7 of [7] (pp. 664–665).

### 2.3 Statement of main theorem

We now present our main theorem in all dimensions.

**Theorem 2.2** *If $F : B^n \to \mathbb{C}$ is a function satisfying the conditions:*

(i) *$F \in \mathcal{C}^\infty(0)$, and*
(ii) *$F$ is holomorphic along every complex integral curve of an aligned holomorphic vector field,*

*then $F$ is holomorphic on a neighborhood of the origin.*

Notice that this theorem includes Theorem 1.3 (the main theorem of [7]).

We are now to present the proof; indeed the rest of the paper is devoted to the proof of this theorem.

## 3 A formal power series analysis

We investigate, at this beginning stage, the proof of Theorem 2.2 on the level of formal power series which establishes the first step toward the complete proof (to be presented in Section 5).

Recall the usual multi-index notation and the ordering as follows: $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $z^\alpha = {z_1}^{\alpha_1} \cdots {z_n}^{\alpha_n}$. We also use the *length* $|\beta| := \beta_1 + \ldots + \beta_n$ and the *lexicographic ordering* $\prec$ defined by:

$$\alpha \prec \beta \Leftrightarrow \exists k : \alpha_s = \beta_s \ \forall s < k \text{ and } \alpha_k < \beta_k.$$

Denote by $\mathbb{N}$ the set of nonnegative integers. For any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, define the set

$$A(\lambda_1, \ldots, \lambda_n) := \left\{ (m_1, \ldots, m_n) \in \mathbb{N}^n : \sum_{j=1}^n m_j \geq 1, \ \sum_{j=1}^n m_j \lambda_j = 0 \right\}.$$

Then we present:

**Proposition 3.1** *Let $X = \sum_{j=1}^n (\lambda_j z_j + g_j(z_1, \ldots, z_{j-1})) \frac{\partial}{\partial z_j}$ be a holomorphic vector field, where each $g_j$ is a holomorphic polynomial with no constant term. If $X$ satisfies the condition*

$$A(\lambda_1, \ldots, \lambda_n) = \varnothing \text{ (the empty set)}, \tag{3.1}$$

*then any formal power series $S = \sum_{\alpha,\beta} C_\alpha^\beta z^\alpha \overline{z^\beta}$ (in the multi-index notation) satisfying $\overline{X} S = 0$ has to be a holomorphic formal power series in the sense that $C_\alpha^\beta = 0$ for every $\beta$ with $|\beta| > 0$.*

*Proof* Consider the term in $\overline{X}S$ of multi-degree $(i_1, \ldots, i_n, j_1, \ldots, j_n)$ where $j_1 = \cdots = j_{n-1} = 0$ and $j_n \neq 0$.

Let $\varphi_\nu = \lambda_\nu z_\nu + g_\nu(z)$ in (2.1), and consider the components of $\overline{X}$ which can now be written as $\bar{\varphi}_\nu \frac{\partial}{\partial \bar{z}_\nu}$ with $\nu < n$. It can only produce terms that contain $\bar{z}_\nu$ with $\nu < n$. So does the $\bar{g}_n$ term in $\bar{\varphi}_n$. Therefore the coefficient of the considered term is equal to

$$\lambda_n j_n C_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}.$$

Since $A = \varnothing$, this coefficient vanishes if and only if $C_{i_1,\ldots,i_n}^{j_1,\ldots,j_n} = 0$ whenever $j_1 = \cdots = j_{n-1} = 0$ and $j_n \neq 0$.

We prove the rest by an induction with respect to the lexicographical ordering $\prec$ on multi-indices $(j_1, \ldots, j_n)$:

Initial $(0, \ldots, 0, j_n)$th step: already established above.

Assuming the steps prior to $(J_1, \ldots, J_n)$, i.e., that $C_{i_1,\ldots,i_n}^{j_1,\ldots,j_n} = 0$ for all $(j_1, \ldots, j_n) \prec (J_1, \ldots, J_n)$, we prove the $(J_1, \ldots, J_n)$th step: Suppose that $J_k \neq 0$ but $J_\nu = 0$ for $\nu < k$. Consider the terms in $\overline{X}S$ of multi-degree $(i_1, \ldots, i_n, J_1 \ldots, J_n)$. We show here that such terms cannot be generated by $\bar{\varphi}_\nu \frac{\partial}{\partial \bar{z}_\nu}$ with $\nu < k$ nor by $\bar{g}_k$ in $\bar{\varphi}_k \frac{\partial}{\partial \bar{z}_k}$.

Similar to the initial step, we see that the $\bar{g}_\nu$ terms in $\bar{\varphi}_\nu \frac{\partial}{\partial \bar{z}_\nu}$ with $\nu \geq k$, if different from zero, either produce terms that contain $\bar{z}_\nu$ with $\nu < k$ or increase the lexicographical multi-degree in $\bar{z}_k, \ldots, \bar{z}_n$. In either case they cannot produce terms of multi-degree $(i_1, \ldots, i_n, J_1 \ldots, J_n)$.

Hence the coefficient of the term in $\overline{X}S$ of multi-degree $(i_1, \ldots, i_n, J_1, \ldots, J_n)$ has to be equal to

$$\sum_{\nu=k}^{n} \lambda_\nu J_\nu C_{i_1,\ldots,i_n}^{J_1,\ldots,J_n}.$$

Since $A = \varnothing$, this coefficient vanishes if and only if $C_{i_1,\ldots,i_n}^{J_1,\ldots,J_n} = 0$. This completes the induction and thus the proof of the proposition. □

*Remark 3.2* The condition (3.1) is essential for the proof. If $X = z_1 \frac{\partial}{\partial z_1} - t z_2 \frac{\partial}{\partial z_2}$ for some positive rational number $t = q/p$, then $X$ understood as a vector field on $\mathbb{C}^2$, it is obvious that $(q, p) \in A(1, t)$ and hence $A(1, t) \neq \varnothing$; this violates Condition (3.1). Also the smooth function $F := |z_1|^{2q} |z_2|^{2p}$ is not holomorphic but satisfies the equation $\overline{X}F \equiv 0$.

All holomorphic vector fields contracting at the origin, and hence in particular any aligned holomorphic vector fields (cf. Definition 2.1), satisfy Condition (3.1). Therefore, Proposition 3.1 implies, in particular, the following real-analytic version of generalized Forelli theorem:

**Theorem 3.3** *If $f : B^n \to \mathbb{C}$ is a real-analytic function satisfying $\overline{X} f = 0$ at every point for a holomorphic vector field X contracting at the origin, then $f$ is holomorphic.*

## 4 A uniqueness theorem

Since it is not known *a priori* whether the function $F$ in the statement of Theorem 2.2 should be real-analytic, showing only the "complex-analyticity" of $F$ on the formal power series level as in Sect. 3 is definitely not sufficient for a proof. In order to show that the function $\partial F/\partial \bar{z}_j$ itself vanishes for all $j = 1, \dots, n$, one needs a new identity theorem for the appropriate class of functions. The goal of this section is indeed to establish such a principle, whose role will become clear in Sect. 5 where we complete the proof of Theorem 2.2.

We begin with introducing the appropriate regions in $\mathbb{C}$. Let $P_1, \dots, P_n$ be polynomials, that are not identically zero, in the single complex variable $\zeta$. Let $\lambda_1, \dots, \lambda_n$ positive real numbers. Consider the open plane-region

$$D(\underline{P}, \underline{\lambda}) := \{\zeta \in \mathbb{C} \colon |P_j(\zeta)| e^{-\lambda_j \operatorname{Re} \zeta} < 1, \;\; j = 1, ..., n\}.$$

We call a sequence $\{\zeta_k\}$ in $D(\underline{P}, \underline{\lambda})$ *admissible*, if

$$\lim_{k \to \infty} |P_j(\zeta_k)| e^{-\lambda_j \operatorname{Re} \zeta_k} = 0$$

for every $j = 1, \dots, n$. It is obvious that the admissible sequences consist of two types of points:

(1) $\zeta_k$ tending to the zeros of $P_j$, and
(2) $\zeta_k$ with $\operatorname{Re} \zeta_k \to +\infty$.

Let $L_r := \{\zeta \in \mathbb{C} \colon \operatorname{Im} \zeta = 0, \operatorname{Re} \zeta > r\}$. For any $D(\underline{P}, \underline{\lambda})$, there exists a positive number $r$ such that $L_r \subset D(\underline{P}, \underline{\lambda})$. Denote by $D^\star(\underline{P}, \underline{\lambda})$ the connected component of $D(\underline{P}, \underline{\lambda})$ containing this ray $L_r$. Needless to say, $D^\star(\underline{P}, \underline{\lambda})$ is an unbounded component of $D(\underline{P}, \underline{\lambda})$.

The unique continuation principle we establish is as follows:

**Proposition 4.1** *Let $f$ be a bounded holomorphic function on $D(\underline{P}, \underline{\lambda})$. If*

$$\lim_{k \to \infty} |f(\zeta_k)| \frac{e^{\lambda_j \ell \operatorname{Re} \zeta_k}}{|P_j(\zeta_k)|^\ell} = 0, \tag{4.1}$$

*for every $j = 1, \dots, n$, for any nonnegative integer $\ell$, and for any admissible sequence $\{\zeta_k\}$ whose members are different from any zeros of any $P_j$'s, then $f \equiv 0$ on $D^\star(\underline{P}, \underline{\lambda})$.*

*Proof* First we notice that we may assume $f$ is continuous up to the boundary of $D(\underline{P}, \underline{\lambda})$ by replacing $P_j$ by $rP_j$ for a constant $r > 1$, $j = 1, \dots, n$.

Since $P_j$'s are nontrivial (i.e., not identically zero) polynomials, there exists $\epsilon > 0$ and $A > 0$ such that

$$\inf_{\substack{\operatorname{Re}\zeta > A \\ 1 \le j \le n}} |P_j(\zeta)| > \epsilon. \tag{4.2}$$

Set $D(\underline{P}, \underline{\lambda}; A) := D(\underline{P}, \underline{\lambda}) \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > A\}$. Re-ordering the coordinate functions $(z_1, \ldots, z_n)$, we may assume without loss of generality that

$$\frac{\deg(P_1)}{\lambda_1} \geq \frac{\deg(P_j)}{\lambda_j}, \quad \forall j = 1, \ldots, n, \tag{4.3}$$

where $\deg(P_j)$ represents the degree of the polynomial $P_j$. Then a sequence $\{\zeta_k\}$ in $D(\underline{P}, \underline{\lambda}; A)$ is admissible if and only if

$$\lim_{k\to\infty} |P_1(\zeta_k)| e^{-\lambda_1 \operatorname{Re}\zeta_k} = 0. \tag{4.4}$$

In fact, since

$$C_j |\zeta|^{d_j} \leq |P_j(\zeta)| \leq \widetilde{C}_j |\zeta|^{d_j}$$

for some positive constants $C_j$ and $\widetilde{C}_j$ as $\zeta \to \infty$, where $d_j = \deg(P_j)$, the inequality (4.3) implies that

$$\begin{aligned}\left(|P_1(\zeta)| e^{-\lambda_1 \operatorname{Re}\zeta}\right)^{\lambda_j/\lambda_1} &\geq C_1^{\lambda_j/\lambda_1} |\zeta|^{d_1\lambda_j/\lambda_1} e^{-\lambda_j \operatorname{Re}\zeta} \\ &\geq C_1^{\lambda_j/\lambda_1} |\zeta|^{d_j} e^{-\lambda_j \operatorname{Re}\zeta} \\ &\geq C_1^{\lambda_j/\lambda_1} \widetilde{C}_j^{-1} |P_j(\zeta)| e^{-\lambda_j \operatorname{Re}\zeta}\end{aligned}$$

for every $\zeta$ with sufficiently large modulus. Therefore, if a sequence $\{\zeta_k\} \subset D(\underline{P}, \underline{\lambda}; A)$ satisfies (4.4), then $\operatorname{Re}\zeta_k \to \infty$, since $|P_j| > \epsilon$ on $D(\underline{P}, \underline{\lambda}; A)$ and hence $\lim_{k\to\infty} |P_j(\zeta_k)| e^{-\lambda_j \operatorname{Re}\zeta_k} = 0$ for every $j$, which means $\{\zeta_k\}$ is admissible.

Consider, for every integer $\ell > 0$, the function

$$g_\ell(\zeta) = f(\zeta) \frac{e^{\lambda_1 \ell \zeta}}{P_1(\zeta)^\ell}.$$

Then the function $g_\ell$ is holomorphic on $D(\underline{P}, \underline{\lambda}; A)$. Now we pose and prove:

**Claim 1.** *$g_\ell$ is bounded on $D(\underline{P}, \underline{\lambda}; A)$ for every $\ell$.*

Suppose the claim is false for some $\ell$. Then there exists a sequence $\{\zeta_k\}$ in $D(\underline{P}, \underline{\lambda}; A)$ such that

$$|g_\ell(\zeta_k)| \to \infty \tag{4.5}$$

as $k \to \infty$. Since $f$ is bounded, it follows that

$$\lim_{k\to\infty} \frac{e^{\lambda_1 \operatorname{Re}\zeta_k}}{|P_1(\zeta_k)|} = \infty,$$

which obviously implies that

$$\lim_{k\to\infty} |P_1(\zeta_k)| e^{-\lambda_1 \operatorname{Re}\zeta_k} = 0.$$

Hence $\{\zeta_k\}$ is an admissible sequence. Then by (4.1), we must have $|g_\ell(\zeta_k)| \to 0$ as $k \to \infty$, a contradiction to (4.5). Therefore Claim 1 is justified.

Next we pose

**Claim 2.** *There exists a constant* $\delta_0 > 0$ *such that* $|P_1(\zeta)|e^{-\lambda_1 \operatorname{Re}\zeta} > \delta_0$ *for every* $\zeta \in \partial D(\underline{P}, \underline{\lambda}) \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq A\}$.

Assume the contrary that there exists a sequence $\{\zeta_k\} \in \partial D(\underline{P}, \underline{\lambda}) \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq A\}$ such that $|P_1(\zeta_k)|e^{-\lambda_1 \operatorname{Re}\zeta_k} \to 0$ as $k \to \infty$. Since $P_j$ has no zeros on $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq A\}$, we see that $\operatorname{Re}\zeta_k \to \infty$ as $k \to \infty$. Therefore, by (4.3), we have

$$|P_j(\zeta_k)|e^{-\lambda_j \operatorname{Re}\zeta_k} \to 0$$

as $k \to \infty$, for every $j = 1, \ldots, n$. On the other hand, whenever $\zeta_k \in \partial D(\underline{P}, \underline{\lambda})$, the definition of the region $D(\underline{P}, \underline{\lambda})$ implies that

$$|P_j(\zeta_k)|e^{-\lambda_j \operatorname{Re}\zeta_k} = 1$$

for some $j$. This contradiction proves Claim 2.

We now finish the proof of Proposition 4.1. It follows by (4.2) that

$$|P_1(\zeta)|e^{-\lambda_1 \operatorname{Re}\zeta} \geq \epsilon e^{-\lambda_1 A}$$

for every $\zeta \in D(\underline{P}, \underline{\lambda}) \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = A\}$. Now, let

$$M := \sup_{\zeta \in D(P,\lambda)} |f(\zeta)|, \quad \text{and} \quad C := \max\{\delta_0^{-1}, \epsilon^{-1}e^{\lambda_1 A}\}.$$

Then $|g_\ell(\zeta)| = |f(\zeta)|\dfrac{e^{\lambda_1 \ell \operatorname{Re}\zeta}}{|P_1(\zeta)|^\ell} \leq MC^\ell$ for every $\zeta \in \partial D(\underline{P}, \underline{\lambda}; A)$. Since each $g_\ell$ is a bounded holomorphic function on $D(\underline{P}, \underline{\lambda}; A)$, the maximum modulus principle implies that

$$|g_\ell(\zeta)| = |f(\zeta)|\frac{e^{\lambda_1 \ell \operatorname{Re}\zeta}}{|P_1(\zeta)|^\ell} \leq MC^\ell$$

for every $\zeta \in D(\underline{P}, \underline{\lambda}; A)$. Here we make use of a version of the maximum modulus principle for unbounded domains which is called the *Phragmén-Lindelöf principle.* (cf., e.g., Corollary 2 in [8], p. 220.) Now let $\Omega$ be a connected open set satisfying:

(a) $\Omega \subset \{\zeta \in D(\underline{P}, \underline{\lambda}; A) \colon |P_1(\zeta)|e^{-\lambda_1 \operatorname{Re}\zeta} < \frac{1}{2C}\}$, and
(b) $\Omega \supset L_B := \{\zeta \in \mathbb{C} \colon \operatorname{Im}\zeta = 0, \operatorname{Re}\zeta > B\}$, for some $B > A$.

Then, for every $\zeta \in \Omega$, we obtain that

$$|f(\zeta)| \leq \frac{M}{2^\ell}$$

for every positive integer $\ell$. This implies that $f(\zeta) = 0$ for every $\zeta \in \Omega$. Thus the unique continuation principle for holomorphic functions in one complex variable implies that $f$ vanishes identically on the unbounded component $D^\star(\underline{P}, \underline{lambda})$ as desired. $\square$

## 5 Proof of Theorem 2.2

We now complete the proof of Theorem 2.2, the main result of this paper. The notation in this section are the same as those used in its statement presented in Sect. 2.

### 5.1 Holomorphic continuation

In the next subsection we shall prove that $F$ is holomorphic on an open neighborhood, say $V$, of the origin. Then, as shown in [7], that property will imply the holomorphicity of $F$ in a larger set called *the saturation set* $\mathrm{Sat}(V, X)$ *for the vector field* $X$, which is defined to be the union of the connected integral curves $L$ of $X$ satisfying $L \cap V \neq \varnothing$. For a presentation that does not depend on the (a priori unknown) neighborhood $V$ one usually considers a neighborhood basis $\mathcal{V}$ of the origin 0 and the set $\mathrm{Sat}(0, X) := \bigcap_{U \in \mathcal{V}} \mathrm{Sat}(U, X)$, called the *saturation set of the origin for the vector field* $X$. When $X$ is a contracting vector field with the isolated singularity at 0, as here, there exists an open neighborhood $U_0$ such that $\mathrm{Sat}(0, X) = \mathrm{Sat}(U_0, X)$, and hence, in particular, the set $\mathrm{Sat}(0, X)$ is open. And, the conclusion can be strengthened to: *$F$ is holomorphic on* $\mathrm{Sat}(0, X)$.

### 5.2 Complex-analyticity of $F$ in an open neighborhood of the origin

Take sufficiently small a neighborhood $\widetilde{V}$ of the origin on which the function $F$ is $\mathcal{C}^2$ smooth. Resizing $\widetilde{V}$, we may assume without loss of generality that $\widetilde{V}$ is the polydisc $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, \quad j = 1, \ldots, n\}$. Our present goal is to establish the complex analyticity of $F$ on an open neighborhood of the origin in $\widetilde{V}$.

We shall now use the normalization of $X$, for instance following [1], p. 187. If $X$ is the *aligned* vector field given in the hypothesis of Theorem 2.2 then, without loss of generality, we may assume that the vector field $X$ now takes the form

$$X = \sum_{j=1}^{n} \big(\lambda_j z_j + h_j(z_1, \ldots, z_{j-1})\big) \frac{\partial}{\partial z_j},$$

with all coefficients $\lambda_j$ of the linear terms in the expression of $X$ positive. Furthermore we have, in addition, the following:

(1) $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and
(2) $h_j$ is a holomorphic polynomial that satisfies

$$h_j(e^{-\lambda_1 \zeta} z_1, \ldots, e^{-\lambda_{j-1} \zeta} z_{j-1}) = \mathrm{e}^{-\lambda_j \zeta}\, h_j(z_1, \ldots, z_{j-1})$$

for every $j = 1, \ldots, n$, any $\zeta \in \mathbb{C}$.

Thus, the complex flow-curve of $X$ passing through $\eta := (\eta_1, \dots, \eta_n)$ can be represented by

$$\zeta \to z(\eta; \zeta) = (z_1(\eta; \zeta), \dots, z_n(\eta; \zeta))$$

where

$$z_j(\eta; \zeta) = e^{-\lambda_j \zeta} \left(\eta_j + g_j(\eta; \zeta)\right)$$

with $g_j$ a holomorphic polynomial in $\eta$ and $\zeta$ satisfying

- $g_1 \equiv 0$,
- $g_j(\eta; 0) \equiv 0$, and
- $g_j(\cdot; \zeta)$ depends only upon $\eta_1, \dots, \eta_{j-1}$ (but not upon $\eta_j, \dots, \eta_n$),

for every $j = 1, \dots, n$.

Note that $\eta = z(\eta; 0)$ for every $\eta \in \widetilde{V}$. Since $X$ is a contracting vector field, there exists a neighborhood $V$, with $V \subset \widetilde{V}$, of the origin such that, for every $\eta \in V$, the real integral curve $z(\eta; t)$ is defined for every real $t \geq 0$ in such a way that its image is contained in $\widetilde{V}$.

Let $G(\eta, \zeta) := F(z(\eta; \zeta))$. Then the hypothesis on $F$ yields that $G(\eta, \zeta)$ is a holomorphic function in the variable $\zeta$ on the region $D_\eta := D(\underline{P_\eta}, \underline{\lambda})$, for any $\eta \in V$. Here, of course,

$$\underline{P_\eta}(\cdot) = (P_{\eta,1}(\cdot), \dots, P_{\eta,n}(\cdot)) = (\eta_1 + g_1(\eta; \cdot), \dots, \eta_n + g_n(\eta; \cdot)),$$

and $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$.

Fix an arbitrary $\eta^o \in V \backslash \{0\}$. Since we are assuming that $F$ is $\mathcal{C}^2$ smooth on $\widetilde{V}$, we have

$$\frac{\partial}{\partial \bar{\zeta}} \left( \frac{\partial G}{\partial \bar{\eta}_j} \right) = \frac{\partial}{\partial \bar{\eta}_j} \left( \frac{\partial G}{\partial \bar{\zeta}} \right) = 0$$

for every $j = 1, \dots, n$. This implies that $\partial G \partial \bar{\eta}_j$ is also a bounded holomorphic function in $\zeta$ defined on $D_{\eta^o}$.

The chain rule yields that

$$\frac{\partial G}{\partial \bar{\eta}_n} = \frac{\partial F}{\partial \bar{z}_n} e^{-\lambda_n \bar{\zeta}}.$$

Since the formal power series of $F$ contains no $\bar{z}$ terms, the function $\partial F / \partial \bar{z}_n$ is a $\mathcal{C}^\infty(0)$-function with the trivial formal power series representation. In particular,

$$\left| \frac{\partial F}{\partial \bar{z}_n}(z) \right| |z_j|^{-\ell} \to 0$$

as $z \to 0$, for every $j = 1, \ldots, n$ and $\ell \geq 0$. Since $e^{-\lambda_n \operatorname{Re} \zeta} \leq C$ on $D_{\eta^o}$ for some constant $C > 0$, this implies that for each $j = 1, \ldots, n$ and $\ell \geq 0$,

$$\left| \frac{\partial G}{\partial \bar{\eta}_n}(\zeta_k) \right| |z_j(\eta^o, \zeta_k)|^{-\ell} \leq C \left| \frac{\partial F}{\partial \bar{z}_n}(z(\eta^o, \zeta_k)) \right| |z_j(\eta^o, \zeta_k)|^{-\ell} \to 0$$

as $k \to \infty$, where $\{\zeta_k\}$ is an arbitrary admissible sequence in $D_{\eta^o}$. This yields that the function $\partial G / \partial \bar{\eta}_n$ satisfies (4.1) on the region $D_{\eta^o}$. Consequently, Proposition 4.1 in Sect. 4 applies here; it follows therefore that $\partial G / \partial \bar{\eta}_n$, as well as $\partial F / \partial \bar{z}_n$, vanishes identically along the flow $\{z(\eta^o, \zeta) : \zeta \in D^\star_{\eta^o}\}$, where $D^\star_{\eta^o}$ is the unbounded component of $D_{\eta^o}$ containing $\mathbb{R}^+ := \{t \in \mathbb{R} : t \geq 0\}$.

Moreover for $z_{n-1}$, the chain rule implies that

$$\begin{aligned} \frac{\partial G}{\partial \bar{\eta}_{n-1}} &= \frac{\partial F}{\partial \bar{z}_{n-1}} e^{-\lambda_{n-1}\bar{\zeta}} + \frac{\partial F}{\partial \bar{z}_n} \cdot \overline{\left( \frac{\partial g_n}{\partial \eta_{n-1}} \right)} \cdot e^{-\lambda_n \bar{\zeta}} \\ &= \frac{\partial F}{\partial \bar{z}_{n-1}} e^{-\lambda_{n-1}\bar{\zeta}} \end{aligned}$$

at every point of the flow curve $\{z(\eta^o, \zeta) : \zeta \in D^\star_{\eta^o}\}$. Proposition 4.1 applies here again to yield that $\partial F / \partial \bar{z}_{n-1} = 0$ on the same flow curve. Repeating this process, we arrive at that $\overline{\partial} F \equiv 0$ on the flow curve $\{z(\eta^o, \zeta) : \zeta \in D^\star_{\eta^o}\}$. Let $\zeta = 0$, in particular, to obtain that $\overline{\partial} F(\eta^o) = 0$.

Since $\eta^o$ is an arbitrarily chosen point of $V \backslash \{0\}$, it follows that $\overline{\partial} F = 0$ at every point of $V \backslash \{0\}$. Hence $F$ is holomorphic on $V$ also, as $F \in \mathcal{C}^2(V)$.

Altogether, the proof of Theorem 2.2 is now complete. □

*Remark 5.1* The proof-arguments just given may appear as if they never used the assumption that the holomorphic vector field $X$ must be aligned. On the contrary, the assumption was used throughout. Notice that Proposition 4.1 in Sect. 4, which has played a crucial role, is valid only for the aligned fields. Furthermore, Theorem 2.2 would not hold if the vector field $X$ were not assumed to be aligned; see the discussion with counterexamples in Sect. 2.2 presented immediately after Definition 2.1.

*Remark 5.2* The arguments presented just now also prove the main theorem of Kim-Poletsky-Schmalz [7], i.e., Theorem 1.3 in Sect. 1 of this paper, but with our condition $F \in \mathcal{C}^\infty(0)$. (N.B. Their original theorem uses only the existence of formal Taylor series at 0. But the method we present in this article needs $F$ to be $\mathcal{C}^2$ in an open neighborhood of the origin 0.) On the other hand the regions $D(\underline{P}, \underline{\lambda})$ for the case of [7] are simpler; they are just half-planes, as their $P_j$'s are constants.

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