Math. Ann. (2016) 364:293–311
DOI 10.1007/s00208-015-1210-3

Mathematische Annalen

Degenerations of amoebae and Berkovich spaces

Mattias Jonsson[1]

Received: 10 August 2014 / Revised: 2 February 2015 / Published online: 19 April 2015

Abstract We prove a continuity result for the fibers of the Berkovich analytification of a complex algebraic variety with respect to the maximum of the Archimedean norm and the trivial norm. As a consequence, we obtain generalizations of a result of Mikhalkin and Rullgård about degenerations of amoebae onto tropical varieties.

Mathematics Subject Classification Primary 14T05; Secondary 32A60 · 32P05 · 14M25

1 Introduction

Let $X \subset (\mathbf{C}^*)^n$ be an algebraic subvariety of the n-dimensional complex algebraic torus. The *amoeba* $A_X \subset \mathbf{R}^n$ of X is the image of X under the map

$$\mathcal{L} \colon (\mathbf{C}^*)^n \to \mathbf{R}^n$$

defined by[1] $\mathcal{L} = (-\log|z_1|, \ldots, -\log|z_n|)$, where $(z_1, \ldots, z_n)$ are coordinates on $(\mathbf{C}^*)^n$. See Fig. 1 for a picture of the amoeba of $X = \{z_1 + z_2 + 1 = 0\}$.

More generally, let $(K, |\cdot|)$ be any complete valued field and let $X \subset K^{*n}$ be an algebraic variety. For any valued field extension L/K, let $X_L \subset L^{*n}$ be the base change. Define the *tropicalization* of X to be the subset $X^{\mathrm{trop}} \subset \mathbf{R}^n$ defined by

[1] We use negative signs to match the standard convention for valuations. All logarithms are natural logarithms.

Partially supported by NSF Grants DMS-1001740 and DMS-1266207.

✉ Mattias Jonsson
mattiasj@umich.edu

1 Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA

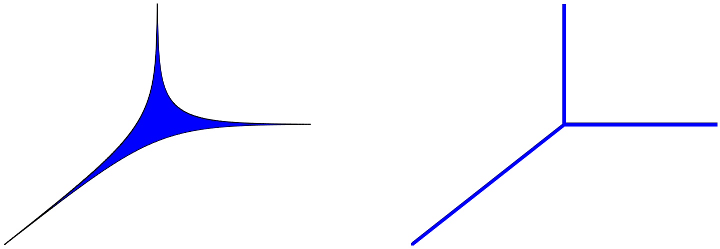

Fig. 1 The amoeba and the tropicalization of the curve $z_1 + z_2 + 1 = 0$ in $\mathbf{C}^{*2}$

$$X^{\text{trop}} = \bigcup_L \mathcal{L}(X_L),$$

where L ranges over all valued field extensions of K and $\mathcal{L}\colon (L^*)^n \to \mathbf{R}^n$ is defined using the same formula as above.

For example, suppose $K = \mathbf{C}$. If $|\cdot|_\infty$ is the usual Archimedean norm on $\mathbf{C}$, then $(\mathbf{C}, |\cdot|_\infty)$ does not admit any nontrivial valued field extensions, so $X^{\text{trop}} = \mathcal{A}_X$ in this case. On the other hand, we can also equip $\mathbf{C}$ with the *trivial norm* $|\cdot|_0$, for which $|a|_0 = 1$ for all $a \in \mathbf{C}^*$. Then the tropicalization of X is equal to the cone over the *logarithmic limit set* of X introduced in [7]. The case $X = \{z_1 + z_2 + 1 = 0\}$ is depicted to the right in Fig. 1. We see that the tropicalization looks like the large scale limit of the amoeba. This is a general fact:

Theorem A *The large scale limit of the amoeba $\mathcal{A}_X$ equals the tropicalization of X:*

$$\lim_{\rho \to 0+} \rho \cdot \mathcal{A}_X = X^{\text{trop}},$$

where the tropicalization is computed using the trivial norm on $\mathbf{C}$.

Here $\rho \cdot \mathcal{A}_X := \{\rho \cdot v \mid v \in \mathcal{A}_X\}$ for $\rho \in \mathbf{R}_+^*$ and the limit can be understood, for example, in the sense of Kuratowski convergence. When X is a hypersurface, Theorem A is a special case of a result by Rullgård and Mikhalkin; see below. The general case of Theorem A is proved in [7] in a slightly different language and conditional on a conjecture that was later establied in [11].

As a more global version, consider a (complex) toric variety Y. There is a natural topological space Y^{trop} canonically associated to Y, see Sect. 4. If $Y = \mathbf{C}^{*n}$, then $Y^{\text{trop}} = \mathbf{R}^n$; in general Y^{trop} contains $\mathbf{R}^n$ as an open dense subset and comes with a multiplicative action by $\mathbf{R}_+^*$ extending the usual action on $\mathbf{R}^n$.

The two absolute values on $\mathbf{C}$ above define two different tropicalization maps of Y onto Y^{trop}. If X is an algebraic subvariety of Y, let $\mathcal{A}_X$ and X^{trop} denote the images of X in Y^{trop} under these two maps. When Y is projective, $\mathcal{A}_X$ is homeomorphic to the *compactified amoeba* defined in [33].

Theorem A′ *We have* $\lim_{\rho \to 0+} \rho \cdot \mathcal{A}_X = X^{\text{trop}}$.

Note that the notation is somewhat abusive since both $\mathcal{A}_X$ and X^{trop} depend on the embedding of X in a toric variety Y. Theorem A is the special case of Theorem A′ when Y is the algebraic torus.

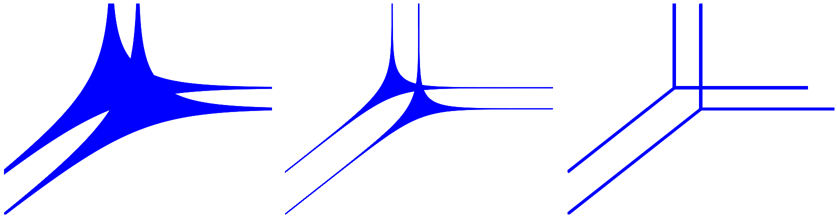

Fig. 2 These pictures illustrating Theorem B show two scaled amoebae and the tropicalization of the curve $V(f_t)$ in the torus $Y = \mathbf{C}^{*2}$, where $f_t = (z_1 + z_2 + 1)(tz_1 + t^{-1}z_2 + 1)$. The first two pictures show the amoeba of the complex curve $V(f_a)$, scaled by a factor $(\log|a|^{-1})^{-1}$, for $a = 0.5$ and $a = 0.2$, respectively. The last picture shows the tropicalization of the curve $V(f_t)$ over $\mathbf{C}((t))$

New we consider one-parameter families of subvarieties. Let $\mathcal{X} \subset \mathbf{C}^* \times Y$ be a closed algebraic subvariety such that the projection of $\mathcal{X}$ onto the first factor $\mathbf{C}^*$ is surjective. Write $X_a \subset Y$ for the fiber of $\mathcal{X}$ above $a \in \mathbf{C}^*$ and $A_{X_a} \subset Y^{\mathrm{trop}}$ for the amoeba as in Theorem A′. Define $\mathcal{X}^{\mathrm{trop}}$ as the tropicalization of the base change $\mathcal{X} \times_{\mathbf{G}_m} \operatorname{Spec} \mathbf{C}((t))$, where $\mathbf{G}_m = \operatorname{Spec} \mathbf{C}[t^{\pm 1}] \simeq \mathbf{C}^*$ and the field $\mathbf{C}((t))$ of formal Laurent series is equipped with the usual non-Archimedean absolute value for which $|t| = e^{-1}$.

Theorem B *We have* $\lim_{a\to 0}(\log|a|^{-1})^{-1} \cdot A_{X_a} = \mathcal{X}^{\mathrm{trop}}$.

See Fig. 2 for an illustration of Theorem B in the case $Y = \mathbf{C}^{*2}$ and $\mathcal{X} = \{z_1+z_2+t = 0\} \subset \mathbf{C}^* \times Y$, and Fig. 3 for the same situation with $Y = \mathbf{P}^2$. When $\mathcal{X} = \mathbf{C}^* \times X$ is a product, Theorem B reduces to Theorem A′.

Theorem B is due to Rullgård [76, Thm. 9] and Mikhalkin [55, Cor 6.4] in the case when $Y = \mathbf{C}^{*n}$ and $\mathcal{X} \subset \mathbf{C}^* \times \mathbf{C}^{*n}$ is a hypersurface; see also [81, Thm. 7.1]. The proofs in *loc. cit.* use the characterization of X^{trop} as the locus where the tropicalization of the Laurent polynomial defining X fails to be affine. The approach in [55] also emphasizes the analogy with the patchworking construction of Viro [86]. As these proofs show, the scaled amoebae in fact converge to the tropicalization in the Hausdorff metric; see also [4].

The higher codimension case of Theorem B for $Y = \mathbf{C}^{*n}$ is stated without proof in [41, Thm. 1.4]. I have not been able to locate a proof nor the general version of Theorem B in the literature. At least in the case $Y = \mathbf{C}^{*n}$, one may in principle reduce Theorem B to the hypersurface case, using, on the one hand, Artin's Approximation Theorem together with the approach in [81, p. 112] and, on the other hand, the fact

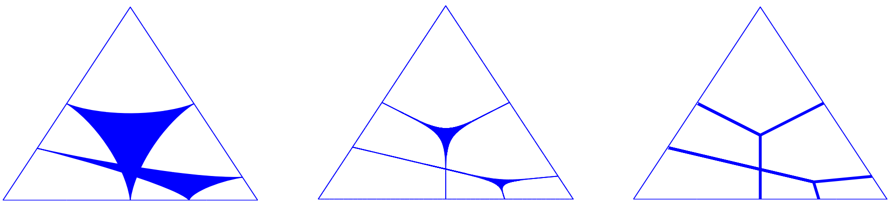

Fig. 3 This picture illustrates Theorem B for the closure in $\mathbf{P}^2$ of the curve $V(f_t)$ in Fig. 2. The *triangle* is the moment polytope of $Y = \mathbf{P}^2$ with its canonical polarization

that the tropicalization of a subvariety is the intersection of the tropicalizations of finitely many hypersurfaces containing the subvariety. However, the latter fact is quite nontrivial, with incomplete proofs appearing in the literature: a correct argument can be found by combining [12] and [15], or in [51].

Our proof of Theorem B is quite different and does not rely on reduction to the hypersurface case. Indeed, the purpose of this paper is to show that these results on degenerations of amoebae are rather direct consequences of a continuity property of the fibers of certain *Berkovich spaces* that were introduced in [10] and contain both Archimedean and non-Archimedean information. Our results give further evidence to the suggestion on p. 51 of *loc. cit.* that such spaces are "worth studying".

Let us explain all this in the context of Theorem A′, leaving the setting of Theorem B to Sect. 5. Consider the field $\mathbf{C}$ equipped with the norm

$$\|\cdot\| := \max\{|\cdot|_\infty, |\cdot|_0\}, \qquad (\blacklozenge)$$

Note that $\|\cdot\|$ is only submultiplicative, but $(\mathbf{C}, \|\cdot\|)$ is nevertheless a Banach ring. Given a complex algebraic variety X, Berkovich introduced in [10] a natural *analytification* X^{An} of X with respect to the norm $\|\cdot\|$ on $\mathbf{C}$. See Sect. 3 for more details on this and on what follows. The space X^{An} is a locally compact Hausdorff space and comes with a natural continuous and surjective map

$$\lambda\colon X^{\mathrm{An}} \to [0, 1].$$

The fiber $\lambda^{-1}(1)$ is the usual complex analytic space X^h associated to X.[2] For $0 < \rho \le 1$, $\lambda^{-1}(\rho)$ is homeomorphic to X^h. Finally, the fiber $\lambda^{-1}(0)$ is the Berkovich analytification of X with respect to $|\cdot|_0$. See Fig. 4 for an illustration of $(\mathbf{P}^1)^{\mathrm{An}}$.

Theorem C *The map* $\lambda\colon X^{\mathrm{An}} \to [0, 1]$ *is open.*

This result essentially says that the Archimedean fibers $\lambda^{-1}(\rho)$ converge to the non-Archimedean fiber $X^{\mathrm{an}} = \lambda^{-1}(0)$ as $\rho \to 0+$. The latter convergence property implies Theorem A′ since there is a natural continuous, proper and surjective *tropicalization map* $Y^{\mathrm{An}} \to Y^{\mathrm{trop}}$ that takes the fiber $\lambda^{-1}(0)$ to X^{trop} and takes any other fiber $\lambda^{-1}(\rho)$ to the scaled amoeba $\rho \cdot \mathcal{A}_X$.

We prove Theorem C using the fact that the points of X^{an} of maximal rational rank are dense. Using resolution of singularities, such points can be realized on a blowup as monomial valuations with rationally independent weights, and then the proof is concluded by a direct computation. See Sect. 3.4 for details. A statement related to Theorem C appears as Corollary 6.8 in [70].

Let us make some bibliographical comments. Amoebae (with the opposite sign convention of ours) were introduced in [33] and have been intensively studied.

When $X = V(f) \subset \mathbf{C}^{*n}$ is a hypersurface, the complement of the amoeba A_X in $\mathbf{R}^n$ is convex and its connected components correspond to Laurent series expansions of $1/f$ at the origin [30,63,75,76,84]. Hypersurface amoebae can also be effectively

[2] The superscript "h" stands for "holomorphic".

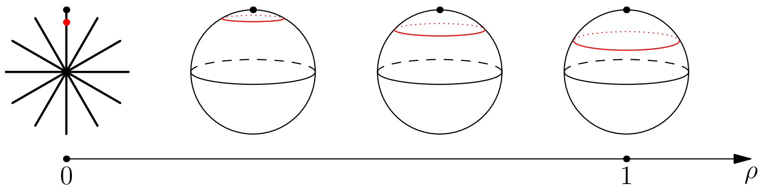

Fig. 4 The analytification $\mathbf{P}^{1,\mathrm{An}}$ of the complex projective line with respect to the norm $\|\cdot\|$ on $\mathbf{C}$, together with the canonical map $\lambda\colon \mathbf{P}^{1,\mathrm{An}} \to [0,1]$. The fiber $\lambda^{-1}(0)$ is the analytification of $\mathbf{P}^1$ with respect to the trivial norm, and is homeomorphic to a cone over $\mathbf{P}^1(\mathbf{C})$. All the other fibers are homeomorphic to a sphere. The *points* on *top* form a continuous section of λ. The *smaller circle* in the fiber $\lambda^{-1}(\rho)$ is of radius $e^{-1/\rho}$; these *circles* converge as $\rho \to 0$ to a unique point in the fiber $\lambda^{-1}(0)$

studied using Ronkin functions [59,64,66,76]. Their boundaries are studied in [52, 54,55,77]

In dimension $n = 2$, there is an inequality between the area of the amoeba A_f defined by a polynomial f and the area of the Newton polygon of f [64]: the case of equality was characterized in [59] as arising from Harnack curves in real algebraic geometry. Further interesting relations between real algebraic curves and amoebae are studied in [54]. The degeneration of amoebae in dimension two onto tropical varieties is used in a striking way in [57] for enumerative problems. Planar amoebae also arise in certain considerations in statistical thermodynamics [46,65].

In higher codimension, amoebae may or may not have finite volume [53] but their complements retain certain weaker convexity properties [38,74]. Computational aspects are studied in [72,82,83,87]. For more information and further references, see the surveys [39,56].

Tropical varieties have appeared in many different contexts. We have defined them here as images under the tropicalization map, but they can also be characterized in terms of so-called *initial degenerations* [19,23,68,79]. They have a polyhedral structure [11, 23] that satisfies a *balancing condition* [35,78,80]. Tropical geometry, especially for curves, can also, to some extent, be developed intrinsically, see [5,6,41,58]. It has seen striking applications to algebraic geometry [17,42,54,57].

The relation between Berkovich spaces (over a valued field) and tropical geometry appears implicitly already in the work of Bieri and Groves [11] which predates the general theory developed by Berkovich himself. Since then, it has been systematically studied by many authors. For finer properties of the tropicalization map, see e.g. [1,2,6,21,34,36,62,73]. In [31,68] it is shown that the Berkovich analytification of an algebraic variety over a non-Archimedean field is the limit of its tropicalizations over all embeddings into toric varieties.

The general idea of using non-Archimedean techniques to study various kinds of limiting behavior of complex analytic objects is also not new. Morgan and Shalen [60] used valuations to compactify complex affine varieties. Favre recently used the space X^{An} to recast and generalize their construction using Berkovich spaces; a statement close to Theorem C (and even closer to Theorem C′ in Sect. 5) appears in [24]. Other examples of how Berkovich spaces, especially analytifications of complex algebraic varieties with respect to the trivial norm, can be used to study complex analytic phenomena can be found in [10,13,26–29,47,60]. Also related—at least in spirit—is the procedure known as Maslov dequantization: see [40,50] and the references therein.

A version of Theorem A for a non-Archimedean absolute value was proved by Gubler, see [35, §8, Cor. 11.13]. The techniques in this paper could likely be adapted to give a new proof of this result, at least in residue characteristic zero, but we leave this for future work. It would also be interesting to study the adelic amoebae associated to varieties defined over a number field, see [23,67]. The results in this paper have recently been used to study the topology of the complements of certain tropical varieties [61].

The organization of the paper is as follows. In Sect. 2 we recall the notion of continuously varying families of spaces in the sense of Kuratowski. In Sect. 3 we discuss various analytification procedures and prove Theorem thmc. Then, in Sect. 4 we study the tropicalization map from Y^{An} to Y^{trop} for a toric variety Y. In particular, we prove Theorem A′. Finally, in Sect. 5 we study one-parameter families of varieties and prove Theorem B, as well as the required fact, Theorem C′, about Berkovich spaces.

2 Continuous families of subspaces

Consider a surjective continuous map $\pi : X \to B$ between topological spaces. Write $X_b := \pi^{-1}(b)$ for $b \in B$. We'd like to study the continuity properties of $b \mapsto X_b$. To this end, suppose that X embeds as a subset of $B \times Y$, for some topological space Y, and that π is the restriction of the projection of $B \times Y$ onto the first factor. We can then view X_b as a subset of Y for all $b \in B$.

Definition 2.1 We say that $b \to X_b$ is *upper semicontinuous* (usc) if given $b_0 \in B$ and $y \in Y \setminus X_{b_0}$, there exist neighborhoods U of y in Y and B_0 of b_0 in B such that $X_b \cap U = \emptyset$ for all $b \in B_0$.

Definition 2.2 We say that $b \to X_b$ is *lower semicontinuous* (lsc) if given $b_0 \in B$ and $y \in X_{b_0}$ and given any neighborhoods U of y in Y and B_0 of b_0 in B, we have $X_b \cap U \neq \emptyset$ for all $b \in B_0$.

Naturally, $b \to X_b$ is *continuous* if it is both usc and lsc. These continuity properties are in the sense of Kuratowski [49]. The proof of the following result is left to the reader.

Lemma 2.3 *The map $b \to X_b$ is usc iff X is closed in $B \times Y$. It is lsc iff $\pi : X \to B$ is open.*

Now suppose Y is a metric space. We can then consider continuity of $b \to X_b$ in the *Hausdorff topology*, which means the following: for every $b_0 \in B$ and every $\varepsilon > 0$ there exists a neighborhood B_0 of b in B such that, whenever $b \in B_0$, any point in X_{b_0} (resp. X_b) is at distance at most ε from some point in X_b (resp. X_{b_0}).

Suppose that X_b is a closed subset of Y for all b. Then continuity of $b \to X_b$ in the Hausdorff topology implies continuity in the sense of Kuratowski. The converse is true when Y is compact.

3 Analytification

We recall a special case of the construction in [10, §2]; see also [8, §1.5] and [69,70]. Consider a Banach ring $(k, \|\cdot\|)$. This means that k is a commutative ring with unit and that $\|\cdot\| : k \to \mathbf{R}_+$ satisfies $\|a\| = 0$ iff $a = 0$; $\|a - b\| \le \|a\| + \|b\|$ and $\|ab\| \le \|a\| \cdot \|b\|$ for all $a, b \in k$; and k is complete in the metric induced by $\|\cdot\|$. In fact, we will only consider the case when k is a *field*.

Let X be a separated scheme of finite type over k. The construction in [10] associates an *analytification* X^{An} of X with respect to the norm $\|\cdot\|$ on k. It is defined as follows.[3] For any affine open subset $U = \operatorname{Spec} A$ of X, where A is a finitely generated k-algebra, let U^{An} be the (nonempty) set of multiplicative seminorms on A whose restrictions to k are bounded by the norm $\|\cdot\|$. The topology on U^{An} is the weakest one for which $U^{\mathrm{An}} \ni |\cdot| \to |f|$ is continuous for every $f \in A$.

It is customary to denote the points in U^{an} by a letter such as x and the corresponding seminorm by $|\cdot|_x$. The latter induces a multiplicative norm on $A/\mathfrak{p}_x$, where $\mathfrak{p}_x$ is the kernel of $|\cdot|_x$. Let $\mathcal{H}(x)$ be the completion of the fraction field of $A/\mathfrak{p}_x$ with respect to this norm.

By gluing together the spaces U^{An} we construct a topological space X^{An}. This space is Hausdorff, locally compact and countable at infinity. The assignment $x \mapsto \mathfrak{p}_x$ above globalizes to a continuous map

$$\pi : X^{\mathrm{An}} \to X,$$

where X is viewed as a scheme, equipped with the Zariski topology. The assignment $X \to X^{\mathrm{An}}$ is functorial. If $X \hookrightarrow Y$ is an open (resp. closed) embedding, then so is $X^{\mathrm{An}} \hookrightarrow Y^{\mathrm{An}}$. If $X \to Y$ is surjective, then so is $X^{\mathrm{An}} \to Y^{\mathrm{An}}$.

The analytification of the zero-dimensional affine space is equal to the Berkovich spectrum $\mathcal{M}(k, \|\cdot\|)$ defined in [8, §1.2]. The canonical map $X \to \mathbf{A}^0 = \operatorname{Spec} k$ induces a surjective, continuous map

$$\lambda\colon X^{\mathrm{An}} \to \mathcal{M}(k, \|\cdot\|).$$

We shall study this general analytification functor $X \mapsto X^{\mathrm{An}}$ for three types of Banach fields $(k, \|\cdot\|)$.

3.1 Archimedean case

First assume that $k = \mathbf{C}$ is the field of complex numbers and that $\|\cdot\| = |\cdot|_\infty$ is the usual Archimedean norm. Denote by X^h the usual complex analytic variety associated to X. Recall that the points of X^h can be identified with the closed points of X.

It turns out that X^h can be identified with the analytification X^{An} above in such a way that π maps a point of X^h to the corresponding closed point of X. To see this, first

[3] While we shall only consider the analytification as a topological space, one can also equip it with a structure sheaf.

note that $\mathcal{M}(\mathbf{C}, |\cdot|_\infty) = \{|\cdot|_\infty\}$ is a singleton. Now consider an open affine subset U. To each point $x \in U^h$ we can associate a seminorm $|\cdot|_x \in U^{\mathrm{An}}$ by $|f|_x := |f(x)|_\infty$. This gives rise to a injective continuous map $U^h \to U^{\mathrm{An}}$ which is surjective by the Gelfand–Mazur Theorem, and easily seen to be a homeomorphism.

3.2 Non-Archimedean case

Next suppose that k is a *non-Archimedean field*. This means that $\|\cdot\| = |\cdot|$, where $|\cdot|$ is a non-Archimedean, multiplicative norm on k, that is $|ab| = |a|\cdot|b|$ and $|a-b| \le \max\{|a|, |b|\}$ for any $a, b \in k$. The analytifications X^{An} are then special cases of the *Berkovich spaces* studied in [8,9].[4] To conform with the notation in *loc. cit.* we write X^{an} instead of X^{An}.

To any non-Archimedean field $(k, |\cdot|)$ is associated a *value group* $|k^*| := \{|a| \mid a \in k^*\}$ as well as its divisible version $\sqrt{|k^*|} := \{r^{1/n} \mid r \in |k^*|,\ n \ge 1\}$. Now suppose $x \in X^{\mathrm{an}}$. We can view $\sqrt{|k^*|}$ and $\sqrt{|\mathcal{H}(x)^*|}$ as $\mathbf{Q}$-vector spaces. Define the *rational rank* $t(x)$ of x as the codimension of $\sqrt{|k^*|}$ in $\sqrt{|\mathcal{H}(x)^*|}$. If X has dimension n, then $t(x) \le n$ for all $x \in X^{\mathrm{an}}$, see [9, Lemma 2.5.2]. In fact, $t(x)$ is bounded by the transcendence degree over k of the residue field of $\pi(x)$. We say that x has *maximal rational rank* if $t(x) = n$. In this case, $\pi(x)$ is the generic point of an irreducible component of dimension n, and x defines a valuation of the residue field at this point.

Our approach to the proof of Theorem C in the introduction is based on

Lemma 3.1 *Assume that X has pure dimension n and that the divisible value group $\sqrt{|k^*|}$ has infinite codimension in $\mathbf{R}_+^*$ as a $\mathbf{Q}$-vector space. Then the set of points in X^{an} with maximal rational rank, $t(x) = n$, is dense in X^{an}.*

In fact, a more general statement is true. I am grateful to V. Berkovich for the following statement and proof. (Closely related results appear as Lemma 10.1.2 of [20] and Corollary 5.7 of [71].) Here we freely use terminology and results from [8] and [9].

Lemma 3.2 *Let k be as in Lemma 3.1. Consider a k-analytic space X of pure dimension n and let X' be the set of points $x \in X$ such that $t(x) = n$. Then X' is dense in X.*

Proof We may assume that X is k-affinoid. Given positive numbers $r_1, \dots, r_m$ whose images in $\mathbf{R}_+^*/\sqrt{|k^*|}$ are linearly independent, define a valued field extension K_r/k as in [8, p.22]. We can pick r such that the base change $Y = X\hat{\otimes}_k K_r$ is strictly K_r-affinoid and of pure dimension n. The image of the analogous subset Y' of Y in X under the continuous canonical map $Y \to X$ lies in X'. This reduces the situation to the case when X is strictly k-affinoid and k is nontrivially valued.

The set X_0 of points $x \in X$ with $[\mathcal{H}(x) : k] < \infty$ is dense in X, and any point of X_0 has a fundamental system of strictly affinoid neighborhood, see Proposition 2.1.15 and its proof in [8]. Hence it suffices to show that every strictly k-affinoid space of pure dimension n contains a point x with $t(x) = n$. By Noether normalization, the situation is reduced to the case when X is a closed polydisc of radii one. By the assumption on k, we can find numbers $0 < r_1, \dots, r_n < 1$ whose images in $\mathbf{R}_+^*/\sqrt{|k^*|}$ are linearly

[4] They are good k-analytic spaces without boundary.

independent. Then the maximal point of the closed polydisc of radii $(r_1, \dots, r_n)$ belongs to X'. □

Now we specialize to the case when $k = \mathbf{C}$ is the field of complex numbers and $|\cdot| = |\cdot|_0$ is the *trivial norm*. Berkovich spaces over this non-Archimedean field has seen a surprising number of applications, see for example [10,13,24,26,28,29,43,85]. Their topological structure is partially described in [13,25,43].

Consider a complex algebraic variety X of pure dimension n. Here is an example of a point $x \in X^{\mathrm{an}}$ of maximal rational rank, $t(x) = n$. Suppose $\xi \in X$ is a closed point, X is smooth at ξ and there exist coordinates $z_1, \dots, z_n$ at ξ and positive numbers $\alpha_i > 0$, $1 \le i \le n$. Then we can define a *monomial valuation* v on $\widehat{\mathscr{O}_{X,\xi}} \simeq \mathbf{C}[[z_1, \dots, z_n]]$ by setting

$$v\left(\sum_{m \in \mathbf{Z}_+^n} a_m z^m\right) := \min\{m_1\alpha_1 + \dots + m_n\alpha_n \mid a_m \neq 0\}.$$

The valuation v defines a point $x = e^{-v}$ in X^{an} with $\pi(x) = \xi$, and we have $t(x) = n$ iff the numbers α_i are linearly independent over $\mathbf{Q}$. We call x a *monomial point*.

Lemma 3.3 *Assume that X has pure dimension n and that $x \in X^{\mathrm{an}}$ has maximal rational rank $t(x) = n$. Then there exists a surjective birational morphism $\varphi : Y \to X$, with Y smooth, and a monomial point $y \in Y^{\mathrm{an}}$ with $t(y) = n$ and $\varphi^{\mathrm{an}}(y) = x$.*

Proof The point x defines a real rank one valuation on the function field of X and the condition $t(x) = n$ implies that this valuation is an Abhyankar valuation. The statement to be proved is then an example of local uniformization of Abhyankar valuations, see [48]. A simple proof using Hironaka's theorem on resolutions of singularities is given in [22, Proposition 2.8]; see also [44, Proposition 3.7]. □

3.3 Hybrid case

Finally we consider the "hybrid" construction of [10, §2] that combines Archimedean and non-Archimedean information. Equip $\mathbf{C}$ with the norm $\|\cdot\|$ defined in (♦), that is,

$$\|\cdot\| := \max\{|\cdot|_\infty, |\cdot|_0\}.$$

The Berkovich spectrum $\mathcal{M}(\mathbf{C}, \|\cdot\|)$ is the set of multiplicative seminorms $|\cdot|$ on $\mathbf{C}$ bounded by $\|\cdot\|$. Such a seminorm has to be of the form $|\cdot|_\infty^\rho$ for some $\rho \in [0, 1]$, where the case $\rho = 0$ is interpreted as the trivial norm. Thus we can identify $\mathcal{M}(\mathbf{C}, \|\cdot\|)$ with the interval $[0, 1]$, so we get a surjective, continuous map

$$\lambda\colon X^{\mathrm{An}} \to [0, 1].$$

Concretely, this map can be defined by $\lambda(x) = \log|e|_x$.

The fiber $\lambda^{-1}(\rho)$ is equal to the analytification of X with respect to the multiplicative norm $|\cdot|_\infty^\rho$ on $\mathbf{C}$ (where $\rho = 0$ is interpreted as the trivial norm).

In view of Sect. 3.1, the fiber $\lambda^{-1}(1)$ is therefore homeomorphic to (and will be identified with) X^h in such a way that π maps a point of X^h to the corresponding closed point of X.

For $0 < \rho \leq 1$, the fiber $\lambda^{-1}(\rho)$ is also homeomorphic to X^h: each seminorm $|\cdot|$ in $X^{\mathrm{An}} \cap \lambda^{-1}(\rho)$ is of the form $|f| = |f(x)|_\infty^\rho$ for some $x \in X^h$. In fact, $\lambda^{-1}(]0,1])$ is homeomorphic to the product $]0,1] \times X^h$, see [10, Lemma 2.1].

Finally, the fiber $\lambda^{-1}(0)$ is the Berkovich analytification of X with respect to the trivial norm on $\mathbf{C}$, as in Sect. 3.2. Following [10], we denote this space by X^{an}.

Any closed point $\eta \in X$ gives rise to a continuous section s_η of λ: if $\eta \in U = \operatorname{Spec} A$, then $s_\eta(\rho)$ is the multiplicative seminorm on A defined by $f \mapsto |f(\eta)|_\infty^\rho$.

See Fig. 4 for a picture of the space X^{An} when $X = \mathbf{P}^1$.

3.4 Proof of Theorem C

We must prove that $\lambda\colon X^{\mathrm{An}} \to [0,1]$ is open. Recall that there exists a homeomorphism $\lambda^{-1}(]0,1]) \xrightarrow{\sim}]0,1] \times X^h$ that commutes with λ, so the restriction of λ to $X^{\mathrm{An}} \setminus X^{\mathrm{an}}$ is open. Therefore, it suffices to prove that for any $x \in X^{\mathrm{an}}$, the pair (X, x) satisfies:

($\star$) for any neighborhood U of x in X^{An}, $\lambda(U)$ is a neighborhood of 0 in $[0, 1]$

In fact, it suffices to prove ($\star$) for x of maximal rational rank, since by Lemma 3.1 such points are dense in X^{an}. Thus assume $t(x) = n$. By Lemma 3.3 we can find a surjective birational morphism $\phi\colon Y \to X$ and a monomial point $y \in Y^{\mathrm{an}}$ such that $\phi^{\mathrm{An}}(y) = x$. Since ϕ^{An} is continuous and surjective, it suffices to prove ($\star$) for the pair (Y, y).

Thus we may assume that X is smooth and that x is a monomial point. By assumption, there exists a closed point $\xi \in X$ such that $v = -\log|\cdot|_x$ is a monomial valuation on $\mathcal{O}_{X,\xi}$ in some local coordinates $z_1, \dots, z_n$ at ξ, say with weights $\alpha_i = v(z_i) > 0$, where $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbf{Q}$. Upon replacing X by an open affine neighborhood, we assume that $X = \operatorname{Spec} A$ is affine and that $z_i \in A$ for all i.

For $0 < \rho \ll 1$, consider the following polycircle in the coordinates z_i

$$Z'_\rho = \left\{\eta \in X^h \ \middle|\ |z_i(\eta)| = e^{-\alpha_i/\rho} \quad \text{for } 1 \leq i \leq n\right\}.$$

Also write Z_ρ for the image of Z'_ρ under the isomorphism $\lambda^{-1}(1) \xrightarrow{\sim} \lambda^{-1}(\rho)$. We claim that if U is any neighborhood of x in X^{An}, and $0 < \varepsilon \ll 1$, then then $Z_\rho \subset U$ for $0 < \rho \leq \varepsilon^2$. This will show that $\lambda(U) \supset [0, \varepsilon^2]$ and hence complete the proof.

To prove the claim, we may assume that U is of the form

$$U^+(f,t) := \left\{y \in X^{\mathrm{An}} \mid |f|_y < t\right\} \text{ or } U^-(f,t) := \left\{y \in X^{\mathrm{An}} \mid |f|_y > t\right\}$$

where $f \in A$ and $t > 0$. Indeed, finite intersections of such sets form a basis of neighborhoods of x in X^{An}. We consider only the case $U = U^+(f,t)$, leaving the case $U = U^-(f,t)$ to the reader. Pick a real number $s > 0$ such that

$$|f|_x < s < t$$

Expand f as a power series

$$f = \sum_{m \in \mathbf{Z}_+^n} a_m z^m$$

in $\widehat{\mathcal{O}_{X,\xi}} \simeq \mathbf{C}[\![z_1, \ldots, z_n]\!]$. This series converges in some neighborhood of ξ in X^h, so there exists $R \geq 1$ such that

$$|a_m|_\infty \leq R^{|m|} \tag{3.1}$$

for all m, where we write $|m| = m_1 + \cdots + m_n$.

Since the α_i are rationally independent, there exists $\bar{m} \in \mathbf{Z}_+^n$ such that $a_{\bar{m}} \neq 0$ and $\langle \bar{m}, \alpha \rangle < \langle m, \alpha \rangle := \sum_{i=1}^n m_i \alpha_i$ for all $m \neq \bar{m}$ such that $a_m \neq 0$. Note that $e^{-\langle \bar{m}, \alpha \rangle} = |f|_x < s$. We choose ε small enough so that if $0 < \rho \leq \varepsilon^2$, then

$$R^{\rho|\bar{m}|} \leq \sqrt{\frac{t}{s}}, \tag{3.2}$$

$$R^{\rho|m|} e^{-\langle m, \alpha \rangle} \leq R^{\rho|\bar{m}|} e^{-\langle \bar{m}, \alpha \rangle - \varepsilon|m|} \quad \text{when } a_m \neq 0 \text{ and } m \neq \bar{m}, \tag{3.3}$$

and

$$\left(\sum_{m \in \mathbf{Z}_+^m} e^{-|m|/\varepsilon} \right)^\rho < \sqrt{\frac{t}{s}}. \tag{3.4}$$

We claim that $Z_\rho \subset U$ for $0 < \rho \leq \varepsilon^2$ for such ε. To see this, pick $y \in Z_\rho$. We use (3.1)–(3.4) to estimate the terms in the series expansion of f. First,

$$|a_{\bar{m}} z(\eta)^{\bar{m}}|_\infty = |a_{\bar{m}}|_\infty \cdot |z^{\bar{m}}|_y^{1/\rho} \leq R^{|\bar{m}|} e^{-\langle \bar{m}, \alpha \rangle / \rho}.$$

Second, if $m \neq \bar{m}$ and $a_m \neq 0$, then

$$\begin{aligned} |a_m z(\eta)^m|_\infty &= |a_m|_\infty \cdot |z^m|_y^{1/\rho} \leq R^{|m|} e^{-\langle m, \alpha \rangle / \rho} \\ &\leq R^{|\bar{m}|} e^{-\langle \bar{m}, \alpha \rangle / \rho} e^{-\varepsilon|m|/\rho} \leq R^{|\bar{m}|} e^{-\langle \bar{m}, \alpha \rangle / \rho} e^{-|m|/\varepsilon}. \end{aligned}$$

Since $m \neq 0$ when $m \neq \bar{m}$ and $a_m \neq 0$, this leads to

$$\begin{aligned} |f|_y = |f(\eta)|_\infty^\rho &\leq \left(\sum_m |a_m z(\eta)^m|_\infty \right)^\rho \\ &\leq \left(R^{|\bar{m}|} e^{-\langle \bar{m}, \alpha \rangle / \rho} \right)^\rho \left(1 + \sum_{m \neq \bar{m}, a_m \neq 0} e^{-|m|/\varepsilon} \right)^\rho \end{aligned}$$

$$\leq R^{|\bar{m}|\rho} e^{-\langle \bar{m}, \alpha \rangle} \left(\sum_{m \in \mathbf{R}_+^n} e^{-|m|/\varepsilon} \right)^{\rho} < \sqrt{\frac{t}{s}} \cdot s \cdot \sqrt{\frac{t}{s}} = t,$$

and hence $y \in U = U^+(f, t)$, completing the proof.

4 Toric varieties and tropicalization

We recall some basic definitions about toric varieties from [32]. Let $N \simeq \mathbf{Z}^n$ be a lattice, $M = \mathrm{Hom}(N, \mathbf{Z})$ the dual lattice, and Σ a fan in N. To each cone $\sigma \in \Sigma$ is associated a finitely generated monoid $S_\sigma := \check{\sigma} \cap M$, a finitely generated algebra $\mathbf{Z}[S_\sigma]$ and an affine variety $U_\sigma = \mathrm{Spec}\, \mathbf{Z}[S_\sigma]$. By suitably gluing together the different affine varieties U_σ over $\sigma \in \Sigma$, we obtain a toric variety $Y = Y_\Sigma$.

We can also associate a tropical object $Y^{\mathrm{trop}} = Y_\Sigma^{\mathrm{trop}}$ to Σ following [32, §4.1] or [3]; see also [45] or [68].[5] Namely, consider the additive monoid $\overline{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ equipped with the natural topology. For each cone $\sigma \in \Sigma$, let $U_\sigma^{\mathrm{trop}} = \mathrm{Hom}(S_\sigma, \overline{\mathbf{R}})$ be the set of monoid homomorphisms, and equip U_σ^{trop} with the topology of pointwise convergence. For example, $U_0^{\mathrm{trop}} = N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^n$. The space Y^{trop} is obtained by gluing together U_σ^{trop} for $\sigma \in \Sigma$ and contains $N_{\mathbf{R}}$ as an open dense subset. It comes with the scaling action by $\mathbf{R}_+^*$ induced by the same action on $\overline{\mathbf{R}}$. For a polarized projective toric variety Y, the moment map gives a homeomorphism of Y^{trop} onto the moment polytope in $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$.

4.1 Tropicalization

As in Sect. 3, let Y^{An} be the analytification of $Y \times_{\mathbf{Z}} \mathbf{C}$ with respect to the norm $\|\cdot\|$ on $\mathbf{C}$. We have a continuous map

$$\mathrm{trop}\colon Y^{\mathrm{An}} \to Y^{\mathrm{trop}}$$

defined as follows. Let σ be a cone in Σ. A point in U_σ^{An} is a multiplicative seminorm $|\cdot|$ on $\mathbf{C}[S_\sigma]$ whose restriction to $\mathbf{C}$ is bounded by $\|\cdot\|$. In particular, $-\log|\cdot|$ defines a monoid homomorphism from S_σ to $\overline{\mathbf{R}}$, and hence an element in U_σ^{trop}. It is easy to verify that the maps $U_\sigma^{\mathrm{An}} \to U_\sigma^{\mathrm{trop}}$ glue together to a globally defined continuous map $\mathrm{trop}\colon Y^{\mathrm{An}} \to Y^{\mathrm{trop}}$.

Let $\lambda\colon Y^{\mathrm{An}} \to [0, 1]$ be the canonical map, and set

$$Y^{\mathrm{Trop}} := [0, 1] \times Y^{\mathrm{trop}} \quad \text{and} \quad \mathrm{Trop} := \lambda \times \mathrm{trop}\,.$$

[5] As with the case of the analytification, the tropicalization Y^{trop} will only be considered as a topological space (together with an action by $\mathbf{R}_+^*$) and not equipped with a structure sheaf.

This leads to a commutative diagram

$$\begin{array}{ccc} Y^{\mathrm{An}} & \xrightarrow{\mathrm{Trop}} & Y^{\mathrm{Trop}} \\ & \searrow^{\lambda} & \downarrow \\ & & [0,1] \end{array}$$

where the map $Y^{\mathrm{Trop}} \to [0, 1]$ is the projection onto the first factor.

Proposition 4.1 *For any toric variety Y, the map*

$$\mathrm{Trop}\colon Y^{\mathrm{An}} \to Y^{\mathrm{Trop}}$$

is continuous, proper and surjective; hence it is also closed.

Proof We basically argue as in Lemma 2.1 and Sect. 3 of [68], but include some details as our setting is slightly different. The statements to be proved are local on either the source or target, so it suffices to consider the case when $Y = U_\sigma$ is affine.

In this case, the continuity of the map $\mathbf{C}[S_\sigma]^{\mathrm{An}} \to U_\sigma^{\mathrm{Trop}}$ is clear from the definition. To prove properness, pick generators $m_1, \ldots, m_N$ of the monoid S_σ. It suffices to prove that if $0 \le \rho \le \rho' \le 1$ and $-\infty < s_i \le t_i \le +\infty$ for $1 \le i \le N$, then the set

$$W := \mathrm{Trop}^{-1}\Big([\rho, \rho'] \times \{v \in U_\sigma^{\mathrm{trop}} \mid s_i \le v(m_i) \le t_i \text{ for } 1 \le i \le N\}\Big)$$

is compact in $\mathbf{C}[S_\sigma]^{\mathrm{An}}$. Now, the characters $z_i := \chi^{m_i}$, $1 \le i \le N$ generate $\mathbf{C}[S_\sigma]$ as a $\mathbf{C}$-algebra; we have

$$\mathbf{C}[S_\sigma] \simeq \mathbf{C}[z_1, \ldots, z_N]/\mathfrak{a}$$

for some (monomial) ideal $\mathfrak{a} \subset \mathbf{C}[z_1, \ldots, z_N]$. Under this identification, W becomes the set of multiplicative seminorms $|\cdot|$ on $\mathbf{C}[z_1, \ldots, z_N]$ whose restrictions to $\mathbf{C}$ are bounded by $\|\cdot\|$, and such that $e^\rho \le |e| \le e^{\rho'}$, $e^{-t_i} \le |z_i| \le e^{-s_i}$ for $1 \le i \le N$, and $|f| = 0$ for all $f \in \mathfrak{a}$. It is then clear that W is compact, as a consequence of Tychonoff's Theorem.

Finally, surjectivity can be established as follows. Pick any $(\rho, v) \in U_\sigma^{\mathrm{Trop}}$ and let m_i, $1 \le i \le N$, be generators of S_σ as before. Set $t_i := v(m_i) \in \overline{\mathbf{R}}$.

First suppose $\rho = 0$. Define a multiplicative seminorm $|\cdot|$ on $\mathbf{C}[z_1, \ldots, z_N]$ by $|\sum_\beta a_\beta z^\beta| = \max\{e^{-\langle t,\beta\rangle} \mid a_\beta \ne 0\}$, where $\langle t, \beta\rangle = \sum_{i=1}^N t_i\beta_i$. This seminorm vanishes on the ideal $\mathfrak{a}$, and hence induces a multiplicative seminorm $|\cdot|$ on $\mathbf{C}[S_\sigma]$ whose restriction to $\mathbf{C}$ is the trivial norm. It is then clear that $\mathrm{Trop}(|\cdot|) = (0, v)$.

Now suppose $0 < \rho \le 1$. Let $\eta \in \operatorname{Spec} \mathbf{C}[z_1, \ldots, z_N]$ be the closed point with coordinates $z_i(\eta) = e^{-t_i}$, $1 \le i \le N$, and define a multiplicative seminorm $|\cdot|$ on $\mathbf{C}[z_1, \ldots, z_N]$ by $|f| = |f(\eta)|_\infty^\rho$. As before, this induces a multiplicative seminorm on $\mathbf{C}[U_\sigma]$ whose restriction to $\mathbf{C}$ is equal to $|\cdot|_\infty^\rho$, so $\mathrm{Trop}(|\cdot|) = (\rho, v)$. □

4.2 Proof of Theorem A′

Let X be a complex algebraic subvariety of $Y \times_{\mathbf{Z}} \mathbf{C}$. Then X^{An} is a closed subset of Y^{An}. Let $X^{\mathrm{trop}} \subset Y^{\mathrm{trop}}$ and $X^{\mathrm{Trop}} \subset Y^{\mathrm{Trop}}$ be the images of X^{An} under the mappings trop and Trop, respectively. By Proposition 4.1, X^{Trop} is closed in Y^{Trop}. We have a commutative diagram

$$\begin{array}{ccc} X^{\mathrm{An}} & \xrightarrow{\mathrm{Trop}} & X^{\mathrm{Trop}} \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \pi_1} \\ & & [0,1] \end{array}$$

The map $\lambda\colon X^{\mathrm{An}} \to [0,1]$ is continuous and surjective, and by Theorem C it is also open. The map $\mathrm{Trop}\colon X^{\mathrm{An}} \to X^{\mathrm{Trop}}$ is surjective by definition and continuous by Proposition 4.1. It follows from these two properties that $\pi_1\colon X^{\mathrm{Trop}} \to [0,1]$ is open and surjective.

Write $\pi_1^{-1}(\rho) = \{\rho\} \times X_\rho^{\mathrm{trop}}$ for $0 \le \rho \le 1$, where $X_\rho^{\mathrm{trop}} \subset Y^{\mathrm{trop}}$. Lemma 2.3 implies that $\rho \mapsto X_\rho^{\mathrm{trop}}$ is continuous. Theorem A′ will thus follow immediately if we can prove that $X_0^{\mathrm{trop}} = X^{\mathrm{trop}}$ and $X_\rho^{\mathrm{trop}} = \rho \cdot A_X$ for $0 < \rho \le 1$.

Now, the fiber $\lambda^{-1}(1)$ of X^{An} is the analytification of X with respect to the Archimedean norm $|\cdot|_\infty$ on $\mathbf{C}$. Hence the fiber X_1^{trop} of X^{Trop} is equal to the amoeba A_X. Similarly, for $0 < \rho \le 1$, $\lambda^{-1}(\rho)$ is the analytification of X with respect to the norm $|\cdot|_\infty^\rho$ on $\mathbf{C}$, and this implies that X_ρ is the scaled amoeba $\rho \cdot A_X$ for $0 < \rho \le 1$. Finally, the fiber X_0^{trop} is the image of $X^{\mathrm{an}} \subset Y^{\mathrm{an}}$ under the tropicalization map $Y^{\mathrm{an}} \to Y^{\mathrm{trop}}$, where the analytifications are defined using the trivial norm on $\mathbf{C}$. This image is equal to the tropicalization X^{trop} of X as defined in [35]. We should check that this image also agrees with the definition of X^{trop} in the introduction. On the one hand, the tropicalization does not change under non-Archimedean field extensions, see [35, Prop. 3.7]. On the other hand, X^{an} may be viewed as the set of equivalence classes of L-valued points, over all valued field extensions $(L, |\cdot|)$ of $(\mathbf{C}, |\cdot|_0)$, see [8, 3.4.2]. This completes the proof.

5 One-parameter families

Consider a complex algebraic variety $\mathcal{X}$ that admits a surjective morphism

$$p\colon \mathcal{X} \to \mathbf{G}_m$$

where $\mathbf{G}_m = \operatorname{Spec} \mathbf{C}[t^{\pm 1}] \simeq \mathbf{C}^*$. We can view $\mathcal{X}$ as a one-parameter family of complex algebraic varieties, and we are interested in the behavior as $t \to 0$.

As in Sect. 3.3, let $\mathcal{X}^{\mathrm{An}}$ be the Berkovich analytification with respect to the norm $\|\cdot\|$ on $\mathbf{C}$, and consider the closed subset $\mathcal{X}^\sharp \subset \mathcal{X}^{\mathrm{An}}$ of seminorms for which $|t| = e^{-1}$. The morphism p gives rise to a continuous surjective map $p^{\mathrm{An}}\colon \mathcal{X}^{\mathrm{An}} \to \mathbf{G}_m^{\mathrm{An}}$ that sends

$\mathcal{X}^\sharp$ to $\mathbf{G}_m^\sharp$, and is equivariant with respect to the continuous maps $\lambda\colon \mathcal{X}^{\mathrm{An}} \to [0,1]$ and $\lambda\colon \mathbf{G}_m^{\mathrm{An}} \to [0,1]$. Write $X_\rho^\sharp$ (resp. $\mathbf{G}_{m,\rho}^\sharp$) for the fiber $\lambda^{-1}(\rho)$ inside $\mathcal{X}^\sharp$ (resp. $\mathbf{G}_m^\sharp$).

Note that $\mathbf{G}_{m,\rho}^\sharp$ consists of all multiplicative seminorms $|\cdot|$ on $\mathbf{C}[t^{\pm 1}]$ such that $|t| = e^{-1}$ and $|a| = |a|_\infty^\rho$ for all $a \in \mathbf{C}^*$. In particular, $\mathbf{G}_{m,0}^\sharp$ is a singleton, consisting of the restriction to $\mathbf{C}[t^{\pm 1}]$ of the multiplicative non-Archimedean norm on $\mathbf{C}((t))$ such that $|t| = e^{-1}$ and $|a| = 1$ for $a \in \mathbf{C}^*$. Now let $0 < \rho \le 1$. Any seminorm $|\cdot|$ in $\mathbf{G}_{m,\rho}^\sharp$ is then of the form $|f| := |f(a)|_\infty^\rho$ for some $a \in \mathbf{C}^*$, and the condition $|t| = e^{-1}$ means exactly that $|a|_\infty = e^{-1/\rho}$. Thus $\mathbf{G}_{m,\rho}^\sharp$ is in bijection with the circle of radius $e^{-1/\rho}$ in $\mathbf{C}$, so $\mathbf{G}_m^\sharp$ can and will be identified with the closed disc $\Delta_{e^{-1}}$ of radius e^{-1} in $\mathbf{C}$. Under this identification we have

$$\lambda(a) = \left(\log |a|^{-1}\right)^{-1} \quad \text{for} \quad a \in \Delta_{e^{-1}}.$$

Write $p^\sharp\colon \mathcal{X}^\sharp \to \Delta_{e^{-1}}$ for the restriction of p^{An} to $\mathcal{X}^\sharp$, and $X_a^\sharp$ for the fiber above $a \in \Delta_{e^{-1}}$. The central fiber $X_0^\sharp$ is isomorphic to the analytification of the base change $\mathcal{X} \times_{\mathbf{G}_m} \mathbf{C}((t))$, with respect to the non-Archimedean norm on $\mathbf{C}((t))$. Any other fiber $X_a^\sharp$, $0 < |a| \le e^{-1}$, is homeomorphic to the fiber above $t = a$ of the complex analytic space $\mathcal{X}^h$.

Theorem C′ *For* $0 < \delta \ll 1$, *the map* $p^\sharp\colon \mathcal{X}^\sharp \to \Delta_{e^{-1}}$ *is open above* Δ_δ.

Remark 5.1 One can check that when $\mathcal{X} = \mathbf{G}_m \times X$ is a product, Theorem C′ implies Theorem C in the introduction.

Proof Using Hironaka's theorem on resolution of singularities, we can find a proper and surjective birational morphism $\mathcal{Y} \to \mathcal{X}$, with $\mathcal{Y}$ smooth. Then $p^\sharp\colon \mathcal{Y}^\sharp \to \Delta_{e^{-1}}$ factors through a continuous surjective map $\mathcal{Y}^\sharp \to \mathcal{X}^\sharp$. Hence, if $\mathcal{Y}^\sharp \to \Delta_\delta$ is open for some $\delta \in (0,1)$, then so is $\mathcal{X}^\sharp \to \Delta_\delta$. We may therefore assume that $\mathcal{X}$ is smooth.

There exists a finite subset $A \subset \mathbf{G}_m$ such that $p\colon \mathcal{X} \to \mathbf{G}_m$ is flat above $\mathbf{G}_m \setminus A$, see for example [37, Ch. III, Prop. 9.7]. By [18, Corollary, p. 73] this implies that $p^h\colon \mathcal{X}^h \to \mathbf{C}^*$, the analytification of p with respect to $|\cdot|_\infty$, is open above $\mathbf{C}^* \setminus A$. Pick $\delta > 0$ small enough so that $|a| > \delta$ for all $a \in A$. Then $p^\sharp\colon \mathcal{X}^\sharp \to \Delta_{e^{-1}}$ is open above $\Delta_\delta \setminus \{0\}$. It remains to see that $p^\sharp$ is open also at points on the non-Archimedean fiber $X_0^\sharp$.

By the Nagata compactification theorem (see [16]) there exists a proper complex algebraic variety $\overline{\mathcal{X}}$, and an open immersion $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$, with dense image, such that p extends to a proper morphism $p\colon \overline{\mathcal{X}} \to \mathbf{P}^1$. Using resolution of singularities, we may assume that $\overline{\mathcal{X}}$ is smooth. Again by [37, Ch. III, Prop. 9.7], $p\colon \overline{\mathcal{X}} \to \mathbf{P}^1$ is automatically flat above $\mathbf{P}^1 \setminus A$.

The general properties of the analytification functor imply that $\mathcal{X}^{\mathrm{An}}$ is an open subset of $\overline{\mathcal{X}}^{\mathrm{An}}$. We need to show that if $x \in X_0^\sharp$ and U is a neighborhood of x in $\mathcal{X}^\sharp$, then $p(U)$ is a neighborhood of 0 in $\Delta_{e^{-1}}$. Since the $\mathbf{Q}$-vector space $\sqrt{|\mathbf{C}((t))^*|}$ is of dimension one, Lemma 3.2 applies. We may therefore assume that x is a point of maximal rational rank, $t(x) = n$, since such points are dense in $X_0^\sharp$. The divisible value group $\sqrt{|\mathcal{H}(x)^*|}$ of x is a $\mathbf{Q}$-vector space of dimension $n+1$; hence x defines an Abhyankar valuation of the function field of $\mathcal{X}$ of rational rank $n+1$. That advantage of having $\overline{\mathcal{X}}$ proper and smooth is now that this valuation admits a unique *center* on $\overline{\mathcal{X}}$, as a consequence of the valuative criterion of properness. The center is a point $\xi \in \overline{\mathcal{X}}_0$ such that the valuation is nonnegative on the local ring $\mathcal{O}_{\overline{\mathcal{X}},\xi}$ and strictly positive on the maximal ideal.

Using [44, Proposition 3.7] or [14, Remark 3.8] we may, after a suitable blowup of $\overline{\mathcal{X}}$ above $0 \in \mathbf{P}^1$, assume that there exist local coordinates $z_1, \dots, z_{n+1}$ at ξ, positive integers $b_1, \dots, b_{n+1}$ and rationally independent positive real numbers $\alpha_1, \dots, \alpha_{n+1}$ such that $t = u \prod_{i=1}^{n+1} z_i^{b_i}$, with u a unit in $\mathcal{O}_{\overline{\mathcal{X}},\xi}$, and the point x defines a monomial valuation v on $\mathcal{O}_{\overline{\mathcal{X}},\xi}$ in these coordinates, with values $v(z_i) = \alpha_i$ for $1 \le i \le n+1$. In particular, $\sum_{i=1}^{n+1} b_i\alpha_i = v(t) = 1$.

For $0 < |a| \ll 1$, set

$$Z_a := X_a^\sharp \cap \{|z_i| = e^{-\alpha_i} \text{ for } 1 \le i \le n+1\}.$$

The same type of estimates as in the proof of Theorem C now show that any open neighborhood U of x in $\mathcal{X}^\sharp$ will contain Z_a for $0 < |a| \ll 1$. Indeed, we may assume $U = \{|f| > t\}$ or $U = \{|f| < t\}$, where $t > 0$ and $f \in \mathcal{O}_{\overline{\mathcal{X}},\xi}$. This proves that $p^\sharp(U)$ is an open neighborhood of 0 in Δ_δ, as was to be shown. □

5.1 Proof of Theorem B

The product $\mathbf{G}_m \times Y$ is a toric variety, and we have $(\mathbf{G}_m \times Y)^{\mathrm{trop}} = \mathbf{R} \times Y^{\mathrm{trop}}$. The image of $(\mathbf{G}_m \times Y)^\sharp$ in $(\mathbf{G}_m \times Y)^{\mathrm{trop}}$ is given by

$$\mathrm{trop}((\mathbf{G}_m \times Y)^\sharp) = \{1\} \times Y^{\mathrm{trop}} \simeq Y^{\mathrm{trop}}.$$

Via the identification $\mathbf{G}_m^\sharp \simeq \Delta := \Delta_{e^{-1}}$ above, this induces a commutative diagram

$$\begin{array}{ccc} (\mathbf{G}_m \times Y)^\sharp & \xrightarrow{p \times \mathrm{trop}} & \Delta \times Y^{\mathrm{trop}} \\ & \searrow^{p} & \downarrow \\ & & \Delta \end{array}$$

Now suppose $\mathcal{X}$ is a closed subvariety of $(\mathbf{G}_m \times Y) \times_{\mathbf{Z}} \mathbf{C} \simeq \mathbf{C}^* \times (Y \times_{\mathbf{Z}} \mathbf{C})$ such that the projection of $\mathcal{X}$ to $\mathbf{C}^*$ is surjective. Let $\mathcal{X}^\dagger \subset \Delta \times Y^{\mathrm{trop}}$ be the image of $\mathcal{X}^\sharp$ under $p \times \mathrm{trop}$. Its fiber over $a \in \Delta$ is then equal to

$$X_a^\dagger := \mathrm{trop}(X_a^\sharp),$$

where, again, $X_a^\sharp = p^{-1}(a)$. If $a \neq 0$, then $X_a^\dagger = \lambda(a) \cdot A_{X_a}$. On the other hand, $X_0^\dagger$ is equal to $\mathcal{X}^{\mathrm{trop}}$, the image of $X_0^\sharp$ in Y^{trop}. This completes the proof.

Acknowledgments I thank V. Berkovich for the proof of Lemma 3.2, and M. Baker, S. Boucksom, A. Ducros, W. Gubler, S. Payne and A. Werner, for comments on a preliminary version of this manuscript. I have also benefitted from discussions with E. Brugallé, C. Favre, G. Mikhalkin and B. Teissier.

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