



Completion, extension, factorization, and lifting of operators

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Abstract The well-known results of M. G. Kreĭn concerning the description of self-adjoint contractive extensions of a hermitian contraction T_1 and the characterization of all nonnegative selfadjoint extensions \tilde{A} of a nonnegative operator A via the inequalities $A_K \leq \tilde{A} \leq A_F$, where A_K and A_F are the Kreĭn–von Neumann extension and the Friedrichs extension of A , are generalized to the situation, where \tilde{A} is allowed to have a fixed number of negative eigenvalues. These generalizations are shown to be possible under a certain minimality condition on the negative index of the operators $I - T_1^*T_1$ and A , respectively; these conditions are automatically satisfied if T_1 is contractive or A is nonnegative, respectively. The approach developed in this paper starts by establishing first a generalization of an old result due to Yu. L. Shmul’yan on completions of nonnegative block operators. The extension of this fundamental result allows us to prove analogs of the above mentioned results of M. G. Kreĭn and, in addition, to solve some related lifting problems for J -contractive operators in Hilbert, Pontryagin and Kreĭn spaces in a simple manner. Also some new factorization results are derived, for instance, a generalization of the well-known Douglas factorization of Hilbert space operators. In the final steps of the treatment some very recent results concerning inequalities between semibounded selfadjoint relations and their inverses turn out to be central in order to treat the ordering of non-contractive selfadjoint operators under Cayley transforms properly.

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1 Introduction

Almost 70 years ago in his famous paper [47] M. G. Kreĭn proved that for a densely defined nonnegative operator A in a Hilbert space there are two extremal extensions of A , the Friedrichs (hard) extension A_F and the Kreĭn–von Neumann (soft) extension A_K , such that every nonnegative selfadjoint extension \tilde{A} of A can be characterized by the following two inequalities:

$$(A_F + a)^{-1} \leq (\tilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0.$$

To obtain such a description he used Cayley transforms of the form

$$T_1 = (I - A)(I + A)^{-1}T = (I - \tilde{A})(I + \tilde{A})^{-1},$$

to reduce the study of unbounded operators to the study of contractive selfadjoint extensions T of a hermitian nondensely defined contraction T_1 . In the study of contractive selfadjoint extensions of T_1 he introduced a notion which is nowadays called “the shortening of a bounded nonnegative operator H to a closed subspace \mathfrak{N} ” of \mathfrak{H} as the (unique) maximal element in the set

$$\{D \in [\mathfrak{H}] : 0 \leq D \leq H, \text{ran } D \subset \mathfrak{N}\}, \quad (1)$$

which is denoted by $H_{\mathfrak{N}}$; cf. [3, 4, 57]. Here and in what follows the notation $[\mathfrak{H}_1, \mathfrak{H}_2]$ stands for the space of all bounded everywhere defined operators acting from \mathfrak{H}_1 to \mathfrak{H}_2 ; if $\mathfrak{H} = \mathfrak{H}_1 = \mathfrak{H}_2$ then the shorter notation $[\mathfrak{H}] = [\mathfrak{H}_1, \mathfrak{H}_2]$ is used. By means of shortening of operators he proved the existence of a minimal and maximal contractive extension T_m and T_M of T_1 and that T is a selfadjoint contractive extension of T_1 if and only if $T_m \leq T \leq T_M$.

Later the study of nonnegative selfadjoint extensions of $A \geq 0$ was generalized to the case of nondensely defined operators $A \geq 0$ by Ando and Nishio [5], as well as to the case of linear relations (multivalued linear operators) $A \geq 0$ by Coddington and de Snoo [22]. Further studies followed this work of M. G. Kreĭn; the approach in terms of “boundary conditions” to the extensions of a positive operator A was proposed by Vishik [63] and Birman [16]; an exposition of this theory based on the investigation of quadratic forms can be found from [2]. An approach to the extension theory of symmetric operators based on abstract boundary conditions was initiated even earlier by Calkin [21] under the name of reduction operators, and later, independently the technique of boundary triplets was introduced to formalize the study of boundary value problems in the framework of general operator theory; see [20, 29, 31, 37, 43, 54]. Later the extension theory of unbounded symmetric Hilbert space operators and related resolvent formulas originating also from the work of Kreĭn [45, 46], see also e.g. [52], was generalized to the spaces with indefinite inner products in the well-known

series of papers by Langer and Kreĭn, see e.g. [49, 50], and all of this has been further investigated, developed, and extensively applied in various other areas of mathematics and physics by numerous other researchers.

In spite of the long time span, natural extensions of the original results of Kreĭn in [47] seem not to be available in the literature. Obviously the most closely related result appears in Constantinescu and Gheondea [24], where for a given pair of a row operator $T_r = (T_{11}, T_{12}) \in [\mathfrak{H}_1 \oplus \mathfrak{H}'_1, \mathfrak{H}_2]$ and a column operator $T_c = \text{col}(T_{11}, T_{21}) \in [\mathfrak{H}_1, \mathfrak{H}_2 \oplus \mathfrak{H}'_2]$ the problem for determining all possible operators $\tilde{T} \in [\mathfrak{H}_1 \oplus \mathfrak{H}'_1, \mathfrak{H}_2 \oplus \mathfrak{H}'_2]$ acting from the Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}'_1$ to the Hilbert space $\mathfrak{H}_2 \oplus \mathfrak{H}'_2$ such that

$$P_{\mathfrak{H}_2} \tilde{T} = T_r, \quad \tilde{T} \upharpoonright \mathfrak{H}_1 = T_c,$$

and such that the following negative index (number of negative eigenvalues) conditions are satisfied

$$\kappa_1 := \nu_-(I - \tilde{T}^* \tilde{T}) = \nu_-(I - T_c^* T_c), \quad \kappa_2 := \nu_-(I - \tilde{T} \tilde{T}^*) = \nu_-(I - T_r T_r^*),$$

is considered. The problem was solved in [24, Theorem 5.1] under the condition $\kappa_1, \kappa_2 < \infty$. In the literature cited therein appears also a reference to an unpublished manuscript [53] by H. Langer and B. Textorius, where a similar problem for a given bounded hermitian column operator T has been investigated; see [53, Theorems 1.1, 2.8]¹ and [24, Section 6]. However, in these papers the existence of possible extremal extensions in the solution set in the spirit of [47], when it is nonempty, have not been investigated. Also possible investigations of analogous results for unbounded symmetric operators with a fixed negative index seem to be unavailable in the literature.

In this paper we study classes of “quasi-contractive” symmetric operators T_1 with $\nu_-(I - T_1^* T_1) < \infty$ as well as “quasi-nonnegative” operators A with $\nu_-(A) < \infty$ and the existence and description of all possible selfadjoint extensions T and \tilde{A} of them which preserve the given negative indices $\nu_-(I - T^2) = \nu_-(I - T_1^* T_1)$ and $\nu_-(\tilde{A}) = \nu_-(A)$, and prove precise analogs of the above mentioned results of M. G. Kreĭn under a minimality condition on the negative indices $\nu_-(I - T_1^* T_1)$ and $\nu_-(A)$, respectively. It is an unexpected fact that when there is a solution then the solution set still contains a minimal solution and a maximal solution which then describe the whole solution set via two operator inequalities, just as in the original paper of M. G. Kreĭn. The approach developed in this paper differs from the approach in [47]. In fact, technique based on nonnegative completions of operators appearing in papers by Kolmanovich and Malamud [44] and Hassi et al. [39] will be successfully generalized. In particular, we introduce a new class of completion problems for Hilbert space operators, whose solutions evidently admit a wider scope of applications than what is appearing in the present paper.

The starting point in our approach is to establish a generalization of an old result due to Shmul'yan [59] on completions of nonnegative block operators where the result

¹ After the Math ArXiv version of the present paper we inquired contents of that work from H. Langer who then kindly provided us their initial work in [53].

was applied for introducing so-called Hellinger operator integrals. Our extension of this fundamental result is given in Sect. 2; see Theorem 1 (for the case $\kappa < \infty$) and Theorem 2 (for the case $\kappa = \infty$). Obviously these two results, already in view of the various consequences appearing in later sections, may be considered as being most useful inventions in the present paper with further possible applications in problems appearing also elsewhere (see e.g. [4, 6, 27, 28, 58]).

In this paper we will extensively apply Theorem 1. In Sect. 3 this result is specialized to a class of block operators to characterize occurrence of a minimal negative index for the so-called Schur complement, see Theorem 3. This result can be also viewed as a factorization result and, in fact, it yields a generalization of the well-known Douglas factorization of Hilbert space operators in [32], see Proposition 1, which is completed by a generalization of Sylvester's criterion on additivity of inertia on Schur complements in Proposition 2. In Sect. 4, Theorem 1, or its special case Theorem 3, is applied to solve lifting problems for J -contractive operators in Hilbert, Pontryagin and Kreĭn spaces in a new simple way, the most general version of which is formulated in Theorem 4: this result was originally proved in Constantinescu and Gheondea [23, Theorem 2.3] with the aid of [13, Theorem 5.3]; for special cases, see also Dritschel and Rovnyak [33, 34]. In the Hilbert space case the problem has been solved in [12, 25, 62], further proofs and facts can be found e.g. from [8, 10, 19, 44, 55].

Section 5 contains the extension of the fundamental result of Kreĭn in [47], see Theorem 5, which characterizes the existence and gives a description of all selfadjoint extensions T of a bounded symmetric operator T_1 satisfying the following minimal index condition $\nu_-(I - T^2) = \nu_-(I - T_1^2)$ by means of two extreme extensions via $T_m \leq T \leq T_M$. In Sect. 6 selfadjoint extensions of unbounded symmetric operators, and symmetric relations, are studied under a similar minimality condition on the negative index $\nu_-(A)$; the main result there is Theorem 8. It is a natural extension of the corresponding result of Kreĭn in [47]. The treatment here uses Cayley type transforms and hence is analogous to that in [47]. However, the existence of two extremal extensions in this setting and the validity of all the operator inequalities appearing therein depend essentially on so-called "antitonicity results" proved only very recently for semibounded selfadjoint relations in [15] concerning correctness of the implication $H_1 \leq H_2 \Rightarrow H_1^{-1} \geq H_2^{-1}$ in the case that H_1 and H_2 have some finite negative spectra. In this section analogs of the so-called Kreĭn's uniqueness criterion for the equality $T_m = T_M$ are also established.

2 A completion problem for block operators

By definition the modulus $|C|$ of a closed operator C is the nonnegative selfadjoint operator $|C| = (C^*C)^{1/2}$. Every closed operator admits a polar decomposition $C = U|C|$, where U is a (unique) partial isometry with the initial space $\overline{\text{ran}} |C|$ and the final space $\overline{\text{ran}} C$, cf. [42]. For a selfadjoint operator $H = \int_{\mathbb{R}} t dE_t$ in a Hilbert space \mathfrak{H} the partial isometry U can be identified with the signature operator, which can be taken to be unitary: $J = \text{sign}(H) = \int_{\mathbb{R}} \text{sign}(t) dE_t$, in which case one should define $\text{sign}(t) = 1$ if $t \geq 0$ and otherwise $\text{sign}(t) = -1$.

2.1 Completion to operator blocks with finite negative index

The following theorem solves a completion problem for a bounded incomplete block operator A^0 of the form

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix} \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix} \tag{2}$$

in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$.

Theorem 1 *Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be an orthogonal decomposition of the Hilbert space \mathfrak{H} and let A^0 be an incomplete block operator of the form (2). Assume that $A_{11} = A_{11}^*$ and $A_{21} = A_{12}^*$ are bounded, $\nu_-(A_{11}) = \kappa < \infty$, where $\kappa \in \mathbb{Z}_+$, and let $J = \text{sign}(A_{11})$ be the (unitary) signature operator of A_{11} . Then:*

- (1) *There exists a completion $A \in [\mathfrak{H}]$ of A^0 with some operator $A_{22} = A_{22}^* \in [\mathfrak{H}_2]$ such that $\nu_-(A) = \nu_-(A_{11}) = \kappa$ if and only if*

$$\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}. \tag{3}$$

- (2) *If (3) is satisfied, then the operator $S = |A_{11}|^{[-1/2]}A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined and $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Moreover, S^*JS is the smallest operator in the solution set*

$$\mathcal{A} := \{A_{22} = A_{22}^* \in [\mathfrak{H}_2] : A = (A_{ij})_{i,j=1}^2 : \nu_-(A) = \kappa\} \tag{4}$$

and this solution set admits a description as the (semibounded) operator interval given by

$$\mathcal{A} = \{A_{22} \in [\mathfrak{H}_2] : A_{22} = S^*JS + Y, Y = Y^* \geq 0\}.$$

Proof (i) Assume that there exists a completion $A_{22} \in \mathcal{A}$. Let $\lambda_\kappa \leq \lambda_{\kappa-1} \leq \dots \leq \lambda_1 < 0$ be all the negative eigenvalues of A_{11} and let ε be such that $|\lambda_1| > \varepsilon > 0$. Then $0 \in \rho(A_{11} + \varepsilon)$ and hence one can write

$$\begin{aligned} & \begin{pmatrix} I & 0 \\ -A_{21}(A_{11} + \varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} + \varepsilon & A_{12} \\ A_{21} & A_{22} + \varepsilon \end{pmatrix} \begin{pmatrix} I - (A_{11} + \varepsilon)^{-1}A_{12} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_{11} + \varepsilon & 0 \\ 0 & A_{22} + \varepsilon - A_{21}(A_{11} + \varepsilon)^{-1}A_{12} \end{pmatrix} \end{aligned} \tag{5}$$

The operator in the righthand side of (5) has κ negative eigenvalues if and only if

$$A_{21}(A_{11} + \varepsilon)^{-1}A_{12} \leq A_{22} + \varepsilon \tag{6}$$

or equivalently

$$\int_{-\|A_{11}\|}^{\|A_{11}\|} (t + \varepsilon)^{-1} d\|E_t A_{12} f\|^2 \leq \varepsilon \|f\|^2 + (A_{22} f, f), \tag{7}$$

where E_t is the spectral family of A_{11} and $f \in \mathfrak{H}_2$. We rewrite (7) in the form

$$\int_{[-\|A_{11}\|, 0)} (t + \varepsilon)^{-1} d\|E_t A_{12} f\|^2 + \int_{[0, \|A_{11}\|]} (t + \varepsilon)^{-1} d\|E_t A_{12} f\|^2 \leq \varepsilon \|f\|^2 + (A_{22} f, f),$$

This yields the estimate

$$\int_{[0, \|A_{11}\|]} (t + \varepsilon)^{-1} d\|E_t A_{12} f\|^2 \leq \varepsilon \|f\|^2 + (A_{22} f, f) - \frac{1}{\lambda_1 + \varepsilon} \|A_{12} f\|^2. \tag{8}$$

By letting $\varepsilon \searrow 0$ in (8) the monotone convergence theorem implies that

$$P_+ A_{12} f \in \text{ran } A_{11+}^{1/2} \subset \text{ran } |A_{11}|^{1/2}$$

for all $f \in \mathfrak{H}_2$; here $A_{11+} = \int_{[0, \|A_{11}\|]} t dE_t$ stands for the nonnegative part of A_{11} and P_+ is the orthogonal projection onto the corresponding closed subspace $\overline{\text{ran } A_{11+}} = \int_{[0, \|A_{11}\|]} dE_t$. Since $\text{ran } (I - P_+)$ is the κ -dimensional spectral subspace of A_{11} corresponding to its negative spectrum, one concludes that

$$(I - P_+) A_{12} f \in \text{ran } A_{11} \subset \text{ran } |A_{11}|^{1/2}$$

for all $f \in \mathfrak{H}_2$. Therefore, $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$.

Conversely, if $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$, then the operator $S := |A_{11}|^{[-1/2]} A_{12}$ is well defined, closed and bounded, i.e., $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Since $A_{12} = |A_{11}|^{1/2} S$, it follows from $A_{21} = S^* |A_{11}|^{1/2}$ and

$$\left(\begin{array}{c} |A_{11}|^{1/2} \\ S^* J \end{array} \right) J (|A_{11}|^{1/2} JS) : \nu_-(A) = \kappa, \tag{9}$$

that the operator $A_{22} = S^* JS$ gives a completion for A^0 .

(ii) The proof of (i) shows that $A_{21} = S^* |A_{11}|^{1/2}$ is well defined and that $S^* JS \in [\mathfrak{H}_2]$ gives a solution to the completion problem (2). Now

$$s - \lim_{\varepsilon \searrow 0} A_{21} (A_{11} + \varepsilon)^{-1} A_{12} = s - \lim_{\varepsilon \searrow 0} S^* |A_{11}|^{1/2} (A_{11} + \varepsilon)^{-1} |A_{11}|^{1/2} S = S^* JS$$

and if A_{22} is an arbitrary operator in the set (4), then by letting $\varepsilon \searrow 0$ one concludes that $S^* JS \leq A_{22}$. Therefore, $S^* JS$ satisfies the desired minimality property.

To prove the last statement assume that $Y \in [\mathfrak{H}_2]$ and that $Y \geq 0$. Then $A_{22} = S^*JS + Y$ inserted in A^0 defines a block operator $A_Y \geq A_{\min}$. In particular, $\nu_-(A_Y) \leq \nu_-(A_{\min}) = \kappa < \infty$. On the other hand, it is clear from the formula

$$A_Y = \begin{pmatrix} |A_{11}|^{1/2} \\ S^*J \end{pmatrix} J (|A_{11}|^{1/2} JS) + \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \tag{10}$$

that the κ -dimensional eigenspace corresponding to the negative eigenvalues of A_{11} is A_Y -negative and, hence, $\nu_-(A_Y) \geq \kappa$. Therefore, $\nu_-(A_Y) = \kappa$ and $Y \in \mathcal{A}$.

Notice that in the factorization $A_{12} = |A_{11}|^{1/2}S$, S is uniquely determined under the condition $\text{ran } S \subset \overline{\text{ran}} A_{11}$ (which implies that $\ker A_{12} = \ker S$); cf. [32].

In the case that $\kappa = 0$, the result in Theorem 1 reduces to the well-known criterion concerning completion of an incomplete block operator to a nonnegative operator; cf. [59]. In the case of matrices acting on a finite dimensional Hilbert space, the result with $\kappa > 0$ has been proved very recently in the appendix of [28], where it was applied in solving indefinite truncated moment problems. In the present paper Theorem 1 will be one of the main tools for further investigations.

2.2 Completion to operator blocks with an infinite negative index.

The completion result in Theorem 1 is of some general interest already by the substantial number of its applications known in the case of nonnegative operators. In this section the completion problem is treated in the case that $\kappa = \infty$. For this purpose some further notions will be introduced.

Recall that a subspace $\mathfrak{M} \subset \mathfrak{H}$ is said to be uniformly A -negative, if there exists a positive constant $\nu > 0$ such that $(Af, f) \leq -\nu\|f\|^2$ for all $f \in \mathfrak{M}$. It is maximal uniformly A -negative, if \mathfrak{M} has no proper uniformly A -negative extension. The completion problem is now extended by claiming from the completions the following maximality property:

There exists a subspace $\mathfrak{M} \subset \mathfrak{H}_1$ which is maximal uniformly A -negative. (11)

Theorem 2 *Let A^0 be an incomplete block operator of the form (2) in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let $A_{11} = A_{11}^*$ and $A_{21} = A_{12}^*$ be bounded, let $J = \text{sign}(A_{11})$ be the (unitary) signature operator of A_{11} , and, in addition, assume that there is a spectral gap $(-\delta, 0) \subset \rho(A_{11})$, $\delta > 0$. Then:*

- (i) *There exists a completion $A \in [\mathfrak{H}]$ of A^0 with some operator $A_{22} = A_{22}^*$ satisfying the condition (11) if and only if*

$$\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}.$$

- (ii) *If the condition in (i) is satisfied, then $S = |A_{11}|^{[-1/2]}A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined and*

$S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Moreover, S^*JS is the smallest operator in the solution set

$$\mathcal{A} := \{A_{22} = A_{22}^* \in [\mathfrak{H}_2] : A = (A_{ij})_{i,j=1}^2 \text{ satisfies (11)}\}$$

and this solution set admits a description as the (semibounded) operator interval given by

$$\mathcal{A} = \{A_{22} \in [\mathfrak{H}_2] : A_{22} = S^*JS + Y, Y = Y^* \geq 0\}.$$

Proof To prove this result suitable modifications in the proof of Theorem 1 are needed. (i) First assume that $A_{22} \in \mathcal{A}$ gives a desired completion for A^0 . If $\varepsilon \in (0, \delta)$ then $0 \in \rho(A_{11} + \varepsilon)$ and therefore the block operator (A_{ij}) satisfies the formula (5). We claim that the condition (11) implies the inequality (6) for all sufficiently small values $\varepsilon > 0$. To see this let $\mathfrak{M} \subset \mathfrak{H}_1$ be a subspace for which the condition (11) is satisfied. Then $(A_{11}f, f) \leq -\nu\|f\|^2$ for some fixed $\nu > 0$ and for all $f \in \mathfrak{M}$. Assume that for some $0 < \varepsilon_0 < \min\{\nu, \delta\}$ (6) is not satisfied. Then $((A_{22} + \varepsilon_0 - A_{21}(A_{11} + \varepsilon_0)^{-1}A_{12})v_0, v_0) < 0$ holds for some vector $v_0 \in \mathfrak{H}_2$. Define $\mathfrak{L} = W_{\varepsilon_0}^{-1}(\mathfrak{M} + \text{span}\{v_0\})$, where

$$W_{\varepsilon_0} = \begin{pmatrix} I - (A_{11} + \varepsilon_0)^{-1}A_{12} \\ 0 \quad I \end{pmatrix}.$$

Clearly, W_{ε_0} is bounded with bounded inverse and it maps \mathfrak{M} bijectively onto \mathfrak{M} , so that \mathfrak{L} is a 1-dimensional extension of \mathfrak{M} . It follows from (5) that for all $f \in \mathfrak{L}$,

$$(Af, f) + \varepsilon_0\|f\|^2 = \left(\begin{pmatrix} A_{11} + \varepsilon_0 & 0 \\ 0 & A_{22} + \varepsilon_0 - A_{21}(A_{11} + \varepsilon_0)^{-1}A_{12} \end{pmatrix} u, u \right) < 0,$$

where $u = W_{\varepsilon_0}f \in \mathfrak{M} + \text{span}\{v_0\}$. Therefore, \mathfrak{L} is a proper uniformly A -negative extension of \mathfrak{M} ; a contradiction, which shows that (6) holds for all $0 < \varepsilon < \min\{\nu, \delta\}$. Then, as in the proof of Theorem 1 it is seen that $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$; note that in the estimate (8) λ_1 is to be replaced by $-\delta$.

Conversely, if $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$, then $S = |A_{11}|^{-1/2}A_{12} \in [\mathfrak{H}_2, \mathfrak{H}_1]$ and the block operator A in (9) gives a completion. To prove that A satisfies (11) observe that if \mathfrak{M} is a uniformly A -negative subspace in \mathfrak{H} , then $(|A_{11}|^{1/2} JS)$ maps it bijectively onto a uniformly J -negative subspace in \mathfrak{H}_1 . The spectral subspace corresponding to the negative spectrum of A_{11} is maximal uniformly J -negative in \mathfrak{H}_1 and also uniformly A -negative in \mathfrak{H} . By the above mapping property this subspace must be maximal uniformly A -negative in \mathfrak{H} .

(ii) If $A_{22} = A_{22}^*$ defines a completion $A \in [\mathfrak{H}]$ of A^0 such that (11) is satisfied then by the proof of (i) the inequality (6) holds for all sufficiently small values $\varepsilon > 0$. Now the minimality property of S^*JS can be obtained in the same manner as in Theorem 1.

As to the last statement again for every $Y \in [\mathfrak{H}_2]$, $Y \geq 0$, the block operator A_Y defined in the proof of Theorem 1 satisfies $A_Y \geq A_{\min}$. Hence, every uniformly A_Y -negative subspace is also uniformly A_{\min} -negative. Now it follows from the formula (10) that the spectral subspace corresponding to the negative spectrum of A_{11} , which

is maximal uniformly A_{\min} -negative, is also maximal uniformly A_Y -negative. Hence, A_Y satisfies (11) and $Y \in \mathcal{A}$.

3 Some factorizations of operators with finite negative index

Theorems 1 and 2 contain a valuable tool in solving a couple of other problems, which initially do not occur as a completion problem of some symmetric incomplete block operator. In this section it is shown that Theorem 1 (a) can be used to characterize the existence of certain J -contractive factorizations of operators via a minimal index condition; (b) implies an extension of the well-known Douglas factorization result with a certain specification to the Bognár–Krámli factorization; (c) yields an extension of a factorization result of Shmul’yan for J -bicontractions; (d) allows an extension of a classical Sylvester’s law of inertia of a block operator, which is originally used in characterizing nonnegativity of a bounded block operator via Schur complement.

Some simple inertia formulas are now recalled. The factorization $H = B^*EB$ clearly implies that $v_{\pm}(H) \leq v_{\pm}(E)$. If H_1 and H_2 are selfadjoint operators, then

$$H_1 + H_2 = \begin{pmatrix} I \\ I \end{pmatrix}^* \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

shows that $v_{\pm}(H_1 + H_2) \leq v_{\pm}(H_1) + v_{\pm}(H_2)$. Consider the selfadjoint block operator $H \in [\mathfrak{H}_1 \oplus \mathfrak{H}_2]$ of the form

$$H = \begin{pmatrix} A & B^* \\ B & J_2 \end{pmatrix}, \tag{12}$$

where $J_2 = J_2^* = J_2^{-1}$. By applying the above mentioned inequalities shows that

$$v_{\pm}(A) \leq v_{\pm}(A - B^*J_2B) + v_{\pm}(J_2). \tag{13}$$

Assuming that $v_{-}(A - B^*J_2B)$ and $v_{-}(J_2)$ are finite, the question when $v_{-}(A)$ attains its maximum in (13), or equivalently, $v_{-}(A - B^*J_2B) \geq v_{-}(A) - v_{-}(J_2)$ attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 1. Recall that if $A = J_A|A|$ is the polar decomposition of A , then one can interpret $\mathfrak{H}_A = (\overline{\text{ran}} A, J_A)$ as a Kreĭn space generated on $\overline{\text{ran}} A$ by the fundamental symmetry $J_A = \text{sgn}(A)$.

Theorem 3 *Let $A \in [\mathfrak{H}_1]$ be selfadjoint, $B \in [\mathfrak{H}_1, \mathfrak{H}_2]$, $J_2 = J_2^* = J_2^{-1} \in [\mathfrak{H}_2]$, and assume that $v_{-}(A), v_{-}(J_2) < \infty$. If the equality*

$$v_{-}(A) = v_{-}(A - B^*J_2B) + v_{-}(J_2) \tag{14}$$

holds, then $\text{ran } B^ \subset \text{ran } |A|^{1/2}$ and $B^* = |A|^{1/2}K$ for a unique operator $K \in [\mathfrak{H}_2, \mathfrak{H}_A]$ which is J -contractive: $J_2 - K^*J_AK \geq 0$.*

Conversely, if the equality $B^ = |A|^{1/2}K$ holds for some J -contractive operator $K \in [\mathfrak{H}_2, \overline{\text{ran}} A]$, then the equality (14) is satisfied.*

Proof Assume that (14) is satisfied. The factorization

$$H = \begin{pmatrix} A & B^* \\ B & J_2 \end{pmatrix} = \begin{pmatrix} I & B^* J_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^* J_2 B & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ J_2 B & I \end{pmatrix}$$

shows that $\nu_-(H) = \nu_-(A - B^* J_2 B) + \nu_-(J_2)$, which combined with the equality (14) gives $\nu_-(H) = \nu_-(A)$. Therefore, by Theorem 1 one has $\text{ran } B^* \subset \text{ran } |A|^{1/2}$ and this is equivalent to the existence of a unique operator $K \in [\mathfrak{H}_2, \overline{\text{dom } A}]$ such that $B^* = |A|^{1/2} K$; i.e. $K = |A|^{[-1/2]} B^*$. Furthermore, $K^* J_A K \leq J_2$ by the minimality property of $K^* J_A K$ in Theorem 1, in other words K is a J -contraction.

Converse, if $B^* = |A|^{1/2} K$ for some J -contraction $K \in [\mathfrak{H}_2, \overline{\text{dom } A}]$, then clearly $\text{ran } B^* \subset \text{ran } |A|^{1/2}$. By Theorem 1 the completion problem for H^0 has solutions with the minimal solution $S^* J_A S$, where

$$S = |A|^{[-1/2]} B^* = |A|^{[-1/2]} |A|^{1/2} K = K.$$

Furthermore, by J -contractivity of K one has $K^* J_A K \leq J_2$, i.e. J_2 is also a solution and thus $\nu_-(H) = \nu_-(A)$ or, equivalently, the equality (14) is satisfied.

While Theorem 3 is obtained as a direct consequence of Theorem 1 it will be shown in the next section that this result yields simple solutions to a wide class of lifting problems for contractions in Hilbert, Pontryagin and Kreĭn space settings.

Before deriving the next result some inertia formulas for a class of selfadjoint block operators are recalled. Consider the following two representations

$$\begin{aligned} \begin{pmatrix} J_1 & T^* \\ T & J_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ T J_1 & I \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 - T J_1 T^* \end{pmatrix} \begin{pmatrix} I & J_1 T^* \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & T^* J_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} J_1 - T^* J_2 T & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ J_2 T & I \end{pmatrix}, \end{aligned}$$

where $J_i = J_i^* = J_i^{-1}$, $i = 1, 2$. Since here the triangular operators are bounded with bounded inverse, one concludes that $\text{ran } (J_2 - T J_1 T^*)$ is closed if and only if $\text{ran } (J_1 - T^* J_2 T)$ is closed. Furthermore, one gets the following inertia formulas; cf. e.g. [13, Proposition 3.1].

Lemma 1 *With the above notations one has*

$$\nu_{\pm}(J_1 - T^* J_2 T) + \nu_{\pm}(J_2) = \nu_{\pm}(J_2 - T J_1 T^*) + \nu_{\pm}(J_1),$$

$$\nu_0(J_1 - T^* J_2 T) = \nu_0(J_2 - T J_1 T^*).$$

The next result contains two general factorization results: assertion (i) contains an extension of the well-known Douglas factorization, see [32,35], and assertion (ii) is a specification of the so-called Bognár–Krámli factorization, see [18]: $A = B^* J_2 B$ holds for some bounded operator B if and only if $\nu_{\pm}(J_2) \geq \nu_{\pm}(A)$.

Proposition 1 *Let $A, B,$ and J_2 be as in Theorem 3, and let $v_-(A) = v_-(J_2) < \infty$. Then:*

(i) *The inequality*

$$A \geq B^* J_2 B \tag{15}$$

holds if and only if $B = C|A|^{1/2}$ for some J -contractive operator $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; in this case C is unique and, in addition, J -bicontractive, i.e., $J_A - C^ J_2 C \geq 0$ and $J_2 - C J_A C^* \geq 0$.*

(ii) *The equality*

$$A = B^* J_2 B \tag{16}$$

holds if and only if $B = C|A|^{1/2}$ for some J -isometric operator $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; again C is unique. In addition, C is unitary if and only if $\text{ran } B$ is dense in \mathfrak{H}_2 .

Proof (i) The inequality (15) means that $v_-(A - B^* J_2 B) = 0$. Hence the assumption $v_-(A) = v_-(J_2) < \infty$ implies the equality (14). Therefore, the desired factorization for B is obtained from Theorem 3. Conversely, if $B = C|A|^{1/2}$ for some J -contractive operator C then (14) holds by Theorem 3 and the assumption $v_-(A) = v_-(J_2) < \infty$ implies that $v_-(A - B^* J_2 B) = 0$.

The fact that C is actually J -bicontractive follows directly from Lemma 1.

(ii) Assume that (16) holds. Then by part (i) it remains to prove that in the factorization $B = C|A|^{1/2}$ the operator C is isometric. Substituting $B = C|A|^{1/2}$ into (16) gives

$$A = |A|^{1/2} C^* J_2 C |A|^{1/2}.$$

Since $\text{dom } C, \text{ran } C^* \subset \overline{\text{ran } A}$ and $A = |A|^{1/2} J_A |A|^{1/2}$, the previous identity implies the equality $J_A = C^* J_2 C$, i.e., C is J -isometric. Conversely, if C is J -isometric then clearly (16) holds.

Since $B = C|A|^{1/2}$ and $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$, it is clear that B has dense range in \mathfrak{H}_2 precisely when the range of C is dense in \mathfrak{H}_2 . The (Kreĭn space) adjoint is a bounded operator with $\text{dom } C^{[*]} = \mathfrak{H}_2$. By isometry one has $C^{-1} \subset C^{[*]}$, and thus C^{-1} is also bounded, densely defined and closed. Thus, the equality $C^{-1} = C^{[*]}$ prevails, i.e., C is J -unitary. Conversely, if C is unitary then $C^{-1} = C^{[*]}$ holds and $\text{ran } C = \text{dom } C^{[*]} = \mathfrak{H}_2$. Consequently, $\text{ran } B = \text{ran } C|A|^{1/2}$ is dense in \mathfrak{H}_2 .

If, in particular, $v_-(A) = v_-(J_2) = 0$ then $0 \leq A \leq B^* B$ and Proposition 1 combined with Theorem 1 yields the factorization and range inclusion results proved in [32, Theorem 1] with A replaced by $A^* A$. In particular, notice that if $\text{ran } B^* \subset \text{ran } |A|^{1/2}$, then already Theorem 1 alone implies that $S = |A|^{[-1/2]} B^*$ is bounded and hence $B^* B = |A|^{1/2} S S^* |A|^{1/2} \leq \|S\|^2 A$.

Assertions in part (ii) of Corollary 1 can be found in the literature with a different proof. In fact, the first statement in (ii) appears in [13, Proposition 2.1, Corollary 2.6] while the second statement in (ii) is proved in [23, Corollary 1.3]. Another extension for Douglas’ factorization result can be found from [58].

For a general treatment of isometric (not necessarily densely defined) operators and isometric relations appearing in the proof of Proposition 1 the reader is referred to [14], [26, Section 2], and [27].

A slightly different viewpoint to Proposition 1 gives the following statement, which can be viewed as an extension of a theorem by Shmul'yan, see [60, Theorem 3], on the factorization of bicontractions on Kreĭn spaces; for a related abstract Leech theorem, see [34, Section 3.4].

Corollary 1 *Let $A \in [\mathfrak{H}_1]$ be selfadjoint, let $B \in [\mathfrak{H}_1, \mathfrak{H}_2]$, and let $J_2 = J_2^* = J_2^{-1} \in [\mathfrak{H}_2]$ with $\nu_-(J_2) < \infty$. Then:*

(i)

$$A \geq B^* J_2 B \quad \text{and} \quad \nu_-(A) = \nu_-(J_2)$$

if and only if $B = C|A|^{1/2}$ for some J -bicontractive operator $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; in this case C is unique.

(ii)

$$A = B^* J_2 B \quad \text{and} \quad \nu_-(A) = \nu_-(J_2)$$

if and only if $B = C|A|^{1/2}$ for some J -bicontractive operator C which is also J -isometric, i.e., $J_A - C^ J_2 C = 0$ and $J_2 - C J_A C^* \geq 0$; again C is unique.*

Proof Observe that if C is J -bicontractive, then an application of Lemma 1 shows that $\nu_-(J_2) = \nu_-(J_A) = \nu_-(A)$. Now the stated equivalences can be obtained from Proposition 1.

This section is finished with an extension of Sylvester's law of inertia, which is actually obtained as a consequence of Theorem 1.

Proposition 2 *Let $A = (A_{ij})_{i,j=1}^2$ be an arbitrary selfadjoint block operator in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, which satisfies the range inclusion (3), and let $S = |A_{11}|^{[-1/2]} A_{12}$. Then $\nu_-(A) < \infty$ if and only if $\nu_-(A_{11}) < \infty$ and $\nu_-(A_{22} - S^* J S) < \infty$; in this case*

$$\nu_-(A) = \nu_-(A_{11}) + \nu_-(A_{22} - S^* J S).$$

In particular, $A \geq 0$ if and only if $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$, $A_{11} \geq 0$, and $A_{22} - S^ J S \geq 0$.*

Proof By the assumption (3) $S = |A_{11}|^{[-1/2]} A_{12}$ is an everywhere defined bounded operator and, since $A_{11} = |A_{11}|^{1/2} J |A_{11}|^{1/2}$ (cf. Theorem 1), the following equality holds:

$$A = \begin{pmatrix} |A_{11}|^{1/2} & 0 \\ S^* J & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & A_{22} - S^* J S \end{pmatrix} \begin{pmatrix} |A_{11}|^{1/2} & J S \\ 0 & I \end{pmatrix},$$

i.e. $A = B^* E B$ where E stands for the diagonal operator with $\nu_-(E) = \nu_-(A_{11}) + \nu_-(A_{22} - S^* J S)$ and the triangular operator B on the right side is bounded and has dense range in $\overline{\text{ran}} A_{11} \oplus \mathfrak{H}_2$. Clearly, $\nu_-(A) \leq \nu_-(E)$ and it remains to prove that if $\nu_-(A) < \infty$ then $\nu_-(A) = \nu_-(E)$.

To see this assume that $\nu_-(A) < \nu_-(E)$. We claim that $\text{ran } B$ contains an E -negative subspace \mathfrak{L} with dimension $\dim \mathfrak{L} > \nu_-(A)$. Assume the converse and let $\mathfrak{L} \subset \text{ran } B$ be a maximal E -negative subspace with $\dim \mathfrak{L} \leq \nu_-(A)$. Then $(E\mathfrak{L})^\perp$ must be E -nonnegative, since if $v \perp E\mathfrak{L}$ and $(Ev, v) < 0$, then $\text{span}\{v + \mathfrak{L}\}$ would be a proper E -negative extension of \mathfrak{L} . Since $E\mathfrak{L}$ is finite dimensional and $\text{ran } B$ is dense in $\overline{\text{ran}} A_{11} \oplus \mathfrak{H}_2$, $\text{ran } B$ has dense intersection with $(\overline{\text{ran}} A_{11} \oplus \mathfrak{H}_2) \ominus E\mathfrak{L}$, and hence the closure of this subspace is also E -nonnegative. Consequently, $\nu_-(E) = \nu_-(\mathfrak{L})$, a contradiction with the assumption $\nu_-(E) > \nu_-(A)$. This proves the claim that $\text{ran } B$ contains an E -negative subspace \mathfrak{L} with $\dim \mathfrak{L} > \nu_-(A)$. However, then the subspace $\mathfrak{L}' = \{u \in \overline{\text{ran}} A_{11} \oplus \mathfrak{H}_2 : Bu \in \mathfrak{L}\}$ satisfies $\dim \mathfrak{L}' \geq \dim \mathfrak{L}$ and, moreover, \mathfrak{L}' is A -negative: $(Au, u) = (EBu, Bu) < 0, u \in \mathfrak{L}', u \neq 0$. Thus, $\nu_-(A) \geq \dim \mathfrak{L}$, a contradiction with $\dim \mathfrak{L} > \nu_-(A)$. This completes the proof.

Proposition 2 completes Theorem 1: if $\text{ran } A_{12} \subset \text{ran } |A_{11}|^{1/2}$ then $A_{11} = J|A_{11}|$ and $A_{12} = |A_{11}|^{1/2}S$ imply that $A_{21}|A_{11}|^{[-1/2]}J|A_{11}|^{[-1/2]}A_{12} = S^*JS$. Hence the negative index of A can be calculated by using the following version of a *generalized Schur complement* or a *shorted operator* (defined initially for a nonnegative operator H in (1))

$$A_{\mathfrak{H}_2} := \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^*JS \end{pmatrix} \tag{17}$$

via the explicit formula

$$\nu_-(A) = \nu_-(A_{11}) + \nu_-(A_{22} - A_{21}|A_{11}|^{[-1/2]}J|A_{11}|^{[-1/2]}A_{12}). \tag{18}$$

The addition made in Proposition 2 concerns selfadjoint operators A_{22} that are not solutions to the original completion problem for A^0 .

The notion of a shorted operator in infinite dimensional Hilbert spaces has been extended to the case of not necessarily selfadjoint block operators in a paper by Antezana et al. [6]. These so-called bilateral shorted operators introduced and studied therein use two range inclusions, see [6, Definitions 3.5, 4.1], which in the selfadjoint case reduce to the single condition (3) appearing in Theorems 1 and 2.

4 Lifting of operators with finite negative index

As a first application of the completion problem solved in Sect. 2 it is shown how nicely some lifting results established in a series of papers by Arsene, Constantinescu, and Gheondea, see [12, 13, 23, 24], as well as in Dritschel and Rovnyak [33, 34] (see also further references appearing in these papers) on contractive operators with finite number of negative squares can be derived from Theorem 1.

For this purpose some standard notations are introduced. Let $(\mathfrak{H}_1, (\cdot, \cdot)_1)$ and $(\mathfrak{H}_2, (\cdot, \cdot)_2)$ be Hilbert spaces and let J_1 and J_2 be symmetries in \mathfrak{H}_1 and \mathfrak{H}_2 , i.e. $J_i = J_i^* = J_i^{-1}$, so that $(\mathfrak{H}_i, (J_i \cdot, \cdot)_i), i = 1, 2$, becomes a Kreĭn space. Then associate with $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ the corresponding defect and signature operators

$$D_T = |J_1 - T^* J_2 T|^{1/2}, \quad J_T = \text{sign}(J_1 - T^* J_2 T), \quad \mathfrak{D}_T = \overline{\text{ran}} D_T,$$

where the so-called defect subspace \mathfrak{D}_T can be considered as a Kreĭn space with the fundamental symmetry J_T . Similar notations are used with T^* :

$$D_{T^*} = |J_2 - T J_1 T^*|^{1/2}, \quad J_{T^*} = \text{sign}(J_2 - T J_1 T^*), \quad \mathfrak{D}_{T^*} = \overline{\text{ran}} D_{T^*}.$$

By definition $J_T D_T^2 = J_1 - T^* J_2 T$ and $J_T D_T = D_T J_T$ with analogous identities for D_{T^*} and J_{T^*} . In addition,

$$\begin{aligned} (J_1 - T^* J_2 T) J_1 T^* &= T^* J_2 (J_2 - T J_1 T^*), \\ (J_2 - T J_1 T^*) J_2 T &= T J_1 (J_1 - T^* J_2 T). \end{aligned} \tag{19}$$

Recall that $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ is said to be a J -contraction if $J_1 - T^* J_2 T \geq 0$, i.e. $\nu_-(J_1 - T^* J_2 T) = 0$. If, in addition, T^* is a J -contraction, T is termed as a J -bicontraction, in which case $\nu_-(J_1) = \nu_-(J_2)$ by Lemma 1. In what follows it is assumed that

$$\kappa_1 := \nu_-(J_1 - T^* J_2 T) < \infty, \quad \kappa_2 := \nu_-(J_2 - T J_1 T^*) < \infty.$$

In this case Lemma 1 shows that

$$\nu_-(J_2) = \nu_-(J_1) + \kappa_2 - \kappa_1. \tag{20}$$

The aim in this section is to show applicability of Theorem 1 in establishing formulas for so-called liftings \tilde{T} of T with prescribed negative indices $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ for the defect subspaces, equivalently, for the associated signature operators. Given a bounded operator $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ the problem is to describe all operators \tilde{T} from the extended Kreĭn space $(\mathfrak{H}_1 \oplus \mathfrak{H}'_1, J_1 \oplus J'_1)$ to the extended Kreĭn space $(\mathfrak{H}_2 \oplus \mathfrak{H}'_2, J_2 \oplus J'_2)$ such that

$$(*) \quad P_2 \tilde{T} \upharpoonright \mathfrak{H}_1 = T \quad \text{and} \quad \nu_-(\tilde{J}_1 - \tilde{T}^* \tilde{J}_2 \tilde{T}) = \tilde{\kappa}_1, \quad \nu_-(\tilde{J}_2 - \tilde{T} \tilde{J}_1 \tilde{T}^*) = \tilde{\kappa}_2,$$

with some fixed values of $\tilde{\kappa}_1, \tilde{\kappa}_2 < \infty$. Here P_i stands for the orthogonal projection from $\tilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}'_i$ onto \mathfrak{H}_i and $\tilde{J}_i = J_i \oplus J'_i, i = 1, 2$. In addition, it is assumed that the exit spaces are Pontryagin spaces, i.e., that

$$\nu_-(J'_1), \nu_-(J'_2) < \infty.$$

Following [13, 23] consider first the following column extension problem:

(*)_c Give a description of all operators $T_c = \text{col} \begin{pmatrix} T \\ C \end{pmatrix} \in [\mathfrak{H}_1, \mathfrak{H}_2 \oplus \mathfrak{H}'_2]$, such that $\nu_-(J_1 - T_c^* \tilde{J}_2 T_c) = \tilde{\kappa}_1 (< \infty)$.

Since $J_1 - T_c^* \tilde{J}_2 T_c = J_1 - T^* J_2 T - C^* J'_2 C$, then necessarily (see Sect. 3)

$$\tilde{\kappa}_1 \geq \kappa_1 - \nu_-(C^* J'_2 C) \geq \kappa_1 - \nu_-(J'_2).$$

Moreover, it is clear that $\tilde{\kappa}_2 \geq \kappa_2$, since $J_2 - TJ_1T^*$ appears as the first diagonal entry of the 2×2 block operator $\tilde{J}_2 - T_c J_1 T_c^*$ when decomposed w.r.t. $\tilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}'_i$, $i = 1, 2$.

With the minimal value of $\tilde{\kappa}_1$ all solutions to this problem will now be described by applying Theorem 1 to an associated 2×2 block operator T_C appearing in the proof below; in fact the result is just a special case of Theorem 3.

Lemma 2 *Let $\tilde{\kappa}_1 = \nu_-(J_1 - T_c^* \tilde{J}_2 T_c)$ and assume that $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J'_2) (\geq 0)$. Then $\text{ran } C^* \subset \text{ran } D_T$ and the formula*

$$T_C = \begin{pmatrix} T \\ K^* D_T \end{pmatrix}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $()_c$ and the set of all J -contractions $K \in [\mathfrak{H}'_2, \mathfrak{D}_T]$.*

Proof To make the argument more explicit consider the following block operator

$$T_C := \begin{pmatrix} J_1 - T^* J_2 T & C^* \\ C & J'_2 \end{pmatrix} = \begin{pmatrix} I & C^* J'_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} J_1 - T_c^* \tilde{J}_2 T_c & 0 \\ 0 & J'_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ J'_2 C & I \end{pmatrix}.$$

Clearly $\nu_-(T_C) = \nu_-(J_1 - T_c^* \tilde{J}_2 T_c) + \nu_-(J'_2) < \infty$, which combined with $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J'_2)$ shows that $\nu_-(T_C) = \kappa_1 = \nu_-(J_1 - T^* J_2 T)$. Now, the statement is obtained from Theorem 1 or, more directly, just by applying Theorem 3.

Remark 1 (i) The above proof, which essentially makes use of an associated 2×2 block operator T_C (being a special case of the block operator H in (12) behind Theorem 3), is new even in the case of Hilbert space contractions. In particular, it shows that the operator K in Lemma 2 coincides with the operator S that gives the minimal solution $S^* J_T S$ to the completion problem associated with T_C ; the J -contractivity of K itself is equivalent to the fact that T_C is also a solution precisely when $\tilde{\kappa} = \kappa - \nu_-(J'_2)$.

(ii) The existence of a solution to Problem $(*)_c$ is proved here using only the condition $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J'_2) (\geq 0)$. The corresponding result in [23, Lemma 2.2] is formulated (and formally also proved) under the additional condition $\tilde{\kappa}_2 = \kappa_2$. In the case that $\nu_-(J_1) < \infty$ the equality $\tilde{\kappa}_2 = \kappa_2$ follows automatically from the equality $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J'_2)$: to see this apply (20) to T and T_c , which leads to $\nu_-(J_1) + \kappa_2 = \nu_-(J_1) + \tilde{\kappa}_2$, so that $\nu_-(J_1) < \infty$ implies $\kappa_2 = \tilde{\kappa}_2$. Naturally, in Lemma 2 the condition $\tilde{\kappa}_2 = \kappa_2$ follows from the condition $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J'_2)$ also in the case where $\nu_-(J_1) = \infty$; see Corollary 3 below.

Finally, it is mentioned that for a Pontryagin space operator T the result in Lemma 2 was proved in [13, Lemma 5.2].

In a dual manner we can treat the following row extension problem; again initially considered in [13, 23]:

$(*)_r$ Give a description of all operators $T_r = (T \ R) \in [\mathfrak{H}_1 \oplus \mathfrak{H}'_1, \mathfrak{H}_2]$, such that $\nu_-(J_2 - T_r \tilde{J}_1 T_r^*) = \tilde{\kappa}_2 (< \infty)$.

Analogous to the case of column operators, $J_2 - T_r \tilde{J}_1 T_r^* = J_2 - T J_1 T^* - R J_1' R^*$ gives the estimate

$$\tilde{\kappa}_2 \geq \kappa_2 - v_-(R J_1' R^*) \geq \kappa_2 - v_-(J_1').$$

Moreover, it is clear that $\tilde{\kappa}_1 \geq \kappa_1$. With the minimal value of $\tilde{\kappa}_2$ all solutions to Problem $(*)_r$ are established by applying Theorem 1 to an associated 2×2 block operator T_r .

Lemma 3 *Let $\tilde{\kappa}_2 = v_-(J_2 - T_r \tilde{J}_1 T_r^*)$ and assume that $\tilde{\kappa}_2 = \kappa_2 - v_-(J_1') (\geq 0)$. Then $\text{ran } R \subset \text{ran } D_{T^*}$ and the formula*

$$T_r = (T \ D_{T^*} B)$$

establishes a one-to-one correspondence between the set of all solutions to Problem $()_r$ and the set of all J -contractions $B \in [\mathfrak{H}'_1, \mathfrak{D}_{T^*}]$.*

Proof To prove the statement via Theorem 1 (cf. Theorem 3) consider

$$T_R := \begin{pmatrix} J_2 - T J_1 T^* & R \\ R^* & J_1' \end{pmatrix} = \begin{pmatrix} I & R J_1' \\ 0 & I \end{pmatrix} \begin{pmatrix} J_2 - T_r \tilde{J}_1 T_r^* & 0 \\ 0 & J_1' \end{pmatrix} \begin{pmatrix} I & 0 \\ J_1' R^* & I \end{pmatrix}.$$

Then clearly $v_-(T_R) = v_-(J_2 - T_r \tilde{J}_1 T_r^*) + v_-(J_1')$ and hence the assumption $\tilde{\kappa}_2 = \kappa_2 - v_-(J_1')$ is equivalent to $v_-(T_R) = \kappa_2 = v_-(J_2 - T J_1 T^*)$. Therefore, again the statement follows from Theorem 1 or directly from Theorem 3.

Remarks similar to those made after Lemma 2 can be done here, too. In particular, the corresponding result in [23, Lemma 2.1] is formulated under the additional condition $\tilde{\kappa}_1 = \kappa_1$: here this equality will be a consequence from the equality $\tilde{\kappa}_2 = \kappa_2 - v_-(J_1')$; cf. Corollary 3 below.

To prove the main result concerning parametrization of all 2×2 liftings in a larger Kreĭn space with minimal signature for the defect operators an indefinite version of the commutation relation of the form $T D_T = D_{T^*} T$ is needed; these involve so-called link operators introduced in [13, Section 4].

We will give a simple proof for the construction of link operators (see [13, Proposition 4.1]) by applying Heinz inequality combined with the basic factorization result from [32]. The first step is formulated in the next lemma, which is connected to a result of Kreĭn [48] concerning continuity of a bounded Banach space operator which is symmetric w.r.t. to a continuous definite inner product; the existence of link operators was proved in [13] via this result of Kreĭn. Here a statement, analogous to that of Kreĭn, is formulated in pure Hilbert space operator language by using the modulus of the product operator; see [34, Lemma B2], where Kreĭn’s result is presented with a proof due to W. T. Reid.

Lemma 4 *Let $S \in [\mathfrak{H}_1, \mathfrak{H}_2]$ and let $H \in [\mathfrak{H}_2]$ be nonnegative. Then*

$$HS = (HS)^* \implies |HS| \leq \mu H \text{ for some } \mu < \infty.$$

Proof Since HS is selfadjoint, one obtains

$$(HS)^2 = HSS^*H \leq \mu^2 H^2, \quad \mu = \|S\| < \infty.$$

Now by Heinz inequality (see e.g. [17, Theorem 10.4.2]) we get

$$|HS| = (HSS^*H)^{1/2} \leq \mu H.$$

Corollary 2 *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ and let J_1 and J_2 be symmetries in \mathfrak{H}_1 and \mathfrak{H}_2 as above. Then there exist unique operators $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ and $L_{T^*} \in [\mathfrak{D}_{T^*}, \mathfrak{D}_T]$ such that*

$$D_{T^*}L_T = TJ_1D_T \upharpoonright \mathfrak{D}_T, \quad D_T L_{T^*} = T^*J_2D_{T^*} \upharpoonright \mathfrak{D}_{T^*};$$

in fact, $L_T = D_{T^}^{[-1]}TJ_1D_T \upharpoonright \mathfrak{D}_T$ and $L_{T^*} = D_T^{[-1]}T^*J_2D_{T^*} \upharpoonright \mathfrak{D}_{T^*}$.*

Proof Denote $S = J_{T^*}J_2TJ_TJ_1T^*$. Then (19) implies that

$$\begin{aligned} D_{T^*}^2S &= (J_2 - TJ_1T^*)J_2TJ_TJ_1T^* \\ &= TJ_1(J_1 - T^*J_2T)J_TJ_1T^* \\ &= TJ_1D_T^2J_1T^* \geq 0, \end{aligned}$$

so that $D_{T^*}^2S$ is nonnegative and, in particular, selfadjoint. By Lemma 4 with $\mu = \|S\|$ one has

$$0 \leq TJ_1D_T^2J_1T^* = D_{T^*}^2S \leq \mu D_{T^*}^2.$$

This last inequality is equivalent to the factorization $TJ_1D_T \upharpoonright \mathfrak{D}_T = D_{T^*}L_T$ with a unique operator $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$, see [32, Theorem 1], which by means of Moore–Penrose generalized inverse can be rewritten as indicated.

The second formula is obtained by applying the first one to T^* .

The following identities can be obtained with direct calculations; see [13, Section 4]:

$$\begin{aligned} L_T^*J_{T^*} \upharpoonright \mathfrak{D}_{T^*} &= J_T L_{T^*}; \\ (J_T - D_TJ_1D_T) \upharpoonright \mathfrak{D}_T &= L_T^*J_{T^*}L_T; \\ (J_{T^*} - D_{T^*}J_2D_{T^*}) \upharpoonright \mathfrak{D}_{T^*} &= L_{T^*}^*J_T L_{T^*}. \end{aligned} \tag{21}$$

The next corollary contains the promised identity $\tilde{\kappa}_1 = \kappa_1$ under the assumption $\tilde{\kappa}_2 = \kappa_2 - \nu_-(J_2') \geq 0$ in Lemma 3. Similarly $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J_1')$ implies $\tilde{\kappa}_2 = \kappa_2$; the general result for the first case can be formulated as follows (and there is similar result for the latter case).

Corollary 3 *Let R be a bounded operator such that $\text{ran } R \subset \text{ran } D_{T^*}$ and let T_r be the corresponding row operator and denote $\tilde{\kappa}_1 = \nu_-(\tilde{J}_1 - T_r^*J_2T_r)$. Then $R = D_{T^*}B$ for a (unique) bounded operator $B \in [\mathfrak{H}'_1, \mathfrak{D}_{T^*}]$ and*

$$\tilde{\kappa}_1 = \kappa_1 + \nu_-(J_1' - B^*J_{T^*}B).$$

In particular, J -contractivity of B is equivalent to $\tilde{\kappa}_1 = \kappa_1$.

Proof Recall that $\text{ran } R \subset \text{ran } D_{T^*}$ is equivalent to the factorization $R = D_{T^*}B$. By applying the commutation relations in Corollary 2 together with the identities (21) one gets the following expression for $J_T D_{T^*}^2$:

$$\begin{aligned} J_T D_{T^*}^2 &= \begin{pmatrix} J_1 - T^* J_2 T & -T^* J_2 D_{T^*} B \\ -B^* D_{T^*} J_2 T & J_1' - B^* D_{T^*} J_2 D_{T^*} B \end{pmatrix} \\ &= \begin{pmatrix} J_T D_T^2 & -D_T L_{T^*} B \\ -B^* L_{T^*}^* D_T & J_B D_B^2 + B^* L_{T^*}^* J_T L_{T^*} B \end{pmatrix}. \end{aligned} \tag{22}$$

Now apply Proposition 2 and calculate the Schur complement, cf. (18),

$$J_B D_B^2 + B^* L_{T^*}^* J_T L_{T^*} B - B^* L_{T^*}^* D_T (D_T^{[-1]} J_T D_T^{[-1]}) D_T L_{T^*} B = J_B D_B^2,$$

to see that $\tilde{\kappa}_1 = \nu_-(J_1 - T^* J_2 T) + \nu_-(J_1' - B^* J_{T^*} B)$.

By means of Lemmas 2, 3 and the link operators in Corollary 2 one can now establish the main result concerning the lifting problem (*).

First notice that if Problem (*) has a solution, then by treating \tilde{T} as a row extension of its first column T_c and as a column extension of its first row T_r one gets from the inequalities preceding Lemmas 2, 3 the estimates

$$\begin{aligned} \tilde{\kappa}_1 &\geq \kappa_1(T_r) - \nu_-(J_2') \geq \kappa_1 - \nu_-(J_2'); \\ \tilde{\kappa}_2 &\geq \kappa_2(T_c) - \nu_-(J_1') \geq \kappa_2 - \nu_-(J_1'). \end{aligned} \tag{23}$$

Under the minimal choice of the indices $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ Problem (*) is already solvable; all solutions are described by the following result, which was initially proved in [23, Theorem 2.3] with the aid of [13, Theorem 5.3]. Here a different proof is presented, again based on an application of Theorem 1.

Theorem 4 *Let \tilde{T} be a bounded operator from $(\mathfrak{H}_1 \oplus \mathfrak{H}_1', J_1 \oplus J_1')$ to $(\mathfrak{H}_2 \oplus \mathfrak{H}_2', J_2 \oplus J_2')$ such that $P_2 \tilde{T} \upharpoonright \mathfrak{H}_1 = T$. Assume that $0 \leq \kappa_1 - \nu_-(J_2') = \tilde{\kappa}_1 < \infty$ and $0 \leq \kappa_2 - \nu_-(J_1') = \tilde{\kappa}_2 < \infty$. Then the Problem (*) is solvable and the formula*

$$\tilde{T} = \begin{pmatrix} T & D_{T^*} \Gamma_1 \\ \Gamma_2 D_T & -\Gamma_2 L_{T^*}^* J_{T^*} \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} \end{pmatrix}$$

establishes a one-to-one correspondence between the set of all solutions to Problem () and the set of triplets $\{\Gamma_1, \Gamma_2, \Gamma\}$ where $\Gamma_1 \in [\mathfrak{H}_1', \mathfrak{D}_{T^*}]$ and $\Gamma_2^* \in [\mathfrak{H}_2', \mathfrak{D}_T]$ are J -contractions and $\Gamma \in [\mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_2^*}]$ is a Hilbert space contraction.*

Proof Assume that there is a solution \tilde{T} to Problem (*) and write it in the form

$$\tilde{T} = \begin{pmatrix} T & R \\ C & X \end{pmatrix}$$

with the first column denoted by T_c and first row denoted by T_r , and assume that $\tilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')$ and $\tilde{\kappa}_2 = \kappa_2 - \nu_-(J_1')$. Then (23) shows that $\kappa_1 = \kappa_1(T_r)$ and

$\kappa_2 = \kappa_2(T_c)$. Hence Lemma 3 can be applied by viewing \tilde{T} as a row extension of T_c to get a range inclusion and then from Corollary 3 one gets the equality $\tilde{\kappa}_1 = \kappa_1(T_c)$. Similarly applying Lemma 2 and the analog of Corollary 3 to column operator \tilde{T} one gets the equality $\tilde{\kappa}_2 = \kappa_2(T_r)$. Thus $\kappa_1(T_c) = \kappa_1 - \nu_-(J'_2)$ and $\kappa_2(T_r) = \kappa_2 - \nu_-(J'_1)$. Consequently, one can apply Lemma 2 to the first column T_c and Lemma 3 to the first row T_r to get the stated factorizations $C = \Gamma_2 D_T$ and $R = D_T^* \Gamma_1$ with unique J -contractions Γ_1 and Γ_2^* .

To establish a formula for X we proceed by considering the block operator

$$H := \begin{pmatrix} J_{T_r} D_{T_r}^2 & T_{r,2}^* \\ T_{r,2} & J'_2 \end{pmatrix},$$

where $T_{r,2}$ denotes the second row of \tilde{T} . It is straightforward to derive the following formula for the Schur complement

$$J_{T_r} D_{T_r}^2 - T_{r,2}^* J'_2 T_{r,2} = \tilde{J}_1 - \tilde{T}^* \tilde{J}_2 \tilde{T}.$$

Thus $\nu_-(H) = \tilde{\kappa}_1 + \nu_-(J'_2) = \kappa_1 = \nu_-(J_{T_r})$ and one can apply Theorem 1 to get the factorization $T_{r,2}^* = D_{T_r} \tilde{K}$ with a unique $\tilde{K} \in [\mathfrak{H}'_2, \mathfrak{D}_{T_r}]$ satisfying $\tilde{K}^* J_{T_r} \tilde{K} \leq J'_2$, i.e., \tilde{K} is a J -contraction; see Theorem 3.

It follows from (22) that

$$J_{T_r} D_{T_r}^2 = \begin{pmatrix} D_T & 0 \\ -\Gamma_1^* L_{T_r}^* J_T & D_{\Gamma_1} \end{pmatrix} \begin{pmatrix} J_T & 0 \\ 0 & I_{D_{\Gamma_1}} \end{pmatrix} \begin{pmatrix} D_T & -J_T L_{T_r}^* \Gamma_1 \\ 0 & D_{\Gamma_1} \end{pmatrix} =: B^* \hat{J} B.$$

Since here $\nu_-(J_{T_r}) = \kappa_1 = \nu_-(J_T)$ and B is a triangular operator whose range is dense in $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ (the diagonal entries D_T and D_{Γ_1} of B have dense ranges by definition), there is a unique Pontryagin space J -unitary operator U from \mathfrak{D}_{T_r} onto $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ such that $B = U D_{T_r}$; see Proposition 1 (ii). It follows that $K^* := (U^{-1})^* \tilde{K}$ is a J -contraction from \mathfrak{H}'_2 to $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ and $K B = \tilde{K}^* D_{T_r} = T_{r,2}$. Now $J'_2 - K \hat{J} K^* \geq 0$ gives

$$0 \leq K_1 K_1^* \leq J'_2 - K_0 J_T K_0^*, \tag{24}$$

where $K = (K_0 K_1)$ is considered as a row operator, and $T_{r,2} = K B$ reads as

$$\Gamma_2 D_T = K_0 D_T, \quad X = -K_0 J_T L_{T_r}^* \Gamma_1 + K_1 D_{\Gamma_1}.$$

Since all contractions that are involved are unique, $K_0 = \Gamma_2$, $J'_2 - K_0 J_T K_0^* = D_{\Gamma_2}^2$, and (24) implies that there is a unique Hilbert space contraction $\Gamma \in [\mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_2}]$ such that $K_1 = D_{\Gamma_2}^* \Gamma$. The desired formula for \tilde{T} is proven (cf. (21)). It is clear from the proof that every operator \tilde{T} of the stated form is a solution and that there is one-to-one correspondence via the triplets $\{\Gamma_1, \Gamma_2, \Gamma\}$ of J -contractions.

Remark 2 (i) By replacing \tilde{T} with its adjoint \tilde{T}^* it is clear that all formulas remain the same and are obtained by changing T with T^* and interchanging the roles of the

indices 1 and 2; see also (21). This connects the considerations with row and column operators to each other.

(ii) If $\kappa_1 = 0$ so that $J_1 - T^* J_2 T \geq 0$, then the above proof becomes slightly simpler since then $J_T, J_T,$ and J'_2 are identity operators and K is a Hilbert space contraction. Then Theorem 4 gives all contractive liftings of a contraction in a Kreĭn space. If in addition $\kappa_2 = 0$, then one gets all bicontractive liftings of a bicontraction in a Kreĭn space with Pontryagin spaces as exit spaces. In the special case that the exit spaces are Hilbert spaces ($v_-(J_1) = v_-(J_2) = 0$ and $\kappa_1 = \kappa_2 = 0$) Theorem 4 coincides with [33, Theorem 3.6]. In fact, the present proof can be seen as a further development of the proof appearing in that paper; see also further references and historical remarks given in [33,34].

5 Contractive extensions of contractions with minimal negative indices

Let \mathfrak{H}_1 be a closed linear subspace of the Hilbert space \mathfrak{H} , let $T_{11} = T_{11}^* \in [\mathfrak{H}_1]$ be an operator such that $v_-(I - T_{11}^2) = \kappa < \infty$. Denote

$$J = \text{sign}(I - T_{11}^2), \quad J_+ = \text{sign}(I - T_{11}), \quad \text{and} \quad J_- = \text{sign}(I + T_{11}), \quad (25)$$

and let $\kappa_+ = v_-(I - T_{11})$ and $\kappa_- = v_-(I + T_{11})$. It is obvious that $J = J_- J_+ = J_+ J_-$. Moreover, there is an equality $\kappa = \kappa_- + \kappa_+$ as stated in the next lemma.

Lemma 5 *Let $T = T^* \in [\mathfrak{H}_1]$ be an operator such that $v_-(I - T^2) = \kappa < \infty$ then $v_-(I - T^2) = v_-(I + T) + v_-(I - T)$.*

Proof Let $E_t(\cdot)$ be the resolution of identity of T . Then by the spectral mapping theorem the spectral subspace corresponding to the negative spectrum of $I - T^2$ is given by $E_t((-\infty; -1) \cup (1; \infty)) = E_t((-\infty; -1)) \oplus E_t((1; \infty))$. Consequently, $v_-(I - T^2) = \dim E_t((-\infty; -1)) + \dim E_t((1; \infty)) = v_-(I + T) + v_-(I - T)$.

The next problem concerns the existence and a description of selfadjoint operators T such that $\tilde{A}_+ = I + T$ and $\tilde{A}_- = I - T$ solve the corresponding completion problems

$$A_{\pm}^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^* \\ \pm T_{21} & * \end{pmatrix}, \quad (26)$$

under *minimal index conditions* $v_-(I + T) = v_-(I + T_{11}), v_-(I - T) = v_-(I - T_{11})$, respectively. Observe, that if $I \pm T$ provides an arbitrary completion to A_{\pm}^0 then clearly $v_-(I \pm T) \geq v_-(I \pm T_{11})$. Thus by Lemma 5 the two minimal index conditions above are equivalent to the single condition $v_-(I - T^2) = v_-(I - T_{11}^2)$.

Unlike with the case of a selfadjoint contraction T_{11} , this problem need not have solutions when $v_-(I - T_{11}^2) > 0$. It is clear from Theorem 1 that the conditions $\text{ran } T_{21}^* \subset \text{ran } |I - T_{11}|^{1/2}$ and $\text{ran } T_{21}^* \subset \text{ran } |I + T_{11}|^{1/2}$ are necessary for the existence of solutions; however alone they are not sufficient.

The next theorem gives a general solvability criterion for the completion problem (26) and describes all solutions to this problem, when the criterion is satisfied. As in

the definite case, there are minimal solutions A_+ and A_- which are connected to two extreme selfadjoint extensions T of

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : \mathfrak{H}_1 \rightarrow \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}, \tag{27}$$

now with finite negative index $\nu_-(I - T^2) = \nu_-(I - T_{11}^2) > 0$. The set of all solutions T to the problem (26) will be denoted by $\text{Ext}_{T_1, \kappa}(-1, 1)$.

Theorem 5 *Let T_1 be a symmetric operator as in (27) with $T_{11} = T_{11}^* \in [\mathfrak{H}_1]$ and $\nu_-(I - T_{11}^2) = \kappa < \infty$, and let $J = \text{sign}(I - T_{11}^2)$. Then the completion problem for A_{\pm}^0 in (26) has a solution $I \pm T$ for some $T = T^*$ with $\nu_-(I - T^2) = \kappa$ if and only if the following condition is satisfied:*

$$\nu_-(I - T_{11}^2) = \nu_-(I - T_1^* T_1). \tag{28}$$

If this condition is satisfied then the following facts hold:

- (i) *The completion problems for A_{\pm}^0 in (26) have minimal solutions A_{\pm} .*
- (ii) *The operators $T_m := A_+ - I$ and $T_M := I - A_- \in \text{Ext}_{T_1, \kappa}(-1, 1)$.*
- (iii) *The operators T_m and T_M have the block forms*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & -I + V(I - T_{11}) J V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & I - V(I + T_{11}) J V^* \end{pmatrix}, \end{aligned} \tag{29}$$

where $D_{T_{11}} := |I - T_{11}^2|^{1/2}$ and V is given by $V := \text{clos}(T_{21} D_{T_{11}}^{[-1]})$.

- (iv) *The operators T_m and T_M are extremal extensions of T_1 :*

$$T \in \text{Ext}_{T_1, \kappa}(-1, 1) \text{ iff } T = T^* \in [\mathfrak{H}], \quad T_m \leq T \leq T_M. \tag{30}$$

- (v) *The operators T_m and T_M are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{31}$$

Proof It is easy to see that $\kappa = \nu_-(I - T_{11}^2) \leq \nu_-(I - T_1^* T_1) \leq \nu_-(I - T^2)$. Hence the condition $\nu_-(I - T^2) = \kappa$ implies (28). The sufficiency of this condition is established while proving the assertions (i)–(iii) below. (i) If the condition (28) is satisfied then $\text{ran } T_{21}^* \subset \text{ran } |I - T_{11}^2|^{1/2}$ by Lemma 2. In fact, this inclusion is equivalent to the inclusions $\text{ran } T_{21}^* \subset \text{ran } |I \pm T_{11}|^{1/2}$, which by Theorem 1 means that both of the completion problems, A_{\pm}^0 in (26), are solvable. Consequently, the following operators

$$S_- = |I + T_{11}|^{[-1/2]} T_{21}^*, \quad S_+ = |I - T_{11}|^{[-1/2]} T_{21}^* \tag{32}$$

are well defined and they provide the minimal solutions A_{\pm} to the completion problems for A_{\pm}^0 in (26). Notice that the assumption that there is a simultaneous solution $I \pm T$ with a *single* selfadjoint operator T is not yet used here.

(ii) & (iii) Proof of (i) shows that the inclusion $\text{ran } T_{21}^* \subset \text{ran } |I - T_{11}^2|^{1/2}$ holds. This last inclusion alone is equivalent to the existence of a (unique) bounded operator $V^* = D_{T_{11}}^{[-1]} T_{21}^*$ with $\ker V \supset \ker D_{T_{11}}$, such that $T_{21}^* = D_{T_{11}} V^*$. The operators $T_m := A_+ - I$ and $T_M := I - A_-$ (see proof of (i)) can be now rewritten as in (29). Observe that

$$S_{\mp} = |I \pm T_{11}|^{[-1/2]} D_{T_{11}} V^* = P_{\mp} |I \mp T_{11}|^{1/2} V^* = |I \mp T_{11}|^{1/2} P_{\mp} V^*,$$

where P_{\mp} are the orthogonal projections onto

$$(\ker |I \pm T_{11}|^{1/2})^{\perp} = (\ker |I \pm T_{11}|)^{\perp} = \overline{\text{ran}} |I \pm T_{11}| = \overline{\text{ran}} |I \pm T_{11}|^{1/2}.$$

Since $\ker V \supset \ker D_{T_{11}}$ implies $\overline{\text{ran}} V^* \subset \overline{\text{ran}} D_{T_{11}} \subset \overline{\text{ran}} |I \pm T_{11}|^{1/2}$, it follows that

$$S_- = |I - T_{11}|^{1/2} V^*, \quad S_+ = |I + T_{11}|^{1/2} V^*.$$

Consequently, see (25),

$$\begin{aligned} S_-^* J_- S_- &= V |I - T_{11}|^{1/2} J_- |I - T_{11}|^{1/2} V^* = V (I - T_{11}) J V^*, \\ S_+^* J_+ S_+ &= V |I + T_{11}|^{1/2} J_+ |I + T_{11}|^{1/2} V^* = V (I + T_{11}) J V^*, \end{aligned}$$

which implies the representations for T_m and T_M in (29). Clearly, T_m and T_M are selfadjoint extensions of T_1 , which satisfy the equalities

$$\nu_-(I + T_m) = \kappa_-, \quad \nu_-(I - T_M) = \kappa_+.$$

Moreover, it follows from (29) that

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - V J V^*) \end{pmatrix}. \tag{33}$$

Now the assumption (28) will be used again. Since $\nu_-(I - T_1^* T_1) = \nu_-(I - T_{11}^2)$ and $T_{21} = V D_{T_{11}}$ it follows from Lemma 2 that $V^* \in [\mathfrak{H}_2, \mathfrak{D}_{T_{11}}]$ is J -contractive: $I - V J V^* \geq 0$. Therefore, (33) shows that $T_M \geq T_m$ and $I + T_M \geq I + T_m$ and hence, in addition to $I + T_m$, also $I + T_M$ is a solution to the problem A_+^0 and, in particular, $\nu_-(I + T_M) = \kappa_- = \nu_-(I + T_m)$. Similarly, $I - T_M \leq I - T_m$ which implies that $I - T_M$ is also a solution to the problem A_-^0 , in particular, $\nu_-(I - T_M) = \kappa_+ = \nu_-(I - T_m)$. Now by applying Lemma 5 we get

$$\begin{aligned} \nu_-(I - T_m^2) &= \nu_-(I - T_m) + \nu_-(I + T_m) = \kappa_+ + \kappa_- = \kappa, \\ \nu_-(I - T_M^2) &= \nu_-(I - T_M) + \nu_-(I + T_M) = \kappa_+ + \kappa_- = \kappa. \end{aligned}$$

Therefore, $T_m, T_M \in \text{Ext}_{T_1, \kappa}(-1, 1)$ which in particular proves that the condition (28) is sufficient for solvability of the completion problem (26).

(iv) Observe, that $T \in \text{Ext}_{T_1, \kappa}(-1, 1)$ if and only if $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_{\mp}$. By Theorem 1 this is equivalent to

$$S_-^* J_- S_- - I \leq T_{22} \leq I - S_+^* J_+ S_+. \tag{34}$$

The inequalities (34) are equivalent to (30).

(v) The relations (31) follow from (32) and (29).

For a Hilbert space contraction T_1 one has $\nu_-(I - T_{11}^2) \leq \nu_-(I - T_1^* T_1) = 0$, i.e., the criterion (28) is automatically satisfied. In this case Theorem 5 has been proved in [39]. As Theorem 5 shows, under the minimal index condition $\nu_-(I - T^2) = \nu_-(I - T_{11}^2)$, the solution set $\text{Ext}_{T_1, \kappa}(-1, 1)$ admits the same attractive description as an operator interval determined by the two extreme extensions T_m and T_M as was originally proved by Kreĭn in his paper [47] when describing all contractive selfadjoint extensions of a Hilbert space contraction. In particular, Theorem 5 shows that if there is a solution to the completion problem (26), i.e. if T_1 satisfies the index condition (28), then all selfadjoint extensions T of T_1 satisfying the equality $\nu_-(I - T^2) = \nu_-(I - T_1^* T_1)$ are determined by the operator inequalities $T_m \leq T \leq T_M$. The original paper [47] of M. G. Kreĭn has never been translated: for some literature in English where many of the original ideas of Kreĭn have been presented we refer to the monographs [1, 9, 51] and the papers [11, 39].

The original proof of Kreĭn in [47] for the description of all contractive selfadjoint extensions of a Hilbert space contraction T_1 as the operator interval in (30) was based on the notion of shortening or shorted operator; cf. (1). To get this result Kreĭn first constructed one contractive selfadjoint extension T for T_1 and then used it together with the following two formulas involving shortening of $I + T$ and $I - T$ to the subspace $\mathfrak{N} = \mathfrak{H} \ominus \text{dom } T_1 = \mathfrak{H}_2$:

$$T_m = T - (I + T)\mathfrak{N}, \quad T_M = T + (I - T)\mathfrak{N},$$

see [47, Theorem 3]. It follows from Theorem 1, see also (10), and the formulas for T_m and T_M in Theorem 5 that these descriptions of T_m and T_M remain true in the present setting: indeed, using the given block formulas one can directly check that

$$I + T = I + T_m + (I + T)\mathfrak{N}, \quad I - T = I - T_M + (I - T)\mathfrak{N},$$

where the shortening is calculated as defined in (17).

Notice that T belongs to the solution set $\text{Ext}_{T_1, \kappa}(-1, 1)$ precisely when $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_{\mp}$. This means that every selfadjoint extension of T_1 for which $(I - T^2) = \nu_-(I - T_1^* T_1)$ admits precisely κ_- eigenvalues on the interval $(-\infty, -1)$ and κ_+ eigenvalues on the interval $(1, \infty)$; in total there are $\kappa = \kappa_- + \kappa_+$ eigenvalues outside the closed interval $[-1, 1]$. The fact that the numbers $\kappa_{\mp} = \nu_-(I \pm T)$ are constant in the solution set $\text{Ext}_{T_1, \kappa}(-1, 1)$ is crucial for dealing properly with the Cayley transforms in the next section.

6 A generalization of M. G. Kreĭn's approach to the extension theory of nonnegative operators

6.1 Some antitonicity theorems for selfadjoint relations

The notion of inertia of a selfadjoint relation in a Hilbert space is defined by means of its associated spectral measure. In what follows the Hilbert space is assumed to be separable.

Definition 1 Let H be a selfadjoint relation in a separable Hilbert space \mathfrak{H} and let $E_t(\cdot)$ be the spectral measure of H . The inertia of H is defined as the ordered quadruplet $i(H) = \{i^+(H), i^-(H), i^0(H), i^\infty(H)\}$, where

$$\begin{aligned} i^+(H) &= \dim \operatorname{ran} E_t((0, \infty)), & i^-(H) &= \dim \operatorname{ran} E_t((-\infty, 0)), \\ i^0(H) &= \dim \ker H, & i^\infty(H) &= \dim \operatorname{mul} H. \end{aligned}$$

In particular, for a selfadjoint relation H in \mathbb{C}^n , the quadruplet $i(H)$ consists of the numbers of positive, negative, zero, and infinite eigenvalues of H ; cf. [15]. Hence, if H is a selfadjoint matrix in \mathbb{C}^n , then $i^\infty(H) = 0$ and the remaining numbers make up the usual inertia of H .

The following theorem characterizes the validity of the implication

$$H_1 \leq H_2 \quad \Rightarrow \quad H_2^{-1} \leq H_1^{-1}$$

for a pair of bounded selfadjoint operators H_1 and H_2 having bounded inverses; in the infinite dimensional case it has been proved independently in [30,40,61]; cf. also [41]. Some extensions of this result, where the condition $\min\{i_2^+, i_1^-\} < \infty$ is relaxed, are also contained in [40,41,61].

Theorem 6 Let H_1 and H_2 be bounded and boundedly invertible selfadjoint operators in a separable Hilbert space \mathfrak{H} . Let $i(H_j) = \{i_j^+, i_j^-, i_j^0, i_j^\infty\}$ be the inertia of H_j , $j = 1, 2$, and assume that $\min\{i_2^+, i_1^-\} < \infty$ and that $H_1 \leq H_2$. Then

$$H_2^{-1} \leq H_1^{-1} \quad \text{if and only if} \quad i(H_1) = i(H_2).$$

Very recently two extensions of Theorem 6 have been established in [15] for a general pair of selfadjoint operators and relations without any invertibility assumptions. For the present purposes we need the second main antitonicity theorem from [15], which reads as follows.

Theorem 7 Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space \mathfrak{H} which are semibounded from below. Let $i(H_j) = \{i_j^+, i_j^-, i_j^0, i_j^\infty\}$ be the inertia of H_j , $j = 1, 2$, and assume that $i_1^- < \infty$ and that $H_1 \leq H_2$. Then

$$H_2^{-1} \leq H_1^{-1} \quad \text{if and only if} \quad i_1^- = i_2^-.$$

The ordering appearing in Theorem 7 is defined via

$$H_1 \leq H_2 \iff 0 \leq (H_2 - aI)^{-1} \leq (H_1 - aI)^{-1},$$

where $a < \min\{\mu(H_1), \mu(H_2)\}$ is fixed and $\mu(H_i) \in \mathbb{R}$ stands for the lower bound of $H_i, i = 1, 2$. Notice that the conditions $H_1 \leq H_2$ and $i_1^- < \infty$ imply $i_2^- < \infty$; in particular these conditions already imply that the inverses H_1^{-1} and H_2^{-1} are also semibounded from below. For further facts on ordering of semibounded selfadjoint operators and relations the reader is referred to [15,42].

6.2 Cayley transforms

Define the linear fractional transformation \mathcal{C} , taking a linear relation A into a linear relation $\mathcal{C}(A)$, by

$$\mathcal{C}(A) = \{ \{f + f', f - f'\} : \widehat{f} = \{f, f'\} \in A \} = -I + 2(I + A)^{-1}. \tag{35}$$

Clearly, \mathcal{C} maps the (closed) linear relations one-to-one onto themselves, $\mathcal{C}^2 = I$, and

$$\mathcal{C}(A)^{-1} = \mathcal{C}(-A), \tag{36}$$

for every linear relation A . Moreover,

$$\begin{aligned} \text{dom } \mathcal{C}(A) &= \text{ran } (I + A), & \text{ran } \mathcal{C}(A) &= \text{ran } (I - A), \\ \text{ker } (\mathcal{C}(A) - I) &= \text{ker } A, & \text{ker } (\mathcal{C}(A) + I) &= \text{mul } A. \end{aligned}$$

In addition, \mathcal{C} preserves closures, adjoints, componentwise sums, orthogonal sums, intersections, and inclusions. The relation $\mathcal{C}(A)$ is symmetric if and only if A is symmetric. It follows from (35) and

$$\|f + f'\|^2 - \|f - f'\|^2 = 4\text{Re}(f', f) \tag{37}$$

that \mathcal{C} gives a one-to-one correspondence between nonnegative (selfadjoint) linear relations and symmetric (respectively, selfadjoint) contractions. Observe the following mapping properties of \mathcal{C} on the extended real line $\mathbb{R} \cup \{\pm\infty\}$:

$$\begin{aligned} \mathcal{C}([0, 1]) &= [0, 1]; & \mathcal{C}([-1, 0]) &= [1, +\infty]; \\ \mathcal{C}([1, +\infty]) &= [-1, 0]; & \mathcal{C}([-\infty, -1]) &= [-\infty, -1]. \end{aligned} \tag{38}$$

If H is a selfadjoint relation then

$$i^-(I + H) = i^-(\mathcal{C}(H) + I), \quad i^-(I - H) = i^-(\mathcal{C}(H)^{-1} + I),$$

and hence

$$\begin{aligned} \sigma(H) \cap (-\infty, -1) &= \sigma(\mathcal{C}(H)) \cap (-\infty, -1), \\ \sigma(H) \cap (1, +\infty) &= \sigma(\mathcal{C}(H)^{-1}) \cap (-\infty, -1) = \sigma(\mathcal{C}(H)) \cap (-1, 0); \end{aligned} \tag{39}$$

which can also be seen from (38).

6.3 M. G. Kreĭn’s approach to the extension theory with a minimal negative index

In M. G. Kreĭn’s approach to the extension theory of nonnegative operators the idea is to make a connection to the selfadjoint contractive extensions of a hermitian contraction T via the Cayley transform in (35). The extension of this approach to the present indefinite situation is based on the fact that the Cayley transform still reverses the ordering of selfadjoint extensions due to the antitonicity result formulated in Theorem 7 and the fact that in Theorem 5 $T \in \text{Ext}_{T_1, \kappa}(-1, 1)$ if and only if $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_{\mp}$.

A semibounded symmetric relation A is said to be quasi-nonnegative if the associated form $a(f, f) := (f', f), \{f, f'\} \in A$, has a finite number of negative squares, i.e. every A -negative subspace $\mathcal{L} \subset \text{dom } A$ is finite dimensional. If the maximal dimension of A -negative subspaces is finite and equal to $\kappa \in \mathbb{Z}_+$, then A is said to be κ -nonnegative; the more precise notations $\nu_-(a), \nu_-(A)$ are used to indicate the maximal number of negative squares of the form a and the relation A , respectively; here $\nu_-(a) = \nu_-(A)$. A selfadjoint extension \tilde{A} of A is said to be a κ -nonnegative extension of A if $\nu_-(\tilde{A}) = \kappa$. The set of all such extension will be denoted by $\text{Ext}_{A, \kappa}(0, \infty)$.

If A is a closed symmetric relation in the Hilbert space \mathfrak{H} with $\kappa_-(A) < \infty$, then the subspace $\mathfrak{H}_1 := \text{ran}(I + A)$ is closed, since the Cayley transform $T_1 = \mathcal{C}(A)$ is a closed bounded symmetric operator in \mathfrak{H} with $\text{dom } T_1 = \mathfrak{H}_1$. Let P_1 be the orthogonal projection onto \mathfrak{H}_1 and let $P_2 = I - P_1$. Then the form

$$a_1(f, f) := (P_1 f', f), \quad \{f, f'\} \in A, \tag{40}$$

is symmetric and it has a finite number of negative squares.

Lemma 6 *Let A be a closed symmetric relation in \mathfrak{H} with $\kappa_-(A) < \infty$ and let $T_1 = \mathcal{C}(A)$. Then the form a_1 is given by*

$$a_1(f, f) = a(f, f) + \|P_2 f\|^2 \tag{41}$$

with $\nu_-(a_1) \leq \nu_-(A)$. Moreover,

$$4a_1(f, f) = \|g\|^2 - \|T_{11}g\|^2, \quad 4a(f, f) = \|g\|^2 - \|T_1g\|^2,$$

where $\{f, f'\} \in A, g = f + f'$, and $T_{11} = P_1 T_1$. In addition, $T_{21} = P_2 T_1$ satisfies $\|T_{21}g\|^2 = \|P_2 f\|^2 = -(P_2 f, f')$.

Proof The formula (37) shows that if $T_1 = \mathcal{C}(A)$ and $\{f, f'\} \in A$, then

$$\|g\|^2 - \|T_1g\|^2 = 4(f', f) = 4a(f, f), \quad g = f + f' \in \text{dom } T_1 = \mathfrak{H}_1.$$

Moreover, $T_{21}g = P_2(f - f') = 2P_2f = -2P_2f'$ gives $(P_2f', f) = -\|P_2f\|^2$ and

$$\|T_{21}g\|^2 = -4(P_2f', P_2f) = -4(P_2f', f).$$

In particular, (41) follows from

$$a(f, f) = (P_1f', f) + (P_2f', f) = a_1(f, f) - \|P_2f\|^2.$$

Finally, (41) combined with $\|T_{21}g\|^2 = 4\|P_2f\|^2$ leads to

$$4a_1(f, f) = \|g\|^2 - \|T_1g\|^2 + \|T_{21}g\|^2 = \|g\|^2 - \|T_{11}g\|^2.$$

The main result in this section concerns the existence and a description of all selfadjoint extensions \tilde{A} of a symmetric relation A for which $\nu_-(\tilde{A}) < \infty$ attains the minimal value $\nu_-(a_1)$. A criterion for the existence of such a selfadjoint extension is established, in which case all such extensions are described in a manner that is familiar from the case of nonnegative operators. To formulate the result assume that the selfadjoint quasi-contractive extensions T_m and T_M of T_1 as in Theorem 5 exist, and denote the corresponding selfadjoint relations A_F and A_K by

$$A_F = X(T_m) = -I + 2(I + T_m)^{-1}, \quad A_K = X(T_M) = -I + 2(I + T_M)^{-1}. \tag{42}$$

Theorem 8 *Let A be a closed symmetric relation in \mathfrak{H} with $\nu_-(A) < \infty$ and denote $\kappa = \nu_-(a_1) (\leq \nu_-(A))$, where a_1 is given by (40). Then $\text{Ext}_{A,\kappa}(0, \infty)$ is nonempty if and only if $\nu_-(A) = \kappa$. In this case A_F and A_K are well defined and they belong to $\text{Ext}_{A,\kappa}(0, \infty)$. Moreover, the formula*

$$\tilde{A} = -I + 2(I + T)^{-1} \tag{43}$$

gives a bijective correspondence between the quasi-contractive selfadjoint extensions $T \in \text{Ext}_{T_1,\kappa}(-1, 1)$ of T_1 and the selfadjoint extensions $\tilde{A} = \tilde{A}^ \in \text{Ext}_{A,\kappa}(0, \infty)$ of A . Furthermore, $\tilde{A} = \tilde{A}^* \in \text{Ext}_{A,\kappa}(0, \infty)$ precisely when*

$$A_K \leq \tilde{A} \leq A_F, \tag{44}$$

or equivalently, when $A_F^{-1} \leq \tilde{A}^{-1} \leq A_K^{-1}$, or

$$(A_F + I)^{-1} \leq (\tilde{A} + I)^{-1} \leq (A_K + I)^{-1}. \tag{45}$$

The set $\text{Ext}_{A^{-1},\kappa}(0, \infty)$ is also nonempty and $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$ if and only if $\tilde{A}^{-1} \in \text{Ext}_{A^{-1},\kappa}(0, \infty)$. The extreme selfadjoint extensions A_F and A_K of A are connected to those of A^{-1} via

$$(A^{-1})_F = (A_K)^{-1}, \quad (A^{-1})_K = (A_F)^{-1}. \tag{46}$$

Proof Since $v_-(A) < \infty$, the Cayley transform $T_1 = \mathcal{C}(A)$ defines a bounded symmetric operator in \mathfrak{H} with $\mathfrak{H}_1 = \text{dom } T_1 = \text{ran } (I + A)$. It follows from Lemma 6 that

$$v_-(A) = v_-(a) = v_-(I - T_1^* T_1), \quad v_-(a_1) = v_-(I - T_{11}^2),$$

and therefore the condition $v_-(A) = \kappa$ is equivalent to solvability criterion (28) in Theorem 5. Moreover, \tilde{A} is a selfadjoint extension of A if and only if $T = \mathcal{C}(\tilde{A})$ is selfadjoint extension of T_1 and by Lemma 6 the equality $v_-(\tilde{A}) = v_-(I - T^2)$ holds. Therefore, it follows from Theorem 5 that the set $\text{Ext}_{A,\kappa}(0, \infty)$ is nonempty if and only if $v_-(A) = \kappa$ and in this case the formula (43) establishes a one-to-one correspondence between the sets $\text{Ext}_{A,\kappa}(0, \infty)$ and $\text{Ext}_{T_1,\kappa}(-1, 1)$.

Next the characterizations (44) and (45) for the set $\text{Ext}_{A,\kappa}(0, \infty)$ are established. Let $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$ and let $T = \mathcal{C}(\tilde{A})$. According to Theorem 7 $T = \mathcal{C}(\tilde{A}) \in \text{Ext}_{T_1,\kappa}(-1, 1)$ if and only if T satisfies the inequalities $T_m \leq T \leq T_M$. It is clear from the formulas (42) and (43) that the inequalities $I + T_m \leq I + T \leq I + T_M$ are equivalent to the inequalities (45).

On the other hand, $v_-(I - T_{11}^2) = v_-(I - T^2)$ and hence the indices $\kappa_+ = v_-(I - T_{11}) = v_-(I - T)$ and $\kappa_- = v_-(I + T_{11}) = v_-(I + T)$ do not depend on $T = \mathcal{C}(\tilde{A})$; cf. (25). The mapping properties (39) of the Cayley transform imply that the number of eigenvalues of \tilde{A} on the open intervals $(-\infty, -1)$ and $(-1, 0)$ are also constant and equal to κ_- and κ_+ , respectively. In particular, since $\kappa_- = v_-(I + T)$ is constant we can apply Theorem 6 to conclude that the inequalities $I + T_m \leq I + T \leq I + T_M$ are equivalent to

$$(I + T_M)^{-1} \leq (I + T)^{-1} \leq (I + T_m)^{-1},$$

which due to the formulas (42) and (43) can be rewritten as $A_F + I \leq \tilde{A} + I \leq A_K + I$, or as $A_F \leq \tilde{A} \leq A_K$. This proves (44). Since $v_-(\tilde{A}) = \kappa = \kappa_- + \kappa_+$ is also constant, an application of Theorem 7 shows that the inequalities (44) are also equivalent to $A_F^{-1} \leq \tilde{A}^{-1} \leq A_K^{-1}$.

As to the inverse A^{-1} , notice that $v_-(A^{-1}) = v_-(A)$. Moreover, since $A^{-1} = \mathcal{C}(-T_1)$ it is clear that $\text{ran } (I + A^{-1}) = \text{dom } T_1$ and thus the form associated to A^{-1} via (40) satisfies $a_1^{(-1)}(f', f') = (P_1 f, f') = (P_1 f', f) = a_1(f, f)$. In particular, $v_-(a_1^{(-1)}) = v_-(a_1)$. Moreover, it is clear that $v_-(A^{-1}) = v_-(A)$. Consequently, the equality $v_-(A) = v_-(a_1)$ is equivalent to the equality $v_-(A^{(-1)}) = v_-(a_1^{(-1)})$. Furthermore, it is clear that $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$ if and only if $\tilde{A}^{-1} \in \text{Ext}_{A^{-1},\kappa}(0, \infty)$.

Finally, the relations (46) are obtained from (31), (36), and (42).

It follows from Theorem 8 that the extensions $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$ admit a uniform lower bound $\mu \leq \mu(\tilde{A})$ ($\mu \leq 0$). This leads to the following inequalities for the resolvents.

Corollary 4 *With the assumptions as in Theorem 8 let $\nu_-(a_1) = \nu_-(A) < \infty$ and $\mu \leq 0$ be a uniform lower bound for the extensions $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$. Then the resolvents of these extensions satisfy the inequalities*

$$(A_F + a)^{-1} \leq (\tilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > -\mu. \tag{47}$$

Proof Let $T = \mathcal{C}(\tilde{A}) \in \text{Ext}_{T_1,\kappa}(-1, 1)$ be the Cayley transform of the extension $\tilde{A} \in \text{Ext}_{A,\kappa}(0, \infty)$ and rewrite the resolvent of \tilde{A} in the form

$$(\tilde{A} + a)^{-1} = \frac{1}{a - 1} I - \frac{2}{(a - 1)^2} \left(T + \frac{a + 1}{a - 1} \right)^{-1}.$$

Since $-a < \mu \leq \mu(\tilde{A})$, T admits precisely κ_- eigenvalues below the number $-(a + 1)/(a - 1) < -1$. Therefore the inequalities $T_m \leq T \leq T_M$ in Theorem 5 or, equivalently, the inequalities

$$T_m + \frac{a + 1}{a - 1} \leq T + \frac{a + 1}{a - 1} \leq T_M + \frac{a + 1}{a - 1}$$

imply the inequalities (47) by Theorem 6.

The antonicity Theorems 6, 7 can be also used as follows. If the inequalities (44) and $A_F^{-1} \leq \tilde{A}^{-1} \leq A_K^{-1}$ hold, then $\kappa = \nu_-(\tilde{A}) = \nu_-(A_K) = \nu_-(A_F)$ is constant. If, in addition, (45) is satisfied, then it follows from (44) that $\kappa_- = \nu_-(I + \tilde{A}) = \nu_-(I + A_K) = \nu_-(I + A_F)$ is constant, so that also $\kappa_+ = \nu_-(I - \tilde{A}) = \nu_-(I - A_K) = \nu_-(I - A_F)$ is constant. However, in this case the equality $\nu_-(a_1) = \nu_-(A)$ need not hold and there can also be selfadjoint extensions \tilde{A} of A with

$$\nu_-(\tilde{A}) = \nu_-(A_K) = \nu_-(A_F) > \nu_-(A) \geq \nu_-(a_1),$$

which neither satisfy the inequalities (44) and (45), nor the equalities $\nu_-(I + \tilde{A}) = \kappa_-$ and $\nu_-(I - \tilde{A}) = \kappa_+$. It is emphasized that the result in Theorem 8 characterizes all selfadjoint extensions in $\text{Ext}_{A,\kappa}(0, \infty)$ under the minimal index condition $\kappa = \nu_-(a_1) = \nu_-(A)$.

In the case that A is nonnegative one has automatically $\kappa = \nu_-(a_1) = \nu_-(A) = 0$. Therefore, Theorem 8 is a precise generalization of the well-known characterization of the class $\text{Ext}_A(0, \infty)$ (with $\kappa = 0$) due to M. G. Kreĭn [47] to the case of a finite negative (minimal) index $\kappa > 0$. The selfadjoint extensions A_F and A_K of A are called the Friedrichs (hard) and the Kreĭn–von Neumann (soft) extension, respectively; these notions go back to [36, 56]. The extremal properties (47) of the Friedrichs and Kreĭn–von Neumann extensions were discovered by Kreĭn [47] in the case when A is a densely defined nonnegative operator. The case when $A \geq 0$ is not densely defined

was considered by Ando and Nishio [5], and Coddington and de Snoo [22]. In the nonnegative case the formulas (46) can be found in [5, 22]. Notice that in view of (43) and (44) the minimal solution of the completion problem for a nonnegative block operator A^0 can be also interpreted by means of the Kreĭn–von Neumann extension of the first column $\text{col}(A_{11}, A_{21})$ in (2); cf. [7, Section 4], [39, Section 4.9].

6.4 Kreĭn's uniqueness criterion

To establish a generalization of Kreĭn's uniqueness criterion for the equality $A_F = A_K$ in Theorem 8, i.e., for $\text{Ext}_{A,K}(0, \infty)$ to consist only of one extension, we first derive some general facts on J -contractions by means of their commutation properties.

Let \mathfrak{H}_1 and \mathfrak{H}_2 be Hilbert spaces with symmetries J_1 and J_2 , respectively, and let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a J -contraction, i.e., $J_1 - T^*J_2T \geq 0$. Let D_T and D_{T^*} be the corresponding defect operators and let J_T and J_{T^*} be their signature operators as defined in Sect. 4. The first lemma connects the kernels of the defect operators D_T and D_{T^*} .

Lemma 7 *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$, let J_i be a symmetry in \mathfrak{H}_i , $i = 1, 2$, and let D_T and D_{T^*} be the defect operators of T and T^* , respectively. Then*

$$J_2T(\ker D_T) = \ker D_{T^*}, \quad T^*J_2(\ker D_{T^*}) = \ker D_T. \quad (48)$$

In particular,

$$\ker D_T = \{0\} \text{ if and only if } \ker D_{T^*} = \{0\}.$$

Proof It suffices to show the first identity in (48). If $\varphi \in \ker D_T = \ker J_T D_T^2$, then the second identity in (19) implies that $J_2T\varphi \in \ker J_{T^*} D_{T^*}^2 = \ker D_{T^*}$. Hence, $J_2T(\ker D_T) \subset \ker D_{T^*}$. Conversely, let $\varphi \in \ker D_{T^*}$. Then $0 = J_{T^*} D_{T^*}^2 \varphi$ or, equivalently, $\varphi = J_2T J_1 T^* \varphi$, and here $J_1 T^* \varphi \in \ker D_T$ by the first identity in (19). This proves the reverse inclusion.

Lemma 8 *Let the notations be as in Lemma 7. Then*

$$\text{ran } T \cap \text{ran } D_{T^*} = \text{ran } T J_1 D_T = \text{ran } D_{T^*} L_T,$$

where L_T is the link operator defined in Corollary 2.

Proof By the commutation formulas in Corollary 2 we have $\text{ran } T J_1 D_T = \text{ran } D_{T^*} L_T \subset \text{ran } T \cap \text{ran } D_{T^*}$. Hence, it suffices to prove the inclusion

$$\text{ran } T \cap \text{ran } D_{T^*} \subset \text{ran } T J_1 D_T.$$

Suppose that $\varphi \in \text{ran } T \cap \text{ran } D_{T^*}$. Then Corollary 2 shows that $T^*J_2\varphi = D_T f$ for some $f \in \mathfrak{D}_T$, while the second identity in (19) implies that

$$(J_2 - T J_1 T^*) J_2 \varphi = T J_1 D_T g,$$

for some $g \in \mathfrak{D}_T$. Therefore,

$$\varphi = (J_2 - T J_1 T^*) J_2 \varphi + T J_1 T^* J_2 \varphi = T J_1 D_T g + T J_1 D_T f = T J_1 D_T (g + f)$$

and this completes the proof.

We can now characterize J -isometric operators $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ as follows.

Proposition 3 *With the notations as in Lemma 7 the following statements are equivalent:*

- (i) T is J -isometric, i.e., $T^* J_2 T = J_1$;
- (ii) $\ker T = \{0\}$ and $\text{ran } T \cap \text{ran } D_{T^*} = \{0\}$;
- (iii) for some, and equivalently for every, subspace \mathfrak{L} with $\text{ran } J_2 T \subset \overline{\mathfrak{L}}$ one has

$$\sup_{f \in \mathfrak{L}} \frac{|(f, T\varphi)|}{\|D_{T^*} f\|} = \infty \text{ for every } \varphi \in \mathfrak{H}_1 \setminus \{0\}, \tag{49}$$

i.e., there is no constant $0 \leq C < \infty$ satisfying $|(f, T\varphi)| \leq C \|D_{T^*} f\|$ for every $f \in \mathfrak{L}$, if $\varphi \neq 0$.

Proof (i) \Rightarrow (iii) Let \mathfrak{L} be an arbitrary subspace with $\text{ran } J_2 T \subset \overline{\mathfrak{L}}$. Assume that the supremum in (49) is finite for some $\varphi = J_1 \psi \in \mathfrak{H}_1$. Then there exists $0 \leq C < \infty$, such that

$$|(f, T J_1 \psi)| \leq C \|D_{T^*} f\| \text{ for every } f \in \mathfrak{L}.$$

Since $\text{ran } J_2 T \subset \overline{\mathfrak{L}}$ and T is J -isometric, also the following inequality holds:

$$\|\psi\|^2 = (J_1 T^* J_2 T \psi, \psi) \leq C \|D_{T^*} J_2 T \psi\|. \tag{50}$$

By taking adjoints (and zero extension for L_{T^*}) in the second identity in Corollary 2 it is seen that $D_{T^*} J_2 T \psi = L_{T^*}^* D_T \psi = 0$, since T is J -isometric. Hence (50) implies $\varphi = J_1 \psi = 0$. Therefore (49) holds for every $\varphi \neq 0$.

(iii) \Rightarrow (ii) Assume that (49) is satisfied with some subspace \mathfrak{L} . If (ii) does not hold, then either $\ker T \neq \{0\}$, in which case (49) does not hold for $0 \neq \varphi \in \ker T$, or $\text{ran } T \cap \text{ran } D_{T^*} \neq \{0\}$. However, then with $0 \neq T\varphi = D_{T^*} h$ the supremum in (49) is finite even if f varies over the whole space \mathfrak{H}_2 . Thus, if (ii) does not hold then (49) fails to be true.

(ii) \Rightarrow (i) Let $\text{ran } T \cap \text{ran } D_{T^*} = \{0\}$. Then by Lemma 8 one has $T J_1 D_T = 0$ and it follows from $\ker T = \{0\}$ that $D_T = 0$, i.e., T is isometric. This completes the proof.

After these preparations we are ready to prove the analog of Kreĭn’s uniqueness criterion for the equality $T_m = T_M$ in the case of quasi-contractions appearing in Theorem 5.

Theorem 9 *Let the Hilbert space \mathfrak{H} be decomposed as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric quasi-contraction satisfying the condition (28) in Theorem 5. Then $T_m = T_M$ if and only if*

$$\sup_{f \in \mathfrak{H}_1} \frac{|(T_1 f, \varphi)|^2}{|(I - T_1^* T_1) f, f|} = \infty \text{ for every } \varphi \in \mathfrak{H}_2 \setminus \{0\}. \tag{51}$$

Proof Let $J = \text{sign}(I - T_1^2)$. According to Theorem 5 there is $V \in [\mathfrak{D}_{T_1}, \mathfrak{H}_2]$, such that $T_{21} = V D_{T_1}$; moreover, V^* is a J -contraction, i.e., $I - V J V^* \geq 0$. This implies that

$$(T_1 f, \varphi) = (T_{21} f, \varphi) = (D_{T_1} f, V^* \varphi), \tag{52}$$

and a direct calculation shows that

$$I - T_1^* T_1 = I - T_{11}^2 - T_{21}^* T_{21} = J D_{T_1}^2 - D_{T_1} V^* V D_{T_1} = D_{T_1} D_V J_V D_V D_{T_1}. \tag{53}$$

By construction $D_V \in [\mathfrak{D}_{T_1}]$ and therefore $\text{ran } D_V D_{T_1}$ is dense in $\mathfrak{D}_V = \overline{\text{ran } D_V}$. Furthermore, since V^* is J -contractive it follows from Lemma 1 that $v_-(J_V) = v_-(J) = v_-(I - T_1^2)$ and, therefore, the assumption (28) shows that $v_-(J_V) = v_-(I - T_1^* T_1)$. Now according to Proposition 1 (ii) it follows from (53) that there is a unique J -unitary operator $C \in [\mathfrak{D}_{T_1}, \mathfrak{D}_V]$ such that $D_V D_{T_1} = C D_{T_1}$.

In view of (33) $T_m = T_M$ if and only if V^* is J -isometric. Since $\text{ran } J V^* \subset \overline{\text{ran } D_{T_1}}$, it follows from (i) and (iii) in Proposition 3 that $T := V^*$ satisfies the condition (49) with $\mathfrak{L} = \text{ran } D_{T_1}$.

On the other hand, it follows from (53) and J -unitarity of $C \in [\mathfrak{D}_{T_1}, \mathfrak{D}_V]$ that

$$\|D_V D_{T_1}\| \leq \|C\| \|D_{T_1}\|, \quad \|D_{T_1}\| \leq \|C^{-1}\| \|D_V D_{T_1}\|.$$

By combining this equivalence between the norms of $\|D_{T_1}\|$ and $\|D_V D_{T_1}\|$ with the equality (52) one concludes that V^* satisfies the condition (49) precisely when T_1 satisfies the condition (51).

Remark 3 In the case of a hermitian contraction acting in a Hilbert space the criterion in Theorem 9 was proved by Kreĭn [47].

As to the geometric interpretation of the condition in Theorem 9, observe that if the supremum (51) is finite for some φ , then $T_{21}^* \varphi \in \text{ran } D_{T_1}$ (see e.g. [38, Corollary 2]) and as the proof shows $D_{T_1} = D_{T_1} D_V C^{-*}$, which gives the equation $D_{T_1} V^* \varphi = D_{T_1} D_V C^{-*} v$ for some v . Consequently, $V^* \varphi = D_V C^{-*} v$ and hence again an application of Proposition 3 to V^* , now using items (i) and (ii), shows that (51) is equivalent to V^* being J -isometric. Here (see (33))

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - V J V^*) \end{pmatrix}.$$

Recall that the minimal and maximal extension T_m and T_M of T_1 are determined via the minimal solutions $A_+ = I + T_m = S_-^* J_- S_-$ and $A_- = I - T_M = S_+^* J_+ S_+$ to the completion problems (26), where

$$S_- = |I + T_{11}|^{[-1/2]}T_{21}^*, \quad S_+ = |I - T_{11}|^{[-1/2]}T_{21}^*.$$

Here $Q_m := S_-^* J_- S_- = V(I - T_{11})JV^*$ and $Q_M := S_+^* J_+ S_+ = V(I + T_{11})JV^*$ appear when calculating the generalized Schur complements of the block operators A_+ and A_- using proper range inclusions; see Proposition 2 and (17). These two operators can be expressed either by limit values or by integrals as follows:

$$Q_m = T_{21}(I + T_{11})^{(-1)}T_{21}^* := \lim_{\varepsilon \uparrow 1} T_{21}(I + \varepsilon T_{11})^{-1}T_{21}^* = \int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}dE_t T_{21}^*}{1 + t},$$

$$Q_M = T_{21}(I - T_{11})^{(-1)}T_{21}^* := \lim_{\varepsilon \uparrow 1} T_{21}(I - \varepsilon T_{11})^{-1}T_{21}^* = \int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}dE_t T_{21}^*}{1 - t},$$

where ε is sufficiently close to 1 (to guarantee proper invertibility of indicated inverses) and E_t stands for the spectral family of T_{11} . With these notations the equality $T_m = T_M$ can be also rewritten in the form $Q_m - I = I - Q_M$, i.e. $2I = Q_m + Q_M = 2VJV^*$ or, equivalently,

$$\int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}dE_t T_{21}^*}{1 - t^2} = I. \tag{54}$$

In the special case of finite defect numbers ($\dim(\text{dom } T_1)^\perp < \infty$) the condition (54) appears in Langer and Textorius [53, Theorem 2.8]. Notice, that using the factorization $T_{21} = VD_{T_{11}}$ and the formula $I - T_{11}^2 = JD_{T_{11}}^2$ the condition (54) can immediately be rewritten in the form $VJV^* = I$.

The criterion in Theorem 9 can be translated to the situation of Theorem 8 via Cayley transform to get the analog of Kreĭn’s uniqueness criterion for the equality $A_F = A_K$.

Corollary 5 *Let A be a closed symmetric relation in \mathfrak{H} satisfying the condition $v_-(A) = v_-(a_1) < \infty$ in Theorem 8. Then the equality $A_F = A_K$ holds if and only if the following condition is fulfilled:*

$$\sup_{g \in \mathfrak{H}_1} \frac{|((A + I)^{-1}g, \varphi)|^2}{(|\widehat{A}|g, g)} = \infty \quad \text{for every } \varphi \in \ker(A^* + I) \setminus \{0\}, \tag{55}$$

where $\widehat{A} = (I + A)^{-*}A(I + A)^{-1}$ is a bounded selfadjoint operator in $\mathfrak{H}_1 = \text{ran}(A + I)$.

Proof Let $T_1 = \mathcal{C}(A)$ so that $\{f, f'\} \in A$ if and only if $\{f + f', 2f\} \in T_1 + I$; see (35). Then with $g = f + f' \in \text{dom } T_1 = \mathfrak{H}_1$ and $\varphi \in \mathfrak{H}_2 = (\text{dom } T_1)^\perp$ one has

$$(T_1 g, \varphi) = ((T_1 + I)g, \varphi) = 2((A + I)^{-1}g, \varphi).$$

Let $A_s = P_s A$ be the operator part of A ; here P_s stands for the orthogonal projection onto $\text{mul } A = (\text{dom } A^*)^\perp = \ker(T_1 + I)$. Then the form $a(f, f) = (f', f)$ associated with A can be rewritten as $a(f, f) = (A_s f, f)$, $f \in \text{dom } A$, and thus

$$((I - T_1^* T_1)g, g) = 4(f', f) = 4(A_s(I + A)^{-1}g, (I + A)^{-1}g),$$

where $2(I + A)^{-1} = T_1 + I$ is a bounded operator from \mathfrak{H}_1 into \mathfrak{H} . Then clearly $\widehat{A} = (I + A)^{-*} A_s (I + A)^{-1}$ is a bounded selfadjoint operator in \mathfrak{H}_1 and, moreover, $v_-(\widehat{A}) = v_-(a) = v_-(I - T_1^* T_1)$; see Lemma 6. Thus, it follows from Proposition 1 that there is a J -unitary operator C from $\overline{\text{ran } \widehat{A}}$ into \mathfrak{D}_{T_1} such that $D_{T_1} = C|\widehat{A}|^{1/2}$. As in the proof of Theorem 8 this implies the equivalence of the conditions (51) and (55).

Observe that if A is nonnegative then with $\{f, f'\} \in A$ and $g = f + f' \in \mathfrak{H}_1$,

$$((A + I)^{-1}g, \varphi) = (f, \varphi), \quad (A_s(I + A)^{-1}g, (I + A)^{-1}g) = (A_s f, f),$$

and, therefore, in this case the condition (55) can be rewritten as

$$\sup_{\{f, f'\} \in A} \frac{|(f, \varphi)|^2}{(f', f)} = \infty \quad \text{for every } \varphi \in \ker(A^* + I) \setminus \{0\},$$

the criterion which for a densely defined operator A was obtained in [47] and for a nonnegative relation A can be found in [38, 39].

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