Math. Ann. (2016) 364:649–655
DOI 10.1007/s00208-015-1230-z

Mathematische Annalen

Sufficient bigness criterion for differences of two nef classes

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Received: 24 December 2014 / Revised: 3 May 2015 / Published online: 14 May 2015

Abstract We prove the qualitative part of Demailly's conjecture on transcendental Morse inequalities for differences of two nef classes satisfying a numerical relative positivity condition on an arbitrary compact Kähler (and even more general) manifold. The result improves on an earlier one by J. Xiao whose constant $4n$ featuring in the hypothesis is now replaced by the optimal and natural n. Our method follows arguments by Chiose as subsequently used by Xiao up to the point where we introduce a new way of handling the estimates in a certain Monge–Ampère equation. This result is needed to extend to the Kähler case and to transcendental classes the Boucksom–Demailly–Paun–Peternell cone duality theorem if one is to follow these authors' method and was conjectured by them.

1 Introduction

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Following various authors (e.g. [2]), Xiao makes in [6] the following assumption

(H) *there exists a Hermitian metric ω on X such that*

$$\partial\bar\partial\omega^k = 0 \quad \text{for all k} = 1, 2, \ldots, n-1.$$

It is clear that (H) holds if X is a Kähler manifold. It is also standard and easy to check that condition (H) is equivalent to either of the following two equivalent conditions:

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$$\partial\bar\partial\omega = 0 \quad \text{and} \quad \partial\bar\partial\omega^2 = 0 \iff \partial\bar\partial\omega = 0 \quad \text{and} \quad \partial\omega\wedge\bar\partial\omega = 0.$$

Following Xiao's method in [6], itself inspired by earlier authors, especially Chiose [2], we prove the following statement in which a real Bott–Chern class of bidegree $(1, 1)$ being *nef* means, as usual, that it contains C^∞ representatives with arbitrarily small negative parts (see inequalities (1)).

Theorem 1.1 *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ satisfying the assumption (H). Then, for any* **nef** *Bott-Chern cohomology classes $\{\alpha\}, \{\beta\} \in H^{1,1}_{BC}(X, \mathbb{R})$, the following implication holds:*

$$\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0 \Longrightarrow \textit{the class } \{\alpha-\beta\} \textit{ contains a Kähler current}.$$

This answers affirmatively the *qualitative* part of a special version (i.e. the one for a difference of two nef classes) of Demailly's *transcendental Morse inequalities conjecture* (see [1, Conjecture 10.1, (ii)]) and will be crucial to the eventual extension of the duality theorem proved in [1, Theorem 2.2] to transcendental classes in the fairly general context of compact Kähler (not necessarily projective) manifolds. Although the method we propose here also produces a lower bound for the volume of the difference class $\{\alpha-\beta\}$, this bound (that we will not present here) is weaker than the lower bound $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\}$ predicted in the *quantitative* part of Conjecture 10.1, (ii) in [1].

Xiao proves in [6] the existence of a Kähler current in the class $\{\alpha-\beta\}$ under the stronger assumption $\{\alpha\}^n - 4n\,\{\alpha\}^{n-1}.\{\beta\} > 0$ and the same assumption (H) on X. The two ingredients he uses are as follows.

Lemma 1.2 (Lamari's duality lemma, [3, Lemme 3.3]) *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let α be any C^∞ real $(1, 1)$-form on X. Then the following two statements are equivalent.*

- (i) *There exists a distribution ψ on X such that $\alpha + i\partial\bar\partial\psi \geq 0$ in the sense of $(1, 1)$-currents on X.*
- (ii) *$\int_X \alpha\wedge\gamma^{n-1} \geq 0$ for any Gauduchon metric γ on X.*

As an aside, we notice that this statement, when applied to d-closed real $(1, 1)$-forms α, translates to the pseudo-effective cone $\mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R})$ of X and the closure of the Gauduchon cone $\overline{\mathcal{G}_X} \subset H^{n-1,n-1}_A(X, \mathbb{R})$ of X being dual under the duality between the Bott-Chern cohomology of bidegree $(1, 1)$ and the Aeppli cohomology of bidegree $(n-1, n-1)$. (See [4] for the definition of the Gauduchon cone.)

Theorem 1.3 (The Tosatti–Weinkove resolution of Hermitian Monge–Ampère equations, [5]) *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let ω be a Hermitian metric on X.*

Then, for any C^∞ function $F : X \to \mathbb{R}$, there exist a unique constant $C > 0$ and a unique C^∞ function $\varphi : X \to \mathbb{R}$ such that

$$(\omega + i\partial\bar\partial\varphi)^n = Ce^F\omega^n, \quad \omega + i\partial\bar\partial\varphi > 0 \quad \textit{and} \quad \sup_X\varphi = 0.$$

As a matter of fact, Yau's classical theorem that solved the Calabi Conjecture, of which Theorem 1.3 is a generalisation to the possibly non-Kähler context, suffices for the proof of Theorem 1.1 whose assumptions imply that X must be Kähler (as already pointed out by Xiao in his situation based on [2, Theorem 0.2]) although this is not used either here or in Xiao's work.

2 Xiao's approach

In this section, we simply reproduce Xiao's arguments (themselves inspired by earlier authors) up to the point where we will branch off in a different direction in the next section to handle certain estimates.

Let us fix a Hermitian metric ω on X such that $\partial\bar\partial\omega^k = 0$ for all k. We also fix nef Bott–Chern $(1,\ 1)$-classes $\{\alpha\}$, $\{\beta\}$. By the nef assumption, for every $\varepsilon > 0$, there exist C^∞ functions $\varphi_\varepsilon, \psi_\varepsilon : X \to \mathbb{R}$ such that

$$\alpha_\varepsilon := \alpha + \varepsilon\,\omega + i\partial\bar\partial\varphi_\varepsilon > 0 \quad \text{and} \quad \beta_\varepsilon := \beta + \varepsilon\,\omega + i\partial\bar\partial\psi_\varepsilon > 0 \quad \text{on} \quad X. \tag{1}$$

Note that α_ε and β_ε need not be d-closed, but the property $\partial\bar\partial\omega^k = 0$ yields

$$\partial\bar\partial\alpha_\varepsilon^k = \partial\bar\partial\beta_\varepsilon^k = 0 \quad \text{and} \quad \partial\bar\partial(\alpha + \varepsilon\,\omega)^k = \partial\bar\partial(\beta + \varepsilon\,\omega)^k = 0 \tag{2}$$

for all $k = 1, 2, \dots, n-1$. We normalise $\sup_X \varphi_\varepsilon = \sup_X \psi_\varepsilon = 0$ for every $\varepsilon > 0$.

Let us fix $\varepsilon > 0$. The existence of a Kähler current in the class $\{\alpha - \beta\} = \{\alpha_\varepsilon - \beta_\varepsilon\}$ is equivalent to

$$\exists\,\delta > 0,\ \exists \text{ a distribution } \theta_\delta \text{ on } X \quad \text{such that} \quad \alpha_\varepsilon - \beta_\varepsilon + i\partial\bar\partial\theta_\delta \geq \delta\,\alpha_\varepsilon,$$

which, in view of Lamari's duality Lemma 1.2, is equivalent to

$$\exists\delta > 0 \quad \text{such that} \quad \int_X (\alpha_\varepsilon - \beta_\varepsilon) \wedge \gamma^{n-1} \geq \delta \int_X \alpha_\varepsilon \wedge \gamma^{n-1}$$

for every Gauduchon metric γ on X. This is, of course, equivalent to

$$\exists\delta > 0 \quad \text{such that} \quad (1-\delta)\int_X \alpha_\varepsilon \wedge \gamma^{n-1} \geq \int_X \beta_\varepsilon \wedge \gamma^{n-1}$$

for every Gauduchon metric γ on X.

Xiao's approach is to prove the existence of a Kähler current in the class $\{\alpha - \beta\} = \{\alpha_\varepsilon - \beta_\varepsilon\}$ by contradiction. Suppose that no such current exists. Then, for every $\varepsilon > 0$ and every sequence of positive reals $\delta_m \downarrow 0$, there exist Gauduchon metrics $\gamma_{m,\,\varepsilon}$ on X such that

$$(1-\delta_m)\int\limits_X \alpha_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1} < \int\limits_X \beta_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1} = 1 \quad \text{for all } m\in\mathbb{N}^\star,\ \varepsilon>0. \tag{3}$$

The last identity is a normalisation of the Gauduchon metrics $\gamma_{m,\,\varepsilon}$ which is clearly always possible by rescaling $\gamma_{m,\,\varepsilon}$ by a positive factor. This normalisation implies that for every $\varepsilon>0$, the positive definite $(n-1,\,n-1)$-forms $(\gamma_{m,\,\varepsilon}^{n-1})_m$ are uniformly bounded in mass, hence after possibly extracting a subsequence we can assume the convergence $\gamma_{m,\,\varepsilon}^{n-1}\to\Gamma_{\infty,\,\varepsilon}$ in the weak topology of currents as $m\to+\infty$, where $\Gamma_{\infty,\,\varepsilon}\geq 0$ is an $(n-1,\,n-1)$-current on X. Taking limits as $m\to+\infty$ in (3), we get

$$\int\limits_X \alpha_\varepsilon\wedge\Gamma_{\infty,\,\varepsilon}\leq 1 \quad \text{for all } \varepsilon>0. \tag{4}$$

Note that the l.h.s. of (3) does not change if α_ε is replaced with any $\alpha_\varepsilon+i\partial\bar\partial u$ (thanks to $\gamma_{m,\,\varepsilon}$ being Gauduchon), while $\alpha_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}$ is (after division by $\gamma_{m,\,\varepsilon}^n$) the trace of α_ε w.r.t. $\gamma_{m,\,\varepsilon}$ divided by n (i.e. the arithmetic mean of the eigenvalues). To find a lower bound for the trace that would contradict (3), it is natural to prescribe the volume form (i.e. the product of the eigenvalues) of some $\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon}$ by imposing that it be, up to a constant factor, the strictly positive $(n,\,n)$-form featuring in the r.h.s. of (3). More precisely, the Tosatti-Weinkove Theorem 1.3 allows us to solve the Monge-Ampère equation

$$(\star)_{m,\,\varepsilon} \qquad (\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon})^n = c_\varepsilon\,\beta_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}$$

for any $\varepsilon>0$ and any $m\in\mathbb{N}^\star$ by ensuring the existence of a unique constant $c_\varepsilon>0$ and of a unique C^∞ function $u_{m,\,\varepsilon}\,:\,X\to\mathbb{R}$ satisfying $(\star)_{m,\,\varepsilon}$ such that

$$\widetilde\alpha_{m,\,\varepsilon}:=\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon}>0, \quad \sup_X(\varphi_\varepsilon+u_{m,\,\varepsilon})=0.$$

Note that c_ε is independent of m since we must have

$$c_\varepsilon=\int\limits_X\widetilde\alpha_{m,\,\varepsilon}^n=\int\limits_X(\alpha+\varepsilon\omega)^n\downarrow\int\limits_X\alpha^n:=c_0>0, \tag{5}$$

where the non-increasing convergence is relative to $\varepsilon\downarrow 0$. Indeed, the second identity in (5) follows from $\partial\bar\partial(\alpha+\varepsilon\omega)^k=0$ for all $k=1,2,\dots,n-1$ (cf. (2)). Thus, it is significant that c_ε does not change if we add any $i\partial\bar\partial u$ to α, i.e. c_ε depends only on the Bott–Chern class $\{\alpha\}$, on ω and on ε. Analogously, one defines

$$M_\varepsilon:=\int\limits_X\widetilde\alpha_{m,\,\varepsilon}^{n-1}\wedge\beta_\varepsilon=\int\limits_X(\alpha+\varepsilon\omega)^{n-1}\wedge(\beta+\varepsilon\omega)\downarrow\int\limits_X\alpha^{n-1}\wedge\beta:=M_0\geq 0, \tag{6}$$

where the non-increasing convergence is relative to $\varepsilon\downarrow 0$. Clearly, M_ε is independent of m and depends only on the Bott-Chern classes $\{\alpha\}$, $\{\beta\}$, on ω and on ε. Note that the second integral in (6) equals $\int_X (\alpha + \varepsilon\omega + i\partial\bar\partial\varphi_\varepsilon)^{n-1} \wedge (\beta + \varepsilon\omega + i\partial\bar\partial\psi_\varepsilon)$ which is positive since $\alpha_\varepsilon, \beta_\varepsilon > 0$ by (1). Since $M_0 \geq 0$, the hypothesis $c_0 - nM_0 > 0$ made in Theorem 1.1 implies $c_0 > 0$. This justifies the final claim in (5).

3 Estimates in the Monge–Ampère equation

We now propose an approach to the details of these estimates that differs from that of Xiao. We start with a very simple, elementary (and probably known) observation.

Lemma 3.1 *For any Hermitian metrics α, β, γ on a complex manifold, the following inequality holds at every point:*

$$(\Lambda_\alpha \beta) \cdot (\Lambda_\beta \gamma) \geq \Lambda_\alpha \gamma. \tag{7}$$

Proof Since (7) is a pointwise inequality, we fix an arbitrary point x and choose local coordinates about x such that

$$\beta(x) = \sum_j i dz_j \wedge d\bar z_j, \quad \alpha(x) = \sum_j \alpha_j\, i dz_j \wedge d\bar z_j \quad \text{and} \quad \gamma(x) = \sum_{j,k} \gamma_{j\bar k}\, i dz_j \wedge d\bar z_k.$$

Then $\alpha_j > 0$ and $\gamma_{j\bar j} > 0$ for every j. If we denote by the same symbol any $(1,\ 1)$-form and its coefficient matrix in the chosen coordinates, we have

$$\alpha^{-1}\gamma = \left(\frac{1}{\alpha_j}\gamma_{j\bar k}\right)_{j,k}, \text{ hence } \mathrm{Tr}(\alpha^{-1}\gamma) = \sum_j \frac{1}{\alpha_j}\gamma_{j\bar j}.$$

Thus (7) translates to $(\sum_j \frac{1}{\alpha_j}) \sum_k \gamma_{k\bar k} \geq \sum_j \frac{1}{\alpha_j}\gamma_{j\bar j}$ which clearly holds since $\sum_{j\neq k} \frac{1}{\alpha_j}\gamma_{k\bar k} > 0$ because all the α_j and all the $\gamma_{k\bar k}$ are positive. □

Our main observation is the following statement.

Lemma 3.2 *For every $m \in \mathbb{N}^\star$ and every $\varepsilon > 0$, we have*

$$\left(\int_X \widetilde\alpha_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1}\right) \cdot \left(\int_X \widetilde\alpha_{m,\varepsilon}^{n-1} \wedge \beta_\varepsilon\right) \geq \frac{1}{n}\int_X \widetilde\alpha_{m,\varepsilon}^n = \frac{c_\varepsilon}{n}. \tag{8}$$

Proof Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, resp. $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, be the eigenvalues of $\widetilde\alpha_{m,\varepsilon}$, resp. β_ε, w.r.t. $\gamma_{m,\varepsilon}$. We have

$$\widetilde\alpha_{m,\varepsilon}^n = \lambda_1 \dots \lambda_n\, \gamma_{m,\varepsilon}^n \quad \text{and} \quad \widetilde\alpha_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1} = \frac{1}{n}(\Lambda_{\gamma_{m,\varepsilon}}\widetilde\alpha_{m,\varepsilon})\,\gamma_{m,\varepsilon}^n = \frac{\lambda_1 + \cdots + \lambda_n}{n}\gamma_{m,\varepsilon}^n.$$

Similarly, $\beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} = \frac{1}{n}(\Lambda_{\gamma_{m,\varepsilon}}\beta_\varepsilon)\,\gamma_{m,\varepsilon}^n = \frac{\mu_1 + \cdots + \mu_n}{n}\gamma_{m,\varepsilon}^n$.

Thus, the Monge–Ampère equation $(\star)_{m,\,\varepsilon}$ translates to

$$\lambda_1 \ldots \lambda_n = c_\varepsilon \, \frac{\mu_1 + \cdots + \mu_n}{n}. \tag{9}$$

In particular, the normalisation $\int_X \beta_\varepsilon \wedge \gamma_{m,\,\varepsilon}^{n-1} = 1$ reads

$$\frac{1}{c_\varepsilon} \int\limits_X \lambda_1 \ldots \lambda_n \, \gamma_{m,\,\varepsilon}^n = \int\limits_X \frac{\mu_1 + \cdots + \mu_n}{n} \, \gamma_{m,\,\varepsilon}^n = 1. \tag{10}$$

Note that we also have

$$\widetilde{\alpha}_{m,\,\varepsilon}^{n-1} \wedge \beta_\varepsilon = \frac{1}{n} \, (\Lambda_{\widetilde{\alpha}_{m,\,\varepsilon}} \beta_\varepsilon) \, \widetilde{\alpha}_{m,\,\varepsilon}^n = \frac{1}{n} \, (\Lambda_{\widetilde{\alpha}_{m,\,\varepsilon}} \beta_\varepsilon) \, \lambda_1 \ldots \lambda_n \, \gamma_{m,\,\varepsilon}^n. \tag{11}$$

Putting all of the above together, we get

$$\begin{aligned}
&\left(\int\limits_X \widetilde{\alpha}_{m,\,\varepsilon} \wedge \gamma_{m,\,\varepsilon}^{n-1} \right) \cdot \left(\int\limits_X \widetilde{\alpha}_{m,\,\varepsilon}^{n-1} \wedge \beta_\varepsilon \right) \\
&= \left(\int\limits_X \frac{1}{n} \, (\Lambda_{\gamma_{m,\,\varepsilon}} \widetilde{\alpha}_{m,\,\varepsilon}) \, \gamma_{m,\,\varepsilon}^n \right) \cdot \left(\int\limits_X \frac{1}{n} \, (\Lambda_{\widetilde{\alpha}_{m,\,\varepsilon}} \beta_\varepsilon) \, \lambda_1 \ldots \lambda_n \, \gamma_{m,\,\varepsilon}^n \right) \\
&\overset{(a)}{\geq} \frac{1}{n^2} \left(\int\limits_X \left[(\Lambda_{\gamma_{m,\,\varepsilon}} \widetilde{\alpha}_{m,\,\varepsilon}) \, (\Lambda_{\widetilde{\alpha}_{m,\,\varepsilon}} \beta_\varepsilon) \right]^{\frac{1}{2}} \, (\lambda_1 \ldots \lambda_n)^{\frac{1}{2}} \, \gamma_{m,\,\varepsilon}^n \right)^2 \\
&\overset{(b)}{\geq} \frac{1}{n^2} \left(\int\limits_X (\Lambda_{\gamma_{m,\,\varepsilon}} \beta_\varepsilon)^{\frac{1}{2}} \, (\lambda_1 \ldots \lambda_n)^{\frac{1}{2}} \, \gamma_{m,\,\varepsilon}^n \right)^2 \overset{(c)}{=} \frac{1}{n^2} \left(\int\limits_X \frac{\sqrt{n}}{\sqrt{c_\varepsilon}} \, \lambda_1 \ldots \lambda_n \, \gamma_{m,\,\varepsilon}^n \right)^2 \\
&= \frac{1}{n \, c_\varepsilon} \left(\int\limits_X \widetilde{\alpha}_{m,\,\varepsilon}^n \right)^2 \overset{(d)}{=} \frac{1}{n \, c_\varepsilon} \left(\int\limits_X c_\varepsilon \, \beta_\varepsilon \wedge \gamma_{m,\,\varepsilon}^{n-1} \right)^2 \overset{(e)}{=} \frac{c_\varepsilon}{n}.
\end{aligned}$$

This proves (8). Inequality (a) is an application of the Cauchy–Schwarz inequality, inequality (b) has followed from (7), identity (c) has followed from (9), identity (d) has followed from $\widetilde{\alpha}_{m,\,\varepsilon}^n = c_\varepsilon \, \beta_\varepsilon \wedge \gamma_{m,\,\varepsilon}^{n-1}$ (which is nothing but the Monge–Ampère equation $(\star)_{m,\,\varepsilon}$), while identity ($e$) has followed from the normalisation $\int_X \beta_\varepsilon \wedge \gamma_{m,\,\varepsilon}^{n-1} = 1$ (cf. (3)). The proof of Lemma 3.2 is complete. □

End of proof of Theorem 1.1. Now, $\widetilde{\alpha}_{m,\,\varepsilon} = \alpha_\varepsilon + i\partial\bar\partial u_{m,\,\varepsilon}$ and $\partial\bar\partial \gamma_{m,\,\varepsilon}^{n-1} = 0$, so

$$\int\limits_X \widetilde{\alpha}_{m,\,\varepsilon} \wedge \gamma_{m,\,\varepsilon}^{n-1} = \int\limits_X \alpha_\varepsilon \wedge \gamma_{m,\,\varepsilon}^{n-1} \longrightarrow \int\limits_X \alpha_\varepsilon \wedge \Gamma_{\infty,\,\varepsilon} \leq 1 \quad \text{for all } \varepsilon > 0, \tag{12}$$

where the above arrow stands for convergence as $m \to +\infty$ and the last inequality is nothing but (4) (which, recall, is a consequence of the assumption that no Kähler current exists in $\{\alpha - \beta\}$—an assumption that we are going to contradict). On the other hand, the second factor on the l.h.s. of (8) is precisely M_ε defined in (6), so in particular it is independent of m. Fixing any $\varepsilon > 0$, taking limits as $m \to +\infty$ in (8) and using (12), we get

$$M_\varepsilon \geq \frac{c_\varepsilon}{n} \quad \text{for every } \varepsilon > 0. \tag{13}$$

Taking now limits as $\varepsilon \downarrow 0$ and using (6) and (5), we get

$$M_0 \geq \frac{c_0}{n}, \qquad \text{i.e.} \quad \{\alpha\}^{n-1}.\{\beta\} \geq \frac{\{\alpha\}^n}{n}.$$

The last identity means that $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} \leq 0$ which is impossible if we suppose that $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$. This is the desired contradiction proving the existence of a Kähler current in the class $\{\alpha - \beta\}$ under the assumption $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$. □

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