Math. Ann. (2015) 361:725–740
DOI 10.1007/s00208-014-1090-y

Mathematische Annalen

Evolution of contractions by mean curvature flow

Andreas Savas-Halilaj · Knut Smoczyk

Received: 6 December 2013 / Revised: 12 August 2014 / Published online: 23 August 2014

Abstract In this article we investigate length decreasing maps $f : M \to N$ between Riemannian manifolds M, N of dimensions $m \geq 2$ and n, respectively. Assuming that M is compact and N is complete such that

$$\sec_M > -\sigma \quad \text{and} \quad \mathrm{Ric}_M \geq (m-1)\sigma \geq (m-1)\sec_N \geq -\mu,$$

where σ, μ are positive constants, we show that the mean curvature flow provides a smooth homotopy of f into a constant map.

Mathematics Subject Classification 53C44 · 53C42 · 57R52 · 35K55

1 Introduction

Let $f : M \to N$ be a smooth map between Riemannian manifolds. To any such f we assign a geometric quantity called *k-dilation*, which measures how much the map stretches k-dimensional volumes. For example the 1-dilation coincides with the Lipschitz constant of the map. The map f is called a *contraction* or *weakly length decreasing* if its 1-dilation is less or equal to 1. Equivalently, the map f is a contraction if

---

The first author is supported financially by the grant $E\Sigma\Pi A$: PE1-417.

A. Savas-Halilaj (✉) · K. Smoczyk
Institut für Differentialgeometrie and Riemann Center for Geometry and Physics,
Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
e-mail: andreas.savas-halilaj@math.uni-hannover.de

K. Smoczyk
e-mail: smoczyk@math.uni-hannover.de

$$f^*\mathrm{g}_N \leq \mathrm{g}_M,$$

where g_M, g_N stand for the Riemannian metrics of M and N, respectively. In particular, the map f will be called *strictly length decreasing* if $f^*\mathrm{g}_N < \mathrm{g}_M$ everywhere and an *isometry* if $f^*\mathrm{g}_N \equiv \mathrm{g}_M$.

If $M = \mathbb{S}^m$ and $N = \mathbb{S}^n$ are unit spheres and $f : \mathbb{S}^m \to \mathbb{S}^n$ is a strictly length decreasing map, then the diameter of $f(\mathbb{S}^m)$ is strictly less than π which implies that the map f is not surjective. Hence, f must be null-homotopic. Tsui and Wang [12] proved that maps $f : \mathbb{S}^m \to \mathbb{S}^n$ between unit spheres with 2-dilation strictly less than 1, or equivalently *strictly area decreasing*, are also homotopic to a constant map. As it was shown by Guth [2,3] this result cannot be extended in the case of k-dilation for $k \geq 3$.

Based on ideas developed in [12,13], Lee and Lee [5] proved that any strictly area decreasing map between compact Riemannian manifolds M and N whose sectional curvatures are bounded by $\sec_M \geq \sigma_1$ and $\sigma_2 \geq \sec_N$, where σ_1, σ_2 are two real constants such that

$$\sigma_1 \geq \sigma_2 > 0 \quad \text{or} \quad \sigma_1 > 0 \geq \sigma_2,$$

is homotopic by mean curvature flow to a constant map. We would like to point out here that the curvature assumptions can be relaxed even much further as it was shown in [7]. The goal of this paper is to show that in the length decreasing case one can drop the compactness assumption on N. More precisely we prove:

Theorem *Let M and N be two Riemannian manifolds with M being compact and N complete. Assume that $m = \dim M \geq 2$ and that there exist positive constants σ, μ such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N and the Ricci curvature Ric_M of M satisfy*

$$\sec_M > -\sigma \quad \textit{and} \quad \mathrm{Ric}_M \geq (m-1)\sigma \geq (m-1)\sec_N \geq -\mu.$$

Let $f : M \to N$ be a strictly length decreasing map. Then the mean curvature flow of the graph of f remains the graph of a strictly length decreasing map, exists for all time and f converges to a constant map.

In the case where N is compact the above result is contained in our previous paper [7]. The key argument to remove the compactness is an estimate on the mean curvature vector field of the evolving graphs. In particular, we prove that the norm of the mean curvature vector field remains uniformly bounded in time. This will imply that the evolving graphs stay in compact regions of $M \times N$ on time intervals $[0, T)$, with $T < \infty$. Using this estimate, the blow-up analysis of Wang [13] and White's regularity theorem [14] we are able to prove that the maximal time T of existence of the flow is ∞. To prove the mean curvature estimate, we introduce a tensor on the normal bundle of the evolving graphs and compare the maximum of the norm of the mean curvature with the biggest eigenvalue of this tensor.

2 Graphs

2.1 Basic facts

We follow here the notations of our previous two papers [7,8]. The product manifold $M \times N$ will always be regarded as a Riemannian manifold equipped with the metric

$$\mathrm{g}_{M\times N} = \langle \cdot\,,\cdot\rangle := \mathrm{g}_M \times \mathrm{g}_N .$$

The *graph* of a map $f : M \to N$ is defined to be the submanifold

$$\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}$$

of the product $M \times N$. The graph $\Gamma(f)$ can be parametrized via the embedding $F : M \to M \times N$, $F := I_M \times f$, where I_M is the identity map of M.

The Riemannian metric induced by F on M will be denoted by

$$\mathrm{g} := F^*\mathrm{g}_{M\times N} .$$

The two natural projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are submersions, that is they are smooth and have maximal rank. The tangent bundle of the product manifold $M \times N$, splits as a direct sum

$$T(M \times N) = TM \oplus TN.$$

The four metric tensors g_M, g_N, $\mathrm{g}_{M\times N}$ and g are related by

$$\begin{aligned} \mathrm{g}_{M\times N} &= \pi_M^*\mathrm{g}_M + \pi_N^*\mathrm{g}_N, \\ \mathrm{g} &= F^*\mathrm{g}_{M\times N} = \mathrm{g}_M + f^*\mathrm{g}_N . \end{aligned}$$

As in [7,8], define the symmetric 2-tensors

$$\begin{aligned} \mathrm{s}_{M\times N} &:= \pi_M^*\mathrm{g}_M - \pi_N^*\mathrm{g}_N, \\ \mathrm{s} &:= F^*\mathrm{s}_{M\times N} = \mathrm{g}_M - f^*\mathrm{g}_N . \end{aligned}$$

The Levi-Civita connection $\nabla^{\mathrm{g}_{M\times N}}$ associated to $\mathrm{g}_{M\times N}$ is related to the Levi-Civita connections ∇^{g_M} on (M, g_M) and ∇^{g_N} on (N, g_N) by

$$\nabla^{\mathrm{g}_{M\times N}} = \pi_M^*\nabla^{\mathrm{g}_M} \oplus \pi_N^*\nabla^{\mathrm{g}_N} .$$

The corresponding curvature operator $\mathrm{R}_{M\times N}$ on $M \times N$ with respect to the metric $\mathrm{g}_{M\times N}$ is related to the curvature operators R_M on (M, g_M) and R_N on (N, g_N) by

$$\mathrm{R}_{M\times N}=\pi_M^*\mathrm{R}_M\oplus\pi_N^*\mathrm{R}_N.$$

The Levi-Civita connection on M with respect to the induced metric g is denoted by ∇, the curvature tensor by R and the Ricci curvature by Ric.

2.2 The second fundamental form

The differential $\mathrm{d}F$ of F can be regarded as a section in the induced bundle $F^*T(M\times N)\otimes T^*M$. In the sequel we will denote all full connections on bundles over M that are induced by the Levi-Civita connection on $M\times N$ via the immersion $F: M\to M\times N$ by the same letter ∇. The covariant derivative of $\mathrm{d}F$ is called the *second fundamental form* of the immersion F and it will be denoted by A. That is

$$A(v,w):=(\nabla\mathrm{d}F)(v,w),$$

for any vector fields $v, w\in TM$. If ξ is a normal vector of the graph, then the symmetric tensor A_ξ given by

$$A_\xi(v,w):=\langle A(v,w),\xi\rangle$$

is called the *second fundamental form with respect to the direction* ξ.

The trace of A with respect to the metric g is called the *mean curvature vector field* of $\Gamma(f)$ and it will be denoted by

$$H:=\mathrm{trace}_{\mathrm{g}}A.$$

Note that H is a section in the normal bundle $\mathcal{N}M$. The graph $\Gamma(f)$ is called *minimal* if H vanishes identically.

Every vector V of $F^*T(M\times N)$ can be decomposed as

$$V=V^\top+V^\perp,$$

where $V^\top$ is the *tangential component* and $V^\perp$ for the *normal component* of V along F. Introduce now the natural projection map $\mathrm{pr}: F^*T(M\times N)\to\mathcal{N}M$, $\mathrm{pr}(V):=V^\perp$. We can express this map locally as

$$\mathrm{pr}(V)=V-\sum_{k,l=1}^m \mathrm{g}^{kl}\langle V,\mathrm{d}F(\partial_k)\rangle\,\mathrm{d}F(\partial_l),$$

where $\{\partial_1,\dots,\partial_m\}$ is the basis of a local coordinate chart defined on an open neighborhood of the manifold M and g^{kl} are the components of the inverse matrix $(\mathrm{g}_{kl})^{-1}$, where $\mathrm{g}_{kl}=\mathrm{g}(\partial_k,\partial_l)$, $1\le k,l\le m$. The connection of the normal bundle will be denoted by the letter $\nabla^\perp$ and is defined by

$$\nabla^\perp_v\xi:=\mathrm{pr}\left(\nabla_v\xi\right),$$

where here $v \in TM$ and $\xi \in \mathcal{N}M$. The Laplacian with respect to $\nabla^\perp$ will be denoted by $\Delta^\perp$.

By *Gauß' equation* the tensors R and $\mathrm{R}_{M\times N}$ are related by the formula

$$\begin{aligned}&\big(\mathrm{R} - F^*\mathrm{R}_{M\times N}\big)(v_1, w_1, v_2, w_2)\\&\quad = \big\langle A(v_1, v_2), A(w_1, w_2)\big\rangle - \big\langle A(v_1, w_2), A(w_1, v_2)\big\rangle,\end{aligned}$$

and the second fundamental form satisfies the *Codazzi equation*

$$\begin{aligned}&(\nabla_u A)(v, w) - (\nabla_v A)(u, w)\\&\quad = \mathrm{R}_{M\times N}\big(\mathrm{d}F(u), \mathrm{d}F(v), \mathrm{d}F(w)\big) - \mathrm{d}F\big(\mathrm{R}(u, v, w)\big),\end{aligned}$$

for any $u, v, w, v_1, v_2, w_1, w_2 \in TM$.

2.3 Singular decomposition

As in [7,8], fix a point $x \in M$ and let $\lambda_1^2 \leq \cdots \leq \lambda_m^2$ be the eigenvalues at x of $f^*\mathrm{g}_N$ with respect to g_M. The corresponding values $\lambda_i \geq 0$, $i \in \{1, \ldots, m\}$, are called *singular values* of the differential $\mathrm{d}f$ of f at the point x. It turns out that the singular values depend continuously on x. Set $r := \operatorname{rank} \mathrm{d}f(x)$. Obviously, $r \leq \min\{m, n\}$ and

$$\lambda_1 = \cdots = \lambda_{m-r} = 0.$$

At the point x consider an orthonormal basis

$$\{\alpha_1, \ldots, \alpha_{m-r}; \alpha_{m-r+1}, \ldots, \alpha_m\}$$

with respect to g_M which diagonalizes $f^*\mathrm{g}_N$. Furthermore, at the point $f(x)$ consider an orthonormal basis

$$\{\beta_1, \ldots, \beta_{n-r}; \beta_{n-r+1}, \ldots, \beta_n\}$$

with respect to g_N such that

$$\mathrm{d}f(\alpha_i) = \lambda_i \beta_{n-m+i},$$

for any $i \in \{m - r + 1, \ldots, m\}$. We may define a special basis for the tangent and the normal space of $\Gamma(f)$ in terms of the singular values. The vectors

$$e_i := \begin{cases} \alpha_i, & 1 \leq i \leq m - r,\\ \frac{1}{\sqrt{1+\lambda_i^2}}\,(\alpha_i \oplus \lambda_i \beta_{n-m+i}), & m - r + 1 \leq i \leq m,\end{cases}$$

form an orthonormal basis with respect to the metric $\mathrm{g}_{M\times N}$ of the tangent space $\mathrm{d}F(T_x M)$ of the graph $\Gamma(f)$ at x. The vectors

$$\xi_i := \begin{cases} \beta_i, & 1 \le i \le n-r, \\ \frac{1}{\sqrt{1+\lambda^2_{i+m-n}}}\left(-\lambda_{i+m-n}\alpha_{i+m-n} \oplus \beta_i\right), & n-r+1 \le i \le n, \end{cases}$$

give an orthonormal basis with respect to $\mathrm{g}_{M\times N}$ of the normal space $\mathcal{N}_x M$ of the graph $\Gamma(f)$ at the point $F(x)$. Note that

$$\mathrm{s}_{M\times N}(e_i, e_j) = \frac{1-\lambda_i^2}{1+\lambda_i^2}\delta_{ij}, \quad 1 \le i, j \le m. \tag{2.1}$$

Thus, the map f is strictly length decreasing if and only if the symmetric 2-tensor s is positive.

Denote by $\mathrm{s}^{\perp}$ the restriction of $\mathrm{s}_{M\times N}$ to the normal bundle of the graph. Then, we can readily check that

$$\mathrm{s}^{\perp}(\xi_i, \xi_j) = \begin{cases} -\delta_{ij}, & 1 \le i \le n-r, \\ -\frac{1-\lambda^2_{i+m-n}}{1+\lambda^2_{i+m-n}}\delta_{ij}, & n-r+1 \le i \le n. \end{cases} \tag{2.2}$$

Hence, if there exists a positive constant ε such that $\mathrm{s} \ge \varepsilon\, \mathrm{g}$, then $\mathrm{s}^{\perp} \le -\varepsilon\, \mathrm{g}^{\perp}$, where $\mathrm{g}^{\perp}$ stands for the restriction of $\mathrm{g}_{M\times N}$ on $\mathcal{N}M$. Furthermore,

$$\mathrm{s}_{M\times N}(e_{m-r+i}, \xi_{n-r+j}) = -\frac{2\lambda_{m-r+i}}{1+\lambda^2_{m-r+i}}\delta_{ij}, \quad 1 \le i, j \le r. \tag{2.3}$$

Moreover, the value of $\mathrm{s}_{M\times N}$ on any other mixed term is zero.

3 Evolution equations

Let M and N be Riemannian manifolds, $f : M \to N$ a smooth map and $F : M \to M \times N$, $F := I_M \times f$, the graph $\Gamma(f)$ of f. Deform $\Gamma(f)$ by mean curvature flow in the product space $M \times N$. That is consider the family of immersions $F : M \times [0, T) \to M \times N$ satisfying the equation

$$\begin{cases} \frac{dF}{dt}(x,t) = H(x,t) \\ F(x,0) = F(x) \end{cases}$$

where $(x,t) \in M \times [0,T)$, $H(x,t)$ is the mean curvature vector field at $x \in M$ of the immersion $F_t : M \to M \times N$ given by $F_t(\,\cdot\,) := F(\,\cdot\,, t)$ and T the maximal time of existence of the solution. From the compactness of M it follows that the evolving submanifold stays a graph on an interval $[0, T_g)$ with $T_g \le T$, that is there exists a family of diffeomorphisms $\phi_t : M \to M$ and a family of maps $f_t : M \to N$ such that $F_t \circ \phi_t = I_M \times f_t$, for any $t \in [0, T_g)$. In the matter of fact, under the assumptions of the Theorem, the singular values of f remain uniformly bounded in time and the solution of the mean curvature flow stays a graph as long as the flow exists. This result follows from the next lemma, which still holds in the case where N is complete.

Lemma 3.1 ([7]) *Let M and N be Riemannian manifolds with M being compact and N complete. Assume that $m = \dim M \geq 2$ and that there exists a positive constant σ such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N and the Ricci curvature Ric_M of M satisfy*

$$\sec_M > -\sigma \quad \textit{and} \quad \mathrm{Ric}_M \geq (m-1)\sigma \geq (m-1)\sec_N.$$

Let $f : M \to N$ be a strictly length decreasing map such that $\mathrm{s} \geq \varepsilon\, \mathrm{g}$, where ε is a positive constant. Then the inequality $\mathrm{s} \geq \varepsilon\, \mathrm{g}$ is preserved under the mean curvature flow. Furthermore, $T_g = T$.

Now we claim that the norm of the mean curvature vector remains bounded in time. Inspired on ideas developed for the Lagrangian mean curvature flow in [11] (see also [1] for the Lagrangian mean curvature flow of non-compact euclidean domains in $\mathbb{C}^m$) we will compare the eigenvalues of the symmetric tensor $H \otimes H$ with the biggest eigenvalue of $-\,\mathrm{s}^{\perp}$.

Lemma 3.2 *Let ξ be a local vector field along the graph of f_{t_0} which is normal to $\Gamma(f_{t_0})$ at a point x_0. The time derivative of* pr *at $\xi \in \mathcal{N}_{x_0}M$, when it is regarded as a bundle map* $\mathrm{pr} : F^*T(M \times N) \to F^*T(M \times N)$, *is given by*

$$(\nabla_{\partial_t}\mathrm{pr})(\xi) = -\sum_{j=1}^{m} \langle \xi, \nabla_{e_j} H \rangle e_j = -\sum_{j=1}^{m} \langle \xi, \nabla^{\perp}_{e_j} H \rangle e_j,$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame in the tangent bundle of the graph. Moreover, the time derivative of the natural projection at $\xi \in \mathcal{N}_{x_0}M$, when it is regarded as a map $\mathrm{pr} : F^*T(M \times N) \to \mathcal{N}M$, *is zero. That is, $(\nabla^{\perp}_{\partial_t}\mathrm{pr})(\xi) = 0$.*

Proof Consider a local coordinate system $(x_1, \ldots, x_m)$ around x_0 and suppose that the vectors $\{\partial_1|_{x_0}, \ldots, \partial_m|_{x_0}\}$ are orthonormal. Extend them now via parallel transport to a frame field $\{\varepsilon_1, \ldots, \varepsilon_m\}$ which is orthonormal with respect to the Riemannian metric $\mathrm{g}(t_0)$. In order to simplify the notation we set $e_i = \mathrm{d}F_{t_0}(\varepsilon_i)$, $1 \leq i \leq m$. Extend also the vector ξ arbitrarily.

Differentiating along the time direction, we get that

$$\begin{aligned}\left(\nabla_{\partial_t}\mathrm{pr}\right)(\xi) = {} & \nabla_{\partial_t}\xi - \mathrm{pr}(\nabla_{\partial_t}\xi) \\ & - \sum_{k,l=1}^{m} \mathrm{g}^{kl} \langle \nabla_{\partial_t}\xi, \mathrm{d}F(\partial_k)\rangle\, \mathrm{d}F(\partial_l) \\ & - \sum_{k,l=1}^{m} \partial_t\left(\mathrm{g}^{kl}\right) \langle \xi, \mathrm{d}F(\partial_k)\rangle\, \mathrm{d}F(\partial_l)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k,l=1}^{m} \mathrm{g}^{kl} \langle \xi, \nabla_{\partial_t} \mathrm{d}F(\partial_k)\rangle \, \mathrm{d}F(\partial_l) \\
&- \sum_{k,l=1}^{m} \mathrm{g}^{kl} \langle \xi, \mathrm{d}F(\partial_k)\rangle \nabla_{\partial_t} \mathrm{d}F(\partial_l).
\end{aligned}
$$

Because,

$$
\partial_t \big(\mathrm{g}^{kl}\big) = 2 \sum_{s,z=1}^{m} \mathrm{g}^{ks} \, \mathrm{g}^{zl} \, A_H(\partial_s, \partial_z)
$$

we deduce that

$$
\begin{aligned}
\big(\nabla_{\partial_t} \mathrm{pr}\big)(\xi) = &- \sum_{k,l=1}^{m} \mathrm{g}^{kl} \langle \xi, \nabla_{\partial_k} H\rangle \, \mathrm{d}F(\partial_l) \\
&- \sum_{k,l=1}^{m} \mathrm{g}^{kl} \langle \xi, \mathrm{d}F(\partial_k)\rangle \nabla_{\partial_l} H \\
&-2 \sum_{k,s,z,l=1}^{m} \mathrm{g}^{ks} \, \mathrm{g}^{zl} \, A_H(\partial_s, \partial_z) \langle \xi, \mathrm{d}F(\partial_k)\rangle \, \mathrm{d}F(\partial_l).
\end{aligned}
$$

Since, $\mathrm{g}_{kl}(x_0, t_0) = \delta_{kl}$, we get that at this point it holds

$$
\begin{aligned}
\big(\nabla_{\partial_t} \mathrm{pr}\big)(\xi) &= - \sum_{j=1}^{m} \langle \xi, \nabla_{e_j} H\rangle e_j - \sum_{j=1}^{m} \langle \xi, e_j\rangle \nabla_{e_j} H \\
&\quad -2 \sum_{i,j=1}^{m} A_H(e_i, e_j) \langle \xi, e_i\rangle e_j \\
&= - \sum_{j=1}^{m} \langle \xi, \nabla_{e_j} H\rangle e_j.
\end{aligned}
$$

Now, since $\mathrm{pr} \circ \mathrm{pr} = \mathrm{pr}$, we have that

$$
\big(\nabla^{\perp}_{\partial_t} \mathrm{pr}\big)(\xi) = \mathrm{pr}\,\big(\nabla_{\partial_t} \mathrm{pr}(\xi)\big) - \mathrm{pr}\,\big(\nabla_{\partial_t} \xi\big) = \mathrm{pr}\,\big\{\big(\nabla_{\partial_t} \mathrm{pr}\big)(\xi)\big\} = 0.
$$

This completes the proof of the lemma. □

In the next lemma we compute the evolution equation of $\mathrm{s}^{\perp}$. For that reason, it is necessary to extend $\mathrm{s}^{\perp}$ on $F^*T(M \times N)$, by defining

$$
\mathrm{s}^{\perp}(V, W) = \mathrm{s}_{M \times N}\big(\mathrm{pr}(V), \mathrm{pr}(W)\big)
$$

for any $V, W \in F^*T(M \times N)$.

Lemma 3.3 *Let ξ be a unit vector normal to the evolving submanifold at a fixed point (x_0, t_0) in space time. Then*

$$\begin{aligned}\left(\nabla^{\perp}_{\partial_t}\,\mathrm{s}^{\perp} - \Delta^{\perp}\,\mathrm{s}^{\perp}\right)(\xi,\xi) = 2\sum_{i,j=1}^{m} A_{\xi}(e_i,e_j)\,\mathrm{s}_{M\times N}\big(A(e_i,e_j),\xi\big)\\ -2\sum_{i,j=1}^{m} \mathrm{R}_{M\times N}(e_i,e_j,e_i,\xi)\,\mathrm{s}_{M\times N}(e_j,\xi)\\ -2\sum_{i,j,k=1}^{m} A_{\xi}(e_i,e_j)A_{\xi}(e_i,e_k)\mathrm{s}_{M\times N}(e_j,e_k),\end{aligned}$$

for any orthonormal basis $\{e_1,\ldots,e_m\}$ of $\mathrm{d}F_{t_0}(T_{x_0}M)$.

Proof Let us compute at first the time derivative of $\mathrm{s}^{\perp}$. Extend ξ locally to a smooth vector field along the graph. Then, using the fact that $\mathrm{s}_{M\times N}$ and pr are parallel tensors, we get that

$$\begin{aligned}(\nabla^{\perp}_{\partial_t}\,\mathrm{s}^{\perp})(\mathrm{pr}(\xi),\mathrm{pr}(\xi)) &= \partial_t\big\{\mathrm{s}_{M\times N}(\mathrm{pr}(\xi),\mathrm{pr}(\xi))\big\}\\ &\quad -2\mathrm{s}_{M\times N}\big(\nabla^{\perp}_{\partial_t}\,\mathrm{pr}(\xi),\mathrm{pr}(\xi)\big)\\ &= 2\mathrm{s}_{M\times N}\big(\nabla_{\partial_t}\,\mathrm{pr}(\xi) - \nabla^{\perp}_{\partial_t}\,\mathrm{pr}(\xi),\mathrm{pr}(\xi)\big)\\ &= 2\mathrm{s}_{M\times N}\big(\nabla_{\partial_t}\,\mathrm{pr}(\xi) - \mathrm{pr}(\nabla_{\partial_t}\xi),\mathrm{pr}(\xi)\big)\\ &= 2\mathrm{s}_{M\times N}\big((\nabla_{\partial_t}\,\mathrm{pr})(\mathrm{pr}(\xi)),\mathrm{pr}(\xi)\big).\end{aligned}$$

By virtue of Lemma 3.2, we deduce that at (x_0, t_0), the time derivative of $\mathrm{s}^{\perp}$ is given by

$$(\nabla^{\perp}_{\partial_t}\,\mathrm{s}^{\perp})(\xi,\xi) = -2\sum_{j=1}^{m}\langle\nabla^{\perp}_{e_j}H,\xi\rangle\mathrm{s}_{M\times N}(e_j,\xi).$$

In the next step we compute the Laplacian of $\mathrm{s}^{\perp}$. As usual, consider two vectors ξ and η on $\mathcal{N}M$ and extend them locally to smooth normal vector fields. At first let us compute the covariant derivative of $\mathrm{s}^{\perp}$ with respect to the direction e_i. Using the fact that $\mathrm{s}_{M\times N}$ is parallel, we have

$$\begin{aligned}\left(\nabla^{\perp}_{e_i}\,\mathrm{s}^{\perp}\right)(\xi,\eta) &= e_i\big\{\mathrm{s}_{M\times N}(\xi,\eta)\big\} - \mathrm{s}_{M\times N}(\nabla^{\perp}_{e_i}\xi,\eta) - \mathrm{s}_{M\times N}(\xi,\nabla^{\perp}_{e_i}\eta)\\ &= \mathrm{s}_{M\times N}\big(\nabla_{e_i}\xi - \nabla^{\perp}_{e_i}\xi,\eta\big) + \mathrm{s}_{M\times N}\big(\xi,\nabla_{e_i}\eta - \nabla^{\perp}_{e_i}\eta\big).\end{aligned}$$

Recall from the Weingarten formulas that

$$\nabla_{e_i}\xi = -\sum_{j=1}^{m} A_{\xi}(e_i,e_j)e_j + \nabla^{\perp}_{e_i}\xi.$$

Hence,

$$\big(\nabla^{\perp}_{e_i}\mathrm{s}^{\perp}\big)(\xi,\eta)=-A_{\xi}(e_i,e_j)\mathrm{s}_{M\times N}(e_j,\eta)-A_{\eta}(e_i,e_j)\mathrm{s}_{M\times N}(e_j,\xi).$$

Differentiating once more in the direction of e_i, we get

$$\begin{aligned}\big(\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}\mathrm{s}^{\perp}\big)(\xi,\xi)=&-2\sum_{j=1}^{m}\langle(\nabla^{\perp}_{e_i}A)(e_j,e_i),\xi\rangle\mathrm{s}_{M\times N}(e_j,\xi)\\&-2\sum_{j=1}^{m}A_{\xi}(e_i,e_j)\mathrm{s}_{M\times N}(A(e_i,e_j),\xi)\\&+2\sum_{j,k=1}^{m}A_{\xi}(e_i,e_j)A_{\xi}(e_i,e_k)\mathrm{s}_{M\times N}(e_j,e_k).\end{aligned}$$

From the Codazzi equation we get

$$\big(\nabla^{\perp}_{e_i}A\big)(e_j,e_i)=\big(\nabla^{\perp}_{e_j}A\big)(e_i,e_i)+\mathrm{pr}\,\big(\mathrm{R}_{M\times N}(e_i,e_j,e_i)\big).$$

Substituting the above relation in the formula of the Hessian of s and then taking a trace, we see that

$$\begin{aligned}\big(\Delta^{\perp}\mathrm{s}^{\perp}\big)(\xi,\xi)=&-2\sum_{j=1}^{m}\langle\nabla^{\perp}_{e_j}H,\xi\rangle\mathrm{s}_{M\times N}(e_j,\xi)\\&-2\sum_{i,j=1}^{m}A_{\xi}(e_i,e_j)\mathrm{s}_{M\times N}(A(e_i,e_j),\xi)\\&+2\sum_{i,j,k=1}^{m}A_{\xi}(e_i,e_j)A_{\xi}(e_i,e_k)\mathrm{s}_{M\times N}(e_j,e_k)\\&+2\sum_{i,j=1}^{m}\mathrm{R}_{M\times N}(e_i,e_j,e_i,\xi)\mathrm{s}_{M\times N}(e_j,\xi).\end{aligned}$$

Combining the above formula for the Laplacian with the formula for the time derivative, we deduce the evolution equation for $\mathrm{s}^{\perp}$. □

Consider the symmetric tensor $\vartheta\in\mathrm{Sym}\,\big(F^{*}T(M\times N)\otimes F^{*}T(M\times N)\big)$, given by

$$\vartheta(V,W):=H_{\mathrm{pr}(V)}\cdot H_{\mathrm{pr}(W)},$$

where $H_{\xi}=\mathrm{trace}\,A_{\xi}$ is the component of the mean curvature vector field in the direction of the normal vector ξ.

Lemma 3.4 *The symmetric* 2*-tensor* ϑ *evolves under the mean curvature flow according to the formula*

$$\big(\nabla^{\perp}_{\partial_t}\vartheta - \Delta^{\perp}\vartheta\big)(\xi,\xi) = 2\sum_{i,j=1}^{m} A_H(e_i,e_j)A_\xi(e_i,e_j)H_\xi - 2\sum_{i=1}^{m}\langle\nabla^{\perp}_{e_i}H,\xi\rangle^2$$
$$-2\sum_{i=1}^{m}\mathrm{R}_{M\times N}\big(H,e_i,e_i,\xi\big)H_\xi$$

for any normal vector ξ in the normal bundle of the submanifold.

Proof At first let us compute the time derivative of ϑ. Fix a point (x_0,t_0) in space-time and consider a unit normal vector ξ of $\Gamma(f_{t_0})$ at the point x_0. Now extend ξ to a local smooth vector field.

Computing and then estimating at (x_0,t_0), we get that

$$\begin{aligned}\big(\nabla^{\perp}_{\partial_t}\vartheta\big)(\xi,\xi) &= \partial_t\big\{\vartheta(\mathrm{pr}(\xi),\mathrm{pr}(\xi))\big\} - 2\vartheta\big(\nabla^{\perp}_{\partial_t}\mathrm{pr}(\xi),\mathrm{pr}(\xi)\big)\\ &= 2\langle\nabla^{\perp}_{\partial_t}H,\mathrm{pr}(\xi)\rangle H_{\mathrm{pr}(\xi)} + 2\langle H,\nabla^{\perp}_{\partial_t}\mathrm{pr}(\xi)\rangle H_{\mathrm{pr}(\xi)}\\ &\quad -2\langle H,\nabla^{\perp}_{\partial_t}\mathrm{pr}(\xi)\rangle H_{\mathrm{pr}(\xi)}\\ &= 2\big\langle\nabla^{\perp}_{\partial_t}H,\xi\big\rangle H_\xi.\end{aligned}$$

From the evolution equation of the mean curvature vector H (see [10, Corollary 3.8]) we deduce that at the point x_0 it holds

$$\nabla^{\perp}_{\partial_t}H - \Delta^{\perp}H = \sum_{i=1}^{m}\mathrm{pr}\left(\mathrm{R}_{M\times N}(H,e_i,e_i)\right) + \sum_{i,j=1}^{m}A_H(e_i,e_j)A(e_i,e_j).$$

Combining the above two equalities, we obtain

$$\begin{aligned}\big(\nabla^{\perp}_{\partial_t}\vartheta\big)(\xi,\xi) &= 2\langle\Delta^{\perp}H,\xi\rangle H_\xi\\ &\quad -2\sum_{i=1}^{m}\mathrm{R}_{M\times N}\big(H,e_i,e_i,\xi\big)H_\xi\\ &\quad +2\sum_{i,j=1}^{m}A_H(e_i,e_j)A_\xi(e_i,e_j)H_\xi.\end{aligned}$$

The next step is to compute the Laplacian of the tensor ϑ. At first let us compute the covariant derivative. Fix a point (x_0,t_0) in space time and let ξ,η be two normal vector fields of $\Gamma(f_{t_0})$ defined in a neighborhood of x_0. Differentiating with respect to the direction e_i, we have

$$\big(\nabla^{\perp}_{e_i}\vartheta\big)(\xi,\eta) = \big\langle\nabla^{\perp}_{e_i}H,\xi\big\rangle H_\eta + \big\langle\nabla^{\perp}_{e_i}H,\eta\big\rangle H_\xi.$$

Differentiating once more with respect to the direction e_i and summing up we deduce that

$$\big(\Delta^{\perp}\vartheta\big)(\xi,\xi) = 2\big\langle\Delta^{\perp}H,\xi\big\rangle H_{\xi} + 2\sum_{i=1}^{m}\big\langle\nabla^{\perp}_{e_i}H,\xi\big\rangle^{2}. \tag{3.1}$$

Combining the relations of the time derivative and of the Laplacian we obtain the desired evolution equation. This completes the proof. ◻

4 Proof of the theorem

During this section we will always assume that the Riemannian manifolds (M, g_M), (N, g_N) and $f : M \to N$ satisfy the assumption of the Theorem. The next lemma will be crucial to deal with the non-compactness of N.

Lemma 4.1 *There exists a uniform positive constant C such that*

$$\|H\|^2(x,t) \le C,$$

for any $(x,t) \in M \times [0,T)$.

Proof Consider the symmetric 2-tensor P, defined on the normal bundles of the evolving graphs and given by

$$P := \kappa\,\vartheta + \mathrm{s}^{\perp},$$

where κ is a sufficiently small positive constant such that $P < 0$ at time $t = 0$. We claim now that, taking if necessary a smaller choice for κ, the tensor P remains negative definite in time. Assume in contrary that this is not true. Then, there will be a first time such that P admits a unit null-eigenvector η at a point (x_0, t_0). Note that η is normal at the graph at the point (x_0, t_0).

According to the second derivative criterion [4], we have

(a) $P(\xi,\eta) = \kappa\,\vartheta(\xi,\eta) + \mathrm{s}^{\perp}(\xi,\eta) = 0$,

(b) $(\nabla P)(\eta,\eta) = 0$,

(c) $\big(\nabla^{\perp}_{\partial_t}P - \Delta^{\perp}P\big)(\eta,\eta) \ge 0$,

for any normal vector ξ of the graph at the point (x_0, t_0).

Estimating at (x_0, t_0) we get from (c) that

$$\begin{aligned} 0 \le &-\sum_{i,j,k=1}^{m} A_{\eta}(e_i,e_j)A_{\eta}(e_i,e_k)\mathrm{s}_{M\times N}(e_j,e_k) - \kappa\sum_{i=1}^{m}\langle\nabla^{\perp}_{e_i}H,\eta\rangle^2 \\ &+\sum_{i,j=1}^{m}\Big\{A_{\eta}(e_i,e_j)\,\mathrm{s}_{M\times N}\big(A(e_i,e_j),\eta\big) + \kappa\,A_H(e_i,e_j)A_{\eta}(e_i,e_j)H_{\eta}\Big\} \\ &-\sum_{i,j=1}^{m}\Big\{\mathrm{R}_{M\times N}(e_i,e_j,e_i,\eta)\,\mathrm{s}_{M\times N}(e_j,\eta) + \kappa\mathrm{R}_{M\times N}\big(H,e_i,e_i,\eta\big)H_{\eta}\Big\}. \end{aligned}$$

Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $\mathcal{N}_{x_0}M$. Then,

$$\begin{aligned}
0 \leq &- \sum_{i,j=1}^{m} A_\eta(e_i, e_j) A_\eta(e_i, e_k) \mathrm{s}_{M\times N}(e_j, e_k) \\
&+ \sum_{l=1}^{n} \sum_{i,j=1}^{m} A_\eta(e_i, e_j) A_{\xi_l}(e_i, e_j) \mathrm{s}^{\perp}(\xi_l, \eta) \\
&+ \sum_{l=1}^{n} \sum_{i,j=1}^{m} \kappa H_{\xi_l} H_\eta A_{\xi_l}(e_i, e_j) A_\eta(e_i, e_j) \\
&- \sum_{i,j=1}^{m} \Big\{ \mathrm{R}_{M\times N}(e_i, e_j, e_i, \eta)\, \mathrm{s}_{M\times N}(e_j, \eta) + \kappa \mathrm{R}_{M\times N}\big(H, e_i, e_i, \eta\big) H_\eta \Big\}.
\end{aligned}$$

Since, for any $l \in \{1, \ldots, n\}$, from (a) it holds

$$\kappa H_{\xi_l} H_\eta = -\, \mathrm{s}^{\perp}(\xi_l, \eta),$$

we finally get that

$$\begin{aligned}
0 \leq &- \sum_{i,j=1}^{m} A_\eta(e_i, e_j) A_\eta(e_i, e_k) \mathrm{s}_{M\times N}(e_j, e_k) \\
&- \sum_{i,j=1}^{m} \Big\{ \mathrm{R}_{M\times N}(e_i, e_j, e_i, \eta)\, \mathrm{s}_{M\times N}(e_j, \eta) + \kappa \mathrm{R}_{M\times N}\big(H, e_i, e_i, \eta\big) H_\eta \Big\}. \quad (4.1)
\end{aligned}$$

Denote by $\mathcal{A}$ the first part of (4.1) whose terms are involving the second fundamental form and by $\mathcal{B}$ the remaining curvature terms. The idea is to show that $\mathcal{A}$ becomes sufficiently negative for small choices of κ and dominates $\mathcal{B}$ that depends only on the singular values and the geometry of M and N.

Fact 1 Since $\mathrm{s}^{\perp}$ remains negative in time, from Lemma 3.1 it follows that there exists a universal positive constant ε such that

$$\varepsilon |\xi|^2 \leq -\, \mathrm{s}^{\perp}(\xi, \xi) \leq |\xi|^2,$$

for any ξ in $\mathcal{N}M$. Note that at (x_0, t_0) it holds,

$$\kappa H_\eta^2 = \kappa\, \vartheta(\eta, \eta) = -\, \mathrm{s}^{\perp}(\eta, \eta) \geq \varepsilon. \quad (4.2)$$

Therefore, as κ becomes smaller H_η^2 becomes larger.

Fact 2 From the relations (2.1) we deduce that

$$\mathcal{A} \leq -\sum_{i,j=1}^{m} A_\eta^2(e_i, e_j) \mathrm{s}_{M\times N}(e_i, e_j) \leq -\varepsilon |A_\eta^2| \leq -\frac{\varepsilon}{m} H_\eta^2,$$

where $\mathrm{s} \geq \varepsilon\,\mathrm{g}$ was applied in the second inequality. Thus for sufficiently small values of κ, $\mathcal{A}$ becomes sufficiently negative.

Fact 3 Note now that the first term of $\mathcal{B}$ depends only on the geometry of (M, g_M) and (N, g_N) as well as on the singular values of f_t which we know are bounded. The second term of $\mathcal{B}$ also depends only on these data, since

$$\begin{aligned}\kappa\,\mathrm{R}_{M\times N}(H, e_i, e_i, \eta)H_\eta &= \sum_{l=1}^{n} \kappa\, H_{\xi_l} H_\eta \mathrm{R}_{M\times N}(\xi_l, e_i, e_i, \eta)\\ &= -\sum_{l=1}^{n} \mathrm{s}^{\perp}(\xi_l, \eta)\mathrm{R}_{M\times N}(\xi_l, e_i, e_i, \eta),\end{aligned}$$

where $\{\xi_1, \ldots, \xi_n\}$ is a local basis on the normal bundle of the graph. Therefore, there exists a universal constant $c := c(M, N, \varepsilon)$ such that $\mathcal{B} \leq c$. Therefore, due to relation (4.2) we get that

$$\mathcal{B} \leq \frac{c}{\varepsilon}\varepsilon \leq \frac{c}{\varepsilon}\kappa H_\eta^2.$$

Thus,

$$\mathcal{A} + \mathcal{B} \leq \left(\frac{c}{\varepsilon}\kappa - \frac{\varepsilon}{m}\right) H_\eta^2.$$

Consequently, for $\kappa < \varepsilon^2/c\,m$, we see that $\mathcal{A} + \mathcal{B} < 0$ which contradicts (4.1). Therefore, the norm of the mean curvature vector field remains bounded in time. This completes the proof of the lemma. □

Remark 4.1 As one can see from the proof, we make use only of the facts that M is compact, N is complete with bounded sectional curvatures and that all the singular values of f_t are bounded from above by a positive universal constant which is less than 1.

The proof of the Theorem will be concluded by exploiting the blow up argument of Wang [13] and White's regularity theorem [14]. Let us recall at first the following crucial inequality for the time dependent angle-type function

$$u := \frac{1}{\sqrt{(1+\lambda_1^2)\cdots(1+\lambda_m^2)}}.$$

Lemma 4.2 ([7]) *The following estimate holds,*

$$\nabla_{\partial_t} \log u \geq \Delta \log u - 2c_0 \log u + \delta\|A\|^2,$$

for some positive real numbers c_0 and δ.

Once this estimate is available one can use White's regularity theorem [14] to exclude finite time singularities as long as on finite time intervals the graphs stay in compact regions of $M \times N$, which clearly is true, if $M \times N$ is compact. In our case N is complete but we may now exploit the mean curvature estimate of the Theorem to get the desired C^0-estimate for the graphs on finite time intervals. To see this, fix a point $x \in M$ and consider the smooth curve $\gamma : [t_0, t_1] \to M \times N$, given by

$$\gamma(x, t) := F(x, t).$$

The length $L(\gamma)$ of γ can be estimated using the bound of the mean curvature vector as follows

$$\begin{aligned} L(\gamma) &= \int_{t_0}^{t_1} \left\| \frac{\mathrm{d}F}{\mathrm{d}t}(x, t) \right\| \mathrm{d}t \\ &\leq \int_{t_0}^{t_1} \|H(x, t)\| \,\mathrm{d}t \leq C(t_1 - t_0) \\ &\leq CT, \end{aligned}$$

Therefore,

$$\operatorname{dist}(F(x, t_0), F(x, t_1)) \leq L(\gamma) \leq CT.$$

Suppose the graphs remain in a compact region W of $M \times N$ on a finite time interval $[0, T)$. By Nash's embedding theorem [6] one can embed W isometrically in some euclidean space $\mathbb{R}^p$ and make sure that the isometric embedding has bounded geometry. The bounded geometry is essential in the application of White's regularity theorem [14] for the mean curvature flow with controlled error terms, which by the compactness of W is applied to the mean curvature flow of $F(M) \subset W \subset \mathbb{R}^p$. Following the same arguments developed in the papers [13, Section 4] or [5, Section 3], we can prove the long-time existence and the convergence of the mean curvature flow to a constant map.

Remark 4.2 We would like to mention that in [9] pointwise curvature estimates are obtained for area decreasing maps between flat compact Riemann surfaces. In general, the problem of obtaining pointwise curvature estimates for the area decreasing mean curvature flow is still open as well as longtime existence of the area decreasing mean curvature flow in case of complete target manifolds.

References

1. Chau, A., Chen, J., He, W.: Lagrangian mean curvature flow for entire Lipschitz graphs. Calc. Var. Partial Differ. Equ. **44**, 199–220 (2012)
2. Guth, L.: Contraction of areas vs. topology of mappings. Geom. Funct. Anal. **23**, 1804–1902 (2013)
3. Guth, L.: Homotopy non-trivial maps with small k-dilation, pp. 1–7. arXiv:0709.1241v1 (2007)
4. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255–306 (1982)
5. Lee, K.-W., Lee, Y.-I.: Mean curvature flow of the graphs of maps between compact manifolds. Trans. Am. Math. Soc. **363**, 5745–5759 (2011)

6. Nash, J.: The imbedding problem for Riemannian manifolds. Ann. Math. **63**(2), 20–63 (1956)
7. Savas-Halilaj, A., Smoczyk, K.: Homotopy of area decreasing maps by mean curvature flow. Adv. Math. **255**, 455–473 (2014)
8. Savas-Halilaj, A., Smoczyk, K.: Bernstein theorems for length and area decreasing minimal maps. Calc. Var. Partial Differ. Equ. **50**, 549–577 (2014)
9. Smoczyk, K., Tsui, M.-P., Wang, M.-T.: Curvature decay estimates of graphical mean curvature flow in higher co-dimensions, pp. 1–17. arXiv:1401.4154 (2014)
10. Smoczyk, K.: Mean curvature flow in higher codimension-Introduction and survey. Global Differential Geometry. Springer Proceedings in Mathematics, vol. 12, pp. 231–274 (2012)
11. Smoczyk, K.: Long-time existence of the Lagrangian mean curvature flow. Calc. Var. Partial Differ. Equ. **20**, 25–46 (2004)
12. Tsui, M.-P., Wang, M.-T.: Mean curvature flows and isotopy of maps between spheres. Comm. Pure Appl. Math. **57**, 1110–1126 (2004)
13. Wang, M.-T.: Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension. Invent. Math. **148**, 525–543 (2002)
14. White, B.: A local regularity theorem for mean curvature flow. Ann. Math. **161**(2), 1487–1519 (2005)