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Heat kernel on smooth metric measure spaces with nonnegative curvature

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Abstract We derive a local Gaussian upper bound for the f-heat kernel on complete smooth metric measure space $(M, g, e^{-f}dv)$ with nonnegative Bakry–Émery Ricci curvature. As applications, we obtain a sharp L^1_f-Liouville theorem for f-subharmonic functions and an L^1_f-uniqueness property for nonnegative solutions of the f-heat equation, assuming f is of at most quadratic growth. In particular, any L^1_f-integrable f-subharmonic function on gradient shrinking and steady Ricci solitons must be constant. We also provide explicit f-heat kernel for Gaussian solitons.

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1 Introduction and main results

In this paper we study Gaussian upper estimates for the f-heat kernel on smooth metric measure spaces with nonnegative Bakry–Émery Ricci curvature and their applications. Recall that a complete smooth metric measure space is a triple $(M, g, e^{-f}dv)$, where (M, g) is an n-dimensional complete Riemannian manifold, dv is the volume element of g, f is a smooth function on M, and $e^{-f}dv$ (for short, $d\mu$) is called the weighted volume element or the weighted measure. The m-Bakry–Émery Ricci curvature [1] associated to $(M, g, e^{-f}dv)$ is defined by

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$$\mathrm{Ric}_f^m = \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,$$

where Ric is the Ricci curvature of the manifold, ∇^2 is the Hessian with respect to the metric g and m is a constant. We refer the readers to [2,20–22] for further details. When $m = \infty$, we write $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty$. Smooth metric measure spaces are closely related to gradient Ricci solitons, the Ricci flow, probability theory, and optimal transport. A smooth metric measure space $(M, g, e^{-f} dv)$ is said to quasi-Einstein if

$$\mathrm{Ric}_f^m = \lambda g$$

for some constant λ. When $m = \infty$, it is exactly a gradient Ricci soliton. A gradient Ricci soliton is called expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. Ricci solitons are natural extensions of Einstein manifolds and have drawn more and more attentions. See [5] for a nice survey and references therein.

The associated f-Laplacian Δ_f on a smooth metric measure space is defined as

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to the weighted measure. On a smooth metric measure space, it is natural to consider the f-heat equation

$$(\partial_t - \Delta_f)u = 0$$

instead of the heat equation. If u is independent of time t, then u is a f-harmonic function. Throughout this paper we denote by $H(x, y, t)$ the f-heat kernel, that is, for each $y \in M$, $H(x, y, t) = u(x, t)$ is the minimal positive solution of the f-heat equation with $\lim_{t\to 0} u(x, t) = \delta_{f,y}(x)$, where $\delta_{f,y}(x)$ is defined by

$$\int_M \phi(x)\delta_{f,y}(x)e^{-f} dv = \phi(y)$$

for $\phi \in C_0^\infty(M)$. Equivalently, $H(x, y, t)$ is the kernel of the semigroup $P_t = e^{t\Delta_f}$ associated to the Dirichlet energy $\int_M |\nabla\phi|^2 e^{-f} dv$, where $\phi \in C_0^\infty(M)$. In general the f-heat kernel always exists on complete smooth metric measure spaces, but it may not be unique.

When f is constant, then $H(x, y, t)$ is just the heat kernel for the Riemannian manifold (M, g). Cheng et al. [10] obtained uniform Gaussian estimates for the heat kernel on Riemannian manifolds with sectional curvature bounded below, which was later extended by Cheeger et al. [9] to manifolds with bounded geometry. In 1986, Li and Yau [19] proved sharp Gaussian upper and lower bounds on Riemannian manifolds of nonnegative Ricci curvature, using the gradient estimate and the Harnack inequality. Grigor'yan and Saloff-Coste [13,27–29] independently proved similar estimates on Riemannian manifolds satisfying the volume doubling property and the Poincaré inequality, using the Moser iteration technique. Davies [12] further developed Gaussian upper bounds under a mean value property assumption. Recently, Li and Xu [16] also obtained some new estimates on complete Riemannian manifolds with Ricci curvature bounded from below by further improving the Li–Yau gradient estimate.

Recently, there have been several work on f-heat kernel estimates on smooth metric measure spaces and its applications. In [20], Li obtained Gaussian estimates for the f-heat kernel, and proved an L^1_f-Liouville theorem, assuming Ric_f^m $(m < \infty)$ bounded below by a negative quadratic function, which generalizes a classical result of Li [17]. He also mentioned that we may not be able to prove an L^1_f-Liouville theorem only assuming a lower bound on Ric_f. The main difficulty is the lack of effective upper bound for the f-heat kernel. In [8], by analyzing the heat kernel for a family of warped product manifolds, Charalambous and Lu also gave f-heat kernel estimates when Ric_f^m $(m < \infty)$ is bounded below. In [31], the first author proved f-heat kernel estimates assuming Ric_f bounded below by a negative constant and f bounded.

In this paper we prove a local Gaussian upper bound for the f-heat kernel on smooth metric measure spaces with $\mathrm{Ric}_f \geq 0$, which generalizes the classical result of Li and Yau [19].

Theorem 1.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Fix a fixed point $o \in M$ and $R > 0$. For any $\epsilon > 0$, there exist constants $c_1(n, \epsilon)$ and $c_2(n)$, such that*

$$H(x, y, t) \leq \frac{c_1(n, \epsilon)\, e^{c_2(n)A(R)}}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \times \exp\left(-\frac{d^2(x, y)}{(4+\epsilon)t}\right) \tag{1.1}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$, where $\lim_{\epsilon\to 0} c_1(n, \epsilon) = \infty$. In particular, there exist constants $c_3(n, \epsilon)$ and $c_4(n)$, such that

$$H(x, y, t) \leq \frac{c_3(n, \epsilon)\, e^{c_4(n)A(R)}}{V_f(B_x(\sqrt{t}))} \cdot \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{\frac{n}{2}} \times \exp\left(-\frac{d^2(x, y)}{(4+\epsilon)t}\right) \tag{1.2}$$

for any $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$, where $\lim_{\epsilon\to 0} c_3(n, \epsilon) = \infty$. Here $A(R) := \sup_{x\in B_o(3R)} |f(x)|$.

As pointed out by Munteanu and Wang [25], only assuming $\mathrm{Ric}_f \geq 0$ may not be sufficient to derive f-heat kernel estimates by classical Li–Yau gradient estimate procedure [19]. But we can derive a Gaussian upper bound using the De Giorgi–Nash–Moser theory and the weighted version of Davies's integral estimate [11].

For Gaussian solitons, the f-heat kernel can be solved explicitly in closed forms.

Example 1.2 f-heat kernel for steady Gaussian soliton.

Let $(\mathbb{R}, g_0, e^{-f}dx)$ be a 1-dimensional steady Gaussian soliton, where g_0 is the Euclidean metric and $f(x) = \pm x$. Then $\mathrm{Ric}_f = 0$. The heat kernel of the operator $\Delta_f = \frac{d^2}{dx^2} \mp \frac{d}{dx}$ is given by

$$H(x,y,t)=\frac{e^{\pm\frac{x+y}{2}}\cdot e^{-t/4}}{(4\pi t)^{1/2}}\times\exp\left(-\frac{|x-y|^2}{4t}\right).$$

This f-heat kernel is solved using the separation of variables method, since it seems not in the literature, for the sake of completeness we include it in the appendix.

Example 1.3 Mehler heat kernel [14] for shrinking Gaussian soliton.

Let $(\mathbb{R},\ g_0, e^{-f}dx)$ be a 1-dimensional shrinking Gaussian soliton, where g_0 is the Euclidean metric and $f(x)=x^2$. Then $\mathrm{Ric}_f=2$. The heat kernel of the operator $\Delta_f=\frac{d^2}{dx^2}-2x\frac{d}{dx}$ is given by

$$H(x,y,t)=\frac{1}{(2\pi\,\sinh 2t)^{1/2}}\times\exp\left(\frac{2xye^{-2t}-(x^2+y^2)e^{-4t}}{1-e^{-4t}}+t\right).$$

Example 1.4 Mehler heat kernel [14] for expanding Gaussian soliton.

Let $(\mathbb{R},\ g_0, e^{-f}dx)$ be a 1-dimensional expanding Gaussian soliton, where g_0 is the Euclidean metric and $f(x)=-x^2$. Then $\mathrm{Ric}_f=-2$. The heat kernel of the operator $\Delta_f=\frac{d^2}{dx^2}+2x\frac{d}{dx}$ is given by

$$H(x,y,t)=\frac{1}{(2\pi\,\sinh 2t)^{1/2}}\times\exp\left(\frac{2xye^{-2t}-(x^2+y^2)}{1-e^{-4t}}-t\right).$$

As applications, we prove an L^1_f-Liouville theorem on complete smooth metric measure spaces with $\mathrm{Ric}_f\geq 0$ and f to be of at most quadratic growth. We say $u\in L^p_f$, if $\int_M|u|^pe^{-f}dv<\infty$.

Theorem 1.5 *Let $(M,g,e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with* $\mathrm{Ric}_f\geq 0$. *Assume there exist nonnegative constants a and b such that*

$$|f|(x)\leq ar^2(x)+b\quad for\ all\ x\in M,$$

where $r(x)$ is the geodesic distance function to a fixed point $o\in M$. Then any nonnegative L^1_f-integrable f-subharmonic function must be identically constant. In particular, any L^1_f-integrable f-harmonic function must be identically constant.

From [6,15] any complete noncompact shrinking or steady gradient Ricci soliton satisfies the assumptions in Theorem 1.5. Hence

Corollary 1.6 *Let $(M,g,e^{-f}dv)$ be a complete noncompact gradient shrinking or steady Ricci soliton. Then any nonnegative L^1_f-integrable f-subharmonic function must be identically constant.*

Remark 1.7 Pigola et al. (see Corollary 23 in [26]) proved that on a complete gradient shrinking Ricci soliton, any locally Lipschitz f-subharmonic function $u\in L^p_f$, $1<p<\infty$, is constant. Our result shows that this is true in the case $p=1$. Brighton

[3], Cao and Zhou [6], Munteanu and Sesum [24], Munteanu and Wang [25], Wei and Wylie [30] have proved several similar results.

The growth condition of f in Theorem 1.5 is sharp as explained by the following simple example.

Example 1.8 Consider the 1-dimensional smooth metric measure space $(\mathbb{R},\ g_0, e^{-f}dx)$, where g_0 is the Euclidean metric and $f(x) = x^{2+2\delta}$, $\delta = \frac{1}{2m+1}$ for $m \in \mathbb{N}$. By direct computation, $\mathrm{Ric}_f \geq 0$. Let

$$u(x) := \int_0^{|x|} e^{t^{2+2\delta}} dt.$$

Then u is f-harmonic. Moreover we claim $u \in L^1(\mu)$. Indeed, the integration by parts implies the identity

$$\int_1^x e^{t^{2+2\delta}} dt = \frac{1}{2+2\delta}\left[\frac{e^{x^{2+2\delta}}}{x^{1+2\delta}} - e + (1+2\delta)\int_1^x \frac{e^{t^{2+2\delta}}}{t^{2+2\delta}} dt\right].$$

Then by L'Hospital rule, when x is large enough,

$$\int_1^x e^{t^{2+2\delta}} dt = \frac{1}{2+2\delta}\frac{e^{x^{2+2\delta}}}{x^{1+2\delta}}(1+o(1)).$$

Therefore

$$\int_{\mathbb{R}} u e^{-f} dx = \int_{-\infty}^{\infty}\left(\int_0^{|x|} e^{t^{2+2\delta}} dt\right) e^{-x^{2+2\delta}} dx < \infty,$$

i.e., $u \in L^1_f$, but $u \notin L^p_f$ for any $p > 1$. On the other hand, if $\delta = 0$ then $u \notin L^1_f$.

By Theorem 1.5, we prove a uniqueness theorem for L^1_f-solutions of the f-heat equation, which generalizes the classical result of Li [17].

Theorem 1.9 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with* $\mathrm{Ric}_f \geq 0$. *Assume there exist nonnegative constants a and b such that*

$$|f|(x) \leq ar^2(x) + b \quad for\ all\ x \in M,$$

where $r(x)$ is the distance function to a fixed point $o \in M$. If $u(x,t)$ is a nonnegative function defined on $M \times [0, +\infty)$ satisfying

$$(\partial_t - \Delta_f)u(x,t) \leq 0, \quad \int_M u(x,t)e^{-f} dv < +\infty$$

for all $t > 0$, *and*

$$\lim_{t\to 0}\int_M u(x,t)e^{-f}dv = 0,$$

then $u(x,t) \equiv 0$ *for all* $x \in M$ *and* $t \in (0,+\infty)$. *In particular, any* L^1_f*-solution of the* f*-heat equation is uniquely determined by its initial data in* L^1_f.

The rest of the paper is organized as follows. In Sect. 2, we provide a relative f-volume comparison theorem for smooth metric measure spaces with nonnegative Bakry–Émery Ricci curvature. Using the comparison theorem, we derive a local f-volume doubling property, a local f-Neumann Poincaré inequality, a local Sobolev inequality, and a f-mean value inequality. In Sect. 3, we prove local Gaussian upper bounds of the f-heat kernel by applying the mean value inequality. In Sects. 4 and 5, we prove the L^1_f-Liouville theorem for f-subharmonic functions and the L^1_f-uniqueness property for nonnegative solutions of the f-heat equation following the argument of Li in [17]. In appendix, we compute the f-heat kernel of 1-dimensional steady Gaussian soliton.

2 Poincaré, Sobolev and mean value inequalities

Let $\Delta_f = \Delta - \nabla f\cdot\nabla$ be the f-Laplacian on a complete smooth metric measure space $(M, g, e^{-f}dv)$. Throughout this section, we will assume

$$\mathrm{Ric}_f \geq 0.$$

For a fixed point $o \in M$ and $R > 0$, we denote

$$A(R) = \sup_{x\in B_o(3R)} |f(x)|.$$

We often write A for short. First we have the relative f-volume comparison theorem proved by Wei and Wylie [30] and Munteanu and Wang [25].

Lemma 2.1 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space. If* $\mathrm{Ric}_f \geq 0$, *then for any* $x \in B_o(R)$,

$$\frac{V_f(B_x(R_1,R_2))}{V_f(B_x(r_1,r_2))} \leq e^{4A}\frac{R_2^n - R_1^n}{r_2^n - r_1^n}, \tag{2.1}$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2 < R$, $r_1 \leq R_1$, $r_2 \leq R_2 < R$, *where* $B_x(R_1,R_2) := B_x(R_2)\backslash B_x(R_1)$.

Proof of Lemma 2.1 Wei and Wylie (see (3.19) in [30]) proved the following f-mean curvature comparison theorem. Recall that the weighted mean curvature $m_f(r)$ is defined as $m_f(r) = m(r) - \nabla f\cdot\nabla r = \Delta_f r$. For any $x \in B_o(R) \subset M$, if $\mathrm{Ric}_f \geq 0$, then

$$m_f(r) \leq \frac{n-1}{r} + \frac{2}{r^2}\int_0^r f(t)dt - \frac{2}{r}f(r),$$

along any minimal geodesic segment from x.

In geodesic polar coordinates, the volume element is written as $dv = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element of the unit sphere S^{n-1}. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$. By the first variation of the area,

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = (\ln(\mathcal{A}(r,\theta)))' = m(r,\theta).$$

Therefore

$$\frac{\mathcal{A}'_f}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta).$$

For $0 < r_1 < r_2 < R$, integrating this from r_1 to r_2 we get

$$\begin{aligned}\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)} &= \exp\left(\int_{r_1}^{r_2} m_f(s,\theta)ds\right)\\ &\le \left(\frac{r_2}{r_1}\right)^{n-1} \exp\left[2\int_{r_1}^{r_2}\frac{1}{r^2}\left(\int_0^r f(t)dt\right)dr - 2\int_{r_1}^{r_2}\frac{f(r)}{r}dr\right].\end{aligned}$$

Since

$$\int_{r_1}^{r_2}\frac{1}{r^2}\left(\int_0^r f(t)dt\right)dr = -\frac{1}{r}\left(\int_0^r f(t)dt\right)\Bigg|_{r_1}^{r_2} + \int_{r_1}^{r_2}\frac{f(r)}{r}dr,$$

then we have

$$\frac{\mathcal{A}_f(r_2,\theta)\cdot\exp\left(\frac{2}{r_2}\int_0^{r_2} f(t)dt\right)}{\mathcal{A}_f(r_1,\theta)\cdot\exp\left(\frac{2}{r_1}\int_0^{r_1} f(t)dt\right)} \le \left(\frac{r_2}{r_1}\right)^{n-1}$$

for $0 < r_1 < r_2 < R$. That is $r^{1-n}\mathcal{A}_f(r,\theta)\exp(\frac{2}{r}\int_0^r f(t)dt)$ is nonincreasing in r. Applying Lemma 3.2 in [32], we get

$$\frac{\int_{R_1}^{R_2}\mathcal{A}_f(r,\theta)\cdot\exp\left(\frac{2}{t}\int_0^t f(\tau)d\tau\right)dt}{\int_{r_1}^{r_2}\mathcal{A}_f(r,\theta)\cdot\exp\left(\frac{2}{t}\int_0^t f(\tau)d\tau\right)dt} \le \frac{\int_{R_1}^{R_2} t^{n-1}dt}{\int_{r_1}^{r_2} t^{n-1}dt}$$

for $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \le R_1$ and $r_2 \le R_2 < R$. Integrating along the sphere direction gives

$$\frac{V_f(B_x(R_1,R_2))}{V_f(B_x(r_1,r_2))} \le e^{4A}\frac{R_2^n - R_1^n}{r_2^n - r_1^n},$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2 < R$, $r_1 \le R_1$, $r_2 \le R_2 < R$, where $B_x(R_1,R_2) = B_x(R_2)\backslash B_x(R_1)$. □

From (2.1), letting $r_1 = R_1 = 0$, $r_2 = r$ and $R_2 = 2r$, we get

$$V_f(B_x(2r)) \leq 2^n e^{4A} \cdot V_f(B_x(r)) \tag{2.2}$$

for any $0 < r < R/2$. This inequality implies that the local f-volume doubling property holds. This property will play a crucial role in our paper. We say that a complete smooth metric measure space $(M, g, e^{-f}dv)$ satisfies the local f-volume doubling property if for any $0 < R < \infty$, there exists a constant $C(R)$ such that

$$V_f(B_x(2r)) \leq C(R) \cdot V_f(B_x(r))$$

for any $0 < r < R$ and $x \in M$. Note that when the above inequality holds with $R = \infty$, then it is called the global f-volume doubling property.

From Lemma 2.1, we have

Lemma 2.2 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. If $\mathrm{Ric}_f \geq 0$, then*

$$\frac{V_f(B_x(s))}{V_f(B_y(r))} \leq 4^n e^{8A} \left(\frac{s}{r}\right)^{\kappa},$$

where $\kappa = \log_2(2^n e^{4A})$, for any $0 < r < s < R/4$ and all $x \in B_o(s)$ and $y \in B_x(s)$. Moreover, we have

$$V_f(B_x(r)) \leq e^{4A} \left(\frac{d(x, y)}{r} + 1\right)^n V_f(B_y(r))$$

for any $x, y \in B_o(\frac{1}{4}R)$ and $0 < r < R/2$.

Proof Choose a real number k such that $2^k < s/r \leq 2^{k+1}$. Since $y \in B_x(s)$,

$$B_x(s) \subset B_y(2s) \subset B_y(2^{k+2}r),$$

and $V_f(B_x(s)) \leq V_f(B_y(2^{k+2}r))$. Moreover, the assumption $\mathrm{Ric}_f \geq 0$ implies the local f-volume doubling property (2.2). So we have

$$V_f(B_x(s)) \leq (2^n e^{4A})^{k+2} V_f(B_y(r)) \leq (2^n e^{4A})^2 (s/r)^{\kappa} V_f(B_y(r)),$$

where $\kappa = \log_2(2^n e^{4A})$. This proves the first part of the lemma.

For the second part, letting $r_1 = 0$, $r_2 = r$, $R_1 = d(x, y) - r$ and $R_2 = d(x, y) + r$ in Lemma 2.1, we have

$$\frac{V_f(B_x(d(x, y) + r)) - V_f(B_x(d(x, y) - r))}{V_f(B_x(r))} \leq e^{4A} \left(\frac{d(x, y)}{r} + 1\right)^n$$

for any $x, y \in B_o(\frac{1}{4}R)$ and $0 < r < R/2$. Therefore we get

$$\begin{aligned} V_f(B_x(r)) &\le V_f(B_y(d(x,y)+r)) - V_f(B_y(d(x,y)-r)) \\ &\le e^{4A}\left(\frac{d(x,y)}{r}+1\right)^n V_f(B_y(r)) \end{aligned}$$

for any $x, y \in B_o(\frac{1}{4}R)$ and $0 < r < R/2$. □

By Lemma 2.1, following Buser's proof [4] or Saloff-Coste's alternate proof (Theorem 5.6.5 in [29]), we get a local Neumann Poincaré inequality on smooth metric measure spaces, see also Munteanu and Wang (see Lemma 3.1 in [25]).

Lemma 2.3 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge 0$. Then for any $x \in B_o(R)$,*

$$\int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 e^{-f} dv \le c_1 e^{c_2 A} \cdot r^2 \int_{B_x(r)} |\nabla\varphi|^2 e^{-f} dv \tag{2.3}$$

for all $0 < r < R$ and $\varphi \in C^\infty(B_x(r))$, where $\varphi_{B_x(r)} = V_f^{-1}(B_x(r)) \int_{B_x(r)} \varphi e^{-f} dv$. The constants c_1 and c_2 depend only on n.

Remark 2.4 When f is constant, this was classical result of Saloff-Coste (see (6) in [28] or Theorem 5.6.5 in [29]).

Combining Lemmas 2.1, 2.2, 2.3 and the argument in [27], we obtain a local Sobolev inequality.

Lemma 2.5 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space with $\mathrm{Ric}_f \ge 0$. Then there exist constants $p > 2$, c_3 and c_4, all depending only on n such that*

$$\left(\int_{B_o(r)} |\varphi|^{\frac{2p}{p-2}} e^{-f} dv\right)^{\frac{p-2}{p}} \le \frac{c_3 e^{c_4 A} \cdot r^2}{V_f(B_o(r))^{\frac{2}{p}}} \int_{B_o(r)} (|\nabla\varphi|^2 + r^{-2}|\varphi|^2) e^{-f} dv \tag{2.4}$$

for any $x \in M$ such that $0 < r(x) < R$ and $\varphi \in C^\infty(B_o(r))$.

Sketch proof of Lemma 2.5 The proof is essentially a weighted version of Theorem 2.1 in [27] (see also Theorem 3.1 in [28]).

Besides, we have an alternate proof by applying the local Neumann Sobolev inequality of Munteanu and Wang (see Lemma 3.2 in [25])

$$\|\varphi - \varphi_{B_o(r)}\|_{\frac{2p}{p-2}} \le \frac{c_3 e^{c_4 A} \cdot r}{V_f(B_o(r))^{\frac{1}{p}}} \|\nabla\varphi\|_2,$$

where $\|f\|_m = (\int_{B_o(r)} |f|^m d\mu)^{1/m}$. Munteanu and Wang proved this inequality holds without the weighted measure, and it is still true by checking their proof when integrals are with respect to the weighted volume element $e^{-f}dv$. Combining this with the Minkowski inequality

$$\|\varphi\|_{\frac{2p}{p-2}} \le \|\varphi - \varphi_{B_o(r)}\|_{\frac{2p}{p-2}} + \|\varphi_{B_o(r)}\|_{\frac{2p}{p-2}},$$

it is sufficient to prove

$$\|\varphi_{B_o(r)}\|_{\frac{2p}{p-2}} \le \frac{c_3 e^{c_4 A}}{V_f(B_o(r))^{\frac{1}{p}}}\|\varphi\|_2,$$

which follows from Cauchy–Schwarz inequality. Hence the lemma follows. □

Lemma 2.5 is a critical step in proving the Harnack inequality by the Moser iteration technique [23]. We apply it to prove a local mean value inequality for the f-heat equation, which is similar to the case when f is constant, obtained by Saloff-Coste [27] and Grigor'yan [13].

Proposition 2.6 *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional complete noncompact smooth metric measure space. Fix* $R > 0$. *Assume that* (2.4) *holds up to* R. *Then there exist constants* $c_5(n, p)$ *and* $c_6(n, p)$ *such that, for any real* s, *for any* $0 < \delta < \delta' \le 1$, *and for any smooth positive solution* u *of the* f*-heat equation in the cylinder* $Q = B_o(r) \times (s - r^2, s)$, $r < R$, *we have*

$$\sup_{Q_\delta}\{u\} \le \frac{c_5 e^{c_6 A}}{(\delta' - \delta)^{2+p}\, r^2\, V_f(B_o(r))} \cdot \int_{Q_{\delta'}} u\, d\mu\, dt, \tag{2.5}$$

where $Q_\delta = B_o(\delta r) \times (s - \delta r^2, s)$ *and* $Q_{\delta'} = B_o(\delta' r) \times (s - \delta' r^2, s)$.

Proof The proof is analogous to Theorem 5.2.9 in [29]. For the readers convenience, we provide a detailed proof. We need to carefully examine the explicit coefficients of the mean value inequality in terms of the Sobolev constants in (2.4).

Without loss of generality we assume $\delta' = 1$. For any nonnegative function $\phi \in C_0^\infty(B)$, $B = B_o(r)$, we have

$$\int_B (\phi u_t + \nabla\phi\nabla u)d\mu = 0.$$

Let $\phi = \psi^2 u$, $\psi \in C_0^\infty(B)$, then

$$\begin{aligned}\int_B (\psi^2 u u_t + \psi^2|\nabla u|^2)d\mu &\le 2\left|\int_B u\psi\nabla u\nabla\psi\, d\mu\right|\\ &\le 3\int_B |\nabla\psi|^2 u^2 d\mu + \frac{1}{3}\int_B \psi^2|\nabla u|^2 d\mu,\end{aligned}$$

so we get that

$$\int_B (2\psi^2 u u_t + |\nabla(\psi u)|^2) d\mu \leq 10\|\nabla\psi\|_\infty^2 \int_{\mathrm{supp}(\psi)} u^2 d\mu.$$

Multiplying a smooth function $\lambda(t)$, which will be determined later, from the above inequality, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_B (\lambda\psi u)^2 d\mu\right) + \lambda^2 \int_B |\nabla(\psi u)|^2 d\mu \\ \leq C\lambda(\lambda\|\nabla\psi\|_\infty^2 + |\lambda'| \sup \psi^2) \int_{\mathrm{supp}(\psi)} u^2 d\mu,\end{aligned}$$

where C is a finite constant which will change from line to line in the following inequalities.

Next we choose ψ and λ such that, for any $0 < \sigma' < \sigma < 1$, $\kappa = \sigma - \sigma'$,

(1) $0 \leq \psi \leq 1$, $\mathrm{supp}(\psi) \subset \sigma B$, $\psi = 1$ in $\sigma' B$ and $|\nabla\psi| \leq 2(\kappa r)^{-1}$;
(2) $0 \leq \lambda \leq 1$, $\lambda = 0$ in $(-\infty, s - \sigma r^2)$, $\lambda = 1$ in $(s - \sigma' r^2, +\infty)$, and $|\lambda'(t)| \leq 2(\kappa r)^{-2}$.

Let $I_\sigma = (s - \sigma r^2, s)$ and $I'_\sigma = (s - \sigma' r^2, s)$. For any $t \in I_{\sigma'}$, integrating the above inequality over $(s - r^2, t)$,

$$\sup_{I_{\sigma'}} \left\{\int_B \psi u^2 d\mu\right\} + \int_{B\times I_{\sigma'}} |\nabla(\psi u)|^2 d\mu dt \leq C\,(r\kappa)^{-2} \int_{Q_\sigma} u^2 d\mu dt. \tag{2.6}$$

On the other hand, by the Hölder inequality and the assumption of proposition, for some $p > 2$, we have

$$\begin{aligned}\int_B \varphi^{2(1+\frac{2}{p})} d\mu &\leq \left(\int_B |\varphi|^{\frac{2p}{p-2}} d\mu\right)^{\frac{p-2}{p}} \cdot \left(\int_B \varphi^2 d\mu\right)^{\frac{2}{p}} \\ &\leq \left(\int_B \varphi^2 d\mu\right)^{\frac{2}{p}} \cdot \left(E(B) \int_B (|\nabla\varphi|^2 + r^{-2}|\varphi|^2) d\mu\right)\end{aligned} \tag{2.7}$$

for all $\varphi \in C_0^\infty(B)$, where $E(B) = c_3 e^{c_4 A} r^2 V_f(B_o(r))^{-2/p}$. Combining (2.6) and (2.7), we get

$$\int_{Q_{\sigma'}} u^{2\theta} d\mu dt \leq E(B)\left[C(r\kappa)^{-2} \int_{Q_\sigma} u^2 d\mu dt\right]^\theta$$

with $\theta = 1 + 2/p$. For any $m \geq 1$, u^m is also a smooth positive solution of $(\partial_t - \Delta_f)u(x,t) \leq 0$. Hence the above inequality indeed implies

$$\int_{Q_{\sigma'}} u^{2m\theta} d\mu dt \leq E(B)\left[C(r\kappa)^{-2} \int_{Q_\sigma} u^{2m} d\mu dt\right]^\theta \tag{2.8}$$

for $m \geq 1$.

Let $\kappa_i = (1-\delta)2^{-i}$, which satisfies $\Sigma_1^{\infty}\kappa_i = 1-\delta$. Let $\sigma_0 = 1, \sigma_{i+1} = \sigma_i - \kappa_i = 1 - \Sigma_1^i \kappa_j$. Applying (2.8) for $m = \theta^i$, $\sigma = \sigma_i$, $\sigma' = \sigma_{i+1}$, we have

$$\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}} d\mu dt \leq E(B)\left[C^{i+1}((1-\delta)r)^{-2}\int_{Q_{\sigma_i}} u^{2\theta^i} d\mu dt\right]^{\theta}.$$

Therefore

$$\left(\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}} d\mu dt\right)^{\theta^{-(i+1)}} \leq C^{\Sigma j\theta^{1-j}} \cdot E(B)^{\Sigma\theta^{-j}} \cdot [(1-\delta)r]^{-2\Sigma\theta^{1-j}} \int_Q u^2 d\mu dt,$$

where Σ denotes the summations from 1 to $i+1$. Letting $i \to \infty$ we get

$$\sup_{Q_\delta}\{u^2\} \leq C \cdot E(B)^{p/2} \cdot [(1-\delta)r]^{-2-p} ||u||^2_{2,Q} \tag{2.9}$$

for some $p > 2$.

Formula (2.9) in fact is a L^2_f-mean value inequality. Next, we will apply (2.9) to prove (2.5) by a different iterative argument. Let $\sigma \in (0,1)$ and $\rho = \sigma + (1-\sigma)/4$. Then (2.9) implies

$$\sup_{Q_\sigma}\{u\} \leq F(B) \cdot (1-\sigma)^{-1-p/2} ||u||_{2,Q_\rho},$$

where $F(B) = c_3 e^{c_4 A} \cdot r^{-1} \cdot V_f(B_o(r))^{-1/2}$. Since

$$\|u\|_{2,Q} \leq \|u\|^{1/2}_{\infty,Q} \cdot \|u\|^{1/2}_{1,Q}$$

for any Q, so we have

$$\|u\|_{\infty,Q_\sigma} \leq F(B) \cdot \|u\|^{1/2}_{1,Q} \cdot (1-\sigma)^{-1-p/2} \|u\|^{1/2}_{\infty,Q_\rho}. \tag{2.10}$$

Now fix $\delta \in (0,1)$ and let $\sigma_0 = \delta$, $\sigma_{i+1} = \sigma_i + (1-\sigma_i)/4$, which satisfy $1-\sigma_i = (3/4)^i(1-\delta)$. Applying (2.10) to $\sigma = \sigma_i$ and $\rho = \sigma_{i+1}$, we have

$$\|u\|_{\infty,Q_{\sigma_i}} \leq (4/3)^{(1+p/2)i} F(B) \cdot \|u\|^{1/2}_{1,Q} \cdot (1-\delta)^{-1-p/2} \|u\|^{1/2}_{\infty,Q_{\sigma_{i+1}}}.$$

Therefore, for any i,

$$\|u\|_{\infty,Q_\delta} \leq (4/3)^{(1+p/2)\Sigma j(\frac{1}{2})^j} \times [F(B) \cdot \|u\|^{1/2}_{1,Q} \cdot (1-\delta)^{-1-p/2}]^{\Sigma(\frac{1}{2})^j} \|u\|^{(\frac{1}{2})^i}_{\infty,Q_{\sigma_i}},$$

where Σ denotes the summations from 0 to $i-1$. Letting $i \to \infty$ we get

$$\|u\|_{\infty,Q_\delta} \leq (4/3)^{(2+p)} [F(B) \cdot \|u\|^{1/2}_{1,Q} \cdot (1-\delta)^{-1-p/2}]^2,$$

that is,

$$\|u\|_{\infty,Q_\delta} \le (4/3)^{(2+p)} c_5 e^{c_6 A} (1-\delta)^{-2-p} \cdot r^{-2} \cdot V_f(B_o(r))^{-1} \cdot \|u\|_{1,Q}$$

and the proposition follows. □

3 Gaussian upper bounds of the f-heat kernel

In this section, we prove Gaussian upper bounds of the f-heat kernel on smooth metric measure spaces with nonnegative Bakry–Émery Ricci curvature by applying Proposition 2.6 and Lemma 2.2. To prove Theorem 1.1, first we need a weighted version of Davies' integral estimate [11].

Lemma 3.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Let $\lambda_1(M) \ge 0$ be the bottom of the L^2_f-spectrum of the f-Laplacian on M. Assume that B_1 and B_2 are bounded subsets of M. Then*

$$\begin{aligned}
&\int_{B_1}\int_{B_2} H(x,y,t)d\mu(x)d\mu(y)\\
&\quad \le V_f(B_1)^{1/2} V_f(B_2)^{1/2} \exp\left(-\lambda_1(M)t - \frac{d^2(B_1,B_2)}{4t}\right),
\end{aligned} \tag{3.1}$$

where $d(B_1, B_2)$ denotes the distance between the sets B_1 and B_2.

Proof of Lemma 3.1 By the approximation argument, it suffices to prove (3.1) for the f-heat kernel H_Ω of any compact set with boundary Ω containing B_1 and B_2. In fact, let Ω_i be a sequence of compact exhaustion of M such that $\Omega_i \subset \Omega_{i+1}$ and $\cup_i \Omega_i = M$. If we prove (3.1) for the f-heat kernel H_{Ω_i} for any i, then the lemma follows by letting $i \to \infty$ and observing that $\lambda_1(\Omega_i) \to \lambda_1(M)$, where $\lambda_1(\Omega_i) > 0$ is the first Dirichlet eigenvalue of the f-Laplacian on Ω_i, and $\lambda_1(M) = \inf_{\Omega_i \subset M} \lambda_1(\Omega_i)$.

We consider the function $u(x,t) = e^{t\Delta_f|_\Omega}\mathbf{1}_{B_1}$ with Dirichlet boundary condition: $u(x,t) = 0$ on $\partial\Omega$. Then

$$\begin{aligned}
\int_{B_2}\int_{B_1} H_\Omega(x,y,t)d\mu(y)d\mu(x) &= \int_{B_2}\left(\int_\Omega H_\Omega(x,y,t)\mathbf{1}_{B_1}d\mu(y)\right)d\mu(x)\\
&= \int_{B_2} u(x,t)d\mu(x)\\
&\le V_f(B_2)^{1/2}\cdot\left(\int_{B_2} u^2(x,t)d\mu(x)\right)^{1/2}.
\end{aligned} \tag{3.2}$$

For some $\alpha > 0$, we define $\xi(x,t) = \alpha d(x, B_1) - \frac{\alpha^2}{2}t$ and consider the function

$$J(t) := \int_\Omega u^2(x,t)e^{\xi(x,t)}d\mu(x).$$

Claim: Function $J(t)$ satisfies

$$J(t) \leq J(t_0) \cdot \exp(-2\lambda_1(\Omega)(t-t_0)) \tag{3.3}$$

for all $0 < t_0 \leq t$.

This claim will be proved later. We now continue to prove Lemma 3.1. If $x \in B_2$, then $\xi(x,t) \geq \alpha d(B_2, B_1) - \frac{\alpha^2}{2}t$. Hence

$$\begin{aligned} J(t) &\geq \int_{B_2} u^2(x,t)e^{\xi(x,t)}d\mu(x) \\ &\geq \exp\left(\alpha d(B_2, B_1) - \frac{\alpha^2}{2}t\right)\int_{B_2} u^2(x,t)d\mu(x). \end{aligned} \tag{3.4}$$

On the other hand, if $x \in B_1$ then $\xi(x,0) = 0$. Using (3.3) and the continuity of $J(t)$ at $t = 0^+$, we have

$$\begin{aligned} J(t) &\leq J(0) \cdot \exp(-2\lambda_1(\Omega)t) \\ &= \int_\Omega e^{\xi(x,0)}\mathbf{1}_{B_1}d\mu(x) \cdot \exp(-2\lambda_1(\Omega)t) \\ &= V_f(B_1) \cdot \exp(-2\lambda_1(\Omega)t) \end{aligned} \tag{3.5}$$

Combining (3.2), (3.4) and (3.5), and choosing $\alpha = d(B_1, B_2)/t$, we get

$$\int_{B_1}\int_{B_2} H_\Omega(x,y,t)d\mu(x)d\mu(y) \leq V_f(B_1)^{1/2}V_f(B_2)^{1/2}\exp\left(-\lambda_1(\Omega)t - \frac{d^2(B_1,B_2)}{4t}\right)$$

for any compact set $\Omega \subset M$. Lemma 3.1 is proved. □

Proof of the claim. Since $\xi_t \leq -\frac{1}{2}|\nabla\xi|^2$ and $u_t = \Delta_f u$, we compute directly

$$\begin{aligned} J'(t) &\leq 2\int_\Omega u\Delta_f u e^\xi d\mu(x) - \frac{1}{2}\int_\Omega u^2 e^\xi|\nabla\xi|^2 d\mu(x) \\ &= -2\int_\Omega |\nabla u|^2 e^\xi d\mu(x) - 2\int_\Omega u\langle\nabla u, \nabla\xi\rangle e^\xi d\mu(x) - \frac{1}{2}\int_\Omega u^2 e^\xi|\nabla\xi|^2 d\mu(x) \\ &= -2\int_\Omega (u\nabla\xi + 2\nabla u)^2 e^\xi d\mu(x) \\ &= -2\int_\Omega |\nabla(ue^{\xi/2})|^2 d\mu(x). \end{aligned} \tag{3.6}$$

Moreover the definition of $\lambda_1(\Omega)$ implies

$$\int_\Omega |\nabla(ue^{\xi/2})|^2 d\mu(x) \geq \lambda_1(\Omega)\int_\Omega |ue^{\xi/2}|^2 d\mu(x) = \lambda_1(\Omega)J(t).$$

Substituting this into (3.6) we get $J'(t) \leq -2\lambda_1(\Omega)J(t)$ and the claim is proved. □

Now we prove the upper bounds of f-heat kernel by modifying the argument of [12] (see also [18]).

Proof of Theorem 1.1 We denote $u : (y,s) \mapsto H(x,y,s)$ be a f-heat kernel. Under the assumption $t \geq r_2^2$, applying u to Proposition 2.6 with a fixed $x \in B_o(R/2)$, we have

$$\begin{aligned}\sup_{(y,s)\in Q_\delta} H(x,y,s) &\leq \frac{c_5 e^{c_6 A}}{r_2^2 V_f(B_2)} \cdot \int_{t-1/4r_2^2}^{t} \int_{B_2} H(x,\zeta,s)d\mu(\zeta)ds \\ &= \frac{c_5 e^{c_6 A}}{4V_f(B_2)} \cdot \int_{B_2} H(x,\zeta,s')d\mu(\zeta) \end{aligned} \tag{3.7}$$

for some $s' \in (t - 1/4r_2^2, t)$, where $Q_\delta = B_y(\delta r_2) \times (t - \delta r_2^2, t)$ with $0 < \delta < 1/4$, and $B_2 = B_y(r_2) \subset B_o(R)$ for $y \in B_o(R/2)$. Applying Proposition 2.6 and the same argument to the positive solution

$$v(x,s) = \int_{B_2} H(x,\zeta,s)d\mu(\zeta)$$

of the f-heat equation, for the variable x with $t \geq r_1^2$, we also have

$$\begin{aligned}\sup_{(x,s)\in \overline{Q}_\delta} \int_{B_2} H(x,\zeta,s)d\mu(\zeta) &\leq \frac{c_5 e^{c_6 A}}{r_1^2 V_f(B_1)} \cdot \int_{t-1/4r_1^2}^{t} \int_{B_1}\int_{B_2} H(\xi,\zeta,s)d\mu(\zeta)d\mu(\xi)ds \\ &= \frac{c_5 e^{c_6 A}}{4V_f(B_1)} \cdot \int_{B_1}\int_{B_2} H(\xi,\zeta,s'')d\mu(\zeta)d\mu(\xi) \end{aligned} \tag{3.8}$$

for some $s'' \in (t-1/4r_1^2, t)$, where $\overline{Q}_\delta = B_x(\delta r_1)\times(t-\delta r_1^2, t)$ with $0 < \delta < 1/4$, and $B_1 = B_x(r_1) \subset B_o(R)$ for $x \in B_o(R/2)$. Now letting $r_1 = r_2 = \sqrt{t}$ and combining (3.7) with (3.8), the f-heat kernel satisfies

$$H(x,y,t) \leq \frac{(c_5 e^{c_6 A})^2}{V_f(B_1)V_f(B_2)} \cdot \int_{B_1}\int_{B_2} H(\xi,\zeta,s'')d\mu(\zeta)d\mu(\xi) \tag{3.9}$$

for all $x, y \in B_o(R/2)$ and $0 < t < R^2/4$. Using Lemma 3.1 and noticing that $s'' \in (\frac{3}{4}t, t)$, (3.9) becomes

$$H(x,y,t) \leq \frac{(c_5 e^{c_6 A})^2}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \times \exp\left(-\frac{3}{4}\lambda_1 t - \frac{d^2(B_1,B_2)}{4t}\right) \tag{3.10}$$

for all $x, y \in B_o(R/2)$ and $0 < t < R^2/4$. Notice that if $d(x,y) \leq 2\sqrt{t}$, then $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0$ and hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = 0 \leq 1 - \frac{d^2(x,y)}{4t},$$

and if $d(x, y) > 2\sqrt{t}$, then $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = d(x, y) - 2\sqrt{t}$, and hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4t} \le -\frac{d^2(x, y)}{4(1+\epsilon)t} + C(\epsilon)$$

for some constant $C(\epsilon)$, where $\epsilon > 0$, and if $\epsilon \to 0$, then the constant $C(\epsilon) \to \infty$. Therefore, by (3.10) we have

$$H(x, y, t) \le \frac{c_7(n, \epsilon)e^{2c_6 A}}{V_f(B_x(\sqrt{t}))^{1/2} V_f(B_y(\sqrt{t}))^{1/2}} \times \exp\left(-\frac{3}{4}\lambda_1 t - \frac{d^2(x, y)}{4(1+\epsilon)t}\right) \quad (3.11)$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$. Recall that by Lemma 2.2

$$V_f(B_x(\sqrt{t})) \le e^{4A}\left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^n V_f(B_y(\sqrt{t}))$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < R^2/4$. Therefore we get

$$H(x, y, t) \le \frac{c_7(n, \epsilon)e^{(2c_6+2)A}}{V_f(B_x(\sqrt{t})} \cdot \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{\frac{n}{2}} \times \exp\left(-\frac{3}{4}\lambda_1 t - \frac{d^2(x, y)}{(4+\epsilon)t}\right)$$

for all $x, y \in B_o(\frac{1}{4}R)$ and $0 < t < R^2/4$. □

4 L^1_f-Liouville theorem

In this section, we will prove L^1_f-Liouville theorem on complete noncompact smooth metric measure spaces by using the f-heat kernel estimates proved in Sect. 3. Our result extends the classical L^1-Liouville theorem obtained by Li [17] and the weighted versions proved by Li [20] and the first author [31].

We start from a useful lemma.

Lemma 4.1 *Under the same assumption as in Theorem 1.5, then the complete smooth metric measure space* $(M, g, e^{-f}dv)$ *is stochastically complete, i.e.,*

$$\int_M H(x, y, t)e^{-f}dv(y) = 1.$$

Proof In Lemma 2.1, letting $r_1 = R_1 = 0$, $r_2 = 1$, $R_2 = R > 1$ and $x = o$, if $|f|(x) \le ar^2(x) + b$, then

$$V_f(B_o(R)) \le C(n, b)R^n e^{c(n,a)R^2}$$

for all $R > 1$. Hence

$$\int_1^\infty \frac{R}{\log V_f(B_o(R))} dR = \infty. \quad (4.1)$$

By Grigor'yan's Theorem 3.13 in [14], this implies that the smooth metric measure space $(M, g, e^{-f}dv)$ is stochastically complete. □

Now we prove Theorem 1.5 following the arguments of Li in [17]. We first prove the following integration by parts formula.

Theorem 4.2 *Under the same assumption as in Theorem 1.5, for any nonnegative L^1_f-integrable f-subharmonic function u, we have*

$$\int_M \Delta_{f\,y} H(x,y,t)u(y)d\mu(y) = \int_M H(x,y,t)\Delta_f u(y)d\mu(y).$$

Proof of Theorem 4.2 Applying Green's theorem to $B_o(R)$, we have

$$\begin{aligned}
&\left|\int_{B_o(R)} \Delta_{f\,y} H(x,y,t)u(y)d\mu(y) - \int_{B_o(R)} H(x,y,t)\Delta_f u(y)d\mu(y)\right| \\
&\quad = \left|\int_{\partial B_o(R)} \frac{\partial}{\partial r}H(x,y,t)u(y)d\mu_{\sigma,R}(y) - \int_{\partial B_o(R)} H(x,y,t)\frac{\partial}{\partial r}u(y)d\mu_{\sigma,R}(y)\right| \\
&\quad \le \int_{\partial B_o(R)} |\nabla H|(x,y,t)u(y)d\mu_{\sigma,R}(y) + \int_{\partial B_o(R)} H(x,y,t)|\nabla u|(y)d\mu_{\sigma,R}(y),
\end{aligned}$$

where $d\mu_{\sigma,R}$ denotes the weighted area measure on $\partial B_o(R)$ induced by $d\mu$. We shall show that the above two boundary integrals vanish as $R \to \infty$. Without loss of generality, we assume $x \in B_o(R/8)$.

Step 1. Let $u(x)$ be a nonnegative f-subharmonic function. Since $\mathrm{Ric}_f \ge 0$ and $|f| \le ar^2(x) + b.$, by Proposition 2.6 we get

$$\sup_{B_o(R)} u(x) \le Ce^{\alpha R^2} V_f^{-1}(2R) \int_{B_o(2R)} u(y)d\mu(y), \tag{4.2}$$

where constants C and α depend on n, a and b. Let $\phi(y) = \phi(r(y))$ be a nonnegative cut-off function satisfying $0 \le \phi \le 1$, $|\nabla\phi| \le \sqrt{3}$ and

$$\phi(r(y)) = \begin{cases} 1 & \text{on } B_o(R+1)\backslash B_o(R), \\ 0 & \text{on } B_o(R-1) \cup (M\backslash B_o(R+2)). \end{cases}$$

Since u is f-subharmonic, by the Cauchy–Schwarz inequality we have

$$\begin{aligned}
0 \le \int_M \phi^2 u\Delta_f u d\mu &= -\int_M \nabla(\phi^2 u)\nabla u d\mu \\
&= -2\int_M \phi u\langle\nabla\phi\nabla u\rangle d\mu - \int_M \phi^2|\nabla u|^2 d\mu \\
&\le 2\int_M |\nabla\phi|^2 u^2 d\mu - \frac{1}{2}\int_M \phi^2|\nabla u|^2 d\mu.
\end{aligned}$$

By (4.2), we have that

$$\begin{aligned}\int_{B_o(R+1)\backslash B_o(R)}|\nabla u|^2 d\mu &\le 4\int_M |\nabla\phi|^2 u^2 d\mu\\ &\le 12\int_{B_o(R+2)} u^2 d\mu\\ &\le 12 \sup_{B_o(R+2)} u\cdot \|u\|_{L^1(\mu)}\\ &\le \frac{Ce^{\alpha(R+2)^2}}{V_f(2R+4)}\cdot\|u\|^2_{L^1(\mu)}.\end{aligned}$$

On the other hand, the Cauchy–Schwarz inequality implies that

$$\int_{B_o(R+1)\backslash B_o(R)}|\nabla u| d\mu \le \left(\int_{B_o(R+1)\backslash B_o(R)}|\nabla u|^2 d\mu\right)^{1/2}\cdot[V_f(R+1)\backslash V_f(R)]^{1/2}.$$

Combining the above two inequalities, we have

$$\int_{B_o(R+1)\backslash B_o(R)}|\nabla u| d\mu \le C_1 e^{\alpha R^2}\cdot\|u\|_{L^1(\mu)}, \tag{4.3}$$

where $C_1 = C_1(n,a,b)$.

Step 2. By letting $\epsilon = 1$ in Theorem 1.1, the f-heat kernel $H(x,y,t)$ satisfies

$$H(x,y,t)\le \frac{c_3}{V_f(B_x(\sqrt{t})}\cdot\left(\frac{d(x,y)}{\sqrt{t}}+1\right)^{\frac{n}{2}}\times\exp\left(c_4R^2-\frac{d^2(x,y)}{5t}\right) \tag{4.4}$$

for any $x, y \in B_o(R)$ and $0 < t < R^2/4$, where $c_3 = c_3(n,b)$ and $c_4 = c_4(n,a)$. Together with (4.3) we get

$$\begin{aligned}J_1 &:= \int_{B_o(R+1)\backslash B_o(R)} H(x,y,t)|\nabla u|(y)d\mu(y)\\ &\le \sup_{y\in B_o(R+1)\backslash B_o(R)} H(x,y,t)\cdot\int_{B_o(R+1)\backslash B_o(R)}|\nabla u| d\mu\\ &\le \frac{C_2\|u\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))}\cdot\left(\frac{R+1+d(o,x)}{\sqrt{t}}+1\right)^{\frac{n}{2}}\times\exp\left(-\frac{(R-d(o,x))^2}{5t}+c_4(R+1)^2\right),\end{aligned}$$

where $C_2 = C_2(n,a,b)$.

Thus, for T sufficiently small and for all $t \in (0,T)$ there exists a fixed constant $\beta > 0$ such that

$$J_1 \le \frac{C_3\|u\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))}\cdot\left(\frac{R+1+d(o,x)}{\sqrt{t}}+1\right)^{\frac{n}{2}}\times\exp\left(-\beta R^2+c\frac{d^2(o,x)}{t}\right),$$

where $C_3 = C_3(n, a, b)$. Hence for all $t \in (0, T)$ and all $x \in M$, J_1 tends to zero as R tends to infinity.

Step 3. Consider the integral with respect to $d\mu$,

$$\begin{aligned}\int_M \phi^2(y)|\nabla H|^2(x,y,t) &= -2\int_M \langle H(x,y,t)\nabla\phi(y), \phi(y)\nabla H(x,y,t)\rangle \\ &\quad - \int_M \phi^2(y) H(x,y,t)\Delta_f H(x,y,t) \\ &\le 2\int_M |\nabla\phi|^2(y) H^2(x,y,t) + \frac{1}{2}\int_M \phi^2(y)|\nabla H|^2(x,y,t) \\ &\quad - \int_M \phi^2(y) H(x,y,t)\Delta_f H(x,y,t).\end{aligned}$$

This implies

$$\begin{aligned}&\int_{B_o(R+1)\backslash B_o(R)} |\nabla H|^2 \\ &\le \int_M \phi^2(y)|\nabla H|^2(x,y,t) \\ &\le 4\int_M |\nabla\phi|^2 H^2 - 2\int_M \phi^2 H\Delta_f H \\ &\le 12\int_{B_o(R+2)\backslash B_o(R-1)} H^2 + 2\int_{B_o(R+2)\backslash B_o(R-1)} H|\Delta_f H| \\ &\le 12\int_{B_o(R+2)\backslash B_o(R-1)} H^2 + 2\left(\int_{B_o(R+2)\backslash B_o(R-1)} H^2\right)^{\frac{1}{2}}\left(\int_M (\Delta_f H)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{4.5}$$

By Lemma 4.1, we have

$$\int_M H(x,y,t)d\mu(y) = 1$$

for all $x \in M$ and $t > 0$. By (4.4) we get

$$\begin{aligned}\int_{B_o(R+2)\backslash B_o(R-1)} H^2(x,y,t)d\mu &\le \sup_{y\in B_o(R+2)\backslash B_o(R-1)} H(x,y,t) \\ &\le \frac{c_3}{V_f(B_x(\sqrt{t}))}\cdot\left(\frac{R+2-d(o,x)}{\sqrt{t}}+1\right)^{\frac{n}{2}} \\ &\quad \times \exp\left[-\frac{(R-1-d(o,x))^2}{5t} + c_4(R+2)^2\right].\end{aligned} \tag{4.6}$$

We *claim* that there exists a constant $C_4 > 0$ such that

$$\int_M (\Delta_f H)^2(x,y,t)d\mu \le \frac{C_4}{t^2} H(x,x,t). \tag{4.7}$$

Because f-heat kernel on M can be obtained by taking the limit of f-heat kernels on a compact exhaustion of M, it suffices to prove the claim for f-heat kernel on any compact subdomain of M. Let $H(x, y, t)$ is a f-heat kernel on a compact subdomain $\Omega \subset M$, by the eigenfunction expansion, we have

$$H(x, y, t) = \sum_{i}^{\infty} e^{-\lambda_i t} \psi_i(x)\psi_i(y),$$

where $\{\psi_i\}$ are orthonormal basis of the space of L^2_f functions with Dirichlet boundary value satisfying the equation

$$\Delta_f \psi_i = -\lambda_i \psi_i.$$

Differentiating with respect to the variable y, we have

$$\Delta_f H(x, y, t) = -\sum_{i}^{\infty} \lambda_i e^{-\lambda_i t} \psi_i(x)\psi_i(y).$$

Notice that $s^2 e^{-2s} \leq C_5 e^{-s}$ for all $0 \leq s < \infty$, therefore

$$\int_M (\Delta_f H)^2 d\mu(y) \leq C_5 t^{-2} \sum_{i}^{\infty} e^{-\lambda_i t} \psi_i^2(x) = C_5 t^{-2} H(x, x, t)$$

and claim (4.7) follows.

Combining (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}\int_{B_o(R+1)\backslash B_o(R)} |\nabla H|^2 d\mu \leq C_6[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t)] \\ \times \left(\frac{R+2-d(o,x)}{\sqrt{t}} + 1\right)^{\frac{n}{2}} \\ \times \exp\left[-\frac{(R-1-d(o,x))^2}{10t} + c_4(R+2)^2\right].\end{aligned}$$

where $V_f = V_f(B_x(\sqrt{t}))$. By the Cauchy–Schwarz inequality we get,

$$\begin{aligned}&\int_{B_o(R+1)\backslash B_o(R)} |\nabla H| d\mu \\ &\leq [V_f(B_o(R+1))\backslash V_f(B_o(R))]^{1/2} \times \left[\int_{B_o(R+1)\backslash B_o(R)} |\nabla H|^2 d\mu\right]^{1/2} \\ &\leq C_6 V_f^{1/2}(B_o(R+1))[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t)]^{1/2} \\ &\quad \times \left(\frac{R+2-d(o,x)}{\sqrt{t}} + 1\right)^{\frac{n}{4}} \times \exp\left[-\frac{(R-1-d(o,x))^2}{20t} + \frac{c_9}{2}(R+2)^2\right]. \qquad (4.8)\end{aligned}$$

Therefore, by (4.2) and (4.8), by Cauchy–Schwarz inequality we have

$$
\begin{aligned}
J_2 &:= \int_{B_o(R+1)\setminus B_o(R)} |\nabla H(x,y,t)| u(y) d\mu(y) \\
&\le \sup_{y\in B_o(R+1)\setminus B_o(R)} u(y) \cdot \int_{B_o(R+1)\setminus B_o(R)} |\nabla H(x,y,t)| d\mu(y) \\
&\le \frac{C_7 \|u\|_{L^1(\mu)}}{V_f^{1/2}(B_o(2R+2))} \cdot [V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x,x,t)]^{1/2} \\
&\quad \times \left(\frac{R+2-d(o,x)}{\sqrt{t}} + 1\right)^{\frac{n}{4}} \times \exp\left[-\frac{(R-1-d(o,x))^2}{20t} + c_{10}(R+2)^2\right],
\end{aligned}
$$

where $V_f = V_f(B_x(\sqrt{t}))$. Similar to the case of J_1, by choosing T sufficiently small, for all $t \in (0, T)$ and all $x \in M$, J_2 also tends to zero when R tends to infinity.

Step 4. By the mean value theorem, for any $R > 0$ there exists $\bar{R} \in (R, R+1)$ such that

$$
\begin{aligned}
J &:= \int_{\partial B_o(\bar{R})} [H(x,y,t)|\nabla u|(y) + |\nabla H|(x,y,t) u(y)] d\mu_{\sigma,\bar{R}}(y) \\
&= \int_{B_o(R+1)\setminus B_p(R)} [H(x,y,t)|\nabla u|(y) + |\nabla H|(x,y,t) u(y)] d\mu(y) \\
&= J_1 + J_2.
\end{aligned}
$$

By step 2 and step 3, we know that by choosing T sufficiently small, for all $t \in (0, T)$ and all $x \in M$, J tends to zero as $\bar{R}$ (and hence R) tends to infinity. Therefore we finish the proof of Theorem 4.2 for T sufficiently small.

Step 5. Using the semigroup property of the f-heat equation,

$$
\begin{aligned}
\frac{\partial}{\partial(s+t)}(e^{(s+t)\Delta_f} u) &= \frac{\partial}{\partial t}(e^{s\Delta_f} e^{t\Delta_f} u) = e^{s\Delta_f} \frac{\partial}{\partial t}(e^{t\Delta_f} u) \\
&= e^{s\Delta_f} e^{t\Delta_f} (\Delta_f u) = e^{(s+t)\Delta_f}(\Delta_f u),
\end{aligned}
$$

we prove Theorem 4.2 for all time $t > 0$. □

Next we prove the L^1_f Liouville theorem, Theorem 1.5.

Proof of Theorem 1.5 Let $u(x)$ be a nonnegative, L^1_f-integrable and f-subharmonic function defined on M. We define a space-time function

$$
u(x,t) = \int_M H(x,y,t) u(y) d\mu(y)
$$

with initial data $u(x,0)=u(x)$. From Theorem 4.2, we conclude that

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \int_M \frac{\partial}{\partial t}H(x,y,t)u(y)d\mu(y)\\ &= \int_M \Delta_{f\,y}H(x,y,t)u(y)d\mu(y)\\ &= \int_M H(x,y,t)\Delta_{f\,y}u(y)d\mu(y)\geq 0,\end{aligned} \tag{4.9}$$

that is, $u(x,t)$ is increasing in t. By Lemma 4.1,

$$\int_M H(x,y,t)d\mu(y)=1$$

for all $x\in M$ and $t>0$. So we have

$$\int_M u(x,t)d\mu(x)=\int_M\int_M H(x,y,t)u(y)d\mu(y)d\mu(x)=\int_M u(y)d\mu(y).$$

Since $u(x,t)$ is increasing in t, so $u(x,t)=u(x)$ and hence $u(x)$ is a nonnegative f-harmonic function, i.e. $\Delta_f u(x)=0$.

On the other hand, for any positive constant a, let us define a new function $h(x)=\min\{u(x),a\}$. Then h satisfies

$$0\leq h(x)\leq u(x),\quad |\nabla h|\leq|\nabla u|\quad\text{and}\quad \Delta_f h(x)\leq 0.$$

In particular, h satisfies estimates (4.2) and (4.3). Similarly we define $h(x,t)$ and

$$\begin{aligned}\frac{\partial}{\partial t}h(x,t) &= \frac{\partial}{\partial t}\int_M H(x,y,t)h(y)d\mu(y)\\ &= \int_M H(x,y,t)\Delta_{f\,y}h(y)d\mu(y)\leq 0.\end{aligned}$$

By the same argument, we have that $\Delta_f h(x)=0$.

By the regularity theory of f-harmonic functions, this is impossible unless $h=u$ or $h=a$. Since a is arbitrary and u is nonnegative, so u must be identically constant. The theorem then follows from the fact that the absolute value of a f-harmonic function is a nonnegative f-subharmonic function. □

5 L^1_f-uniqueness property

For the completeness we provide a detailed proof of Theorem 1.9 following the arguments of Li in [17].

Proof of Theorem 1.9 Let $u(x,t)\in L^1_f$ be a nonnegative function satisfying the assumption in Theorem 1.9. For $\epsilon>0$, let $u_\epsilon(x)=u(x,\epsilon)$. Define

$$e^{t\Delta_f}u_\epsilon(x) = \int_M H(x, y, t)u_\epsilon(y)d\mu(y) \tag{5.1}$$

and

$$F_\epsilon(x, t) = \min\{0, u(x, t+\epsilon) - e^{t\Delta_f}u_\epsilon(x)\}.$$

Then $F_\epsilon(x, t)$ is nonnegative and satisfies

$$\lim_{t\to 0} F_\epsilon(x, t) = 0 \quad \text{and} \quad (\partial_t - \Delta_f)F_\epsilon(x, t) \le 0.$$

Let $T > 0$ be fixed. Let $h(x) = \int_0^T F_\epsilon(x, t)dt$, which satisfies

$$\begin{aligned}\Delta_f h(x) &= \int_0^T \Delta_f F_\epsilon(x, t)dt \\ &\ge \int_0^T \partial_t F_\epsilon(x, t)dt = F_\epsilon(x, T) \ge 0.\end{aligned} \tag{5.2}$$

Moreover,

$$\begin{aligned}\int_M h(x)d\mu &= \int_0^T \int_M F_\epsilon(x, t)d\mu dt \\ &\le \int_0^T \int_M |u(x, t+\epsilon) - e^{t\Delta_f}u_\epsilon(x)|d\mu dt \\ &\le \int_0^T \int_M u(x, t+\epsilon)d\mu dt + \int_0^T \int_M e^{t\Delta_f}u_\epsilon(x)d\mu dt < \infty,\end{aligned}$$

where the first term on the right hand side is finite from our assumption, and the second term is finite because $e^{t\Delta_f}$ is a contractive semigroup in L^1_f. Therefore, $h(x)$ is a nonnegative L^1_f-integrable f-subharmonic function. By Theorem 1.5, $h(x)$ must be constant. Combining with (5.2) we have $F_\epsilon(x, t) = 0$. Hence $F_\epsilon(x, T) \equiv 0$ for all $x \in M$ and $T > 0$, which implies

$$e^{t\Delta_f}u_\epsilon(x) \ge u(x, t+\epsilon). \tag{5.3}$$

Next we estimate the function $e^{t\Delta_f}u_\epsilon(x)$ in (5.1). Applying the upper bound estimate of the heat kernel $H(x, y, t)$ and letting $R = 2d(x, y) + 1$, we have

$$\begin{aligned}e^{t\Delta_f}u_\epsilon(x) \le &\frac{C}{V_f(B_x(\sqrt{t}))} \cdot \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{\frac{n}{2}} \\ &\times \int_M \left[\exp\left(\tilde{C}d^2(x, y) - \frac{d^2(x, y)}{5t}\right) u(y, \epsilon)\right] d\mu(y).\end{aligned}$$

Thus there exists a sufficiently small $t_0 > 0$ such that for all $0 < t < t_0$, we have $\lim_{\epsilon\to 0} e^{t\Delta_f} u_\epsilon(x) = 0$ by the assumption

$$\lim_{\epsilon\to 0}\int_M u(x,\epsilon)d\mu(x) = 0.$$

Therefore by the semigroup property, we conclude that $\lim_{\epsilon\to 0} e^{t\Delta_f} u_\epsilon(x) = 0$ for all $x \in M$ and $t > 0$. Combining with (5.3) we get $u(x,t) \le 0$. Therefore $u(x,t) \equiv 0$. □

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Appendix

In the appendix we solve for the f-heat kernel of 1-dimensional steady Gaussian soliton $(\mathbb{R},\ g_0, e^{-f}dx)$, where g_0 is the Euclidean metric, and $f = kx$ with $k = \pm 1$. The method is standard separation of variables. Suppose the f-heat kernel is of the form

$$H(x,y,t) = \varphi(y)\phi(x)\psi(t) \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

For a fixed y, we get

$$\begin{aligned} H_t &= \varphi\phi e^{-\frac{|x-y|^2}{4t}}\left(\psi_t + \psi\frac{|x-y|^2}{4t^2}\right),\\ H_x &= \varphi\psi e^{-\frac{|x-y|^2}{4t}}\left(\phi_x - \phi\frac{x-y}{2t}\right),\\ H_{xx} &= \varphi\psi e^{-\frac{|x-y|^2}{4t}}\left(\phi_{xx} + \phi\frac{|x-y|^2}{4t^2} - \phi_x\frac{x-y}{t} - \phi\frac{1}{2t}\right). \end{aligned}$$

So $H_t = H_{xx} - f_x H_x$ implies

$$\begin{aligned} \phi\left(\psi_t + \psi\frac{|x-y|^2}{4t^2}\right) = \psi\left(\phi_{xx} + \phi\frac{|x-y|^2}{4t^2} - \phi_x\frac{x-y}{t} - \frac{\phi}{2t}\right)\\ - k\psi\left(\phi_x - \phi\frac{x-y}{2t}\right). \end{aligned}$$

That is,

$$\frac{\psi_t}{\psi} = \frac{\phi_{xx} - k\phi_x}{\phi} - \frac{x-y}{2t}\cdot\frac{2\phi_x - k\phi}{\phi} - \frac{1}{2t}.$$

Therefore

$$\frac{\phi_{xx} - k\phi_x}{\phi} = C_1, \qquad \frac{(2\phi_x - k\phi)(x-y)}{\phi} = C_2, \qquad \frac{\psi_t}{\psi} = C_1 - \frac{1+C_2}{2t},$$

From above, their solutions are

$$\phi = C_3 e^{\frac{1}{2}kx},$$
$$\psi = C_4 \frac{1}{\sqrt{t}} e^{-4/t},$$

where C_1, C_2, C_3, C_4 are constants.

By the initial condition $\lim_{t\to 0} u(x,t) = \delta_{f,y}(x)$ we get $\varphi(y) = e^{\frac{1}{2}ky}$, and $C_3C_4 = \frac{1}{2\sqrt{\pi}}$. Therefore the f-heat kernel is

$$H(x,y,t) = \frac{e^{\pm\frac{x+y}{2}} \cdot e^{-t/4}}{(4\pi t)^{1/2}} \times \exp\left(-\frac{|x-y|^2}{4t}\right).$$

It is easy to check that $\int_{\mathbb{R}} H(x,y,t)e^{-f(x)}dx = 1$, which confirms the stochastic completeness proved in Lemma 4.1. □

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