



Glanon groupoids

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Abstract We introduce the notions of Glanon groupoids, which are Lie groupoids equipped with multiplicative generalized complex structures, and of Glanon algebroids, their infinitesimal counterparts. Both symplectic and holomorphic Lie groupoids are particular instances of Glanon groupoids. We prove that there is a bijection between Glanon algebroids on one hand and source connected and source-simply connected Glanon groupoids on the other. As a consequence, we recover various known integrability results and obtain the integration of holomorphic Lie bialgebroids to holomorphic Poisson groupoids.

1 Introduction

In their study of quantization, Karasëv [19], Weinstein [34], and Zakrzewski [40,41] independently introduced the notion of symplectic groupoids. By a symplectic groupoid, we mean a Lie groupoid equipped with a multiplicative symplectic 2-form

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on the space of morphisms. It is a classical theorem that the unit space of a symplectic groupoid is naturally a Poisson manifold [7]. The Lie algebroid of a symplectic groupoid $\Gamma \rightrightarrows M$ is naturally isomorphic to $(T^*M)_\pi$, the canonical Lie algebroid associated to the Poisson manifold (M, π) . Conversely, Mackenzie–Xu [29] proved that, for a given Poisson manifold (M, π) , if the Lie algebroid $(T^*M)_\pi$ integrates to an \mathfrak{s} -connected and \mathfrak{s} -simply connected Lie groupoid $\Gamma \rightrightarrows M$, then Γ is naturally a symplectic groupoid. As a consequence, they recovered the following theorem of Karasev–Weinstein: every Poisson manifold of dimension n admits a symplectic realization of dimension $2n$. The symplectic groupoid structure on Γ was also obtained by Cattaneo–Felder [6] using the Poisson sigma model. The full integrability criterion for Poisson manifolds was obtained later by Crainic–Fernandes [10]. In fact, symplectic groupoids constitute a particular class of a more general type of structures called Poisson groupoids, which were discovered by Weinstein [35] and also comprise Drinfeld’s Poisson groups [11]. In a Poisson groupoid, the Poisson bivector field and the groupoid multiplication are required to be compatible: the Poisson bivector field must be “multiplicative.” It was proved in [28] that a Poisson bivector field on a Lie groupoid $G \rightrightarrows M$ with Lie algebroid A is multiplicative if and only if it induces a morphism of Lie groupoids from the cotangent groupoid $T^*G \rightrightarrows A^*$ to the tangent groupoid $TG \rightrightarrows TM$. For instance, a Poisson bivector field on a Lie group G is multiplicative in the sense of Drinfeld [11] if and only if it induces a morphism of Lie groupoids from $T^*G \rightrightarrows \mathfrak{g}^*$ to $TG \rightrightarrows \{*\}$. A Poisson groupoid whose Poisson bivector field is nondegenerate is a symplectic groupoid as the inverse of the Poisson bivector is a multiplicative symplectic form in the sense of Coste–Dazord–Weinstein [7].

Two of the authors have recently been interested in holomorphic Lie algebroids and holomorphic Lie groupoids [24]. Finding out which holomorphic Lie algebroids can be integrated is a very natural problem. In [23], together with Laurent-Gengoux, they studied holomorphic Lie algebroids and their relation with real Lie algebroids. A holomorphic Lie algebroid is a real Lie algebroid structure on a holomorphic vector bundle $A \rightarrow X$ such that (1) the sheaf \mathcal{A} of holomorphic sections of A is stable under the Lie bracket of (all smooth) sections of A and (2) the restriction of the Lie bracket to \mathcal{A} is \mathbb{C} -linear. Laurent-Gengoux et al. proved in particular that a holomorphic Lie algebroid A can be integrated to a holomorphic Lie groupoid if and only if its underlying real Lie algebroid A_R is integrable as a real Lie algebroid [24].

In this paper, we introduce the notion of Glanon groupoids: Lie groupoids equipped with a multiplicative generalized complex structure. Recall that a generalized complex structure in the sense of Hitchin [15] on a manifold M is a smooth bundle map $\mathcal{J}: TM \oplus T^*M \rightarrow TM \oplus T^*M$ such that $\mathcal{J}^2 = -\text{id}_{TM \oplus T^*M}$, \mathcal{J} preserves the natural nondegenerate symmetric bilinear form on the fibers of $TM \oplus T^*M$, and the $+i$ -eigenbundle of \mathcal{J} is involutive with respect to the Courant bracket (or, equivalently, the Nijenhuis tensor of \mathcal{J} vanishes). A generalized complex structure \mathcal{J} on a Lie groupoid $\Gamma \rightrightarrows M$ is said to be multiplicative if \mathcal{J} is a Lie groupoid automorphism of the Courant groupoid $T\Gamma \oplus T^*\Gamma \rightrightarrows TM \oplus A^*$ in the sense of Mehta [30]. When \mathcal{J} is the generalized complex structure determined by a symplectic structure on Γ , it is clear that \mathcal{J} is multiplicative if and only if the symplectic 2-form is multiplicative. In this case, the Glanon groupoid is simply a symplectic groupoid. On the other hand,

when \mathcal{J} is the generalized complex structure determined by a complex structure on Γ , \mathcal{J} is multiplicative if and only if the complex structure on Γ is multiplicative. In this case, the Glanon groupoid is simply a holomorphic Lie groupoid. On the infinitesimal level, to each Glanon groupoid corresponds a Glanon Lie algebroid: a Lie algebroid A equipped with a generalized complex structure $\mathcal{J}_A: TA \oplus T^*A \rightarrow TA \oplus T^*A$ which is also an automorphism of the Lie algebroid $TA \oplus T^*A \rightarrow TM \oplus A^*$. More precisely, we prove the following main result:

Theorem A *If Γ is an \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid with Lie algebroid A , then there is a bijection between Glanon groupoid structures on Γ and Glanon Lie algebroid structures on A .*

As a consequence, we recover the following standard results [24, 29]:

Theorem B *Let (M, π) be a Poisson manifold. If Γ is a \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid integrating the Lie algebroid $(T^*M)_\pi$, then Γ automatically admits a symplectic groupoid structure.*

Theorem C *If Γ is a \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid integrating the real Lie algebroid A_R underlying a holomorphic Lie algebroid A , then Γ is a holomorphic Lie groupoid.*

When the matrix representation of the generalized complex structure \mathcal{J} on a Glanon groupoid Γ relative to the direct sum decomposition $T\Gamma \oplus T^*\Gamma$ of the Pontryagin bundle¹ has the special form

$$\mathcal{J} = \begin{pmatrix} N & \pi^\sharp \\ 0 & -N^* \end{pmatrix},$$

Γ is simply a holomorphic Poisson groupoid and it is no surprise that its Lie algebroid is necessarily part of a holomorphic Lie bialgebroid. We prove the following result.

Theorem D *Given a holomorphic Lie bialgebroid (A, A^*) , if the real Lie algebroid A_R underlying A integrates to a \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid Γ , then Γ is a holomorphic Poisson groupoid.*

This theorem was proved in [24] using a different method in the special case of the holomorphic Lie bialgebroid $((T^*X)_\pi, TX)$ determined by a holomorphic Poisson manifold (X, π) . More precisely, it was proved that when the underlying real Lie algebroid of $(T^*X)_\pi$ integrates to a \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid Γ , then Γ is automatically a holomorphic symplectic groupoid. To the best of our knowledge, the integration problem for arbitrary holomorphic Lie bialgebroids had remained open to this day. Solving it constituted one of the motivations behind our study of Glanon groupoids.

¹ Yoshimura and Marsden refer to the Whitney sum $TM \oplus T^*M$ as the ‘‘Pontryagin bundle’’ of M because of the fundamental role it plays in the geometric interpretation of Pontryagin’s maximum principle. Izu Vaisman calls it the ‘‘big tangent bundle’’ of M . For more details, see [22].

It is known that a generalized complex structure on a manifold determines on it a Poisson bivector field [2, 14]. Therefore a Glanon groupoid is automatically a (real) Poisson groupoid and we can consider the ‘Glanon to Poisson’ forgetful functor. On the other hand, every Glanon Lie algebroid A admits a linear Poisson structure so that its dual A^* is also a Lie algebroid. We prove that, for any Glanon algebroid A , the pair (A, A^*) automatically constitutes a Lie bialgebroid. Finally, we prove that the ‘groupoid to algebroid’ Lie functor, which takes Glanon groupoids and Poisson groupoids, respectively, to Glanon algebroids and Lie bialgebroids, commutes with the forgetful functor, which takes Glanon groupoids and Glanon algebroids, respectively, to Poisson groupoids and Lie bialgebroids.

Note that in this paper we confine ourselves to the standard Courant groupoid $T\Gamma \oplus T^*\Gamma$. Instead, one could have considered the twisted Courant groupoid $(T\Gamma \oplus T^*\Gamma)_H$, where H is a multiplicative closed 3-form. This will be discussed elsewhere. Note also that a multiplicative generalized complex structure on a groupoid induces an endomorphism $j: TM \oplus A^* \rightarrow TM \oplus A^*$ of the unit space and an endomorphism $j_A: A \oplus T^*M \rightarrow A \oplus T^*M$ of the core² of the VB-groupoid $T\Gamma \oplus T^*\Gamma$. Thinking of multiplicative generalized complex structures as pairs of complex conjugate multiplicative Dirac structures, we can conclude from results in [18] that a multiplicative generalized complex structure on a groupoid is equivalent to a pair of complex conjugate Dirac bialgebroids on its space of units. The detailed study of properties of the maps j_A and j , the Dirac bialgebroids, and the associated Dorfman connections will be investigated in the spirit of [23] in a future project.

Notation

In the following, $\Gamma \rightrightarrows M$ will always be a Lie groupoid with set of arrows Γ , set of objects M , source and target $\mathbf{s}, \mathbf{t}: \Gamma \rightarrow M$, object inclusion map $\epsilon: M \rightarrow \Gamma$ and inversion map $\mathbf{i}: \Gamma \rightarrow \Gamma$. The product of $g, h \in \Gamma$ with $\mathbf{s}(g) = \mathbf{t}(h)$ will be written $\mathbf{m}(g, h) = g \star h$ or simply gh .

The Lie functor that sends a Lie groupoid to its Lie algebroid and Lie groupoid morphisms to Lie algebroid morphisms is \mathbf{A} . For simplicity, we will write $\mathbf{A}(\Gamma) = A$. The Lie algebroid $q_A: A \rightarrow M$ is identified with $T_M^S \Gamma$, the bracket $[\cdot, \cdot]_A$ is defined with the right invariant vector fields and the anchor $\rho_A = \rho$ is the restriction of $T\mathbf{t}$ to A . Hence, as a manifold, A is embedded in $T\Gamma$. The inclusion is $\iota: A \rightarrow T\Gamma$. Given $a \in \Gamma(A)$, the right-invariant section corresponding to a will simply be written a^r , i.e. $a^r(g) = TR_g a(\mathbf{t}(g))$ for all $g \in \Gamma$. We will write a^l for the left-invariant vector field defined by a , i.e. $a^l(g) = -T(L_g \circ \mathbf{i})(a(\mathbf{s}(g)))$ for all $g \in \Gamma$.

The projection map of a vector bundle $A \rightarrow M$ will always be written $q_A: A \rightarrow M$, unless specified otherwise. For a smooth manifold M , we fix once and for all the notation $p_M := q_{TM}: TM \rightarrow M$ and $c_M := q_{T^*M}: T^*M \rightarrow M$. We will write \mathbf{P}_M

² The Whitney sum $T\Gamma \oplus T^*\Gamma$, which is simultaneously a vector bundle with Γ as base manifold and a Lie groupoid with $TM \oplus A^*$ as unit space, is a VB-groupoid with the vector bundle $A \oplus T^*M$ over M as core [26, 32]. It is well known that a morphism of VB-groupoids determines a morphism of their cores.

for the Pontryagin bundle $TM \oplus T^*M$ over M , and pr_M for the canonical projection $q_{P_M} : P_M \rightarrow M$.

A bundle morphism $P_M \rightarrow P_M$, for a manifold M , will always be meant to be over the identity on M .

2 Preliminaries

2.1 Dirac structures

Let $A \rightarrow M$ be a vector bundle with dual bundle $A^* \rightarrow M$. The natural pairing $A \oplus A^* \rightarrow \mathbb{R}$, $(a_m, \xi_m) \mapsto \xi_m(a_m)$ will be written $\langle \cdot, \cdot \rangle_A$ or $\langle \cdot, \cdot \rangle_{q_A}$ if the vector bundle structure needs to be specified. The direct sum $A \oplus A^*$ is endowed with a canonical fiberwise pairing $(\cdot, \cdot)_A$ given by

$$((a_m, \xi_m), (b_m, \eta_m))_A = \langle b_m, \xi_m \rangle_A + \langle a_m, \eta_m \rangle_A,$$

for all $m \in M$, $a_m, b_m \in A_m$ and $\xi_m, \eta_m \in A_m^*$.

In particular, the Pontryagin bundle $P_M = TM \oplus T^*M$ of a smooth manifold M is endowed with the pairing $(\cdot, \cdot)_{TM}$, which will be written as usual $\langle \cdot, \cdot \rangle_M$.

The orthogonal of a subbundle $E \subseteq A \oplus A^*$ relative to the pairing $(\cdot, \cdot)_A$ will be written E^\perp . An almost Dirac structure [8] on M is a Lagrangian vector subbundle $D \subset P_M$. That is, D coincides with its orthogonal relative to $\langle \cdot, \cdot \rangle_M$, $D = D^\perp$, so its fibers are necessarily $\dim M$ -dimensional.

The set of sections $\Gamma(P_M)$ of the Pontryagin bundle of M is endowed with the Courant–Dorfman bracket, given by

$$\llbracket X + \alpha, Y + \beta \rrbracket = [X, Y] + (\mathbf{L}_X \beta - \mathbf{i}_Y \mathbf{d}\alpha),$$

for all $X + \alpha, Y + \beta \in \Gamma(P_M)$.

An almost Dirac structure D on a manifold M is a Dirac structure if its set of sections is closed under this bracket, i.e. $\llbracket \Gamma(D), \Gamma(D) \rrbracket \subset \Gamma(D)$.

2.2 Generalized complex structures

Let V be a vector space. Consider a linear endomorphism \mathcal{J} of $V \oplus V^*$ such that $\mathcal{J}^2 = -\text{id}_{V \oplus V^*}$ and \mathcal{J} is orthogonal with respect to the inner product

$$(X + \xi, Y + \eta)_V = \xi(Y) + \eta(X), \quad \forall X, Y \in V, \xi, \eta \in V^*.$$

Such a linear map is called a linear generalized complex structure by Hitchin [15]. The complexified vector space $(V \oplus V^*) \otimes \mathbb{C}$ decomposes as the direct sum

$$(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$$

of the eigenspaces of \mathcal{J} corresponding to the eigenvalues $\pm i$ respectively, i.e.

$$E_{\pm} = \{ (X + \xi) \mp i\mathcal{J}(X + \xi) \mid X + \xi \in V \oplus V^* \}.$$

Both eigenspaces are maximal isotropic with respect to $\langle \cdot, \cdot \rangle_V$ and they are complex conjugate to each other.

The following lemma is obvious.

Lemma 1 *The linear generalized complex structures are in bijection with the splittings $(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$ with E_{\pm} maximal isotropic and $E_- = \overline{E_+}$.*

Definition 1 Let M be a manifold and \mathcal{J} a bundle endomorphism of $\mathbf{P}_M = TM \oplus T^*M$ such that $\mathcal{J}^2 = -\text{id}_{\mathbf{P}_M}$, and \mathcal{J} is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$. Then \mathcal{J} is a *generalized almost complex structure*. In the associated eigenbundle decomposition

$$T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M = E_+ \oplus E_-,$$

if $\Gamma(E_+)$ is closed under the (complexified) Courant bracket, then E_+ is a (complex) Dirac structure on M and one says that \mathcal{J} is a *generalized complex structure* [14, 15].

If E_+ is a Dirac structure, then E_- must also be a Dirac structure since $E_- = \overline{E_+}$. Indeed (E_+, E_-) is a complex Lie bialgebroid in the sense of Mackenzie–Xu [28], in which E_+ and E_- are complex conjugate to each other.

Definition 2 Let $\mathcal{J}: \mathbf{P}_M \rightarrow \mathbf{P}_M$ be a vector bundle morphism. Then the *generalized Nijenhuis torsion* associated to \mathcal{J} is the map

$$\mathcal{N}_{\mathcal{J}}: \mathbf{P}_M \times_M \mathbf{P}_M \rightarrow \mathbf{P}_M$$

defined by

$$\mathcal{N}_{\mathcal{J}}(\xi, \eta) = \llbracket \mathcal{J}\xi, \mathcal{J}\eta \rrbracket + \mathcal{J}^2 \llbracket \xi, \eta \rrbracket - \mathcal{J} (\llbracket \mathcal{J}\xi, \eta \rrbracket + \llbracket \xi, \mathcal{J}\eta \rrbracket)$$

for all $\xi, \eta \in \Gamma(\mathbf{P}_M)$, where the bracket is the Courant–Dorfman bracket.

Note that if \mathcal{J} in the last definition is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$ and satisfies $\mathcal{J}^2 = -\text{id}$ (i.e. if \mathcal{J} is an almost complex structure), then its Nijenhuis torsion is a tensor.

The following proposition gives two equivalent definitions of a generalized complex structure.

Proposition 1 *A generalized complex structure is equivalent to any of the following:*

1. A bundle endomorphism \mathcal{J} of \mathbf{P}_M such that \mathcal{J} is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$, $\mathcal{J}^2 = -\text{id}_{\mathbf{P}_M}$ and $\mathcal{N}_{\mathcal{J}} = 0$.
2. A complex Lie bialgebroid (E_+, E_-) whose double is the standard Courant algebroid $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$, and E_+ and E_- are complex conjugate.

For a two-form ω on M we denote by $\omega^b: TM \rightarrow T^*M$ the bundle map $X \mapsto i_X\omega$, while for a bivector π on M we denote by $\pi^\sharp: T^*M \rightarrow TM$ the contraction with π . Also if π is a Poisson bivector, we denote by $[\cdot, \cdot]_\pi$ the Lie algebroid bracket defined on the space of 1-forms on M by

$$[\xi, \eta]_\pi = L_{\pi^\sharp\xi}\eta - L_{\pi^\sharp\eta}\xi - d\pi(\xi, \eta)$$

for all $\xi, \eta \in \Omega^1(M)$.

A generalized complex structure $\mathcal{J}: \mathbb{P}_M \rightarrow \mathbb{P}_M$ can be written

$$\mathcal{J} = \begin{pmatrix} N & \pi^\sharp \\ \omega^b & -N^* \end{pmatrix}$$

with π is a bivector field on M and $\pi^\sharp: T^*M \rightarrow TM$, $\pi^\sharp(\alpha) = \pi(\alpha, \cdot)$ is the vector bundle morphism defined by π , ω is a two-form on M and $\omega^b: TM \rightarrow T^*M$, $X \mapsto i_X\omega$ is the vector bundle morphism defined by ω , and $N: TM \rightarrow TM$ is a bundle map. The geometric structures N , π and ω have to satisfy together a list of identities [9]. In particular π is a Poisson bivector field.

Let $\mathcal{I}_M: \mathbb{P}_M \rightarrow \mathbb{P}_M$ be the endomorphism

$$\mathcal{I}_M = \begin{pmatrix} \text{id}_{TM} & 0 \\ 0 & -\text{id}_{T^*M} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \langle \mathcal{I}_M(\cdot), \mathcal{I}_M(\cdot) \rangle_M &= -\langle \cdot, \cdot \rangle_M, \\ [[\mathcal{I}_M(\cdot), \mathcal{I}_M(\cdot)]] &= \mathcal{I}_M[[\cdot, \cdot]] \end{aligned}$$

and the following proposition follows.

Proposition 2 *If \mathcal{J} is a generalized almost complex structure on M , then*

$$\bar{\mathcal{J}} := \mathcal{I}_M \circ \mathcal{J} \circ \mathcal{I}_M \tag{1}$$

is a generalized almost complex structure. Furthermore,

$$\mathcal{N}_{\bar{\mathcal{J}}} = \mathcal{I}_M \circ \mathcal{N}_{\mathcal{J}} \circ (\mathcal{I}_M, \mathcal{I}_M).$$

Hence, $\bar{\mathcal{J}}$ is a generalized complex structure if and only if \mathcal{J} is a generalized complex structure.

The following are two standard examples [15].

1. Let J be an almost complex structure on M . Then

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

is $\langle \cdot, \cdot \rangle_M$ -orthogonal and satisfies $\mathcal{J}^2 = -\text{id}$. \mathcal{J} is a generalized complex structure if and only if J is integrable.

2. Let ω be a nondegenerate 2-form on M . Then

$$\mathcal{J} = \begin{pmatrix} 0 & -(\omega^b)^{-1} \\ \omega^b & 0 \end{pmatrix}$$

is a generalized complex structure if and only if $d\omega = 0$, i.e. if and only if ω is a symplectic 2-form.

2.3 Pontryagin bundle over a Lie groupoid

The tangent prolongation of a Lie groupoid Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Applying the tangent functor to each of the maps defining Γ yields a Lie groupoid structure on $T\Gamma$ with base TM , source $T\mathfrak{s}$, target $T\mathfrak{t}$, multiplication $T\mathfrak{m}: T(\Gamma \times_M \Gamma) \rightarrow T\Gamma$ and inversion $T\mathfrak{i}: T\Gamma \rightarrow T\Gamma$. The identity at $v_p \in T_p M$ is $1_{v_p} = T_p \in v_p$. This defines the *tangent prolongation* $T\Gamma \rightrightarrows TM$ of $\Gamma \rightrightarrows M$ or the *tangent groupoid associated to* $\Gamma \rightrightarrows M$.

The cotangent Lie groupoid defined by a Lie groupoid If $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid $A \rightarrow M$, then there is also an induced Lie groupoid structure on $T^*\Gamma \rightrightarrows A^*$. The source map $\hat{\mathfrak{s}}: T^*\Gamma \rightarrow A^*$ is given by

$$\hat{\mathfrak{s}}(\alpha_g) \in A_{\mathfrak{s}(g)}^* \quad \text{for } \alpha_g \in T_g^*\Gamma, \quad \hat{\mathfrak{s}}(\alpha_g)(a(\mathfrak{s}(g))) = \alpha_g(a^l(g))$$

for all $a \in \Gamma(A)$, and the target map $\hat{\mathfrak{t}}: T^*\Gamma \rightarrow A^*$ is given by

$$\hat{\mathfrak{t}}(\alpha_g) \in A_{\mathfrak{t}(g)}^*, \quad \hat{\mathfrak{t}}(\alpha_g)(a(\mathfrak{t}(g))) = \alpha_g(a^r(g))$$

for all $a \in \Gamma(A)$. If $\hat{\mathfrak{s}}(\alpha_g) = \hat{\mathfrak{t}}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by

$$(\alpha_g \star \alpha_h)(v_g \star v_h) = \alpha_g(v_g) + \alpha_h(v_h)$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(\Gamma \times_M \Gamma)$.

This Lie groupoid structure was introduced in [7], see also [27,32]. Note that the original definition was the following: let Λ_Γ be the graph of the partial multiplication \mathfrak{m} in Γ , i.e.

$$\Lambda_\Gamma = \{(g, h, g \star h) \mid g, h \in \Gamma, \mathfrak{s}(g) = \mathfrak{t}(h)\}.$$

The isomorphism $\psi: (T^*\Gamma)^3 \rightarrow (T^*\Gamma)^3$, $\psi(\alpha, \beta, \gamma) = (\alpha, \beta, -\gamma)$ sends the conormal space $(T\Lambda_\Gamma)^\circ \subseteq (T^*\Gamma)^3|_{\Lambda_\Gamma}$ to a submanifold Λ_* of $(T^*\Gamma)^3$. It is shown in [7] that Λ_* is the graph of a groupoid multiplication on $T^*\Gamma$, which is exactly the multiplication defined above.

The ‘‘Pontryagin groupoid’’ of a Lie groupoid If $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid $A \rightarrow M$, according to [30], there is hence an induced VB-Lie groupoid

structure on $\mathbf{P}_\Gamma = T\Gamma \oplus T^*\Gamma$ over $TM \oplus A^*$, namely, the product groupoid, where $T\Gamma \oplus T^*\Gamma$ and $TM \oplus A^*$ are identified with the fiber products $T\Gamma \times_\Gamma T^*\Gamma$ and $TM \times_M A^*$, respectively. It is called a Courant groupoid by Mehta [30].

Proposition 3 *Let $\Gamma \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \rightarrow M$. Then the Pontryagin bundle $\mathbf{P}_\Gamma = T\Gamma \oplus T^*\Gamma$ is a Lie groupoid over $TM \oplus A^*$, and the canonical projection $\mathbf{P}_\Gamma \rightarrow \Gamma$ is a Lie groupoid morphism.*

We will write $\mathbb{T}t$ for the target map $\mathbf{P}_\Gamma \rightarrow TM \oplus A^*$ defined by $\mathbb{T}t(v_g, \alpha_g) = (Tt(v_g), \hat{t}(\alpha_g))$, $\mathbb{T}s$ for the source map $\mathbf{P}_\Gamma \rightarrow TM \oplus A^*$, and $\mathbb{T}\epsilon, \mathbb{T}i, \mathbb{T}m$ for the embedding of the units, the inversion map, and the multiplication of this Lie groupoid.

2.4 Pontryagin bundle over a Lie algebroid

Given any vector bundle $q_A : A \rightarrow M$, the map $Tq_A : TA \rightarrow TM$ has a vector bundle structure obtained by applying the tangent functor to the operations in $A \rightarrow M$. The operations in $TA \rightarrow TM$ are consequently vector bundle morphisms with respect to the tangent bundle structures in $TA \rightarrow A$ and $TM \rightarrow M$ and TA with these two structures is therefore a double vector bundle which we call the *tangent double vector bundle of $A \rightarrow M$* (see [26] and references given there).

If $(q_A : A \rightarrow M, [\cdot, \cdot]_A, \rho_A)$ is a Lie algebroid, then there is a Lie algebroid structure on $Tq_A : TA \rightarrow TM$ defined in [28], with respect to which $p_A : TA \rightarrow A$ is a Lie algebroid morphism over $p_M : TM \rightarrow M$; this is the *tangent prolongation of $A \rightarrow M$* .

For a general vector bundle $q : A \rightarrow M$, there is also a double vector bundle

$$\begin{array}{ccc}
 T^*A & \xrightarrow{r_A} & A^* \\
 c_A \downarrow & & \downarrow q^* \\
 A & \xrightarrow{q_A} & M
 \end{array}$$

Here the map r_A is the composition of the Legendre transformation $T^*A \rightarrow T^*A^*$ with the projection $T^*A^* \rightarrow A^*$. Elements of T^*A can be represented locally as (ω, a, ϕ) where $\omega \in T_m^*M$, $a \in A_m$, $\phi \in A_m^*$ for some $m \in M$. The Legendre transformation

$$T^*A \ni (\omega, \phi, a) \mapsto (-\omega, a, \phi) \in T^*A^*$$

is an isomorphism of double vector bundles preserving the side bundles; that is to say, it is a vector bundle morphism over both A and A^* . An intrinsic definition of the Legendre transformation can be found in [28].

Since A is a Lie algebroid, its dual A^* has a linear Poisson structure, and the cotangent space T^*A^* has a Lie algebroid structure over A^* . Hence, there is a unique Lie algebroid structure on $r_A : T^*A \rightarrow A^*$ with respect to which the projection $c_A : T^*A \rightarrow A$ is a Lie algebroid morphism over $A^* \rightarrow M$.

Now one can form the fibered product Lie algebroid $TA \times_A T^*A \rightarrow TM \times_M A^*$.

Proposition 4 *Let $A \rightarrow M$ be a Lie algebroid. Then the Pontryagin bundle P_A is naturally a Lie algebroid. Moreover, the canonical projection $P_A \rightarrow A$ is a Lie algebroid morphism.*

The double vector bundle $(P_A, TM \oplus A^*, A, M)$ is a VB-Lie algebroid in the sense of Gracia-Saz and Mehta [13].

2.5 Canonical identifications

The canonical pairing $\lrcorner \cdot, \cdot \int_A : A \oplus A^* \rightarrow \mathbb{R}$ induces a nondegenerate pairing $\langle\langle \cdot, \cdot \rangle\rangle_A = \text{pr}_2 \circ T \lrcorner \cdot, \cdot \int_A$ on $TA \times_{TM} TA^*$, where $\text{pr}_2 : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the map which ‘forgets’ the base point of a tangent vector (see [27]):

$$\begin{array}{ccc}
 A \oplus A^* & & TA \times_{TM} TA^* \\
 \lrcorner \cdot, \cdot \int_A \downarrow & & T \lrcorner \cdot, \cdot \int_A \downarrow \searrow \langle\langle \cdot, \cdot \rangle\rangle_A \\
 \mathbb{R} & & T\mathbb{R} \xrightarrow{\text{pr}_2} \mathbb{R}
 \end{array}$$

That is, if $\xi \in TA$ and $\chi \in TA^*$ are such that $Tq_A(\xi) = Tq_{A^*}(\chi)$, then $\xi = \frac{d}{dt}|_{t=0}a(t) \in TA$ and $\chi = \frac{d}{dt}|_{t=0}\varphi(t) \in TA^*$ for some curves $a : (-\varepsilon, \varepsilon) \rightarrow A$ and $\varphi : (-\varepsilon, \varepsilon) \rightarrow A^*$ such that $q_{A^*} \circ \varphi = q_A \circ a$ and $\langle\langle \xi, \chi \rangle\rangle_A = \frac{d}{dt}|_{t=0} \lrcorner a(t), \varphi(t) \int_A$. For instance, if $X \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$, then $TX \in \Gamma_{TM}(TA)$ and $T\alpha \in \Gamma_{TM}(TA^*)$ are such that $Tq_A(TX) = \text{id}_{TM} = Tq_{A^*}(T\alpha)$ and we have for all $v_p = \dot{c}(0) \in T_pM$:

$$\langle\langle TX(v_p), T\alpha(v_p) \rangle\rangle_A = \frac{d}{dt} \Big|_{t=0} \lrcorner X, \alpha \int_A (c(t)) = \text{pr}_2(T_p(\alpha(X))(v_p)). \tag{2}$$

If $(Tq_A)^\vee : (TA)^\vee \rightarrow TM$ is the vector bundle that is dual to the vector bundle $Tq_A : TA \rightarrow TM$, there is an induced isomorphism I

$$\begin{array}{ccc}
 TA^* & \xrightarrow{I} & (TA)^\vee \\
 Tq_{A^*} \downarrow & & \downarrow (Tq_A)^\vee \\
 TM & \longrightarrow & TM
 \end{array}$$

that is defined by

$$\lrcorner \xi, I(\chi) \int_{Tq_A} = \langle\langle \xi, \chi \rangle\rangle_A$$

for all $\chi \in TA^*$ and $\xi \in TA$ such that $Tq_{A^*}(\chi) = Tq_A(\xi)$. That is, the following diagram commutes:

$$\begin{array}{ccc}
 TA \times_{TM} TA^* & \xrightarrow{T\{\cdot, \cdot\}_A} & T\mathbb{R} \\
 \downarrow (id, I) & \searrow \langle \langle \cdot, \cdot \rangle \rangle_A & \downarrow pr_2 \\
 TA \times_{TM} (TA)^\vee & \xrightarrow{\{\cdot, \cdot\}_{Tq_A}} & \mathbb{R}
 \end{array}$$

Applying the construction of I above to the case $q_A = p_{TM}$, we get an isomorphism

$$\begin{array}{ccc}
 T(T^*M) & \xrightarrow{I} & (TTM)^\vee \\
 Tc_M \downarrow & & \downarrow (Tp_M)^\vee \\
 TM & \longrightarrow & TM
 \end{array}$$

We also have the canonical involution (see for instance [27])

$$\begin{array}{ccc}
 TTM & \xrightarrow{\sigma} & TTM \\
 Tp_M \downarrow & & \downarrow p_{TM} \\
 TM & \longrightarrow & TM
 \end{array}$$

Recall that for $V \in \mathfrak{X}(M)$ the map $TV: TM \rightarrow TTM$ is a section of $Tp_M: TTM \rightarrow TM$ and $\sigma(TV)$ is a section of $p_{TM}: TTM \rightarrow TM$, i.e. a vector field on TM .

We get an isomorphism $\varsigma := \sigma^* \circ I: T(T^*M) \rightarrow T^*(TM)$

$$\begin{array}{ccc}
 T(T^*M) & \xrightarrow{\varsigma} & T^*(TM) \\
 Tc_M \downarrow & & \downarrow c_{TM} \\
 TM & \longrightarrow & TM
 \end{array}$$

Proposition 5 *The map $\Sigma := (\sigma, \varsigma): TP_M \rightarrow P_{TM}$*

$$\begin{array}{ccc}
 TP_M & \xrightarrow{\Sigma} & P_{TM} \\
 Tpr_M \downarrow & & \downarrow pr_{TM} \\
 TM & \longrightarrow & TM
 \end{array}$$

where $pr_M: P_M \rightarrow M$ is the projection, establishes an isomorphism of vector bundles.

2.6 Lie functor from \mathbf{P}_Γ to \mathbf{P}_A

Proposition 6 *Let $\Gamma \rightrightarrows M$ be a Lie groupoid with Lie algebroid A . Then the Lie algebroid of \mathbf{P}_Γ is canonically isomorphic to \mathbf{P}_A .*

Moreover, the pairing $\langle \cdot, \cdot \rangle_\Gamma$ is a groupoid morphism: $\mathbf{P}_\Gamma \times_\Gamma \mathbf{P}_\Gamma \rightarrow \mathbb{R}$. Its corresponding Lie algebroid morphism coincides under the canonical isomorphism $\mathbf{A}(\mathbf{P}_\Gamma) \cong \mathbf{P}_A$ with the pairing $\langle \cdot, \cdot \rangle_A: \mathbf{P}_A \times_A \mathbf{P}_A \rightarrow \mathbb{R}$.

This is a standard result, but we recall its proof because it will be useful later on.

For any Lie groupoid $\Gamma \rightrightarrows M$ with Lie algebroid $A \rightarrow M$, the tangent bundle projection $p_\Gamma: T\Gamma \rightarrow \Gamma$ is a groupoid morphism over $p_M: TM \rightarrow M$ and applying the Lie functor gives a canonical morphism $\mathbf{A}(p_\Gamma): \mathbf{A}(T\Gamma) \rightarrow A$. This acquires a vector bundle structure by applying $\mathbf{A}(\cdot)$ to the operations in $T\Gamma \rightarrow \Gamma$. This yields a system of vector bundles

$$\begin{array}{ccc} \mathbf{A}(T\Gamma) & \xrightarrow{q_{\mathbf{A}(T\Gamma)}} & TM \\ \mathbf{A}(p_\Gamma) \downarrow & & \downarrow p_M \\ A & \xrightarrow{q_A} & M \end{array}$$

in which $\mathbf{A}(T\Gamma)$ has two vector bundle structures, the maps defining each being morphisms with respect to the other. In other words, $\mathbf{A}(T\Gamma)$ is a double vector bundle.

Associated with the vector bundle $q_A: A \rightarrow M$ is the tangent double vector bundle

$$\begin{array}{ccc} TA & \xrightarrow{Tq_A} & TM \\ p_A \downarrow & & \downarrow p_M \\ A & \xrightarrow{q_A} & M \end{array}$$

It is shown in [28] that the canonical involution $\sigma: T(T\Gamma) \rightarrow T(T\Gamma)$ restricts to a canonical map $\sigma_\Gamma: \mathbf{A}(T\Gamma) \rightarrow TA$ which is an isomorphism of double vector bundles preserving the side bundles. The tangent prolongation $TA \rightarrow TM$ of the Lie algebroid A and the Lie algebroid $\mathbf{A}(T\Gamma) \rightarrow TM$ of $T\Gamma \rightrightarrows TM$ are isomorphic via σ_Γ [28].

Similarly, the cotangent groupoid structure $T^*\Gamma \rightrightarrows A^*$ is defined by maps which are vector bundle morphisms and, reciprocally, the operations in the vector bundle $c_\Gamma: T^*\Gamma \rightarrow \Gamma$ are groupoid morphisms. Taking the Lie algebroid of $T^*\Gamma \rightrightarrows A^*$ we get a double vector bundle

$$\begin{array}{ccc} \mathbf{A}(T^*\Gamma) & \xrightarrow{q_{\mathbf{A}(T^*\Gamma)}} & A^* \\ \mathbf{A}(c_\Gamma) \downarrow & & \downarrow p_M \\ A & \xrightarrow{q_A} & M \end{array}$$

where the vector bundle operations in $\mathbf{A}(T^*\Gamma) \rightarrow A$ are obtained by applying the Lie functor to those in $T^*\Gamma \rightarrow \Gamma$.

It follows from the definitions of the operations in $T^*\Gamma \rightrightarrows A^*$ that the canonical pairing

$$\langle \cdot, \cdot \rangle_{T\Gamma} : T\Gamma \times_{\Gamma} T^*\Gamma \rightarrow \mathbb{R}$$

is a groupoid morphism into the additive group(oid) \mathbb{R} . Hence $\langle \cdot, \cdot \rangle_{T\Gamma}$ induces a Lie algebroid morphism

$$\mathbf{A}(\langle \cdot, \cdot \rangle_{T\Gamma}) : \mathbf{A}(T\Gamma) \times_A \mathbf{A}(T^*\Gamma) \rightarrow \mathbf{A}(\mathbb{R}) = \mathbb{R}.$$

Note that $\mathbf{A}(\langle \cdot, \cdot \rangle_{T\Gamma})$ is the restriction to $\mathbf{A}(T\Gamma) \times_A \mathbf{A}(T^*\Gamma)$ of

$$\langle \langle \cdot, \cdot \rangle \rangle_{T\Gamma} : T(T\Gamma) \times_{T\Gamma} T(T^*\Gamma) \rightarrow \mathbb{R}.$$

As noted in [28], $\mathbf{A}(\langle \cdot, \cdot \rangle_{T\Gamma})$ is nondegenerate, and therefore induces an isomorphism $I_{\Gamma} : \mathbf{A}(T^*\Gamma) \rightarrow \mathbf{A}(T\Gamma)^{\vee}$ of double vector bundles, where $\mathbf{A}(T\Gamma)^{\vee}$ is the dual of $\mathbf{A}(T\Gamma) \rightarrow A$. Now dualizing $\sigma_{\Gamma}^{-1} : TA \rightarrow \mathbf{A}(T\Gamma)$ over A , we define

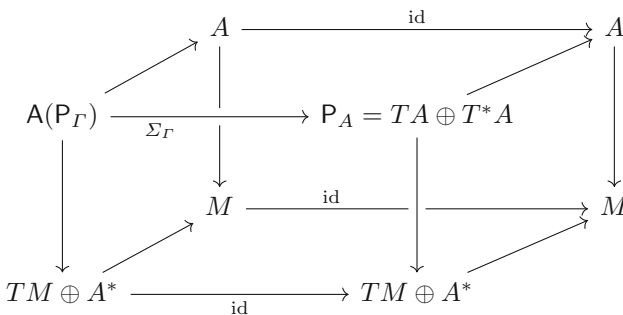
$$\zeta_{\Gamma} = (\sigma_{\Gamma}^{-1})^* \circ I_{\Gamma} : \mathbf{A}(T^*\Gamma) \rightarrow T^*A.$$

This is an isomorphism of double vector bundles preserving the side bundles. The Lie algebroids $T^*A \rightarrow A^*$ and $\mathbf{A}(T^*\Gamma) \rightarrow A^*$ are isomorphic via ζ_{Γ} [28].

The Lie algebroid of the direct sum $\mathbf{P}_{\Gamma} = T\Gamma \oplus T^*\Gamma$ is equal to

$$\mathbf{A}(\mathbf{P}_{\Gamma}) = T_U^{\mathbb{T}^s} \mathbf{P}_{\Gamma},$$

where we write U for the unit space of \mathbf{P}_{Γ} ; i.e. $U := TM \oplus A^*$. By the considerations above, we have a Lie algebroid morphism $\Sigma_{\Gamma} = (\sigma_{\Gamma}, \zeta_{\Gamma})$:



preserving the side bundles A and $TM \oplus A^*$.

Recall that we also have a map

$$\Sigma = (\sigma, \zeta) : T\mathbf{P}_{\Gamma} \rightarrow \mathbf{P}_{T\Gamma}.$$

Lemma 2 *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and u an element of $\mathbf{A}(\mathbf{P}_\Gamma) \subseteq T\mathbf{P}_\Gamma$ projecting to $a_m \in A$ and $(v_m, \alpha_m) \in T_m M \times A_m^*$. Then, if $\Sigma_\Gamma(u) = (v_{a_m}, \alpha_{a_m}) \in \mathbf{P}_A(a_m)$ and $\Sigma(u) = (\tilde{v}_{a_m}, \tilde{\alpha}_{a_m}) \in \mathbf{P}_{T\Gamma}(a_m)$, we have $T\iota v_{a_m} = \tilde{v}_{a_m}$ and $\alpha_{a_m} = \tilde{\alpha}_{a_m}|_{T_{a_m} A}$.*

See Remark 2 below for an interpretation of this lemma.

Proof The first equality follows immediately from the definition of σ_Γ .

Choose $T_{a_m} A \ni w_{a_m} = \sigma_\Gamma(y)$ for some $y \in \mathbf{A}(T\Gamma) \subseteq T(T\Gamma)$ and $\tilde{w}_{a_m} := T\iota w_{a_m}$. Write also $u = (x, \xi)$ with $x \in \mathbf{A}(T\Gamma)$ and $\xi \in \mathbf{A}(T^*\Gamma) \subseteq T(T^*\Gamma)$, i.e. $\alpha_{a_m} = \zeta_\Gamma(\xi)$. Then we have $\mathbf{A}(\mathbf{p}_\Gamma)(y) = \mathbf{A}(c_\Gamma)(\xi) = a_m$ and we can compute

$$\begin{aligned} \wr(w_{a_m}, \alpha_{a_m}) \int_{TA} &= \wr(y, I_\Gamma(\xi)) \int_{\mathbf{A}(T\Gamma)} = \mathbf{A}(\wr(\cdot, \cdot) \int_{T\Gamma})(y, \xi) \\ &= \langle \langle y, \xi \rangle \rangle_{T\Gamma} = \wr(y, I(\xi)) \int_{T\mathbf{P}_\Gamma} \\ &= \wr(\sigma(y), \zeta(\xi)) \int_{\mathbf{P}_{T\Gamma}} = \wr(\tilde{w}_{a_m}, \tilde{\alpha}_{a_m}) \int_{\mathbf{P}_{T\Gamma}} = \wr(w_{a_m}, \iota^* \tilde{\alpha}_{a_m}) \int_{\mathbf{P}_A}. \end{aligned}$$

□

The pairing $\langle \cdot, \cdot \rangle_\Gamma$ is a groupoid morphism $\mathbf{P}_\Gamma \times_\Gamma \mathbf{P}_\Gamma \rightarrow \mathbb{R}$. Hence, we can consider the Lie algebroid morphism

$$\mathbf{A}(\langle \cdot, \cdot \rangle_\Gamma): \mathbf{A}(\mathbf{P}_\Gamma) \times_A \mathbf{A}(\mathbf{P}_\Gamma) \rightarrow \mathbf{A}(\mathbb{R}) = \mathbb{R}.$$

We have

$$\mathbf{A}(\langle \cdot, \cdot \rangle_\Gamma) = (\text{pr}_2 \circ T\langle \cdot, \cdot \rangle_\Gamma) |_{\mathbf{A}(\mathbf{P}_\Gamma) \times_A \mathbf{A}(\mathbf{P}_\Gamma)}.$$

We can see from the proof of the last lemma that $\mathbf{A}(\wr(\cdot, \cdot) \int_{T\Gamma})$ coincides with $\wr(\cdot, \cdot) \int_{TA}$ under the isomorphism Σ_Γ . Hence, $\mathbf{A}(\langle \cdot, \cdot \rangle_\Gamma)$ coincides with the pairing $\langle \cdot, \cdot \rangle_A: \mathbf{P}_A \times_A \mathbf{P}_A \rightarrow \mathbb{R}$, under the canonical isomorphism $\mathbf{A}(\mathbf{P}_\Gamma) \cong \mathbf{P}_A$.

3 Multiplicative generalized complex geometry

3.1 Glanon groupoids

Definition 3 Let $\Gamma \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \rightarrow M$. A generalized almost complex structure \mathcal{J} on Γ is *multiplicative* if it is an automorphism of the Lie groupoid $T\Gamma \oplus T^*\Gamma \rightrightarrows TM \oplus A^*$. We call the pair $(\Gamma \rightrightarrows M, \mathcal{J})$ a *generalized almost complex groupoid*. We use the symbol j to denote the induced automorphism of the unit space $TM \oplus A^*$:

$$\begin{array}{ccc} T\Gamma \oplus T^*\Gamma & \xrightarrow{\mathcal{J}} & T\Gamma \oplus T^*\Gamma \\ \text{Tt} \downarrow \parallel \downarrow \text{Ts} & & \text{Tt} \downarrow \parallel \downarrow \text{Ts} \\ TM \oplus A^* & \xrightarrow{j} & TM \oplus A^* \end{array}$$

If \mathcal{J} is a generalized complex structure, then we call the pair $(\Gamma \rightrightarrows M, \mathcal{J})$ a *Glanon groupoid*.

This is equivalent to $D_{\mathcal{J}}$ and $\overline{D_{\mathcal{J}}}$, the eigenspaces of $\mathcal{J}: (T\Gamma \oplus T^*\Gamma) \otimes \mathbb{C} \rightarrow (T\Gamma \oplus T^*\Gamma) \otimes \mathbb{C}$ corresponding to the eigenvalues i and $-i$ being multiplicative Dirac structures on $\Gamma \rightrightarrows M$ [31].

Note also that \mathcal{J} is a Lie groupoid morphism if and only if the maps $N: T\Gamma \rightarrow T\Gamma$, $\pi^{\sharp}: T^*\Gamma \rightarrow T\Gamma$, $N^*: T^*\Gamma \rightarrow T^*\Gamma$ and $\omega^b: T\Gamma \rightarrow T^*\Gamma$ such that

$$\mathcal{J} = \begin{pmatrix} N & \pi^{\sharp} \\ \omega^b & -N^* \end{pmatrix}$$

are all Lie groupoid morphisms. In particular, we have the following proposition.

Proposition 7 *If Γ is a Glanon groupoid, then Γ is naturally a Poisson groupoid.*

Example 1 (Glanon groups) Let $G \rightrightarrows \{*\}$ be a Glanon Lie group. Since any multiplicative two-form must vanish (see for instance [17]), the underlying generalized complex structure on G is equivalent to a multiplicative holomorphic Poisson structure. Therefore, Glanon Lie groups are in bijection with complex Poisson Lie groups.

Example 2 (Symplectic groupoids) Consider a Lie groupoid $\Gamma \rightrightarrows M$ equipped with a non-degenerate two-form ω . Then the map

$$\mathcal{J}_{\omega} = \begin{pmatrix} 0 & -(\omega^b)^{-1} \\ \omega^b & 0 \end{pmatrix},$$

where $\omega^b: T\Gamma \rightarrow T^*\Gamma$ is the bundle map $X \mapsto \mathbf{i}_X\omega$, defines a Glanon groupoid structure on $\Gamma \rightrightarrows M$ if and only if (Γ, ω) is a symplectic groupoid.

Example 3 (Holomorphic Lie groupoids) Let Γ be a holomorphic Lie groupoid and $J_{\Gamma}: T\Gamma \rightarrow T\Gamma$ its complex structure. Then the map

$$\mathcal{J} = \begin{pmatrix} J_{\Gamma} & 0 \\ 0 & -J_{\Gamma}^* \end{pmatrix},$$

defines a Glanon groupoid structure on Γ .

3.2 Glanon Lie algebroids

Let A be a Lie algebroid over M . Recall that $\mathbf{P}_A = TA \oplus T^*A$ has the structure of a Lie algebroid over $TM \oplus A^*$.

Definition 4 A Glanon Lie algebroid is a Lie algebroid A endowed with a generalized complex structure $\mathcal{J}_A: \mathbf{P}_A \rightarrow \mathbf{P}_A$ that is a Lie algebroid morphism.

The induced Poisson structure π on A in that case is linear, and the map $\pi^\sharp : T^*A \rightarrow TA$ is a Lie algebroid morphism over some map $A^* \rightarrow TM$. The linear Poisson structure on A is then equivalent to a Lie algebroid structure on A^* , such that (A, A^*) is a Lie bialgebroid [28].

Proposition 8 *If A is a Glanon Lie algebroid, then (A, A^*) is a Lie bialgebroid.*

Example 4 (Glanon Lie algebras) Let \mathfrak{g} be a Lie algebra. Then $P_{\mathfrak{g}} = \mathfrak{g} \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ is a Lie algebroid over \mathfrak{g}^* . Hence, a map

$$\begin{aligned} \mathcal{J} : P_{\mathfrak{g}} &\rightarrow P_{\mathfrak{g}}, \\ \mathcal{J}(x, y, \xi) &= (x, \mathcal{J}_{x, \mathfrak{g}}(y, \xi), \mathcal{J}_{x, \mathfrak{g}^*}(y, \xi)) \end{aligned}$$

can only be a Lie algebroid morphism over $j : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ if $\mathcal{J}_{x, \mathfrak{g}^*}$ does not depend on the \mathfrak{g} -component. That is, the map $\mathcal{J}_x : \{x\} \times \mathfrak{g} \times \mathfrak{g}^* \rightarrow \{x\} \times \mathfrak{g} \times \mathfrak{g}^*$ has the matrix

$$\begin{pmatrix} n_x & \pi_x^\sharp \\ 0 & -n_x^* \end{pmatrix}.$$

It thus follows that it must be equivalent to a complex Lie bialgebra.

Example 5 (Symplectic Lie algebroids) Let (M, π) be a Poisson manifold. Let A be the cotangent Lie algebroid $A = (T^*M)_\pi$. Then the map

$$\begin{pmatrix} 0 & (\omega_A^\flat)^{-1} \\ \omega_A^\flat & 0 \end{pmatrix}$$

where ω_A is the canonical cotangent symplectic structure on A , defines a Glanon Lie algebroid structure on A .

Example 6 (Holomorphic Lie algebroids) Let A be a holomorphic Lie algebroid. Let A_R be its underlying real Lie algebroid and $j : TA_R \rightarrow TA_R$ the corresponding complex structure. Then the map

$$\begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

defines a Glanon Lie algebroid structure on A_R .

3.3 Main theorem

Now we are ready to state the main theorem of this paper.

Theorem 1 *If Γ is a Glanon groupoid with Lie algebroid A , then A is a Glanon Lie algebroid.*

Conversely, given a Glanon Lie algebroid A , if Γ is a \mathfrak{s} -connected and \mathfrak{s} -simply connected Lie groupoid integrating A , then Γ is a Glanon groupoid.

Remark 1 Ortiz shows in his thesis [31] that multiplicative Dirac structures on a Lie groupoid $\Gamma \rightrightarrows M$ are in one-one correspondence with morphic Dirac structures on its Lie algebroid, i.e. Dirac structures $D_A \subseteq P_A$ such that D_A is a subalgebroid of $P_A \rightarrow TM \oplus A^*$ over a set $U \subseteq TM \oplus A^*$.

By extending this result to complex Dirac structures and using the fact that a multiplicative generalized complex structure on $\Gamma \rightrightarrows M$ is the same as a pair of transversal, complex conjugate, multiplicative Dirac structures in the complexified P_Γ , one finds an alternative method for proving our main theorem. This alternative proof relies crucially on the integration of VB-algebroids to VB-groupoids described in [5] and hence is far more technical than our approach.

Applying Theorem 1 to Example 2 and Example 5, we obtain immediately the following

Theorem 2 *Let (P, π) be a Poisson manifold. If Γ is a \mathfrak{s} -connected and \mathfrak{s} -simply connected Lie groupoid integrating the Lie algebroid $(T^*P)_\pi$, then Γ admits a symplectic groupoid structure.*

Similarly, applying Theorem 1 to Examples 3 and 6, we obtain immediately the following theorem.

Theorem 3 *If Γ is an \mathfrak{s} -connected and \mathfrak{s} -simply connected Lie groupoid integrating the underlying real Lie algebroid A_R of a holomorphic Lie algebroid A , then Γ is a holomorphic Lie groupoid.*

A Glanon groupoid is automatically a Poisson groupoid, while a Glanon Lie algebroid must be a Lie bialgebroid. The following result reveals their connection

Theorem 4 *Let Γ be a Glanon groupoid with its Glanon Lie algebroid A , (Γ, π) and (A, A^*) their induced Poisson groupoid and Lie bialgebroid respectively. Then the corresponding Lie bialgebroid of (Γ, π) is isomorphic to (A, A^*) .*

3.4 Tangent Courant algebroid

In [3], Boumaiza–Zaalani proved that the tangent bundle of a Courant algebroid is naturally a Courant algebroid. In this section, we study the Courant algebroid structure on P_{TM} in terms of the isomorphism $\Sigma: TP_M \rightarrow P_{TM}$ defined in Proposition 5.

First we need to introduce some notations. For every $V \in \mathfrak{X}(M)$, set $\mathbb{T}V = \sigma(TV) \in \mathfrak{X}(TM)$. For every $\alpha \in \Omega^1(M)$, set $\mathbb{T}\alpha = \zeta(T\alpha) \in \Omega^1(TM)$. For every $f \in C^\infty(M)$, set $\mathbb{T}f = \text{pr}_2 \circ Tf \in C^\infty(TM, \mathbb{R})$.

Note that if $v_m = \dot{c}(0) \in T_mM$, then

$$\mathbb{T}f(v_m) = \mathbb{T}f(\dot{c}(0)) = \left. \frac{d}{dt} \right|_{t=0} (f \circ c(t)), \tag{3}$$

that is, $\mathbb{T}f = \mathbf{d}f: TM \rightarrow \mathbb{R}$.

Now introduce the map $\mathbb{T}: \Gamma(\mathbb{P}_M) \rightarrow \Gamma(\mathbb{P}_{TM})$ given by

$$\mathbb{T}(V, \alpha) = \Sigma(TV, T\alpha) = (\mathbb{T}V, \mathbb{T}\alpha),$$

for all $(V, \alpha) \in \Gamma(\mathbb{P}_M)$.

The main result of this section is the following:

Proposition 9 For any $e_1, e_2 \in \Gamma(\mathbb{P}_M)$, we have

$$\begin{aligned} \llbracket \mathbb{T}e_1, \mathbb{T}e_2 \rrbracket &= \mathbb{T} \llbracket e_1, e_2 \rrbracket, & (4) \\ \langle \mathbb{T}e_1, \mathbb{T}e_2 \rangle_{TM} &= \mathbb{T}(\langle e_1, e_2 \rangle_M). & (5) \end{aligned}$$

The following results show that $\mathbb{T}f \in C^\infty(TM)$, $\mathbb{T}V \in \mathfrak{X}(TM)$ and $\mathbb{T}\alpha \in \Omega^1(TM)$ are the complete lifts of $f \in C^\infty(M)$, $V \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$ in the sense of [39].

Lemma 3 For all $f \in C^\infty(M)$ and $V \in \mathfrak{X}(M)$, we have $\mathbb{T}V(\mathbb{T}f) = \mathbb{T}(V(f))$.

Proof Let ϕ be the flow of V . For any $v_m = \dot{c}(0) \in T_m M$, we have

$$\begin{aligned} (\mathbb{T}V(\mathbb{T}f))(v_m) &= (\mathbb{T}V)(v_m)(\mathbb{T}f) \\ &= \sigma \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \phi_s(c(t)) \right) \mathbb{T}f \\ &= \left(\frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} \phi_s(c(t)) \right) (\mathbb{T}f) \\ &= \left(\frac{d}{ds} \Big|_{s=0} \mathbb{T}f \left(\frac{d}{dt} \Big|_{t=0} \phi_s(c(t)) \right) \right) \\ &\stackrel{(3)}{=} \frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} (f \circ \phi_s)(c(t)) \\ &= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} (f \circ \phi_s)(c(t)) \\ &= \frac{d}{dt} \Big|_{t=0} V(f)(c(t)) \stackrel{(3)}{=} \mathbb{T}(V(f))(v_m). \end{aligned}$$

□

The following lemma characterizes the sections $\mathbb{T}\alpha$ of $\Omega^1(TM)$.

Lemma 4 1. For all $\alpha \in \Omega^1(M)$ and $V \in \mathfrak{X}(M)$, we have

$$\lrcorner \mathbb{T}V, \mathbb{T}\alpha \rceil_{TTM} = \mathbb{T}(\lrcorner V, \alpha \rceil_{TM}). \tag{6}$$

2. Given $\xi \in \Omega^1(TM)$, we have $\xi = 0$ if and only if

$$\lrcorner \mathbb{T}V, \xi \rceil_{TTM} = 0, \quad \forall V \in \mathfrak{X}(M).$$

Proof 1. Using $\sigma^2 = \text{id}_{TM}$, we get:

$$\begin{aligned} \lrcorner \mathbb{T}V, \mathbb{T}\alpha \rceil_{pTM} &= \lrcorner \sigma(TV), (\sigma^* \circ I)(T\alpha) \rceil_{pTM} \\ &= \lrcorner TV, I(T\alpha) \rceil_{pTM} \\ &= \langle \lrcorner TV, T\alpha \rangle_{TM} \stackrel{(2)}{=} T \lrcorner V, \alpha \rceil_{TM}. \end{aligned}$$

2. Let $\xi \in \Omega^1(TM)$ be such that $\xi(\mathbb{T}V) = 0$ for all $V \in \mathfrak{X}(M)$. For any $u \in TM$ with $u \neq 0$ and $v \in T_u(TM)$, there exists a vector field $V \in \mathfrak{X}(M)$ such that $\mathbb{T}V(u) = v$. This yields $\xi(v) = 0$. Therefore, ξ vanishes at all points of TM except for the zero section of TM . By continuity, we get $\xi = 0$. \square

Using this, we can show the following formulas.

Lemma 5 1. For all $V, W \in \mathfrak{X}(M)$, we have

$$[\mathbb{T}V, \mathbb{T}W] = \mathbb{T}[V, W].$$

2. For any $\alpha, \beta \in \Omega^1(M)$ and $V, W \in \mathfrak{X}(M)$, we have

$$\mathfrak{L}_{\mathbb{T}V}\mathbb{T}\beta - \mathbf{i}_{\mathbb{T}W}\mathbf{d}\mathbb{T}\alpha = \mathbb{T}(\mathfrak{L}_V\beta - \mathbf{i}_W\mathbf{d}\alpha).$$

Proof 1. This is an easy computation, using the fact that if ϕ is the flow of the vector field V , then $T\phi$ is the flow of $\mathbb{T}V$ (alternatively, see [27]).

2. For any $U \in \mathfrak{X}(M)$, we can compute

$$\begin{aligned} \lrcorner \mathbb{T}U, \mathfrak{L}_{\mathbb{T}V}\mathbb{T}\beta - \mathbf{i}_{\mathbb{T}W}\mathbf{d}\mathbb{T}\alpha \rceil_{pTM} &= \mathbb{T}V \lrcorner \mathbb{T}\beta, \mathbb{T}U \rceil_{pTM} - \lrcorner \mathbb{T}\beta, \mathbb{T}[V, U] \rceil_{pTM} - \mathbb{T}W \lrcorner \mathbb{T}\alpha, \mathbb{T}U \rceil_{pTM} \\ &\quad + \mathbb{T}U \lrcorner \mathbb{T}\alpha, \mathbb{T}W \rceil_{pTM} + \lrcorner \mathbb{T}\alpha, \mathbb{T}[W, U] \rceil_{pTM} \\ &= \mathbb{T}V (\mathbb{T} \lrcorner \beta, U \rceil_{pM}) - \mathbb{T} \lrcorner \beta, [V, U] \rceil_{pM} - \mathbb{T}W (\mathbb{T} \lrcorner \alpha, U \rceil_{pM}) \\ &\quad + \mathbb{T}U (\mathbb{T} \lrcorner \alpha, W \rceil_{pM}) + \mathbb{T} \lrcorner \alpha, [W, U] \rceil_{pM} \\ &= \mathbb{T} (V \lrcorner \beta, U \rceil_{pM}) - \mathbb{T} \lrcorner \beta, [V, U] \rceil_{pM} - \mathbb{T} (W \lrcorner \alpha, U \rceil_{pM}) \\ &\quad + \mathbb{T} (U \lrcorner \alpha, W \rceil_{pM}) + \mathbb{T} \lrcorner \alpha, [W, U] \rceil_{pM} \\ &= \mathbb{T} \lrcorner U, \mathfrak{L}_V\beta - \mathbf{i}_W\mathbf{d}\alpha \rceil_{pM} \\ &= \lrcorner \mathbb{T}U, \mathbb{T}(\mathfrak{L}_V\beta - \mathbf{i}_W\mathbf{d}\alpha) \rceil_{pTM}. \end{aligned}$$

We get

$$\lrcorner \mathbb{T}U, \mathbb{T}(\mathfrak{L}_V\beta - \mathbf{i}_W\mathbf{d}\alpha) - (\mathfrak{L}_{\mathbb{T}V}\mathbb{T}\beta - \mathbf{i}_{\mathbb{T}W}\mathbf{d}\mathbb{T}\alpha) \rceil_{pTM} = 0$$

for all $U \in \mathfrak{X}(M)$ and we can conclude using Lemma 4. \square

Proof of Proposition 9 Equation (5) follows immediately from (6).

Formula (4) for the Dorfman bracket on sections of $\mathbf{P}_{TM} = T(TM) \oplus T^*(TM)$ follows from Lemma 5. \square

3.5 Nijenhuis torsion

Now let $\mathcal{J} : P_M \rightarrow P_M$ be a vector bundle morphism over the identity. Consider the map $T\mathcal{J} : TP_M \rightarrow TP_M$ and the map $\mathbb{T}\mathcal{J}$ defined by the commutative diagram

$$\begin{CD} TP_M @>T\mathcal{J}>> TP_M \\ @V\Sigma VV @VV\Sigma V \\ P_{TM} @>>\mathbb{T}\mathcal{J}>> P_{TM} \end{CD}$$

i.e. $\mathbb{T}\mathcal{J} = \Sigma \circ T\mathcal{J} \circ \Sigma^{-1}$. Then, by definition, we get, for all $e \in \Gamma(P_M)$,

$$\mathbb{T}\mathcal{J}(\mathbb{T}e) = (\Sigma \circ T\mathcal{J})(T e) = \Sigma(T(\mathcal{J}(e))) = \mathbb{T}(\mathcal{J}(e)). \tag{7}$$

The following lemma is immediate.

- Lemma 6**
1. $\mathbb{T}(\text{id}_{P_M}) = \text{id}_{P_{TM}}$.
 2. We have $\mathbb{T}(\mathcal{J}^2) = (\mathbb{T}(\mathcal{J}))^2$ for every base-preserving endomorphism \mathcal{J} of the vector bundle P_M .

If $\mathcal{J} : P_M \rightarrow P_M$ is now a generalized complex structure, the Nijenhuis torsion is a tensor and hence can be seen as a vector bundle map

$$\mathcal{N}_{\mathcal{J}} : P_M \times_M P_M \rightarrow P_M.$$

We consider as above

$$T\mathcal{N}_{\mathcal{J}} : TP_M \times_{TM} TP_M \rightarrow TP_M.$$

Define $\mathbb{T}\mathcal{N}_{\mathcal{J}} : P_{TM} \times_{TM} P_{TM} \rightarrow P_{TM}$ by the following commutative diagram:

$$\begin{CD} TP_M \times_{TM} TP_M @>T\mathcal{N}_{\mathcal{J}}>> TP_M \\ @V\Sigma \times \Sigma VV @VV\Sigma V \\ P_{TM} \times_{TM} P_{TM} @>>\mathbb{T}\mathcal{N}_{\mathcal{J}}>> P_{TM} \end{CD}$$

An easy computation using (7) and (4) yields

$$\mathbb{T}\mathcal{N}_{\mathcal{J}}(\mathbb{T}e_1, \mathbb{T}e_2) = \mathcal{N}_{\mathbb{T}\mathcal{J}}(\mathbb{T}e_1, \mathbb{T}e_2)$$

for all $e_1, e_2 \in \Gamma(P_M)$. As in the proof of Lemma 4, this implies that $\mathbb{T}\mathcal{N}_{\mathcal{J}}$ and $\mathcal{N}_{\mathbb{T}\mathcal{J}}$ coincide at all points of TM outside the zero section of TM . By continuity, we obtain the following theorem.

Theorem 5 *Let $\mathcal{J} : \mathbf{P}_M \rightarrow \mathbf{P}_M$ be a vector bundle morphism. Then $\mathbb{T}\mathcal{N}_{\mathcal{J}} = \mathcal{N}_{\mathbb{T}\mathcal{J}}$.*

Let $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$ be Lie groupoids. Recall that a map $\Phi : \Gamma \rightarrow \Gamma'$ is a groupoid morphism if and only if the map $\Phi \times \Phi \times \Phi$ restricts to a map $\Lambda_{\Gamma} \rightarrow \Lambda_{\Gamma'}$, where Λ_{Γ} and $\Lambda_{\Gamma'}$ are the graphs of the multiplications in $\Gamma \rightrightarrows M$ and respectively $\Gamma' \rightrightarrows M'$.

Consider the graphs of the multiplications on $T\Gamma$ and $T^*\Gamma$:

$$\Lambda_{T\Gamma} = \{(v_g, v_h, v_g \star v_h) \mid v_g, v_h \in T\Gamma, T\mathfrak{t}(v_h) = T\mathfrak{s}(v_g)\} = T\Lambda_{\Gamma}$$

and

$$\Lambda_{T^*\Gamma} = \{(\alpha_g, \alpha_h, \alpha_g \star \alpha_h) \mid \alpha_g, \alpha_h \in T^*\Gamma, \hat{\mathfrak{t}}(\alpha_h) = \hat{\mathfrak{s}}(\alpha_g)\}.$$

Recall that if

$$(\Lambda_{T^*\Gamma})^{\text{op}} = \{(\alpha_g, \alpha_h, -(\alpha_g \star \alpha_h)) \mid \alpha_g, \alpha_h \in T^*\Gamma, \hat{\mathfrak{t}}(\alpha_h) = \hat{\mathfrak{s}}(\alpha_g)\},$$

i.e.

$$\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} (\Lambda_{T^*\Gamma})^{\text{op}} = (\text{id}_{\mathbf{P}_{\Gamma}} \times \text{id}_{\mathbf{P}_{\Gamma}} \times \mathcal{I}_{\Gamma}) (\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}),$$

then

$$(\Lambda_{T^*\Gamma})^{\text{op}} = (T\Lambda_{\Gamma})^{\circ}.$$

A map $\mathcal{J} : \mathbf{P}_{\Gamma} \rightarrow \mathbf{P}_{\Gamma}$ is a groupoid morphism if and only if $\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}$ is stable under the map $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$. This yields:

Lemma 7 *The map \mathcal{J} is a groupoid morphism if and only if $\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} (\Lambda_{T^*\Gamma})^{\text{op}}$ is stable under the map $\mathcal{J} \times \mathcal{J} \times \bar{\mathcal{J}}$, where $\bar{\mathcal{J}}$ is defined by (1).*

Since $\Lambda_{T\Gamma} = T\Lambda_{\Gamma}$ and $(\Lambda_{T^*\Gamma})^{\text{op}} \cong (T\Lambda_{\Gamma})^{\circ}$, we get that \mathcal{J} is multiplicative if and only if $T\Lambda_{\Gamma} \oplus (T\Lambda_{\Gamma})^{\circ}$ is stable under $\mathcal{J} \times \mathcal{J} \times \bar{\mathcal{J}}$.

Further, if \mathcal{J} is orthogonal relative to $\langle \cdot, \cdot \rangle_{\Gamma}$ and satisfies $\mathcal{J}^2 = -\text{id}_{\mathbf{P}_{\Gamma}}$ (i.e. if \mathcal{J} is a generalized almost complex structure), then we say for short that the Nijenhuis torsion $\mathcal{N}_{\mathcal{J}} : \mathbf{P}_{\Gamma} \times_{\Gamma} \mathbf{P}_{\Gamma} \rightarrow \mathbf{P}_{\Gamma}$ is *multiplicative* if it is a Lie groupoid morphism. Similarly as above, we find that the Nijenhuis torsion $\mathcal{N}_{\mathcal{J}}$ is multiplicative if and only if $\mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\bar{\mathcal{J}}}$ restricts to a map

$$(\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}) \times_{\Lambda_{\Gamma}} (\Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}) \rightarrow \Lambda_{T\Gamma} \oplus_{\Lambda_{\Gamma}} \Lambda_{T^*\Gamma}.$$

Lemma 8 *The map $\mathcal{N}_{\mathcal{J}}$ is multiplicative if and only if $\mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\bar{\mathcal{J}}}$ restricts to a map*

$$(T\Lambda_{\Gamma} \oplus_{\Lambda_{\Gamma}} (T\Lambda_{\Gamma})^{\circ}) \times_{\Lambda_{\Gamma}} (T\Lambda_{\Gamma} \oplus_{\Lambda_{\Gamma}} (T\Lambda_{\Gamma})^{\circ}) \rightarrow T\Lambda_{\Gamma} \oplus_{\Lambda_{\Gamma}} (T\Lambda_{\Gamma})^{\circ}.$$

The following lemma is easy to prove.

Lemma 9 *If M is a smooth manifold and N a submanifold of M , then the Courant–Dorfman bracket on \mathbf{P}_M restricts to sections of $TN \oplus (TN)^\circ$.*

Note that $TN \oplus (TN)^\circ$ is a generalized Dirac structure in the sense of [1]. We get the following theorem.

Theorem 6 *Let $(\Gamma \rightrightarrows M, \mathcal{J})$ be a generalized almost complex groupoid. Then the Nijenhuis tensor $\mathcal{N}_{\mathcal{J}}$ is a Lie groupoid morphism $\mathcal{N}_{\mathcal{J}}: \mathbf{P}_\Gamma \times_\Gamma \mathbf{P}_\Gamma \rightarrow \mathbf{P}_\Gamma$.*

Proof Choose sections $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in \Gamma(\mathbf{P}_\Gamma)$ such that $(\xi_1, \xi_2, \xi_3)|_{\Lambda_\Gamma}$ and $(\eta_1, \eta_2, \eta_3)|_{\Lambda_\Gamma}$ belong to $\Gamma(T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^\circ)$. Then $(\mathcal{J}\xi_1, \mathcal{J}\xi_2, \mathcal{J}\xi_3)|_{\Lambda_\Gamma}$ and $(\mathcal{J}\eta_1, \mathcal{J}\eta_2, \mathcal{J}\eta_3)|_{\Lambda_\Gamma}$ belong to $\Gamma(T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^\circ)$. From Lemma 9, it follows that

$$(\mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\mathcal{J}} \times \mathcal{N}_{\mathcal{J}})((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3))$$

takes values in

$$T\Lambda_\Gamma \oplus_{\Lambda_\Gamma} (T\Lambda_\Gamma)^\circ$$

on Λ_Γ . By Lemma 8, the proof is complete. □

3.6 Infinitesimal multiplicative Nijenhuis tensor

Definition 5 Let $\mathcal{J}: \mathbf{P}_\Gamma \rightarrow \mathbf{P}_\Gamma$ be a Lie groupoid morphism. The map

$$\mathfrak{A}(\mathcal{J}): \mathbf{P}_A \rightarrow \mathbf{P}_A$$

is defined by the commutative diagram

$$\begin{CD} A(\mathbf{P}_\Gamma) @>\Sigma_\Gamma>> \mathbf{P}_A \\ @V\mathfrak{A}(\mathcal{J})VV @VV\mathfrak{A}(\mathcal{J})V \\ A(\mathbf{P}_\Gamma) @>\Sigma_\Gamma>> \mathbf{P}_A \end{CD}$$

The following lemma can be found in [5].

Lemma 10 *Let $\mathcal{J}: \mathbf{P}_\Gamma \rightarrow \mathbf{P}_\Gamma$ be a multiplicative map. Then $\mathfrak{A}(\mathcal{J})$ is an endomorphism of the vector bundle \mathbf{P}_A if and only if \mathcal{J} is an endomorphism of the vector bundle \mathbf{P}_Γ .*

Now assume that \mathcal{J} is orthogonal relative to $\langle \cdot, \cdot \rangle_\Gamma$ and that $\mathcal{J}^2 = -\text{id}_{\mathbf{P}_\Gamma}$. Since the map $\mathcal{N}_{\mathcal{J}}: \mathbf{P}_\Gamma \times_\Gamma \mathbf{P}_\Gamma \rightarrow \mathbf{P}_\Gamma$ is then also a Lie groupoid morphism by Theorem 6, we can also consider

$$\mathfrak{A}(\mathcal{N}_{\mathcal{J}}): \mathbf{P}_A \times_A \mathbf{P}_A \rightarrow \mathbf{P}_A$$

defined by

$$\begin{array}{ccc}
 A(\mathbb{P}_\Gamma) \times_A A(\mathbb{P}_\Gamma) & \xrightarrow{\Sigma_\Gamma^2} & \mathbb{P}_A \times_A \mathbb{P}_A \\
 A(\mathcal{N}_\mathcal{J}) \downarrow & & \downarrow A(\mathcal{N}_\mathcal{J}) \\
 A(\mathbb{P}_\Gamma) & \xrightarrow{\Sigma_\Gamma} & \mathbb{P}_A
 \end{array}$$

The main result of this section is the following.

Theorem 7 *Suppose that a map $\mathcal{J}: \mathbb{P}_\Gamma \rightarrow \mathbb{P}_\Gamma$ is simultaneously a vector bundle morphism and a Lie groupoid morphism.*

1. *Then the map $\mathfrak{A}(\mathcal{J})$ is $\langle \cdot, \cdot \rangle_A$ -orthogonal if and only if \mathcal{J} is $\langle \cdot, \cdot \rangle_\Gamma$ -orthogonal.*
2. *Moreover, if \mathcal{J} is orthogonal w.r.t. $\langle \cdot, \cdot \rangle_\Gamma$ and $\mathcal{J}^2 = -\text{id}_{\mathbb{P}_\Gamma}$, then $\mathfrak{A}(\mathcal{N}_\mathcal{J}) = \mathcal{N}_{\mathfrak{A}(\mathcal{J})}$.*

For the proof, we need a couple of lemmas.

Definition 6 Let M be a smooth manifold and $\iota: N \hookrightarrow M$ a submanifold of M .

1. A section $e_N =: X_N + \alpha_N$ is ι -related to $e_M = X_M + \alpha_M$ if $\iota_* X_N = X_M|_N$ and $\alpha_N = \iota^* \alpha_M$. We write then $e_N \sim_\iota e_M$.
2. Two vector bundle morphisms $\mathcal{J}_N: \mathbb{P}_N \rightarrow \mathbb{P}_N$ and $\mathcal{J}_M: \mathbb{P}_M \rightarrow \mathbb{P}_M$ are said to be ι -related if for each section e_N of \mathbb{P}_N , there exists a section $e_M \in \mathbb{P}_M$ such that $e_N \sim_\iota e_M$ and $\mathcal{J}_N(e_N) \sim_\iota \mathcal{J}_M(e_M)$.
3. Two vector bundle morphisms $\mathcal{N}_N: \mathbb{P}_N \times_N \mathbb{P}_N \rightarrow \mathbb{P}_N$ and $\mathcal{N}_M: \mathbb{P}_M \times_M \mathbb{P}_M \rightarrow \mathbb{P}_M$ are ι -related if for each pair of sections $e_N, f_N \in \Gamma(\mathbb{P}_N)$, there exist sections $e_M, f_M \in \Gamma(\mathbb{P}_M)$ such that $e_N \sim_\iota e_M, f_N \sim_\iota f_M$ and $\mathcal{N}_N(e_N, f_N) \sim_\iota \mathcal{N}_M(e_M, f_M)$.

Lemma 11 *Let M be a manifold and $N \subseteq M$ a submanifold. If $e_N, f_N \in \Gamma(\mathbb{P}_N)$ are ι -related to $e_M, f_M \in \Gamma(\mathbb{P}_M)$, then $\llbracket e_N, f_N \rrbracket \sim_\iota \llbracket e_M, f_M \rrbracket$, for the Courant–Dorfman bracket.*

Proof This is an easy computation, see also [33]. □

Lemma 12 *Let M be a manifold, $N \subseteq M$ a submanifold and $\mathcal{J}_N: \mathbb{P}_N \rightarrow \mathbb{P}_N$ and $\mathcal{J}_M: \mathbb{P}_M \rightarrow \mathbb{P}_M$ two ι -related generalized almost complex structures. Then the generalized Nijenhuis tensors $\mathcal{N}_{\mathcal{J}_N}$ and $\mathcal{N}_{\mathcal{J}_M}$ are ι -related.*

Proof This follows immediately from Lemma 11 and the definition. □

Recall that the Lie algebroid A of a Lie groupoid $\Gamma \rightrightarrows M$ is an embedded submanifold of $T\Gamma$, $\iota: A \hookrightarrow T\Gamma$.

Remark 2 Note that Lemma 2 states that if $u \in \Gamma_A(A(\mathbb{P}_\Gamma))$ and $\tilde{u} \in \Gamma_{T\Gamma}(T\mathbb{P}_\Gamma)$ is an extension of u , then $\Sigma_\Gamma \circ u \sim_\iota \Sigma \circ \tilde{u}$.

Lemma 13 *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\mathcal{J}: \mathbb{P}_\Gamma \rightarrow \mathbb{P}_\Gamma$ a vector bundle morphism. If \mathcal{J} is multiplicative, then*

1. $\mathfrak{A}(\mathcal{J}) : \mathbf{P}_A \rightarrow \mathbf{P}_A$ and $\mathbb{T}\mathcal{J} : \mathbf{P}_{T\Gamma} \rightarrow \mathbf{P}_{T\Gamma}$ are ι -related, and
2. if \mathcal{J} is a generalized almost complex structure, then $\mathfrak{A}(\mathcal{N}_{\mathcal{J}})$ and $\mathbb{T}\mathcal{N}_{\mathcal{J}}$ are ι -related.

Proof 1. Choose a section e_A of \mathbf{P}_A . Then we have $\Sigma_{\Gamma}^{-1}(e_A) =: u \in \Gamma(\mathbf{A}(\mathbf{P}_{\Gamma}))$ and since $\mathbf{A}(\mathbf{P}_{\Gamma}) \subseteq T\mathbf{P}_{\Gamma}$, we find a section \tilde{u} of $T\mathbf{P}_{\Gamma}$ such that \tilde{u} restricts to u . Set $e_{T\Gamma} := \Sigma \circ \tilde{u}$. By Lemma 2, we have then $e_A \sim_{\iota} e_{T\Gamma}$. Furthermore, by construction of $\mathbf{A}(\mathcal{J})$, we know that $\mathbf{A}(\mathcal{J}) \circ u = (T\mathcal{J}) \circ \tilde{u}$ on $TM \oplus A^* = TM \oplus TM^{\circ} \subseteq \mathbf{P}_{\Gamma}$. We have then

$$\mathfrak{A}(\mathcal{J})(e_A) = \Sigma_{\Gamma} \circ \mathbf{A}(\mathcal{J}) \circ u \sim_{\iota} \Sigma \circ T\mathcal{J} \circ \tilde{u} = \mathbb{T}\mathcal{J}(e_{T\Gamma}).$$

2. By definition of $\mathfrak{A}(\mathcal{N}_{\mathcal{J}})$, this can be shown in the same manner. □

Proof of Theorem 7 1. The map $\mathfrak{A}(\mathcal{J})$ is $\langle \cdot, \cdot \rangle_A$ -orthogonal if and only if

$$\langle \cdot, \cdot \rangle_A \circ (\mathfrak{A}(\mathcal{J}), \mathfrak{A}(\mathcal{J})) = \langle \cdot, \cdot \rangle_A.$$

We have

$$\langle \cdot, \cdot \rangle_A = \mathfrak{A}(\langle \cdot, \cdot \rangle_{\Gamma}) = \mathbf{A}(\langle \cdot, \cdot \rangle_{\Gamma}) \circ (\Sigma_{\Gamma}^{-1}, \Sigma_{\Gamma}^{-1})$$

by definition and so

$$\begin{aligned} \langle \cdot, \cdot \rangle_A \circ (\mathfrak{A}(\mathcal{J}), \mathfrak{A}(\mathcal{J})) &= \mathbf{A}(\langle \cdot, \cdot \rangle_{\Gamma}) \circ (\Sigma_{\Gamma}^{-1}, \Sigma_{\Gamma}^{-1}) \circ (\Sigma_{\Gamma}, \Sigma_{\Gamma}) \circ (\mathbf{A}(\mathcal{J}), \mathbf{A}(\mathcal{J})) \circ (\Sigma_{\Gamma}^{-1}, \Sigma_{\Gamma}^{-1}) \\ &= \mathbf{A}(\langle \cdot, \cdot \rangle_{\Gamma} \circ (\mathcal{J}, \mathcal{J})) \circ (\Sigma_{\Gamma}^{-1}, \Sigma_{\Gamma}^{-1}). \end{aligned}$$

Since $\Sigma_{\Gamma} : \mathbf{A}(\mathbf{P}_{\Gamma}) \rightarrow \mathbf{P}_A$ is an isomorphism, we get that $\mathfrak{A}(\mathcal{J})$ is $\langle \cdot, \cdot \rangle_A$ -orthogonal if and only if

$$\mathbf{A}(\langle \cdot, \cdot \rangle_{\Gamma}) = \mathbf{A}(\langle \cdot, \cdot \rangle_{\Gamma} \circ (\mathcal{J}, \mathcal{J}))$$

and we can conclude.

2. Choose sections $e_A, f_A \in \Gamma(\mathbf{P}_A)$ and $u, v \in \Gamma_A(\mathbf{A}(\mathbf{P}_{\Gamma}))$ such that $e_A = \Sigma_{\Gamma} \circ u$, $f_A = \Sigma_{\Gamma} \circ v$. Choose as in the proof of Lemma 13 two extensions \tilde{u} and $\tilde{v} \in \Gamma_{T\Gamma}(T\mathbf{P}_{\Gamma})$ of u and v and set $e_{T\Gamma} := \Sigma \circ \tilde{u}$ and $f_{T\Gamma} := \Sigma \circ \tilde{v} \in \Gamma(\mathbf{P}_{T\Gamma})$. Then we have

$$\begin{aligned} e_A \sim_{\iota} e_{T\Gamma}, \quad f_A \sim_{\iota} f_{T\Gamma}, \\ \mathfrak{A}(\mathcal{J})(e_A) \sim_{\iota} \mathbb{T}\mathcal{J}(e_{T\Gamma}), \quad \mathfrak{A}(\mathcal{J})(f_A) \sim_{\iota} \mathbb{T}\mathcal{J}(f_{T\Gamma}), \\ \text{and } \mathfrak{A}(\mathcal{N}_{\mathcal{J}})(e_A, f_A) \sim_{\iota} \mathbb{T}\mathcal{N}_{\mathcal{J}}(e_{T\Gamma}, f_{T\Gamma}). \end{aligned} \tag{8}$$

But by Lemma 12, (8) yields also

$$\mathcal{N}_{\mathfrak{A}(\mathcal{J})}(e_A, f_A) \sim_{\iota} \mathcal{N}_{\mathbb{T}\mathcal{J}}(e_{T\Gamma}, f_{T\Gamma}).$$

Since $\mathcal{N}_{\mathbb{T}\mathcal{J}} = \mathbb{T}\mathcal{N}_{\mathcal{J}}$ by Theorem 5, we get that

$$\mathfrak{A}(\mathcal{N}_{\mathcal{J}})(e_A, f_A) \sim_i \mathcal{N}_{\mathbb{T}\mathcal{J}}(e_{T\Gamma}, f_{T\Gamma}) \quad \text{and} \quad \mathcal{N}_{\mathfrak{A}(\mathcal{J})}(e_A, f_A) \sim_i \mathcal{N}_{\mathbb{T}\mathcal{J}}(e_{T\Gamma}, f_{T\Gamma}).$$

This yields

$$\mathfrak{A}(\mathcal{N}_{\mathcal{J}})(e_A, f_A) = \mathcal{N}_{\mathfrak{A}(\mathcal{J})}(e_A, f_A)$$

and the proof is complete. □

The following corollary is immediate.

Corollary 1 *Let $\mathcal{J} : P_{\Gamma} \rightarrow P_{\Gamma}$ be a multiplicative generalized almost complex structure. Then $\mathcal{N}_{\mathfrak{A}(\mathcal{J})} = 0$ if and only if $\mathcal{N}_{\mathcal{J}} = 0$.*

3.7 Proof of the integration theorem

Proof of Theorem 1 By Lemma 10, the map $\mathfrak{A}(\mathcal{J}) : P_A \rightarrow P_A$ is a vector bundle morphism. Since $\mathcal{J}^2 = -\text{id}_{P_{\Gamma}}$, it follows from Lemma 6 that $(\mathfrak{A}(\mathcal{J}))^2 = -\text{id}_{P_A}$. From Corollary 1 and Theorem 7, we infer that $\mathcal{N}_{\mathfrak{A}(\mathcal{J})} = 0$ and $\mathfrak{A}(\mathcal{J})$ is $\langle \cdot, \cdot \rangle_A$ -orthogonal.

Since

$$\begin{array}{ccc} \mathbf{A}(P_{\Gamma}) & \xrightarrow{\mathbf{A}(\mathcal{J})} & \mathbf{A}(P_{\Gamma}) \\ \downarrow & & \downarrow \\ TM \oplus A^* & \longrightarrow & TM \oplus A^* \end{array}$$

is a Lie algebroid morphism and

$$\begin{array}{ccc} \mathbf{A}(P_{\Gamma}) & \xrightarrow{\Sigma|_{\mathbf{A}(P_{\Gamma})}} & P_A \\ \downarrow & & \downarrow \\ TM \oplus A^* & \longrightarrow & TM \oplus A^* \end{array}$$

is a Lie algebroid isomorphism over the identity, the map

$$\mathfrak{A}(\mathcal{J}) = \Sigma|_{\mathbf{A}(P_{\Gamma})} \circ \mathbf{A}(\mathcal{J}) \circ (\Sigma|_{\mathbf{A}(P_{\Gamma})})^{-1} = \Sigma|_{\mathbf{A}(P_{\Gamma})} \circ \mathbf{A}(\mathcal{J}) \circ \Sigma^{-1}|_{P_A},$$

is a Lie algebroid morphism

$$\begin{array}{ccc} P_A & \xrightarrow{\mathfrak{A}(\mathcal{J})} & P_A \\ \downarrow & & \downarrow \\ TM \oplus A^* & \longrightarrow & TM \oplus A^* \end{array} .$$

For the second part, consider the map

$$A_{\mathcal{J}} := \Sigma^{-1}|_{P_A} \circ \mathcal{J}_A \circ \Sigma|_{A(P_\Gamma)} : A(P_\Gamma) \rightarrow A(P_\Gamma).$$

Since $\mathcal{J}_A : P_A \rightarrow P_A$ is a Lie algebroid morphism, $A_{\mathcal{J}}$ is a Lie algebroid morphism and there is a unique Lie groupoid morphism $\mathcal{J} : P_\Gamma \rightarrow P_\Gamma$ such that $A_{\mathcal{J}} = A(\mathcal{J})$. By Lemma 10, \mathcal{J} is a morphism of vector bundles.

We get then immediately $\mathcal{J}_A = \mathfrak{A}(\mathcal{J})$. Since $\mathcal{J}_A^2 = -\text{id}_A$, we get $\mathfrak{A}(\mathcal{J}^2) = -\text{id}_A = \mathfrak{A}(-\text{id}_{P_\Gamma})$ and we can conclude by Theorem 7. □

4 Application

4.1 Holomorphic Lie bialgebroids

Given a complex manifold X , let Θ_X denote the sheaf of holomorphic vector fields on X .

Let $A \rightarrow X$ be a holomorphic vector bundle and let $\rho : A \rightarrow T_X$ be a holomorphic vector bundle map, which we call anchor. When the sheaf \mathcal{A} of holomorphic sections of $A \rightarrow X$ is a sheaf of complex Lie algebras, the anchor map ρ induces a homomorphism of sheaves of complex Lie algebras from \mathcal{A} to Θ_X , and the Leibniz identity

$$[V, fW] = (\rho(V)f) \cdot W + f[V, W]$$

holds for all $V, W \in \mathcal{A}(U)$, $f \in \mathcal{O}_X(U)$, and all open subsets U of X , we say that A is a holomorphic Lie algebroid. Holomorphic Lie algebroids were studied in various contexts, see for instance [4, 12, 16, 23, 36].

Since the sheaf \mathcal{A} locally generates the $C^\infty(X)$ -module of all smooth sections of A , each holomorphic Lie algebroid structure on a holomorphic vector bundle $A \rightarrow X$ determines a unique smooth real Lie algebroid structure on A .

Proposition 10 [23] *Let $A \rightarrow X$ be a holomorphic vector bundle and let $\rho : A \rightarrow T_X$ be a holomorphic vector bundle map. Given a structure of holomorphic Lie algebroid on A with anchor ρ , there exists a unique structure of smooth real Lie algebroid on A with the same anchor ρ such that the inclusion of the sheaf of holomorphic sections into the sheaf of smooth sections is a morphism of sheaves of Lie algebras over \mathbb{R} .*

Conversely, given a real Lie algebroid $A \rightarrow X$, it is a holomorphic Lie algebroid if $A \rightarrow X$ is a holomorphic vector bundle, with the sheaf of holomorphic sections denoted \mathcal{A} , such that the Lie bracket on smooth sections induces a \mathbb{C} -linear bracket on $\mathcal{A}(U)$, for all open subsets $U \subset X$. We write $A_{\mathbb{R}}$ to denote the real Lie algebroid underlying a holomorphic Lie algebroid A .

Assume that $(A \rightarrow X, \rho, [\cdot, \cdot])$ is a holomorphic Lie algebroid. Multiplication by the scalar $\sqrt{-1}$ in each fiber of A determines an automorphism j of the vector bundle A . It is simple to see that the Nijenhuis torsion of j vanishes [23]. Hence one can define a new (real) Lie algebroid structure on A (see [21]), with anchor $\rho \circ j$ and Lie bracket

$$[V, W]_j := [jV, W] + [V, jW] - j[V, W] = -j[jV, jW], \quad \forall V, W \in \Gamma(A), \tag{9}$$

which we call *underlying imaginary Lie algebroid* of A and write A_I . It follows immediately that $j: A_I \rightarrow A_R$ is an isomorphism of Lie algebroids [21].

Given a holomorphic vector bundle $A \rightarrow X$, we use the symbols \mathcal{A} , \mathcal{A}^k , and \mathcal{A}^\bullet to denote the sheaves of holomorphic sections of the holomorphic vector bundle $A \rightarrow X$, its k -th exterior power $\wedge^k A \rightarrow X$, and the Whitney sum $\bigoplus_{k \geq 0} (\wedge^k A)$ respectively.

Definition 7 If a holomorphic vector bundle A (with sheaf of holomorphic sections \mathcal{A}) and its dual A^* are both holomorphic Lie algebroids and the Chevalley–Eilenberg differential d_* of the Lie algebroid A^* is a derivation of the (sheaf of) complex Lie algebras \mathcal{A}^\bullet , i.e.

$$d_*[V, W] = [d_*V, W] + [V, d_*W], \quad \forall V, W \in \mathcal{A}^\bullet(U) \tag{10}$$

for all open subsets U of the base manifold X , we say that the pair (A, A^*) is a holomorphic Lie bialgebroid.

As in the smooth case, a holomorphic vector bundle $A \rightarrow X$ is a holomorphic Lie algebroid if and only if $(\mathcal{A}^\bullet, \wedge, [\cdot, \cdot])$ is a sheaf of Gerstenhaber algebras [23].

Proposition 11 *Let $A \rightarrow X$ be a holomorphic vector bundle. The pair (A, A^*) is a holomorphic Lie bialgebroid if and only if $(\mathcal{A}^\bullet, \wedge, [\cdot, \cdot], d_*)$ is a sheaf of differential Gerstenhaber algebras over X .*

Proof Since the proof is exactly the same as in the smooth case [20, 25, 38], we only sketch it briefly. If (A, A^*) is a holomorphic Lie bialgebroid, the holomorphic Lie algebroid structure on A^* induces a complex of sheaves $d_*: \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$ over X . Since d_* is a derivation with respect to the exterior multiplication, it follows immediately, as in [20], from the compatibility condition (10) that

$$d_*[X, f] = [d_*X, f] + [X, d_*f], \quad \forall X \in \mathcal{A}(U), f \in \mathcal{O}_X(U). \tag{11}$$

Therefore, since the exterior algebra \mathcal{A}^\bullet is generated by its homogeneous elements of degree 0 and 1, we have

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \mathcal{A}^\bullet(U).$$

Thus $(\mathcal{A}^\bullet, \wedge, [\cdot, \cdot], d_*)$ is a sheaf of differential Gerstenhaber algebras over X . The converse is obvious. □

Proposition 12 *Let (A, A^*) be a holomorphic Lie bialgebroid with anchors $\rho: A \rightarrow T_X$ and $\rho_*: A^* \rightarrow T_X$. Then*

1. $L_{df}V = -[d_*f, V]$ for any $f \in \mathcal{O}_X(U)$ and $V \in \mathcal{A}(U)$;
2. $[d_*f, d_*g] = d_*\{f, g\}$, $\forall f, g \in \mathcal{O}_X(U)$;

3. $\rho \circ (\rho_*)^* = -\rho_* \circ \rho^*$. Therefore, the holomorphic bundle map

$$\pi_X^\# = \rho \circ (\rho_*)^* : T_X^* \rightarrow T_X.$$

is skew-symmetric and defines a holomorphic Poisson bivector on X .

Proof The proof is similar to the proofs of Proposition 3.4, Corollary 3.5 and Proposition 3.6 in [28]. □

4.2 Associated real Lie bialgebroids

Given a holomorphic Lie algebroid A , we denote its underlying real and imaginary Lie algebroids by A_R and A_I and their respective Chevalley–Eilenberg differentials by d^R and d^I . When the dual A^* of A is also a holomorphic Lie algebroid, its underlying real and imaginary Lie algebroids are written A_R^* and A_I^* and their respective Chevalley–Eilenberg differentials d_*^R and d_*^I .

Lemma 14 *Let A be a holomorphic Lie algebroid over a complex manifold X . Then*

1. $d^I \alpha = -(j^* \circ d^R \circ j^*) \alpha$, for all $\alpha \in \Gamma(A^*)$;
2. $d^I f = (j^* \circ d^R) f$, for all $f \in C^\infty(M)$.

Proof For all $V, W \in \Gamma(A)$, we have

$$(d^I f)(V) = \rho_I(V)(f) = \rho(jV)(f) = (d^R f)(jV)$$

and

$$\begin{aligned} (d^I(j^* \alpha))(V, W) &= \rho_I(V)\alpha(jW) - \rho_I(W)\alpha(jV) - \alpha(j[V, W]_j) \\ &= \rho(j(V))\alpha(jW) - \rho(j(W))\alpha(jV) - \alpha([j(V), j(W)]) \\ &= (d^R \alpha)(j(V), j(W)). \end{aligned}$$

□

If the dual A^* of a holomorphic Lie algebroid $A \rightarrow X$ is also endowed with a holomorphic Lie algebroid structure, we can conclude that

$$\begin{aligned} d_*^I V &= -(j \circ d_*^R \circ j) V, \quad \forall V \in \Gamma(A), \\ d_*^I f &= (j \circ d_*^R) f, \quad \forall f \in C^\infty(X), \end{aligned} \tag{12}$$

since j^* is the multiplication by the scalar $\sqrt{-1}$ in each fiber of A^* .

Proposition 13 *Let A be a holomorphic vector bundle over a complex manifold X . Assume that A and its dual A^* are both holomorphic Lie algebroids. The following assertions are equivalent:*

- (1) (A, A^*) is a holomorphic Lie bialgebroid;
- (2) (A_R, A_R^*) is a Lie bialgebroid;
- (3) (A_R, A_I^*) is a Lie bialgebroid.

Proof (b) \Rightarrow (a) It is clear that if (A_R, A_R^*) is a real Lie bialgebroid then the compatibility condition for (A, A^*) to be a holomorphic Lie bialgebroid is automatically satisfied.

(a) \Rightarrow (b) Fix an arbitrary open subset U of X and an arbitrary holomorphic section $V \in \mathcal{A}(U)$. Consider the operator $C^\infty(U, \mathbb{C}) \rightarrow \Gamma(A_R|_U \otimes \mathbb{C})$ defined by

$$\mathcal{L}_V f = d_*^R[V, f] - [d_*^R V, f] - [V, d_*^R f] \tag{13}$$

for all $f \in C^\infty(U, \mathbb{C})$. Here $d_*^R: \Gamma(\wedge^\bullet A_R \otimes \mathbb{C}) \rightarrow \Gamma(\wedge^{\bullet+1} A_R \otimes \mathbb{C})$ is the Chevalley–Eilenberg differential with the trivial complex coefficients of the Lie algebroid A_R^* , and V is seen as a section of $A_R|_U$. It is simple to check that \mathcal{L}_V is a derivation, i.e.

$$\mathcal{L}_V(fg) = f\mathcal{L}_Vg + g\mathcal{L}_Vf.$$

Since (A, A^*) is a holomorphic Lie bialgebroid, it follows from (11) that $\mathcal{L}_V f = 0$, for all $f \in \mathcal{O}_X(U)$. Here we use the fact that $d_* f = d_*^R f$ for all $f \in \mathcal{O}_X(U)$. On the other hand, we also have $\mathcal{L}_V \bar{f} = 0$ for all $f \in \mathcal{O}_X(U)$ since each term of (13) vanishes [23]. Therefore, we have $\mathcal{L}_V f = 0$ for all $f \in C^\infty(U, \mathbb{C})$. Finally, since the restricted vector bundle $A|_U$ is trivial and $\Gamma(A|_U \otimes \mathbb{C})$ is $C^\infty(U, \mathbb{C})$ -linearly spanned by $\mathcal{A}(U)$ when the subset U is contractible, it follows that $d_*[X, Y] = [d_* X, Y] + [X, d_* Y]$ for any $X, Y \in \Gamma(A|_U \otimes \mathbb{C})$. Hence (A_R, A_R^*) is a real Lie bialgebroid.

(a) \Leftrightarrow (c) The equivalence between (a) and (c) can be proved similarly, using the equality $d_*^I f = i \cdot d_* f$ for all $f \in \mathcal{A}(U)$. □

It is well known that Lie bialgebroids are symmetric, viz. (A_R, A_R^*) is a Lie bialgebroid if and only if (A_R^*, A_R) is a Lie bialgebroid. This is obviously still true in the holomorphic setting.

Proposition 14 *The pair (A, A^*) is a holomorphic Lie bialgebroid iff the pair (A^*, A) is a holomorphic Lie bialgebroid.*

Proposition 15 *Let A be a holomorphic vector bundle over a complex manifold X . Assume A and its dual A^* are both holomorphic Lie algebroids.*

- 1. (A_R, A_R^*) is a Lie bialgebroid if and only if (A_I, A_I^*) is a Lie bialgebroid,
- 2. (A_R, A_I^*) is a Lie bialgebroid if and only if (A_I, A_R^*) is a Lie bialgebroid.

Proof Recall from (9) that the Lie bracket on A_I is given by $[V, W]_j = -j[jV, jW]$ for all $V, W \in \Gamma(A_I)$.

1. Assume that (A_R, A_R^*) is a Lie bialgebroid. Then, for all $V, W \in \Gamma(A_I)$, we have

$$\begin{aligned} d_*^I[V, W]_j &\stackrel{(12)}{=} (j \circ d_*^R \circ j)(-j[jV, jW]) = (j \circ d_*^R)([jV, jW]) \\ &= j \left([d_*^R(jV), jW] + [jV, d_*^R(jW)] \right) \\ &\stackrel{(12)}{=} -j \left([j(d_*^I V), jW] + [jV, j(d_*^I W)] \right) \\ &= [d_*^I V, W]_j + [V, d_*^I W]_j, \end{aligned}$$

which shows that (A_I, A_I^*) is a Lie bialgebroid. The converse can be shown in the same manner.

2. Assume that (A_I, A_R^*) is a Lie bialgebroid. Then, for all $V, W \in \Gamma(A_R)$, we have

$$\begin{aligned} d_*^I[V, W] &= -(j \circ d_*^R \circ j)([V, W]) = -(j \circ d_*^R \circ j)(-j[j(V), j(W)]_j) \\ &= -(j \circ d_*^R)([j(V), j(W)]_j) \\ &= -j[d_*^R(jV), jW]_j - j[jV, d_*^R(jW)]_j \\ &= -j[j(d_*^I V), jW]_j - j[jV, j(d_*^I W)]_j \\ &= [d_*^I V, W] + [V, d_*^I W], \end{aligned}$$

which shows that (A_R, A_I^*) is a Lie bialgebroid. The converse can be proved similarly. □

Proposition 16 *Let A be a holomorphic vector bundle over a complex manifold X . Assume A and its dual A^* are both holomorphic Lie algebroids. Then (A_R, A_I^*) is a Lie bialgebroid if and only if A_R is a Glanon Lie algebroid when endowed with the generalized complex structure $\mathcal{J}_{A_R} : \mathbf{P}_{A_R} \rightarrow \mathbf{P}_{A_R}$ with block-matrix representation*

$$\mathcal{J}_{A_R} = \begin{pmatrix} J_{A_R} & \pi_{A_I^*}^\sharp \\ 0 & -J_{A_R}^* \end{pmatrix}$$

(where $\pi_{A_I^*}$ is the Poisson structure on A_R that is induced by the Lie algebroid structure on A_I^*).

Proof Assume first that (A_R, \mathcal{J}_{A_R}) is a Glanon Lie algebroid. Then the map $\pi_{A_I^*}^\sharp$ is a morphism of Lie algebroids $T^*A \rightarrow TA$, and it follows that (A_R, A_I^*) is a Lie bialgebroid [29].

Conversely, if (A_R, A_I^*) is a Lie bialgebroid, the map $\pi_{A_I^*}^\sharp$ is a morphism of Lie algebroids $T^*A \rightarrow TA$. According to Proposition 3.12 in [24] up to a scalar, $\pi_A(\cdot, \cdot) = \pi_{A_I^*}(\cdot, \cdot) + i\pi_{A_I^*}(J_{A_R}^*\cdot, \cdot)$ is the holomorphic Lie Poisson structure on A induced by the holomorphic Lie algebroid A^* . By [23, Theorem 2.7], \mathcal{J}_{A_R} is hence a generalized complex structure. Since A is a holomorphic Lie algebroid, the map $J_{A_R} : TA_R \rightarrow TA_R$ is a morphism of Lie algebroids according to [24]. □

We summarize our results in the following:

Theorem 8 *Let A be a holomorphic vector bundle over a complex manifold X . Assume A and its dual A^* are both holomorphic Lie algebroids. The following assertions are equivalent:*

1. (A, A^*) is a holomorphic Lie bialgebroid;
2. (A_R, A_R^*) is a Lie bialgebroid;
3. (A_R, A_I^*) is a Lie bialgebroid;
4. (A_I, A_R^*) is a Lie bialgebroid;
5. (A_I, A_I^*) is a Lie bialgebroid;
6. the Lie algebroid A_R endowed with the map $\mathcal{J}_{A_R} : \mathbf{P}_{A_R} \rightarrow \mathbf{P}_{A_R}$,

$$\mathcal{J}_{A_R} = \begin{pmatrix} J_{A_R} & \pi_{A_I}^\sharp \\ 0 & -J_{A_R}^* \end{pmatrix}$$

is a Glanon Lie algebroid.

Example 7 Given a holomorphic Poisson tensor $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0}X)$ on a complex manifold X , let A denote the canonical holomorphic Lie algebroid structure on the tangent bundle of X and let A^* denote the holomorphic Lie algebroid structure associated to π on the cotangent bundle of X . Then (A, A^*) is a holomorphic Lie bialgebroid with $A_R = T_X$, $A_I = (T_X)_J$, $A_R^* = (T_X)_{4\pi_R}^*$ and $A_I^* = (T_X)_{4\pi_I}^*$.

4.3 Holomorphic Poisson groupoids

Definition 8 [28,35,37] A holomorphic Poisson groupoid is a holomorphic Lie groupoid $\Gamma \rightrightarrows M$ endowed with a holomorphic Poisson tensor $\pi_\Gamma \in \Gamma(\wedge^2 T^{1,0}\Gamma)$ such that the graph Λ of the groupoid multiplication is a coisotropic submanifold of $\Gamma \times \Gamma \times \bar{\Gamma}$, where $\bar{\Gamma}$ stands for Γ endowed with the opposite Poisson structure.

Many properties of (smooth) Poisson groupoids generalize in a straightforward manner to the holomorphic setting. In particular, if $\Gamma \rightrightarrows X$ is a holomorphic Poisson groupoid, then X is naturally a holomorphic Poisson manifold. More precisely, there exists a unique holomorphic Poisson tensor on X with respect to which the source map $\mathbf{s} : \Gamma \rightarrow X$ is a holomorphic Poisson map and the target map is an anti-Poisson map.

Theorem 9 [25,37] *Let $\Gamma \rightrightarrows X$ be a holomorphic Lie groupoid with associated Lie algebroid $A \rightarrow X$ and let π_Γ be a holomorphic Poisson tensor on Γ . Then π_Γ is multiplicative if and only if $\pi_\Gamma^\sharp : T^*\Gamma \rightarrow T\Gamma$ is a morphism of holomorphic groupoids. In this case, the restriction of the groupoid morphism $\pi_\Gamma^\sharp : T^*\Gamma \rightarrow T\Gamma$ to the unit spaces is a map $A^* \rightarrow TX$.*

For any open subset $U \subset M$ and $X \in \mathcal{A}^k(U)$, it follows as in [37, Theorem3.1] that $[X^r, \pi_\Gamma]$ is a right-invariant holomorphic $(k + 1)$ -vector field on Γ_U^U . Hence it defines an element, denoted d_*X , in $\mathcal{A}^{k+1}(U)$, i.e. $(d_*X)^r = [X^r, \pi_\Gamma]$. As in [38],

one proves that $(\mathcal{A}^\bullet, \wedge, [\cdot, \cdot], d_*)$ is a sheaf of differential Gerstenhaber algebras over M . This proves the following proposition.

Proposition 17 *Let $\Gamma \rightrightarrows X$ be a holomorphic Lie groupoid and let $A \rightarrow X$ be the associated holomorphic Lie algebroid. If $\Gamma \rightrightarrows X$ is a holomorphic Poisson groupoid, then the pair (A, A^*) is a holomorphic Lie bialgebroid.*

The notation $\mathbf{A}(\Gamma \rightrightarrows X, \pi_\Gamma) = (A, A^*)$ means that $(\Gamma \rightrightarrows X, \pi_\Gamma)$ is a Poisson groupoid and (A, A^*) is its associated Lie bialgebroid.

Proposition 18 *If π_Γ is a multiplicative Poisson tensor on a holomorphic groupoid $\Gamma \rightrightarrows X$, π_R and π_I are the real and imaginary parts of $\pi_\Gamma \in \Gamma(\wedge^2 T^{1,0}\Gamma)$, and $\mathbf{A}(\Gamma \rightrightarrows X, \pi_\Gamma) = (A, A^*)$, then $(\Gamma \rightrightarrows X, \pi_R)$ and $(\Gamma \rightrightarrows X, \pi_I)$ are smooth Poisson groupoids, $\mathbf{A}(\Gamma \rightrightarrows X, \pi_R) = (A_R, A_{\frac{1}{4}\cdot R}^*)$, and $\mathbf{A}(\Gamma \rightrightarrows X, \pi_I) = (A_R, A_{\frac{1}{4}\cdot I}^*)$. Here $A_{\frac{1}{4}\cdot R}^*$ (respectively $A_{\frac{1}{4}\cdot I}^*$) stands for the Lie algebroid $(A_R^*, \frac{1}{4}[\cdot, \cdot]_{A_R^*}, \frac{1}{4}\rho_{A_R^*})$ (respectively $(A_I^*, \frac{1}{4}[\cdot, \cdot]_{A_I^*}, \frac{1}{4}\rho_{A_I^*})$).*

Proof If π_Γ is a multiplicative holomorphic Poisson structure on $\Gamma \rightrightarrows M$, then its real and imaginary parts are also multiplicative. It is shown in [23] that both π_R and π_I are Poisson bivector fields.

We have $\text{Im}((T^*\Gamma)_{\pi_\Gamma}) = (T^*\Gamma)_{4\pi_I}$ and $\text{Re}((T^*\Gamma)_{\pi_\Gamma}) = (T^*\Gamma)_{4\pi_R}$ [23]. The Lie algebroid structure on $(T^*\Gamma)_{\pi_\Gamma}$ restricts to the holomorphic Lie algebroid structure on A^* as follows; the map $\rho: A^* \rightarrow TM$ is just the restriction of π_Γ^\sharp to $A^* = TM^\circ$ seen as a subbundle of $T_M^*\Gamma$, and the bracket on $T^*\Gamma$ restricts to a bracket on A^* . In the same manner, the Lie groupoid $(\Gamma \rightrightarrows M, \pi_R^\sharp)$ induces a Lie algebroid structure on $(A_R)^*$ that is the restriction of the Lie algebroid structure on $(T^*\Gamma)_{\pi_R}$. Hence, we can conclude easily. □

Proposition 19 1. *Every holomorphic Poisson groupoid $(\Gamma \rightrightarrows X, \pi)$ inherits a canonical Glanon groupoid structure: the automorphism \mathcal{J}_π of \mathbf{P}_Γ given by the matrix*

$$\begin{pmatrix} J_\Gamma & \pi_I^\sharp \\ 0 & -J_\Gamma^* \end{pmatrix}$$

(where J_Γ denotes the complex structure of Γ) is a multiplicative generalized complex structure on Γ .

2. *The matrix representation of the Lie algebroid morphism $\mathfrak{A}(\mathcal{J}_\pi): \mathbf{P}_A \rightarrow \mathbf{P}_A$ (see Definition 5) is*

$$\begin{pmatrix} J_{A_R} & \pi_{A_{\frac{1}{4}\cdot I}^*}^\sharp \\ 0 & -J_{A_R}^* \end{pmatrix},$$

where $\pi_{A_{\frac{1}{4}\cdot I}^*}$ is the linear Poisson structure on A_R determined by the Lie algebroid $A_{\frac{1}{4}\cdot I}^*$.

Proof 1. By [23, Theorem 2.7], \mathcal{J}_π is a generalized complex structure. Since $(\Gamma \rightrightarrows X, J_\Gamma)$ is a holomorphic Lie groupoid and $(\Gamma \rightrightarrows X, \pi_I)$ is a Poisson groupoid, $J_\Gamma: T\Gamma \rightarrow T\Gamma$, its dual $J_\Gamma^*: T^*\Gamma \rightarrow T^*\Gamma$, and $\pi_I^\sharp: T^*\Gamma \rightarrow T\Gamma$ are multiplicative maps.

2. It is shown in [24] that $\sigma_\Gamma \circ A(J_\Gamma) \circ \sigma_\Gamma^{-1} = J_{A_R}$, and in [29] that $\zeta_\Gamma \circ A(\pi_I^\sharp) \circ \zeta_\Gamma^{-1} = \pi_{A^*}^\sharp$, since $(A_R, A_{\frac{1}{4}, I}^*)$ is the Lie bialgebroid of $(\Gamma \rightrightarrows M, \pi_I)$, where $\sigma_\Gamma: A(T\Gamma) \rightarrow TA$ and $\zeta_\Gamma: A(T^*\Gamma) \rightarrow T^*A$ are the morphisms of Lie algebroids defined in Sect. 2.6. □

A holomorphic Lie bialgebroid (A, A^*) is said to be *integrable* if there exists a holomorphic Poisson groupoid $(\Gamma \rightrightarrows X, \pi)$ such that $\mathbf{A}(\Gamma \rightrightarrows X, \pi) = (A, A^*)$.

As a consequence of Theorem 1 and Proposition 19, we finally obtain the main result of this section:

Theorem 10 *Given a holomorphic Lie bialgebroid (A, A^*) , if the underlying real Lie algebroid A_R integrates to a \mathfrak{S} -connected and \mathfrak{S} -simply connected Lie groupoid Γ , then Γ is a holomorphic Poisson groupoid.*

Remark 3 This result was proved in [24] in the special case where (A, A^*) is the holomorphic Lie bialgebroid $((T^*X)_\pi, TX)$ associated to a holomorphic Poisson manifold (X, π) .

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