



# Weighted composition operators on the Dirichlet space: boundedness and spectral properties

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**Abstract** Boundedness of weighted composition operators  $W_{u,\varphi}$  acting on the classical Dirichlet space  $\mathcal{D}$  as  $W_{u,\varphi}f = u(f \circ \varphi)$  is studied in terms of the *multiplier space* associated to the symbol  $\varphi$ , i.e.,  $\mathcal{M}(\varphi) = \{u \in \mathcal{D} : W_{u,\varphi} \text{ is bounded on } \mathcal{D}\}$ . A prominent role is played by the multipliers of the Dirichlet space. As a consequence, the spectrum of  $W_{u,\varphi}$  in  $\mathcal{D}$  whenever  $\varphi$  is an automorphism of the unit disc is studied, extending a recent work of Hyvärinen et al. (J. Funct. Anal. 265:1749–1777, 2013) to the context of the Dirichlet space.

**Keywords** Dirichlet space · Weighted composition operator

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## 1 Introduction and preliminaries

Let  $\mathbb{D}$  denote the open unit disc in the complex plane  $\mathbb{C}$ . The Dirichlet space  $\mathcal{D}$  consists of analytic functions  $f$  on  $\mathbb{D}$  such that

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

is finite. Here  $A$  stands for the normalized Lebesgue area measure of the unit disc. Observe that for a univalent function  $f$ , the integral above is just the area of  $f(\mathbb{D})$ .

It is well known that  $\mathcal{D} \subset \mathcal{H}^2 \subset \mathcal{A}^2$ , where  $\mathcal{H}^2$  and  $\mathcal{A}^2$  denote respectively the Hardy and Bergman spaces on  $\mathbb{D}$ , and that  $f \in \mathcal{D}$  if and only if  $f' \in \mathcal{A}^2$ . The recent monograph [8] is an excellent source to learn about the Dirichlet space and its particular issues.

If  $\varphi$  is an analytic function on  $\mathbb{D}$  with  $\varphi(\mathbb{D}) \subset \mathbb{D}$ , then the equation

$$C_{\varphi}f = f \circ \varphi$$

defines a composition operator  $C_{\varphi}$  on the space of all holomorphic functions on the unit disc  $\mathcal{H}(\mathbb{D})$ . On the Dirichlet space  $\mathcal{D}$ , a necessary condition for  $C_{\varphi}$  to be bounded is that  $\varphi \in \mathcal{D}$ . Nevertheless, not all the Dirichlet functions induce bounded composition operators on  $\mathcal{D}$ . Such functions were characterized in 1980 by Voas [20] in his Ph.D. thesis.

In this work, we shall be concerned with weighted composition operators on  $\mathcal{D}$ : for  $u \in \mathcal{D}$  and  $\varphi$  a holomorphic self-map of  $\mathbb{D}$  we define the weighted composition operator  $W_{u,\varphi}$  on  $\mathcal{D}$  by

$$(W_{u,\varphi}f)(z) = u(z)f(\varphi(z)),$$

noting that  $W_{u,\varphi}$  is not, in principle, a bounded operator on  $\mathcal{D}$ . It is clear that if  $C_{\varphi}$  is a bounded operator on  $\mathcal{D}$  and  $u$  is a *multiplier* of  $\mathcal{D}$ , that is, the analytic Toeplitz operator  $T_u : f \mapsto uf$  is defined everywhere on  $\mathcal{D}$  and hence bounded, the weighted composition operator  $W_{u,\varphi}$  on  $\mathcal{D}$  is obviously bounded.

A well known fact about the Dirichlet space is that the algebra  $\mathcal{M}(\mathcal{D})$  consisting of the multipliers of  $\mathcal{D}$  is not that easy to describe. Indeed, their elements were characterized by Stegenga [19] in a remarkable paper in terms of a condition involving the logarithmic capacity of their boundary values. In particular, the strict inclusion  $\mathcal{M}(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{H}^{\infty}$  holds. Here  $\mathcal{H}^{\infty}$  denotes the space of bounded analytic functions in  $\mathbb{D}$  endowed with the sup-norm. A straightforward reformulation in terms of Carleson measures for  $\mathcal{D}$  (that is, there is a continuous injection from  $\mathcal{D}$  into  $L^2(\mathbb{D}, \mu)$ ), yields the fact that  $u \in \mathcal{M}(\mathcal{D})$  if and only if  $u$  is bounded and the measure  $\mu$  defined by  $d\mu(z) = |u'(z)|^2 dA(z)$  is a Carleson measure for  $\mathcal{D}$ . We refer to [21] for multipliers and Carleson measures in Dirichlet spaces (and to [1, 8] for more on the subject of multipliers).

Concerning boundedness of weighted composition operators on  $\mathcal{D}$ , let us remark that one may construct self-maps of the unit disc  $\varphi$  such that  $\varphi \notin \mathcal{D}$  and a multiplier

$u \in \mathcal{M}(\mathcal{D})$  such that  $W_{u,\varphi}$  is bounded in the Dirichlet space. For instance, let  $u(z) = (1 - z)^2$  and let  $\varphi$  be the infinite Blaschke product with zeroes  $(1 - 1/n^2)_{n \geq 1}$ . Now  $\varphi \notin \mathcal{D}$ , so  $C_\varphi$  is clearly unbounded. However, for  $f \in \mathcal{D}$  we have

$$((1 - z)^2(f \circ \varphi))' = -2(1 - z)(f \circ \varphi) + (1 - z)^2(f' \circ \varphi)\varphi';$$

now the first term is clearly in the Bergman space  $\mathcal{A}^2$ , while for the second term we observe that  $(1 - z)^2\varphi'$  is a bounded analytic function in  $\mathbb{D}$  and  $(f' \circ \varphi) \in \mathcal{A}^2$ , so that it also lies in  $\mathcal{A}^2$  (with control of norms), showing that  $W_{u,\varphi}$  is bounded on  $\mathcal{D}$ .

Therefore, facing the problem of describing the weighted composition operators taking  $\mathcal{D}$  boundedly into itself deals not only with the multipliers of  $\mathcal{D}$  but also with those self-maps of the unit disc that may induce unbounded composition operators in  $\mathcal{D}$ .

At this regards, for a self-map  $\varphi$  of the unit disc  $\mathbb{D}$ , we define the *multiplier space*  $\mathcal{M}(\varphi)$  associated to  $\varphi$  by

$$\mathcal{M}(\varphi) = \{u \in \mathcal{D} : W_{u,\varphi} \text{ is bounded on } \mathcal{D}\}.$$

It is clear that if  $C_\varphi$  is bounded on  $\mathcal{D}$ , then  $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{M}(\varphi) \subseteq \mathcal{D}$ . Moreover, if  $\varphi$  induces an unbounded  $C_\varphi$  in  $\mathcal{D}$ , then  $\mathcal{M}(\mathcal{D})$  is no longer contained in  $\mathcal{M}(\varphi)$  since, in such a case, this latter space does not contain the constant functions.

The aim of this work is twofold. On one hand, we are interested in identifying the multiplier space  $\mathcal{M}(\varphi)$  for self-maps  $\varphi$  of  $\mathbb{D}$ . Indeed, we will be able to characterize the extreme cases whenever  $\varphi$  is a self-map of  $\mathbb{D}$  belonging to  $\mathcal{D}$ . Let us remark here that in the case of the Hardy space  $\mathcal{H}^2$ , where  $C_\varphi$  is automatically bounded and the multiplier space is  $\mathcal{M}(\mathcal{H}^2) = \mathcal{H}^\infty$ , Gallardo-Gutiérrez et al. proved that  $\mathcal{M}_{\mathcal{H}^2}(\varphi) = \mathcal{H}^2$  if and only if  $\|\varphi\|_\infty < 1$  (see [9]) and  $\mathcal{M}_{\mathcal{H}^2}(\varphi) = \mathcal{H}^\infty$  if and only if  $\varphi$  is a finite Blaschke product; this latter statement was showed previously in a different way in [5] and [16].

Let us also point out that in the course of our findings, we will prove a *Decomposition Theorem* for the Dirichlet space (cf. Theorem 2.1), which is interesting in its own and whose proof is based on the theory of model spaces for the shift operator in the Hardy space (see [17] for more information about model spaces). As far as we know, this is the first time model spaces come into play with the Dirichlet space.

On the other hand, we are interested in the spectral properties of weighted composition operators in  $\mathcal{D}$ . In [12], Higdon computed the spectrum of composition operators in  $\mathcal{D}$  induced by linear fractional self-maps of  $\mathbb{D}$ . The techniques developed there were quite different from those carried over by Cowen in [6] in the corresponding case of the Hardy space  $\mathcal{H}^2$  (see also [7, Chapter 7]), due to the particular nature of  $\mathcal{D}$ .

In a very recent work, Hyvärinen et al. [13] have described the spectra of invertible weighted composition operators  $W_{u,\varphi}$  acting on a large class of analytic function spaces including the weighted Bergman and the weighted Hardy spaces; generalizing previous results obtained in [11]. Nevertheless, as they also remark, their results do not apply directly to the Dirichlet space since they rely on the fact that the algebra of the multipliers of the spaces considered is  $\mathcal{H}^\infty$ . Our aim is to extend Hyvärinen, Lindström, Nieminen and Saukko's results to the context of the Dirichlet space  $\mathcal{D}$ , pointing out that their techniques are no longer working in  $\mathcal{D}$ .

The last section of the paper gives a description of the spectra of invertible weighted composition operators. We first note (see Proposition 3.1) that a bounded weighted composition operator  $W_{u,\varphi}$  in the Dirichlet space  $\mathcal{D}$  is invertible if and only if  $u$  is a multiplier bounded away from zero in  $\mathbb{D}$  and  $\varphi$  is an automorphism of the unit disc. Consequently, three separate cases are considered, depending on the nature of the disc automorphism  $\varphi$ : elliptic, parabolic or hyperbolic. When  $\varphi$  is parabolic, causal operators will play a prominent role in order to determine explicitly the spectrum of  $W_{u,\varphi}$ .

## 2 Boundedness of weighted composition operators

In this section, we study boundedness of weighted composition operators in the Dirichlet space. In order to show the results at this respect, we prove a *Decomposition Theorem* for  $\mathcal{D}$  based on model spaces.

Let  $B$  be a finite Blaschke product and write  $K_B$  for the model space  $K_B = \mathcal{H}^2 \ominus B\mathcal{H}^2$ , which is finite-dimensional; indeed  $\dim K_B = \deg B$ . Observe that if  $g_k \in K_B$ , then it does not matter which norm we use, since  $K_B$  is finite-dimensional and all norms are equivalent. We proceed to state the *Decomposition Theorem* in its full generality, since the main arguments of the proof also work for the Bergman space.

**Theorem 2.1** (Decomposition Theorem) *Let  $B$  be a finite Blaschke product such that  $B(0) = 0$ . Then*

- (1)  $f \in \mathcal{H}^2$  if and only if  $f = \sum_{k=0}^{\infty} g_k B^k$  (convergence in  $\mathcal{H}^2$  norm) with  $g_k \in K_B$  and  $\sum_{k=0}^{\infty} \|g_k\|^2 < \infty$ .
- (2)  $f \in \mathcal{D}$  if and only if  $f = \sum_{k=0}^{\infty} g_k B^k$  (convergence in  $\mathcal{D}$  norm) with  $g_k \in K_B$  and  $\sum_{k=0}^{\infty} (k + 1)\|g_k\|^2 < \infty$ .
- (3)  $f \in \mathcal{A}^2$  if and only if  $f = \sum_{k=0}^{\infty} g_k B^k$  (convergence in  $\mathcal{A}^2$  norm) with  $g_k \in K_B$  and  $\sum_{k=0}^{\infty} \|g_k\|^2/(k + 1) < \infty$ .

Statement (1) is included for the sake of completeness since it is a standard fact that  $\mathcal{H}^2 = K_B \oplus BK_B \oplus B^2K_B \oplus \dots$  as an orthonormal direct sum.

*A word about notation.* Throughout this work,  $a \lesssim b$  will denote that there exists an independent constant  $C$  such that  $a \leq Cb$ ; this constant may be different in each instance.

*Proof* We proceed to prove (2) and (3). We claim that for finite sums

$$\left\| \sum_{k=0}^N g_k B^k \right\|_{\mathcal{D}}^2 \lesssim \sum_{k=0}^N (k + 1)\|g_k\|^2 \quad \text{and} \quad \left\| \sum_{k=0}^N g_k B^k \right\|_{\mathcal{A}^2}^2 \lesssim \sum_{k=0}^N \|g_k\|^2/(k + 1)$$

with the implied constants independent of  $N$ .

Let  $e_1, \dots, e_n$  be a basis of the space  $K_B$ . Then, writing  $g_k = \sum_{\ell=1}^n a_{k\ell} e_\ell$  we have, since  $C_B$  is a bounded composition operator in  $\mathcal{D}$ , that

$$\begin{aligned} \left\| \sum_{k=0}^N \sum_{\ell=1}^n a_{k\ell} e_\ell B^k \right\|_{\mathcal{D}} &= \left\| \sum_{\ell=1}^n e_\ell \sum_{k=0}^N a_{k\ell} B^k \right\|_{\mathcal{D}} \\ &\leq \sum_{\ell=1}^n \|e_\ell\|_{\mathcal{M}(\mathcal{D})} \left\| \sum_{k=0}^N a_{k\ell} B^k \right\|_{\mathcal{D}} \\ &\lesssim \sum_{\ell=1}^n \left( \sum_{k=0}^N (k+1) |a_{k\ell}|^2 \right)^{1/2} \\ &\lesssim \left( \sum_{\ell=1}^n \sum_{k=0}^N (k+1) |a_{k\ell}|^2 \right)^{1/2}, \text{ by equivalence of norms on } \mathbb{C}^n, \\ &\lesssim \left( \sum_{k=0}^N (k+1) \|g_k\|^2 \right)^{1/2}, \end{aligned}$$

where the notation  $\|e_\ell\|_{\mathcal{M}(\mathcal{D})}$  represents  $\sup\{\|e_\ell f\|_{\mathcal{D}} : \|f\|_{\mathcal{D}} \leq 1\}$ . A similar calculation can be made in  $\mathcal{A}^2$ .

To obtain the converse inequality, we use the dual pairing between  $\mathcal{D}$ , equipped with the equivalent norm  $\|\sum_{k=0}^\infty a_k z^k\|_X^2 = \sum_{k=0}^\infty (k+1) |a_k|^2$  and  $\mathcal{A}^2$ , equipped with the equivalent norm  $\|\sum_{k=0}^\infty a_k z^k\|_Y^2 = \sum_{k=0}^\infty |a_k|^2 / (k+1)$ , namely

$$\left\langle \sum_{k=0}^\infty a_k z^k, \sum_{k=0}^\infty b_k z^k \right\rangle = \sum_{k=0}^\infty a_k \bar{b}_k.$$

For a finite sum  $\sum_{k=0}^N g_k B^k$  we take  $h_k \in K_B$  with  $h_k = (k+1)g_k$  for each  $k$ . Now we use the  $\mathcal{H}^2$  orthogonality of  $B^r K_B$  and  $B^s K_B$  for positive integers  $r$  and  $s$  with  $r \neq s$ , to deduce that

$$\left\langle \sum_{k=0}^N g_k B^k, \sum_{k=0}^N h_k B^k \right\rangle = \sum_{k=0}^N (k+1) \|g_k\|_{\mathcal{H}^2}^2.$$

Since

$$\left\| \sum_{k=0}^N h_k B^k \right\|_Y^2 \lesssim \sum_{k=0}^N \|h_k\|^2 / (k+1) = \sum_{k=0}^N (k+1) \|g_k\|^2,$$

it follows that

$$\left\| \sum_{k=0}^N g_k B^k \right\|_X^2 \gtrsim \sum_{k=0}^N (k+1) \|g_k\|^2,$$

and so we have a uniform equivalence of the Dirichlet norm and the quantity

$$\left( \sum_{k=0}^N (k + 1)^2 \|g_k\|^2 \right)^{1/2},$$

at least for finite sums. Since the Dirichlet space is contained in the Hardy space we may make the obvious extension to the whole of  $\mathcal{D}$  using infinite sums.

The argument for the Bergman space is analogous: once more we have an equivalence of norms, and since the Hardy space is dense in the Bergman space we obtain the required result.  $\square$

Recall that if  $C_\varphi$  is bounded then  $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{M}(\varphi) \subseteq \mathcal{D}$ . For  $\varphi$  a finite Blaschke product the space of weighted composition operators is as small as possible, as the following result shows.

**Theorem 2.2** *Let  $\varphi$  be an inner function. Then  $\mathcal{M}(\varphi) = \mathcal{M}(\mathcal{D})$  if and only if  $\varphi$  is a finite Blaschke product.*

*Proof* If  $\varphi$  is inner but not a finite Blaschke product, then  $\varphi \notin \mathcal{D}$ , and so  $C_\varphi$  is unbounded. Thus, taking  $u(z) \equiv 1$ , we have that  $u \in \mathcal{M}(\mathcal{D})$  but  $u \notin \mathcal{M}(\varphi)$ .

Now suppose that  $\varphi = B$ , a finite Blaschke product. Let  $T_u$  denote the map  $f \in \mathcal{D} \mapsto uf$ . We must show that  $T_u C_B$  is bounded if and only if  $u \in \mathcal{M}(\mathcal{D})$ . It is clear that if  $u \in \mathcal{M}(\mathcal{D})$  then  $T_u C_B$  is bounded on  $\mathcal{D}$ , since both  $T_u$  and  $C_B$  are bounded.

For the converse, note that without loss of generality we may take  $B(0) = 0$ , since if for some  $a$  we have  $B(a) = 0$ , then, setting

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

we have  $B \circ \varphi_a(0) = 0$  and

$$C_{\varphi_a} T_u C_B = T_{u \circ \varphi_a} C_{B \circ \varphi_a}.$$

Now  $T_u C_B$  is bounded if and only if  $T_{u \circ \varphi_a} C_{B \circ \varphi_a}$  is bounded, and showing that  $u \in \mathcal{M}(\mathcal{D})$  is equivalent to showing that  $u \circ \varphi_a \in \mathcal{M}(\mathcal{D})$ .

Now, given that  $T_u C_B$  is bounded, let  $f \in \mathcal{D}$ . By Theorem 2.1 we may write

$$f = \sum_{k=0}^{\infty} g_k B^k = \sum_{k=0}^{\infty} \sum_{\ell=1}^n a_{k\ell} e_\ell B^k,$$

where each  $g_k \in K_B$ ,  $\{e_1, \dots, e_n\}$  is a basis of  $K_B$ , and for each  $\ell$  the scalars  $(a_{k\ell})$  satisfy

$$\sum_{k=0}^{\infty} (k + 1) |a_{k\ell}|^2 \lesssim \sum_{k=0}^{\infty} (k + 1) \|g_k\|^2 \lesssim \|f\|_{\mathcal{D}}^2.$$

Write  $f_\ell(z) = \sum_{k=0}^{\infty} a_{k\ell} z^k$  so that  $\|f_\ell\|_{\mathcal{D}} \lesssim \|f\|_{\mathcal{D}}$ . Then

$$\left\| u \sum_{k=0}^{\infty} a_{k\ell} B^k \right\|_{\mathcal{D}} = \|T_u C_B f_{\ell}\|_{\mathcal{D}} \lesssim \|T_u C_B\| \|f\|_{\mathcal{D}}.$$

Now  $e_1, \dots, e_n$  lie in  $\mathcal{M}(\mathcal{D})$  and we conclude that

$$\|uf\|_{\mathcal{D}} = \left\| \sum_{\ell=1}^n e_{\ell} T_u C_B f_{\ell} \right\|_{\mathcal{D}} \lesssim \|T_u C_B\| \|f\|_{\mathcal{D}},$$

so that  $u \in \mathcal{M}(\mathcal{D})$  and the Theorem is proved. □

Next example shows that the assumption about  $\varphi$  being inner cannot be relaxed; even if  $\|\varphi\|_{\infty} = 1$  and  $C_{\varphi}$  is bounded in  $\mathcal{D}$ .

*Example 2.1* We can have  $\mathcal{M}(\varphi) \neq \mathcal{M}(\mathcal{D})$  even if  $\|\varphi\|_{\infty} = 1$  and  $C_{\varphi}$  is bounded in  $\mathcal{D}$ . Let us consider

$$\varphi(z) = \frac{1-z}{2} \quad \text{and} \quad u(z) = \sum_{k=2}^{\infty} \frac{z^k}{k(\log k)^{3/4}}.$$

Note that  $u \in \mathcal{D} \setminus \mathcal{M}(\mathcal{D})$  (see [8, Thm. 5.1.6]). Nonetheless,  $T_u C_{\varphi}$  is bounded; that is,  $u \in \mathcal{M}(\varphi)$ .

In order to show that, let  $f \in \mathcal{D}$ . We have

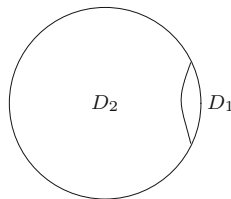
$$(u(f \circ \varphi))' = u'(f \circ \varphi) + u(f' \circ \varphi)\varphi'.$$

It suffices if we prove that each of these terms lies in the Bergman space  $\mathcal{A}^2$ .

Let us split the disc into disjoint measurable sets  $\mathbb{D} = D_1 \cup D_2$ , where  $D_1$  is a small neighbourhood of 1, mapped by  $\varphi$  into a small disc about 0 (Fig. 1).

On  $D_1$  both  $u$  and  $u'$  are square-integrable with respect to Lebesgue measure, while  $f \circ \varphi$  and  $f' \circ \varphi$  are uniformly bounded by constants depending only on the norm of  $f$ . Likewise, on  $D_2$ ,  $u$  and  $u'$  are bounded, while  $f \circ \varphi$  and  $f' \circ \varphi$  are square-integrable. Our conclusion is that  $T_u C_{\varphi}$  is bounded on the Dirichlet space  $\mathcal{D}$ .

At the other extreme, we have the following result.



**Fig. 1** A partition of the unit disc

**Theorem 2.3** *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then  $\mathcal{M}(\varphi) = \mathcal{D}$  if and only if*

- (1)  $\|\varphi\|_\infty < 1$ , and
- (2)  $\varphi \in \mathcal{M}(\mathcal{D})$ .

*Proof* Suppose first that  $\varphi$  satisfies conditions (1) and (2). Let  $u$  be a Dirichlet function. In order to prove that  $T_u C_\varphi$  is a bounded operator on  $\mathcal{D}$ , let  $f \in \mathcal{D}$  and consider

$$(u(f \circ \varphi))' = u'(f \circ \varphi) + u(f' \circ \varphi)\varphi'.$$

It suffices to show that each of these terms lies in the Bergman space  $\mathcal{A}^2$ .

We have that  $\|f \circ \varphi\|_\infty \lesssim \|f\|_{\mathcal{D}}$  since, for  $w \in \mathbb{D}$ ,

$$(f \circ \varphi)(w) = \langle f, k_{\varphi(w)} \rangle_{\mathcal{D}},$$

where  $k_{\varphi(w)}$  denotes the reproducing kernel at  $\varphi(w)$  in  $\mathcal{D}$ , which is bounded in norm independently of  $w$  since  $\|\varphi\|_\infty < 1$ . Hence, since  $u \in \mathcal{D}$ , we have

$$\|u'(f \circ \varphi)\|_{\mathcal{A}^2} \lesssim \|f\|_{\mathcal{D}}. \tag{1}$$

Also,  $\|f' \circ \varphi\|_\infty \lesssim \|f\|_{\mathcal{D}}$ , since the derivative kernels  $k'_{\varphi(w)}$  satisfying

$$f'(\varphi(w)) = \langle f, k'_{\varphi(w)} \rangle_{\mathcal{D}}$$

are also uniformly bounded in norm when  $\|\varphi\|_\infty < 1$ . Now by means of condition (2),  $\varphi \in \mathcal{M}(\mathcal{D})$ , and therefore the measure  $|\varphi'(z)|^2 dA(z)$  is a Carleson measure for  $\mathcal{D}$ . Thus, it follows that

$$\|u(f' \circ \varphi)\varphi'\|_{\mathcal{A}^2} \lesssim \|f\|_{\mathcal{D}}. \tag{2}$$

From (1) and (2), one gets that  $T_u C_\varphi$  is a bounded operator for all  $u \in \mathcal{D}$ .

Conversely, if  $T_u C_\varphi$  is bounded for all  $u \in \mathcal{D}$ , then by the Closed Graph Theorem  $C_\varphi$  maps  $\mathcal{D}$  boundedly into  $\mathcal{M}(\mathcal{D})$  and hence into  $\mathcal{H}^\infty$ . Suppose that  $\|\varphi\|_\infty = 1$ ; then we may find an unbounded  $f \in \mathcal{D}$ , and by considering functions  $f_n(z) = f(e^{i\theta_n} z)$  for suitable angles  $\theta_n$ , obtain a sequence of normalized functions  $f_n$  in  $\mathcal{D}$  and  $(z_n) \subset \mathbb{D}$  with  $|z_n| \rightarrow 1$  such that  $|f_n(\varphi(z_n))| \rightarrow \infty$ . This is a contradiction, so we conclude that  $\|\varphi\|_\infty < 1$ .

We now see that, for  $f \in \mathcal{D}$  fixed, we have  $\|u(f' \circ \varphi)\varphi'\|_{\mathcal{A}^2} \lesssim \|u\|_{\mathcal{D}}$ . So let  $f(z) = z$ . We conclude that  $T_{\varphi'} : \mathcal{D} \rightarrow \mathcal{A}^2$  is bounded, or equivalently  $|\varphi'(z)|^2 dA(z)$  is a Carleson measure for  $\mathcal{D}$ . Given that  $\varphi$  is bounded, this condition implies that  $\varphi \in \mathcal{M}(\mathcal{D})$ , as mentioned in the introduction. This concludes the proof of the Theorem. □

### 3 Spectral properties

In this section, we are interested in describing the spectra of invertible weighted composition operators in the Dirichlet space. As we pointed out in Sect. 1, the techniques in [13] depends strongly on the fact that the algebra of the multipliers contains  $\mathcal{H}^\infty$ .



The next result identifies invertible weighted composition operators in the Dirichlet space. It can be found, for example, in [3, Thm 3.3] and [22, Cor. 11].

**Proposition 3.1** *Let  $W_{h,\varphi}$  be a bounded weighted composition operator in the Dirichlet space  $\mathcal{D}$ . Then  $W_{h,\varphi}$  is invertible in  $\mathcal{D}$  if and only if  $h \in \mathcal{M}(\mathcal{D})$ , bounded away from zero in  $\mathbb{D}$  and  $\varphi$  is an automorphism of  $\mathbb{D}$ . In such a case, the inverse operator of  $W_{h,\varphi} : \mathcal{D} \rightarrow \mathcal{D}$  is also a weighted composition operator and*

$$(W_{h,\varphi})^{-1} = \frac{1}{h \circ \varphi^{-1}} C_{\varphi^{-1}}.$$

Recall that an automorphism  $\varphi$  of  $\mathbb{D}$  can be expressed in the form

$$\varphi(z) = e^{i\theta} \frac{p - z}{1 - \bar{p}z} \quad (z \in \mathbb{D}),$$

where  $p \in \mathbb{D}$  and  $-\pi < \theta \leq \pi$ . Recall that  $\varphi$  is called *hyperbolic* if  $|p| > \cos(\theta/2)$  (thus,  $\varphi$  fixes two points on  $\partial\mathbb{D}$ ); *parabolic* if  $|p| = \cos(\theta/2)$  (so,  $\varphi$  fixes just one point, located on  $\partial\mathbb{D}$ ) and *elliptic* if  $|p| < \cos(\theta/2)$  (therefore,  $\varphi$  fixes two points, one of them in  $\mathbb{D}$  and the other in the exterior of  $\overline{\mathbb{D}}$ ). See [18, Chapter 0], for instance.

*Notation* Assume  $\varphi$  is a self-map of the unit disc  $\mathbb{D}$ . In what follows,  $\varphi_n$  will denote the  $n$ th iterate of the map  $\varphi$ , that is,

$$\varphi_n = \varphi \circ \varphi \circ \dots \circ \varphi \quad (n \text{ times}),$$

for any  $n \geq 0$ , where  $\varphi_0$  is the identity function. It is clear that  $C_\varphi^n = C_{\varphi_n}$  for any  $n \geq 0$ . If  $W_{h,\varphi}$  is a bounded weighted composition operator in  $\mathcal{D}$ , it is rather straightforward that

$$W_{h,\varphi}^n f(z) = h(z) \cdots h(\varphi_{n-1}(z)) f(\varphi_n(z))$$

for any  $f \in \mathcal{D}$  and  $z \in \mathbb{D}$ . Following [13], we will denote

$$h_{(n)} = \prod_{k=0}^{n-1} h \circ \varphi_k;$$

where  $h_{(0)} = 1$  for convenience.

In what follows, we restrict our attention to weighted composition operators  $W_{h,\varphi}$  acting on  $\mathcal{D}$  induced by disc automorphisms  $\varphi$ . By Theorem 2.2, this implies that  $h$  is a multiplier of  $\mathcal{D}$ .

### 3.1 Elliptic case

In [13, Section 4.3], the authors describe the spectrum of  $W_{h,\varphi}$  acting on a large class of spaces of analytic functions whenever  $h$  is in the disc algebra  $A(\mathbb{D})$  and  $\varphi$  is an

elliptic automorphism. Our hypotheses on  $h$  in the context of  $\mathcal{D}$  is rather more general, since  $W_{h,\varphi}$  is bounded if and only if  $h \in \mathcal{M}(\mathcal{D})$  (and the spaces  $\mathcal{M}(\mathcal{D})$  and  $A(\mathbb{D})$  are not contained in each other). Nevertheless, it is possible to take a bit further some of the ideas developed in [13] and show the following result in a similar way.

**Theorem 3.1** *Suppose that  $\varphi$  is an elliptic automorphism of  $\mathbb{D}$  with fixed point  $a \in \mathbb{D}$  and  $W_{h,\varphi}$  a weighted composition operator on  $\mathcal{D}$ . Then*

- (1) *either there exists a positive integer  $j$  such that  $\varphi_j(z) = z$  for all  $z \in \mathbb{D}$ , in which case, if  $m$  is the smallest such integer, then*

$$\sigma(W_{h,\varphi}) = \overline{\{\lambda : \lambda^m = h_{(m)}(z), z \in \mathbb{D}\}},$$

- (2) *or  $\varphi_n \neq I$  for every  $n$  and, if  $W_{h,\varphi}$  is invertible, then*

$$\sigma(W_{h,\varphi}) = \{\lambda : |\lambda| = |h(a)|\}.$$

*Proof* The proof of (1) goes as in [13, Theorem 4.11]. The only minor change concerns the inclusion

$$\sigma(W_{h,\varphi}) \subset \overline{\{\lambda : \lambda^m = h_{(m)}(z), z \in \mathbb{D}\}},$$

where a similar argument applies taking into account the fact that if  $g \in \mathcal{M}(\mathcal{D})$  and it is bounded away from zero, then  $1/g$  is also in  $\mathcal{M}(\mathcal{D})$ . With respect to (2), just observe that  $W_{h,\varphi}^{-1} = \frac{1}{h \circ \varphi^{-1}} C_{\varphi^{-1}}$  is bounded, and hence  $1/(h \circ \varphi^{-1})$  is in  $\mathcal{M}(\mathcal{D})$  since  $\varphi^{-1}$  is a disc automorphism (see Theorem 2.2). Therefore, we refer the reader to [13] once more. □

### 3.2 Parabolic case

Now, let us assume that  $W_{h,\varphi}$  is an invertible weighted composition operator on  $\mathcal{D}$ , where  $\varphi$  is a parabolic disc automorphism. The previous ideas in [13] to determine the spectrum of  $W_{h,\varphi}$  made an extensive use of the fact that the sequence orbit  $\{\varphi_n(z_0)\}$  of a point  $z_0 \in \mathbb{D}$  is an interpolating sequence for  $\mathcal{H}^\infty$ , a space which is assumed to be contained in the multipliers of the spaces considered [see condition (C3) in [13], for instance]. In the case of the Dirichlet space, the interpolating sequences for the multiplier spaces  $\mathcal{M}(\mathcal{D})$  were characterized by Marshall and Sundberg [15], and independently by Bishop [2]. Nevertheless,  $\{\varphi_n(z_0)\}, z_0 \in \mathbb{D}$ , is no longer interpolating in  $\mathcal{M}(\mathcal{D})$  and therefore, our proof completely differs from the previous ones. Likewise, the work in [11] made use of inner functions, which are inappropriate in the context of  $\mathcal{D}$ .

**Theorem 3.2** *Suppose that  $\varphi$  is a parabolic automorphism of  $\mathbb{D}$  with fixed point  $a \in \mathbb{T}$  and  $W_{h,\varphi}$  a weighted composition operator on  $\mathcal{D}$ , determined by an  $h \in \mathcal{M}(\mathcal{D})$  that is continuous at  $a$ . If  $W_{h,\varphi}$  is invertible, then*

$$\sigma(W_{h,\varphi}) = \{\lambda \in \mathbb{C} : |\lambda| = |h(a)|\}.$$

*Proof* We begin by showing that the spectrum of  $W_{h,\varphi}$  is contained in the circle  $\{\lambda : |\lambda| = |h(a)|\}$ . Recall that  $h \in \mathcal{M}(\mathcal{D})$  implies that  $h \in \mathcal{H}^\infty$  and that  $|h'|^2 dA$  is a Carleson measure for  $\mathcal{D}$ ; that is, that there exists a constant  $K > 0$  such that

$$\int_{\mathbb{D}} |h'(z)|^2 |f|^2 dA(z) \leq K^2 \|f\|_{\mathcal{D}}^2$$

for all  $f \in \mathcal{D}$ . We write  $\|h'\|_{\mathcal{C}}$  for the least such  $K$ . Moreover, we see that  $1/h \in \mathcal{M}(\mathcal{D})$  since  $W_{h,\varphi}$  is invertible, which implies that  $h(a) \neq 0$ .

Recalling that  $(W_{h,\varphi})^n = W_{h_{(n)},\varphi_n} = T_{h_{(n)}}C_{\varphi_n}$ , we estimate its norm. Since the spectral radius of  $C_{\varphi}$  is 1 (see [12]) so that for each  $\varepsilon > 0$  we have  $\|C_{\varphi_n}\| \leq (1 + \varepsilon)^n$  for  $n$  sufficiently large, it will be sufficient to consider the operator of multiplication by  $h_{(n)}$ . For  $f \in \mathcal{D}$  we have

$$(h_{(n)}f)' = h_{(n)}f' + h'_{(n)}f.$$

The  $\mathcal{A}^2$  norm of the first term can be estimated using the fact that for each  $\varepsilon > 0$  there is an  $m$  such that

$$\|h_{(n)}\|_{\infty} \leq \|h\|_{\infty}^m [(1 + \varepsilon)|h(a)|]^{n-m} \tag{3}$$

for all  $n \geq m$ , which is given in the proof of [13, Lem. 4.2].

Also

$$\|h'_{(n)}\|_{\mathcal{C}} \leq \sum_{j=0}^{n-1} \|h_{(n),j}(h \circ \varphi_j)'\|_{\mathcal{C}},$$

where  $h_{(n),j} = h_{(n)}/(h \circ \varphi_j)$ .

Hence, as in (3) we have for  $j < m$

$$\|h_{(n),j}\|_{\infty} \leq \|h\|_{\infty}^{m-1} [(1 + \varepsilon)|h(a)|]^{n-m},$$

while for  $j \geq m$  we have

$$\|h_{(n),j}\|_{\infty} \leq \|h\|_{\infty}^m [(1 + \varepsilon)|h(a)|]^{n-m-1}.$$

Also

$$\|(h \circ \varphi_j)'\|_{\mathcal{C}} = \|\varphi'_j(h' \circ \varphi_j)\|_{\mathcal{C}} \leq \|\varphi'_j\|_{\infty} \|h'\|_{\mathcal{C}}.$$

Now, we estimate  $\|\varphi'_j\|_{\infty}$  for any  $j$ . Let us suppose without loss of generality that  $a = -1$ . Hence

$$\varphi_n(z) = \frac{(2 - niy)z - niy}{niyz + 2 + niy},$$

where  $y \in \mathbb{R} \setminus \{0\}$  (see, for instance, [10] for a similar computation when  $a = 1$ ). For a Möbius map  $\psi(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$  we have

$$|\psi'(z)| = \frac{1 - |\alpha|^2}{|1 - \bar{\alpha}z|^2},$$

and in the case of  $\varphi_n$  we have  $\alpha = 1 + O(1/n)$ , so that  $\|\varphi'_n\|_\infty = O(n)$ .

Putting all this together we conclude that

$$\|h'_{(n)}\|_{\mathcal{C}} \leq Cn^2[(1 + \varepsilon)|h(a)]^n,$$

where  $C$  does not depend on  $n$ , and hence (since we have this for all  $\varepsilon > 0$ )

$$\limsup_{n \rightarrow \infty} \|T_{h_{(n)}}\|^{1/n} \leq |h(a)|.$$

Having bounded the spectral radius of  $W_{h,\varphi}$  by  $|h(a)|$ , we may similarly bound the spectral radius of its inverse by  $1/|h(a)|$ , using Proposition 3.1. Thus the spectrum of  $W_{h,\varphi}$  is contained in the circle of radius  $|h(a)|$ .

We now prove that the spectrum of  $W_{h,\varphi}$  is the entire circle, by showing that for each  $\lambda$  with  $|\lambda| = |h(a)|$  the spectral radius  $\rho(W_{h,\varphi} - \lambda I)$  is at least  $2|h(a)|$ . This technique was also used in [13], but there are extra complications in our case. As we already assumed, we may choose the fixed point  $a$  of  $\varphi$  to be  $-1$ .

There is a unitary mapping  $J$  from  $\mathcal{D} = \mathcal{D}(\mathbb{D})$  onto  $\mathcal{D}(\mathbb{C}_+)$ , the Dirichlet space of the right half-plane, induced by the conformal involution

$$M : z \mapsto \frac{1 - z}{1 + z};$$

an easy calculation shows that  $JW_{h,\varphi}J^{-1} = \tilde{W}_{h \circ M, \tau}$ , the weighted composition operator on  $\mathcal{D}(\mathbb{C}_+)$  induced by a parabolic automorphism  $\tau = M \circ \varphi \circ M$ , which takes the form  $\tau : s \rightarrow s + ic$  for some  $c > 0$  since it fixes  $\infty$ .

The following is a straightforward Corollary of [4, Thm. 3.2], given that the operator  $Z := (\tilde{W}_{h \circ M, \tau} - \lambda I)^n$  is causal (which may be most easily stated by saying that if the inverse Laplace transform of a function  $u$  is supported on  $(T, \infty)$  for some  $T > 0$ , then so is the inverse Laplace transform of its image  $Zu$ ).

- Let  $\tilde{h} : \mathbb{C}_+ \rightarrow \mathbb{C}$  be holomorphic and  $\tau : \mathbb{C}_+ \rightarrow \mathbb{C}_+$  a causal holomorphic function such that the weighted composition  $\tilde{W}_{\tilde{h}, \tau}$  is bounded on  $\mathcal{D}(\mathbb{C}_+)$ . Then for all  $\lambda \in \mathbb{C}$  and  $n \geq 1$  the inequality

$$\|(\tilde{W}_{\tilde{h}, \tau} - \lambda I)^n\|_{\mathcal{H}^2(\mathbb{C}_+)} \leq \|(\tilde{W}_{\tilde{h}, \tau} - \lambda I)^n\|_{\mathcal{D}(\mathbb{C}_+)}$$

holds, and hence  $\rho(\tilde{W}_{\tilde{h}, \tau} - \lambda I)_{\mathcal{H}^2(\mathbb{C}_+)} \leq \rho(\tilde{W}_{\tilde{h}, \tau} - \lambda I)_{\mathcal{D}(\mathbb{C}_+)}$ .

Now consider the operator  $\tilde{W}_{h \circ M, \tau}$  on  $\mathcal{H}^2(\mathbb{C}_+)$ , and note that it is unitarily equivalent to the weighted composition operator  $W_{\frac{1+\varphi}{1+z}h, \varphi}$  on  $\mathcal{H}^2(\mathbb{D})$  (see, for example, [14]). We then have

$$\begin{aligned} \rho(W_{h, \varphi} - \lambda I)_{\mathcal{D}} &= \rho(\tilde{W}_{h, \tau} - \lambda I)_{\mathcal{D}(\mathbb{C}_+)} \\ &\geq \rho(\tilde{W}_{h, \tau} - \lambda I)_{\mathcal{H}^2(\mathbb{C}_+)} \\ &= \rho\left(W_{\frac{1+\varphi}{1+z}h, \varphi} - \lambda I\right)_{\mathcal{H}^2(\mathbb{D})} \\ &\geq 2 \left| \frac{1 + \varphi(z)}{1 + z} \right|_{z=-1} |h(-1)| = 2|h(-1)|, \end{aligned}$$

where the last assertion is a direct consequence of an observation made in the proof of [13, Thm. 4.3] and the fact that

$$\frac{1 + \varphi(z)}{1 + z} \rightarrow 1 \quad \text{as } z \rightarrow -1,$$

since  $\varphi$  is parabolic. Since  $\rho(W_{h, \varphi} - \lambda I)_{\mathcal{D}} \geq 2|h(-1)|$ , we have

$$\sigma(W_{h, \varphi}) = \{\lambda \in \mathbb{C} : |\lambda| = |h(-1)|\},$$

as required. □

### 3.3 Hyperbolic case

The same method as we adopted for the parabolic case can be used to show that the spectrum of  $W_{h, \varphi}$  is contained in an annulus, in the case that  $\varphi$  is a hyperbolic automorphism.

**Theorem 3.3** *Suppose that  $\varphi$  is a hyperbolic automorphism of  $\mathbb{D}$  with attractive fixed point  $a \in \mathbb{T}$  and repelling fixed point  $b \in \mathbb{T}$ . Let  $W_{h, \varphi}$  be a weighted composition operator on  $\mathcal{D}$ , determined by an  $h \in \mathcal{M}(\mathcal{D})$  that is continuous at  $a$  and  $b$ . If  $W_{h, \varphi}$  is invertible, then*

$$\rho(W_{h, \varphi}) \leq \max\{|h(a)|, |h(b)|\} / \mu,$$

where  $\varphi$  is conjugate to the automorphism

$$\psi(z) = \frac{(1 + \mu)z + (1 - \mu)}{(1 - \mu)z + (1 + \mu)},$$

with  $0 < \mu < 1$ . Hence  $\sigma(W_{h, \varphi})$  is contained in the annulus with radii  $\max\{|h(a)|, |h(b)|\} / \mu$  and  $\min\{|h(a)|, |h(b)|\} \mu$ .

*Proof* For each  $\varepsilon > 0$  there is an  $m$  such that the estimate

$$\|h_{(n)}\|_\infty \leq \|h\|_\infty^m [(1 + \varepsilon) \max\{|h(a)|, |h(b)|\}]^{n-m}$$

holds, as in [13].

Similarly, for the derivative, we have

$$\|h'_{(n)}\|_{\mathcal{C}} \leq \sum_{j=0}^{n-1} \|h_{(n),j}(h \circ \varphi_j)'\|_{\mathcal{C}},$$

where  $h_{(n),j} = h_{(n)}/(h \circ \varphi_j)$ , and where  $\|\cdot\|_{\mathcal{C}}$  has been defined in the proof of Theorem 3.2. Now, however,

$$\|(h \circ \varphi_j)'\|_{\mathcal{C}} = \|\varphi'_j(h' \circ \varphi_j)\|_{\mathcal{C}} \leq \|\varphi'_j\|_\infty \|h'\|_{\mathcal{C}},$$

where  $\|\varphi'_j\|_\infty = O(\mu^{-j})$ , using the fact that

$$\psi_j(z) = \frac{(1 + \mu^j)z + (1 - \mu^j)}{(1 - \mu^j)z + (1 + \mu^j)},$$

as in [10, p. 38]. Also

$$\|h_{(n),j}\|_\infty \leq \|h\|_\infty^{m-1} [(1 + \varepsilon) \max\{|h(a)|, |h(b)|\}]^{n-m}$$

for  $j < m$  and

$$\|h_{(n),j}\|_\infty \leq \|h\|_\infty^m [(1 + \varepsilon) \max\{|h(a)|, |h(b)|\}]^{n-m-1}$$

for  $j \geq m$ .

By similar arguments to those used in the proof of Theorem 3.2, we conclude that

$$\limsup_{n \rightarrow \infty} \|T_{h_{(n)}}\|^{1/n} \leq \max\{|h(a)|, |h(b)|\}/\mu.$$

The final assertion follows on considering the spectral radius of  $W_{h,\varphi}^{-1}$ . □

**A final remark** Finally, regarding Theorem 3.3, we would like to pose the following open question:

*If  $\varphi$  is a hyperbolic automorphism of  $\mathbb{D}$  with attractive fixed point  $a \in \mathbb{T}$  and repelling fixed point  $b \in \mathbb{T}$  conjugated to the automorphism  $\psi(z) = \frac{(1+\mu)z+(1-\mu)}{(1-\mu)z+(1+\mu)}$ , with  $0 < \mu < 1$  and  $W_{h,\varphi}$  is a weighted composition operator on  $\mathcal{D}$ , determined by an  $h \in \mathcal{M}(\mathcal{D})$  continuous at both  $a$  and  $b$ , does it follow that*

$$\sigma(W_{h,\varphi}) = \{z \in \mathbb{C} : \min\{|h(a)|, |h(b)|\}\mu \leq |z| \leq \max\{|h(a)|, |h(b)|\}/\mu\}$$

*whenever  $W_{h,\varphi}$  is invertible?*

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