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Filtrations and test-configurations

With an appendix by Sebastien Boucksom

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Abstract We introduce a strengthening of K-stability, based on filtrations of the homogeneous coordinate ring. This allows for considering certain limits of families of test-configurations, which arise naturally in several settings. We prove that if a manifold with no automorphisms admits a cscK metric, then it satisfies this stronger stability notion. We also discuss the relation with the birational transformations in the definition of b-stability.

1 Introduction

Given a compact complex manifold X with an ample line bundle L, the notion of a test-configuration is central to the definition of K-stability, which in turn is conjecturally related to the existence of a constant scalar curvature Kähler metric in the first Chern class $c_1(L)$, by the Yau–Tian–Donaldson conjecture [9,30,32]. Roughly speaking, test-configurations for (X, L) are $\mathbf{C}^*$-equivariant flat degenerations of X into possibly singular schemes. It was shown by Witt Nyström [31] that test-configurations for (X, L) give rise to filtrations of the homogeneous coordinate ring and in this paper we explore the converse direction of this. The first observation is that every suitable filtration gives rise to a family of test-configurations living in larger and larger projective spaces, and that the filtration should in some sense be thought of as the limit of this family. See Sect. 3 for the detailed definitions.

It is natural to extend the class of test-configurations to these limiting objects for several reasons. For instance every convex function on the moment polytope of a toric variety can be thought of as a filtration, but only the rational piecewise linear convex

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functions give rise to test-configurations by Donaldson's work [9]. Another reason is that Apostolov–Calderbank–Gauduchon–Tønnesen–Friedman [1] have found an example of a manifold that does not admit an extremal metric, but does not appear to be destabilized by a test-configuration. Rather it is destabilized by a $\mathbf{C}^*$-equivariant degeneration which is equipped with an irrational polarization, and this can be thought of as a filtration. Note that by the work of Chen–Donaldson–Sun [5] this issue does not arise in the case of Kähler–Einstein metrics. Finally in [29] we studied minimizing sequences for the Calabi functional on a ruled surface, and found that the limiting behavior of the metrics has an algebro-geometric counterpart, as a sequence of test-configurations. In general there is no limiting test-configuration, since in the sequence we need embeddings into larger and larger projective spaces, but once again we can think of the limit as a filtration. We will describe these examples in more detail in Sect. 4. Note that Ross and Witt Nyström [25] have done related work in a more analytic direction. Starting with a suitable filtration, they define an "analytic test-configuration", which is a geodesic ray in the space of metrics in a weak sense. For more in this direction see for example Phong–Sturm [22].

We define a notion of Futaki invariant for filtrations, extending the usual definition. Our main result, in Sect. 6 is the following.

Theorem A *Suppose that X admits a cscK metric in $c_1(L)$, and the automorphism group of (X, L) is finite. Then if χ is a filtration for (X, L) such that $\|\chi\|_2 > 0$, then the Futaki invariant of χ satisfies* $\mathrm{Fut}(\chi) > 0$.

Here $\|\chi\|_2$ is a norm of the filtration, and the filtrations with zero norm play the role of the trivial test-configuration. This result is a strengthening of Stoppa's result [26], whose conclusion under the same assumptions is that (X, L) is K-stable, since it implies that the Futaki invariant has to be bounded away from zero uniformly along certain families of test-configurations. In addition, similarly to Stoppa's argument, we use the existence result for cscK metrics on blowups due to Arezzo–Pacard [2], and the asymptotic Chow stability of cscK manifolds with no discrete automorphism group due to Donaldson [8].

A key new ingredient in the proof is the Okounkov body [21], and the concave (in our case convex) transform of a filtration introduced by Boucksom–Chen [4], which was also used in the context of test-configurations by Witt Nyström [31]. We review these constructions in Sect. 5.

In addition, the proof relies on the following result, which was stated as a conjecture in an earlier version of this paper. The result is due to S. Boucksom, and the proof is presented in the appendix as Theorem 20.

Theorem B *Suppose that $S \subset \bigoplus_{k\geqslant 0} H^0(X, L^k)$ is a graded subalgebra which contains an ample series (see Definition 17). In addition suppose that*

$$\lim_{k\to\infty} k^{-n} \dim S_k < \lim_{k\to\infty} k^{-n} \dim H^0(X, L^k),$$

where n is the dimension of X. Then there is a point $p \in X$ and a number $\varepsilon > 0$, such that

$$S_k \subset H^0(X, L^k \otimes I_p^{\lceil k\varepsilon \rceil}),$$

for all k, where I_p is the ideal sheaf of the point p.

In [7] Donaldson introduced a new notion of stability, called b-stability, which is a similar strengthening of K-stability, but it allows for more general families of test-configurations (and even more general degenerations) than what we are able to encode using filtrations so far. In Sect. 7 we make some basic observations about the relation with filtrations. In particular we will show that Proposition 11, which is a variant Theorem A above, gives a strengthening of the main theorem in [6].

2 Test-configurations, the Futaki invariant and the Chow weight

We briefly recall the notion of test-configuration and their Futaki invariants from Donaldson [9]. Given a polarized variety (X, L), a test-configuration for (X, L) is a flat, polarized, $\mathbf{C}^*$-equivariant family $(\mathcal{X}, \mathcal{L}) \to \mathbf{C}$, where the generic fiber is isomorphic to (X, L^r) for some $r > 0$. The number r is called the exponent of the test-configuration. The Futaki invariant and the Chow weight are both computed in terms of the induced $\mathbf{C}^*$-action on the central fiber (X_0, L_0). Namely let us write d_{rk} for the dimension of, and w_{rk} for the total weight of the action on $H^0_{X_0}(L_0^k)$. For large k, using the equivariant Riemann-Roch theorem, we have expansions

$$\begin{aligned} d_{rk} &= a_0(rk)^n + a_1(rk)^{n-1} + \dots \\ w_{rk} &= b_0(rk)^{n+1} + b_1(rk)^n + \dots, \end{aligned} \tag{1}$$

where n is the dimension of X. We write the expansions in terms of rk instead of k, because we think of the numbers d_{rk} and w_{rk} as being related to the line bundles L^{rk} on X. For instance this way the number a_0 is the volume of (X, L), and does not depend on the exponent r of the test-configuration. The Futaki invariant of the family is defined to be

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2}.$$

Note that the Futaki invariant remains unchanged if we replace the line bundle $\mathcal{L}$ on $\mathcal{X}$ by a power. The Chow weight of the family is

$$\mathrm{Chow}_r(\mathcal{X}, \mathcal{L}) = \frac{r b_0}{a_0} - \frac{w_r}{d_r}. \tag{2}$$

In the notation for the Chow weight, the subscript r means that the test-configuration has exponent r. We emphasize this, since unlike for the Futaki invariant, it makes a difference if we replace $\mathcal{L}$ by a power, and later on we will not have the line bundle explicit in the notation. In fact we have

$$\mathrm{Chow}_{rk}(\mathcal{X}, \mathcal{L}^k) = \frac{krb_0}{a_0} - \frac{w_{kr}}{d_{kr}},$$

from which it is easy to check that

$$\mathrm{Fut}(\mathcal{X}) = \lim_{k\to\infty} \mathrm{Chow}_{rk}(\mathcal{X}, \mathcal{L}^k). \tag{3}$$

For the record we state the following definitions (see for example Ross–Thomas [24]).

Definition 1 The polarized manifold (X, L) is K-stable, if the Futaki invariant is positive for every test-configuration, for which the central fiber is not isomorphic to X.

The polarized manifold (X, L) is asymptotically Chow stable, if there is some k_0, such that the Chow weight is positive for all test-configurations with exponent greater than k_0, and whose central fiber is not isomorphic to X.

We will need to define a norm for test-configurations. There are various options for this, analogous to various L^p norms for functions. Given a test-configuration as above, write A_{rk} for the generator of the $\mathbf{C}^*$-action on $H^0_{X_0}(L_0^k)$. So $\mathrm{Tr}(A_{rk}) = w_{rk}$ in our notation above. We then have an expansion

$$\mathrm{Tr}(A_{rk}^2) = c_0(rk)^{n+2} + \dots \tag{4}$$

for large k, and we define the norm $\|\mathcal{X}\|_2$ of the test-configuration by

$$\|\mathcal{X}\|_2^2 = c_0 - \frac{b_0^2}{a_0}. \tag{5}$$

This is analogous to the L^2-norm of functions, normalized to be zero on constants. Note that the norm is unchanged if we replace $\mathcal{L}$ by a power.

In what follows, it will be natural to think of test-configurations slightly differently. Recall that all test-configurations of exponent r for (X, L) can be obtained by embedding $X \hookrightarrow \mathbf{P}(V^*)$ for $V = H^0(X, L^r)$, and then choosing a $\mathbf{C}^*$-action on V^*. The test-configuration is then obtained by taking the $\mathbf{C}^*$-orbit of X, and completing this family across the origin with the flat limit. Let us assume that the weights of the dual action on V are all positive (we can modify the original $\mathbf{C}^*$-action by another action with constant weights, without changing any of the invariants of the test-configuration). The weight decomposition under this $\mathbf{C}^*$-action gives rise to a flag

$$\{0\} = V_0 \subset V_1 \subset \dots \subset V_k = V, \tag{6}$$

where V_i is spanned by the eigenvectors with weight at most i. The point we want to make is that the test-configuration is determined by this flag. This can be seen as follows. Suppose that $\lambda_1, \lambda_2 : \mathbf{C}^* \to GL(V)$ are two one-parameter subgroups, with the same flag (6). Let $v \in V$ be such that $\lambda_1(t) \cdot v = t^i v$ for all t, and let $v = w_1 + \cdots + w_i$ be the weight decomposition of v with respect to λ_2. Note that only weights up to i occur in this decomposition since λ_2 has the same flag as λ_1. It follows that

$$\lambda_2(t)^{-1}\lambda_1(t) \cdot v = t^i(t^{-1}w_1 + \ldots + t^{-i}w_i),$$

and so

$$\lim_{t\to 0} \lambda_2(t)^{-1}\lambda_1(t) \cdot v = w_i.$$

Applying this to each weight vector for λ_1, we see that $M(t) = \lambda_2(t)^{-1}\lambda_1(t)$ extends to a map $M : \mathbf{C} \to GL(V)$ (the fact that $M(0)$ is invertible follows by interchanging λ_1, λ_2 in the above argument). It then follows that the families in $\mathbf{P}(V^*)$ defined by the orbits of X under the dual actions of λ_1 and λ_2 are equivalent. Because of this, we will often speak of the test-configuration induced by a flag in $H^0(X, L^r)$, and also we will make use of the matrices A_k as above, as if we have already picked a $\mathbf{C}^*$-action giving rise to the flag. The point of view of flags is useful more generally in GIT, see for example Sect. 2.2 in Mumford–Fogarty–Kirwan [20].

3 Filtrations

Let (X, L) be a polarized manifold. Let us write $R_k = H^0(X, L^k)$, and

$$R = \bigoplus_{k\geqslant 0} R_k = \bigoplus_{k\geqslant 0} H^0(X, L^k)$$

for the homogeneous coordinate ring of (X, L). We will assume throughout the paper that R_1 generates R.

Definition 2 A *filtration* of R is a chain of finite dimensional subspaces

$$\mathbf{C} = F_0R \subset F_1R \subset F_2R \subset \ldots \subset R,$$

such that the following conditions hold:

1. The filtration is multiplicative, i.e. $(F_iR)(F_jR) \subset F_{i+j}R$ for all $i, j \geqslant 0$,
2. The filtration is compatible with the grading R_k of R, i.e. if $f \in F_iR$ for some $i \geqslant 0$ then each homogeneous piece of f is in F_iR,
3. We have

$$\bigcup_{i\geqslant 0} F_iR = R.$$

This notion of filtration is more or less equivalent to the one used in Witt Nyström [31]. The main difference is that our indices are the negative of his, and in addition our filtration is "scaled" so that each nontrivial piece has positive index. In analogy to [31] we could allow more general filtrations, where $F_i R$ can be non-empty for negative i as well, assuming a boundedness condition. Namely we assume that for some constant C, the filtration $F_i R_k$ on the degree k piece of R satisfies $F_{-Ck} R_k = \{0\}$. In this case we could define a new filtration by letting $F_i' R_k = F_{i-Ck} R_k \oplus \mathbf{C}$ for all $i \geqslant 0$, and it would satisfy our conditions. In addition in [31] the filtered pieces are indexed by real numbers, while ours are integers, but this is also not a significant restriction.

Given a filtration χ of R, the Rees algebra of χ is defined by

$$\mathrm{Rees}(\chi) = \bigoplus_{i \geqslant 0} (F_i R) t^i \subset R[t].$$

This is a flat $\mathbf{C}[t]$-subalgebra of $R[t]$, since it is a torsion-free $\mathbf{C}[t]$-module (see Corollary 6.3 in Eisenbud [11]). In addition the associated graded algebra of χ is

$$\mathrm{gr}(\chi) = \bigoplus_{i \geqslant 0} (F_i R)/(F_{i-1} R),$$

where $F_{-1} R = \{0\}$. Note that both of these algebras have two gradings. One grading comes from the grading of R, while another, denoted by i here, comes from the filtration. The fiber of the Rees algebra of χ at non-zero t is isomorphic to R, while the fiber at $t = 0$ is isomorphic to $\mathrm{gr}(\chi)$.

3.1 Finitely generated filtrations

Let us call a filtration finitely generated, if its Rees algebra is finitely generated. In this case the filtration gives rise to a test-configuration for (X, L), whose total space is $\mathrm{Proj}_{\mathbf{C}[t]} \mathrm{Rees}(\chi)$, where the grading in the Proj construction is the grading coming from R (which is suppressed in the notation). The central fiber of the test-configuration is $\mathrm{Proj}_{\mathbf{C}}(\mathrm{gr}(\chi))$, where again we are using the grading induced by the grading of R. The grading given by the filtration is the one which induces a $\mathbf{C}^*$-action on the family as well as on its central fiber. In order for the action to be compatible with multiplication on $\mathbf{C}$, the function t must have weight -1. This implies that in terms of sections on the central fiber, the sections in $(F_i R)/(F_{i-1} R)$ have weight $-i$. It is these weights that are used in the calculation of the Futaki invariant.

Finitely generated filtrations therefore give rise to test-configurations. Conversely, Witt Nyström [31] showed that every test-configuration gives rise to a finitely generated filtration of R. Let us recall the construction briefly. We are thinking of a test-configuration as a $\mathbf{C}^*$-equivariant flat family $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbf{C}$, such that the generic fiber is isomorphic to (X, L^r) for some power $r > 0$. If $s \in R_r$, then we can think of s as a section of $\mathcal{L}$ over the fiber $\pi^{-1}(1)$. Using the $\mathbf{C}^*$-action we can extend

s to a meromorphic section $\overline{s}$ of $\mathcal{L}$ over the whole of $\mathcal{X}$. We then define

$$F_i R_r = \{s \in R_r \,:\, t^i \overline{s} \text{ is holomorphic on } \mathcal{X}\}. \tag{7}$$

Note that Witt Nyström uses $t^{-i}\overline{s}$ instead of $t^i\overline{s}$, so his filtration is the opposite of ours. This filtration may not satisfy that $F_0 R_r$ is empty (which we require of our filtrations), but this can easily be achieved by first modifying the $\mathbf{C}^*$-action on $\mathcal{L}$ by an action with constant weights. We can then extend this filtration of R_r to a filtration of R as follows. Let N be such that $F_N R_r = R_r$. Then let $\mathcal{R} \subset R[t]$ be the $\mathbf{C}[t]$-subalgebra generated by

$$R_1 t^N \oplus \left(\bigoplus_{i=1}^{N} (F_i R_r) t^i \right). \tag{8}$$

We can then define a filtration

$$F_i R = \{s \in R \,:\, t^i s \in \mathcal{R}\}. \tag{9}$$

The point of adding in the generators $R_1 t^N$ is to ensure that for every $s \in R$ there is some i such that $s \in F_i R$, i.e. that Condition (3) in Definition 2 holds. At the same time because of the choice of N, the induced filtration on R_{kd} for any $k > 0$ coincides with that obtained by the construction in Eq. (7) applied to sections of $\mathcal{L}^k$. It follows from this that $\mathrm{Proj}_{\mathbf{C}[t]}\mathcal{R}$ is isomorphic to the test-configuration $\mathcal{X}$ that we started with.

3.2 General filtrations

The main point of considering filtrations instead of test-configurations is that filtrations are more general, since they are not all finitely generated. At the same time any filtration can be approximated by finitely generated filtrations in the following sense. Suppose that $\mathcal{R}$ is the Rees algebra corresponding to a filtration χ, and in addition let $\mathcal{R}_i$ be a sequence of finitely generated $\mathbf{C}[t]$-subalgebras of $\mathcal{R}$, such that

$$\mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R},$$

and $\bigcup_{i>0} \mathcal{R}_i = \mathcal{R}$. Then using the construction in Eq. (9) we obtain a family of induced filtrations χ_i, and we think of χ as the limit of the sequence χ_i.

Given a filtration χ it will be convenient to choose one specific approximating sequence in our constructions.

Definition 3 Given a filtration χ, the approximating sequence $\chi^{(k)}$ is the sequence of finitely generated filtrations defined as follows. For each k we let $\chi^{(k)}$ be the finitely generated filtration induced by the filtration χ restricted to R_k as above, in Eqs. (8) and (9).

Equivalently, we can think of $\chi^{(k)}$ as the test-configuration of exponent k, corresponding to the filtration on R_k as described at the end of the last section.

We will use the following comparison between $\chi^{(k)}$ and χ many times. For any l, let us write $F_i' R_{kl}$ and $F_i R_{kl}$ for the filtrations on R_{kl} given by $\chi^{(k)}$ and χ respectively. Then by construction $F_i' R_k = F_i R_k$ for all i, and $F_i' R_{kl} \subset F_i R_{kl}$ for $l > 1$. Indeed, once we fix the filtration $\chi^{(k)}$ on R_k, then for all $l > 1$ and i, the space $F_i' R_{kl}$ is the smallest possible subspace of R_{kl}, which is compatible with the multiplicative property of $\chi^{(k)}$.

Definition 4 Given a filtration χ, we define the Futaki invariant, and k^{th} Chow weight of χ to be

$$\mathrm{Fut}(\chi) = \liminf_{k\to\infty} \mathrm{Fut}(\chi^{(k)}, \mathcal{L})$$
$$\mathrm{Chow}_k(\chi) = \mathrm{Chow}_k(\chi^{(k)}, \mathcal{L}),$$

where $(\chi^{(k)}, \mathcal{L})$ is the test-configuration of exponent k defined by the filtration on R_k induced by χ. We also define a norm of the filtration by

$$\|\chi\|_2 = \liminf_{k\to\infty} \|\chi^{(k)}\|_2.$$

We will see in Lemma 7 that the lim inf in the definition of the norm is actually a limit.

There are other possible numerical invariants of a filtration, related to the Futaki invariant. For instance in Donaldson's work [6] the relevant quantity is the asymptotic Chow weight of a filtration, which is $\liminf_{k\to\infty} \mathrm{Chow}_k(\chi)$. We will explain this in Sect. 7.1. Note that if the filtration is finitely generated, then the asymptotic Chow weight is equal to the Futaki invariant, because of Eq. (3).

Example 1 For filtrations, the role of trivial test-configurations is played by filtrations with zero norm. This includes filtrations which are limits of non-trivial test-configurations. For example on $\mathbf{P}^1$, we can define the filtration (where $R_k = H^0(\mathcal{O}(k))$)

$$F_i R_k = \{\text{all sections vanishing at } (0:1)\},$$

for $0 < i < k$, and

$$F_i R_k = R_k,$$

for $i \geqslant k$. It is not hard to check that the norm of this filtration is 0. The corresponding sequence of test-configurations is simply deformation to the normal cone of the point $(0:1)$, with smaller and smaller parameters as $k \to \infty$ (see Ross–Thomas [23]). While none of these test-configurations is trivial, it is reasonable that their limit should be thought of as being trivial, and in particular the Futaki invariant of this filtration is zero.

Example 2 On the other hand there are also non-trivial test-configurations which have zero norm. For example the test-configuration for $\mathbf{P}^1$, whose central fiber is a double line (i.e. the family of conics $z^2 - txy = 0$ as $t \to 0$) has zero norm, even though it has non-zero Futaki invariant. Note that after taking the normalization of the total space, the test-configuration becomes a product configuration.

We say that a filtration χ is *destabilizing*, if $\|\chi\|_2 > 0$, and $\mathrm{Fut}(\chi) \leqslant 0$. We expect that if X admits a cscK metric in the class $c_1(L)$ and has no holomorphic vector fields, then no destabilizing filtration exists. This is a slightly stronger statement than saying that (X, L) is K-stable, since certain limiting objects are also required to have positive Futaki invariant. On the other hand the condition $\|\chi\|_2 > 0$ does exclude some non-trivial test-configurations which are considered in K-stability, like the one in Example 2. At the same time it was pointed out by Li–Xu [19] that even in the definition of K-stability one should not consider test-configurations such as these by restricting attention to test-configurations with normal total space. The reason is that there are always certain non-normal test-configurations, which are non-trivial, but have zero Futaki invariant. We therefore believe that the condition $\|\chi\|_2 > 0$ is very natural even for test-configurations.

4 Examples

For toric varieties Donaldson [9] showed that any rational piecewise linear convex function on the moment polytope gives rise to a test-configuration of the variety. We will show that at the same time any positive convex function on the polytope gives rise to a filtration of the homogeneous coordinate ring. Since adding a constant to a rational piecewise linear convex function only changes the test-configuration by an action on the line bundle with constant weights, it is not restrictive to only consider positive functions.

Suppose that $f : \Delta \to \mathbf{R}$ is a positive convex function, where Δ is the moment polytope corresponding to the polarized toric variety (X, L). For us Δ is closed, so f is automatically bounded, although in Donaldson's work [9] some unbounded convex functions also play a role. At the same time we can allow functions which are not continuous at the boundary of Δ. A basis of sections of $H^0(X, L^k)$ can be identified with the rational lattice points in $\Delta \cap \frac{1}{k}\mathbf{Z}^n$. If

$$\alpha \in \Delta \cap \frac{1}{k}\mathbf{Z}^n,$$

write s_α for the corresponding section of L^k. Now on $R_k = H^0(X, L^k)$ define the filtration as follows:

$$F_i R_k = \mathrm{span}\,\{s_\alpha \,:\, kf(\alpha) \leqslant i\}\,. \tag{10}$$

The convexity of f ensures that the filtration of the graded ring of (X, L) defined in this way will satisfy the multiplicative property. The other two conditions in Definition 2 also follow easily.

We can also see what the sequence of test-configurations are, which approximate the filtration defined by f. Let $f_k : \Delta \to \mathbf{R}$ be the largest convex function which on the points $\alpha \in \Delta \cap \frac{1}{k}\mathbf{Z}^n$ is defined by

$$f_k(\alpha) = \frac{1}{k}\lceil kf(\alpha)\rceil.$$

Then the filtration defined on R_k by (10) using the function f is the same as that obtained by the same formula, but using the function f_k. So the test-configuration obtained from the filtration on the piece R_k can be seen as the toric test-configuration defined by the function f_k, which is a rational piecewise-linear approximation to the function f. As for the Futaki invariants, Donaldson showed that the test-configuration corresponding to f_k has Futaki invariant up to a constant factor given by

$$\mathrm{Fut}(f_k) = \int_{\partial\Delta} f_k\, d\sigma - a\int_{\Delta} f_k\, d\mu,$$

where $d\sigma$ is a certain measure on the boundary, and a is a normalizing constant ($a = a_1/a_0$ in the notation of Eq. (1)). Since f_k is a decreasing sequence of functions converging to f pointwise, we have

$$\lim_{k\to\infty} \mathrm{Fut}(f_k) = \int_{\partial\Delta} f\, d\sigma - a\int_{\Delta} f\, d\mu.$$

In [9] this functional plays an important role even when defined on convex functions which are not piecewise linear. It is therefore useful that it can still be interpreted algebro-geometrically, as the Futaki invariant of a non-finitely generated filtration.

Another instance where more general convex functions appear is in the study of optimal test-configurations for toric varieties [28]. Note that the optimal destabilizing convex functions constructed in that paper are not known to be bounded, so the filtration given by Eq. 10 might not satisfy Condition (3) in Definition 2. We hope that with more work one can show that the optimal destabilizing convex functions are actually bounded, but in any case this filtration should be thought of as being analogous to the Harder–Narasimhan filtration of an unstable vector bundle. It is tempting to speculate that in general, on any unstable manifold (X, L) one can define such an optimal destabilizing filtration.

This picture can be extended to bundles of toric varieties, in particular to ruled surfaces, following [27]. In this way, the "optimal destabilizing test-configurations" that we found in [29] can also be seen as filtrations. In addition Apostolov–Calderbank–Gauduchon–Tønnesen–Friedman [1] found an example of a $\mathbf{P}^1$-bundle over a threefold that does not admit an extremal metric, but appears to be only destabilized by a non-algebraic degeneration (it has not been shown that there are no destabilizing test-configurations). This also fits into the above picture applied to toric bundles, and thus can also be thought of as a filtration.

5 The Okounkov body

The Okounkov body [21] is a convenient way to package some information about the graded ring R and its filtrations, as shown by Boucksom–Chen [4], and Witt

Nyström [31]. In this section we briefly recall the main points of this, but see [4] and also Lazarsfeld–Mustaţă [18] for more details.

First we recall the construction of the Okounkov body. Choose a point $p \in X$ and a set of local holomorphic coordinates $z_1, \dots, z_n$ centered at p. Let $s \in H^0(X, L)$ be a section which does not vanish at p. Then every section $f \in H^0(X, L^k)$ can be written near p as

$$f = s^k \cdot (\text{power series in } z_1, \dots, z_n). \tag{11}$$

We use the graded lexicographic order on monomials. This means that monomials with larger total degree are larger, and monomials with the same degree are ordered using the lexicographic order. Writing $R = \bigoplus H^0(X, L^k)$, we can define a map

$$\nu : R \mapsto \mathbf{Z}^n,$$

such that $\nu(f)$ is equal to the exponent of the lowest order term in the expansion (11). For every $k > 0$ we then define the subset $P_k \subset \mathbf{Z}^n$ given by

$$P_k = \{\nu(f) \,:\, f \in R_k\} \subset \mathbf{Z}^n.$$

The Okounkov body is defined to be the closure

$$P = \overline{\bigcup_{k \geqslant 1} \frac{1}{k} P_k}.$$

The property that $\nu(fg) = \nu(f) + \nu(g)$ can be used to show that P is a convex body in the positive orthant of $\mathbf{R}^n$. Note that the Okounkov body P will in general depend on the choice of the point p and the choice of local coordinates z_i.

Let us write $\Delta_\varepsilon \subset \mathbf{R}^n$ for the n-simplex

$$\Delta_\varepsilon = \left\{(a_1, \dots, a_n) \,:\, a_i \geqslant 0, \sum a_i \leqslant \varepsilon\right\}.$$

It will be useful to know that P contains Δ_ε for small ε and for this it is important that we are using the graded lexicographic order and not the ungraded version.

Lemma 5 *For sufficiently small $\varepsilon > 0$ we have $\Delta_\varepsilon \subset P$. More precisely there exists some $\varepsilon > 0$ such that for sufficiently large k we have $\Delta_{k\varepsilon - 1} \cap \mathbf{Z}^n \subset P_k$.*

Proof Let $\varepsilon > 0$ be a small rational number, smaller than the Seshadri constant of p with respect to L (in other words the $\mathbf{Q}$-line bundle $L - \varepsilon E$ on the blowup $Bl_p X$ is ample). Let $\mathcal{I}_p$ be the ideal sheaf of p. If k is such that $k\varepsilon$ is an integer, consider the exact sequence

$$0 \longrightarrow \mathcal{I}_p^{k\varepsilon} L^k \longrightarrow L^k \longrightarrow \mathcal{O}_{k\varepsilon p} \otimes L^k|_p \longrightarrow 0.$$

For large k the cohomology group $H^1(X, \mathcal{I}_p^{\varepsilon k} L^k)$ vanishes, so the map

$$H^0(X, L^k) \longrightarrow H^0(X, \mathcal{O}_{k\varepsilon p} \otimes L^k|_p)$$

is surjective. On the other hand this simply maps a section of L^k to its $(k\varepsilon - 1)$-jet at p. It follows that for any n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ with $a_i \geqslant 0$ and $\sum a_i \leqslant k\varepsilon - 1$ there exists a section $f \in H^0(X, L^k)$ such that $\nu(f) = \mathbf{a}$. This implies that the Okounkov body P contains Δ_ε. □

Now suppose that we have a filtration $\{F_i R\}$ on R as in Definition 2. Boucksom–Chen [4] showed how this gives rise to a convex function on the Okounkov body (or concave in their case, since our conventions differ). Briefly the construction goes as follows. For every $t \geqslant 0$ we can define a graded subalgebra $R^{\leqslant t} \subset R$ whose degree k piece is

$$R_k^{\leqslant t} = F_{\lfloor tk \rfloor} R_k. \tag{12}$$

Using only sections of $R^{\leqslant t}$ we can repeat the construction of the Okounkov body, and we will obtain a closed convex subset $P^{\leqslant t} \subset P$, which will be non-empty as long as $t > t_0$ for some constant t_0. The convex transform of the filtration is defined to be the function $G : P \to \mathbf{R}$ given by

$$G(x) = \inf\{t \,:\, x \in P^{\leqslant t}\}. \tag{13}$$

Then G is convex, because of the following convexity property:

$$tP^{\leqslant s_1} + (1-t)P^{\leqslant s_2} \subset P^{\leqslant ts_1 + (1-t)s_2}.$$

It follows that G is continuous on the interior of P, and in [4] it is shown that G is lower semicontinuous on the whole of P. The restriction of G to the simplex Δ_ε from Lemma 5 is also upper semicontinuous (see Gale-Klee-Rockafellar [12]), so in fact G is continuous near the corner $0 \in P$.

We can arrive at the convex function G in a slightly different way too. Namely for each k, we let $G_k : P \to \mathbf{R}$ be the convex envelope of the function

$$\begin{aligned} g_k : \frac{1}{k} P_k &\to \mathbf{R} \\ \alpha &\mapsto \min\{i/k \,:\, \text{there is } f \in F_i R_k \text{ such that } \nu(f) = k\alpha\}, \end{aligned} \tag{14}$$

where we can let $G_k = \infty$ outside the convex hull of $\frac{1}{k} P_k$. It can then be shown that $G_k \geqslant G$ for all k, and $G_k \to G$ uniformly on compact subsets of the interior of P, but G_k might not converge to G on the boundary of P.

A crucial point (see [31, Lemma 3.3]) is that for each $k > 0$ and any function T we have

$$\sum_{i \geqslant 1} T(i/k) \cdot (\dim F_i R_k - \dim F_{i-1} R_k) = \sum_{\alpha \in \frac{1}{k} P_k} T(g_k(\alpha)). \tag{15}$$

In particular, if the filtration comes from a test-configuration, and we write A_k for the generator of the induced $\mathbf{C}^*$-action on on the sections over the central fiber, then

$$\mathrm{Tr}(A_k) = \sum_{i \geqslant 1} -i \cdot (\dim F_i R_k - \dim F_{i-1} R_k) = -k \sum_{\alpha \in \frac{1}{k} P_k} g_k(\alpha). \tag{16}$$

At the same time for continuous T, by [4, Theorem A] and (15), we have the asymptotic result

$$\lim_{k\to\infty} \frac{1}{k^n} \sum_{\alpha\in\frac{1}{k}P_k} T(g_k(\alpha)) = \int_P T\circ G\, d\mu, \tag{17}$$

where μ is the Lebesgue measure on P. This shows for instance that if χ was induced by a test-configuration, then in the expansions (1) we have

$$a_0 = \mathrm{Vol}(P), \quad b_0 = -\int_P G_\chi\, d\mu, \tag{18}$$

where G_χ is the convex transform of the filtration χ. Note that the coefficients a_1 and b_1 cannot be expressed in terms of the Okounkov body and the convex transform in general. This is only possible for very special filtrations, for example the filtrations on toric varieties that we discussed in Sect. 4.

We will often start with a filtration χ, and look at the corresponding sequence of test-configurations $\chi^{(k)}$ obtained from the induced filtration on R_k. The following lemma gives some simple properties of the corresponding convex transforms.

Lemma 6 *Let χ be a filtration on R, and for each k, let $\chi^{(k)}$ be the test-configuration given by the filtration on R_k. Let us also write $\chi^{(k)}$ for the corresponding filtration that we defined in Sect. 3, which is canonically defined on the Veronesi subalgebra $\bigoplus_{i\geqslant 0} R_{ki}$. For each l we can then construct functions*

$$g_l, g_l^{(k)} : \frac{1}{l}P_l \to \mathbf{R},$$

according to (14), and also we have the convex transforms $G, G^{(k)}$. These functions satisfy the following properties:

1. *We have $g_k^{(k)} = g_k$, and $g_{kl}^{(k)} \geqslant g_{kl}$ for each k, l.*
2. *If the filtration χ satisfies $R_1 \subset F_N R$, then $g_{kl}^{(k)} \leqslant N$ for all k, l. In addition $G^{(k)} \leqslant N$ for each k.*
3. *$G^{(k)} \geqslant G$ for all k, and $G^{(k)} \to G$ uniformly on compact subsets of the interior of P.*

Proof Let $F_i R$ be the filtration χ, and for a fixed k write $F_i' R$ for the filtration $\chi^{(k)}$. Then by the construction of $\chi^{(k)}$ we have $F_i' R_k = F_i R_k$ for each i since the filtrations on R_k induced by χ and $\chi^{(k)}$ coincide. In addition, for each $l > 1$ and i, $F_i' R_{kl}$ is the smallest possible subspace, such that the multiplicative property holds for the filtration $\chi^{(k)}$. It follows that

$$F_i' R_{kl} \subset F_i R_{kl} \text{ for each } i, l \geqslant 1. \tag{19}$$

We now prove the 3 statements that we need.

1. Since $F_i' R_{kl} \subset F_i R_{kl}$ for all $i, l \geqslant 1$, we have $g_{kl}^{(k)} \geqslant g_{kl}$. In addition equality holds for $l = 1$ since $F_i' R_k = F_i R_k$ for all i.

2. If $R_1 \subset F_N R$, then the multiplicative property implies $R_k \subset F_{kN} R$. On R_k the filtrations $\chi^{(k)}$ and χ coincide, so we also have $R_k \subset F'_{kN} R$. Using the multiplicative property again, $R_{kl} \subset F'_{klN} R$. This implies that $g_{kl}^{(k)} \leqslant N$ for all k, l. At the same time, using the notation (12) for the filtration $\chi^{(k)}$ we have $R_{kl}^{\leqslant N} = R_{kl}$, so from the construction of the convex transform $G^{(k)}$ we have $G^{(k)} \leqslant N$.
3. The fact that $G^{(k)} \geqslant G$ follows from (19) and the definition of the convex transform. Moreover $G^{(k)}$ is bounded above by the convex envelope of $g_k^{(k)} = g_k$, but on compact subsets of the interior of P, the convex envelopes of g_k converge to G as $k \to \infty$. □

One consequence is the following formula for the norm of a filtration χ.

Lemma 7 *Given a filtration χ, its norm $\|\chi\|_2$ can be expressed in terms of the convex transform G_χ as follows:*

$$\|\chi\|_2^2 = \int_P (G_\chi - \overline{G}_\chi)^2 \, d\mu, \tag{20}$$

where $\overline{G}_\chi$ is the average of G_χ on P.

Proof Recall that we defined the norm $\|\chi\|_2$ by approximating χ using finitely generated filtrations $\chi^{(k)}$, induced by the filtration χ on R_k. Let us write $c_0^{(k)}$ for the constant in the expansion (4) corresponding to the test-configuration $\chi^{(k)}$, and $G^{(k)}$ for the convex transform of $\chi^{(k)}$. From (15) and (17) applied to $T(x) = x^2$, we get

$$c_0^{(k)} = \int_P (G^{(k)})^2 \, d\mu.$$

Using also the formulas analogous to (18) for $\chi^{(k)}$ and the definition of the norm in (5), we get

$$\|\chi^{(k)}\|_2^2 = \int_P (G^{(k)})^2 \, d\mu - \frac{1}{\mathrm{Vol}(P)} \left(\int_P G^{(k)} \, d\mu \right)^2.$$

By Lemma 6 we have $G^{(k)} \to G_\chi$ uniformly on compact subsets of the interior of P, and also all the functions are uniformly bounded by the same constant. Therefore the formula (20) follows by letting $k \to \infty$. □

It is important to note that the Okounkov body P and the convex transform G_χ will in general depend on the point and local coordinates chosen in the construction of the Okounkov body. The volume of P and the integrals in (18) and (20) are however independent of these choices.

We record the following lemmas, which we will use in the next section.

Lemma 8 *Suppose that χ is a filtration for (X, L), and G_χ is its convex transform. The essential supremum of G_χ depends only on χ and not on the data (the point p and local coordinates z_i) used in constructing the Okounkov body.*

Proof Recall that by the definition in Eq. (13), we have

$$G_\chi(x) = \inf\{t \,:\, x \in P^{\leqslant t}\},$$

in terms of the convex sets $P^{\leqslant t} \subset P$ of the Okounkov body, corresponding to the graded subalgebras $R^{\leqslant t} \subset R$ defined in (12). We claim that the essential supremum of G_χ is given by

$$T = \inf\{t \,:\, P^{\leqslant t} = P\}. \tag{21}$$

Indeed it is clear that $\operatorname{ess\,sup} G_\chi \leqslant T$. On the other hand if $s < T$, then $P^{\leqslant s} \neq P$ and these being closed convex bodies there must be an interior point $y \in P$ such that $y \notin P^{\leqslant s}$. In particular $G_\chi(y) \geqslant s$ and since G_χ is convex, this implies that $\operatorname{ess\,sup} G_\chi \geqslant s$. Since $s < T$ was arbitrary, we get $\operatorname{ess\,sup} G_\chi \geqslant T$.

To see that T, defined by (21), is independent of the choice of Okounkov body, note that $P^{\leqslant t} = P$ if and only if $\operatorname{vol} P^{\leqslant t} = \operatorname{vol} P$, since both are closed convex bodies. Moreover the volumes of these convex bodies can be computed as the asymptotic volume of the corresponding linear series (cf. Eq. (17) with $T = 1$):

$$\operatorname{vol} P^{\leqslant t} = \lim_{k\to\infty} \frac{1}{k^n} \dim R_k^{\leqslant t}.$$

In turn the latter asymptotic volume is clearly independent of the choice of Okounkov body. □

Lemma 9 *Suppose that χ is a filtration for (X, L). Write G_χ for the convex transform, and g_k for the function defined in (14). If*

$$\sum_{\alpha\in\frac{1}{k}P_k} g_k(\alpha) - \overline{G}_\chi \dim R_k < 0 \tag{22}$$

for infinitely many k, then (X, L) is asymptotically Chow unstable.

Proof As in Lemma 6, consider the test-configuration $\chi^{(k)}$ given by the induced filtration on R_k. Let us also write A_{kl} for the generator of the $\mathbf{C}^*$-action on R_{kl} given by the test-configuration $\chi^{(k)}$. Writing $g_l^{(k)}$ for the functions corresponding to $\chi^{(k)}$ as in Lemma 6, we have

$$\operatorname{Tr}(A_{kl}) = -kl \sum_{\alpha\in\frac{1}{kl}P_{kl}} g_{kl}^{(k)}(\alpha),$$

from Eq. (16). From Lemma 6 we then get

$$\operatorname{Tr}(A_{kl}) \leqslant -kl \sum_{\alpha\in\frac{1}{kl}P_{kl}} g_{kl}(\alpha),$$

but crucially, equality holds for $l = 1$. It then follows from Eq. (17), that

$$\lim_{k\to\infty} \frac{1}{(kl)^{n+1}} \mathrm{Tr}(A_{kl}) \leqslant -\int_P G_\chi \, d\mu.$$

From the defining formula (2) for the Chow weight of this test-configuration, we get

$$\mathrm{Chow}_k(\chi^{(k)}) \leqslant -\frac{k}{\mathrm{Vol(P)}} \int_P G_\chi \, d\mu + \frac{k}{\dim R_k} \sum_{\alpha \in \frac{1}{k} P_k} g_k(\alpha).$$

Since this is the Chow weight of a test-configuration with exponent k, and by assumption this expression is negative for infinitely many k, it follows that (X, L) is asymptotically Chow unstable. □

6 Extending Stoppa's argument

In this section we will prove Theorem A, which we state again here.

Theorem 10 *Suppose that X admits a cscK metric in $c_1(L)$ and the automorphism group of (X, L) is finite. If χ is a filtration such that $\|\chi\|_2 > 0$, then* $\mathrm{Fut}(\chi) > 0$.

Proof We will first assume that the dimension $n > 1$. Choose a point in X and local coordinates so that we can construct the Okounkov body P of (X, L), and the convex transform G_χ of the filtration. If $\|\chi\|_2 > 0$, then according to the formula (20), the function G_χ is not constant. Let M be the essential supremum of G_χ, and $\overline{G}_\chi$ its average. Let us write

$$\Lambda = \frac{9}{10} M + \frac{1}{10} \overline{G}_\chi,$$

and consider the subalgebra $R^{\leqslant \Lambda} \subset R$. As before, write $P^{\leqslant \Lambda}$ for the convex subset of P obtained by performing the Okounkov body construction using only sections of $R^{\leqslant \Lambda}$. By the construction of G_χ and the choice of Λ, the subset $P^{\leqslant \Lambda} \subset P$ is a proper subset. It follows that

$$\lim_{k\to\infty} k^{-n} \dim R_k^{\leqslant \Lambda} < \lim_{k\to\infty} k^{-n} \dim R_k,$$

since these limits are just the volumes of $P^{\leqslant \Lambda}$ and P. In addition it is shown in [4] that $R^{\leqslant \Lambda}$ contains an ample series (see Definition 17). Applying Theorem 20 we find a point $p \in X$ and a number $\varepsilon > 0$, such that

$$R_k^{\leqslant \Lambda} \subset H^0(X, L^k \otimes I_p^{\lceil k\varepsilon \rceil}), \tag{23}$$

for all k.

We can now go back and use the point p and any choice of local coordinates to construct the Okounkov body P, noting that the statement (23) is independent of these choices. In addition the essential supremum M is unchanged by Lemma 8. We can also assume that ε is small enough such that the simplex Δ_ε satisfies $\Delta_\varepsilon \subset P$ according to Lemma 5. Note that in constructing the Okounkov body, the sections $f \in R_k$ which vanish to order at least $\lceil k\varepsilon \rceil$ at p all satisfy

$$\frac{1}{k}\nu(f) \in \overline{P \backslash \Delta_\varepsilon},$$

so the convex transform (constructed again with the new choice of p) satisfies

$$G_\chi(x) \geqslant \Lambda \text{ for } x \in \Delta_\varepsilon. \tag{24}$$

Now consider the sequence of test-configurations obtained by restricting the filtration χ to R_k for each k, and write $\chi^{(k)}$ for the corresponding filtrations. We will argue by contradiction, assuming that

$$\liminf_{k>0} \mathrm{Fut}(\chi^{(k)}) = 0. \tag{25}$$

Following [26] the key step is to obtain from this a test-configuration for the blowup of X at a suitable point. Let $\delta > 0$ be small. Then we can choose k as large as we like, such that $\mathrm{Fut}(\chi^{(k)}) < \delta$, and to simplify notation, we let $\eta = \chi^{(k)}$. Write G_η for the convex transform of η. Given the point p and parameter ε, we can consider the filtration induced by η on the subalgebra

$$\bigoplus_{k\geqslant 0} H^0(X, L^k \otimes I_p^{\lceil k\varepsilon \rceil}) \subset \bigoplus_{k\geqslant 0} R_k.$$

If ε is rational and less than the Seshadri constant of p in (X, L), then this gives rise to a filtration on the blowup $(Bl_pX, L - \varepsilon E)$, where E is the exceptional divisor. Our goal is to prove that if δ and ε are sufficiently small, then we can use Lemma 9 applied to this filtration to show that the blowup is not asymptotically Chow stable. This will give us the required contradiction, since by Arezzo-Pacard's result [2] the blowup admits a cscK metric for small ε, and so is asymptotically Chow stable by Donaldson's result [8].

To compute the expression (22) on the blowup, note that we can simply work on the part of the Okounkov body P given by $\overline{P \backslash \Delta_\varepsilon}$. We want to show that the numbers

$$Ch_m = \sum_{\alpha \in \overline{P\backslash\Delta_\varepsilon} \cap \frac{1}{m} P_m} g_m(\alpha) - \frac{\int_{P\backslash\Delta_\varepsilon} G_\eta \, d\mu}{\mathrm{Vol}(P\backslash\Delta_\varepsilon)} \dim H^0(X, L^m \otimes I_p^{\lceil m\varepsilon \rceil}) \tag{26}$$

are negative for large m, where the functions g_m are constructed from the filtration η according to (14). We will focus on those m for which $m\varepsilon \in \mathbf{Z}$. At this point is it convenient to introduce normalizations $\widetilde{G}_\eta = G_\eta - \overline{G}_\eta$, and $\widetilde{g}_m = g_m - \overline{G}_\eta$, so that

$\widetilde{G}_\eta$ has zero average. It is easy to see that we can then compute Ch_m using $\widetilde{g}_m$ and $\widetilde{G}_\eta$, and we get the same formula:

$$Ch_m = \sum_{\alpha \in \overline{P \backslash \Delta_\varepsilon} \cap \frac{1}{m} P_m} \widetilde{g}_m(\alpha) - \frac{\int_{P\backslash\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu}{\mathrm{Vol}(P\backslash\Delta_\varepsilon)} \dim H^0(X, L^m \otimes I_p^{\lceil m\varepsilon \rceil}). \tag{27}$$

Replacing g_m by $\widetilde{g}_m$ corresponds to changing the $\mathbf{C}^*$-action on the test-configuration η by an action with constant weights, and this leaves the Futaki invariant unchanged. The advantage is that now in the expansion (1) for η we have $b_0 = 0$, and $\mathrm{Fut}(\eta) = -b_1/a_0$, where b_1 is given by (see (16))

$$\sum_{\alpha \in \frac{1}{k} P_k} \widetilde{g}_m(\alpha) = -b_1 m^{n-1} + O(k^{n-2}). \tag{28}$$

At the same time from the Riemann-Roch Theorem we have

$$\dim H^0(X, L^m \otimes I_p^{\lceil m\varepsilon \rceil}) = (a_0 - \mathrm{Vol}(\Delta_\varepsilon)) m^n + O(m^{n-1}). \tag{29}$$

It will be useful to define two boundary pieces of Δ_ε, namely let $\partial_0 \Delta_\varepsilon$ consist of those faces which meet in the origin, and let $\partial_1 \Delta_\varepsilon$ be the remaining face. In addition we define a boundary measure $d\sigma$, which equals the Lebesgue measure on the faces in $\partial_0 \Delta_\varepsilon$, and is a scaling of the Lebesgue measure on the remaining face $\partial_1 \Delta_\varepsilon$, such that the volume of each face is $\varepsilon^{n-1}/(n-1)!$. Using that $\widetilde{g}_m \geqslant \widetilde{G}_\eta$, we have

$$\begin{aligned} \sum_{\alpha \in \overline{P \backslash \Delta_\varepsilon} \cap \frac{1}{m} P_m} \widetilde{g}_m(\alpha) &= \sum_{\alpha \in \frac{1}{m} P_m} \widetilde{g}_m(\alpha) - \sum_{\alpha \in (\Delta_\varepsilon \backslash \partial_1 \Delta_\varepsilon) \cap \frac{1}{m} P_m} \widetilde{g}_m(\alpha) \\ &\leqslant \sum_{\alpha \in \frac{1}{m} P_m} \widetilde{g}_m(\alpha) - \sum_{\alpha \in (\Delta_\varepsilon \backslash \partial_1 \Delta_\varepsilon) \cap \frac{1}{m} P_m} \widetilde{G}_\eta(\alpha) \\ &= -m^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu + m^{n-1} \left(-b_1 - \frac{1}{2} \int_{\partial_0 \Delta_\varepsilon} \widetilde{G}_\eta \, d\sigma \right. \\ &\quad \left. + \frac{1}{2} \int_{\partial_1 \Delta_\varepsilon} \widetilde{G}_\eta \, d\sigma \right) + O(m^{n-2}). \end{aligned} \tag{30}$$

Here we used an Euler–Maclaurin type formula for the sum of $\tilde{G}_\eta$ over lattice points, see for example Guillemin–Sternberg [13]. Note that the sign of the integral over $\partial_1 \Delta_\varepsilon$ is different because we need to compensate for the fact that the lattice points on $\partial_1 \Delta_\varepsilon$ are missing from the sum.

It will now be convenient to write $M = \overline{G}_\chi + 10\lambda$, and so $\Lambda = \overline{G}_\chi + 9\lambda$, where G_χ is the convex transform of the filtration we started with. From Lemma 6, $G_\eta \to G_\chi$ uniformly on compact subsets of the interior of P as $k \to \infty$, but also $G_\eta \geqslant G_\chi$, so if k is chosen to be large enough, we have

$$\begin{aligned} &G_\eta(x) \geqslant \overline{G}_\chi + 9\lambda \ \text{ for } x \in \Delta_\varepsilon, \\ &\int_{\partial_1 \Delta_\varepsilon} G_\eta \, d\sigma \leqslant (M+\delta)\mathrm{Vol}(\partial_1 \Delta_\varepsilon) = (\overline{G}_\chi + 10\lambda + \delta)\frac{\varepsilon^{n-1}}{(n-1)!}, \end{aligned} \tag{31}$$

where we also used (24). Since $\overline{G}_\eta \to \overline{G}_\chi$ as $k \to \infty$, we can choose k large enough so that (31) implies

$$\begin{aligned} &\widetilde{G}_\eta(x) \geqslant 9\lambda - \delta \ \text{ for } x \in \Delta_\varepsilon, \\ &\int_{\partial_1 \Delta_\varepsilon} \widetilde{G}_\eta \, d\sigma \leqslant (10\lambda + 2\delta)\frac{\varepsilon^{n-1}}{(n-1)!}, \end{aligned} \tag{32}$$

Using these bounds in (30), we have, assuming $n \geqslant 2$ and δ is sufficiently small,

$$\begin{aligned} \sum_{\alpha \in \overline{P\backslash\Delta_\varepsilon} \cap \frac{1}{m}P_m} \widetilde{g}_m(\alpha) &\leqslant -m^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu + m^{n-1}\left(\delta - \frac{4\lambda\varepsilon^{n-1}}{(n-1)!} + \delta\frac{(n+2)\varepsilon^{n-1}}{2(n-1)!}\right) \\ &\quad + O(m^{n-2}) \\ &\leqslant -m^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu + m^{n-1}\left(\delta - \frac{\lambda\varepsilon^{n-1}}{(n-1)!}\right) + O(m^{n-2}). \end{aligned} \tag{33}$$

For the other term in the expression (27) for Ch_m, we have (using that $\widetilde{G}_\eta$ has integral zero)

$$\frac{\int_{P\backslash\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu}{\mathrm{Vol}(\mathrm{P}\backslash\Delta_\varepsilon)}\left[\mathrm{Vol}(P\backslash\Delta_\varepsilon)m^n + O(m^{n-1})\right] \geqslant -m^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu - C\varepsilon^n m^{n-1}, \tag{34}$$

for some C, at least for large enough m. Combining (33) and (34) in the formula (27) we have

$$Ch_m \leqslant m^{n-1}\left(\delta - \frac{\lambda\varepsilon^{n-1}}{(n-1)!} + C\varepsilon^n\right) + O(m^{n-2}).$$

Choosing ε sufficiently small, it follows that if δ is small enough (i.e. we chose k large enough when setting $\eta = \chi^{(k)}$), then $Ch_m < 0$ for all large m. This concludes the proof, in the case when X has dimension $n > 1$.

Suppose now that $n = 1$. We then take the product of X with any cscK manifold, which has finite automorphism group. For example we can take $Y = X \times X$, with the polarization $L_Y = \pi_1^* L \otimes \pi_2^* L$, where π_1, π_2 are the two projection maps. Writing $R^Y = \bigoplus R_k^Y$ for the homogeneous coordinate ring of (Y, L_Y), we have $R_k^Y = R_k \otimes R_k$. A filtration χ for R naturally induces a filtration χ^Y for R^Y, simply by letting

$$F_i R_k^Y = (F_i R_k) \otimes R_k,$$

for each i, k. Moreover this operation commutes with taking the sequence of finitely generated filtrations induced by a given filtration. In other words, the filtration

$(\chi^{(i)})^Y$ coincides with the filtration $(\chi^Y)^{(i)}$. Now suppose that χ is given by a test-configuration, and χ^Y is the induced test-configuration for Y. Writing A_k and A_k^Y for the generators of the corresponding $\mathbf{C}^*$-actions, we can calculate that

$$\mathrm{Tr}(A_k^Y) = (\dim R_k)\mathrm{Tr}(A_k),$$

and

$$\mathrm{Tr}((A_k^Y)^2) = (\dim R_k)\mathrm{Tr}(A_k^2).$$

From these it is straight forward to calculate that

$$\begin{aligned}\mathrm{Fut}(\chi^Y) &= \mathrm{Fut}(\chi)\\ \|\chi^Y\|_2 &= \sqrt{a_0}\|\chi\|_2,\end{aligned}$$

where a_0 is the volume of (X, L) as usual. It follows that the $n = 1$ case is a consequence of the $n = 2$ case that we already proved. □

As we mentioned before, there are many alternative possibilities for defining a Futaki type invariant of a filtration. In the next section we will consider the relation of our work to Donaldson's notion of b-stability, and for this the relevant numerical invariant is the *asymptotic Chow weight*, which we define as

$$\mathrm{Chow}_\infty(\chi) = \liminf_{k\to\infty} \mathrm{Chow}(\chi^{(k)}). \tag{35}$$

Here as in Definition 3, $\chi^{(k)}$ is the test-configuration induced by the filtration χ by restricting χ to R_k. Note that if χ is a finitely generated filtration, then because of (3) we have $\mathrm{Chow}_\infty(\chi) = \mathrm{Fut}(\chi)$, but in general it is not clear what the relationship is between the two invariants.

Proposition 11 *Suppose that X admits a cscK metric in $c_1(L)$ and the automorphism group of (X, L) is finite. Then if χ is a filtration for (X, L) such that $\|\chi\|_2 > 0$, then* $\mathrm{Chow}_\infty(\chi) > 0$.

Proof of Proposition 11 The proof of this proposition is not too different from the proof of Theorem 10. In fact we can follow the proof of Theorem 10 word for word up to Eq. 30, except in Eq. 25 we use the Chow weight instead of the Futaki invariant, and now we will have to control Ch_m for $m = k$. In other words we will not be able to take m much larger than k, as was done in the proof of Theorem 10. This makes the proof more difficult and the convexity of the convex transform plays a crucial role when we apply Lemma 12 below.

Let us fix a small $\delta > 0$, and suppose initially that $n > 1$. We can then find arbitrarily large k, such that the test-configuration $\eta = \chi^{(k)}$ satisfies $\mathrm{Chow}(\eta) < \delta$. As in the proof of Theorem 10, we introduce normalized functions $\widetilde{G}_\eta = G_\eta - \overline{G}_\eta$, and $\widetilde{g}_k = g_k - \overline{G}_\eta$. Then the Chow weight of η is given by

$$\mathrm{Chow}(\eta) = \frac{k}{\dim H^0(X, L^k)} \sum_{\alpha \in \frac{1}{k} P_k} \widetilde{g}_k(\alpha) < \delta. \tag{36}$$

Moreover using the notation from the proof of Theorem 10, if we choose k large enough, then we can assume that $\widetilde{G}_\eta$ satisfies similar bounds to (31):

$$\begin{aligned} \widetilde{G}_\eta(x) &\geqslant 9\lambda - \delta \ \text{ for } x \in \Delta_\varepsilon, \\ \int_{\Delta_\varepsilon \setminus \Delta_{\varepsilon - n/k}} \widetilde{G}_\eta \, d\sigma &\leqslant (10\lambda + 2\delta)\mathrm{Vol}(\Delta_\varepsilon \setminus \Delta_{\varepsilon-n/k}) \leqslant (10\lambda + 2\delta)\frac{n\varepsilon^{n-1}}{k(n-1)!}, \end{aligned} \tag{37}$$

for some $\lambda > 0$. As before, we want to control Ch_k, given by the formula (27), with k instead of m. We also have the inequality (34) as before, so if k is large enough, then

$$Ch_k \leqslant \sum_{\alpha \in \overline{P \setminus \Delta_\varepsilon} \cap \frac{1}{k} P_k} \widetilde{g}_k(\alpha) + k^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu + C\varepsilon^n k^{n-1}. \tag{38}$$

In this equation we have

$$\sum_{\alpha \in \overline{P \setminus \Delta_\varepsilon} \cap \frac{1}{k} P_k} \widetilde{g}_k(\alpha) = \sum_{\alpha \in \frac{1}{k} P_k} \widetilde{g}_k(\alpha) - \sum_{\alpha \in (\Delta_\varepsilon \setminus \partial_1 \Delta_\varepsilon) \cap \frac{1}{k} P_k} \widetilde{g}_k(\alpha), \tag{39}$$

and now we bound the last sum in a different way from what we did before, using Lemma 12 below. Note that if k is large enough, then by changing ε slightly, we can assume that $k\varepsilon \in \mathbf{Z}$. For example we can replace ε by $\frac{1}{k}\lceil k\varepsilon \rceil$ without changing the last sum in (39). Then

$$(\Delta_\varepsilon \setminus \partial_1 \Delta_\varepsilon) \cap \frac{1}{k} P_k = \Delta_{\varepsilon - 1/k} \cap \frac{1}{k} P_k.$$

Using the bound (37) together with Lemma 12 applied to the simplex $\Delta_{\varepsilon-1/k}$, and that $\widetilde{g}_k \geqslant \widetilde{G}_\eta$ on $\frac{1}{k} P_k$, we have

$$\sum_{\alpha \in \Delta_{\varepsilon-1/k} \cap \frac{1}{k} P_k} \widetilde{g}_k(\alpha) \geqslant k^n \int_{\Delta_{\varepsilon-n/k}} \widetilde{G}_\eta \, d\mu + k^{n-1}(9\lambda - \delta)\frac{(3n-1)\varepsilon^{n-1}}{2(n-1)!} - C_1 k^{n-2},$$

where we can choose C_1 to be independent of ε and k. Using (37) again, we get

$$\begin{aligned} \sum_{\alpha \in \Delta_{\varepsilon-1/k} \cap \frac{1}{k} P_k} \widetilde{g}_k(\alpha) \geqslant{}& k^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu - k^{n-1}(10\lambda + 2\delta)\frac{n\varepsilon^{n-1}}{(n-1)!} \\ &+ k^{n-1}(9\lambda - \delta)\frac{(3n-1)\varepsilon^{n-1}}{2(n-1)!} - C_1 k^{n-2} \\ \geqslant{}& k^n \int_{\Delta_\varepsilon} \widetilde{G}_\eta \, d\mu + k^{n-1}\left(\frac{5\lambda}{2} - \frac{7n-1}{2}\delta\right)\frac{\varepsilon^{n-1}}{(n-1)!} - C_1 k^{n-2}, \end{aligned}$$

where we used that $n \geqslant 2$. Putting this together with (39) into the bound (38) for Ch_k, if δ is sufficiently small we get

$$Ch_k \leqslant \sum_{\alpha \in \frac{1}{k} P_k} \widetilde{g}_k(\alpha) - k^{n-1}\left(2\lambda \frac{\varepsilon^{n-1}}{(n-1)!} + C\varepsilon^n\right) + C_1 k^{n-2}.$$

Using the bound (36) on the Chow weight of η, this implies

$$Ch_k \leqslant k^{n-1}\left[\delta \mathrm{Vol}(P) - 2\lambda \frac{\varepsilon^{n-1}}{(n-1)!} + C\varepsilon^n\right] + C_2 k^{n-2},$$

where C_2 can be chosen to be independent of δ. Now if we choose ε, and then δ sufficiently small, then the leading coefficient is negative. So if k is sufficiently large we will have $Ch_k < 0$, and just as in Theorem 10, this gives a contradiction. In addition just as before, the $n = 1$ case can be reduced to the higher dimensional result. □

We used the following lemma.

Lemma 12 *Suppose that for some rational $c \in (0, 1)$, the function f is convex on the simplex*

$$\Delta_c = \{(x_1, \ldots, x_n) : x_i \geqslant 0,\ x_1 + \ldots + x_n \leqslant c\} \subset \mathbf{R}^n,$$

and $f(x) \geqslant L$ for all $x \in \Delta_c$. There is a constant $C(n)$ depending only on the dimension such that for all large k for which $kc \in \mathbf{Z}$ we have

$$\sum_{\alpha \in \Delta_c \cap \frac{1}{k}\mathbf{Z}^n} f(\alpha) \geqslant k^n \int_{\Delta_{c-\frac{n-1}{k}}} f\, d\mu + k^{n-1} L \frac{(3n-1)c^{n-1}}{2(n-1)!} - k^{n-2} C(n) L.$$

With some more work it is likely that the integral can be taken over Δ_c, with a corresponding change in the k^{n-1} term, but for us this simpler result is enough. Such expansions for Riemann sums over polytopes are well known (see e.g. Guillemin-Sternberg [13]), but usually the error term depends on derivatives of the function. The point of this result is that if f is convex, then we have better control on the error term.

Proof First let us assume that $f \geqslant 0$. If Q is a cube with volume $1/k^n$, then Jensen's inequality implies that

$$\frac{1}{2^n} \sum_{v \text{ vertex of } Q} f(v) \geqslant k^n \int_Q f\, d\mu. \tag{40}$$

Now the key point is that we can cover the simplex $\Delta_{c-\frac{n-1}{k}}$ with cubes whose vertices are in $\Delta_c \cap \frac{1}{k}\mathbf{Z}^n$. Applying (40) to all of these cubes, we obtain

$$\sum_{\alpha \in \Delta_c \cap \frac{1}{k}\mathbf{Z}^n} f(\alpha) \geqslant k^n \int_{\Delta_{c-\frac{n-1}{k}}} f\, d\mu, \tag{41}$$

since we will have to count each vertex at most 2^n times. Vertices near the boundary only need to be counted fewer times, but since $f \geqslant 0$, counting them more times just increases the sum.

In general if $f \geqslant L$, then we apply (41) to $f - L$, and we get

$$\sum_{\alpha \in \Delta_c \cap \frac{1}{k}\mathbf{Z}^n} f(\alpha) \geqslant k^n \int_{\Delta_{c-\frac{n-1}{k}}} f\, d\mu - k^n L \mathrm{Vol}(\Delta_{c-\frac{n-1}{k}}) + L \cdot \#(\Delta_c \cap \frac{1}{k}\mathbf{Z}^n), \quad (42)$$

where we know that the number of lattice points in $\Delta_c \cap \frac{1}{k}\mathbf{Z}^n$ is given by

$$\#(\Delta_c \cap \frac{1}{k}\mathbf{Z}^n) = k^n \frac{c^n}{n!} + k^{n-1} \frac{(n+1)c^{n-1}}{2(n-1)!} + O(k^{n-2}).$$

At the same time

$$\mathrm{Vol}(\Delta_{c-\frac{n-1}{k}}) = \frac{c^n}{n!} - \frac{c^{n-1}}{k(n-2)!} + O(k^{-2}).$$

Using these expansions in (42), we get the required result. □

7 Relation to *b*-stability

7.1 Birationally transformed test-configurations

In this section we will show how the main result in Donaldson [6] can be improved using Proposition 11. This result shows that a manifold admitting a cscK metric satisfies a weak version of *b*-stability, which is a notion introduced in Donaldson [7]. We quickly recall one ingredient in the definition of *b*-stability.

The starting point is a test-configuration $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbf{C}$ for the pair (X, L), and for simplicity we assume that the exponent of the test-configuration is 1. In addition, suppose that the central fiber X_0 has a distinguished component B. Using this data, Donaldson defines a family of test-configurations $(\mathcal{X}_i, \mathcal{L}_i) \to \mathbf{C}$, which we will recall below. Given the same data, we can also define a filtration for the homogeneous coordinate ring, similarly to the construction of Witt Nyström in Eq. 7. As before, given any $s \in H^0(X, L^k)$ we extend this as a $\mathbf{C}^*$-invariant meromorphic section $\overline{s}$ of $\mathcal{L}^k$, and now we define for all i, k

$$F_i^B R_k = \{s \in R_k \ :\ t^i \overline{s} \text{ has no pole at the generic point of } B\}. \quad (43)$$

We might need to modify the $\mathbf{C}^*$-action on $\mathcal{L}$ by an action with constant weights to ensure that this filtration satisfies $F_0 R = \mathbf{C}$. Let us write χ for the resulting filtration. We claim that the filtrations of R_k for $k \geqslant 1$ induce a sequence of test-configurations $\chi^{(k)}$, which coincide with the birationally modified test-configurations defined by Donaldson.

One way to see this is using the point of view of the Rees algebras. Let us write $F_i R$ for the filtration corresponding to our test-configuration $\mathcal{X}$. Then we can think of

$$\bigoplus_{i\geqslant 0}(F_i R_k)t^i$$

as all the holomorphic sections of $\mathcal{L}^k$ over $\mathcal{X}$, and

$$\bigoplus_{i\geqslant 0}(F_i^B R_k)t^i$$

as those meromorphic sections of $\mathcal{L}^k$, which only have poles on $X_0\backslash B$. In the notation of [6], we can write this as the sections of $\mathcal{L}^k \otimes \Lambda^m$ for some large enough m, where Λ^m is the sheaf of meromorphic functions with poles of order at most m along $X_0\backslash B$. In Donaldson's construction we need to take sections $\overline{\sigma}_a$ which give a basis in each fiber of $\pi_*(\mathcal{L}^k \otimes \Lambda^m)$. These sections give an embedding of $X \times \mathbf{C}^*$ into $\mathbf{P}^N \times \mathbf{C}$ where $\dim R_k = N+1$, and the new family $(\mathcal{X}_k, \mathcal{L}_k)$ is the closure of the image of this embedding. More explicitly, let us choose a decomposition of R_k as a direct sum

$$R_k = \bigoplus R_{k,i},$$

where for each i we have

$$F_i^B R_k = \bigoplus_{j\leqslant i} R_{k,j}$$

Then choose a basis $\{\sigma_a\}$ for R_k such that each σ_a is in one of the $R_{k,i}$, i.e. $\sigma_a \in R_{k,i_a}$ for some i_a. We can then define $\overline{\sigma}_a = t^{i_a}\sigma_a$ for each a. Since these span the space of sections of $\mathcal{L}^k \otimes \Lambda^m$ over the central fiber under the restriction map

$$\bigoplus_{i\geqslant 0}(F_i^B R_k)t^i \to \bigoplus_{i\geqslant 1}(F_i^B R_k)/(F_{i-1}^B R_k),$$

they give a basis of sections for $\pi_*(\mathcal{L}^k \otimes \Lambda^m)$ at each point. The embedding of $X \times \mathbf{C}^* \to \mathbf{P}^N \times \mathbf{C}$ is then given by

$$(x,t) \mapsto ([t^{a_0}\sigma_0(x) : \ldots : t^{a_N}\sigma_N(x)], t).$$

The closure of this is precisely the test-configuration for X given by the $\mathbf{C}^*$-action with weights $a_0, \ldots, a_N$, which is the same as the test-configuration given by the filtration F_i^B on R_k. Therefore the sequence of birationally transformed test-configuration $(\mathcal{X}_k, \mathcal{L}_k)$ coincides with our test-configurations $\chi^{(k)}$.

From this point of view, the main result of [6] can be rephrased as follows. Write A_k for the generator of the $\mathbf{C}^*$-action on the central fiber of the test-configuration $\chi^{(k)}$, and let N_k be the difference between the maximum and minimum eigenvalues of A_k. Then the result in [6] is the following

Theorem 13 *(Donaldson [6]) Suppose that X admits a cscK metric in $c_1(L)$, and the automorphism group of (X, L) is finite. Assume that central fiber X_0 above is reduced, and the component B does not lie in a hyperplane in $\mathbf{P}(H^0(X_0, L_0)^*)$. Moreover, suppose that for each k, the power $\mathcal{I}_B^k$ of the ideal sheaf of B in $\mathcal{X}$ coincides with the sheaf of holomorphic functions vanishing to order k at the generic point of B. Then there is a constant $C > 0$, such that for all k we have*

$$\mathrm{Chow}(\chi^{(k)}) \geqslant Ck^{-1}N_k. \tag{44}$$

It is natural to define a norm $\|\chi\|_\infty$ of the filtration χ by

$$\|\chi\|_\infty = \liminf_{k\to\infty} \frac{1}{k} N_k.$$

Then (44) is equivalent to saying that if $\|\chi\|_\infty > 0$, then $\mathrm{Chow}_\infty(\chi) > 0$, using the asymptotic Chow weight we defined in Eq. (35).

We will now show that Proposition 11 implies this theorem, even without the condition on the powers $\mathcal{I}_B^k$ of the ideal sheaf of B.

Proposition 14 *Suppose that X admits a cscK metric in $c_1(L)$, and the automorphic group of (X, L) is finite. Suppose that we have a test-configuration for X with reduced central fiber X_0. Suppose that X_0 contains an irreducible component B, which is not contained in a hyperplane in $\mathbf{P}(H^0(X_0, L_0)^*)$. Construct the filtration χ as above. If $\|\chi\|_\infty > 0$, then $\mathrm{Chow}_\infty(\chi) > 0$.*

Proof We just need to show that $\|\chi\|_2 > 0$ in order to apply Proposition 11. If $\|\chi\|_\infty > 0$, then the test-configuration is necessarily non-trivial, and since B is not contained in any hyperplane the $\mathbf{C}^*$-action on $H^0(B, L_0)$ is non-trivial (i.e. it does not have constant weights). We can choose a $\mathbf{C}^*$-invariant complement of the space of sections vanishing on B inside $H^0(X_0, L_0^k)$. Let us write

$$H^0(B, L_0^k) \subset H^0(X_0, L_0^k)$$

for this complementary subspace. By the construction, the weights of the $\mathbf{C}^*$-action of the birationally modified test-configuration $\chi^{(k)}$ on this subspace are the same as the weights of the original test-configuration. Therefore the norm $\|\chi^{(k)}\|_2$ is bounded below by the norm of the $\mathbf{C}^*$-action on (B, L_0) given by the original test-configuration χ. So we just need to check that this $\mathbf{C}^*$-action on (B, L_0) has positive norm. Since χ is non-trivial, the corresponding $\mathbf{C}^*$-action on $H^0(B, L_0)$ does not have constant weights, so the smallest weight λ_{min} differs from the largest weight λ_{max}. Let s_{min} and s_{max} be corresponding $\mathbf{C}^*$-equivariant sections. For any k divisible by 3 we have an inclusion

$$H^0(B, L_0^{k/3}) \hookrightarrow H^0(B, L_0^k),$$

where the map is multiplication by $s_{min}^{2k/3}$. This implies that in the weight decomposition of $H^0(B, L_0^k)$ there will be at least dim $H^0(B, L_0^{k/3})$ sections with weights at most $\frac{k}{3}\lambda_{max} + \frac{2k}{3}\lambda_{min}$. Writing λ_k for the average weight on $H^0(B, L_0^{k/3})$ we then have

$$\mathrm{Tr}\left[\left(A_k - \frac{\mathrm{Tr}(A_k)}{d_k}\right)^2\right] \geqslant c_0 k^n \left(\lambda_k - \frac{k}{3}\lambda_{max} - \frac{2k}{3}\lambda_{min}\right)_+^2$$

for some $c_0 > 0$, where we are writing $(x)_+ = \max\{x, 0\}$.

In an similar way we can also get

$$\mathrm{Tr}\left[\left(A_k - \frac{\mathrm{Tr}(A_k)}{d_k}\right)^2\right] \geqslant c_0 k^n \left(\frac{2k}{3}\lambda_{max} + \frac{k}{3}\lambda_{min} - \lambda_k\right)_+^2$$

Since

$$(\lambda_k - a)_+^2 + (b - \lambda_k)_+^2 \geqslant \frac{1}{2}(b - a)^2$$

for any $a < b$, it follows that

$$\mathrm{Tr}\left[\left(A_k - \frac{\mathrm{Tr}(A_k)}{d_k}\right)^2\right] \geqslant \frac{1}{2} c_0 k^{n+2} \left(\frac{\lambda_{max} - \lambda_{min}}{3}\right)^2.$$

In particular $\|\chi\|_2 > 0$, so we can apply Proposition 11. □

7.2 Filtrations from arcs

In the definition of b-stability, in addition to families of birationally modified test-configurations, one also needs to consider more general degenerations which Donaldson calls arcs.

Just like for test-configurations, we first embed X into a projective space $X \subset \mathbf{P}^N$ using sections of L^r for some r. Then instead of acting by a one-parameter subgroup, we choose a meromorphic map $g : D \to GL(N+1)$, where D is the disk of radius 2 in $\mathbf{C}$ (by rescaling we could use any disk), such that g restricts to a holomorphic map on $\mathbf{C}^*$, and $g(1) = \mathrm{Id}$. Looking at the family $g(t) \cdot X$ for $t \neq 0$, and taking the closure across zero in the Hilbert scheme, we obtain a flat family $\pi : (\mathcal{X}, \mathcal{L}) \to D$, such that the fibers away from 0 are isomorphic to (X, L^r). Conversely any such family can be seen using a meromorphic map $g : D \to GL(N+1)$ once it is embedded into a projective space.

Such degenerations also give rise to filtrations in a similar way to test-configurations. For simplicity we assume that $r = 1$. Thinking of a section $u \in H^0(L^k)$ as a section of $\mathcal{L}^k$ over $\pi^{-1}(1)$, we can extend any section $u \in H^0(L^k)$ to a meromorphic section

$\overline{u}$ of $\mathcal{L}^k$ over $\mathcal{X}$. We define a filtration of $R = \bigoplus_{k\geqslant 0} H^0(L^k)$ by

$$F_i R = \left\{ u \in R \; : \; \begin{array}{l} \text{there exists a holomorphic family of sections} \\ v(t) \in R \text{ such that} \\ t^i(\overline{u} + t\overline{v(t)}) \text{ is holomorphic on } \mathcal{X} \end{array} \right\}, \tag{45}$$

where to ensure that $F_0 R = \mathbf{C}$, we may need to multiply $g(t)$ by a power of t. Note that if $t^i(\overline{u_1} + t\overline{v_1(t)})$ and $t^j(\overline{u_2} + t\overline{v_2(t)})$ are holomorphic, then so is their product

$$t^{i+j}\left[\overline{u_1 u_2} + t\overline{(u_1 v_2(t) + u_2 v_1(t) + t v_1(t) v_2(t))}\right],$$

so $u_1 u_2 \in F_{i+j} R$, and we get a filtration.

This filtration χ gives rise to a sequence of test-configurations $\chi^{(k)}$ as usual. More concretely, for each k, our arc induces a meromorphic family of linear maps on $R_k = H^0(X, L^k)$, which we can think of as a meromorphic family of matrices $g_k(t)$, invertible for $t \neq 0$. As explained in [7, Proposition 2], this family can be factored in the form

$$g_k(t) = L_k(t) t^{A_k} R_k(t), \tag{46}$$

where A_k is a diagonal matrix with entries $t^{\lambda_0}, t^{\lambda_1}, \ldots, t^{\lambda_{N_k}}$, and $L_k(t), R_k(t)$ are holomorphic and invertible for all t. We can then define a flag in R_k, by letting $x \in F_i' R_k$ if t^{A_k} acts on $R_k(0)x$ with weights at least $-i$.

Lemma 15 *The filtrations F_i' and F_i on R_k defined using the factorization* (46) *and by* (45) *respectively coincide.*

Proof We will do this for $k = 1$, and we will drop the k subscript. For any $u \in R_1$, the extension $\overline{u}$ is just given by $g(t)u$. If t^A acts on $R(0)u$ with weights at least $-i$, then $t^i L(t) t^A R(0) u$ is holomorphic, which means that

$$t^i g(t) R(t)^{-1} R(0) u$$

is holomorphic. But $R(t) = R(0) + tS(t)$ for some holomorphic family of matrices $S(t)$, so

$$R(t)^{-1} R(0) u = u - t R(t)^{-1} S(t) u.$$

Letting $v(t) = R(t)^{-1} S(t) u$ we see that $u \in F_i R_1$.

Conversely suppose that we have $v(t)$ such that $t^i g(t)(u + tv(t))$ is holomorphic. Since

$$t^i g(t)(u + tv(t)) = t^i L(t) t^A R(0) u + t^{i+1} L(t) t^A \tilde{v}(t)$$

for some $\tilde{v}(t)$, we see that $R(0)u$ cannot have a non-zero component in a weight space less than $-i$, since the resulting singularity cannot be cancelled using the other term. Therefore $u \in F_i' R_1$. □

Given an arc, an extension of the Chow weight is defined in [7], which coincides with the usual Chow weight if the arc is actually a test-configuration. We will see that this can be computed from the filtration χ. Let us take $r = 1$ again for simplicity. We think of the degeneration as a map $f : D \to \mathrm{Hilb}$, and pull back the Chow line bundle L_{Chow} to D. Picking any element x in the fiber over 1, we can use the map $g(t)$ to define a meromorphic section of L_{Chow} over D, which is holomorphic away from the origin. If this section has a pole of order $-w$, then the Chow weight is essentially w, once we normalize so that each $g(t)$ is in $SL(N+1)$. To compute this, we just need to know that according to Knudsen-Mumford [17], the Chow line bundle can be defined as the leading term λ_{n+1} of the expansion

$$\det \pi_*(\mathcal{L}^k) = \lambda_{n+1}^{\binom{k}{n+1}} \otimes \ldots \otimes \lambda_0, \tag{47}$$

for large k, where λ_i are certain natural $\mathbf{Q}$-line bundles on the base of the family $\pi : (\mathcal{X}, \mathcal{L}) \to D$ (in fact they are pulled back from the Hilbert scheme, under the map f). Note that if we were not only considering matrices in $SL(N+1)$ then we would need an extra factor involving $\det \pi_*(\mathcal{L}^k)$ in the definition of the Chow line bundle.

In terms of the matrices $g_k(t)$ above, we are interested in the asymptotics as $k \to \infty$ of the order of the pole of $\det g_k(t)$ at $t = 0$, where $g_1(t)$ is normalized to be in $SL(N+1)$. From the factorization (46) it is clear that the order of the pole is $-\lambda_0 - \ldots - \lambda_N = -\mathrm{Tr}(A_k)$. The Chow weight is then given up to a positive multiple by the asymptotic formula

$$b_0 = \lim_{k \to \infty} k^{-(n+1)} \mathrm{Tr}(A_k).$$

If $g_1(t)$ were not normalized to be in $SL(N+1)$, then we could compensate for this to get the general formula

$$\widetilde{\mathrm{Chow}}_1(\chi) = \frac{b_0}{a_0} - \frac{w_1}{N+1},$$

where a_0 is the volume of (X, L) as usual. This is analogous to the formula we had in the case of a test-configuration, in Eq. 2. The subscript 1 means that the original test-configuration had exponent 1 (in general the formula changes just like for the usual Chow weight in Eq. (2)). In addition we put a tilde on top to distinguish this Chow weight from the Chow weights $\mathrm{Chow}_k(\chi)$ of the filtration in Definition 4.

In general these two Chow weights are not equal, and in fact for each k we have

$$\widetilde{\mathrm{Chow}}_k(\chi) \geqslant \mathrm{Chow}_k(\chi). \tag{48}$$

This is very similar to what we used in Lemma 9. Indeed, focusing on the case when $k = 1$, recall that $\mathrm{Chow}_1(\chi)$ is the Chow weight of the test-configuration induced by the filtration on R_1. As in Lemma 6, let us write $\chi^{(1)}$ for the corresponding finitely-generated filtration. If we write G_χ and $G_\chi^{(1)}$ for the convex transforms of χ and $\chi^{(1)}$ (corresponding to a fixed Okounkov body), then the two Chow weights are given by

$$\mathrm{Chow}_1(\chi) = -\overline{G}_\chi^{(1)} - \frac{w_1}{N+1},$$
$$\widetilde{\mathrm{Chow}}_1(\chi) = -\overline{G}_\chi - \frac{w_1}{N+1},$$

where we used the relations (18) for both χ and $\chi^{(1)}$. From Lemma 6 we know that $G_\chi^{(1)} \geqslant G_\chi$, so the inequality (48) on the Chow weights follows. It should not be surprising that we get a smaller Chow weight by looking at the corresponding test-configuration, since by the Hilbert-Mumford criterion we know that in testing for Chow stability, it is enough to look at test-configurations and we do not need general arcs.

Let us now combine arcs with the construction from the previous section, so let us suppose that we have a distinguished component B in the central fiber of our arc $\mathcal{X}$. Just as in the case of test-configurations, Donaldson constructs a sequence of arcs $\mathcal{X}_i$. At the same time, we can also obtain a filtration χ just like in Eq. (43), by letting

$$F_i^B R_k = \left\{ u \in R_k \ : \ \begin{array}{l} t^i(\overline{u} + t\overline{v(t)}) \text{ has no pole at} \\ \text{the generic point of } B \text{ for some } v(t) \end{array} \right\},$$

Now the sequence of test-configurations $\chi^{(i)}$ induced by χ are certainly not the same as the arcs $\mathcal{X}_i$. Instead for each i, the test-configuration $\chi^{(i)}$ is simply the test-configuration given by the filtration on $H^0(X, L^i)$ which is induced by the arc $\mathcal{X}_i$. It follows then in the same way as above, that the Chow weight of the arc $\mathcal{X}_i$ is bounded from below by the Chow weight $\mathrm{Chow}_i(\chi)$ of the test-configuration $\chi^{(i)}$. In other words

$$\liminf_{i\to\infty} \widetilde{\mathrm{Chow}}_i(\mathcal{X}_i) \geqslant \mathrm{Chow}_\infty(\chi),$$

in terms of the asymptotic Chow weight of the filtration.

The conclusion from all this is that Proposition 11 can be used to obtain a result analogous to Proposition 14 for arcs instead of just test-configurations.

7.3 Webs of descendants

The full definition of b-stability in [7] focuses more on the possible central fibers rather than the degenerations themselves. This leads to extra complications, since a given scheme could be the central fiber of several different degenerations. It is not clear whether filtrations are versatile enough to encode this richer data of what Donaldson calls a "web of descendants", so we leave a more detailed examination of this to future studies.

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Appendix: Asymptotic vanishing orders of graded linear series—S. Boucksom[1]

Iitaka dimension and multiplicity

Let X be a projective variety over an algebraically closed field k (of any characteristic), set $n := \dim X$, and let L be a line bundle on X. Denote by

$$R = R(X, L) := \bigoplus_{m \in \mathbb{N}} H^0(X, mL)$$

the algebra of sections of L. Given a graded subalgebra S of R (aka *graded linear series* of L), set

$$\mathbb{N}(S) := \{m \in \mathbb{N} \mid S_m \neq 0\},$$

which is a sub-semigroup of of $\mathbb{N}$, hence coincides outside a finite set with the multiples of the gcd $m(S) \in \mathbb{N}$ of $\mathbb{N}(S)$, sometimes known as the *exponent* of S. Define also the *Iitaka dimension* of S as $\kappa(S) := \operatorname{tr.deg}(S/k) - 1$ if $S \neq k$, and $\kappa(S) := -\infty$ otherwise, so that $\kappa(S) \in \{-\infty, 0, 1, ..., n\}$.

In this generality, the following result is due to Kaveh and Khovanskii [16] (see also [3]).

Theorem 16 *Let $S \neq k$ be a graded subalgebra of $R(X, L)$, and write $\kappa = \kappa(S)$.*

(i) The multiplicity

$$e(S) = \lim_{m \in \mathbb{N}(S),\, m \to \infty} \frac{\kappa!}{m^\kappa} \dim S_m$$

exists in $]0, +\infty[$.

(ii) For each $m \in \mathbb{N}(S)$, let $\Phi_m : X \dashrightarrow \mathbb{P}(S_m^*)$ be the rational map defined by linear series S_m, and denote by Y_m its image. Then we have $\dim Y_m = \kappa$ *for all $m \in \mathbb{N}(S)$ large enough, and*

$$e(S) = \lim_{m \in \mathbb{N}(S),\, m \to \infty} \frac{\deg Y_m}{m^\kappa}.$$

Note that L is big iff $\kappa(X, L) := \kappa(R)$ is equal to $n := \dim X$, and we then have $e(R) = \operatorname{vol}(L)$, the *volume* of L.

Definition 17 We say that *S contains an ample series* if

(i) $S_m \neq 0$ for all $m \gg 1$, i.e. S has exponent $m(S) = 1$.
(ii) There exists a decomposition $L = A + E$ into $\mathbb{Q}$-divisors with A ample and E effective such that $H^0(X, mA) \subset S_m \subset H^0(X, mL)$ for all sufficiently divisible $m \in \mathbb{N}$.

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This condition immediately implies that the rational map $\Phi_m : X \dashrightarrow \mathbb{P}(S_m^*)$ defined by S_m in birational onto its image Y_m for all $m \gg 1$.

Assuming this, let $\mathfrak{b}_m \subset \mathcal{O}_X$ be the *base-ideal* of S_m, i.e. the image of the evaluation map $S_m \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X$. Let $\mu_m : X_m \to X$ be any birational morphism with X_m normal and projective and such that $\mathfrak{b}_m \cdot \mathcal{O}_{X_m}$ is locally principal, hence of the form $\mathcal{O}_{X_m}(-F_m)$ for an effective Cartier divisor F_m on X_m. We then set

$$P_m := \mu_m^* L - \tfrac{1}{m} F_m, \tag{49}$$

which is a nef $\mathbb{Q}$-Cartier divisor on X_m. If m divides l, then we may choose X_l to dominate X_m, and we have $P_l \geq P_m$ after pulling back to X_l (in the sense that the difference is an effective $\mathbb{Q}$-divisor). Note also that the intersection number (P_m^n) does not depend on the choice of X_m by the projection formula, and that $(P_l^n) \geq (P_m^n)$ when m divides l, since P_m and P_l are nef with $P_l \geq P_m$.

As a consequence of Theorem 16 above (see also [15, Theorem C]), we get the following version of the Fujita approximation theorem:

Corollary 18 *Let S be a graded subalgebra of R, and assume that S contains an ample series. Then $e(S) = \lim_{m\to\infty}(P_m^n)$.*

Proof With the notation of Theorem 16, the rational map Φ_m lifts to a morphism $f_m : X_m \to \mathbb{P}(S_m^*)$ which is birational onto its image Y_m and such that $f_m^*\mathcal{O}(1) = \mu_m^*(mL) - F_m = mP_m$. We thus see that

$$(P_m^n) = \frac{\deg Y_m}{m^n},$$

and the result follows from (ii) in Theorem 16. □

Remark 19 The special case of Theorem 16 where S contains an ample series, which is what is being used in the previous corollary, was first established in [18].

Asymptotic vanishing orders and multiplicities

Our goal is to prove the following result.

Theorem 20 *Let X be a smooth projective variety over an algebraically closed field k, and let L be a line bundle on X. Let S be a graded subalgebra of $R = R(X, L)$, and assume that S contains an ample series. Assume also that $e(S) < e(R) = \mathrm{vol}(L)$. Then there exists $\varepsilon > 0$ and a (closed) point $x \in X$ with maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ such that $S_m \subset H^0\left(X, mL \otimes \mathfrak{m}_x^{\lfloor m\varepsilon \rfloor}\right)$ for all m.*

Recall that a *divisorial valuation* (aka *discrete valuation of rank* 1) on X is a valuation $v : k(X)^* \to \mathbb{R}$ of the form $v = c\,\mathrm{ord}_E$ with $c > 0$ and E a prime divisor on a birational model X' of X, which can always be assumed to be normal, projective and to dominate X. In particular, since X is smooth, every scheme theoretic point

$\xi \in X$ defines a divisorial valuation ord_ξ. If we denote by $V = \overline{\{\xi\}}$ the subvariety of X having ξ as its generic point, then we have for all $f \in \mathcal{O}_{X,x}$

$$\operatorname{ord}_\xi(f) = \min_{x \in V} \operatorname{ord}_x(f). \tag{50}$$

If we still denote by $\mathfrak{b}_m$ the base-ideal of S_m, then each divisorial valuation v on X defines a subadditive sequence

$$v(\mathfrak{b}_m) := \min\{v(f) \mid f \in \mathfrak{b}_m \setminus \{0\}\},$$

and we may thus define the *asymptotic vanishing order of S along v* (cf. [10]) as

$$v(S) := \lim_{m \to \infty} \frac{v(\mathfrak{b}_m)}{m} \in [0, +\infty[.$$

In this language, the conclusion of Theorem 20 amounts to the existence of a closed point $x \in X$ such that $\operatorname{ord}_x(S) > 0$. We begin with the following consequence of Izumi's theorem on divisorial valuations.

Lemma 21 *If there exists a divisorial valuation v on X such that $v(S) > 0$, then* $\operatorname{ord}_x(S) > 0$ *for some closed point $x \in X$.*

Proof Let $\xi \in X$ be the center of v on X (concretely, there exists a birational morphism $\mu : X' \to X$ with X' projective and a prime divisor $E \subset X'$ such that $v = c \operatorname{ord}_E$, $c > 0$, and ξ is then the generic point of $\mu(E) \subset X$). Since the divisorial valuations ord_ξ and v share the same center ξ on X, the version of Izumi's theorem proved in [14, Theorem 1.2] implies that there exists $C > 0$ such that

$$C^{-1} v(f) \leq \operatorname{ord}_\xi(f) \leq C v(f)$$

for all $f \in \mathcal{O}_{X,\xi}$. Applying this to $f \in \mathfrak{b}_m$ yields in the limit as $m \to \infty$

$$\operatorname{ord}_\xi(S) \geq C^{-1} v(S) > 0.$$

But for any closed point $x \in \overline{\{\xi\}}$ we also have $\operatorname{ord}_x \geq \operatorname{ord}_\xi$ on $\mathcal{O}_{X,x}$ by (50), and this similarly implies $\operatorname{ord}_x(S) \geq \operatorname{ord}_\xi(S)$, hence $\operatorname{ord}_x(S) > 0$. □

As a consequence of Corollary 18, we next prove:

Lemma 22 *Let S, S' be two graded subalgebras of R containing an ample series. If $v(S) \geq v(S')$ for all divisorial valuations v, then $e(S) \leq e(S')$.*

Proof Let $\mathfrak{b}_m, \mathfrak{b}'_m \subset \mathcal{O}_X$ be the base-ideals of S_m and S'_m respectively, and let P_m and P'_m be the nef $\mathbb{Q}$-Cartier divisors they determine on some high enough model X_m over X, as in (49).

Given $\varepsilon > 0$, Corollary 18 allows to find $m_0 \in \mathbb{N}$ such that $e(S) \leq (P^n_{m_0}) + \varepsilon$, and hence

$$e(S) \leq (P_{m_1} \cdot P^{n-1}_{m_0}) + \varepsilon \tag{51}$$

for any multiple m_1 of m_0, since P_{m_0} is nef and $P_{m_0} \le P_{m_1}$. By the projection formula and the definition of P_{m_1} and P'_{m_1}, we have

$$(P_{m_1} \cdot P_{m_0}^{n-1}) - (P'_{m_1} \cdot P_{m_0}^{n-1}) = \sum_{E \subset X_{m_0}} \left(\frac{\mathrm{ord}_E(F'_{m_1})}{m_1} - \frac{\mathrm{ord}_E(F_{m_1})}{m_1} \right) (E \cdot P_{m_0}^{n-1}),$$

where the sum runs over prime divisors E of X_{m_0} and any E actually contributing to the sum is contained in the support of $F_{m_0} + F'_{m_0}$, hence belongs to a finite set of prime divisors of X_{m_0} independent of m_1. Since we have by assumption

$$\lim_{m_1 \to \infty} \frac{\mathrm{ord}_E(F_{m_1})}{m_1} = \mathrm{ord}_E(S) \ge \mathrm{ord}_E(S') = \lim_{m_1 \to \infty} \frac{\mathrm{ord}_E(F'_{m_1})}{m_1}$$

for any such E, we may thus choose m_1 a large enough multiple of m_0 to guarantee that

$$(P_{m_1} \cdot P_{m_0}^{n-1}) \le (P'_{m_1} \cdot P_{m_0}^{n-1}) + \varepsilon,$$

and hence

$$e(S) \le (P'_{m_1} \cdot P_{m_2} \cdot P_{m_0}^{n-2}) + 2\varepsilon \tag{52}$$

for any multiple m_2 of m_1, by (51) and the fact that $P_{m_0}, P'_{m_1}, P_{m_2}$ are nef with $P_{m_0} \le P_{m_2}$. We similarly have

$$\begin{aligned}(P'_{m_1} \cdot P_{m_2} \cdot P_{m_0}^{n-2}) - (P'_{m_1} \cdot P'_{m_2} \cdot P_{m_0}^{n-2}) = \\ = \sum_{E \subset X_{m_1}} \left(\frac{\mathrm{ord}_E(F'_{m_2})}{m_2} - \frac{\mathrm{ord}_E(F_{m_2})}{m_2} \right) (P'_{m_1} \cdot E \cdot P_{m_0}^{n-1}) \le \varepsilon\end{aligned}$$

for m_2 large enough, hence

$$e(S) \le (P'_{m_1} \cdot P'_{m_2} \cdot P_{m_3} \cdot P_{m_0}^{n-3}) + 3\varepsilon$$

for any multiple m_3 of m_2, using (52) and $P_{m_0} \le P_{m_3}$. Continuing in this way, we finally obtain positive integers m_1, ..., m_n with m_i dividing m_{i+1} and such that

$$e(S) \le (P'_{m_1} \cdot \ldots \cdot P'_{m_n}) + (n+1)\varepsilon,$$

hence

$$e(S) \le (P'^n_{m_n}) + (n+1)\varepsilon$$

since $P_{m_i} \le P_{m_n}$. But m_n can be taken to be as large as desired, thus $(P'^n_{m_n})$ is as close to $e(S')$ as we like by Corollary 18, and we conclude as desired that $e(S) \le e(S')$. □

Proof of Theorem 20 By Lemma 22, the assumption $e(S) < e(R)$ implies that $v(S) > v(R) \ge 0$ for some divisorial valuation v. We conclude using Lemma 21. □

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