Math. Ann. (2014) 359:211–238
DOI 10.1007/s00208-013-1000-8

Mathematische Annalen

On the spectrum of the Laplacian

Nelia Charalambous · Zhiqin Lu

Received: 27 May 2013 / Revised: 23 October 2013 / Published online: 15 December 2013

Abstract In this article we prove a generalization of Weyl's criterion for the essential spectrum of a self-adjoint operator on a Hilbert space. We then apply this criterion to the Laplacian on functions over open manifolds and get new results for its essential spectrum.

Mathematics Subject Classification (2000) Primary 58J50; Secondary 58E30

1 Introduction

Let M be a complete noncompact Riemannian manifold of dimension n and denote by Δ the Laplacian acting on $\mathcal{C}_0^\infty(M)$. It is well known that the self-adjoint extension of Δ on $L^2(M)$ exists and is a unique nonpositive definite and densely defined linear operator. We will also use Δ to denote this extension for the remaining paper.

The spectrum of $-\Delta$, $\sigma(-\Delta)$, consists of all points $\lambda \in \mathbb{C}$ for which $\Delta + \lambda I$ fails to be invertible. Since $-\Delta$ is nonnegative definite, its L^2-spectrum is contained in $[0, \infty)$. The essential spectrum of $-\Delta$, $\sigma_{\text{ess}}(-\Delta)$, consists of the cluster points in the spectrum and of isolated eigenvalues of infinite multiplicity. The following result is due to Donnelly [5]: if there exists an infinite dimensional subspace G in the domain

N. Charalambous was partially supported by CONACYT of Mexico and is thankful to the Asociación Mexicana de Cultura A.C. and Z. Lu is partially supported by the DMS-12-06748.

N. Charalambous
Department of Mathematics and Statistics, University of Cyprus, 1678 Nicosia, Cyprus
e-mail: nelia@ucy.ac.cy

Z. Lu (✉)
Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA
e-mail: zlu@uci.edu

of Δ such that

$$\|\Delta u + \lambda u\|_{L^2} \leq \sigma \|u\|_{L^2} \tag{1}$$

for all $u \in G$, then

$$\sigma_{\text{ess}}(-\Delta) \cap (\lambda - \sigma, \lambda + \sigma) \neq \emptyset.$$

The functions u are referred to as the approximate eigenfunctions corresponding to the eigenvalue λ. The above criterion is simple to apply and has directed the study of the essential spectrum of the Laplacian for the last three decades. A related result of the above is as follows: let u be a nonzero smooth function with compact support. If (1) is satisfied, then

$$\sigma(-\Delta) \cap (\lambda - \sigma, \lambda + \sigma) \neq \emptyset.$$

We remark that for the above criteria to be valid, we do not have to assume the completeness of the manifold M. They can be applied to both compact and noncompact manifolds with boundary (with either Dirichlet or Neumann boundary conditions) as well as complete compact and noncompact manifolds. If M is compact, the criterion gives the eigenvalue estimates.

In most problems the ideal space to work on is the L^2 function space when compared to the L^q spaces. However, this is not the case when considering the spectrum of the Laplacian. On a Riemannian manifold, most of the approximate eigenfunctions we can write out explicitly must be related to the distance function. It is well known however, that the Laplacian of the distance function is locally bounded in L^1, but not in L^2.

We can see this in the following simple example. Take $M = S^1 \times (-\infty, \infty)$, letting (θ, x) be the coordinates. Then the radial function with respect to the point (0,0) is given by

$$r = \sqrt{x^2 + (\min(\theta, 2\pi - \theta))^2}.$$

A straightforward computation gives

$$\Delta r = -\frac{2\pi}{\sqrt{x^2 + \pi^2}} \delta_{\{\theta = \pi\}} + \text{ a smooth function,}$$

where $\delta_{\{\theta=\pi\}}$ is the Delta function along the submanifold $\{\theta = \pi\}$. Therefore Δr is not locally L^2.

The failure of the L^2 integrability of the Laplacian of the distance function was one of the main difficulties in applying the classical criterion above. In fact, it was not possible to prove that the L^2 essential spectrum of the Laplacian on a manifold with nonnegative Ricci curvature is $[0, \infty)$ by directly computing the L^2 spectrum. Additional assumptions on the curvature and geometry of the manifold were necessary (see for example [2,6,9,10,13,20]).

Donnelly [6] proved that the essential spectrum of the Laplacian is $[0, \infty)$ for manifolds of nonnegative Ricci curvature and maximal volume growth. Wang [19], by using the seminal theorem of Sturm [18], removed the maximal volume growth condition. Wang's result confirms the conjecture that the spectrum of manifolds with nonnegative Ricci curvature is $[0, \infty)$. In [16], Lu-Zhou gave a technical generalization of Wang's result which includes the case of manifolds of finite volume.

In this article, we introduce a new method for computing the spectrum of a self-adjoint operator on a Hilbert space (see Theorem 2.3) which has the following application in the case of the Laplacian:

Theorem 1.1 *Let M be a Riemannian manifold and let Δ be the Laplacian. Assume that for $\lambda \in \mathbb{R}^+$, there exists a nonzero function u in the domain of Δ such that*

$$\|u\|_{L^\infty} \cdot \|\Delta u + \lambda u\|_{L^1} \le \delta \|u\|_{L^2}^2 \tag{2}$$

for some positive number $\delta > 0$. Then

$$\sigma(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

where

$$\varepsilon = \min(1, (\lambda + 2)\delta^{1/3}).$$

Moreover,

$$\sigma_{\text{ess}}(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

if for any compact subset K of M, there exists a nonzero function u in the domain of Δ satisfying (2) whose support is outside K.

We expect the above result to have many applications in spectrum theory (for example, on manifolds with corners, where the test functions are usually not smooth). In this paper, we concentrate on applying the criterion to the computation of the essential spectrum of complete noncompact manifolds.

Theorem 1.1 proves to be a powerful tool in expanding the set of manifolds for which the L^2 essential spectrum is the nonnegative real line. In the case of shrinking Ricci solitons, we are able to prove the following result.

Theorem 1.2 *The L^2 essential spectrum of a complete shrinking Ricci soliton is* $[0, \infty)$.

Note that no curvature assumption is needed here.

For a large class of manifolds (for example, the shrinking Ricci solitons), we are able to control the volume growth about a fixed point, but it is difficult to prove the uniformness of their volume growth. Without the uniform volume growth property, the theorem of Sturm does not apply and the results of Wang [19] and Lu-Zhou [16] cannot be used. Therefore, the following result may be practically useful:

Theorem 1.3 *Let M be a complete noncompact Riemannian manifold. Suppose that, with respect to a fixed point p, the radial Ricci curvature is asymptotically nonnegative (see Lemma 5.2). If the volume of the manifold is finite we additionally assume that its volume does not decay exponentially at p. Then the L^2 spectrum of the Laplace operator on functions is* $[0, \infty)$.

We shall also use Theorem 1.1 to modify a result of Elworthy-Wang [8] on manifolds that posses an exhaustion function (Theorem 8.2). We replace the L^2 norm assumption by an L^1 norm assumption.

In the last section, we show that it is possible to work with continuous test functions in Theorem 1.1. By using them instead we avoid the repetitive choosing of cut-off functions.

The essential spectrum of the Laplacian on noncompact Riemannian manifolds is interesting and important as it reveals a lot of information about the geometry of the manifold. Although there are lot of interesting open problems in this direction, the authors believe that answering the following conjecture is the most important one.

Conjecture 1.4 *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded below. Then the L^2 essential spectrum of the Laplacian on functions is a connected subset of the real line. In other words, the essential spectrum set is of the form $[a, \infty)$, where a is a nonnegative real number.*

As is well-known, the essential spectrum of a Schrödinger operator could be very complicated (cf. [17, Chapter XIII]) and it certainly need not be a connected set. For the case of the Laplacian on a complete manifold however, in all known examples the L^2 essential spectrum is a connected set. In this paper, we answer the above conjecture in some special cases. We believe that the analysis of the wave kernel is needed to answer the conjecture in full.

2 The Weyl criterion for quadratic forms

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The norm and inner product in $\mathcal{H}$ are respectively denoted by $\|\cdot\|$ and $(\cdot, \cdot)$. Let $\sigma(H)$, $\sigma_{\text{ess}}(H)$ be the spectrum and the essential spectrum of H, respectively. Let $\mathfrak{D}(H)$ be the domain of H. The Classical Weyl criterion states that

Theorem 2.1 (Classical Weyl's criterion) *A nonnegative real number λ belongs to $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
(2) $(H - \lambda)\psi_n \to 0,$ *as $n \to \infty$ in $\mathcal{H}$.*

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

(3) $\psi_n \to 0$ *weakly as $n \to \infty$ in $\mathcal{H}$.*

Remark 2.2 The above theorem is still true if the convergence in (2) is replaced by weak convergence, the statement of which can be found (without proof) in [4] and later in [15]. This version of the Weyl criterion was applied for the first time to the

Laplacian on curved Euclidean domains in [15]. The authors are grateful to David Krejčiřík for informing them of the results.

We have the following functional analytic result, which generalizes the weak Weyl criterion. To the authors' knowledge, this result seems to be new.

Theorem 2.3 *Let f be a bounded positive continuous function over* $[0, \infty)$. *A non-negative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
(2) $(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \to 0,$ *as $n \to \infty$ and*
(3) $(\psi_n, (H-\lambda)\psi_n) \to 0,$ *as $n \to \infty$.*

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

(4) $\psi_n \to 0,$ *weakly as $n \to \infty$ in $\mathcal{H}$.*

Proof Since H is a densely defined self-adjoint operator, the spectral measure E exists and we can write

$$H = \int_0^\infty \lambda \, dE. \tag{3}$$

Assume that $\lambda \in \sigma(H)$. Then by Weyl's criterion, there exists a sequence $\{\psi_n\}$ such that

$$\|(H-\lambda)\psi_n\| \to 0, \quad \|\psi_n\| = 1$$

as $n \to \infty$.

We write

$$\psi_n = \int_0^\infty dE(t)\psi_n$$

as its spectral decomposition. Then

$$(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) = \int_0^\infty f(t)(t-\lambda)^2 d\|E(t)\psi_n\|^2.$$

Since f is a bounded positive function, we have

$$(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \leq C \int_0^\infty (t-\lambda)^2 d\|E(t)\psi_n\|^2 = C\|(H-\lambda)\psi_n\|^2.$$

Moreover,

$$(\psi_n, (H-\lambda)\psi_n) \le C\,\|\psi_n\| \cdot \|(H-\lambda)\psi_n\|.$$

Thus the necessary part of the theorem is proved.

Now assume that $\lambda > 0$ and $\lambda \notin \sigma(H)$. Then there is a $\lambda > \varepsilon > 0$ such that $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) = 0$. We write

$$\psi_n = \psi_n^1 + \psi_n^2, \tag{4}$$

where

$$\psi_n^1 = \int_0^{\lambda-\varepsilon} dE(t)\psi_n,$$

and $\psi_n^2 = \psi_n - \psi_n^1$.

Then

$$\begin{aligned}
&(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n)\\
&\quad = \left(f(H)(H-\lambda)\psi_n^1, (H-\lambda)\psi_n^1\right) + \left(f(H)(H-\lambda)\psi_n^2, (H-\lambda)\psi_n^2\right)\\
&\quad \ge c_1\|\psi_n^1\|^2 + (f(H)(H-\lambda)\psi_n^2, (H-\lambda)\psi_n^2) \ge c_1\|\psi_n^1\|^2,
\end{aligned}$$

where the positive number c_1 is the infimum of the function $f(t)(t-\lambda)^2$ on $[0, \lambda-\varepsilon]$. Therefore

$$\|\psi_n^1\| \to 0$$

by *(2)*. On the other hand, we similarly get

$$(\psi_n, (H-\lambda)\psi_n) \ge \varepsilon\|\psi_n^2\|^2 - \lambda\|\psi_n^1\|^2.$$

If the criteria *(2), (3)* are satisfied, then, by the two estimates above, we conclude that both ψ_n^1, ψ_n^2 go to zero. This contradicts $\|\psi_n\| = 1$, and the theorem is proved.

Note that for $\lambda = 0$, ψ_n^1 is automatically zero, and the second half of the proof would give the contradiction. □

We apply Theorem 2.3 to the Laplacian on functions. In this setting two particular cases of the function f will be of interest.

Corollary 2.1 *A nonnegative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
(2) $((H+1)^{-1}\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$ *and*

(3) $(\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$.

Moreover, λ *belongs to* $\sigma_{\text{ess}}(H)$ *of* H *if, and only if, in addition to the above properties*

(4) $\psi_n \to 0,$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

Proof We take $f(x) = (x+1)^{-1}$. The corollary follows from the identity

$$(H+1)^{-1}(H-\lambda) = 1 - (\lambda+1)(H+1)^{-1}.$$

□

In a similar spirit, taking $f(x) = (x+\alpha)^{-(N+1)}$ for a natural number N and a positive number $\alpha > 1$, we also obtain the following generalization

Corollary 2.2 *A nonnegative real number* λ *belongs to the spectrum* $\sigma(H)$ *if, and only if, there exists a sequence* $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ *such that*

(1) $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
(2) $((H+\alpha)^{-i}\psi_n, (H-\lambda)\psi_n) \to 0$ *as* $n \to \infty$ *for two consecutive natural numbers i=N, N+1, and*
(3) $(\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$.

Moreover, λ *belongs to* $\sigma_{\text{ess}}(H)$ *of* H *if, and only if, in addition to the above properties*

(4) $\psi_n \to 0,$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

Remark 2.4 Using the Cauchy inequality, the above two corollaries reduce to Donnelly's criterion (1) when we consider the case $H = -\Delta$.

3 A spectrum estimate result

In this section we will prove a special version of Theorem 2.3 for the Laplacian on functions. We begin with the fact that its resolvent is always bounded on L^∞.

Lemma 3.1 *We have*

$$(-\Delta+1)^{-1}$$

is bounded from $L^\infty(M)$ *to itself and the operator norm is no more than 1.*

The lemma follows from the proof of Lemma 3.1 in [1]. The resolvent is bounded on L^∞ because the heat kernel is bounded on L^∞. This is a property that Davies proves for any nonnegative self-adjoint operator that satisfies Kato's inequality like the Laplacian [3, Theorems 1.3.2, 1.3.3]. It is due to the well-known fact that the Laplacian on functions is a self-adjoint operator that satisfies Kato's inequality. Together with Corollary 2.1 this lemma allows us to obtain an even simpler criterion for the essential spectrum of the Laplacian on functions:

Proof of Theorem 1.1 Assume that

$$\sigma(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \emptyset.$$

By the above lemma, we have

$$|(u, (-\Delta - \lambda)u)| \leq \|u\|_{L^\infty} \cdot \|(-\Delta - \lambda)u\|_{L^1}$$
$$|((-\Delta + 1)^{-1}(-\Delta - \lambda)u, (-\Delta - \lambda)u)| \leq (\lambda + 2)\|u\|_{L^\infty} \cdot \|(-\Delta - \lambda)u\|_{L^1}$$

We write

$$u = u_1 + u_2$$

according to the spectral decomposition of the operator $-\Delta$ (cf. 4). Then we have

$$\varepsilon\|u_2\|_{L^2}^2 - \lambda\|u_1\|_{L^2}^2 \leq \delta\|u\|_{L^2}^2;$$
$$\|u_1\|_{L^2}^2 \leq \frac{(\lambda + 2)(\lambda + 1)}{\varepsilon^2}\delta\|u\|_{L^2}^2.$$

Thus we have

$$\varepsilon\|u_2\|_{L^2}^2 + \varepsilon\|u_1\|_{L^2}^2 \leq \left(\frac{(\lambda + 2)(\lambda + 1)(\lambda + \varepsilon)}{\varepsilon^2} + 1\right)\delta\|u\|_{L^2}^2.$$

Since $\|u_2\|_{L^2}^2 + \|u_1\|_{L^2}^2 = \|u\|_{L^2}^2$, we get

$$\varepsilon \leq \left(\frac{(\lambda + 2)(\lambda + 1)(\lambda + \varepsilon)}{\varepsilon^2} + 1\right)\delta$$

which is a contradiction. The essential spectrum result of the theorem follows from the classical Weyl criterion (Theorem 2.1, (3)). □

4 An approximation theorem

Let M be a complete noncompact Riemannian manifold. Let $p \in M$ be a fixed point. Define

$$r(x) = d(x, p)$$

to be the radial function on M. It is well known that

(1) $r(x)$ is continuous;
(2) $|\nabla r(x)| = 1$ almost everywhere and $r(x)$ is a Lipschitz function;
(3) Δr exists on $M \backslash \{p\}$ in the sense of distribution.

In general, it is not possible to find smooth approximations of a Lipschitz function in the $\mathcal{C}^1$ norm. The following Proposition, which is a more precise version of [16, Proposition 1], implies that this can be done up to a function with small L^1 norm. Such kind of result is essential in Riemannian geometry and should be well-known, but given that we were not able to find a reference, we include a proof.

Proposition 4.1 *For any positive continuous decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\lim_{r\to\infty} \eta(r) = 0,$$

there exist C^∞ functions $\tilde{r}(x)$ and $b(x)$ on M such that

(a) $\|b\|_{L^1(M\setminus B_p(R))} \le \eta(R-1)$;
(b) $\|\nabla\tilde{r} - \nabla r\|_{L^1(M\setminus B_p(R))} \le \eta(R)$

and for any $x \in M$ with $r(x) > 2$

(c) $|\tilde{r}(x) - r(x)| \le \eta(r(x))$ *and* $|\nabla\tilde{r}(x)| \le 2$;
(d) $\Delta\tilde{r}(x) \le \max_{y\in B_x(1)} \Delta r(y) + \eta(r(x)) + |b(x)|$ *in the sense of distribution.*

Proof Without loss of generality, we assume that $\eta(r) < 1$. Let $\{U_i\}$ be a locally finite cover of M and let $\{\psi_i\}$ be the partition of unity subordinate to the cover. Let $\mathbf{x_i} = (x_i^1, \ldots, x_i^n)$ be the local coordinates of U_i. Define $r_i = r|_{U_i}$.

Let $\xi(\mathbf{x})$ be a non-negative smooth function on $\mathbb{R}^n$ whose support is within the unit ball. Assume that

$$\int_{\mathbb{R}^n} \xi = 1.$$

Without loss of generality, we assume that each U_i is an open subset of the unit ball of $\mathbb{R}^n$ with coordinates $\mathbf{x_i}$. Then for any $\varepsilon > 0$,

$$r_{i,\varepsilon} = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \xi\left(\frac{\mathbf{x_i}-\mathbf{y_i}}{\varepsilon}\right) r_i(\mathbf{y_i})d\mathbf{y_i}$$

is a smooth function on U_i and hence on M. Let $\{\sigma_i\}$ be a sequence of positive numbers such that

$$\sum_i \sigma_i(|\Delta\psi_i(x)| + 4|\nabla\psi_i(x)| + \psi_i(x)) \le \eta(r(x)). \tag{5}$$

By [11, Lemma 7.1, 7.2], for each i, we can choose $\varepsilon_i < 1$ small enough so that

$$\begin{aligned} &|r_{i,\varepsilon_i}(x) - r_i(x)| \le \sigma_i;\\ &\|\nabla r_{i,\varepsilon_i} - \nabla r_i\|_{L^1(U_i)} \le \sigma_i. \end{aligned} \tag{6}$$

We also have

$$\Delta r_{i,\varepsilon_i}(x) \leq \max_{y\in B_x(1)} \Delta r_i(y). \tag{7}$$

Define

$$\tilde{r} = \sum_i \psi_i r_{i,\varepsilon_i}, \quad b = 2\sum_i \nabla\psi_i \cdot \nabla r_{i,\varepsilon_i}.$$

Since $\sum_i(\nabla\psi_i \cdot \nabla r_i) = (\sum_i \nabla\psi_i)\cdot\nabla r = 0$ almost everywhere on M, we have

$$b = 2\sum_i \nabla\psi_i \cdot (\nabla r_{i,\varepsilon_i} - \nabla r_i)$$

almost everywhere. Thus *(a)* follows. Similarly, observing that

$$\tilde{r} - r = \sum_i \psi_i(r_{i,\varepsilon_i} - r_i), \quad \text{and } |\nabla r_{i,\varepsilon_i}| < 2,$$

we obtain *(b), (c)*.

To prove *(d)*, we compute

$$\Delta\tilde{r} = \sum_i [(\Delta\psi_i)\, r_{i,\varepsilon_i} + 2\nabla\psi_i \nabla r_{i,\varepsilon_i} + \psi_i \Delta r_{i,\varepsilon_i}],$$

and since

$$\sum_i (\Delta\psi_i) r_i = \sum_i (\Delta\psi_i) r = 0,$$

we have

$$\Delta\tilde{r} = \sum_i [\Delta\psi_i(r_{i,\varepsilon_i} - r_i) + b + \psi_i \Delta r_{i,\varepsilon_i}].$$

By (7), we obtain *(d)* and the Proposition is proved. □

5 Manifolds with Δr asymptotically nonpositive

As we have mentioned in the previous section, the Laplacian of the radial function $r(x) = d(x, p)$ exists in the sense of distribution (except at p). That is, for any nonnegative smooth function f with compact support in $M\backslash\{p\}$, the integral

$$\int_M f\,\Delta r$$

is defined. The following simple observation is due to Wang [19] and is crucial in our estimates.

Lemma 5.1 *The function Δr is locally integrable away from p.*

Proof Let W be any compact set of the form $B_p(R) - B_p(r)$ for $R > r > 0$. Then by the Laplacian comparison theorem, there is a constant C, depending only on the dimension, r, R, and the lower bound of the Ricci curvature on $B_p(R)$, such that

$$\Delta r \leq C$$

on W in the sense of distribution. Thus we have

$$|\Delta r| = |C - \Delta r - C| \leq 2C - \Delta r$$

and therefore

$$\int_W |\Delta r| \leq 2C \operatorname{vol}(W) - \int_W \Delta r.$$

Using Stokes' Theorem, we obtain

$$\int_W |\Delta r| \leq 2C \operatorname{vol}(W) - \int_{\partial W} \frac{\partial r}{\partial n} \leq 2C \operatorname{vol}(W) + \operatorname{vol}(\partial W),$$

and the lemma is proved. □

In this section, we study manifolds with the following property

$$\overline{\lim_{r\to\infty}} \Delta r \leq 0 \tag{8}$$

in the sense of distribution, where $r(x)$ is the radial distance of x to a fixed point p. We shall give a precise estimate of the L^1 norm of Δr in terms of the volume growth of the manifold. But before we do that, we first provide an important example where the above technical condition holds.

We note that for a fixed point $p \in M$ the cut locus $\operatorname{Cut}(p)$ is a set of measure zero in M. The manifold can be written as the disjoint union $M = \Omega \cup \operatorname{Cut}(p)$, where Ω is star-shaped with respect to p. That is, if $x \in \Omega$, then the geodesic line segment $\overline{px} \subset \Omega$. $\partial r = \partial/\partial r$ is well defined on Ω. We have the following result:

Lemma 5.2 *Let $r(x)$ be the radial function with respect to p. Suppose that there exists a continuous function $\delta(r)$ on $\mathbb{R}^+$ such that*

(i) $\lim_{r\to\infty} \delta(r) = 0$;
(ii) $\delta(r) > 0$ *and*
(iii) $Ric(\partial r, \partial r) \geq -(n-1)\delta(r)$ *on* Ω.

Then (8) is valid in the sense of distribution.

Proof On Ω, we have the following Bochner formula

$$0 = \frac{1}{2}\Delta|\nabla r|^2 = |\nabla^2 r|^2 + \nabla r \cdot \nabla(\Delta r) + \mathrm{Ric}(\partial r, \partial r).$$

Since $\nabla^2 r(\partial r, \partial r) = 0$, using the Cauchy inequality, we have

$$0 \geq \frac{1}{n-1}(\Delta r)^2 + \frac{\partial}{\partial r}(\Delta r) - (n-1)\delta(r). \tag{9}$$

Given that Ω is star-shaped, for any fixed direction $\partial/\partial r$, we obtain (8) by comparing the above differential inequality with the Riccati equation.

On the points where r is not smooth, we may use the trick of Gromov as in Proposition 1.1 of [14] to conclude the result in the sense of distribution. □

5.1 Volume comparison theorems

Let p be the fixed point of the manifold. Denote

$$B(r) = B_p(r), \quad V(r) = \mathrm{vol}\,(B_p(r))$$

the geodesic ball of radius r at p and its volume respectively.

The following volume comparison theorem is well-known.

Lemma 5.3 *Let $r(x)$ be the radial function defined above. Assume that (8) is valid in the sense of distribution. Then the manifold has subexponential volume growth at p. In other words, for all $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$, depending only on ε and the manifold, such that for all $R > 0$*

$$V(R) \leq C(\varepsilon)\, e^{\varepsilon R}.$$

Proof Let $m(r)$ be a nonnegative continuous function such that

$$\lim_{r\to\infty} m(r) = 0,$$

and

$$\Delta r \leq m(r)$$

in the sense of distribution. It follows that

$$\int_{B(R)\setminus B(1)} \Delta r \leq \int_{B(R)\setminus B(1)} m(r)$$

which, by Stokes' Theorem, implies that

$$\operatorname{vol}(\partial B(R)) - \operatorname{vol}(\partial B(1)) \leq \int_{B(R)\setminus B(1)} m(r).$$

Let $\varepsilon > 0$. Then we can find an R_ε such that $m(r) < \varepsilon$ for $r > R_\varepsilon$. Setting $f(R) = V(R)$, we obtain

$$f'(R) \leq \operatorname{vol}(\partial B(1)) + \int_{B(R_\varepsilon)\setminus B(1)} m(r) + \varepsilon(f(R) - f(R_\varepsilon))$$

for any $R > R_\varepsilon$. Thus

$$(e^{-\varepsilon R}(f(R) - f(R_\varepsilon)))' \leq C_\varepsilon e^{-\varepsilon R}$$

for $R > R_\varepsilon$, where C_ε is a constant depending on ε and the manifold M. Integrating from R_ε to R, we obtain

$$f(R) < f(R_\varepsilon) + C_\varepsilon \varepsilon^{-1} e^{-\varepsilon R_\varepsilon} e^{\varepsilon R}$$

for $R > R_\varepsilon$. Thus for any R, we have

$$V(R) = f(R) < C(\varepsilon) e^{\varepsilon R}$$

for

$$C(\varepsilon) = f(R_\varepsilon) + C_\varepsilon \varepsilon^{-1} e^{-\varepsilon R_\varepsilon}.$$

□

In other words, whenever the Laplacian of the radial function $r(x) = d(x, p)$ is asymptotically nonnegative in the sense of distribution, the manifold has subexponential volume growth with respect to the point p. In the case of finite volume for the manifold M, we will also need an assumption on the decay rate of the volume of a ball of radius r. We say that the volume of M *decays exponentially at* p, if there exists an $\varepsilon_o > 0$ such that

$$\operatorname{vol}(M) - V(r) \leq e^{-\varepsilon_o r}$$

for all r large. For the purpose of computing the L^2 essential spectrum, we will need that the volume does not decay exponentially.

5.2 L^1 estimates for $\Delta\tilde{r}$

We set $\tilde{r}$ to be the smoothing of r from Proposition 4.1. The following lemma is a more precise version of [16, Lemma 2].

Lemma 5.4 *Let $r(x)$ be the radial function to a fixed point p on M, and suppose that* (8) *is valid in the sense of distribution. Then we have the following two cases*

(a) Whenever vol (M) *is infinite, for any $\varepsilon > 0$ and $r_1 > 0$ large enough, there exists a $K = K(\varepsilon, r_1)$ such that for any $r_2 > K$, we have*

$$\int_{B(r_2)\backslash B(r_1)} |\Delta\tilde{r}| \leq \varepsilon\, V(r_2 + 1); \tag{10}$$

(b) Whenever vol (M) *is finite, for any $\varepsilon > 0$ there exists a $K(\varepsilon) > 0$ such that for any $r_2 > K$, we have*

$$\int_{M\backslash B(r_2)} |\Delta\tilde{r}| \leq \varepsilon\,(\mathrm{vol}\,(M) - V(r_2)) + 2\mathrm{vol}\,(\partial B(r_2)).$$

Proof By Proposition 4.1 and using the idea in the proof of Lemma 5.1, we obtain

$$|\Delta\tilde{r}(x)| \leq 2\left(\max_{y\in B_x(1)} \Delta r(y) + \eta(r(x)) + |b(x)|\right) - \Delta\tilde{r}(x)$$

in the sense of distribution. Using our assumptions on Δr and η, we see that for any $\varepsilon > 0$ we can find an $r_1 > 0$ large enough such that whenever $r(x) > r_1$, then

$$2\left(\max_{y\in B_x(1)} \Delta r(y) + \eta(r(x))\right) < \varepsilon/2$$

also in the sense of distribution. Therefore for $r > r_1 + 2$,

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{\varepsilon}{2}(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| - \int_{B(r)\backslash B(r_1)} \Delta\tilde{r}.$$

Using Stokes' Theorem, we get

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{\varepsilon}{2}(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| - \int_{\partial B(r)} \frac{\partial\tilde{r}}{\partial n} + \int_{\partial B(r_1)} \frac{\partial\tilde{r}}{\partial n},$$

where $\partial/\partial n$ is the outward normal direction on the boundary. Obviously, the above implies that

$$\int_{B(r)\backslash B(r_1)} |\Delta\tilde{r}| \leq \frac{\varepsilon}{2}(V(r) - V(r_1)) + 2\int_{M\backslash B(r_1)} |b| + \int_{\partial B(r)} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right| + \int_{\partial B(r_1)} \frac{\partial\tilde{r}}{\partial n}. \tag{11}$$

We first consider the case when the volume of M is infinite. By Proposition 4.1, choosing r_1 large enough we obtain

$$\int_{M\setminus B(r_1)} |b| < \frac{\varepsilon}{4}$$

and

$$\|\nabla\tilde{r} - \nabla r\|_{L^1(M\setminus B(r_1))} \leq 1 \tag{12}$$

Since the volume of M is infinite, then there exists $K = K(\varepsilon, r_1) > r_1 + 2$ such that whenever $r > K$

$$\int_{B(r)\setminus B(r_1)} |\Delta\tilde{r}| \leq \frac{3\varepsilon}{4}\,(V(r){-}V(r_1)) + \int_{\partial B(r)} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right|. \tag{13}$$

We choose an r' such that $|r' - r| < \frac{1}{2}$ and

$$\int_{\partial B(r')} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right| \leq \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \int_{\partial B(t)} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right| dt.$$

By (12), we have

$$\int_{\partial B(r')} \left|\frac{\partial\tilde{r}}{\partial n} - 1\right| < 2.$$

Therefore,

$$\int_{B(r')\setminus B(r_1)} |\Delta\tilde{r}| \leq \frac{3\varepsilon}{4}\,(V(r') - V(r_1)) + 2.$$

Choosing a possibly larger $K(\varepsilon, r_1)$ we get *(a)*.

The proof of *(b)* is similar. We choose $\eta(r)$ decreasing to zero so fast so that

$$\int_{M\setminus B(r_1)} |b| \leq \frac{\varepsilon}{8}(\mathrm{vol}\,(M) - V(r_1)).$$

Since the volume of M is finite, sending $r \to \infty$ in (11) we have

$$\int_{M\setminus B_p(r_1)} |\Delta\tilde{r}| \leq \varepsilon\,(\mathrm{vol}\,(M) - V(r_1)) + \int_{\partial B_p(r_1)} \frac{\partial\tilde{r}}{\partial n}.$$

Since $|\partial\tilde{r}/\partial n| \leq 2$ by *(c)* of Proposition 4.1, the lemma follows. □

Corollary 5.1 *Suppose that* (i), (ii), (iii) *hold on* M *as in Lemma* 5.2. *Then the same integral estimates for* $\Delta\tilde{r}$ *hold as in Lemma* 5.4.

6 The L^2 spectrum

In this section,we let $\tilde{r}(x)$ be the smoothing function defined in Proposition 4.1 of the radial function $r(x) = d(x, p)$. For each $i \in \mathbb{N}$, let x_i, y_i, R_i, μ_i be large positive numbers such that $x_i > 2R_i > 2\mu_i + 4$ and $y_i > x_i + 2R_i$. We take the cut-off functions $\chi_i : \mathbb{R}^+ \to \mathbb{R}^+$, smooth with support on $[x_i/R_i - 1, y_i/R_i + 1]$ and such that $\chi_i = 1$ on $[x_i/R, y_i/R]$ and $|\chi_i'|, |\chi_i''|$ bounded. Let $\lambda > 0$ be a positive number. We let

$$\phi_i(x) = \chi_i(\tilde{r}/R_i)\, e^{\sqrt{-1}\sqrt{\lambda}\,\tilde{r}}. \tag{14}$$

Setting $\phi = \phi_i,\ R = R_i,\ x = x_i$ and $\chi = \chi_i$, we compute

$$\begin{aligned}\Delta\phi + \lambda\phi = {} & \left(R^{-2}\chi''(\tilde{r}/R) + 2i\sqrt{\lambda}R^{-1}\chi'(\tilde{r}/R)\right) e^{\sqrt{-1}\sqrt{\lambda}\tilde{r}}|\nabla\tilde{r}|^2 \\ & - \lambda\phi(|\nabla\tilde{r}|^2 - 1) + (R^{-1}\chi'(\tilde{r}/R) + i\sqrt{\lambda}\chi)e^{\sqrt{-1}\sqrt{\lambda}\tilde{r}}\Delta\tilde{r}.\end{aligned}$$

Then we have

$$|\phi| \le 1, \quad |\Delta\phi + \lambda\phi| \le \frac{C}{R} + C|\Delta\tilde{r}| + C|\nabla\tilde{r} - \nabla r|, \tag{15}$$

where C is a constant depending only on λ and M.

Denote the inner product on $L^2(M)$ by $(\cdot\,,\ \cdot)$. We have the following key estimates:

Lemma 6.1 *Suppose that* (8) *is valid for the radial function* r *in the sense of distribution. In the case that the volume of* M *is finite, we make the further assumption that its volume does not decay exponentially at* p. *Then there exist sequences of large numbers* x_i, y_i, R_i, μ_i *such that the supports of the* ϕ_i *are disjoint and*

$$\frac{\|(\Delta + \lambda)\phi_i\|_{L^1}}{(\phi_i, \phi_i)} \to 0$$

as $i \to \infty$.

Proof The proof is similar to that of [16]. We define x_i, y_i, R_i, μ_i inductively. If $(x_{i-1}, y_{i-1}, R_{i-1}, \mu_{i-1})$ are defined, then we only need to let μ_i large enough so that the support of ϕ_i is disjoint with the previous ϕ_j's. For simplicity we suppress the i in our notation. The upper bound estimates for $|\phi|$ and $|\Delta\phi + \lambda\phi|$ given in (15) imply that

$$\int_M (\phi, \Delta\phi + \lambda\phi) \leq \frac{C}{R}[V(y+R) - V(x-R)] + C \int_{B(y+R)\setminus B(x-R)} |\Delta\tilde{r}| + \eta(x-R). \quad (16)$$

When the volume of M is infinite, we choose a function η as in Proposition 4.1 such that $\eta \leq 1$. By Lemma 5.4, if we choose R, x large enough but fixed, then for any $y > 0$ large enough we have

$$\int_M (\phi, \Delta\phi + \lambda\phi) \leq 2\varepsilon\, V(y+R+1).$$

Since $\|\phi\|_2^2 \geq V(y) - V(x)$, if we choose y large enough, $\|\phi\|_2^2 \geq \frac{1}{2}V(y)$. The subexponential volume growth of M at p that was proved in Lemma 5.3 implies that there exists a sequence of $y_k \to \infty$ such that $V(y_k + R + 1) \leq 2\,V(y_k)$. If not, then for a fixed number y and for all $k \in \mathbb{N}$ we have that

$$V(y + k(R+1)) > 2^k\, V(y).$$

However, by the subexponential volume growth of the manifold

$$2^k\, V(y) < V(y + k(R+1)) \leq C(\varepsilon_1)\, e^{\varepsilon_1 y}\, e^{k\,\varepsilon_1 (R+1)}$$

for any $\varepsilon_1 > 0$ and k large. This leads to a contradiction when we choose ε_1 such that $\varepsilon_1 R < \log 2$. Therefore, there exists a y such that

$$V(y+R+1) \leq 2\,V(y) \leq 4\|\phi\|_2^2.$$

Combing the above inequalities, we have

$$\int_M (\phi, \Delta\phi + \lambda\phi) \leq 8\varepsilon\|\phi\|_2^2.$$

We now consider the finite volume case. Using equation (16) and Lemma 5.4 we obtain for $x - R > K(\varepsilon)$

$$\int_M (\phi, \Delta\phi + \lambda\phi) \leq (R^{-1} + \varepsilon)\,[\mathrm{vol}\,(M) - V(x-R)] + 2C\,\mathrm{vol}\,(\partial B(x-R)) + \eta(x-R).$$

We set $h(r) = \mathrm{vol}\,(M) - V(r)$, a decreasing function. We choose $\eta(r)$ as in Proposition 4.1 so that $\eta(r) \leq \frac{\varepsilon}{8}h(r)$. Making ε even smaller and choosing R and $x - R$ large enough, we get

$$\int_M (\phi, \Delta\phi + \lambda\phi) \le \varepsilon\, h(x - R) - 2C\, h'(x - R).$$

Given that $\|\phi\|_2^2 \ge h(x) - h(y)$ and the volume of M is finite, we can choose y large enough so that

$$\|\phi\|_2^2 \ge \frac{1}{2} h(x).$$

We would like to prove in this case that there exists a sequence of $x_k \to \infty$ such that

$$\varepsilon\, h(x_k - R) - 2C\, h'(x_k - R) \le 2\varepsilon h(x_k).$$

If the above inequality does not hold, then for all x large enough

$$\varepsilon\, h(x - R) - 2C\, h'(x - R) > 2\varepsilon h(x).$$

Replacing ε by $\varepsilon/2C$, we obtain

$$\varepsilon\, h(x - R) - \, h'(x - R) > 2\varepsilon h(x).$$

This implies that

$$-\big(e^{-\varepsilon x} h(x - R)\big)' > 2\varepsilon h(x)\, e^{-\varepsilon x}.$$

Integrating from x to $x + R$ and using the monotonicity of h we have

$$h(x - R) > 2(1 - e^{-\varepsilon R}) h(x + R).$$

Choosing R even larger, we can make $2(1 - e^{-\varepsilon R}) > 5/4$, therefore

$$h(x - R) > \frac{5}{4} h(x + R)$$

for all x large enough. By iterating this inequality, we get for all positive integers k

$$h(x - R) > \left(\frac{5}{4}\right)^k h(x + (2k - 1)R).$$

Therefore

$$\mathrm{vol}\,(M) - V(x - R) > \left(\frac{5}{4}\right)^k [\mathrm{vol}\,(M) - V(x + (2k - 1)R)]$$

which gives

$$\mathrm{vol}\,(M) - V(x + (2k-1)R) \leq \left(\frac{4}{5}\right)^k \mathrm{vol}\,(M).$$

Sending $k \to \infty$ this contradicts the nonexponential decay assumption on the volume. □

Corollary 5.1 gives

Corollary 6.1 *Suppose that (i), (ii), (iii) hold on M as in Lemma 5.2. In the case that the volume of M is finite, we make the further assumption that its volume does not decay exponentially at p. Then there exist sequences of large numbers x_i, y_i, R_i, μ_i and cut-off functions χ_i such that the supports of the ϕ_i are disjoint and*

$$\frac{\|(\Delta + \lambda)\phi_i\|_{L^1}}{(\phi_i, \phi_i)} \to 0$$

as $i \to \infty$.

Now we prove Theorem 1.3. In fact we will be able to prove a more general, albeit more technical result:

Theorem 6.2 *Let M be a complete noncompact Riemannian manifold. Suppose that, with respect to a fixed point p, the radial function $r(x) = d(x, p)$ satisfies*

$$\overline{\lim_{r\to\infty}}\ \Delta r \leq 0$$

in the sense of distribution, and if the volume of the manifold is finite, we additionally assume that its volume does not decay exponentially at p. Then the L^2 spectrum of the Laplace operator on functions is $[0, \infty)$.

Proof Let ϕ_i be the sequence of functions as defined in (14). Then by the construction of the functions and Corollary 6.1, the assumptions of Theorem 1.1 are satisfied. This completes the proof of the theorem. □

Remark 6.3 We note that a similar result should hold on warped product manifolds $M = \mathbb{R} \times_J \tilde{M}$ with metric $g = d\rho^2 + J^2(\rho, \theta)\,\tilde{g}$, where $(\tilde{M}, \tilde{g})$ is a compact $(n-1)$-dimensional submanifold of M and ρ is the distance function from this submanifold. Under the same asymptotically nonnegative assumption on $\mathrm{Ric}(\partial\rho, \partial\rho)$ as in Lemma 5.2, we also get that the L^2 spectrum of the Laplace operator on functions is $[0, \infty)$.

7 Complete shrinking Ricci solitons

A noncompact complete Riemannian manifold M with metric g is called a gradient shrinking Ricci soliton if there exists a smooth function f such that the Ricci tensor of the metric g is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho\, g_{ij}$$

for some positive constant $\rho > 0$. By rescaling the metric we may rewrite the soliton equation as

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2}\, g_{ij}.$$

The scalar curvature R of a gradient shrinking Ricci soliton is nonnegative, and the volume growth of such manifolds (with respect to the Riemannian metric) is Euclidean. Hamilton [12] proved that the scalar curvature of a gradient shrinking Ricci soliton satisfies the equations

$$\nabla_i R = 2\, R_{ij}\, \nabla_j f,$$
$$R + |\nabla f|^2 - f = C_o$$

for some constant C_o. We may add a constant to f so that

$$R + |\nabla f|^2 - f = 0.$$

In [16], the authors proved that

(1) the L^1 essential spectrum of the Laplacian contains $[0, \infty)$;
(2) the L^2 essential spectrum of the Laplacian is $[0, \infty)$, if the scalar curvature has sub-quadratic growth.

Using our new Weyl Criterion, we are able to remove the curvature condition.

Proof of Theorem 1.2 It can be shown that $f(x) \geq 0$ and the key idea is to use $\rho(x) = 2\sqrt{f(x)}$ as an approximate distance function on the manifold, because of the special properties that it satisfies.

We define

$$D(r) = \{x \in M : \rho(x) < r\}$$

and set $V(r) = \mathrm{vol}\,(D(r))$. For some positive number y sufficiently large we consider the cut-off function $\chi : \mathbb{R}^+ \to \mathbb{R}$, smooth with support in $[0, y + 2]$ and such that $\chi = 1$ on $[1, y + 1]$ and $|\chi'|, |\chi''| \leq C$. For any $\lambda > 0$ and large enough constants b, l we let

$$\phi(\rho) = \chi\left(\frac{\rho - b}{l}\right) e^{\sqrt{-1}\sqrt{\lambda}\,\rho}$$

which has support on $[b + l, b + l(y + 1)]$. Lu and Zhou [16, page 3289] demonstrate that for sufficiently large l and b

$$\int_M |\Delta\phi + \lambda\phi| \leq \varepsilon V(b + (y + 2)l).$$

At the same time

$$\|\phi\|_{L^2}^2 \geq V(b + (y+1)l) - V(b+l)$$

(note that the same holds true for the L^1 norm of ϕ). Arguing as in [16, Theorem 6] we conclude that there exists a y large enough such that

$$\int_M |\Delta\phi + \lambda\phi| \leq 4\varepsilon \|\phi\|_{L^2}^2.$$

As in the previous section, we may also choose appropriate sequences of b_i, l_i such that the supports of the ψ_i are disjoint and condition *(2)* of Theorem 1.1 holds. Condition *(1)* is verified by the estimate above and the fact that $\|\phi_i\|_{L^\infty} = 1$. □

8 Exhaustion functions on complete manifolds

From what we have seen so far, it is apparent that two things are important when computing the essential spectrum of the Laplacian:

(1) The control of the L^1 norm of Δr;
(2) The control of the volume growth and decay of geodesic balls.

The same idea can be used for manifolds whose essential spectrum is not the half real line.

In the spirit of the results above, we are also able to modify a theorem of Elworthy and Wang [8]. We now consider manifolds on which there exists a continuous exhaustion function $\gamma \in \mathcal{C}(M)$ such that

(a) γ is unbounded above and is $\mathcal{C}^2$ smooth in the domain $\{\gamma > R\}$ for some $R > 0$ and
(b) $\mathrm{vol}\,(\{m_o < \gamma < n\}) < \infty$ for some m_o and any $n > m_o$ where the volume is measured with respect to the Riemannian metric.

For $t > 0$ and $c \in \mathbb{R}$ we define $B_t = \{\gamma(x) < t\}$ and set $dv_c = e^{-c\gamma} dv$. For $t \geq s$, let $U_c(s,t) = \mathrm{vol}_c(B_t \backslash B_s)$ where vol_c is the volume with respect to the measure dv_c.

We begin by stating the result of Elworthy and Wang for the sake of comparison.

Theorem 8.1 ([8, Theorem 1.1]) *Suppose that there exists a function $\gamma \in \mathcal{C}(M)$ that satisfies (a) and (b) and a constant $c \in \mathbb{R}$ such that*

$$\lim_{s\to\infty} \overline{\lim_{t\to\infty}}\, U_c(s,t)^{-1} \int_{B_t\backslash B_s} [(\Delta\gamma - c)^2 + (|\nabla\gamma|^2 - 1)^2]\, dv_c = 0 \tag{17}$$

and

$$\lim_{t\to\infty} \max\{U_c(m_o, t), U_c(t,\infty)^{-1}\}\, e^{-\varepsilon t} = 0 \qquad \text{for any } \varepsilon > 0. \tag{18}$$

Then $\sigma(-\Delta) \supset [c^2/4, \infty)$. When the above hold for $c = 0$, then $\sigma(-\Delta) = [0,\infty)$.

Note that condition (18) implies that when $c = 0$ the volume of the manifold grows and decays subexponentially, as was the case for us in the previous sections. The assumption here is that the weighted volume grows and decays subexponentially.

Our result is as follows:

Theorem 8.2 *Suppose that there exists a function $\gamma \in \mathcal{C}(M)$ that satisfies (a) and (b) and a constant $c \in \mathbb{R}$ such that*

$$\lim_{s\to\infty} \overline{\lim_{t\to\infty}}\ U_c(s,t)^{-1} \int\limits_{B_t\setminus B_s} (|\Delta\gamma - c| + ||\nabla\gamma|^2 - 1|)\, dv_c = 0 \tag{19}$$

and

$$\lim_{t\to\infty} \max\{U_c(m_o,t), U_c(t,\infty)^{-1}\}\, e^{-\varepsilon t} = 0 \qquad \text{for any } \varepsilon > 0. \tag{20}$$

If (19) and (20) hold for $c = 0$, then $\sigma(-\Delta) = [0,\infty)$.

In the case they hold for $c \neq 0$, we make the additional assumptions that the heat kernel of the Laplacian satisfies the pointwise bound

$$p_t(x,y) \le Ct^{-m}\, e^{-\frac{(\gamma(x)-\gamma(y))^2}{4C_1 t} - \frac{d(x,y)^2}{4C_2 t} + \beta_1|\gamma(x)-\gamma(y)| + \beta_2 d(x,y) + \beta_3 t} \tag{21}$$

for some positive constants $m, C_1, C_2, \beta_1, \beta_2, \beta_3$, and that the Ricci curvature of the manifold is bounded below $Ric(M) \ge -(n-1)K$ for a nonnegative number K. Then $\sigma(-\Delta) \supset [c^2/4, \infty)$.

In the case $c = 0$, the main difference between our result and Theorem 1.1 of [8] is that we only need to control the L^1 norms of $|\Delta\gamma - c|$ and $||\nabla\gamma|^2 - 1|$ as in (19), instead of their L^2 norms (compare to 17). Our assumption is weaker in various cases, for example when γ is the radial function where we know that its Laplacian is not locally L^2 integrable whenever the manifold has a cut-locus, but it is locally L^1 integrable.

In the case $c \neq 0$, the additional assumption (21) is similar to requiring a uniform Gaussian bound for the heat kernel, but now with respect to the γ function as well. Such a bound is certainly true in the case of hyperbolic space with γ the radial function.

The proof uses similar estimates to those of Elworthy and Wang for the measures of annuli along the exhaustion function γ. We provide an outline of the argument with the necessary modifications.

Proof Set $\lambda \ge c^2/4$ be a fixed number. For any $t > s$ we let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, be a smooth cut-off function with support on $[s-1, t+1]$ and such that $\chi = 1$ on $[s,t]$ and $|\chi'|, |\chi''|$ bounded. Let $\lambda_c = \sqrt{\lambda - c^2/4}$ and define for $s \ge 0$

$$f(s) = e^{(i\lambda_c - c/2)\, s}.$$

Consider the test function

$$\phi_{s,t}(x) = \chi(\gamma(x))\, f(\gamma(x)).$$

We compute

$$\Delta\phi_{s,t} + \lambda\phi_{s,t} = (\chi'' f + 2\chi' f' + \chi f'')|\nabla\gamma|^2 + (\chi' f + \chi f')\Delta\gamma + \lambda\,\chi f.$$

Using the fact that $f'' + cf' + \lambda f = 0$ we obtain

$$\begin{aligned}\Delta\phi_{s,t} + \lambda\phi_{s,t} = {} & (\chi'' f + 2\chi' f')|\nabla\gamma|^2 + (\chi' f)\Delta\gamma + \chi f'(\Delta\gamma - c|\nabla\gamma|^2)\\ & + \lambda\,\chi f(1 - |\nabla\gamma|^2).\end{aligned}$$

Therefore there exists a constant C such that

$$|\Delta\phi_{s,t} + \lambda\phi_{s,t}| \le Ce^{-c/2\gamma}\left[(|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)1_{\mathrm{spt}(B_{t+1}\setminus B_{s-1})} + 1_{\mathrm{spt}(\chi')}\right]. \tag{22}$$

For the rest of the estimates, we will repeatedly use

$$\lim_{s,t\to\infty}(U_c(s-1,s) + U_c(t,t+1))/U_c(s,t) = 0, \tag{23}$$

which follows from (20).

Using (22), we have

$$\begin{aligned}|(\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})| \le C & \int\limits_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\,dv_c\\ & + C(U_c(s-1,s) + U_c(t,t+1)). \end{aligned} \tag{24}$$

We observe that

$$\begin{aligned}&\frac{1}{U_c(s,t)}\int\limits_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\,dv_c\\ &\quad = \left[1 + \frac{U_c(s-1,s) + U_c(t,t+1)}{U_c(s,t)}\right]\\ &\qquad \cdot\frac{1}{U_c(s-1,t+1)}\int\limits_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\,dv_c,\end{aligned}$$

which tends to zero as $s,t \to \infty$ by (23) and assumption (19). Since $\|\phi_{s,t}\|^2_{L^2} \ge U_c(s,t)$, inequality (24), the above estimate and (23) imply that

$$\lim_{s,t\to\infty}|(\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})|/\|\phi_{s,t}\|^2_{L^2} = 0. \tag{25}$$

When $c = 0$, we choose appropriate sequences of $s_n, t_n \to \infty$ such that condition *(2)* of Theorem 1.1 holds. Condition *(1)* of the Corollary follows from (25) and the fact that the functions ϕ_{s_n,t_n} are bounded. Therefore, $\lambda_0 = \sqrt{\lambda}$ belongs to the essential L^2 spectrum. Given that λ is any nonnegative number, the result follows.

In the case $c \neq 0$, we will apply Corollary 2.2. For a fixed natural number $i > m$ and $\alpha > 0$ we have that the integral kernel of $(-\Delta + \alpha)^{-i}$, $g_\alpha^i(x, y)$, is given by

$$g_\alpha^i(x, y) = C(n) \int_0^\infty p_t(x, y)\, t^{i-1}\, e^{-\alpha t}\, dt.$$

On the other hand, it is a property of the exponential function that for any $\beta_4, \beta_5 \in \mathbb{R}$

$$e^{-\frac{(\gamma(x)-\gamma(y))^2}{4C_1 t}} \leq e^{-\beta_4|\gamma(x)-\gamma(y)|} e^{C_1\, \beta_4^2 t}$$

and

$$e^{-\frac{d(x,y)^2}{4C_2 t}} \leq e^{-\beta_5 d(x,y)} e^{C_2\, \beta_5^2 t}.$$

Combining the above, we have that for any $N > m$ and $\beta_4, \beta_5 > 0$ there exists an $\alpha > 0$ large enough, and a constant C such that

$$g_\alpha^i(x, y) \leq C\, e^{-\beta_4\, |\gamma(x)-\gamma(y)| - \beta_5 d(x,y)}$$

for $i = N, N+1$. As a result, for any $t > s > 2$

$$\begin{aligned} \int_{B_{t+1}\setminus B_{s-1}} g_\alpha^i(x, y) e^{-c/2\gamma(y)} dy &\leq C \int_{B_{t+1}\setminus B_{s-1}} e^{-\beta_4|\gamma(x)-\gamma(y)|-\beta_5 d(x,y)}\, e^{-c/2\gamma(y)} dy \\ &\leq C\, e^{-c/2\gamma(x)} \end{aligned}$$

after choosing $\beta_4 = |c|/2$ and $\beta_5 > \sqrt{K}$. This estimate together with (23) also give

$$\begin{aligned} |((-\Delta+\alpha)^{-i}\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})| \leq C & \int_{B_{t+1}\setminus B_{s-1}} (|\Delta\gamma - c| + |\,|\nabla\gamma|^2 - 1|)\, dv_c \\ & + C(U_c(s-1, s) + U_c(t, t+1)). \end{aligned}$$

As a result,

$$\lim_{s,t\to\infty} |((-\Delta+\alpha)^{-i}\phi_{s,t}, \Delta\phi_{s,t} + \lambda\phi_{s,t})| / \|\phi_{s,t}\|_{L^2}^2 = 0. \tag{26}$$

Choosing appropriate sequences of $s_n, t_n \to \infty$ and setting $\psi_n = \phi_{s_n,t_n}/\|\phi_{s_n,t_n}\|_{L^2}$, conditions *(1)* and *(4)* of Corollary 2.2 hold for the functions ψ_n. That *(2)* and *(3)* also hold follows from (25) and (26) respectively. □

9 The use of continuous test functions

In this section we will see that it is not necessary to use cut-off functions in our test functions. We will do that by first proving yet another version of the generalized Weyl's criterion (Corollary 9.1). This version of Weyl's Criterion sometimes provides a cleaner method for computing the essential spectrum.

Let D be a bounded domain of M with smooth boundary. We use the notation $\mathcal{C}_0^\infty(D)$ to denote the set of smooth functions on the closure $\bar{D}$ which vanish on the boundary ∂D. Let $\rho : D \to \mathbb{R}$ be the distance function to the boundary ∂D.

Definition 9.1 We define $\mathcal{C}_0^+(D)$ to be the set of functions f on D with the properties

(1) f is continuous, vanishing on ∂D;
(2) f is Lipschitz, ∇f is essentially bounded, and $|\Delta f|$ exists in the sense of distribution;
(3) As $\varepsilon \to 0$, $\int_{\{\rho\le\varepsilon\}} |f| \le \frac{1}{2}\varepsilon^2(\int_{\partial D} |\nabla f| + o(1))$, and $\int_{\{\rho\le\varepsilon\}} |\nabla f| \le \varepsilon(\int_{\partial D} |\nabla f| + o(1))$.

Let $\mathcal{C}_0^+(M)$ be the set of continuous functions whose support is a bounded domain of M with smooth boundary and

$$f \in \mathcal{C}_0^+(\mathrm{supp} f).$$

We have the following:

Corollary 9.1 *A nonnegative real number λ belongs to the spectrum $\sigma(-\Delta)$, if there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of functions in $\mathcal{C}_0^+(M)$ such that*

(1) $$\frac{\|\psi_n\|_{L^\infty(D_n)} \cdot (\|(-\Delta-\lambda)\psi_n\|_{L^1(D_n)} + \|\nabla\psi_n\|_{L^1(\partial D_n)})}{\|\psi_n\|^2_{L^2(D_n)}} \to 0, \ \text{as } n \to \infty,$$

where $D_n = \mathrm{supp}\,\psi_n$. Moreover, λ belongs to $\sigma_{\mathrm{ess}}(-\Delta)$ of Δ, if

(2) For any compact subset K of M, there exists an n such that the support of ψ_n is outside K.

The above corollary can be proved using the following approximation result:

Proposition 9.1 *Let $f \in \mathcal{C}_0^+(M)$. Then for any $\varepsilon > 0$, there exists a smooth function h of M such that*

(a) $\mathrm{supp}\,(h) \subset \mathrm{supp}\,(f)$;
(b) $\|f - h\|_{L^1} + \|f - h\|_{L^2} \le \varepsilon$;
(c) $\|(-\Delta - \lambda)h\|_{L^1} \le C(\|(-\Delta - \lambda)f\|_{L^1(D)} + \|\nabla f\|_{L^1(\partial D)})$,

where C is a constant independent of f, and $D = \mathrm{supp}\,(f)$.

Proof Let $\chi(t)$ be a cut-off function that vanishes in a neighborhood of 0 and is 1 for $t \ge 1$. Let $\delta > 0$ be a small number. Consider

$$g_\delta(x) = \chi\left(\frac{\rho(x)}{\delta}\right) f(x).$$

It is not difficult to prove *(a), (b)* in the Proposition when we replace h by g_δ. To prove *(c)* we compute

$$(-\Delta-\lambda)g_\delta=\chi(-\Delta-\lambda)f-2\delta^{-1}\chi'\nabla\rho\nabla f-(\delta^{-2}\chi''+\delta^{-1}\chi'\Delta\rho)f.$$

Since ∂D is smooth, ρ is a smooth function near ∂D. Therefore by (3) of Definition 9.1 we have

$$\|(-\Delta-\lambda)g_\delta\|_{L^1}\leq C(\|(-\Delta-\lambda)f\|_{L^1(D)}+\|\nabla f\|_{L^1(\partial D)})$$

for δ sufficiently small.

The proof that g_δ can be approximated by a smooth function is similar to that of Proposition 4.1. We sketch the proof here.

Let $D=\cup U_i$ be a finite cover of D. Without loss of generality, we assume that those U_i's which intersect with ∂D are outside the support of g_δ. Let $\mathbf{x_i}=(x_i^1,\ldots,x_i^n)$ be the local coordinates of U_i. Define $g_i=g_\delta|_{U_i}$.

Let $\xi(\mathbf{x})$ be a non-negative smooth function of $\mathbb{R}^n$ whose support is within the unit ball. Assume that

$$\int_{\mathbb{R}^n}\xi=1.$$

Without loss of generality, we assume that each U_i is an open subset of the unit ball of $\mathbb{R}^n$ with coordinates $\mathbf{x_i}$. Then for any $\varepsilon>0$,

$$g_{i,\varepsilon}=\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\xi\left(\frac{\mathbf{x_i}-\mathbf{y_i}}{\varepsilon}\right)g_i(\mathbf{y_i})d\mathbf{y_i}$$

is a smooth function on U_i and hence on M. Let $\{\sigma_i\}$ be a sequence of positive numbers such that

$$\sum_i\sigma_i(|\Delta\psi_i(x)|+4|\nabla\psi_i(x)|+\psi_i(x)) \tag{27}$$

is sufficiently small. By [11, Lemma 7.1, 7.2], for each i, we can choose $\varepsilon_i<1$ small enough so that

$$\begin{aligned}&|g_{i,\varepsilon_i}(x)-g_i(x)|\leq\sigma_i;\\&\|\nabla g_{i,\varepsilon_i}-\nabla g_i\|_{L^1(U_i)}\leq\sigma_i.\end{aligned} \tag{28}$$

We also have

$$\|\Delta g_{i,\varepsilon_i}\|_{L^1}\leq\|\Delta g_i\|_{L^1}. \tag{29}$$

Define

$$h=\sum_i \psi_i g_{i,\varepsilon_i},\quad b=2\sum_i \nabla\psi_i\cdot\nabla g_{i,\varepsilon_i}.$$

Since $\sum_i(\nabla\psi_i\cdot\nabla g_i)=(\sum_i\nabla\psi_i)\cdot\nabla g_\delta=0$ almost everywhere on D, we have

$$b=2\sum_i \nabla\psi_i\cdot(\nabla g_{i,\varepsilon_i}-\nabla g_i).$$

We compute

$$\Delta h=\sum_i[(\Delta\psi_i)\,g_{i,\varepsilon_i}+2\nabla\psi_i\nabla g_{i,\varepsilon_i}+\psi_i\Delta g_{i,\varepsilon_i}],$$

and since

$$\sum_i(\Delta\psi_i)g_i=\sum_i(\Delta\psi_i)g_\delta=0,$$

we have

$$\Delta h=\sum_i[\Delta\psi_i(g_{i,\varepsilon_i}-g_i)+2\sum_i\nabla\psi_i\cdot(\nabla g_{i,\varepsilon_i}-\nabla g_i)+\psi_i\Delta g_{i,\varepsilon_i}].$$

By (28), (29), we may choose ε_i to be sufficiently small so that

$$\|(-\Delta-\lambda)h\|_{L^1(D)}\le 2\|(-\Delta-\lambda)g_\delta\|_{L^1(D)}.$$

□

Acknowledgments The authors thank Rafe Mezzeo and Jiaping Wang for their interest in and discussions of the essential spectrum problem. They particularly thank David Krejčiřík for the discussion on the alternative versions of Weyl's Criterion which led to the proof of Theorem 2.3.

References

1. Charalambous, N.: Nelia on the equivalence of heat kernel estimates and logarithmic Sobolev inequalities for the Hodge Laplacian. J. Differ. Equ. **233**(1), 291–312 (2007)
2. Chen, Z.H., Lu, Z.Q.: Essential spectrum of complete Riemannian manifolds. Sci. China Ser. A **35**(3), 276–282 (1992) (MR 1183713) (93k:58221)
3. Davies, E.B.: Heat Kernels and Spectral Theory, Cambridge Tracts in Mathematics, vol. 92. Cambridge University Press, Cambridge (1989)
4. Dermenjian, Y., Durand, M., Iftimie, V.: Spectral analysis of an acoustic multistratified perturbed cylinder. Comm. Partial Differ. Equ. **23**(1–2), 141–169 (1998) (MR 1608508) (99a:35035)
5. Donnelly, H.: On the essential spectrum of a complete Riemannian manifold. Topology **20**(1), 1–14 (1981). doi:10.1016/0040-9383(81)90012-4 (MR 592568) (81j:58081)
6. Donnelly, H.: Exhaustion functions and the spectrum of Riemannian manifolds. Indiana Univ. Math. J. **46**(2), 505–527 (1997). doi:10.1512/iumj.1997.46.1338 (MR 1481601) (99b:58230)

7. Donnelly, H.: Spectrum of the Laplacian on asymptotically Euclidean spaces. Mich. Math. J. **46**(1), 101–111 (1999)
8. Elworthy, K.D., Wang, F-Y.: Essential Spectrum on Riemannian Manifolds. Recent Developments in Stochastic Analysis and Related Topics. World Sci. Publ., Hackensack, NJ, pp. 151–165 (2004)
9. Escobar, J.F.: On the spectrum of the Laplacian on complete Riemannian manifolds. Comm. Partial Differ. Equ. **11**(1), 63–85 (1986)
10. Escobar, J.F., Freire, A.: The spectrum of the Laplacian of manifolds of positive curvature. Duke Math. J. **65**(1), 1–21 (1992)
11. Gilbarg, D.A., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics, Reprint of the 1998 edition. Springer, Berlin (2001)
12. Hamilton, R.S.: The Formation of Singularities in the Ricci Flow. Surveys in differential geometry, vol. II, (Cambridge, MA, 1993). Int. Press, Cambridge, pp. 7–136 (1995)
13. Li, J.Y.: Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole. J. Math. Soc. Jpn. **46**(2), 213–216 (1994). doi:10.2969/jmsj/04620213 (MR 1264938) (95g:58248)
14. Schoen, R., Yau, S.-T.: Lectures on differential geometry. In: Conference Proceedings and Lecture Notes in Geometry and Topology, I. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. International Press, Cambridge, MA (1994) (Translated from the Chinese by Ding and S. Y. Cheng; Preface translated from the Chinese by Kaising Tso)
15. Krejčiřík, D., Kříž, J.: On the spectrum of curved quantum waveguides. Publ. RIMS Kyoto Univ. **41**, 757–791 (2005)
16. Lu, Z., Zhou, D.: On the essential spectrum of complete non-compact manifolds. J. Funct. Anal. **260**(11), 3283–3298 (2011). doi:10.1016/j.jfa.2010.10.010 (MR 2776570) (2012e:58058)
17. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV. Analysis of operators, Academic Press [Harcourt Brace Jovanovich Publishers], New York (1978) (MR 0493421) (58 #12429c)
18. Sturm, K.-T.: On the L^p-spectrum of uniformly elliptic operators on Riemannian manifolds. J. Funct. Anal. **118**(2), 442–453 (1993). doi:10.1006/jfan.1993.1150. (MR 1250269) (94m:58227)
19. Wang, J.: The spectrum of the Laplacian on a manifold of nonnegative Ricci curvature. Math. Res. Lett. **4**(4), 473–479 (1997)
20. Zhou, D.T.: Essential spectrum of the Laplacian on manifolds of nonnegative curvature. Internat. Math. Res. Notices, **5**, 209. doi:10.1155/S1073792894000231. (MR 1270134) (95g:58250) (1994) (electronic) (to appear in print)