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Badly approximable points on planar curves and a problem of Davenport

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Abstract Let $\mathcal{C}$ be two times continuously differentiable curve in $\mathbb{R}^2$ with at least one point at which the curvature is non-zero. For any $i, j \geqslant 0$ with $i+j=1$, let $\mathbf{Bad}(i,j)$ denote the set of points $(x,y) \in \mathbb{R}^2$ for which $\max\{\|qx\|^{1/i}, \|qy\|^{1/j}\} > c/q$ for all $q \in \mathbb{N}$. Here $c = c(x,y)$ is a positive constant. Our main result implies that any finite intersection of such sets with $\mathcal{C}$ has full Hausdorff dimension. This provides a solution to a problem of Davenport dating back to the sixties.

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1 Introduction

A real number x is said to be *badly approximable* if there exists a positive constant $c(x)$ such that

$$\|qx\| \ > \ c(x)\, q^{-1} \quad \forall\, q \in \mathbb{N}\,.$$

Here and throughout $\|\cdot\|$ denotes the distance of a real number to the nearest integer. It is well known that set **Bad** of badly approximable numbers is of Lebesgue measure zero but of maximal Hausdorff dimension; i.e. $\dim \mathbf{Bad} = 1$. In higher dimensions

Dedicated to our mathematical grandparents: Harold Davenport and Maurice Dodson.

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there are various natural generalizations of **Bad**. Restricting our attention to the plane $\mathbb{R}^2$, given a pair of real numbers i and j such that

$$0 \leqslant i, j \leqslant 1 \quad \text{and} \quad i + j = 1, \tag{1}$$

a point $(x, y) \in \mathbb{R}^2$ is said to be (i, j)*-badly approximable* if there exists a positive constant $c(x, y)$ such that

$$\max\{ \|qx\|^{1/i}, \ \|qy\|^{1/j} \} \ > \ c(x, y)\, q^{-1} \quad \forall\, q \in \mathbb{N}.$$

Denote by $\mathbf{Bad}(i, j)$ the set of (i, j)-badly approximable points in $\mathbb{R}^2$. If $i = 0$, then we use the convention that $x^{1/i} := 0$ and so $\mathbf{Bad}(0, 1)$ is identified with $\mathbb{R} \times \mathbf{Bad}$. That is, $\mathbf{Bad}(0, 1)$ consists of points (x, y) with $x \in \mathbb{R}$ and $y \in \mathbf{Bad}$. The roles of x and y are reversed if $j = 0$. In the case $i = j = 1/2$, the set under consideration is the standard set $\mathbf{Bad}_2$ of simultaneously badly approximable points. It easily follows from classical results in the theory of metric Diophantine approximation that $\mathbf{Bad}(i, j)$ is of (two-dimensional) Lebesgue measure zero and it was shown in [11] that $\dim \mathbf{Bad}(i, j) = 2$.

1.1 The problem

Badly approximable numbers obeying various functional relations were first studied in the works of Cassels, Davenport and Schmidt from the fifties and sixties. In particular, Davenport [7] in 1964 proved that for any $n \geq 2$ there is a continuum set of $\alpha \in \mathbb{R}$ such that each of the numbers $\alpha, \alpha^2, \ldots, \alpha^n$ are all in **Bad**. In the same paper, Davenport [7, p. 52] states "Problems of a much more difficult character arise when the number of independent parameters is less than the dimension of simultaneous approximation. I do not know whether there is a set of α *with the cardinal of the continuum such that the pair* (α, α^2) *is badly approximable for simultaneous approximation.*" Thus, given the parabola $\mathcal{V}_2 := \{(x, x^2) : x \in \mathbb{R}\}$, Davenport is asking the question:

Is the set $\mathcal{V}_2 \cap \mathbf{Bad}_2$ *uncountable*?

The goal of this paper is to answer this specific question for the parabola and consider the general setup involving an arbitrary planar curve $\mathcal{C}$ and $\mathbf{Bad}(i, j)$. Without loss of generality, we assume that $\mathcal{C}$ is given as a graph

$$\mathcal{C}_f := \{(x, f(x)) : x \in I\}$$

for some function f defined on an interval $I \subset \mathbb{R}$. It is easily seen that some restriction on the curve is required to ensure that $\mathcal{C} \cap \mathbf{Bad}(i, j)$ is not empty. For example, let L_α denote the vertical line parallel to the y-axis passing through the point $(\alpha, 0)$ in the (x, y)-plane. Then, it is easily verified, see [4, §1.3] for the details, that

$$\mathrm{L}_\alpha \cap \mathbf{Bad}(i, j) = \emptyset$$

for any $\alpha \in \mathbb{R}$ satisfying $\liminf_{q\to\infty} q^{1/i}\|q\alpha\| = 0$. Note that the lim inf under consideration is zero if α is a Liouville number. On the other hand, if the lim inf is strictly positive, which it is if $\alpha \in \mathbf{Bad}$, then

$$\dim(\mathrm{L}_\alpha \cap \mathbf{Bad}(i, j)) = 1.$$

This result is much harder to prove and is at the heart of the proof of Schmidt's Conjecture recently established in [4]. The upshot of this discussion regarding vertical lines is that to build a general, coherent theory for badly approximable points on planar curves we need that the curve $\mathcal{C}$ under consideration is in some sense 'genuinely curved'. With this in mind, we will assume that $\mathcal{C}$ is two times continuously differentiable and that there is at least one point on $\mathcal{C}$ at which the curvature is non-zero. We shall refer to such a curve as a $C^{(2)}$ *non-degenerate* planar curve. In other words and more formally, a planar curve $\mathcal{C} := \mathcal{C}_f$ is $C^{(2)}$ non-degenerate if $f \in C^{(2)}(I)$ and there exits at least one point $x \in I$ such that

$$f''(x) \neq 0.$$

For these curves, it is reasonable to suspect that

$$\dim(\mathcal{C} \cap \mathbf{Bad}(i, j)) = 1.$$

If true, this would imply that $\mathcal{C} \cap \mathbf{Bad}(i, j)$ is uncountable and since the parabola $\mathcal{V}_2$ is a $C^{(2)}$ non-degenerate planar curve we obtain a positive answer to Davenport's question. To the best of our knowledge, there has been no progress with Davenport's question to date. More generally, for planar curves (non-degenerate or not) the results stated above for vertical lines constitute the first and essentially only contribution. The main result proved in this paper shows that any finite intersection of $\mathbf{Bad}(i, j)$ sets with a $C^{(2)}$ non-degenerate planar curve is of full dimension.

1.2 The results

Theorem 1 *Let $(i_1, j_1), \ldots, (i_d, j_d)$ be a finite number of pairs of real numbers satisfying (1). Let $\mathcal{C}$ be a $C^{(2)}$ non-degenerate planar curve. Then*

$$\dim\left(\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t) \cap \mathcal{C}\right) = 1.$$

A consequence of this theorem is the following statement regarding the approximation of real numbers by algebraic numbers. As usual, the *height* $H(\alpha)$ of an algebraic number is the maximum of the absolute values of the integer coefficients in its minimal defining polynomial.

Corollary 1 *The set of $x \in \mathbb{R}$ for which there exists a positive constant $c(x)$ such that*

$$|x - \alpha| > c(x)\, H(\alpha)^{-3} \quad \forall \text{ real algebraic numbers } \alpha \text{ of degree } \leq 2$$

is of full Hausdorff dimension.

The corollary represents the 'quadratic' analogue of Jarník's classical dim **Bad** $= 1$ statement and complements the well approximable results of Baker and Schmidt [5] and Davenport and Schmidt [8]. It also makes a contribution to Problems 24, 25 and 26 in [6, §10.2]. To deduce the corollary from the theorem, we exploit the equivalent dual form representation of the set $\mathbf{Bad}(i, j)$. A point $(x, y) \in \mathbf{Bad}(i, j)$ if there exists a positive constant $c(x, y)$ such that

$$\max\{|A|^{1/i}, |B|^{1/j}\} \, \|Ax - By\| > c(x, y) \qquad \forall \, (A, B) \in \mathbb{Z}^2 \backslash \{(0, 0)\}. \tag{2}$$

Then with $d = 1$, $i = j = 1/2$ and $\mathcal{C} = \mathcal{V}_2$, the theorem implies that

$$\dim \left\{ x \in \mathbb{R} \ : \ \max\{|A|^2, |B|^2\} \, \|Ax - Bx^2\| > c(x) \; \forall \, (A, B) \in \mathbb{Z}^2 \backslash \{(0, 0)\} \right\} = 1.$$

It can be verified that this is the statement of the corollary formulated in terms of integer polynomials.

Straight lines are an important class of $C^{(2)}$ planar curves not covered by Theorem 1. In view of the discussion in Sect. 1.1, this is to be expected since the conclusion of the theorem is false for lines in general. Indeed, it is only valid for a vertical line L_α if α satisfies the Diophantine condition $\liminf_{q \to \infty} q^{1/i} \|q\alpha\| > 0$. The following result provides an analogous statement for non-vertical lines.

Theorem 2 *Let $(i_1, j_1), \ldots, (i_d, j_d)$ be a finite number of pairs of real numbers satisfying (1). Given $\alpha, \beta \in \mathbb{R}$, let $\mathrm{L}_{\alpha,\beta}$ denote the line defined by the equation $y = \alpha x + \beta$. Suppose there exists $\epsilon > 0$ such that*

$$\liminf_{q \to \infty} q^{\frac{1}{\sigma} - \epsilon} \|q\alpha\| > 0 \qquad if \qquad \sigma := \max\{\min\{i_t, j_t\} : 1 \leq t \leq d\} > 0 \, .$$

If $\sigma = 0$, suppose that $\beta \in \mathbf{Bad}$ when $\alpha = 0$. Then

$$\dim \left(\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_{\alpha,\beta} \right) = 1.$$

Note that when $\sigma = 0$, we are considering the intersection of $\mathbf{Bad}(0, 1) := \mathbb{R} \times \mathbf{Bad}$ and/or $\mathbf{Bad}(1, 0) := \mathbf{Bad} \times \mathbb{R}$ with $\mathrm{L}_{\alpha,\beta}$ and the result is essentially known. When $\alpha = 0$, the intersection of $\mathbf{Bad}(0, 1)$ with the horizontal line $\mathrm{L}_{0,\beta}$ given by $y = \beta$ is empty unless $\beta \in \mathbf{Bad}$ in which case the full dimension statement is obvious. When $\alpha \neq 0$, the statement is easily verified for the intersection of $\mathbf{Bad}(0, 1)$ or $\mathbf{Bad}(1, 0)$ with $\mathrm{L}_{\alpha,\beta}$. The non-trivial situation corresponds to when considering $\mathbf{Bad}(0, 1) \cap \mathbf{Bad}(1, 0) \cap \mathrm{L}_{\alpha,\beta}$. The fact this intersection is uncountable is a simple consequence of Davenport's result in [7] and it is not difficult to modify Davenport's argument to obtain the full dimension statement.

In all likelihood Theorem 2 is best possible apart from the ϵ appearing in the Diophantine condition on the slope α of the line. Indeed, this is the case for vertical lines—see [4, Theorem 2]. Note that we always have that $\sigma \leqslant 1/2$, so Theorem 2 is always valid for $\alpha \in \mathbf{Bad}$. Also we point out that as a consequence of the Jarník–Besicovitch theorem, the Hausdorff dimension of the exceptional set of α for which the conclusion of the theorem is not valid is bounded above by $2/3$.

Remark 1 The proofs of Theorem 1 and Theorem 2 make use of a general Cantor framework developed in [3]. The framework is essentially extracted from the 'raw' construction used in [4] to establish Schmidt's Conjecture. It will be apparent during the course of the proofs that constructing the right type of general Cantor set in the $d = 1$ case is the main substance. Adapting the construction to deal with finite intersections is not difficult and will follow on applying the explicit 'finite intersection' theorem stated in [3]. However, we point out that by utilizing the arguments in [4, §7.1] for countable intersections it is possible to adapt the $d = 1$ construction to obtain the following strengthening of the theorems.

Theorem 1' *Let (i_t, j_t) be a countable number of pairs of real numbers satisfying (1) and suppose that*

$$\liminf_{t\to\infty} \min\{i_t, j_t\} > 0 \,. \tag{3}$$

Let $\mathcal{C}$ be a $C^{(2)}$ non-degenerate planar curve. Then

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathcal{C}\right) = 1.$$

Theorem 2' *Let (i_t, j_t) be a countable number of pairs of real numbers satisfying (1) and (3). Given $\alpha, \beta \in \mathbb{R}$, let $\mathrm{L}_{\alpha,\beta}$ denote the line defined by the equation $y = \alpha x + \beta$. Suppose there exists $\epsilon > 0$ such that*

$$\liminf_{q\to\infty} q^{\frac{1}{\sigma}-\epsilon}\|q\alpha\| > 0 \qquad \text{where } \sigma := \sup\{\min\{i_t, j_t\} : t \in \mathbb{N}\}.$$

Then

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_{\alpha,\beta}\right) = 1.$$

These statements should be true without the lim inf condition (3). Indeed, without assuming (3) the nifty argument developed by Nesharim in [10] can be exploited to show that the countable intersection of the sets under consideration are non-empty. Unfortunately, the argument fails to show positive dimension let alone full dimension.

Remark 2 This manuscript has taken a very long time to produce. During its slow gestation, An [1] circulated a paper in which he shows that $\mathrm{L}_\alpha \cap \mathbf{Bad}(i, j)$ is winning (in the sense of Schmidt games—see [13, Chp.3]) for any vertically line L_α with $\alpha \in \mathbb{R}$ satisfying the Diophantine condition $\liminf_{q\to\infty} q^{1/i}\|q\alpha\| > 0$. An immediate consequence of this is that $\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\alpha$ is of full dimension as long as α satisfies the Diophantine condition with $i = \sup\{i_t : t \in \mathbb{N}\}$. The point is that this is a statement free of (3) unlike the countable intersection result obtained in [4]. In view of An's work it is very tempting and not at all outrageous to assert that $\mathbf{Bad}(i, j) \cap \mathcal{C}$ *is winning* at least on the part of the curve that is genuinely curved.[1] If true this would imply Theorem 1′ without assuming (3). It is worth stressing that currently we do not even know if $\mathbf{Bad}_2 \cap \mathcal{C}$ is winning.

1.3 Davenport in higher dimensions: what can we expect?

For any n-tuple of nonnegative real numbers $\mathbf{i} := (i_1, \ldots, i_n)$ satisfying $\sum_{s=1}^n i_s = 1$, denote by $\mathbf{Bad}(\mathbf{i})$ the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a positive constant $c(x_1, \ldots, x_n)$ such that

$$\max\{ \|qx_1\|^{1/i_1}, \ldots \|qx_n\|^{1/i_n} \} > c(x_1, \ldots, x_n)\, q^{-1} \quad \forall\, q \in \mathbb{N}\,.$$

The name of the game is to investigate the intersection of these n-dimensional badly approximable sets with manifolds $\mathcal{M} \subset \mathbb{R}^n$. A good starting point is to consider Davenport's problem for arbitrary curves $\mathcal{C}$ in $\mathbb{R}^n$. To this end and without loss of generality, we assume that $\mathcal{C}$ is given as a graph

$$\mathcal{C}_{\mathbf{f}} := \{(f_1(x), \ldots, f_n(x)) : x \in I\}$$

where $\mathbf{f} := (f_1, \ldots, f_n) : I \to \mathbb{R}^n$ is a map defined on an interval $I \subset \mathbb{R}$. As in the planar case, to avoid trivial empty intersection with $\mathbf{Bad}(\mathbf{i})$ sets we assume that the curve is genuinely curved. A curve $\mathcal{C} := \mathcal{C}_{\mathbf{f}} \subset \mathbb{R}^n$ is said to be $C^{(n)}$ non-degenerate if $\mathbf{f} \in C^{(n)}(I)$ and there exists at least one point $x \in I$ such that the Wronskian

$$w(f_1', \ldots, f_n')(x) := \det(f_s^{(t)}(x))_{1 \le s,t \le n} \ne 0.$$

In the planar case ($n = 2$), this condition on the Wronskian is precisely the same as saying that there exits at least one point on the curve at which the curvature is non-zero. Armed with the notion of $C^{(n)}$ non-degenerate curves, there is no reason not to believe in the truth of the following statements.

Conjecture A *Let $\mathbf{i}_t := (i_{1,t} \ldots, i_{n,t})$ be a countable number of n-tuples of non-negative real numbers satisfying $\sum_{s=1}^n i_{s,t} = 1$. Let $\mathcal{C} \subset \mathbb{R}^n$ be a $C^{(n)}$ non-degenerate curve. Then*

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(\mathbf{i}_t) \cap \mathcal{C}\right) = 1.$$

[1] **Added in proof:** An, Beresnevich and the second author have recently proved this winning statement. In fact, winning within the more general inhomogeneous setup is established. A manuscript entitled 'Badly approximable points on planar curves and winning' is in preparation.

Conjecture B *Let* $\mathbf{i} := (i_1, \ldots, i_n)$ *be an* n*-tuple of non-negative real numbers satisfying* $\sum_{s=1}^{n} i_s = 1$*. Let* $\mathcal{C} \subset \mathbb{R}^n$ *be a* $C^{(n)}$ *non-degenerate curve. Then* $\mathbf{Bad}(\mathbf{i}) \cap \mathcal{C}$ *is winning on some arc of* $\mathcal{C}$*.*

Remark 3 In view of the fact that a winning set has full dimension and that the intersection of countably many winning sets is winning, it follows that Conjecture B implies Conjecture A.

Remark 4 Conjecture A together with known results/arguments from fractal geometry implies the strongest version (arbitrary countable intersection plus full dimension) of Schmidt's Conjecture in higher dimension:

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(\mathbf{i}_t)\right) = n.$$

In the case $n = 2$, this follows from An's result mentioned above (Remark 2 in Sect. 1.2)—see also his subsequent paper [2].

Remark 5 Given that we basically know nothing in dimension $n > 2$, a finite intersection version (including the case $t = 1$) of Conjecture A would be a magnificent achievement. In all likelihood, any successful approach based on the general Cantor framework developed in [3] as in this paper would yield Conjecture A, under the extra assumption involving the natural analogue of the lim inf condition (3).

We now turn our attention to general manifolds $\mathcal{M} \subset \mathbb{R}^n$. To avoid trivial empty intersection with $\mathbf{Bad}(\mathbf{i})$ sets, we assume that the manifolds under consideration are non-degenerate. Essentially, these are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane. Formally, a manifold $\mathcal{M}$ of dimension m embedded in $\mathbb{R}^n$ is said to be non-degenerate if it arises from a non–degenerate map $\mathbf{f} : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^m$ and $\mathcal{M} := \mathbf{f}(U)$. The map $\mathbf{f} : U \to \mathbb{R}^n : \mathbf{u} \mapsto \mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u}))$ is said to be *non–degenerate at* $\mathbf{u} \in U$ if there exists some $l \in \mathbb{N}$ such that $\mathbf{f}$ is l times continuously differentiable on some sufficiently small ball centered at $\mathbf{u}$ and the partial derivatives of $\mathbf{f}$ at $\mathbf{u}$ of orders up to l span $\mathbb{R}^n$. *If there exists at least one such non-degenerate point, we shall say that the manifold* $\mathcal{M} = \mathbf{f}(U)$ *is non–degenerate.* Note that in the case that the manifold is a curve $\mathcal{C}$, this definition is absolutely consistent with that of $\mathcal{C}$ being $C^{(n)}$ non-degenerate. Also notice, that any real, connected analytic manifold not contained in any hyperplane of $\mathbb{R}^n$ is non–degenerate. The following are the natural versions of Conjectures A & B for manifolds.

Conjecture C *Let* $\mathbf{i}_t := (i_{1,t} \ldots, i_{n,t})$ *be a countable number of* n*-tuples of non-negative real numbers satisfying* $\sum_{s=1}^{n} i_{s,t} = 1$*. Let* $\mathcal{M} \subset \mathbb{R}^n$ *be a non-degenerate manifold. Then*

$$\dim\left(\bigcap_{t=1}^{\infty} \mathbf{Bad}(\mathbf{i}_t) \cap \mathcal{M}\right) = \dim \mathcal{M}.$$

Conjecture D *Let* $\mathbf{i} := (i_1, \ldots, i_n)$ *be an n-tuple of non-negative real numbers satisfying* $\sum_{s=1}^{n} i_s = 1$. *Let* $\mathcal{M} \subset \mathbb{R}^n$ *be a non-degenerate manifold. Then* $\mathbf{Bad}(\mathbf{i}) \cap \mathcal{M}$ *is winning on some patch of* $\mathcal{M}$.

Remark 6 Conjecture A together with the fibering technique of Pyartly [12] should establish Conjecture C for non-degenerate manifolds that can be foliated by non-degenerate curves. In particular, this includes any non-degenerate analytic manifold.[2]

Beyond manifolds, it would be desirable to investigate Davenport's problem within the more general context of friendly measures [9]. We suspect that the above conjectures for manifolds remain valid with $\mathcal{M}$ replaced by a subset X of $\mathbb{R}^n$ that supports a friendly measure.

2 Preliminaries

Concentrating on Theorem 1, since any subset of a planar curve $\mathcal{C}$ is of dimension less than or equal to one we immediately obtain that

$$\dim\left(\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t) \cap \mathcal{C}\right) \leq 1. \tag{4}$$

Thus, the proof of Theorem 1 reduces to establishing the complementary lower bound statement and as already mentioned in Sect. 1 (Remark 1) the crux is the $d = 1$ case. Without loss of generality, we assume that $i \leqslant j$. Also, the case that $i = 0$ is relatively straight forward to handle so let us assume that

$$0 < i \leqslant j < 1 \quad \text{and} \quad i + j = 1, \tag{5}$$

Then, formally the key to establishing Theorem 1 is the following statement.

Theorem 3 *Let* (i, j) *be a pair of real numbers satisfying* (5). *Let* $\mathcal{C}$ *be a* $C^{(2)}$ *non-degenerate planar curve. Then*

$$\dim \mathbf{Bad}(i, j) \cap \mathcal{C} \geq 1 .$$

The hypothesis that $\mathcal{C} = \mathcal{C}_f := \{(x, f(x)) : x \in I\}$ is $C^{(2)}$ non-degenerate implies that there exist positive constants $C_0, c_0 > 0$ so that

$$c_0 \leqslant |f'(x)| < C_0 \quad \text{and} \quad c_0 \leqslant |f''(x)| < C_0 \quad \forall\, x \in I. \tag{6}$$

[2] A few days before completing this paper, Victor Beresnevich communicated to us that he has established Conjecture A under the extra assumption involving the natural analogue of (3). In turn, under this assumption, by making use of Pyartly's technique he has proved Conjecture C for non-degenerate analytic manifolds. This in our opinion represents a magnificent achievement—see Remark 5.
Added in proof: V. Beresnevich: *Badly approximable points on manifolds*. Pre-print: arXiv:1304.0571.

To be precise, in general we can only guarantee (6) on a sufficiently small sub-interval I_0 of I. Nevertheless, establishing Theorem 3 for the 'shorter' curve $\mathcal{C}_f^* = \{(x, f(x)) : x \in I_0\}$ corresponding to f restricted to I_0 clearly implies the desired dimension result for the curve $\mathcal{C}_f$.

To simplify notation the Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. Unless stated otherwise, the unspecified constant will at most be dependant on i, j, C_0 and c_0 only. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable.

2.1 Geometric interpretation of $\mathbf{Bad}(i, j) \cap \mathcal{C}$

We will work with the dual form of $\mathbf{Bad}(i, j)$ consisting of points $(x, y) \in \mathbb{R}^2$ satisfying (2). In particular, for any constant $c > 0$, let $\mathbf{Bad}_c(i, j)$ denote the set of points $(x, y) \in \mathbb{R}^2$ such that

$$\max\{|A|^{1/i}, |B|^{1/j}\} \, \|Ax - By\| > c \qquad \forall\, (A, B) \in \mathbb{Z}^2 \backslash \{(0, 0)\}. \tag{7}$$

It is easily seen that $\mathbf{Bad}_c(i, j) \subset \mathbf{Bad}(i, j)$ and

$$\mathbf{Bad}(i, j) = \bigcup_{c>0} \mathbf{Bad}_c(i, j).$$

Geometrically, given integers A, B, C with $(A, B) \neq (0, 0)$ consider the line $L = L(A, B, C)$ defined by the equation

$$Ax - By + C = 0.$$

The set $\mathbf{Bad}_c(i, j)$ simply consists of points in the plane that avoid the

$$\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\} \sqrt{A^2 + B^2}}$$

thickening of each line L—alternatively, points in the plane that lie within any such neighbourhood are removed. A consequence of (6) is that this thickening intersects $\mathcal{C}$ in at most two closed arcs. Either of these arcs will be denoted by $\Delta(L)$. Let $\mathcal{R}_0$ be the collection of arcs $\Delta(L)$ on $\mathcal{C}$ arising from lines $L = L(A, B, C)$ with integer coefficients and $(A, B) \neq (0, 0)$.

The upshot of the above analysis is that the set $\mathbf{Bad}_c(i, j) \cap \mathcal{C}$ can be described as the set of all points on $\mathcal{C}$ that survive after removing the arcs $\Delta(L) \in \mathcal{R}_0$. Formally,

$$\mathbf{Bad}_c(i, j) \cap \mathcal{C} = \{(x, f(x)) \in \mathcal{C} : (x, f(x)) \notin \Delta(L) \ \ \forall\, \Delta(L) \in \mathcal{R}_0\}.$$

For reasons that will become apparent later, it will be convenient to remove all but finitely many arcs. With this in mind, let $\mathcal{S}$ be a finite sub-collection of $\mathcal{R}_0$ and consider the set

$$\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}=\{(x,f(x))\in\mathcal{C}\ :\ (x,f(x))\notin\Delta(L)\ \ \forall\,\Delta(L)\in\mathcal{R}_0\backslash\mathcal{S}\}.$$

Clearly, since we are removing fewer arcs $\mathbf{Bad}_{c,\mathcal{S}}(i,j)\supset\mathbf{Bad}_c(i,j)$. On the other hand,

$$S:=\{(x,f(x))\in\mathcal{C}\ :\ Ax-Bf(x)+C=0 \text{ for some } L(A,B,C) \text{ with } \Delta(L)\in\mathcal{S}\}$$

is a finite set of points and it is easily verified that

$$\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}\subset(\mathbf{Bad}_c(i,j)\cap\mathcal{C})\cup S.$$

Since $\dim S=0$ for any finite set S of points, Theorem 3 will follow on showing that

$$\dim\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}\to 1\quad\text{as}\quad c\to 0. \tag{8}$$

In Sect. 2.2.1 we will specify exactly the finite collection of arcs $\mathcal{S}$ that are not to be removed and put $\mathcal{R}:=\mathcal{R}_0\backslash\mathcal{S}$ for this choice of $\mathcal{S}$.

Remark 7 Without loss of generality, when considering lines $L=L(A,B,C)$ we will assume that

$$(A,B,C)=1.$$

Otherwise we can divide the coefficients of L by their common divisor. Then the resulting line L' will satisfy the required conditions and moreover $\Delta(L')\supseteq\Delta(L)$. Therefore, removing the arc $\Delta(L')$ from $\mathcal{C}$ takes care of removing $\Delta(L)$.

2.1.1 Working with the projection of $\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}$

Recall that $\mathcal{C}=\mathcal{C}_f:=\{(x,f(x)):x\in I\}$ where $I\subset\mathbb{R}$ is an interval. Let $\mathbf{Bad}^f_{c,\mathcal{S}}(i,j)$ denote the set of $x\in I$ such that $(x,f(x))\in\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}$. In other words $\mathbf{Bad}^f_{c,\mathcal{S}}(i,j)$ is the orthogonal projection of $\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}$ onto the x-axis. Now notice that in view of (6) the function f is Lipschitz; i.e. for some $\lambda>1$

$$|f(x)-f(x')|\leqslant\lambda|x-x'|\quad\forall x,x'\in I.$$

Thus, the sets $\mathbf{Bad}^f_{c,\mathcal{S}}(i,j)$ and $\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}$ are related by a bi-Lipschitz map and so

$$\dim\mathbf{Bad}_{c,\mathcal{S}}(i,j)\cap\mathcal{C}=\dim\mathbf{Bad}^f_{c,\mathcal{S}}(i,j).$$

Hence establishing (8) is equivalent to showing that

$$\dim\mathbf{Bad}^f_{c,\mathcal{S}}(i,j)\to 1\quad\text{as}\quad c\to 0. \tag{9}$$

Next observe that $\mathbf{Bad}^f_{c,\mathcal{S}}(i,j)$ can equivalently be written as the set of $x\in I$ such that $x\notin\Pi(\Delta(L))$ for all $\Delta(L)\in\mathcal{R}_0\backslash\mathcal{S}$ where the interval $\Pi(\Delta(L))\subset I$ is the orthogonal projection of the arc $\Delta(L) \subset \mathcal{C}$ onto the x-axis. Throughout the paper, we use the fact that the sets under consideration can be viewed either in terms of arcs $\Delta(L)$ on the curve $\mathcal{C}$ or sub-intervals $\Pi(\Delta(L))$ of I. In order to minimize unnecessary and cumbersome notation, we will simply write $\Delta(L)$ even in the case of intervals and always refer to $\Delta(L)$ as an interval. It will be clear from the context whether $\Delta(L)$ is an arc on a curve or a genuine interval on $\mathbb{R}$. However, we stress that by the length of $\Delta(L)$ we will always mean the length of the interval $\Pi(\Delta(L))$. In other words,

$$|\Delta(L)| := |\Pi(\Delta(L))|.$$

2.2 An estimate for the size of $\Delta(L)$

Given a line $L = L(A, B, C)$, consider the function

$$F_L \ : \ I \to \mathbb{R} \ : \ x \to F_L(x) := Ax - Bf(x) + C.$$

To simplify notation, if there is no risk of ambiguity we shall simply write $F(x)$ for $F_L(x)$. Now given an interval $\Delta(L) = \Delta(L(A, B, C))$ let

$$V_L(\Delta) := \min_{x\in\Delta(L)} \{|F_L'(x)|\} = \min_{x\in\Delta(L)} \{|A - Bf'(x)|\}.$$

Since $\Delta(L)$ is closed and F_L is continuous the minimum always exists. If there is no risk of ambiguity we shall simply write V_L for $V_L(\Delta)$. In short, the quantity V_L plays a crucial role in estimating the size of $\Delta(L)$.

Lemma 1 *There exists an absolute constant $K \geq 1$ dependent only on i, j, C_0 and c_0 such that*

$$|\Delta(L)| \leqslant K \min\left\{\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\} \cdot V_L}, \left(\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\} \cdot |B|}\right)^{1/2}\right\}. \tag{10}$$

Proof The statement is essentially a consequence of Pyartly's Lemma [12]: *Let $\delta, \mu > 0$ and $I \subset \mathbb{R}$ be some interval. Let $f(x) \in C^n(I)$ be function such that $|f^{(n)}(x)| > \delta$* for all $x \in I$. *Then there exists a constant $c(n)$ such that*

$$|\{x \in I \ : \ |f(x)| < \mu\}| \leq c(n) \left(\frac{\mu}{\delta}\right)^{1/n}.$$

Armed with this, the first estimate for $|\Delta(L)|$ follows from the fact that

$$|F_L'(x)| \geqslant \delta := V_L \quad \text{and} \quad |F_L(x)| \leqslant \mu := \frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\}}$$

for all $x \in \Delta(L)$. The second makes use of the fact that

$$|F_L''(x)| = |Bf''(x)| > c_0|B| \quad \forall\, x \in \Delta(L).$$

□

Remark 8 The second term inside the minimum on the r.h.s. of (10) is absolutely crucial. It shows that the length of $\Delta(L)$ can not be arbitrary large even when the quantity V_L is small or even equal to zero. The second term is not guaranteed if the curve is degenerate. However, for the lines (degenerate curves) $\mathrm{L}_{\alpha,\beta}$ considered in Theorem 2 the Diophantine condition on α guarantees that V_L is not too small and hence allows us to adapt the proof of Theorem 3 to this degenerate situation.

2.2.1 Type 1 and Type 2 intervals

Consider an interval $\Delta(L) = \Delta(L(A, B, C)) \in \mathcal{R}$. Then Lemma 1 implies that

$$\Delta(L) \subseteq \Delta_1^*(L) \qquad \text{and} \qquad \Delta(L) \subseteq \Delta_2^*(L)$$

where the intervals $\Delta_1^*(L)$ and $\Delta_2^*(L)$ have the same center as $\Delta(L)$ and length given

$$|\Delta_1^*(L)| := \frac{2K \cdot c}{\max\{|A|^{1/i}, |B|^{1/j}\} \cdot V_L},$$
$$|\Delta_2^*(L)| := 2K \left(\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\} \cdot |B|} \right)^{1/2}.$$

We say that the interval $\Delta_1^*(L)$ is of **Type 1** and $\Delta_2^*(L)$ is of **Type 2**. *For obvious reasons, we assume that* $B \neq 0$ *in the case of Type 2*. For each type of interval we define its *height* in the following way:

$$H(\Delta_1^*) = H(A, B) := c^{-1/2} \cdot V_L \cdot \max\{|A|^{1/i}, |B|^{1/j}\};$$
$$H(\Delta_2^*) = H(A, B) := (\max\{|A|^{1/i}, |B|^{1/j}\} \cdot |B|)^{1/2}.$$

So if $\Delta^*(L)$ denotes an interval of either type we have that

$$|\Delta^*(L)| = 2Kc^{1/2} \cdot (H(\Delta^*))^{-1}.$$

Remark 9 Notice that for each positive number H_0 there are only finitely many intervals $\Delta_2^*(L)$ of Type 2 such that $H(\Delta_2^*) \leqslant H_0$.

Recall, geometrically $\mathbf{Bad}_{c,\mathcal{S}}(i, j) \cap \mathcal{C}$ (resp. its projection $\mathbf{Bad}^f_{c,\mathcal{S}}(i, j)$) is the set of points on $\mathcal{C}$ (resp. I) that survive after removing the intervals $\Delta(L) \in \mathcal{R}_0 \backslash \mathcal{S}$. We now consider the corresponding subsets obtained by removing the larger intervals $\Delta^*(L)$. Given $\Delta(L) \in \mathcal{R}_0$, the criteria for which type of interval $\Delta^*(L)$ represents is as follows. Let $R \geqslant 2$ be a large integer and λ be a constant satisfying

$$\lambda > \max\left\{4, \frac{1}{i}, \frac{1+i}{j}\right\}. \tag{11}$$

Furthermore, assume that the constant $c > 0$ satisfies

$$c < \min\left\{(8(C_0+1)R^{-1-ij/2-\lambda})^2, ((C_0+1)C_0R^2)^{-2}\right\}. \tag{12}$$

Given $\Delta(L)$ consider the associated Type 1 interval $\Delta_1^*(L)$. There exists a unique $d \in \mathbb{Z}$ such that

$$R^d \leqslant H(\Delta_1^*) < R^{d+1}. \tag{13}$$

Choose l_0 to be the largest integer such that

$$\lambda l_0 \leqslant \max\{d, 0\}. \tag{14}$$

Then we choose $\Delta^*(L)$ to be the interval $\Delta_1^*(L)$ of Type 1 if

$$V_L > (C_0 + 1) R^{-\lambda(l_0+1)} \max\{|A|, |B|\}.$$

Otherwise, we take $\Delta^*(L)$ to be the interval $\Delta_2^*(L)$ of Type 2. Formally

$$\Delta^*(L) := \begin{cases} \Delta_1^*(L) & \text{if} \quad V_L > (C_0 + 1) R^{-\lambda(l_0+1)} \max\{|A|, |B|\}. \\ \Delta_2^*(L) & \text{otherwise.} \end{cases} \tag{15}$$

Remark 10 It is easily verified that for either type of interval, we have that

$$H(\Delta^*) \geqslant 1.$$

For Type 2 intervals $\Delta_2^*(L)$ this follows by definition. For Type 1 intervals $\Delta_1^*(L)$ assume that $H(\Delta_1) < 1$. It then follows that $d < 0$ and $l_0 = 0$. In turn this implies that

$$\begin{aligned} H(\Delta_1) &:= c^{-1/2} V_L \max\{|A|^{1/i}, |B|^{1/j}\} \\ &\geqslant c^{-1/2} (C_0 + 1) R^{-\lambda} \max\{|A|, |B|\} \max\{|A|^{1/i}, |B|^{1/j}\} \\ &\overset{(12)}{\geqslant} \max\{|A|, |B|\} \max\{|A|^{1/i}, |B|^{1/j}\} \geqslant 1. \end{aligned}$$

This contradicts our assumption and thus we must have that $H(\Delta_1) \geqslant 1$.

We now specify the finite sub-collection $\mathcal{S}$ of intervals from $\mathcal{R}_0$ which are not to be removed. Let $n_0 = n_0(c, R)$ be the minimal positive integer satisfying

$$c^{1/2} \cdot R^{n_0} \cdot C_0 \geqslant 1. \tag{16}$$

Then, define $\mathcal{S}$ to be the collection of intervals $\Delta(L) \in \mathcal{R}_0$ so that $\Delta^*(L)$ is of Type 2 and $H(\Delta^*) < R^{3n_0}$. Clearly $\mathcal{S}$ is a finite collection of intervals—see Remark 9 above. For this particular collection $\mathcal{S}$ we put

$$\mathcal{R} := \mathcal{R}_0 \backslash \mathcal{S}.$$

Armed with this criteria for choosing $\Delta^*(L)$ given $\Delta(L)$ and indeed the finite collection $\mathcal{S}$ we consider the set

$$\mathbf{Bad}_c^*(i, j) \cap \mathcal{C} := \{(x, f(x)) \in \mathcal{C} \ : \ (x, f(x)) \cap \Delta^*(L) = \emptyset \ \ \forall\ \Delta(L) \in \mathcal{R}\}. \tag{17}$$

Clearly,

$$\mathbf{Bad}^*_c(i, j) \cap \mathcal{C} \subset \mathbf{Bad}_{c,\mathcal{S}}(i, j) \cap \mathcal{C}$$

and so Theorem 3 will follow on showing (8) with $\mathbf{Bad}_{c,\mathcal{S}}(i, j) \cap \mathcal{C}$ replaced by $\mathbf{Bad}^*_c(i, j) \cap \mathcal{C}$. Indeed, from this point onward we will work with set defined by (17). In view of this and to simplify notation we shall simply redefine $\mathbf{Bad}_c(i, j) \cap \mathcal{C}$ to be $\mathbf{Bad}^*_c(i, j) \cap \mathcal{C}$ and write $\Delta(L)$ for $\Delta^*(L)$. Just to make it absolutely clear, the intervals $\Delta(L) := \Delta^*(L)$ are determined via the criteria (15) and $\mathcal{R}$ is the collection of such intervals arising from lines $L = L(A, B, C)$ apart from those associated with $\mathcal{S}$. Also, the set $\mathbf{Bad}^f_c(i, j)$ is from this point onward the orthogonal projection of the redefined set $\mathbf{Bad}_c(i, j) \cap \mathcal{C} := \mathbf{Bad}^*_c(i, j) \cap \mathcal{C}$. With this in mind, the key to establishing (9), which in turn implies (8) and therefore Theorem 3, lies in constructing a Cantor-type subset $K_c(i, j)$ of $\mathbf{Bad}^f_c(i, j)$ such that

$$\dim K_c(i, j) \to 1 \quad \text{as} \quad c \to 0\,.$$

3 Cantor sets and applications

The proof of Theorem 1 and indeed Theorem 2 makes use of a general Cantor framework developed in [3]. This is what we now describe.

3.1 A general Cantor framework

The parameters. Let I be a closed interval in $\mathbb{R}$. Let

$$\mathbf{R} := (R_n) \quad \text{with} \quad n \in \mathbb{Z}_{\geqslant 0}$$

be a sequence of natural numbers and

$$\mathbf{r} := (r_{m,n}) \quad \text{with} \quad m, n \in \mathbb{Z}_{\geqslant 0} \;\text{ and }\; m \leqslant n$$

be a two parameter sequence of non-negative real numbers.

The construction. We start by subdividing the interval I into R_0 closed intervals I_1 of equal length and denote by $\mathcal{I}_1$ the collection of such intervals. Thus,

$$\#\mathcal{I}_1 = R_0 \qquad \text{and} \qquad |I_1| = R_0^{-1}\,|\mathrm{I}|.$$

Next, we remove at most $r_{0,0}$ intervals I_1 from $\mathcal{I}_1$. Note that we do not specify which intervals should be removed but just give an upper bound on the number of intervals to be removed. Denote by $\mathcal{J}_1$ the resulting collection. Thus,

$$\#\mathcal{J}_1 \geqslant \#\mathcal{I}_1 - r_{0,0}. \tag{18}$$

For obvious reasons, intervals in $\mathcal{J}_1$ will be referred to as (level one) survivors. It will be convenient to define $\mathcal{J}_0 := \{J_0\}$ with $J_0 := \mathrm{I}$.

In general, for $n \geqslant 0$, given a collection $\mathcal{J}_n$ we construct a nested collection $\mathcal{J}_{n+1}$ of closed intervals J_{n+1} using the following two operations.

- *Splitting procedure.* We subdivide each interval $J_n \in \mathcal{J}_n$ into R_n closed sub-intervals I_{n+1} of equal length and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$\#\mathcal{I}_{n+1} = R_n \times \#\mathcal{J}_n \qquad \text{and} \qquad |I_{n+1}| = R_n^{-1}\,|J_n|.$$

- *Removing procedure.* For each interval $J_n \in \mathcal{J}_n$ we remove at most $r_{n,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within J_n. Note that the number of intervals I_{n+1} removed is allowed to vary amongst the intervals in $\mathcal{J}_n$. Let $\mathcal{I}_{n+1}^n \subseteq \mathcal{I}_{n+1}$ be the collection of intervals that remain. Next, for each interval $J_{n-1} \in \mathcal{J}_{n-1}$ we remove at most $r_{n-1,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}^n$ that lie within J_{n-1}. Let $\mathcal{I}_{n+1}^{n-1} \subseteq \mathcal{I}_{n+1}^n$ be the collection of intervals that remain. In general, for each interval $J_{n-k} \in \mathcal{J}_{n-k}$ $(1 \leqslant k \leqslant n)$ we remove at most $r_{n-k,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}^{n-k+1}$ that lie within J_{n-k}. Also we let $\mathcal{I}_{n+1}^{n-k} \subseteq \mathcal{I}_{n+1}^{n-k+1}$ be the collection of intervals that remain. In particular, $\mathcal{J}_{n+1} := \mathcal{I}_{n+1}^0$ is the desired collection of (level $n+1$) survivors. Thus, the total number of intervals I_{n+1} removed during the removal procedure is at most $r_{n,n}\#\mathcal{J}_n + r_{n-1,n}\#\mathcal{J}_{n-1} + \cdots + r_{0,n}\#\mathcal{J}_0$ and so

$$\#\mathcal{J}_{n+1} \geqslant R_n\#\mathcal{J}_n - \sum_{k=0}^{n} r_{k,n}\#\mathcal{J}_k. \tag{19}$$

Finally, having constructed the nested collections $\mathcal{J}_n$ of closed intervals we consider the limit set

$$\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J.$$

The set $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ will be referred to as a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ *Cantor set.* For further details and examples see [3, §2.2]. The following result ([3, Theorem 4] enables us to estimate the Hausdorff dimension of $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$. It is the key to establishing Theorem 1.

Theorem 4 *Given $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$, suppose that $R_n \geqslant 4$ for all $n \in \mathbb{Z}_{\geqslant 0}$ and that*

$$\sum_{k=0}^{n} \left(r_{n-k,n} \prod_{i=1}^{k} \left(\frac{4}{R_{n-i}} \right) \right) \leqslant \frac{R_n}{4}. \tag{20}$$

Then

$$\dim \mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}) \geqslant \liminf_{n \to \infty} (1 - \log_{R_n} 2).$$

Here we use the convention that the product term in (20) is one when $k = 0$ and by definition $\log_{R_n} 2 := \log 2/\log R_n$.

The next result [3, Theorem 5] enables us to show that the intersection of finitely many sets $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$ is yet another $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set for some appropriately chosen $\mathbf{r}$. This will enable us to establish Theorem 1.

Theorem 5 *For each integer $1 \leqslant i \leqslant k$, suppose we are given a set $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$. Then*

$$\bigcap_{i=1}^{k} \mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}_i)$$

is a $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ Cantor set where

$$\mathbf{r} := (r_{m,n}) \quad \textit{with} \quad r_{m,n} := \sum_{i=1}^{k} r_{m,n}^{(i)}.$$

3.2 The applications

We wish to construct an appropriate Cantor-type set $K_c(i, j) \subset \mathbf{Bad}_c^f(i, j)$ which fits within the general Cantor framework of Sect. 3.1. With this in mind, let $R \geqslant 2$ be a large integer and

$$c_1 := c^{\frac{1}{2}} R^{1+\omega} \quad \text{where} \quad \omega := \frac{ij}{4}$$

and the constant $c > 0$ satisfies (12). Take an interval $J_0 \subset I$ of length c_1. With reference to Sect. 3.1 we denote by $\mathcal{J}_0 := \{J_0\}$. We establish, by induction on n, the existence of the collection $\mathcal{J}_n$ of closed intervals J_n such that $\mathcal{J}_n$ is nested in $\mathcal{J}_{n-1}$; that is, each interval J_n in $\mathcal{J}_n$ is contained in some interval J_{n-1} in $\mathcal{J}_{n-1}$. The length of an interval J_n will be given by

$$|J_n| := c_1 R^{-n},$$

and each interval J_n will satisfy the condition

$$J_n \cap \Delta(L) = \emptyset \qquad \forall\, L \;\; with \;\; H(\Delta) < R^{n-1}. \tag{21}$$

In particular we put

$$K_c(i, j) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J$$

By construction, we have that

$$K_c(i, j) \subset \mathbf{Bad}_c^f(i, j)$$

Now let

$$\epsilon := \frac{ijw}{2} = \frac{(ij)^2}{8} \qquad and \qquad R > R_0(\epsilon)$$

be sufficiently large. Recall that we are assuming that $j \geqslant i > 0$ and so ϵ is strictly positive—we deal with the $i = 0$ case later in Sect. 5.1. Let $n_0 = n_0(c, R)$ be the minimal positive integer satisfying (16); i.e.

$$c^{1/2} \cdot R^{n_0} \cdot C_0 \geqslant 1.$$

It will be apparent from the construction of the collections of $\mathcal{J}_n$ described in Sect. 5 that $K_c(i, j)$ is in fact a $(J_0, \mathbf{R}, \mathbf{r})$ Cantor set $\mathcal{K}(J_0, \mathbf{R}, \mathbf{r})$ with

$$\mathbf{R} := (R_n) = (R, R, R, \ldots)$$

and

$$\mathbf{r} := (r_{m,n}) = \begin{cases} 4R^{1-\epsilon} & \text{if } \ m = n; \\ 2R^{1-\epsilon} & \text{if } \ m < n, \ n - m \neq n_0 \\ 3R^{1-\epsilon} & \text{if } \ n - m = n_0, \ n \geqslant 3n_0 \end{cases}$$

By definition, note that for $R > R_0(\epsilon)$ large enough we have that

$$\text{l.h.s. of (20) } = \sum_{k=0}^{n} r_{n-k,n} \left(\frac{4}{R}\right)^k \leqslant 4R^{1-\epsilon} \frac{1}{1 - 4/R} \leqslant \frac{R}{4} = \text{r.h.s. of (20)}.$$

Also note that $R_n \geqslant 4$ for R large enough. Then it follows via Theorem 4 that

$$\dim \mathbf{Bad}_c^f(i, j) \geqslant \dim K_c(i, j) = \dim \mathcal{K}(J_0, \mathbf{R}, \mathbf{r}) \geqslant 1 - \log_R 2.$$

This is true for all R large enough (equivalently all $c > 0$ small enough) and so on letting $R \to \infty$ we obtain that

$$\dim \mathbf{Bad}(i, j) \cap \mathcal{C} \geq \dim \mathbf{Bad}_c^f(i, j) \to 1.$$

This proves Theorem 3 modulo the construction of the collections $\mathcal{J}_n$ and dealing with $i = 0$. Moreover, Theorem 5 implies that

$$\bigcap_{t=1}^{d} (\mathbf{Bad}(i_t, j_t)) \cap \mathcal{C}$$

contains the Cantor-type set $\mathcal{K}(J_0, \mathbf{R}, \tilde{\mathbf{r}})$ with

$$\tilde{\mathbf{r}} := (\tilde{r}_{m,n}) = \begin{cases} 4dR^{1-\tilde{\epsilon}} & \text{if } \ m = n; \\ 2dR^{1-\tilde{\epsilon}} & \text{if } \ m < n, \ n - m \neq n_0 \\ 3dR^{1-\tilde{\epsilon}} & \text{if } \ n - m = n_0, \ n \geqslant 3n_0. \end{cases}$$

where

$$\tilde{\epsilon} := \min_{1 \leqslant t \leqslant d} \left(\frac{(i_t j_t)^2}{8} \right).$$

On applying Theorem 4 to the set $\mathcal{K}(J_0, \mathbf{R}, \tilde{\mathbf{r}})$ and letting $R \to \infty$ implies that

$$\dim \left(\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t) \cap \mathcal{C} \right) \geqslant 1.$$

This together with the upper bound statement (4) establishes Theorem 1 modulo of course the construction of the collections $\mathcal{J}_n$ and the assumption that $i > 0$.

4 Preliminaries for constructing $\mathcal{J}_n$

In order to construct the appropriate collections $\mathcal{J}_n$ described in Sect. 3.2, it is necessary to partition the collection $\mathcal{R}$ of intervals $\Delta(L)$ into various classes. The aim is to have sufficiently good control on the parameters $|A|$, $|B|$ and V_L within each class. Throughout, $R \geqslant 2$ is a large integer.

- *Firstly we partition all Type 1 intervals $\Delta(L) \in \mathcal{R}$ into classes $C(n)$ and $C(n, k, l)$.*

  A Type 1 interval $\Delta(L) \in C(n)$ if

$$R^{n-1} \ \leqslant \ H(\Delta) \ < \ R^n \, . \tag{22}$$

Furthermore, $\Delta(L) \in C(n, k, l) \subset C(n)$ if

$$2^k R^{n-1} \leqslant H(\Delta) < 2^{k+1} R^{n-1} \qquad 0 \leqslant k < \log_2 R, \tag{23}$$

$$R^{-\lambda(l+1)}(C_0 + 1) \max\{|A|, |B|\} < V_L \leqslant R^{-\lambda l}(C_0 + 1) \max\{|A|, |B|\} \, . \tag{24}$$

and $\Delta(L) \not\subset \Delta(L')$ for any previous $\Delta(L') \in C(n', k', l')$ with $(n', k') < (n, k)$. Here by $(n', k') < (n, k)$ we mean either $n' < n$ or $n' = n$ and $k' < k$.

Note that since the intervals $\Delta(L)$ are of Type 1, it follows from (14) that $l \leqslant l_0$. Moreover

$$V_L = |A - Bf'(x_0)| \overset{(6)}{\leqslant} |A| + C_0|B| \leqslant (1 + C_0) \max\{|A|, |B|\}$$

so l is also nonnegative. Here and throughout x_0 is the point at which $|F_L'(x)| = |A - Bf'(x)|$ attains its minimum with $x \in \Delta(L)$. We let

$$C(n,l) := \bigcup_{k=0}^{\log_2 R} C(n,k,l).$$

- *Secondly we partition all Type 2 intervals $\Delta(L) \in \mathcal{R}$ into classes $C^*(n)$ and $C^*(n,k)$.*

A Type 2 interval $\Delta(L) \in C^*(n)$ if (22) is satisfied. Furthermore, $\Delta(L) \in C^*(n,k) \subset C^*(n)$ if (23) is satisfied and also $\Delta(L) \not\subset \Delta(L')$ for any previous $\Delta(L') \in C^*(n',k')$ with $(n',k') < (n,k)$.

Note that since $H(\Delta) \geqslant 1$, we have the following the complete split of $\mathcal{R}$:

$$\mathcal{R} = \left(\bigcup_{n=0}^{\infty} C(n)\right) \cup \left(\bigcup_{n=0}^{\infty} C^*(n)\right).$$

We now investigate the consequences of the above classes on the parameters $|A|$, $|B|$ and V_L and introduce further subclasses to gain tighter control.

4.1 Estimates for $|A|$, $|B|$ and V_L within a given class

4.1.1 Class $C(n,k,l)$ with $l \geqslant 1$

Suppose $\Delta(L(A,B,C)) \in C(n,k,l)$ for some $l \geqslant 1$. By definition each of these classes corresponds to the case that the derivative $V_L = |F_L'(x_0)|$ satisfies (24). In other words the derivative is essentially smaller than the expected value $\max\{|A|,|B|\}$. Now observe that the r.h.s. of (24) implies either

$$|A - f'(x_0)B| < \frac{C_0+1}{R^\lambda}|A| \Leftrightarrow \left(1 - \frac{C_0+1}{R^\lambda}\right) < \frac{|f'(x_0)B|}{|A|} < \left(1 + \frac{C_0+1}{R^\lambda}\right)$$

or

$$|A - f'(x_0)B| < \frac{C_0+1}{R^\lambda}|B| \Leftrightarrow \left(1 - \frac{C_0+1}{|f'(x_0)|R^\lambda}\right) < \frac{|A|}{|f'(x_0)B|} < \left(1 + \frac{C_0+1}{|f'(x_0)|R^\lambda}\right).$$

Since $|f'(x_0)| \geqslant c_0 > 0$ then in both cases, for R large enough we have that

$$2^{-1}|A| < |f'(x_0)B| < 2|A| \quad \text{or} \quad |A| \asymp |B|. \tag{25}$$

On substituting the estimate (24) for V_L into the definition of the height $H(\Delta)$ we obtain that

$$c^{-\frac{1}{2}} \cdot |A|^{\max\{\frac{i+1}{i},\frac{j+1}{j}\}} R^{-\lambda(l+1)} \ll H(\Delta) \ll c^{-\frac{1}{2}} \cdot |A|^{\max\{\frac{i+1}{i},\frac{j+1}{j}\}} R^{-\lambda l}.$$

This together with (23) and the fact that $i \leqslant j$, implies that

$$\left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda l}\right)^{\frac{i}{i+1}} \ll |A|, |B| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda(l+1)}\right)^{\frac{i}{i+1}}. \tag{26}$$

4.1.2 Class $C(n, k, 0)$

By (23) and (24), we have that in this case

$$c^{-\frac{1}{2}} \cdot \frac{\max\{|A|, |B|\}}{R^\lambda} \max\{|A|^{1/i}, |B|^{1/j}\} \ll H(\Delta) \ll \frac{2^k}{R} R^n.$$

Therefore,

$$|A| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda}\right)^{\frac{i}{i+1}} \tag{27}$$

$$|B| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda}\right)^{\frac{j}{j+1}}. \tag{28}$$

Unfortunately these bounds for $|A|$ and $|B|$ are not strong enough for our purpose. Thus, we partition the class $C(n, k, 0)$ into the following subclasses:

$$C_1(n,k) := \{\Delta(L(A, B, C)) \in C(n,k,0) \ : \ |A| \geqslant \frac{1}{2}|f'(x_0)||B|\}$$

$$C_2(n,k) := \{\Delta(L(A, B, C)) \in C(n,k,0) \ : \ |A| < \frac{1}{2}|f'(x_0)||B|, \ |A|^{1/i} \leqslant |B|^{1/j}\}$$

$$C_3(n,k) := \{\Delta(L(A, B, C)) \in C(n,k,0) \ : \ |A| < \frac{1}{2}|f'(x_0)||B|, \ |A|^{1/i} > |B|^{1/j}\}.$$

- **Subclass $C_1(n, k)$ of $C(n, k, 0)$.** By (27) we have the following bounds for $|B|$ and V_L:

$$|B|, \ V_L \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda}\right)^{\frac{i}{i+1}}. \tag{29}$$

  Note that this bound for $|B|$ is stronger than (28).
- **Subclass $C_2(n, k)$ of $C(n, k, 0)$.** We can strengthen the bound (27) for $|A|$ by the following:

$$|A| \leqslant |B|^{i/j} \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda}\right)^{\frac{i}{j+1}}. \tag{30}$$

Since $|A| < \frac{1}{2}|f'(x_0)||B|$ we have that $V_L \asymp |B|$, therefore

$$V_L \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda}\right)^{\frac{j}{j+1}}. \tag{31}$$

Also we get that $\max\{|A|^{1/i}, |B|^{1/j}\} = |B|^{1/j}$ which together with (23) implies that for any two $\Delta(L_1(A_1, B_1, C_1)), \Delta(L_2(A_2, B_2, C_2)) \in C_2(n,k)$,

$$V_{L_1} \asymp B_1 \asymp B_2 \asymp V_{L_2}. \tag{32}$$

- **Subclass** $C_3(n,k)$ **of** $C(n,k,0)$. As with the previous subclass $C_2(n,k)$ we have that

$$V_L \asymp |B| \qquad \forall \quad \Delta(L(A_2, B_2, C_2)) \in C_3(n,k).$$

We partition $C_3(n,k)$ into subclasses $C_3(n,k,u,v)$ consisting of intervals $\Delta(L(A, B, C)) \in C_3(n,k)$ with

$$2^v R^{\lambda u}|B|^{1/j} < |A|^{1/i} \leqslant 2^{v+1} R^{\lambda u}|B|^{1/j} \qquad u \geqslant 0 \qquad \lambda \log_2 R \geqslant v \geqslant 0. \tag{33}$$

Then

$$|B|^{\frac{j+1}{j}} R^{\lambda u} < |B||A|^{1/i} \asymp V_L \max\{|A|^{1/i}, |B|^{1/j}\} = c^{\frac{1}{2}} H(\Delta) \overset{(23)}{<} \frac{2^{k+1} c^{\frac{1}{2}}}{R} R^n.$$

Therefore

$$V_L \asymp |B| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^n\right)^{\frac{j}{j+1}} R^{-\frac{\lambda u j}{j+1}} \tag{34}$$

and

$$|A| \overset{(33)}{\ll} R^{\lambda(u+1)i}|B|^{i/j} \ll \left(\frac{2^k c^{\frac{1}{2}}}{R} R^n\right)^{\frac{i}{j+1}} R^{\frac{\lambda u i j}{j+1} + \lambda i}. \tag{35}$$

We proceed with estimating the size of the parameter u. The fact that $|A| < \frac{1}{2}|f'(x_0)||B|$ together with (33) and (34) implies that

$$R^{\lambda u} \overset{(33)}{<} \frac{|A|^{1/i}}{|B|^{1/j}} \ll |B|^{\frac{j-i}{ij}} \overset{(34)}{\ll} R^{\frac{(j-i)n}{i(j+1)}}.$$

Therefore for large R, if $C_3(n,k,u,v)$ is nonempty then u satisfies

$$0 \leqslant \lambda u \leqslant \frac{j-i}{i(1+j)} \cdot n + 1. \tag{36}$$

In particular, this shows that u is smaller than n if $\lambda > 1/i$. Finally, it can be verified that the inequalities given by (32) are valid for any two intervals $\Delta(L_1), \Delta(L_2) \in C_3(n, k, u, v)$.

4.1.3 Class $C^*(n, k)$

By the definition (14) of l_0, we have that

$$V_L \leqslant R^{-\lambda}(C_0 + 1) \max\{|A|, |B|\}.$$

This corresponds to the r.h.s. of (24) with $l = 1$ and thus the same arguments as in Sect. 4.1.1 can be utilized to show that (25) is satisfied. By substituting this into the definition of the height we obtain that

$$H(\Delta) \asymp |A|^{\frac{i+1}{2i}}$$

which in view of (23) implies that

$$|A| \asymp |B| \asymp \left(\frac{2^k}{R} \cdot R^n\right)^{\frac{2i}{i+1}}. \tag{37}$$

A consequence of this estimate is that all intervals $\Delta(L) \in C^*(n, k)$ have comparable coefficients A and B. In other words, if $\Delta(L_1), \Delta(L_2) \in C^*(n, k)$ then

$$|A_1| \asymp |B_1| \asymp |B_2| \asymp |A_2|.$$

To estimate the size of V_L we make use of the fact that

$$\begin{aligned} V_L &\leqslant (C_0 + 1) R^{-\lambda(l_0+1)} \max\{|A|, |B|\} \overset{(14)}{\leqslant} (C_0 + 1) R^{-d} \max\{|A|, |B|\} \\ &\overset{(13)}{\leqslant} (C_0 + 1) c^{1/2} \cdot \frac{\max\{|A|, |B|\}}{V_L \max\{|A|^{1/i}, |B|^{1/j}\}} \end{aligned}$$

This together with (12) and (37) enables us to verify that

$$V_L \leqslant \frac{|B|}{R \cdot H(\Delta)} \ll \left(\frac{2^k}{R} \cdot R^n\right)^{-\frac{j}{i+1}}. \tag{38}$$

4.2 Additional subclasses $C(n, k, l, m)$ of $C(n, k, l)$

It is necessary to partition each class $C(n, k, l)$ of Type 1 intervals $\Delta(L)$ into the following subclasses to provide stronger control on V_L. For $m \in \mathbb{Z}$, let

$$C(n, k, l, m) := \left\{ \Delta(L(A, B, C)) \in C(n, k, l) \,\middle|\, \begin{array}{l} 2^{-m-1} R^{-\lambda l} (C_0 + 1) \max\{|A|, |B|\} < V_L \\ V_L \leqslant 2^{-m} R^{-\lambda l} (C_0 + 1) \max\{|A|, |B|\} \end{array} \right\}. \quad (39)$$

In view of (24), it is easily verified that

$$0 \leq m \leq \lambda \log_2 R \asymp \log R.$$

An important consequence of introducing these subclasses is that for any two intervals $\Delta(L_1)$, $\Delta(L_2)$ from $C(n, k, l, m)$ with $l \geqslant 1$ or from $C_1(n, k) \cap C(n, k, 0, m)$, we have that

$$V_{L_1} \asymp V_{L_2} \quad \text{and} \quad |A_1| \asymp |A_2|. \quad (40)$$

5 Defining the collection $\mathcal{J}_n$

We describe the procedure for constructing the collections $\mathcal{J}_n (n = 0, 1, 2, \ldots)$ that lie at the heart of the construction of the Cantor-type set $K_c(i, j) = \mathcal{K}(J_0, \mathbf{R}, \mathbf{r})$ of Sect. 3.2. Recall that each interval $J_n \in \mathcal{J}_n$ is to be nested in some interval J_{n-1} in $\mathcal{J}_{n-1}$ and satisfy (21). We define $\mathcal{J}_n$ by induction on n.

For $n = 0$, we trivially have that (21) is satisfied for any interval $J_0 \subset I$. The point is that $H(\Delta) \geqslant 1$ and so there are no intervals $\Delta(L)$ satisfying the height condition $H(\Delta) < 1$. So take $\mathcal{J}_0 := \{J_0\}$. For the same reason (21) with $n = 1$ is trivially satisfied for any interval J_1 obtained by subdividing J_0 into R closed intervals of equal length $c_1 R^{-1}$. Denote by $\mathcal{J}_1$ the resulting collection of intervals J_1.

In general, given $\mathcal{J}_n$ satisfying (21) we wish to construct a nested collection $\mathcal{J}_{n+1}$ of intervals J_{n+1} for which (21) is satisfied with n replaced by $n + 1$. By definition, any interval J_n in $\mathcal{J}_n$ avoids intervals $\Delta(L)$ arising from lines L with height $H(\Delta)$ bounded above by R^{n-1}. Since any 'new' interval J_{n+1} is to be nested in some J_n, it is enough to show that J_{n+1} avoids intervals $\Delta(L)$ arising from lines L with height $H(\Delta)$ satisfying (22); that is

$$R^{n-1} \leqslant H(\Delta) < R^n.$$

The collection of intervals $\Delta(L) \in \mathcal{R}$ satisfying this height condition is precisely the class $C(n) \cup C^*(n)$ introduced at the beginning of Sect. 4. In other words, it the precisely the collection $C(n) \cup C^*(n)$ of intervals that come into play when attempting to construct $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$. We now proceed with the construction.

Assume that $n \geqslant 1$. We subdivide each J_n in $\mathcal{J}_n$ into R closed intervals I_{n+1} of equal length $c_1 R^{-(n+1)}$ and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$|I_{n+1}| = c_1 R^{-(n+1)} \qquad \text{and} \qquad \#\mathcal{I}_{n+1} \, = \, R \, \times \, \#\mathcal{J}_n .$$

It is obvious that the construction of $\mathcal{I}_{n+1}$ corresponds to the splitting procedure associated with the construction of a (I, **R**, **r**) Cantor set.

In view of the nested requirement, the collection $\mathcal{J}_{n+1}$ which we are attempting to construct will be a sub-collection of $\mathcal{I}_{n+1}$. In other words, the intervals I_{n+1} represent possible candidates for J_{n+1}. The goal now is simple—it is to remove those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \, \cap \, \Delta(L) \, \neq \, \emptyset \quad \text{for some} \;\; \Delta(L) \in C(n) \cup C^*(n) \; . \tag{41}$$

The sought after collection $\mathcal{J}_{n+1}$ consists precisely of those intervals that survive. Formally, for $n \geqslant 1$ we let

$$\mathcal{J}_{n+1} := \{ I_{n+1} \in \mathcal{I}_{n+1} \;\; : \;\; I_{n+1} \, \cap \, \Delta(L) = \emptyset \;\; \text{for any} \; \Delta(L) \in C(n) \cup C^*(n) \}.$$

We claim that these collections of surviving intervals satisfy the following key statement. It implies that the act of removing 'bad' intervals from $\mathcal{I}_{n+1}$ is exactly in keeping with the removal procedure associated with the construction of a (J_0, **R**, **r**) Cantor set with **R** and **r** as described in Sect. 3.2.

Proposition 1 *Let* $\epsilon := (ij)^2/8$ *and with reference to Sect.* 4 *let*

$$C(n,l) := \bigcup_{k=0}^{\log_2 R} C(n,k,l) \; , \; C_1(n) := \bigcup_{k=0}^{\log_2 R} C_1(n,k),$$
$$C_2(n) := \bigcup_{k=0}^{\log_2 R} C_2(n,k) \quad \textit{and} \quad \widetilde{C}_3(n,u) := \bigcup_{k=0}^{\log_2 R} \; \bigcup_{v=0}^{\lambda \log_2 R} C_3(n,k,u,v).$$

Then, for $R > R_0(\epsilon)$ *large enough the following four statements are valid.*

1. *For any fixed interval* $J_{n-l} \in \mathcal{J}_{n-l}$*, the intervals from class* $C(n,l)$ *with* $n/\lambda \geqslant l \geqslant 1$ *intersect no more than* $R^{1-\epsilon}$ *intervals* $I_{n+1} \in \mathcal{I}_{n+1}$ *with* $I_{n+1} \subset J_{n-l}$.
2. *For any* $n \geqslant 3n_0$ *where* n_o *is defined by* (16) *and any fixed interval* $J_{n-n_0} \in \mathcal{J}_{n-n_0}$*, the intervals from class* $C^*(n)$ *intersect no more than* $R^{1-\epsilon}$ *intervals* $I_{n+1} \in \mathcal{I}_{n+1}$ *with* $I_{n+1} \subset J_{n-n_0}$.
3. *For any fixed interval* $J_n \in \mathcal{J}_n$*, the intervals from class* $C_1(n)$ *or* $C_2(n)$ *intersect no more than* $R^{1-\epsilon}$ *intervals* $I_{n+1} \in \mathcal{I}_{n+1}$ *with* $I_{n+1} \subset J_n$.
4. *For any fixed interval* $J_{n-u} \in \mathcal{J}_{n-u}$*, the intervals from class* $\widetilde{C}_3(n,u)$ *intersect no more than* $R^{1-\epsilon}$ *intervals* $I_{n+1} \in \mathcal{I}_{n+1}$ *with* $I_{n+1} \subset J_{n-u}$.

Remark 11 Note that in Part 1 we have that $l < n/\lambda$ and in Part 2 we have that u is bounded above by (36). So in either part we have that $l, u \leqslant n$ for all positive values n. Therefore the collections $\mathcal{J}_{n-l}$ and $\mathcal{J}_{n-u}$ are well defined.

Remark 12 By definition, a planar curve $\mathcal{C} := \mathcal{C}_f$ is $C^{(2)}$ non-degenerate if $f \in C^{(2)}(I)$ and there exits at least one point $x \in I$ such that $f''(x) \neq 0$. It will be apparent during the course of establishing Proposition 1 that the condition on the curvature is only required when considering Part 2. For the other parts only the two times continuously differentiable condition is required. Thus, Parts 1, 3 and 4 of the proposition remain valid even when the curve is a line. The upshot is that Proposition 1 remains valid for any $C^{(2)}$ curve for which V_L is not too small and for such curves we are able to establish the analogue of Theorem 1. We will use this observation when proving Theorem 2.

5.1 Dealing with $\mathbf{Bad}(0, 1) \cap \mathcal{C}$

The construction of the collections $\mathcal{J}_n$ satisfying Proposition 1 requires that $i > 0$. However, by making use of the fact that $\mathbf{Bad}(0, 1) \cap \mathcal{C} = (\mathbb{R} \times \mathbf{Bad}) \cap \mathcal{C}$, the case $(i, j) = (0, 1)$ can be easily dealt with.

Let $R \geqslant 2$ be a large integer, and let

$$c_1 := \frac{2cR^2}{c_0} \quad \text{where} \quad 0 < c < \frac{1}{2R^2} . \tag{42}$$

For a given rational number $p/q (q \geqslant 1)$, let $\Delta_{\mathcal{C}}(p/q)$ be the "interval" on $\mathcal{C}$ defined by

$$\Delta_{\mathcal{C}}(p/q) := \left[f^{-1}\left(\frac{p}{q} \pm \frac{c}{H(p/q)}\right)\right] \quad \text{where} \quad H(p/q) := q^2.$$

In view of (6) the inverse function f^{-1} is well defined. Next observe that the orthogonal projection of $\Delta_{\mathcal{C}}(p/q)$ onto the x-axis is contained in the interval $\Delta(p/q)$ centered at the point $f^{-1}(p/q)$ with length

$$|\Delta(p/q)| := \frac{2c}{c_0 H(p/q)}.$$

By analogy with Sect. 2.1.1 the set $\mathbf{Bad}_c^f(0, 1)$ can be described as the set of $x \in I$ such that $x \notin \Delta(p/q)$ for all rationals p/q. For the sake of consistency with the $i > 0$ situation, for $n \geqslant 0$ let

$$\mathcal{C}(n) := \left\{ \Delta(p/q) \ : \ p/q \in \mathbb{Q} \ \text{ and } \ R^{n-1} \leqslant H(p/q) < R^n \right\}.$$

Since $\mathcal{C}(n) = \emptyset$ for $n = 0$, the following analogue of Proposition 1 allows us to deal with the $i = 0$ case. *For $R \geqslant 4$ and any interval $J_n \in \mathcal{J}_n$, we have that*

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} : I_{n+1} \subset J_n \text{ and } \Delta(p/q) \cap I_{n+1} \neq \emptyset \ \text{ for some } \ \Delta(p/q) \in \mathcal{C}(n)\} \leqslant 3. \tag{43}$$

In short, it allows us to construct a $(J_0, \mathbf{R}, \mathbf{r})$ Cantor subset of $\mathbf{Bad}_c^f(0, 1)$ with

$$\mathbf{R} := (R_n) = (R, R, R, \ldots)$$

and

$$\mathbf{r} := (r_{m,n}) = \begin{cases} 3 & \text{if } m = n; \\ 0 & \text{if } m < n. \end{cases}$$

To establish (43) we proceed as follows. First note that in view of (42), we have that

$$\frac{|\Delta(p/q)|}{|I_{n+1}|} \leqslant 1.$$

Thus, any single interval $\Delta(p/q)$ removes at most three intervals I_{n+1} from $\mathcal{I}_{n+1}$. Next, for any two rationals $p_1/q_1, p_2/q_2 \in \mathcal{C}(n)$ we have that

$$\left| f^{-1}\left(\frac{p_1}{q_1}\right) - f^{-1}\left(\frac{p_2}{q_2}\right) \right| \geqslant \frac{1}{|f'(\xi)| q_1 q_2} \geqslant \frac{1}{c_0} R^{-n} > c_1 R^{-n} := |J_n|$$

where ξ is some number between p_1/q_1 and p_2/q_2. Thus, there is at most one interval $\Delta(p/q)$ that can possibly intersect any given interval J_n from $\mathcal{J}_n$. This together with the previous fact establishes (43).

6 Forcing lines to intersect at one point

From this point onwards, all our effort is geared towards establishing Proposition 1. Fix a generic interval $J \subset I$ of length $c_1' R^{-n}$. Note that the position of J is not specified and sometimes it may be more illuminating to picture J as an interval on $\mathcal{C}$. Consider all intervals $\Delta(L)$ from the same class (either $C(n, k, l, m)$, $C^*(n, k)$, $C_1(n, k) \cap C(n, k, 0, m)$, $C_2(n, k)$ or $C_3(n, k, u, v)$) with $\Delta(L) \cap J \neq \emptyset$. The overall aim of this section is to determine conditions on the size of c_1' so that the associated lines L necessarily intersect at single point.

6.1 Preliminaries: estimates for F_L and F_L'

Let

$$c_1' \geqslant 2Kc^{1/2} \cdot 2^{-k} R. \tag{44}$$

This condition guarantees that any interval $\Delta(L) \in C(n, k, l)$ (or $\Delta(L) \in C^*(n, k)$) has length smaller than $|J|$. Indeed,

$$|\Delta(L)| = 2Kc^{1/2} \cdot (H(\Delta))^{-1} \overset{(23)}{\leqslant} 4Kc^{1/2} R \cdot 2^{-k} R^{-n} \leqslant |J|.$$

In this section we obtain various estimates for $|F_L(x)|$ and $|F'_L(x)|$ that are valid for any $x \in J$. Recall, x_0 is as usual the point at which $|F'_L(x)|$ attains its minimum with $x \in \Delta(L)$.

Lemma 2 *Let* $0 \le m \le \lambda \log_2 R$, $l \geqslant 0$ *and* c'_1 *be a positive parameter such that*

$$8C_0c'_1R^{-n} \leqslant 2^{-m}R^{-\lambda l}. \tag{45}$$

Let $J \subset I$ *be an interval of length* c'_1R^{-n}. *Let* $\Delta(L)$ *be any interval from class* $C(n,k,l,m)$ *such that* $\Delta(L) \cap J \ne \emptyset$. *Then for any* $x \in J$ *we have* $|F'_L(x)| \asymp V_L$ *and*

$$|F_L(x)| \leqslant 5|J|V_L. \tag{46}$$

Proof A consequence of Taylor's formula is that

$$\begin{aligned} |F'_L(x) - V_L| &= |A - Bf'(x) - V_L| = |x - x_0| \cdot |-Bf''(\tilde{x})| \\ &\leqslant (c'_1 + 2Kc^{1/2}R)R^{-n} \cdot C_0 \max\{|A|,|B|\} \\ &\overset{(44)}{\leqslant} 2c'_1R^{-n} \cdot C_0 \max\{|A|,|B|\} \end{aligned} \tag{47}$$

where $\tilde{x}$ is some point between x and x_0. Then by (44) and (45) together with the fact that $\Delta(L) \in C(n,k,l,m)$ we get that

$$|F'_L(x) - V_L| \leqslant \frac{1}{2} \cdot 2^{-m-1}R^{-\lambda l} \max\{|A|,|B|\} \overset{(39)}{\leqslant} \frac{1}{2}V_L.$$

In other words, $|F'_L(x)| \asymp V_L$. Then

$$|F_L(x)| \leqslant |F_L(x_1)| + |x - x_1| \cdot |F'_L(\tilde{x})| \leqslant \frac{c}{\max\{|A|^{1/i},|B|^{1/j}\}} + 4|J|V_L$$

where x_1 is the center of $\Delta(L)$ and $\tilde{x}$ is some point between x and x_1. However

$$c\,(\max\{|A|^{1/i},|B|^{1/j}\})^{-1} = c^{1/2}V_L(H(\Delta))^{-1} \overset{(39)}{\leqslant} c^{1/2}R \cdot R^{-n}V_L \leqslant |J|V_L$$

and as a consequence, (46) follows. □

Lemma 3 *Assume* c'_1 *does not satisfy* (45). *Let* $J \subset I$ *be an interval of length* c'_1R^{-n}. *Let* $\Delta(L(A,B,C)) \in C(n,k,l,m)$ *such that* $\Delta(L) \cap J \ne \emptyset$. *Then for any* $x \in J$ *we have*

$$|F_L(x)| \leqslant 30C_0|J|^2 \max\{|A|,|B|\} \tag{48}$$

and

$$|F'_L(x)| \leqslant 10C_0|J| \max\{|A|,|B|\}. \tag{49}$$

Proof In view of (47) it follows that

$$|F_L'(x)| \leqslant 2c_1' R^{-n} C_0 \max\{|A|, |B|\} + V_L \, .$$

By (39) we have that

$$V_L \leqslant 2^{-m} R^{-\lambda l} \max\{|A|, |B|\} \leqslant 8C_0|J| \max\{|A|, |B|\}.$$

Combining these estimates gives (49).

To establish inequality (48) we use Taylor's formula. The latter implies the existence of some point $\tilde{x}$ between x and x_1 such that

$$\begin{aligned} |F_L(x)| &\leqslant |F_L(x_1)| + |x - x_1||F_L'(x_1)| + \frac{1}{2}|x - x_1|^2| - Bf''(\tilde{x})| \\ &\overset{(49)}{\leqslant} \frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\}} + 20C_0|J|^2 \max\{|A|, |B|\} + 2C_0|J|^2 \max\{|A|, |B|\}. \end{aligned}$$

This together with the fact that the first of the three terms on the r.h.s. is bounded above by $c^{1/2} V_L (H(\Delta))^{-1} \leqslant 8C_0|J|^2 \max\{|A|, |B|\}$ yields (48). □

The next lemma provides an estimate for $F_L(x)$ and $F_L'(x)$ in case $\Delta(L)$ is of Type 2.

Lemma 4 *Let c_1' be a positive parameter such that*

$$1 \leqslant C_0 c_1' \quad \textit{and} \quad R^2 c \leqslant C_0 c_1'^2. \tag{50}$$

Let $J \subset I$ be an interval of length $c_1' R^{-n}$. Let $\Delta(L)$ be any interval from class $C^*(n,k)$ such that $\Delta(L) \cap J \neq \emptyset$. Then for any $x \in J$ we have

$$|F_L(x)| \leqslant 9C_0|J|^2 \max\{|A|, |B|\} \tag{51}$$

and

$$|F_L'(x)| \leqslant 3C_0|J| \max\{|A|, |B|\}. \tag{52}$$

Proof As in the previous two lemmas a simple consequence of Taylor's formula is that there exists $\tilde{x}$ between x and x_0 such that:

$$|F_L'(x)| \leqslant V_L + |x - x_0| \cdot | - Bf''(\tilde{x})| \overset{(38)}{\leqslant} R^{-n} \max\{|A|, |B|\} + 2C_0|J| \max\{|A|, |B|\}$$

which by (50) leads to (52). For the first inequality, by Taylor's formula we have that

$$|F_L(x)| \leqslant \frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\}} + 8C_0|J|^2 \max\{|A|, |B|\} \tag{53}$$

On the other hand by (23) we have that

$$H(\Delta) = (\max\{|A|^{1/i}, |B|^{1/j}\}|B|)^{1/2} \geqslant R^{n-1}$$

and so

$$\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\}} \leqslant \frac{R^2 c|B|}{R^{2n}} \overset{(50)}{\leqslant} C_0|J|^2 \max\{|A|, |B|\}\,.$$

This together with (53) yields (51) . □

6.2 Avoiding parallel lines

Consider all lines $L_1, L_2, \ldots$ such that the corresponding intervals $\Delta(L_1), \Delta(L_2), \ldots$ belong to the same class and intersect J. Recall, $|J| := c_1' R^{-n}$. In this section, we determine conditions on c_1' which ensure that none of the lines L_i are parallel to one another.

Remark 13 For the sake of clarity and to minimize notation, throughout the rest of the paper we will often write $V_1, V_2, \ldots$ instead of $V_{L_1}, V_{L_2}, \ldots$ when there is no risk of ambiguity.

Lemma 5 *Assume that there are at least two parallel lines $L_1(A_1, B_1, C_1)$, $L_2(A_2, B_2, C_2)$ such that $\Delta(L_1) \cap J \neq \emptyset$ and $\Delta(L_2) \cap J \neq \emptyset$. If $\Delta(L_1), \Delta(L_2) \in C(n, k, l, m)$ and (45) is satisfied then*

$$c_1' V_1 \min\{|A_1|, |B_1|\} \gg R^n. \tag{54}$$

If $\Delta(L_1), \Delta(L_2) \in C(n, k, l, m)$ and (45) is false or $\Delta(L_1), \Delta(L_2) \in C^*(n, k)$ and (50) is true then

$$c_1' \sqrt{|A_1||B_1|} \gg R^n. \tag{55}$$

Proof Assuming that $L(A_1, B_1, C_1)$, $L(A_2, B_2, C_2)$ are parallel implies that $A_2 = tA_1$, $B_2 = tB_1$, $t \in \mathbb{Q}$. Without loss of generality, assume that $|t| \leqslant 1$. This implies that $|A_1| \geqslant |A_2|$ and $|B_1| \geqslant |B_2|$. Then for an arbitrary point $x \in J$, we have

$$|tC_1 - C_2| = |tF_{L_1}(x) - F_{L_2}(x)|\,. \tag{56}$$

The denominator of t divides both A_1 and B_1 so t is at most $\min(|A_1|, |B_1|)$. Therefore the l.h.s. of (56) is at least $(\min\{|A_1|, |B_1|\})^{-1}$.

If c_1' satisfies (45) then the conditions of Lemma 2 are true. Therefore $V_1 \asymp V_2$ and r.h.s. of (56) is at most $5|J|(V_1 + V_2) \ll c_1' V_1 R^{-n}$. This together with the previous estimate for the l.h.s. of (56) gives (54). To establish the remaining part of the lemma, we exploit either Lemma 3 or Lemma 4 to show that

$$\text{r.h.s. of (56)} \ll |J|^2 \max\{|A_1|, |B_1|\} = (c_1' R^{-n})^2 \max\{|A_1|, |B_1|\}.$$

This together with the previous estimate for the l.h.s. of (56) gives (55). □

The upshot of Lemma 5 is that there are no parallel lines in the same class passing through a generic J of length $c_1' R^{-n}$ if c_1' is chosen to be sufficiently small so that (54) and (55) are violated; namely

$$0 < c_1' < \min\left\{ \frac{a\,R^n}{V_1 \min\{|A_1|, |B_1|\}}, \frac{b\,R^n}{\sqrt{|A_1||B_1|}} \right\}$$

where a and b are the implied positive constants associated with (54) and (55) respectively.

6.3 Ensuring lines intersect at one point

Recall, our aim is to determine conditions on c_1' which ensure that all lines L associated with intervals $\Delta(L)$ from the same class with $\Delta(L) \cap J \neq \emptyset$ intersect at one point. We will use the following well-known fact. For $i = 1, 2, 3$, let $L_i(A_i, B_i, C_i)$ be a line given by the equation $A_i x - B_i y + C_i = 0$. The lines do not intersect at a single point if and only if

$$\det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \neq 0.$$

Suppose there are at least three intervals $\Delta(L_1), \Delta(L_2), \Delta(L_3)$ from the same class (either $C(n, k, l)$, $C_1(n, k)$, $C_2(n, k)$, $C_3(n, k, u, v)$ or $C^*(n, k)$) that intersect J but the corresponding lines L_1, L_2 and L_3 do not intersect at a single point. Then

$$\left| \det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \right| \geqslant 1.$$

Choose an arbitrary point $x \in J$. Firstly assume that the length $c_1' R^{-n}$ of J satisfies (45) and that the intervals $\Delta(L_1), \Delta(L_2), \Delta(L_3)$ are of Type 1. Then Lemma 2 implies that

$$|F_{L_1}(x)| \ll |J| V_1.$$

The same inequalities are true for $F_{L_2}(x)$ and $F_{L_3}(x)$. We write this formally as

$$\left| \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ f(x) \\ 1 \end{pmatrix} \right| \ll \begin{pmatrix} |J|\,V_1 \\ |J|\,V_2 \\ |J|\,V_3 \end{pmatrix}.$$

where $|(x_1, x_2, x_3)|$ denotes the vector $(|x_1|, |x_2|, |x_3|)$ and $(x_1, x_2, x_3) \ll (y_1, y_2, y_3)$ means that $x_1 \ll y_1$, $x_2 \ll y_2$ and $x_3 \ll y_3$. We shall make use of the following useful fact that is a consequence of the triangle inequality. If two vectors $\mathbf{x}$ and $\mathbf{y}$ from $\mathbb{R}^3$

satisfy $|\mathbf{x}| \ll |\mathbf{y}|$ then for any 3×3 real matrix M we have $|M\mathbf{x}| \ll |M| \cdot |\mathbf{y}|$ where the entries of $|M|$ are the absolute values of the correspondent entries in M. On applying this with

$$\mathbf{x} = \begin{pmatrix} x \\ f(x) \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} |J|\, V_1 \\ |J|\, V_2 \\ |J|\, V_3 \end{pmatrix}, \quad M = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}^{-1}$$

and by using Cramer's rule we obtain the following inequality for the third row:

$$|J|\ (|V_1(A_2B_3 - A_3B_2)| + |V_2(A_1B_3 - A_3B_1)| + |V_3(A_1B_2 - A_2B_1)|) \gg 1.$$

Without loss of generality assume that the first term on the l.h.s. of this inequality is the largest of the three terms. Then

$$c_1'|V_1(A_2B_3 - A_3B_2)| \gg R^n. \tag{57}$$

In other words, if the lines L_1, L_2 and L_3 do not intersect at one point and (45) is true for a given c_1' then (57) must also hold.

If (45) is not true or the intervals $\Delta(L_1)$, $\Delta(L_2)$, $\Delta(L_3)$ are of Type 2 then we apply either Lemma 3 or Lemma 4. Together with Cramer's rule, we obtain that

$$\begin{aligned}|J|^2(&\max\{|A_1|, |B_1|\}|A_2B_3 - A_3B_2| + \max\{|A_2|, |B_2|\}|A_1B_3 - A_3B_1| \\ &+ \max\{|A_3|, |B_3|\}|A_1B_2 - A_2B_1|) \gg 1.\end{aligned}$$

Without loss of generality assume that the first of the three terms on the l.h.s. of this inequality is the largest. Then, we obtain that

$$c_1'\sqrt{|\max\{|A_1|, |B_1|\}(A_2B_3 - A_3B_2)|} \gg R^n. \tag{58}$$

We now investigate the ramifications of the conditions (57) and (58) on specific classes of intervals.

6.3.1 Case $\Delta(L_1), \Delta(L_2), \Delta(L_3) \in C(n, k, l, m), l \geqslant 1$

We start by estimating the difference between $\frac{A_1}{B_1}$ and $\frac{A_2}{B_2}$. By (24) we have that

$$\left|\frac{A_1}{B_1} - \frac{A_2}{B_2}\right| \leqslant \left|\frac{A_1}{B_1} - f'(x_{01})\right| + |f'(x_{01}) - f'(x_{02})| + \left|f'(x_{02}) - \frac{A_2}{B_2}\right| \ll R^{-\lambda l} + |J| \tag{59}$$

where x_{01} and x_{02} are given by $V_1 := |A_1 - B_1 f'(x_{01})|$ and $V_2 := |A_2 - B_2 f'(x_{02})|$ respectively.

- *Assume that* (45) *is satisfied*. This means that $|J| \ll R^{-\lambda l}$. We rewrite (57) as

$$c_1'|V_1B_2B_3|\left|\frac{A_2}{B_2} - \frac{A_3}{B_3}\right| \gg R^n.$$

Then in view of (26), (24) and (59) it follows that

$$R^n \ll c_1' R^{-\lambda l}\left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda(l+1)}\right)^{\frac{3i}{i+1}} \cdot R^{-\lambda l}$$
$$\overset{(45)}{\ll} c_1' R^{n-\frac{j-i}{i+1}n} \cdot R^{-\frac{2-i}{i+1}\lambda l} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{\frac{3i}{i+1}}.$$

Since by assumption $i \leqslant j$, the last inequality implies that if (57) holds then

$$c_1' \gg R^{l\lambda\frac{2-i}{i+1}} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-\frac{3i}{i+1}}.$$

Hence, the condition

$$c_1' \ll R^{l\lambda\frac{2-i}{i+1}} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-\frac{3i}{i+1}}$$

will contradict the previous inequality and imply that (57) is not satisfied. Note that similar arguments imply that if (54) holds then

$$R^n \ll c_1' R^{-\lambda l}\left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda(l+1)}\right)^{\frac{2i}{i+1}} = c_1' R^{n-\frac{j}{i+1}n} R^{-\frac{j}{i+1}\lambda l} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{\frac{2i}{i+1}}.$$

It follows that the condition

$$c_1' \ll R^{l\lambda\frac{j}{i+1}} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-\frac{3i}{i+1}}.$$

will contradict the previous inequality and imply that (54) is not satisfied.

The upshot is that for λ satisfying (11) the following condition on c_1'

$$c_1' \leqslant \delta \cdot R^l \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-\frac{3i}{i+1}} \tag{60}$$

will contradict both (57) and (54). Here $\delta = \delta(i, j, c_0, C_0) > 0$ is the absolute unspecified constant within the previous inequalities involving the Vinogradov symbols. In other words, if c_1' satisfies (60), then the lines L_i associated with the intervals $\Delta(L_i) \in C(n, k, l, m)$ with $l \geqslant 1$ such that $\Delta(L_i) \cap J \neq \emptyset$ intersect at a single point.

- *Assume that* (45) *is false*. In this case $R^{-\lambda l} \ll R^{\lambda}|J|$. In view of (25) we have that $|A_1| \asymp |B_1|$ and inequality (58) implies that

$$c_1' \sqrt{|B_1 B_2 B_3| \left| \frac{A_2}{B_2} - \frac{A_3}{B_3} \right|} \gg R^n.$$

In view of (26) and (59), it follows that to

$$R^n \ll (c_1')^{\frac{3}{2}} R^{\lambda/2 - n/2} \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{n+\lambda(l+1)} \right)^{\frac{3i}{2(i+1)}}$$

which is equivalent to

$$R^n \ll c_1' R^{\lambda/3} \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{\frac{i}{(i+1)}} (R^{n+\lambda l})^{\frac{i}{i+1}}.$$

This together with that fact that $i \leqslant 1/2$ and $\lambda l \leqslant n$ implies that

$$c_1' \gg R^{\frac{j}{i+1}\lambda l} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{i}{i+1}} R^{-\frac{\lambda}{3}}.$$

By similar arguments, estimate (55) implies that

$$c_1' \gg R^{\frac{j}{i+1}\lambda l} \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{i}{i+1}}.$$

The upshot is that for λ satisfying (11), we obtain a contradiction to both these upper bound inequalities for c_1' and thus to (58) and (55), if

$$c_1' \leqslant \delta \cdot R^l \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{i}{i+1}} R^{-\frac{\lambda}{3}}. \tag{61}$$

In other words, if c_1' satisfies (61) but not (45), then the lines L_i associated with the intervals $\Delta(L_i) \in C(n, k, l, m)$ with $l \geqslant 1$ such that $\Delta(L_i) \cap J \neq \emptyset$ intersect at a single point.

6.3.2 Case $\Delta(L_1), \Delta(L_2), \Delta(L_3) \in C^*(n, k)$

For this class of intervals we will eventually make use of Lemma 4. With this in mind, *we assume that* (50) *is valid.* A consequence of (50) is that $R^{-n} \ll |J|$. It is readily verified that in the case under consideration, the analogy to (59) is given by

$$\left|\frac{A_1}{B_1}-\frac{A_2}{B_2}\right| \ll \frac{V_1}{B_1}+|J|+\frac{V_2}{B_2} \overset{(38)}{\ll} |J|.$$

Then by using (37), we find that inequality (58) implies that

$$c_1' \gg R^{\frac{j}{1+i}n}\cdot\left(\frac{2^k}{R}\right)^{-\frac{2i}{i+1}}.$$

Similarly, inequality (55) implies the same upper bound for c_1'. Thus if c_1' satisfies the condition

$$c_1' \leqslant \delta\cdot R^{\frac{j}{1+i}n}\cdot\left(\frac{2^k}{R}\right)^{-\frac{2i}{(i+1)}}, \tag{62}$$

we obtain a contradiction to both (58) and (55).

6.3.3 Case $\Delta(L_1), \Delta(L_2), \Delta(L_3)\in C_1(n,k)\cap C(n,k,0,m)$

In view of (27) and (29), inequality (57) implies that

$$R^n \ll c_1'\left(\frac{2^k c^{\frac{1}{2}}}{R}R^{n+\lambda}\right)^{\frac{3i}{i+1}} \overset{i\leqslant 1/2}{\ll} c_1'\left(\frac{2^k c^{\frac{1}{2}}}{R}R^{n+\lambda}\right).$$

Hence, if c_1' satisfies the condition

$$c_1' \leqslant \delta\cdot\left(\frac{2^k c^{\frac{1}{2}}}{R}R^{\lambda}\right)^{-1} \tag{63}$$

we obtain a contradiction to (57). Note that the same upper bound inequality for c_1' will also contradict (54).

For the class $C_1(n,k)$ as well as all other subclasses of $C(n,k,0)$, when consider the intersection with a generic interval J of length $c_1'R^{-n}$ the constant c_1' will always satisfy (45). *Therefore, without loss of generality we assume that* c_1' *satisfies* (45).

6.3.4 Case $\Delta(L_1), \Delta(L_2), \Delta(L_3)\in C_2(n,k)$

By (28), (30) and (31), inequality (57) implies that

$$R^n \ll c_1'\left(\frac{2^k c^{\frac{1}{2}}}{R}R^{n+\lambda}\right)^{\frac{i}{j+1}+\frac{2j}{j+1}} = c_1'\left(\frac{2^k c^{\frac{1}{2}}}{R}R^{n+\lambda}\right).$$

It is now easily verified that if c_1' satisfies inequality (63) then we obtain a contradiction to (57).

6.3.5 Case $\Delta(L_1), \Delta(L_2), \Delta(L_3) \in C_3(n, k, u, v)$

By (34) and (35), inequality (57) implies that

$$R^n \ll c_1' \left(\frac{2^k c^{\frac{1}{2}}}{R} R^n \right)^{\frac{i}{j+1} + \frac{2j}{j+1}} \cdot R^{-\frac{2\lambda uj}{j+1} + \frac{\lambda uij}{j+1}} R^{\lambda i} \ll c_1' \left(\frac{2^k c^{\frac{1}{2}}}{R} R^n \right) R^{\lambda i - \lambda uj}.$$

Hence, if c_1' satisfies the condition

$$c_1' \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} \right)^{-1} R^{\lambda uj - \lambda i},$$

we obtain a contradiction to (57). It is easily verified that if c_1' satisfies this lower bound inequality, then we also obtain a contradiction to (54) as well.

It follows by (11) that $\lambda \geqslant 1/j$ and therefore the above lower bound inequality for c_1' is true if

$$c_1' \leqslant \delta \cdot R^u \cdot \frac{R^{1-\lambda i}}{2^k c^{\frac{1}{2}}}. \tag{64}$$

The upshot of this section is as follows. Assume that $\Delta(L_1), \Delta(L_2), \Delta(L_3)$ all intersect J and belong to the same class. Then for each class, specific conditions for c_1' have been determined that force the corresponding lines L_1, L_2 and L_3 to intersect at a single point. These conditions are (45), (50), (60), (61), (62), (63) and (64).

7 Geometrical properties of pairs (A, B)

Consider two intervals $\Delta(L_1), \Delta(L_2) \in \mathcal{R}$ where the associated lines $L_1(A_1, B_1, C_1)$ and $L_2(A_2, B_2, C_2)$ are not parallel. Denote by P the point of intersection $L_1 \cap L_2$. To begin with we investigate the relationship between P, $\Delta(L_1)$ and $\Delta(L_2)$.

It is easily seen that

$$P = \left(\frac{p}{q}, \frac{r}{q} \right) = \left(\frac{C_2 B_1 - C_1 B_2}{A_1 B_2 - A_2 B_1}, \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1} \right); \quad (p, r, q) = 1.$$

Therefore

$$A_1 B_2 - A_2 B_1 = tq, \quad C_1 B_2 - C_2 B_1 = -tp, \quad A_1 C_2 - A_2 C_1 = tr \tag{65}$$

for some integer t. Let x_1 and x_2 be two arbitrary points on $\Delta(L_1)$ and $\Delta(L_2)$. Since $P \in L_1 \cap L_2$, it follows that

$$A_1 \left(x_1 - \frac{p}{q} \right) - B_1 \left(f(x_1) - \frac{r}{q} \right) = F_{L_1}(x_1),$$

$$A_2\left(x_2-\frac{p}{q}\right)-B_2\left(f(x_2)-\frac{r}{q}\right)=F_{L_2}(x_2).$$

By Taylor's formula the second equality can be written as

$$A_2\left(x_1-\frac{p}{q}\right)-B_2\left(f(x_1)-\frac{r}{q}\right)=F_{L_2}(x_2)+(x_1-x_2)F'_{L_2}(\tilde{x}),$$

where $\tilde{x}$ is some point between x_1 and x_2. This together with the first equality gives

$$\begin{pmatrix} A_1 & -B_1\\ A_2 & -B_2\end{pmatrix}\cdot\begin{pmatrix} x_1-\frac{p}{q}\\ f(x_1)-\frac{r}{q}\end{pmatrix}=\begin{pmatrix} F_{L_1}(x_1)\\ F_{L_2}(x_2)+(x_1-x_2)F'_{L_2}(\tilde{x})\end{pmatrix}.$$

which on applying Cramer's rule leads to

$$x_1-\frac{p}{q}=\frac{B_1(F_{L_2}(x_2)+(x_1-x_2)F'_{L_2}(\tilde{x}))-B_2F_{L_1}(x_1)}{\det\mathbf{A}} \tag{66}$$

and

$$f(x_1)-\frac{r}{q}=\frac{A_1(F_{L_2}(x_2)+(x_1-x_2)F'_{L_2}(\tilde{x}))-A_2F_{L_1}(x_1)}{\det\mathbf{A}}. \tag{67}$$

Here

$$\det\mathbf{A}:=-A_1B_2+A_2B_1\overset{(65)}{=}-tq.$$

Now assume that both intervals $\Delta(L_1)$ and $\Delta(L_2)$ belong to the same class and intersect a fixed generic interval J of length c'_1R^{-n}. Then, we exploit the fact that x_1, x_2 can both be taken in J. Firstly consider the case that J satisfies (45) and $\Delta(L_1)$, $\Delta(L_2)$ are of Type 1. Then by Lemma 2

$$F'_{L_2}(\tilde{x})\asymp V_2,\ F_{L_1}(x_1)\ll|J|V_1,\ F_{L_2}(x_2)\ll|J|V_2 \text{ and } |x_1-x_2|\leqslant|J|=c'_1R^{-n}.$$

This together with (66) and (67) implies that

$$\begin{aligned}\frac{|B_1|V_2+|B_2|V_1}{R^n}&\gg\frac{|qx_1-p|}{c'_1},\\ \frac{|A_1|V_2+|A_2|V_1}{R^n}&\gg\frac{|qf(x_1)-r|}{c'_1}.\end{aligned} \tag{68}$$

If J does not satisfy (45) and $\Delta(L_1)$, $\Delta(L_2)$ are of Type 1 we make use of Lemma 3 to estimate the size of $F_{L_2}(x_2)$, $F'_{L_2}(\tilde{x})$ and $F_{L_1}(x_1)$. This together with (66) and (67) implies that

$$\begin{aligned}\frac{(|B_1|\max\{|A_2|,|B_2|\}+|B_2|\max\{|A_1|,|B_1|\})}{R^{2n}} &\gg \frac{|qx_1-p|}{(c_1')^2},\\ \frac{(|A_1|\max\{|A_2|,|B_2|\}+|A_2|\max\{|A_1|,|B_1|\})}{R^{2n}} &\gg \frac{|qf(x_1)-r|}{(c_1')^2}.\end{aligned} \tag{69}$$

On making use of Lemma 4, it is easily verified that the same inequalities are valid when $\Delta(L_1)$, $\Delta(L_2)$ are of Type 2 and J satisfies (50).

7.1 The case P is close to $\mathcal{C}$

We consider the situation when the point $P = (p/q, r/q)$ is situated close to the curve $\mathcal{C}$. More precisely, assume that there exists at least one point $(x, f(x)) \in \mathcal{C}$ such that,

$$\left|x - \frac{p}{q}\right| < \frac{c}{2}\cdot q^{-1-i}, \quad \left|f(x) - \frac{r}{q}\right| < \frac{c}{2}\cdot q^{-1-j}.$$

We show that every such point x is situated inside $\Delta(L_0)$ for some line L_0 passing through P. Indeed, each line $L(A, B, C)$ which passes through P will satisfy the equation $Ap - Br + Cq = 0$. By Minkowski's Theorem there exists an integer non-zero solution A_0, B_0, C_0 to this equation such that

$$|A_0| < q^i; \quad |B_0| < q^j.$$

Then

$$|F_{L_0}(x)| = |A_0x - B_0f(x) + C_0| = \left|A_0\left(x - \frac{p}{q}\right) - B_0\left(f(x) - \frac{r}{q}\right)\right| \leqslant cq^{-1}$$

since $|A_0\cdot\frac{p}{q} - B_0\cdot\frac{r}{q} + C_0| = 0$. In other words, the point $x \in \Delta(L_0)$.

7.2 The figure F

Consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from the same class (either $C(n, k, l, m)$ with $l \geqslant 1$, $C^*(n, k)$, $C_1(n, k)\cap C(n, k, 0, m)$, $C_2(n, k)$ or $C_3(n, k, u, v)$) which intersect a generic interval J of length $c_1'R^{-n}$. In this section we investigate the implication of this on the coefficients of the corresponding lines L_t.

In Sect. 6 we have shown that under certain conditions on c_1' all the corresponding lines L_t intersect at one point. Assume now that the appropriate conditions are satisfied—this depends of course on the class of intervals under consideration. Let $P = (p/q, r/q)$ denote the point of intersection of the lines L_t. Then the triple (A_t, B_t, C_t) will satisfy the equation

$$A_tp - B_tr + C_tq = 0 \qquad A_t, B_t, C_t \in \mathbb{Z}.$$

Hence the points $(A_t, B_t) \in \mathbb{Z}^2$ lie in a lattice $\mathbf{L}$ with fundamental domain of area equal to q.

Let x_t be the point of minimum of $|F'_{L_t}(x)|$ on $\Delta(L_t)$. Define

$$\omega_x(P, J) := \max_t\{|qx_t - p|\} \quad \text{and} \quad \omega_y(P, J) := \max_t\{|qf(x_t) - r|\}.$$

Furthermore, let t_1 (resp. t_2) be the integer at which the maximum associated with ω_x (resp. ω_x) is attained; i.e.

$$|qx_{t_1} - p| = \omega_x(P, J) \quad \text{and} \quad |qf(x_{t_2}) - r| = \omega_y(P, J).$$

We now consider several cases.

7.2.1 Interval J satisfies (45) and intervals $\Delta(L_1)$ are of Type 1

Assume that the interval J satisfies (45). Then on applying (68) with respect to the pair of intervals $(\Delta(L_t), \Delta(L_{t_1}))$ and $(\Delta(L_t), \Delta(L_{t_2}))$, we find that the following two conditions are satisfied:

$$\frac{|B_{t_1}V_t| + |B_tV_{t_1}|}{R^n} \geqslant v_x := \frac{\omega_x(P, J)}{c'_1c_x(C_0, c_0, i, j)} \qquad t \neq t_1 \tag{70}$$

$$\frac{|A_{t_2}V_t| + |A_tV_{t_2}|}{R^n} \geqslant v_y := \frac{\omega_y(P, J)}{c'_1c_y(C_0, c_0, i, j)} \qquad t \neq t_2\,, \tag{71}$$

where $c_x(C_0, c_0, i, j)$ and $c_y(C_0, c_0, i, j)$ are constants dependent only on C_0, c_0, i and j.

Firstly consider inequality (70). Since all intervals $\Delta(L_t)$ lie in the same class $(C(n, k, l, m)$ with $l \geqslant 1$, $C_1(n, k) \cap C_(n, k, 0, m)$, $C_2(n, k)$ or $C_3(n, k, u, v))$, then by either (32) or (40) we have $V_{t_1} \asymp V_t$. This together with (23) substituted into (70) gives

$$v_x \leqslant \frac{|B_{t_1}V_t| + |B_tV_{t_1}|}{R^n} \ll \frac{2^{k+1}}{R} \cdot \frac{(|B_{t_1}| + |B_t|)V_t}{H(A_t, B_t)}.$$

In other words,

$$v_x \ll \frac{2^kc^{\frac{1}{2}}}{R} \cdot \frac{|B_t| + |B_{t_1}|}{\max\{|A_t|^{1/i}, |B_t|^{1/j}\}}. \tag{72}$$

This means that all pairs (A_t, B_t) under consideration are situated within some figure defined by (72) which we denote by F_x. Note that F_x depends on B_{t_1} and c'_1 which in turn is defined by the point P, interval J and the class of intervals $\Delta(L_t)$. The upshot is that if all lines L_t intersect at one point P and all intervals $\Delta(L_t)$ intersect J then all pairs (A_t, B_t), except possibly one with $t = t_1$, lie in the set $F_x \cap \mathbf{L}$.

When considering inequality (71), similar arguments enable us to conclude that all pairs (A_t, B_t), except possibly one, lie in the set $F_y \cap \mathbf{L}$ where F_y is the figure defined by

$$v_y \ll \frac{2^kc^{\frac{1}{2}}}{R} \cdot \frac{|A_t| + |A_{t_2}|}{\max\{|A_t|^{1/i}, |B_t|^{1/j}\}}. \tag{73}$$

This together with the previous statement for F_x implies that all pairs (A_t, B_t), except possibly two, lie in the set $F_x \cap F_y \cap \mathbf{L}$.

7.2.2 Interval J does not satisfy (45) and intervals $\Delta(L_t)$ are of Type 1

Now assume that interval J does not satisfy (45). Then by applying (69) for the pair of intervals $(\Delta(L_t), \Delta(L_{t_1}))$ and $(\Delta(L_t), \Delta(L_{t_2}))$ we obtain the following two conditions:

$$\frac{|B_{t_1}|\max\{|A_t|, |B_t|\} + |B_t|\max\{|A_{t_1}|, |B_{t_1}|\}}{R^{2n}} \geqslant \sigma_x := \frac{\omega_x(P, J)}{(c_1')^2 c_x(C_0, i, j)} \quad t \neq t_1$$
$$\frac{|A_{t_2}|\max\{|A_t|, |B_t|\} + |A_t|\max\{|A_{t_2}|, |B_{t_2}|\}}{R^{2n}} \geqslant \sigma_y := \frac{\omega_y(P, J)}{(c_1')^2 c_y(C_0, i, j)} \quad t \neq t_2,$$

which play the same role as (70) and (71) in the previous case. By similar arguments as before, we end up with two figures F_x' and F_y' defined as follows:

$$\sigma_x \ll \frac{2^k c^{\frac{1}{2}}}{R^{n+1}} \cdot \frac{(|B_t| + |B_{t_1}|)\max\{|A_t|, |B_t|, |A_{t_1}|, |B_{t_1}|\}}{V_t \max\{|A_t|^{1/i}, |B_t|^{1/j}\}} \tag{74}$$

and

$$\sigma_y \ll \frac{2^k c^{\frac{1}{2}}}{R^{n+1}} \cdot \frac{(|A_t| + |A_{t_2}|)\max\{|A_t|, |B_t|, |A_{t_2}|, |B_{t_2}|\}}{V_t \max\{|A_t|^{1/i}, |B_t|^{1/j}\}}. \tag{75}$$

The upshot being that when J does not satisfy (45) all pairs (A_t, B_t), except possibly two, lie in the set $F_x' \cap F_y' \cap \mathbf{L}$.

7.2.3 Intervals $\Delta(L_t)$ are of Type 2

As usual, for Type 2 intervals we assume that (50) is satisfied. With appropriate changes, such as the definition of $H(\Delta)$, the same arguments as above can be utilised to show that all pairs (A_t, B_t), except possibly two, lie in the set $F_x^* \cap F_y^* \cap \mathbf{L}$ where the figures F_x^* and F_y^* are defined as follows:

$$\sigma_x \ll \frac{2^{2k}}{R^2} \cdot \frac{|B_t|}{\max\{|A_t|^{1/i}, |B_t|^{1/j}\}} \tag{76}$$

and

$$\sigma_y \ll \frac{2^{2k}}{R^2} \cdot \frac{|A_t|}{\max\{|A_t|^{1/i}, |B_t|^{1/j}\}}. \tag{77}$$

Indeed, the calculations are somewhat simplified since for intervals of Type 2 we have that $|A_t| \asymp |A_{t_1}| \asymp |A_{t_2}|$ and $|B_t| \asymp |B_{t_1}| \asymp |B_{t_2}|$.

7.3 Restrictions to $F_x \cap F_y$ in each class

We now use the specific properties of each class to reduce the size of $F_x \cap F_y$ in each case.

- **Class** $C(n,k,l,m)$ **with** $l \geqslant 1$ **and interval** J **satisfies** (45). Consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C(n,k,l,m)$ such that the corresponding coordinates (A_t, B_t) lie within the figure F_x defined by (72). First of all notice that by (25) we have $|A_t| \asymp |B_t|$. Then by (39) we obtain that

$$\frac{|A_t|}{|V_t|} \asymp 2^m R^{\lambda l} \tag{78}$$

which together with (40) and (72) implies that

$$|B_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R v_x}\right)^{j/i} ; \quad |A_t| \ll \left(\frac{2^k c^{\frac{1}{2}} B_t}{R v_x}\right)^{i} \ll \frac{2^k c^{\frac{1}{2}}}{R v_x}; \quad V_t \ll \frac{2^k c^{\frac{1}{2}}}{R v_x} 2^{-m} R^{-\lambda l}.$$

If we consider the coordinates (A_t, B_t) within the figure F_y defined by (73), we obtain the analogous inequalities:

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R v_y}\right)^{\frac{i}{j}} ; \quad V_t \ll \left(\frac{2^k c^{\frac{1}{2}}}{R v_y}\right)^{\frac{i}{j}} 2^{-m} R^{-\lambda l}.$$

Hence, it follows that all coordinates $(A_t, B_t) \in F_x \cap F_y$ lie inside the box defined by

$$|A_t| \ll \eta := \min\left\{\frac{2^k c^{\frac{1}{2}}}{R v_x}, \left(\frac{2^k c^{\frac{1}{2}}}{R v_y}\right)^{\frac{i}{j}}\right\}; \quad |V_t| \ll |A_t| 2^{-m} R^{-\lambda l}. \tag{79}$$

- **Class** $C(n,k,l,m)$ **with** $l \geqslant 1$ **and interval** J **does not satisfy** (45). Consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C(n,k,l,m)$ such that the corresponding coordinates (A_t, B_t) lie inside F'_x. As in previous case, (78) is valid which together with (40) and (74) implies that

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{\sigma_x R^{n+1}} \cdot 2^m R^{\lambda l} |B_t|\right)^{i} \ll \frac{2^k c^{\frac{1}{2}}}{\sigma_x R} \cdot 2^m R^{\lambda l - n}; \quad V_t \ll |A_t| 2^{-m} R^{-\lambda l}.$$

If we consider the coordinates (A_t, B_t) within the figure F'_y defined by (75), we obtain the analogous inequalities:

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{\sigma_y R} \cdot 2^m R^{\lambda l - n}\right)^{i/j} ; \quad V_t \ll |A_t| 2^{-m} R^{-\lambda l}.$$

Denote by η' the following minimum

$$\eta' := \min\left\{\frac{2^{k+m}c^{\frac{1}{2}}}{\sigma_x R}, \left(\frac{2^{k+m}c^{\frac{1}{2}}}{\sigma_y R}\right)^{i/j}\right\}.$$

Then, since for intervals of Type 1 the parameter l is always at most l_0 which in turn satisfies (14), it follows that all coordinates $(A_t, B_t) \in F'_x \cap F'_y$ lie inside the box defined by

$$|A_t| \ll \eta'; \quad |V_t| \ll \eta' 2^{-m} R^{-\lambda l}. \tag{80}$$

- **Class** $C^*(n,k)$. Consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C^*(n,k)$ such that the corresponding coordinates $(A_t, B_t) \in F^*_x \cap F^*_y$. A consequence of that fact that we are considering Type 2 intervals is that $|B_t| \asymp |B_{t_1}|$. This together with (76) and (77) implies that

$$|B_t| \ll \left(\frac{2^{2k}}{R^2\sigma_x}\right)^{j/i}; \qquad |A_t| \ll \left(\frac{2^{2k}}{R^2\sigma_x}|B_t|\right)^{i} \ll \frac{2^{2k}}{R^2\sigma_x}; \quad |V_t| \overset{(38)}{\ll} |A_t|R^{-n}$$

and

$$|A_t| \ll \left(\frac{2^{2k}}{R^2\sigma_y}\right)^{i/j}.$$

Denote by η^* the following minimum

$$\eta^* := \min\left\{\frac{2^{2k}}{R^2\sigma_x}, \left(\frac{2^{2k}}{R^2\sigma_y}\right)^{i/j}\right\}.$$

The upshot is that all coordinates $(A_t, B_t) \in F^*_x \cap F^*_y$ lie inside the box defined by

$$|A_t| \ll \eta^*; \quad |V_t| \ll \eta^* \cdot R^{-n}. \tag{81}$$

- **Class** $C_1(n,k) \cap C(n,k,0,m)$. As mentioned in Sect. 6.3.3, for all subclasses of $C(n,k,0)$, when consider the intersection with a generic interval J of length $c'_1 R^{-n}$ the constant c'_1 satisfies (45). In other words, J always satisfies (45). With this in mind, consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C_1(n,k) \cap C(n,k,0,m)$ such that the corresponding coordinates (A_t, B_t) lie within the figure F_x defined by (72). Then, the analogue of (78) is

$$2^m|V_t| \asymp |A_t|.$$

Although we cannot guarantee that $|B_t| \asymp |B_{t_1}|$, by (40) we have $V_t \asymp V_{t_1}$ and $|A_t| \asymp |A_{t_1}|$ which in turn implies that $\max\{|A_t|^{1/i}, |B_t|^{1/j}\} \asymp \max\{|A_{t_1}|^{1/i}, |B_{t_1}|^{1/j}\}$. So if $|B_t| \leqslant |B_{t_1}|$, it follows that

$$\frac{|B_t| + |B_{t_1}|}{\max\{|A_t|^{1/i}, |B_t|^{1/j}\}} \asymp \frac{|B_{t_1}|}{\max\{|A_{t_1}|^{1/i}, |B_{t_1}|^{1/j}\}}\,.$$

This together with the previously displayed equation and (72) implies that

$$|B_t| \leqslant |B_{t_1}| \ll \left(\frac{2^k c^{\frac{1}{2}}}{Rv_x}\right)^{j/i}.$$

On the other hand, if $|B_{t_1}| < |B_t|$ we straightforwardly obtain the same estimate for $|B_t|$. So in both cases, we have that

$$|B_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{Rv_x}\right)^{j/i}; \quad |A_t| \ll \left(\frac{2^k c^{\frac{1}{2}} \max\{|B_t|, |B_{t_1}|\}}{Rv_x}\right)^{i} \ll \frac{2^k c^{\frac{1}{2}}}{Rv_x}\,; \quad V_t \ll \frac{2^k c^{\frac{1}{2}}}{Rv_x} 2^{-m}.$$

If we consider the coordinates (A_t, B_t) within the figure F_y, similar arguments together with inequality (73) yield the inequalities:

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{Rv_y}\right)^{\frac{i}{j}}; \quad V_t \ll \left(\frac{2^k c^{\frac{1}{2}}}{Rv_y}\right)^{\frac{i}{j}} 2^{-m} R^{-\lambda l}.$$

Notice that these inequalities are exactly the same as when considering ‘Class $C(n, k, l, m)$ with $l \geqslant 1$, interval J satisfies (45)’ above. The upshot is that all coordinates $(A_t, B_t) \in F_x \cap F_y$ lie inside the box defined by

$$|A_t| \ll \eta\,; \qquad |V_t| \ll 2^{-m}|A_t|\,. \tag{82}$$

Here η is as in (79) and notice that (82) is indeed equal to (79) with $l = 0$.

- **Class** $C_2(n, k)$. In view of (32), for intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C_2(n, k)$ we have that $|B_t| \asymp |B_{t_1}|$. Moreover, although we cannot guarantee that $|A_t| \asymp |A_{t_2}|$, we still have that $\max\{|A_t|^{1/i}, |B_t|^{1/j}\} \asymp \max\{|A_{t_1}|^{1/i}, |B_{t_1}|^{1/j}\}$ and therefore one can apply the same arguments as when considering class $C_1(n, k) \cap C(n, k, 0, m)$ above. As a consequence of (72) and (73), it follows that all coordinates $(A_t, B_t) \in F_x \cap F_y$ lie inside the box defined by

$$|A_t| \ll \eta\,; \qquad |B_t| \ll \eta^{j/i}. \tag{83}$$

- **Class** $C_3(n, k, u, v)$. Consider all intervals $\Delta(L_t(A_t, B_t, C_t))$ from $C_3(n, k, u, v)$ such that the corresponding coordinates (A_t, B_t) lie within the figure F_x. In view of (32), we have that $|A_t| \asymp |B_t| \asymp |B_{t_1}| \asymp |A_{t_2}|$ and (33) implies that $\max\{|A_t|^{1/i}, |B_t|^{1/j}\} > R^{\lambda u}|B_t|^{1/j}$. This together with (72) implies that

$$\frac{Rv_x}{2^k c^{\frac{1}{2}}} \ll B_t^{-i/j} R^{-\lambda u} \Rightarrow B_t \ll \left(\frac{2^k c^{\frac{1}{2}}}{Rv_x}\right)^{j/i} R^{-\lambda u j/i}$$

and

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R v_x}|B_t|\right)^i \ll \frac{2^k c^{\frac{1}{2}}}{R v_x} \cdot R^{-\lambda j u}.$$

If we consider the coordinates (A_t, B_t) within the figure F_y defined by (73), we obtain the analogous inequalities:

$$|A_t| \ll \left(\frac{2^k c^{\frac{1}{2}}}{R v_y}\right)^{i/j} \; ; \quad |B_t| \ll \frac{2^k c^{\frac{1}{2}}}{R v_y} \cdot R^{-\lambda j u}.$$

The upshot is that all coordinates (A_t, B_t) from $F_x \cap F_y$ lie inside the box defined by

$$\begin{aligned} |A_t| &\ll \eta_3 := \min\left\{\frac{2^k c^{\frac{1}{2}}}{R v_x} \cdot R^{-\lambda j u}, \left(\frac{2^k c^{\frac{1}{2}}}{R v_y}\right)^{i/j}\right\} \\ |B_t| &\ll \eta_3^{j/i} R^{-\lambda j u} = \min\left\{\left(\frac{2^k c^{\frac{1}{2}}}{R v_x}\right)^{j/i} R^{-\frac{\lambda j}{i} u}, \frac{2^k c^{\frac{1}{2}}}{R v_y} \cdot R^{-\lambda j u}\right\}. \end{aligned} \tag{84}$$

8 The finale

The aim of this section is to estimate the number of intervals $\Delta(L_t)$ from a given class (either $C(n,k,l,m)$, $C^*(n,k)$, $C_1(n,k) \cap C(n,k,0,m)$, $C_2(n,k)$ or $C_3(n,k,u,v)$) that intersect a fixed generic interval J of length $c_1' R^{-n}$. Roughly speaking, the idea is to show that one of the following two situations necessarily happens:

- All intervals $\Delta(L_t)$ (except possibly at most two) intersect the thickening $\Delta(L_0)$ of some line L_0.
- There are not 'too many' intervals $\Delta(L_t)$.

As in the previous section we assume that all the corresponding lines $L_1, L_2, \cdots$ intersect at one point $P = (p/q, r/q)$. Then the quantities $\omega_x(P, J)$ and $\omega_y(P, J)$ are well defined and the results from Sect. 6.3 are applicable.

8.1 Point P is close to $\mathcal{C}$

Assume that

$$\omega_x(P, J) < \frac{c}{2} q^{-i} \quad \text{and} \quad \omega_y(P, J) < \frac{c}{2} q^{-j}. \tag{85}$$

Then, by the definition of ω_x and ω_y, we have that for each $\Delta(L_t)$

$$\left|x_t - \frac{p}{q}\right| < \frac{c}{2} q^{-1-i} \quad \text{and} \quad \left|f(x_t) - \frac{r}{q}\right| < \frac{c}{2} q^{-1-j}.$$

As usual, x_t is the point in $\Delta(L_t)$ at which $|F'_{L_t}(x)|$ attains its minimum. In Sect. 7.1, it was shown that this implies that all points x_t lie inside $\Delta(L_0)$ for some line L_0. It follows that all intervals $\Delta(L_t)$ intersect $\Delta(L_0)$.

- Assume that $\Delta(L_0)$ has already been removed by the construction described in Sect. 5. In other words, $\Delta(L_0) \in C(n_0, k_0)$ or $\Delta(L_0) \in C^*(n_0, k_0)$ with $(n_0, k_0) < (n, k)$. Recall that by $(n_0, k_0) < (n, k)$ we mean either $n_0 < n$ or $n_0 = n$ and $k_0 < k$. Then by the definition of the classes $C(n, k)$ and $C^*(n, k)$ each interval $\Delta(L_t) \subset \Delta(L_0)$ can be ignored. Hence, the intervals $\Delta(L_t)$ can in total remove at most two intervals of length

$$\frac{R}{2^k} \cdot \frac{Kc^{\frac{1}{2}}}{R^n}$$

on either side of $\Delta(L_0)$.
- Otherwise, by (25) the length of $\Delta(L_0)$ is bounded above by

$$\frac{R}{2^k} \cdot \frac{2Kc^{\frac{1}{2}}}{R^n}.$$

This implies that all the intervals $\Delta(L_t)$ together do not remove more than a single interval $\Delta^+(L_0)$ centered at the same point as $\Delta(L_0)$ but of twice the length. Hence, the length of the removed interval is bounded above by

$$\frac{R}{2^k} \cdot \frac{4Kc^{\frac{1}{2}}}{R^n}. \tag{86}$$

The upshot is that in either case, the total length of the intervals removed by $\Delta(L_t)$ is bounded above by (86).

8.2 Number of intervals $\Delta(L_t)$ intersecting J

We investigate the case when at least one of the bounds in (85) for ω_x or ω_y is not valid. This implies the following for the quantities v_x and v_y:

$$v_x \geqslant \frac{cq^{-i}}{2c'_1 c_x(C_0, i, j)} \quad \text{or} \quad v_y \geqslant \frac{cq^{-j}}{2c'_1 c_y(C_0, i, j)}. \tag{87}$$

The corresponding inequalities for σ_x σ_y are as follows:

$$\sigma_x \geqslant \frac{cq^{-i}}{2(c'_1)^2 c_x(C_0, i, j)} \quad \text{or} \quad \sigma_y \geqslant \frac{cq^{-j}}{2(c'_1)^2 c_y(C_0, i, j)}. \tag{88}$$

We now estimate the number of intervals $\Delta(L_t)$ from the same class which intersect J.

A consequence of Sect. 7.2 is that when considering intervals $\Delta(L_t(A_t, B_t, C_t)$ from the same class which intersect J, all except possibly at most two of the corresponding coordinates (A_t, B_t) lie in the set $F_x \cap F_y \cap \mathbf{L}$ or $F'_x \cap F'_y \cap \mathbf{L}$, or $F^*_x \cap F^*_y \cap \mathbf{L}$—depending on the class of intervals under consideration. Note that for any two associated lines $L_1$and L_2, the coordinates (A_1, B_1), (A_2, B_2) and $(0, 0)$ are not co-linear. To see this, suppose that the three points did lie on a line. Then $A_1/B_1 = A_2/B_2$ and so L_1 and L_2 are parallel. However, this is impossible since the lines L_1 and L_2 intersect at the rational point $P = (p/q, r/q)$.

Now let M be the number of intervals $\Delta(L_t)$ from the same class intersecting J and let F denote the convex 'box' which covers $F_x \cap F_y$ or $F'_x \cap F'_y$ or $F^*_x \cap F^*_y$—depending on the class of intervals under consideration. In view of the discussion above, it then follows that the lattice points of interest in $F \cap \mathbf{L}$ together with the lattice point $(0, 0)$ form the vertices of $(M - 1)$ disjoint triangles lying within F. Since the area of the fundamental domain of $\mathbf{L}$ is equal to q, the area of each of these disjoint triangles is at least $q/2$ and therefore we have that

$$\frac{q}{2}(M-1) \leqslant \textbf{area}(F). \tag{89}$$

We proceed to estimate M for each class separately.

- **Classes** $C(n, k, l, m), l \geqslant 1$ **and** $C_1(n, k) \cap C(n, k, 0, m)$ **and** J **satisfies** (45). By using either (79) for class $C(n, k, l, m), l \geqslant 1$ or (82) for class $C_1(n, k) \cap C(n, k, 0, m)$, it follows that

$$\textbf{area}(F) \ll \eta^2 2^{-m} R^{-\lambda l} \overset{(87)}{\ll} \max\left\{\left(\frac{2^k c'_1}{Rc^{\frac{1}{2}}}\right)^2, \left(\frac{2^k c'_1}{Rc^{\frac{1}{2}}}\right)^{2i/j}\right\} \cdot q^{2i} 2^{-m} R^{-\lambda l}.$$

This combined with (89) gives the following estimate

$$M \ll \max\{D^2, D^{2i/j}\} \cdot 2^{-m} R^{-\lambda l} \quad \text{where} \quad D := \frac{2^k c'_1}{Rc^{\frac{1}{2}}}. \tag{90}$$

- **Class** $C(n, k, l, m),\ l \geqslant 1$ **and** J **does not satisfy** (45). By (80), it follows that

$$\begin{aligned}\textbf{area}(F) &\ll (\eta')^2 2^{-m} R^{-\lambda l}\\ &\overset{(88)}{\ll} \max\left\{\left(\frac{2^{k+m}(c'_1)^2}{Rc^{\frac{1}{2}}}\right)^2, \left(\frac{2^{k+m}(c'_1)^2}{Rc^{\frac{1}{2}}}\right)^{2i/j}\right\} q^{2i} \cdot 2^{-m} R^{-\lambda l}.\end{aligned}$$

This combined with (89) gives the following estimate

$$M \ll \max\{(D')^2, (D')^{2i/j}\}\, 2^m R^{-\lambda l} \quad \text{where} \quad D' := \frac{2^k (c'_1)^2}{Rc^{\frac{1}{2}}}. \tag{91}$$

- **Class** $C^*(n,k)$. By (81), it follows that

$$\mathbf{area}(F) \ \ll\ (\eta^*)^2 R^{-n} \ll \max\left\{\left(\frac{2^k c_1'}{Rc^{\frac{1}{2}}}\right)^4, \left(\frac{2^k c_1'}{Rc^{\frac{1}{2}}}\right)^{4i/j}\right\} q^{2i} R^{-n}.$$

This combined with (89) gives the following estimate

$$M \ \ll\ \max\{(D^*)^4, (D^*)^{4i/j}\}\cdot R^{-n} \quad \text{where} \quad D^* := \frac{2^k c_1'}{Rc^{1/2}}. \tag{92}$$

- **Class** $C_2(n,k)$. By (83), it follows that

$$\mathbf{area}(F) \ \ll\ \eta^{1+\frac{j}{i}} \ \ll\ \max\{D^{1/i}, D^{1/j}\}\, q.$$

This combined with (89) gives the following estimate

$$M \ll \max\{D^{1/i}, D^{1/j}\}. \tag{93}$$

- **Class** $C_3(n,k,u,v)$. By (84), it follows that

$$\begin{aligned}\mathbf{area}(F) \ll \eta_3^{1/i} R^{-\lambda uj} \ &\ll\ \max\left\{(D\cdot R^{-\lambda uj} q^i)^{1/i}, (D\cdot q^j)^{1/j}\right\} R^{-\lambda uj}\\ &\ll \max\{D^{1/i} R^{-\frac{\lambda uj(1+i)}{i}}, D^{1/j} R^{-\lambda uj}\}\cdot q.\end{aligned}$$

This combined with (89) gives the following estimate

$$M \ \ll\ \max\{D^{1/i} R^{-\frac{\lambda uj(1+i)}{i}}, D^{1/j} R^{-\lambda uj}\}. \tag{94}$$

8.3 Number of subintervals removed by a single interval $\Delta(L)$

Let $c_1 := c^{\frac{1}{2}} R^{1+\omega}$ and $\omega := ij/4$ be as in (12). Consider the nested intervals $J_n \subset J_{n-1} \subset J_{n-2} \subset \cdots \subset J_0$ where $J_k \in \mathcal{J}_k$ with $0 \leqslant k \leqslant n$. Consider an interval $\Delta(L) \in C(n) \cap C^*(n)$ such that $\Delta(L) \cap J_n \neq \emptyset$. We now estimate the number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ such that $\Delta(L) \cap I_{n+1} \neq \emptyset$ with $I_{n+1} \subset J_n$. With reference to the construction of $\mathcal{J}_{n+1}$, the desired estimate is exactly the same as the number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ which are removed by the interval $\Delta(L)$. By definition, the length of any interval I_{n+1} is $c_1 R^{-n-1}$ and the length of $\Delta(L)$ is $2Kc^{\frac{1}{2}}(H(\Delta))^{-1}$. Thus, the number of removed intervals is bounded above by

$$2\frac{Kc^{\frac{1}{2}} R^{n+1}}{c_1 H(\Delta)} + 2 \ =\ \frac{2KR^{n-\omega}}{H(\Delta)} + 2. \tag{95}$$

Since $R^{n-1} \leqslant H(\Delta) < R^n$, the above quantity varies between 2 and $[2KR^{1-\omega}] + 2$.

8.4 Condition on l so that J_{n-l} satisfies (45)

Consider an interval J_{n-l}. Recall, by definition

$$|J_{n-l}| = c_1 R^{-n+l} = (c_1 R^l) \cdot R^{-n}.$$

So in this case the parameter c_1' associated with the generic interval J is equal to $c_1 R^l$ and by the choice of c_1 it clearly satisfies (44). We now obtain a condition on l so that (45) is valid when considering the intersection of intervals from $C(n, k, l, m)$ with J_{n-l}. With this in mind, on using the fact that $m \le \lambda \log_2 R$, it follows that

$$8C_0 \cdot c_1 R^{-n+l} \leqslant R^{-\lambda(l+1)} \leqslant 2^{-m} R^{-\lambda l}.$$

Thus, (45) is satisfied if

$$8C_0 \cdot c_1 R^{\lambda} \cdot R^{l(\lambda+1)} \leqslant R^n.$$

By the choice of c_1 and in view of (12), we have that for R sufficiently large

$$c_1 < \frac{1}{8C_0 R^{\lambda}}. \tag{96}$$

Therefore, (45) is satisfied for J_{n-l} if

$$l \leqslant \frac{n}{\lambda + 1}.$$

Notice that this is always the case when $l = 0$.

8.5 Proof of Proposition 1

Define the parameters $\epsilon := \frac{1}{2}(ij)\omega = \frac{1}{8}(ij)^2$ and

$$\tilde{c}(k) := \begin{cases} \frac{c_1 R^{\epsilon-\omega}}{2^k} & \text{if } \ 2^k < R^{1-\omega} \\ c_1 R^{\epsilon-1} & \text{if } \ 2^k \geqslant R^{1-\omega}. \end{cases} \tag{97}$$

Consider an interval $J_{n-l} \in \mathcal{J}_{n-l}$. Cover J_{n-l} by intervals $J_{l,1}, \ldots, J_{l,d}$ of length $\tilde{c}(k)R^{-n+l}$. Note that by the choice of c_1 and R sufficiently large the quantity $c_1' =: \tilde{c}(k)R^l$ satisfies (44). It is easily seen that the number d of such intervals is estimated as follows:

$$\begin{cases} d \leqslant 2^k R^{\omega-\epsilon} & \text{if } \ 2^k < R^{1-\omega} \\ d \leqslant R^{1-\epsilon} & \text{if } \ 2^k \geqslant R^{1-\omega}. \end{cases} \tag{98}$$

8.5.1 Part 1 of Proposition 1

A consequence of Sect. 6.3 is that if $c_1' = c_1 R^l$ satisfies either (60) or (61), depending on whether inequality (45) holds or not , then all lines L associated with intervals $\Delta(L) \in C(n,k,l,m)$ such that $\Delta(L) \cap J_{n-l} \neq \emptyset$ intersect at a single point. This statement remains valid if the interval J_{n-l} is replaced by any nested interval $J_{l,t}$. Inequality (60) is equivalent to

$$c_1 \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{3i}{i+1}} \quad \text{or} \quad c^{\frac{1}{2}} \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{3i}{i+1}} R^{-1-\omega}$$

and inequality (61) is equivalent to

$$c_1 \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{i}{i+1}} R^{-\lambda/3} \quad \text{or} \quad c^{\frac{1}{2}} \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda} \right)^{-\frac{i}{i+1}} R^{-1-\omega-\lambda/3}.$$

In view of (12), for R large enough both of these upper bound inequalities on c are satisfied. Thus the coordinates (A, B) associated with intervals $\Delta(L(A, B, C)) \in C(n,k,l,m)$ intersecting $J_{l,t}$ where $1 \leqslant t \leqslant d$, except possibly at most two, lie within the figure $F := F_x \cap F_y \cap \mathbf{L}$ or $F := F_x' \cap F_y' \cap \mathbf{L}$—depending on whether or not $J_{l,t}$ satisfies (45). Moreover, note that the figure F is the same for $1 \leqslant t \leqslant d$; i.e. it is independent of t.

If (85) is valid, then all intervals $\Delta(L)$ that intersect $J_{l,t}$ can remove at most two intervals of total length bounded above by

$$\frac{R}{2^k} \cdot \frac{4Kc^{\frac{1}{2}}}{R^n}.$$

Then, it follows that the number of removed intervals $I_{n+1} \subset J_{n-l}$ is bounded above by

$$\left(\frac{R}{2^k} \cdot \frac{4Kc^{\frac{1}{2}}}{R^n} \cdot \frac{1}{|I_{n+1}|} + 4 \right) \cdot d = 4 \left(\frac{K\ R^{1-\omega}}{2^k} + 1 \right) \cdot d \ll \left(\frac{R^{1-\omega}}{2^k} + 1 \right) \cdot d \overset{(98)}{\ll} R^{1-\epsilon}. \tag{99}$$

Otherwise, if (85) is false then the number M of intervals $\Delta(L) \in C(n,k,l,m)$ that intersect some $J_{l,t}$ $(1 \leqslant t \leqslant d)$ can be estimated by (90) if $J_{l,t}$ satisfies (45) and by (91) if (45) is not satisfied. This leads to the following estimates.

- M **is bounded by** (90) **and** $2^k < R^{1-\omega}$. Then

$$M \ll \left(\frac{2^k c^{\frac{1}{2}} R^{1+\epsilon} R^l}{2^k R c^{\frac{1}{2}}} \right)^2 2^{-m} R^{-\lambda l} \leqslant (R^{\epsilon})^2 \cdot R^{(2-\lambda)l}.$$

By (11), $\lambda > 2$ and therefore $M \ll R^{2\epsilon}$.

- **M is bounded by (90) and $2^k \geqslant R^{1-\omega}$.** Then

$$M \ll \left(\frac{2^k c^{\frac{1}{2}} R^{\omega+\epsilon} R^l}{R c^{\frac{1}{2}}} \right)^2 2^{-m} R^{-\lambda l} \leqslant (R^{\omega+\epsilon})^2.$$

because $R \geqslant 2^k$ and $\lambda > 2$.
- **M is bounded by (91) and $2^k < R^{1-\omega}$.** Then

$$M \ll \left(\frac{2^k c R^{2+2\epsilon} R^{2l}}{2^{2k} R c^{\frac{1}{2}}} \right)^2 2^m R^{-\lambda l} \leqslant c \cdot \frac{2^m R^2}{2^{2k}} R^{4\epsilon} \cdot R^{(4-\lambda)l}.$$

Since $\lambda > 4$ by (11) and $c < R^{-2-\lambda}$ by (12), it follows that $M \ll R^{4\epsilon}$.
- **M is bounded by (91) and $2^k \geqslant R^{1-\omega}$.** Then

$$M \ll \left(\frac{2^k c R^{2\epsilon+2\omega} R^{2l}}{R c^{\frac{1}{2}}} \right)^2 2^m R^{-\lambda l} \leqslant c \cdot 2^m \cdot R^{4(\epsilon+\omega)} \cdot R^{(4-\lambda)l}.$$

Again, by the choice of λ and c it follows that $M \ll R^{4(\epsilon+\omega)}$.

The upshot of the above upper bounds on M is that

$$M \ll \begin{cases} (R^{\epsilon})^4 & \text{if } 2^k < R^{1-\omega} \\ (R^{\omega+\epsilon})^4 & \text{if } 2^k \geqslant R^{1-\omega}. \end{cases} \tag{100}$$

In addition to these M intervals, we can have at most another $2d$ intervals—two for each $1 \leqslant t \leqslant d$ corresponding to the fact that there may be up to two exceptional intervals $\Delta(L(A, B, C))$ with associated coordinates (A, B) lying outside the figure F. By analogy with (99), these intervals remove at most $R^{1-\epsilon}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ with $I_{n+1} \subset J_{n-l}$.

On multiplying M by the number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ removed by each $\Delta(L)$ from $C(n, k, l, m)$, we obtain via (95) that the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ with $I_{n+1} \subset J_{n-l}$ removed by $\Delta(L) \in C(n, k, l, m)$ is bounded above by

$$\begin{aligned} 2\,R^{1-\epsilon} + \left(\frac{2K R^{n-\omega}}{H(\Delta)} + 2 \right) \cdot (R^{\epsilon})^4 &\overset{(23)}{\ll} R^{1-\epsilon} + \left(\frac{R^{1-\omega}}{2^k} + 1 \right) \cdot (R^{\epsilon})^4 \\ &\ll R^{1-\epsilon} + R^{1-\omega+4\epsilon} \quad \text{if} \quad 2^k < R^{1-\omega} \end{aligned}$$

and by

$$\begin{aligned} 2\,R^{1-\epsilon} + \left(\frac{2K R^{n-\omega}}{H(\Delta)} + 2 \right) \cdot (R^{\omega+\epsilon})^4 &\overset{(23)}{\ll} R^{1-\epsilon} + \left(\frac{R^{1-\omega}}{2^k} + 1 \right) \cdot (R^{\epsilon+\omega})^4 \\ &\ll R^{1-\epsilon} + R^{4(\omega+\epsilon)} \quad \text{if} \quad 2^k \geqslant R^{1-\omega}. \end{aligned}$$

Since $\omega = \frac{1}{4}ij$ and $\epsilon = \frac{1}{2}(ij)\omega$, in either case the number of removed intervals I_{n+1} is $\ll R^{1-\epsilon}$. Now recall that the parameters k and m can only take on a constant times $\log R$ values. Hence, it follows that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} : I_{n+1} \subset J_{n-l}, \exists \Delta(L) \in C(n,l), \Delta(L) \cap I_{n+1} \neq \emptyset\} \ll \log^2 R \cdot R^{1-\epsilon}.$$

For R large enough the r.h.s. is bounded above by $R^{1-\epsilon/2}$.

8.5.2 Part 2 of Proposition 1

Consider an interval $J_{n-n_0} \in \mathcal{J}_{n-n_0}$, where n_0 is defined by (16) and $n \geqslant 3n_0$. Cover J_{n-n_0} by intervals $J_{n_0,1}, \ldots, J_{n_0,d}$ of length $\tilde{c}(k)R^{-n+n_0}$ where $\tilde{c}(k)$ is defined by (97). Notice that d satisfies (98). Also, in view of (16) it follows that $c_1' := \tilde{c}(k)R$ satisfies (50). Therefore, Lemma 4 is applicable to the intervals $J_{n_0,t}$ with $1 \leqslant t \leqslant d$ and indeed is applicable to the whole interval J_{n-n_0}.

To ensure that all lines associated with $\Delta(L) \in C^*(n,k)$ such that $\Delta(L) \cap J_{n-n_0} \neq \emptyset$ intersect at one point, we need to guarantee that (62) is satisfied for $c_1' := c_1 R^{n_0}$. This is indeed the case if

$$c_1 R^{n_0} \leqslant \delta \cdot R^{\frac{j}{1+i}n} \left(\frac{2^k}{R}\right)^{-\frac{2i}{i+1}}. \tag{101}$$

Since $i \leqslant j$ we have that $\frac{j}{1+i} \geqslant \frac{1}{3}$ which together with the fact that $n \geqslant 3n_0$ implies that (101) is true if

$$c^{\frac{1}{2}} \leqslant \delta \cdot \left(\frac{2^k}{R}\right)^{-\frac{2i}{i+1}} R^{-1-\omega}$$

In view of (12), for R large enough this upper bound inequality on c is satisfied. Thus the coordinates (A, B) of all except possibly at most two lines $L(A, B, C)$ associated with intervals $\Delta(L(A, B, C)) \in C^*(n,k)$ with $\Delta(L) \cap J_{n_0,t} \neq \emptyset$ lie within the figure $F := F_x^* \cap F_y^* \cap \mathbf{L}$. By analogy with Part 1, if (85) is valid then the number of intervals $I_{n+1} \subset J_{n-n_0}$ removed by intervals $\Delta(L)$ is bounded above by $R^{1-\epsilon}$. Otherwise, the number M of intervals $\Delta(L) \in C^*(n,k)$ that intersect some $J_{n_0,t}$ $(1 \leqslant t \leqslant d)$ with associated coordinates $(A, B) \in F$ can be estimated by (92). Thus

$$M \ll \left(\frac{2^k \tilde{c}(k)}{Rc^{\frac{1}{2}}}\right)^4 R^{-n} \leqslant \begin{cases} (R^{\epsilon})^4 R^{-n} & \text{if } 2^k < R^{1-\omega} \\ (R^{\epsilon+\omega})^4 R^{-n} & \text{if } 2^k \geqslant R^{1-\omega}. \end{cases}$$

Since $n \geqslant 1$ and $\omega + \epsilon < 1/4$, it follows that $M \ll 1$. Now the same arguments as in Part 1 above can be utilized to verify that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} : I_{n+1} \subset J_{n-l}, \exists \Delta(L) \in C^*(n), \Delta(L) \cap I_{n+1} \neq \emptyset\} \ll \log R \cdot R^{1-\epsilon}.$$

For R large enough the r.h.s. is bounded above by $R^{1-\epsilon/2}$.

8.5.3 Part 3 of Proposition 1

Consider an interval $J_n \in \mathcal{J}_n$. Cover J_n by intervals $J_{0,1}, \ldots, J_{0,d}$ of length $\tilde{c}(k)R^{-n}$ where $\tilde{c}(k)$ is defined by (97). As before, d satisfies (98).

First we consider intervals $\Delta(L)$ from class $C_1(n,k) \cap C(n,k,0,m)$ such that $\Delta(L) \cap J_n \neq \emptyset$. In this case, the conditions (82) on the convex 'box' containing the figure $F_x \cap F_y \cap \mathbf{L}$ and the conditions (90) on M are the same as those when dealing with the class $C(n,k,l,m)$ in Part 1 above. Thus, analogous arguments imply that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ I_{n+1} \subset J_n, \exists \Delta(L) \in C_1(n), \Delta(L) \cap I_{n+1} \neq \emptyset\} \ll R^{1-\epsilon/2}.$$

Next we consider intervals $\Delta(L)$ from class $C_2(n,k)$ such that $\Delta(L) \cap J_n \neq \emptyset$. A consequence of Sect. 6.3 is that if $c_1' := c_1$ satisfies (63), then all lines L associated with intervals $\Delta(L) \in C_2(n,k)$ such that $\Delta(L) \cap J_n \neq \emptyset$ intersect at a single point. Inequality (63) is equivalent to

$$c_1 \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-1} \quad \text{or} \quad c^{\frac{1}{2}} \leqslant \delta \cdot \left(\frac{2^k c^{\frac{1}{2}}}{R} R^{\lambda}\right)^{-1} R^{-1-\omega}.$$

In view of (12), for R large enough this upper bound inequality on c is satisfied. Thus the coordinates (A,B) associated with intervals $\Delta(L(A,B,C)) \in C_2(n,k)$ intersecting $J_{0,t}$ where $1 \leqslant t \leqslant d$, except possibly at most two, lie within the figure $F := F_x \cap F_y \cap \mathbf{L}$. We now follow the arguments from Part 1. If (85) is valid, then we deduce that the total number of intervals $I_{n+1} \subset J_n$ removed by intervals $\Delta(L)$ is bounded above by (99). Otherwise, the number M of intervals $\Delta(L) \in C_2(n,k)$ that intersect some $J_{0,t}$ $(1 \leqslant t \leqslant d)$ with associated coordinates $(A,B) \in F$ can be estimated by (93). Thus, with $c_1' := \tilde{c}(k)$ given by (97) we obtain that

$$M \ll \left(\frac{2^k \tilde{c}(k)}{R c^{\frac{1}{2}}}\right)^{1/i} \leqslant \begin{cases} (R^{\epsilon})^{1/i} & \text{if } 2^k < R^{1-\omega} \\ (R^{\epsilon+\omega})^{1/i} & \text{if } 2^k \geqslant R^{1-\omega}. \end{cases}$$

It follows via (95) that the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ with $I_{n+1} \subset J_n$ removed by $\Delta(L) \in C_2(n,k)$ is bounded above by

$$\begin{aligned} 2\,R^{1-\epsilon} + \left(\frac{2R^{n-\omega}}{H(\Delta)} + 2\right) \cdot (R^{\epsilon})^{1/i} &\overset{(23)}{\ll} R^{1-\epsilon} + \left(\frac{R^{1-\omega}}{2^k} + 1\right) \cdot (R^{\epsilon})^{1/i} \\ &\ll R^{1-\epsilon} + R^{1-\omega+\epsilon/i} \quad \text{if} \quad 2^k < R^{1-\omega} \end{aligned}$$

and

$$\begin{aligned} 2\,R^{1-\epsilon} + \left(\frac{2R^{n-\omega}}{H(\Delta)} + 2\right) \cdot (R^{\omega+\epsilon})^{1/i} &\overset{(23)}{\ll} R^{1-\epsilon} + \left(\frac{R^{1-\omega}}{2^k} + 1\right) \cdot (R^{\epsilon+\omega})^{1/i} \\ &\ll R^{1-\epsilon} + R^{(\omega+\epsilon)/i} \quad \text{if} \quad 2^k \geqslant R^{1-\omega}. \end{aligned}$$

Since $\omega = \frac{1}{4}ij$ and $\epsilon = \frac{1}{2}(ij)\omega$, in either case the number of removed intervals I_{n+1} is $\ll R^{1-\epsilon}$. Hence, we obtain that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ I_{n+1} \subset J_n, \exists \Delta(L) \in C_2(n), \Delta(L) \cap I_{n+1} \neq \emptyset\} \ll \log R \cdot R^{1-\epsilon}.$$

For R large enough the r.h.s. is bounded above by $R^{1-\epsilon/2}$.

8.5.4 Part 4 of Proposition 1

The proof is pretty much the same as for Parts 1-3. Consider an interval $J_{n-u} \in \mathcal{J}_{n-u}$. Cover J_{n-u} by intervals $J_{u,1}, \ldots, J_{u,d}$ of length $\tilde{c}(k)R^{-n+u}$ where $\tilde{c}(k)$ is given by (97). As usual, d satisfies (98). Recall, that $C_3(n, k, u, v) \subset C(n, k, 0)$ and for all subclasses of $C(n, k, 0)$ when consider the intersection with a generic interval J of length $c_1' R^{-n}$ we require the constant c_1' to satisfy (45)—see Sect. 6.3.3. Thus, to begin with we check that the interval J_{n-u} satisfies (45). Now, with $c_1' := c_1 R^u$ and $l = 0$, together with the fact that $m \leqslant \lambda \log_2 R$, the desired inequality (45) would hold if

$$8C_0 c_1 R^{u-n} \leqslant R^{-\lambda}\,.$$

It is easily verified that this is indeed true by making use of the inequalities (36) and (96) concerning u and c_1 respectively.

A consequence of Sect. 6.3 is that if $c_1' := c_1 R^u$ satisfies (64), then all lines L associated with intervals $\Delta(L) \in C_3(n, k, u, v)$ such that $\Delta(L) \cap J_{n-u} \neq \emptyset$ intersect at a single point. Inequality (64) is equivalent to

$$c_1 \leqslant \delta \cdot \frac{R^{1-\lambda i}}{2^k c^{\frac{1}{2}}} \quad \text{or} \quad c^{\frac{1}{2}} \leqslant \delta \cdot \frac{R^{-\lambda i-\omega}}{2^k c^{\frac{1}{2}}}.$$

In view of (12), for R large enough this upper bound inequality on c is satisfied. Thus the coordinates (A, B) associated with intervals $\Delta(L(A, B, C)) \in C_3(n, k, u, v)$ intersecting $J_{u,t}$ where $1 \leqslant t \leqslant d$, except possibly at most two, lie within the figure $F := F_x \cap F_y \cap \mathbf{L}$.

We now follow the arguments from Part 1. If (85) is valid, then we deduce that the total number of intervals $I_{n+1} \subset J_n$ removed by intervals $\Delta(L)$ is bounded above by (99). Otherwise, the number M of intervals $\Delta(L) \in C_3(n, k, u, v)$ that intersect $J_{u,t}$ with associated coordinates $(A, B) \in F$ can be estimated by (94). Thus, with $c_1' := \tilde{c}(k)$ given by (97) we obtain that

$$M \ll \left(\frac{2^k c^{\frac{1}{2}} R^{1+\epsilon} R^u}{2^k R c^{\frac{1}{2}}}\right)^{1/i} R^{u(1-\min\{\frac{\lambda j(1+i)}{i}, \lambda j\})} \overset{(11)}{\leqslant} (R^{\epsilon})^{1/i} \quad \text{if} \quad 2^k < R^{1-\omega}$$

and

$$M \ll \left(\frac{2^k c^{\frac{1}{2}} R^{\omega+\epsilon} R^u}{R c^{\frac{1}{2}}}\right)^{1/i} R^{u(1-\min\{\frac{\lambda j(1+i)}{i},\lambda j\})} \leqslant (R^{\omega+\epsilon})^{1/i} \quad \text{if} \quad 2^k \geqslant R^{1-\omega}.$$

Note that these are exactly the same estimates for M obtained in Part 3 above. Then as before, we deduce that the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ with $I_{n+1} \subset J_n$ removed by $\Delta(L) \in C_3(n, k, u, v)$ is bounded above by $R^{1-\epsilon}$. Hence, it follows that

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} : I_{n+1} \subset J_{n-u}, \exists \Delta(L) \in \widetilde{C}_3(n, u), \Delta(L) \cap I_{n+1} \neq \emptyset\} \ll \log^2 R \cdot R^{1-\epsilon}.$$

For R large enough the r.h.s. is bounded above by $R^{1-\epsilon/2}$.

9 Proof of Theorem 2

The basic strategy of the proof of Theorem 1 also works for Theorem 2. The key is to establish the analogue of Theorem 3. In this section we outline the main differences and modifications. Let (i, j) be a pair of real numbers satisfying (5). Given a line $\mathrm{L}_{\alpha,\beta} : x \to \alpha x + \beta$ we have that

$$F_L(x) := (A - B\alpha)x + C - B\beta \qquad \text{and} \quad V_L := |F_L'(x)| = |A - B\alpha| \,,$$

Thus, with in the context of Theorem 2 the quantity V_L is independent of x. Furthermore, note that the Diophantine condition on α implies that there exists an $\epsilon > 0$ such that

$$V_L \gg B^{-\frac{1}{i}+\epsilon}. \tag{102}$$

Also, $|F_L''(x)| \equiv 0$ and the analogue of Lemma 1 is the following statement.

Lemma 6 *There exists an absolute constant* $K \geq 1$ *dependent only on* i, j, α *and* β *such that*

$$|\Delta(L)| \leqslant K \frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\} \cdot V_L}.$$

A consequence of the lemma is that there are only Type 1 intervals to consider. Next note that for c small enough $H(\Delta) > 1$ for all intervals $\Delta(L)$. Indeed

$$H(\Delta) = c^{-1/2} V_L \max\{|A|^{1/i}, |B|^{1/j}\}.$$

So if $|A| < \frac{|\alpha|}{2}|B|$, then $V_L \asymp B$ and $H(\Delta) > 1$ follows immediately. Otherwise,

$$H(\Delta) \overset{(102)}{\gg} c^{-1/2} \left(\frac{|A|}{|B|}\right)^{1/i}$$

which is also greater than 1 for c sufficiently small.

As in the case of non-degenerate curves, we partition the intervals $\Delta(L) \in \mathcal{R}$ into classes $C(n,k,l)$ according to (23) and (24). Unfortunately, we can not guarantee that $\lambda l \leqslant n$ as in the case of curves. However, we still have the bound $l \leqslant n$. To see that this is the case, suppose that $l > n$. Then (25) is satisfied and

$$|V_L| > R^{-\lambda n}(|\alpha|+1)\max\{|A|,|B|\}. \tag{103}$$

By (23), we have that

$$R^{n-1} \leqslant H(\Delta) \leqslant R^n.$$

On combining the previous two displayed inequalities we get that

$$|A-\alpha B| \ll |B|^{\frac{i-\lambda}{i(1+\lambda)}} \cdot (Rc^{1/2})^{\frac{\lambda.}{1+\lambda}}.$$

Then by choosing λ and c such that

$$\lambda > \frac{i+1}{\epsilon\, i} - 1 \quad \text{and} \quad (Rc^{1/2})^{\frac{\lambda.}{1+\lambda}} < \inf_{q\in\mathbb{N}}\{q^{\frac{1}{i}-\epsilon}||q\alpha||\} := \tau \tag{104}$$

implies that

$$|A-\alpha B| < \tau |B|^{-\frac{1}{i}+\epsilon}.$$

This contradicts the Diophantine condition imposed on α and so we must have that $l \leqslant n$.

With the above differences/changes in mind, it is possible to establish the analogue of Proposition 1 for lines $\mathrm{L}_{\alpha,\beta}$ by following the same arguments and ideas as in the case of $C^{(2)}$ non-degenerate planar curves. The key differences in the analogous statement for lines is that in Part 1 we have $l \leqslant n$ instead of $\lambda l \leqslant n$ and that Part 2 disappears all together since there are no Type 2 intervals to consider. Recall, that even when establishing Proposition 1 for curves, Part 1, 3 and 4 only use the fact that the curve is two times differentiable—see Sect. 5 Remark 12. The analogue of Proposition 1 enables us to construct the appropriate Cantor set $\mathcal{K}(J_0, \mathbf{R}, \mathbf{r})$ which in turn leads to the desired analogue of Theorem 3.

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