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Global $W^{2,p}$ estimates for solutions to the linearized Monge–Ampère equations

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Abstract In this paper, we establish global $W^{2,p}$ estimates for solutions to the linearized Monge–Ampère equations under natural assumptions on the domain, Monge–Ampère measures and boundary data. Our estimates are affine invariant analogues of the global $W^{2,p}$ estimates of Winter for fully nonlinear, uniformly elliptic equations, and also linearized counterparts of Savin's global $W^{2,p}$ estimates for the Monge–Ampère equations.

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Contents

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1 Introduction and statement of the main results

In this paper we consider the linearized Monge–Ampère equations and investigate global L^p estimates for the second derivatives of their solutions. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and ϕ be a locally uniformly convex function on Ω. The linearized Monge–Ampère equation corresponding to ϕ is

$$\mathcal{L}_\phi u := \sum_{i,j=1}^n \Phi^{ij} u_{ij} = f \quad \text{in}\, \Omega, \tag{1.1}$$

where $\Phi = \left(\Phi^{ij}\right)_{1\le i,j\le n} := (\det D^2\phi)\ (D^2\phi)^{-1}$ is the matrix of cofactors of the Hessian matrix $D^2\phi$. As the coefficient matrix Φ is positive semi-definite, $\mathcal{L}_\phi$ is a linear elliptic partial differential operator, possibly degenerate. The operator $\mathcal{L}_\phi$ appears in several contexts including affine maximal surface equation in affine geometry [30–33], Abreu's equation in the context of existence of Kähler metric of constant scalar curvatures in complex geometry [10–13,36], and semigeostrophic equations in fluid mechanics [1,5,25]. Solutions of many important problems in these contexts require a deep understanding of interior and boundary behaviors of solutions to (1.1).

The regularity theory for the linearized Monge–Ampère equation was initiated in the fundamental paper [4] by Caffarelli and Gutiérrez. They established an interior Harnack inequality for nonnegative solutions to the homogeneous equation $\mathcal{L}_\phi u = 0$ in terms of the pinching of the Hessian determinant $\lambda \le \det D^2\phi \le \Lambda$. Their theory is an affine invariant version of the classical Harnack inequality for uniformly elliptic equations with measurable coefficients. This result played a crucial role in Trudinger–Wang's resolution [31] of Chern's conjecture in affine geometry concerning affine maximal hypersurfaces in $\mathbb{R}^3$ and in Donaldson's interior estimates for Abreu's equation in complex geometry [11]. Another contribution to the regularity theory comes from [18] where Gutiérrez and Tournier derived interior $W^{2,\delta}$ estimates for small δ. The interior regularity for Eq. (1.1) was further developed by Gutiérrez and the second author in [16,17] where the (sharp) interior $C^{1,\alpha}$ and $W^{2,p}$ estimates, respectively, were obtained.

Regarding the global regularity, by using Caffarelli–Gutiérrez's interior Harnack estimates and Savin's localization theorem, Savin and the first author [23] established boundary Hölder gradient estimates for solutions to the linearized Monge–Ampère equation. Furthermore, the first author [21] proved global Hölder estimates for solutions to (1.1) in uniformly convex domains, which are the global counterpart of Caffarelli–Gutiérrez's interior Hölder estimates [4].

As mentioned above, Gutiérrez and the second author derived in [17] the interior $W^{2,p}$ estimates for solutions of (1.1) in terms of the L^q-norm of f where $q > \max\{n, p\}$, the pinching of the Hessian determinant $\lambda \leq \det D^2\phi \leq \Lambda$ and the continuity of the Monge–Ampère measure $\det D^2\phi$. The purpose of our paper is to establish global $W^{2,p}$ estimates for solutions to the linearized Monge–Ampère equation (1.1) under natural assumptions on the domain, Monge–Ampère measures and boundary data.

Our first main theorem is concerned with global $W^{2,p}$ estimates for the linearized equation (1.1) when the Monge–Ampère measure $\det D^2\phi$ is close to a constant.

Theorem 1.1 *Let Ω be a bounded, uniformly convex domain with $\partial\Omega \in C^3$, and let $\phi \in C(\overline{\Omega})$ be a convex function satisfying $\phi = 0$ on $\partial\Omega$. Let $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ be the solution to the linearized Monge–Ampère equation*

$$\begin{cases} \mathcal{L}_\phi u = f & in\,\Omega, \\ u = 0 & on\,\partial\Omega, \end{cases}$$

where $f \in L^q(\Omega)$ with $n < q < \infty$. Then, for any $p \in (1, q)$, there exist $0 < \epsilon < 1$ and $C > 0$ depending only on n, p, q and Ω such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^q(\Omega)}$$

provided that the Monge–Ampère measure of ϕ satisfies

$$1 - \epsilon \leq \det D^2\phi \leq 1 + \epsilon \quad in\,\Omega.$$

As a corollary of our method of the proof of Theorem 1.1, we obtain global $W^{2,p}$ estimates for Eq. (1.1) when the Monge–Ampère measure $\det D^2\phi$ is continuous. Our second main theorem states as follows.

Theorem 1.2 *Let Ω be a bounded, uniformly convex domain with $\partial\Omega \in C^3$, and let $\phi \in C(\overline{\Omega})$ be the convex solution to the Monge–Ampère equation*

$$\begin{cases} \det D^2\phi = g & in\,\Omega, \\ \phi = 0 & on\,\partial\Omega, \end{cases}$$

where $g \in C(\overline{\Omega})$ is a continuous function satisfying $0 < \lambda \leq g(x) \leq \Lambda$ in Ω. Let $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ be the solution to the linearized Monge–Ampère equation

$$\begin{cases} \mathcal{L}_\phi u = f & in\,\Omega, \\ u = \varphi & on\,\partial\Omega, \end{cases}$$

where $\varphi \in W^{2,s}(\Omega)$, $f \in L^q(\Omega)$ with $n < q < s < \infty$. Then, for any $p \in (1, q)$, there exists $C > 0$ depending only on $\lambda, \Lambda, n, p, q, s, \Omega$ and the modulus of continuity of g such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|\varphi\|_{W^{2,s}(\Omega)} + \|f\|_{L^q(\Omega)}\right).$$

Our estimates are affine invariant analogues of the global $W^{2,p}$ estimates of Winter [35] for fully nonlinear, uniformly elliptic equations, and are also linearized counterparts of Savin's global $W^{2,p}$ estimates for the Monge–Ampère equation [28]. We note that the continuity condition on the Monge–Ampère measure in Theorem 1.2 is sharp in view of Wang's counterexample [34] for solutions to the Monge–Ampère equation and the fact that $\mathcal{L}_\phi \phi = n \det D^2\phi = ng$. The global second derivative estimates in Theorems 1.1 and 1.2 depend only on the bounds on the Hessian determinant $\det D^2\phi$ and its continuity or closeness to a constant, the geometry of Ω and the quadratic separation of ϕ from its tangent planes on the boundary $\partial\Omega$. This quadratic separation is guaranteed by the C^3 character of boundary domain $\partial\Omega$, data $\phi\mid_{\partial\Omega}$ and the uniform convexity of Ω (see Proposition 2.4). Under the assumptions in the main theorems, the linearized Monge–Ampère operator $\mathcal{L}_\phi$ is not uniformly elliptic, i.e., the eigenvalues of $\Phi = (\Phi^{ij})$ are not necessarily bounded away from 0 and ∞. Moreover, $\mathcal{L}_\phi$ can be possibly singular near the boundary. The degeneracy and singularity of $\mathcal{L}_\phi$ are the main difficulties in establishing our boundary regularity results. We handle the degeneracy of $\mathcal{L}_\phi$ by working as in [4,16,17,21,23] with sections of solutions to the Monge–Ampère equations. These sections have the same role as Euclidean balls have in the classical theory. To overcome the singularity of $\mathcal{L}_\phi$ near the boundary, we use a Localization Theorem at the boundary for solutions to the Monge–Ampère equations which was obtained by Savin [26,27]. In order to obtain the desired global second derivative estimates for solutions u of $\mathcal{L}_\phi$, we need to have good global decay estimates for the distribution function of the second derivatives of u. To this end, we approximate u by solutions of $\mathcal{L}_w$ where w solves the standard Monge–Ampère equation $\det D^2 w = 1$ with appropriate boundary conditions, and use fine geometric properties of boundary sections for solutions to the Monge–Ampère equation which were obtained recently in [22].

Though the statements of our main theorems are rather succinct, their proofs are quite delicate. There are essentially two main steps for the proof of the main estimates:

Step 1: We consider the quasi distance $d(x, \bar{x})$ induced by the solution ϕ to the Monge–Ampère equation and is defined by $d(x, \bar{x})^2 := \phi(x) - \phi(\bar{x}) - \nabla\phi(\bar{x}) \cdot (x - \bar{x})$. We then bound the distribution function of the second derivative D^2u by the Lebesgue measures of the "bad" sets on whose complements the quasi distance $d(x, \bar{x})$ is comparable to the Euclidean distance $|x - \bar{x}|$ in a controllable manner and the graph of u is touched from above and below by "quasi paraboloids" generated by the quasi distance. Intuitively, the better the regularity of ϕ is, the faster these decay estimates can be expected. When $\phi(x) = |x|^2/2$, the Monge–Ampère measure $\det D^2\phi$ is the usual Lebesgue measure and $d(x, \bar{x})$ corresponds to the Euclidean distance. In this step, we establish preliminary power decay estimates for the bad sets under natural assumptions on the domain Ω and the boundary data of ϕ. As a result, we obtain global $W^{2,\delta}(\Omega)$ estimates for u where $\delta > 0$ is small under these natural assumptions provided that the Monge–Ampère measure $\det D^2\phi$ is close to a constant. We also give a more direct proof of global $W^{2,\delta}$ estimates for solutions to the linearized Monge–Ampère equations when the Monge–Ampère measure is only assumed to be bounded away from 0 and ∞. This direct proof is based on interior estimates without resorting to decay estimates of the distribution function of the second derivatives. These estimates, that are of independent interest, are global counterparts of Gutiérrez-

Tournier's interior $W^{2,\delta}$ estimates for solutions to the linearized equation (1.1). Our idea, which is similar to Savin's arguments in [28], is rather simple but useful for the second step and can be roughly described as follows:

$$\text{local estimates} + \text{appropriate covering results} \Longrightarrow \text{global estimates.}$$

Step 2: We improve the power decay estimates obtained in **Step 1** assuming in addition that $\det D^2\phi$ is sufficiently close to 1. This will involve two main auxiliary results:

(1) a global stability of cofactor matrices: we prove that the cofactor matrices of the Hessian matrices of two convex functions defined on the same domain are close if their Monge–Ampère measures and boundary values are close in the L^∞ norm;
(2) a global approximation result: we approximate the solution u by smooth solutions of linearized Monge–Ampère equations associated with convex functions whose Monge–Ampère measures and boundary data are close to those of ϕ.

The main estimates will then follow from a covering theorem for boundary sections and a strong-type $p-p$ estimate for the maximal function corresponding to boundary sections.

Without going into details, we now indicate key technical points that entail for getting global $W^{2,p}$ estimates. First, we show that the distribution function $|\{x : |D^2u| > \beta\}|$ of the second derivatives of the solution u to $\mathcal{L}_\phi u = f$ has some decay of the form $C\beta^{-\tau}$ with $\tau > 0$ small and $C > 0$ depending only on the structural constants in our equation; see Propositions 3.6 and 3.7. In the next step, we refine these decay estimates by working in very small regions of the domain and by rescaling our equation and domain. In this rescaled setting, the constant C above can be improved, roughly by a factor of $\|\Phi - W\|_{L^n} + (⨍ |f|^n)^{1/n}$; see Lemma 5.1. Here W is the matrix of the cofactors of D^2w where w is the solution to the standard Monge–Ampère equation $\det D^2w = 1$ having the same boundary values as ϕ in small regions. When $\det D^2\phi$ is close to 1, the term $\|\Phi - W\|_{L^n}$ can be made as small as we want thanks to the stability of cofactor matrices in Proposition 3.14. The term $(⨍ |f|^n)^{1/n}$ is invariant under a rescaling of our equation that almost preserves the L^∞-norm of the second derivative D^2u. There are two natural rescalings of our equation to be explained in Sect. 2 but the aforementioned rescaling is the most crucial. As a consequence, $(⨍ |f|^n)^{1/n}$ can be made as small as we want provided that f has higher integrability than L^n, but this is the assumption in our main theorems.

The rest of the paper is organized as follows. In Sect. 2, we recall the main tool used in our proof: the Localization Theorem at the boundary for solutions to the Monge–Ampère equation, and state relevant results on the geometry of their sections. We also discuss properties of solutions to the Monge–Ampère equation and its linearization under suitable rescalings using the Localization Theorem. In addition, we establish boundary $C^{2,\alpha}$ estimates for solutions to the standard Monge–Ampère equations $\det D^2w = 1$ having the same boundary values as ϕ on its rescaled sections at the boundary. In Sect. 3, we derive preliminary power decay estimates for the distribution function of the second derivatives of solutions to the linearized Monge–Ampère equations (1.1). We also establish the global $W^{2,\delta}$ estimates for solutions to (1.1), paving the way for proving the global stability of cofactor matrices in Sect. 3.4. Moreover, applying the global stability of cofactor matrices, we obtain in Sect. 3.5 global $W^{2,1+\epsilon}$ estimates for convex solutions to the linearized Monge–Ampère equations when the Monge–Ampère measure is only assumed to be bounded away from zero and infinity. These estimates can be viewed as affine invariant versions of results obtained by De Phillipis–Figalli–Savin and Schmidt. In Sect. 4.1, we prove the global Hölder continuity property of solutions to (1.1). This property together with the boundary $C^{2,\alpha}$ estimates in Sect. 2 will be instrumental in the global approximation lemmas in Sect. 4.2. In the last section, Sect. 5, by combining these approximation lemmas with the preliminary power decay estimates, we obtain density estimates, which improve the power decay estimates in Sect. 3, when the Monge–Ampère measure $\det D^2\phi$ is close to a constant. The proofs of the main results will be given at the end of this section using these density estimates, a covering theorem and a strong-type $p-p$ estimate for the maximal function with respect to sections.

2 The localization theorem and geometry of the Monge–Ampère equation

The results in this section hold under the following global information on the convex domain Ω and the convex function ϕ. We assume there exists $\rho > 0$ such that

$$\Omega \subset B_{1/\rho}, \text{ and for each } y \in \partial\Omega \text{ there is a ball } B_\rho(z) \subset \Omega \text{ that is tangent to } \partial\Omega \text{ at } y. \tag{2.1}$$

Let $\phi : \overline{\Omega} \to \mathbb{R}$, $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying

$$\det D^2\phi = g, \qquad 0 < \lambda \le g \le \Lambda \quad \text{in } \Omega. \tag{2.2}$$

Assume further that on $\partial\Omega$, ϕ separates quadratically from its tangent planes, namely

$$\rho\,|x-x_0|^2 \le \phi(x) - \phi(x_0) - \nabla\phi(x_0)\cdot(x-x_0) \le \rho^{-1}\,|x-x_0|^2\,,\ \forall x, x_0 \in \partial\Omega. \tag{2.3}$$

The section of ϕ centered at $x \in \overline{\Omega}$ with height h is defined by

$$S_\phi(x,h) := \Big\{ y \in \overline{\Omega} : \quad \phi(y) < \phi(x) + \nabla\phi(x)\cdot(y-x) + h \Big\}.$$

For $x \in \Omega$, we denote by $\bar{h}(x)$ the maximal height of all sections of ϕ centered at x and contained in Ω, that is,

$$\bar{h}(x) := \sup\left\{ h \ge 0 |\quad S_\phi(x,h) \subset \Omega \right\}.$$

In this case, $S_\phi(x,\bar{h}(x))$ is called the maximal interior section of ϕ with center $x \in \Omega$.

Remark 2.1 In this paper, we denote by $c, \bar{c}, C, C_1, C_2, \theta_0, \theta_*, \ldots$, positive constants depending only on $\rho, \lambda, \Lambda, n$, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants*. Small universal constants decrease when λ decreases and/or Λ increases. Large universal constants increase when λ decreases and/or Λ increases, etc. Therefore, when $1-\epsilon \leq \det D^2\phi \leq 1+\epsilon$ with $0 < \epsilon < 1/2$, we can suppress the dependence of universal constants on ϵ.

2.1 The localization theorem

In this subsection, we recall the main tool to study geometric properties of boundary sections of solutions to the Monge–Ampère equation: the Localization theorem at the boundary for solution to the Monge–Ampère equation (Theorem 2.2). Throughout this subsection, we assume that the convex domain Ω and the convex function ϕ satisfy (2.1)–(2.3). We now focus on sections centered at a point on the boundary $\partial\Omega$ and describe their geometry. Assume this boundary point to be 0 and by (2.1), we can also assume that

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}}, \tag{2.4}$$

where $\rho > 0$ is the constant given by condition (2.1). After subtracting a linear function, we can assume further that

$$\phi(0) = 0 \quad \text{and} \quad \nabla\phi(0) = 0. \tag{2.5}$$

If the boundary data has quadratic growth near $\{x_n = 0\}$ then, as $h \to 0$, $S_\phi(0,h)$ is equivalent to a half-ellipsoid centered at 0. This is the content of the Localization Theorem proved by Savin [26,27]. Precisely, this theorem reads as follows.

Theorem 2.2 (Localization Theorem [26,27]) *Assume that Ω satisfies* (2.4) *and ϕ satisfies* (2.2)*,* (2.5)*, and*

$$\rho|x|^2 \leq \phi(x) \leq \rho^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \leq \rho\}.$$

Then there exists a constant $k = k(\rho, \lambda, \lambda, n) > 0$ such that for each $h \leq k$ there is an ellipsoid E_h of volume $\omega_n h^{n/2}$ satisfying

$$kE_h \cap \overline{\Omega} \subset S_\phi(0,h) \subset k^{-1}E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$\begin{aligned} A_h E_h = h^{1/2} B_1,\ \det A_h = 1,\ A_h(x) = x - \tau_h x_n, \\ \tau_h = (\tau_1, \tau_2, \ldots, \tau_{n-1}, 0) \quad \text{and} \quad |\tau_h| \leq k^{-1}|\log h|. \end{aligned}$$

From Theorem 2.2 we also control the shape of sections that are tangent to $\partial\Omega$ at the origin.

Proposition 2.3 *Let ϕ and Ω satisfy the hypotheses of the Localization Theorem 2.2 at the origin. Assume that for some $y \in \Omega$ the section $S_\phi(y,h) \subset \Omega$ is tangent to $\partial\Omega$ at 0, i.e., $\partial S_\phi(y,h) \cap \partial\Omega = \{0\}$, for some $h \leq c$ with c universal. Then there exists a small positive constant $k_0 < k$ depending on λ, Λ, ρ and n such that*

$$\nabla\phi(y) = ae_n \quad \textit{for some} \quad a \in [k_0 h^{1/2}, k_0^{-1} h^{1/2}],$$
$$k_0 E_h \subset S_\phi(y,h) - y \subset k_0^{-1} E_h, \qquad k_0 h^{1/2} \leq \mathit{dist}(y, \partial\Omega) \leq k_0^{-1} h^{1/2},$$

with E_h and k the ellipsoid and constant defined in Theorem 2.2.

Proposition 2.3 is a consequence of Theorem 2.2 and was proved in [28].

The quadratic separation from tangent planes on the boundary for ϕ is a crucial assumption in the Localization Theorem (Theorem 2.2). This is the case for solutions to the Monge–Ampère equation with the right hand side bounded away from 0 and ∞ on uniformly convex domains and smooth boundary data as proved in [27, Proposition 3.2].

Proposition 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex domain satisfying (2.1) and $\|\partial\Omega\|_{C^3} \leq 1/\rho$. Let $\phi : \overline{\Omega} \to \mathbb{R}$, $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying $\phi \mid_{\partial\Omega} \in C^3$ and*

$$0 < \lambda \leq \det D^2\phi \leq \Lambda < \infty \quad \textit{in}\ \Omega.$$

Then, on $\partial\Omega$, ϕ separates quadratically from its tangent planes, that is,

$$\rho_0 \left|x - x_0\right|^2 \leq \phi(x) - \phi(x_0) - \nabla\phi(x_0)\cdot(x - x_0) \leq \rho_0^{-1} \left|x - x_0\right|^2, \ \forall x_0, x \in \partial\Omega$$

for some constant $\rho_0 > 0$ depending only on $n, \rho, \lambda, \Lambda$, $\|\phi\|_{C^3(\partial\Omega)}$ and the uniform convexity of Ω.

2.2 Properties of the rescaled functions and boundary regularity estimates

In this subsection, we discuss properties of solutions to the Monge–Ampère equation and its linearization under suitable rescalings and then use these properties to establish a boundary $C^{2,\alpha}$ estimates for solutions to the standard Monge–Ampère equation $\det D^2 w = 1$ in our rescaled setting.

Let Ω and ϕ satisfy the hypotheses of the Localization Theorem at the origin. We know that for all $h \leq k$, $S_\phi(0,h)$ satisfies

$$kE_h \cap \overline{\Omega} \subset S_\phi(0,h) \subset k^{-1} E_h \cap \overline{\Omega}, \tag{2.6}$$

with A_h being a linear transformation and

$$\det A_h = 1, \quad E_h = A_h^{-1} B_{h^{1/2}}, \quad A_h x = x - \tau_h x_n, \quad \tau_h \cdot e_n = 0,$$
$$\|A_h^{-1}\|, \; \|A_h\| \le k^{-1} |\log h|.$$

This gives for all $h \le k$

$$\overline{\Omega} \cap B^+_{h^{2/3}} \subset \overline{\Omega} \cap B^+_{ch^{1/2}/|\log h|} \subset S_\phi(0,h) \subset \overline{\Omega} \cap B^+_{Ch^{1/2}|\log h|} \subset B^+_{h^{1/3}}. \tag{2.7}$$

We denote the rescaled function of ϕ and the rescaled domain of Ω by

$$\phi_h(x) := \frac{\phi\left(h^{1/2} A_h^{-1} x\right)}{h} \quad \text{and} \quad \Omega_h := h^{-1/2} A_h \Omega. \tag{2.8}$$

The function ϕ_h, defined in $\overline{\Omega}_h$, is continuous and solves the Monge–Ampère equation

$$\det D^2 \phi_h = g_h(x), \qquad \lambda \le g_h(x) := g\left(h^{1/2} A_h^{-1} x\right) \le \Lambda.$$

By (2.6), the section of ϕ_h centered at the origin and with height 1 satisfies

$$B_k^+ \cap \overline{\Omega}_h \subset S_{\phi_h}(0,1) = h^{-1/2} A_h S_\phi(0,h) \subset B^+_{k^{-1}} \cap \overline{\Omega_h}. \tag{2.9}$$

In what follows, we denote

$$U_h = S_{\phi_h}(0,1). \tag{2.10}$$

Now, we discuss two natural rescalings for the linearized Monge–Ampère equation

$$\mathcal{L}_\phi u := \Phi^{ij} u_{ij} = f \quad \text{in } \Omega.$$

We focus on the boundary section $S_\phi(0,h)$ in the present setting of Theorem 2.2. L^∞**-norm preserving rescaling.** Consider the following rescaling of functions:

$$u_h(x) := u\left(h^{1/2} A_h^{-1} x\right) \quad \text{and} \quad f_h(x) := hf\left(h^{1/2} A_h^{-1} x\right), \quad \text{for } x \in \Omega_h.$$

Simple computation gives

$$D^2 \phi_h = \left(A_h^{-1}\right)^t D^2 \phi A_h^{-1}, \quad D^2 u_h = h\left(A_h^{-1}\right)^t D^2 u A_h^{-1},$$

and

$$\Phi_h := \left(\det D^2 \phi_h\right) (D^2 \phi_h)^{-1} = (\det D^2 \phi) A_h (D^2 \phi)^{-1} (A_h)^t = A_h \Phi (A_h)^t.$$

Therefore, we find that

$$\mathcal{L}_{\phi_h} u_h = trace(\Phi_h D^2 u_h) = f_h \text{ in } \Omega_h, \quad \text{and} \quad \|u_h\|_{L^\infty(\Omega_h)} = \|u\|_{L^\infty(\Omega)}.$$

Thus this rescaling preserves the L^∞-norm of u. Since $\|f_h\|_{L^n(\Omega_h)} = h^{1/2}\|f\|_{L^n(\Omega)}$ is small if $f \in L^n(\Omega)$ and h small, we can expect that u_h has some nice second derivative estimates, say their boundedness. Given this and as

$$D^2 u(h^{1/2} A_h^{-1} x) = h^{-1}(A_h)^t \, D^2 u_h(x) \, A_h,$$

it is again quite natural to expect that $|D^2 u|$ behaves like $\frac{1}{h}$ in some part of the section $S_\phi(0, h)$. This is what we will prove in Lemma 5.2.

Almost $W^{2,\infty}$-norm preserving rescaling. The next rescaling almost preserves the L^∞-norm of $D^2 u$. Under the following rescaling of functions

$$\tilde{u}_h(x) := h^{-1} u\left(h^{1/2} A_h^{-1} x\right) \quad \text{and} \quad \tilde{f}_h(x) := f\left(h^{1/2} A_h^{-1} x\right) \quad \text{for } x \in \Omega_h,$$

we have $\mathcal{L}_{\phi_h} \tilde{u}_h = \tilde{f}_h$ in Ω_h with

$$\fint_{\Omega_h} \left|\tilde{f}_h\right|^n = \fint_{\Omega} |f|^n,$$

by changing variables and recalling that $\det A_h = 1$. As

$$D^2 \tilde{u}_h(x) = \left(A_h^{-1}\right)^t D^2 u\left(h^{1/2} A_h^{-1} x\right) A_h^{-1},$$

the present rescaling almost preserves the L^∞-norm of $D^2 u$ since

$$\|D^2 \tilde{u}_h\|_{L^\infty(\Omega_h)} \le k^{-2} \, |\log h|^2 \, \|D^2 u\|_{L^\infty(\Omega)}.$$

In principle, the L^∞-norm preserving rescaling allows us to find some good points with controlled second derivatives for u. Having found them, we would like to propagate them by finding more similar points near by, maybe at the cost of a slightly larger bound on the second derivatives. This is the key technical point of the paper and almost $W^{2,\infty}$-norm preserving rescaling is the means for this; see Lemmas 5.2 and 5.4.

A variant of the **L^∞-norm preserving rescaling** is the following which applies to sections tangent to the boundary.

L^∞-norm preserving rescaling in a section tangent to the boundary. Consider a prototype section $S_\phi(y, h)$ with $h := \bar{h}(y) \le c$. By applying Proposition 2.3 to $S_\phi(y, h)$, we see that it is equivalent to an ellipsoid E_h, i.e.,

$$k_0 E_h \subset S_\phi(y, h) - y \subset k_0^{-1} E_h,$$

where

$$E_h := h^{1/2} A_h^{-1} B_1 \quad \text{with} \quad \det A_h = 1, \quad \|A_h\|, \ \|A_h^{-1}\| \le C|\log h|.$$

We use the following rescalings:

$$\tilde{\Omega}_h := h^{-1/2} A_h(\Omega - y),$$

and for $x \in \tilde{\Omega}_h$

$$\tilde{u}_h(x) := u\left(y + h^{1/2} A_h^{-1} x\right),$$
$$\tilde{\phi}_h(x) := h^{-1}\left[\phi\left(y + h^{1/2} A_h^{-1} x\right) - \phi(y) - \nabla\phi(y) \cdot \left(h^{1/2} A_h^{-1} x\right) - h\right].$$

Then

$$B_{k_0} \subset \tilde{U}_h \equiv S_{\tilde{\phi}_h}(0,1) \equiv h^{-1/2} A_h\big(S_\phi(y,h) - y\big) \subset B_{k_0^{-1}}.$$

We have

$$\det D^2\tilde{\phi}_h(x) = \tilde{g}_h(x) := g\left(y + h^{1/2} A_h^{-1} x\right), \quad \tilde{\phi}_h = 0 \text{ on } \partial S_{\tilde{\phi}_h}(0,1)$$

and

$$\min_{S_{\tilde{\phi}_h}(0,1)} \tilde{\phi}_h = -1 = \tilde{\phi}_h(0).$$

Also

$$\tilde{\Phi}_h^{ij}(\tilde{u}_h)_{ij} = \tilde{f}_h(x) := hf\left(y + h^{1/2} A_h^{-1} x\right).$$

Some properties of the rescaled function ϕ_h was established in [27] and [23, Lemma 4.2, Lemma 5.4]. For later use, we record them here.

Lemma 2.5 *There exists a small constant $c = c(n, \rho, \lambda, \Lambda) > 0$ such that if $h \le c$, then*

(a) for any $x, x_0 \in \partial\Omega_h \cap B_{2/k}$ we have

$$\frac{\rho}{4}|x - x_0|^2 \le \phi_h(x) - \phi_h(x_0) - \nabla\phi_h(x_0) \cdot (x - x_0) \le 4\rho^{-1}|x - x_0|^2. \quad (2.11)$$

(b) if $r \le c$ small, we have

$$|\nabla\phi_h| \le Cr|\log r|^2 \quad \text{in} \quad \overline{\Omega}_h \cap B_r.$$

(c) $\partial\Omega_h \cap B_{2/k}$ is a graph in the e_n direction whose $C^{1,1}$ norm is bounded by $Ch^{1/2}$.

(d) *ϕ_h satisfies in $U_h \equiv S_{\phi_h}(0,1)$ the hypotheses of Theorem 2.2 at all points on $\partial U_h \cap B_c$.*
(e) *If $y \in U_h \cap B_{c^2}$ then the maximal interior section $S_{\phi_h}(y, \bar{h}(y))$ of ϕ_h in U_h satisfies:*

$$c \geq \bar{h}(y) \geq k_0^2\, dist^2(y, \partial U_h) \quad and \quad S_{\phi_h}(y, \bar{h}(y)) \subset U_h \cap B_c.$$

Proof [23, Lemma 4.2] contains (a)–(c) while its proof implies (d). The statement (e) can be proved as in [23, Lemma 5.4] and we give a complete proof here. Let $y \in U_h \cap B_{c^2}$. Then it follows from property (d) and (2.7) that $y \in S_{\phi_h}(0, c^3)$. Hence, $\phi_h(y) \leq c^3$. By [22, Lemma 4.1] we obtain $S_{\phi_h}(0, c^3) \subset S_{\phi_h}(y, \theta_0 c^3)$ and consequently

$$\bar{h}(y) \leq \theta_0 c^3. \tag{2.12}$$

Thus, $\bar{h}(y) \leq c$ if c is small. Since $S_{\phi_h}(y, \bar{h}(y))$ is balanced around y, we can use Theorem 3.3.8 in [15] to conclude that

$$S_{\phi_h}(y, \bar{h}(y)) \subset B(y, K\, \bar{h}(y)^b) \tag{2.13}$$

for some universal constants $K, b > 0$.

From (2.12) and (2.13) we see that for c small the section $S_{\phi_h}(y, \bar{h}(y))$ is tangent to $\partial\Omega_h$. Let $x_0 \in \partial S_{\phi_h}(y, \bar{h}(y)) \cap \partial\Omega_h$. Applying (2.11) to x_0 and 0, and using property (b) and (2.13), we have

$$\begin{aligned} \frac{\rho}{4}|x_0|^2 &\leq \phi_h(x_0) = \phi_h(y) + \nabla\phi_h(y)\cdot(x_0 - y) + \bar{h}(y) \\ &\leq c^3 + CK\, |y|\, \bar{h}(y)^b \big| \log|y| \big|^2 + \bar{h}(y). \end{aligned}$$

This together with the assumption $|y| < c^2$ and (2.12) implies that $|x_0| < c$. Now, thanks to (d) we can apply Proposition 2.3 at x_0 and obtain

$$k_0^2 \mathrm{dist}^2(y, \partial U_h) \leq \bar{h}(y) \leq k_0^{-2} \mathrm{dist}^2(y, \partial U_h).$$

Since $S_{\phi_h}(y, \bar{h}(y)) - y \subset k_0^{-1} E_h$, we find from the definition of E_h and $\bar{h}(y) \leq \theta_0 c^3$ that

$$S_{\phi_h}(y, \bar{h}(y)) \subset y + k_0^{-1} E_h \subset B_{c^2 + k_0^{-1} k^{-1} |\bar{h}(y)|^{1/2} |\log \bar{h}(y)|} \subset B_c$$

if c is universally small. □

Remark 2.6 From now on, we fix a universally small constant $c \leq k/2$, $c \ll 1$ depending only on $n, \rho, \lambda, \Lambda$ as in the Lemma 2.5.

The rest of this subsection is devoted to establishing boundary $C^{2,\alpha}$ estimates for the convex solution w to the standard Monge–Ampère equation

$$\begin{cases} \det D^2 w = 1 & \text{in } U_h := S_{\phi_h}(0,1), \\ w = \phi_h & \text{on } \partial U_h. \end{cases} \tag{2.14}$$

For this, we first show in the next lemma that w separates quadratically from its tangent planes on the boundary of U_h.

Lemma 2.7 *Let Ω_h, ϕ_h and U_h be as in (2.8) and (2.10) with $h \le c$. Let $w \in C(\overline{U_h})$ be the convex solution to (2.14). Then there exist universal constants $\delta, \theta > 0$ depending only on $n, \rho, \lambda, \Lambda$ such that for any $x_0 \in \partial U_h \cap B_c$,*

$$x_{n+1} = \phi_h(x_0) + \langle \nabla \phi_h(x_0) - 2\delta^{1-n} k^{-1} \nu_{x_0}, x - x_0 \rangle =: \bar{l}_{x_0}(x)$$

is a supporting hyperplane in $\overline{U_h}$ to w at x_0, and

$$\theta\, |x - x_0|^2 \le w(x) - \bar{l}_{x_0}(x) \le \theta^{-1}\, |x - x_0|^2 \quad \textit{for all} \quad x \in \partial U_h. \tag{2.15}$$

Here ν_{x_0} denotes the unit inner normal to $\partial\Omega_h$ at x_0.

Proof For $x_0 \in \partial U_h \cap B_c$, let $l_{x_0}(x) := \phi_h(x_0) + \nabla \phi_h(x_0) \cdot (x - x_0)$. Then by Lemma 2.5(a),

$$\frac{\rho}{4}|x - x_0|^2 \le \phi_h(x) - l_{x_0}(x) \le \frac{4}{\rho}|x - x_0|^2 \quad \forall x \in \partial U_h \cap \partial\Omega_h. \tag{2.16}$$

By Lemma 2.5(d) and a consequence of the Localization Theorem 2.2 (see (2.7)), there is $r_0 > 0$ universally small depending only on $n, \rho, \lambda, \Lambda$ such that

$$S_{\phi_h}(x_0, r_0) \subset B\left(x_0, \frac{k}{2}\right) \cap \overline{U_h} \subset B_k \cap \overline{U_h}.$$

This gives $\phi_h(x) \ge l_{x_0}(x) + r_0$ for all $x \in \partial U_h \backslash \partial\Omega_h$, and consequently, by (2.9)

$$\phi_h(x) \ge l_{x_0}(x) + \frac{k^2 r_0}{4}|x - x_0|^2 \quad \forall x \in \partial U_h \backslash \partial\Omega_h. \tag{2.17}$$

Define

$$\begin{aligned} w^-(x) := l_{x_0}(x) &+ \delta \left[|x - x_0|^2 - |(x - x_0) \cdot \nu_{x_0}|^2 \right] \\ &+ \delta^{1-n} \left[|(x - x_0) \cdot \nu_{x_0}|^2 - 2k^{-1}(x - x_0) \cdot \nu_{x_0} \right] \quad \forall x \in \overline{U_h}, \end{aligned}$$

where

$$\delta := \min\left\{ \frac{\rho}{4}, \frac{k^2 r_0}{4} \right\}.$$

Then w^- is a convex function in $\overline{U_h}$ satisfying $D^2 w^- = 2\delta\,[I + (\delta^{-n} - 1)\,\nu_{x_0} \otimes \nu_{x_0}]$. Therefore,

$$\det D^2 w^- = (2\delta)^n \delta^{-n} = 2^n > 1 = \det D^2 w \quad \text{in} \quad U_h. \tag{2.18}$$

For $x \in \partial U_h \cap \partial\Omega_h$, we obtain from $0 \le (x - x_0)\cdot \nu_{x_0} \le 2k^{-1}$ and the first inequality in (2.16) that

$$w^-(x) \le l_{x_0}(x) + \delta|x - x_0|^2 \le \phi_h(x) - \frac{\rho}{4}|x - x_0|^2 + \delta|x - x_0|^2 \le \phi_h(x) = w(x).$$

On the other hand, for $x \in \partial U_h \setminus \partial\Omega_h$ by using (2.17) we have

$$w^-(x) \le l_{x_0}(x) + \delta|x - x_0|^2 \le l_{x_0}(x) + \frac{k^2 r_0}{4}|x - x_0|^2 \le \phi_h(x) = w(x).$$

Therefore, $w \ge w^-$ on ∂U_h. It follows from this, (2.18) and the comparison principle that $w(x) \ge w^-(x)$ in $\overline{U_h}$. Hence,

$$\begin{aligned} w(x) &\ge \bar{l}_{x_0}(x) + \delta\left[|x - x_0|^2 - |(x - x_0)\cdot \nu_{x_0}|^2\right] + \delta^{1-n}\,|(x - x_0)\cdot \nu_{x_0}|^2 \\ &\ge \bar{l}_{x_0}(x) + \delta|x - x_0|^2 \quad \text{in} \quad \overline{U_h}. \end{aligned} \tag{2.19}$$

In particular, $w(x) \ge \bar{l}_{x_0}(x)$ for all $x \in \overline{U_h}$. Since $\bar{l}_{x_0}(x_0) = \phi_h(x_0) = w(x_0)$, we then conclude that $x_{n+1} = \bar{l}_{x_0}(x)$ is a supporting hyperplane in $\overline{U_h}$ to w at x_0.

We now show the second inequality in (2.15). For this, we first recall that $0 \le \phi_h \le 1$ in U_h and by Lemma 2.5(b), we find that for $M := 1 + 2k^{-1} C\, c\, |\log c|^2$,

$$\phi_h(x) \le \phi_h(x_0) + \nabla\phi_h(x_0)\cdot(x - x_0) + M \equiv l_{x_0}(x) + M \quad \forall x \in \overline{U_h}. \tag{2.20}$$

We now compare w with w^+ defined by

$$w^+(x) := l_{x_0}(x) + 2\Theta k^{-1}\,(x - x_0)\cdot \nu_{x_0} + \Theta\left[|x - x_0|^2 - |(x - x_0)\cdot \nu_{x_0}|^2\right] \quad \forall x \in \overline{U_h},$$

where

$$\Theta := \max\left\{\frac{4}{\rho},\ \frac{4M}{k^2}\right\}.$$

Clearly, w^+ is a convex function in $\overline{U_h}$ satisfying

$$\det D^2 w^+ = 0 < 1 = \det D^2 w \quad \text{in}\, U_h. \tag{2.21}$$

For $x \in \partial U_h \cap \partial \Omega_h$, we obtain from the second inequality in (2.16) and $\Theta \geq \frac{4}{\rho}$ that

$$\begin{aligned} w^+(x) &= l_{x_0}(x) + \Theta|x-x_0|^2 + \Theta\Big[2k^{-1}(x-x_0)\cdot\nu_{x_0} - |(x-x_0)\cdot\nu_{x_0}|^2\Big] \\ &\geq \phi_h(x) - \frac{4}{\rho}|x-x_0|^2 + \Theta|x-x_0|^2 \geq \phi_h(x) = w(x). \end{aligned}$$

For $x \in \partial U_h \backslash \partial \Omega_h$, we have $|x-x_0| \geq k/2$ and thus, by using (2.20) we obtain

$$w^+(x) \geq l_{x_0}(x) + \Theta|x-x_0|^2 \geq \phi_h(x) - M + \frac{k^2\Theta}{4} \geq \phi_h(x) = w(x).$$

Therefore, $w \leq w^+$ on ∂U_h. It follows from this, (2.21) and the comparison principle that $w \leq w^+$ in $\overline{U_h}$. In particular,

$$\begin{aligned} w(x) &\leq l_{x_0}(x) + 2\Theta k^{-1}(x-x_0)\cdot\nu_{x_0} + \Theta\,|x-x_0|^2 \\ &= \bar{l}_{x_0}(x) + 2k^{-1}\big(\delta^{1-n}+\Theta\big)(x-x_0)\cdot\nu_{x_0} + \Theta\,|x-x_0|^2 \quad \forall x \in \overline{U_h}. \end{aligned}$$

We then use Lemma 2.5(c) for $x \in \partial U_h \cap \partial \Omega_h$ and the fact that $k/2 \leq |x-x_0| \leq 2/k$ for $x \in \partial U_h \backslash \partial \Omega_h$, to conclude that

$$w(x) \leq \bar{l}_{x_0}(x) + C|x-x_0|^2 \quad \forall x \in \partial U_h.$$

This together with (2.19) gives the quadratic separation in (2.15). □

Thanks to the quadratic separation property of w in Lemma 2.7, we can now apply Savin's boundary $C^{2,\alpha}$ estimates for solutions to the Monge–Ampère equations [27] to get boundary $C^{2,\alpha}$ estimates for w when $\partial\Omega \cap B_\rho$ and $\phi\mid_{\partial\Omega\cap B_\rho}$ are $C^{2,\alpha}$ and h is small.

Proposition 2.8 *Let Ω and ϕ satisfy the hypotheses of the Localization Theorem 2.2 at the origin. Assume in addition that $\partial\Omega \cap B_\rho$ is $C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial\Omega \cap B_\rho)$ for some $\alpha \in (0,1)$. Let Ω_h, ϕ_h, U_h and w be as in Lemma 2.7. Then there exists $h_0 > 0$ depending on $n, \lambda, \Lambda, \rho, \alpha$, $\|\partial\Omega \cap B_\rho\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega\cap B_\rho)}$ such that for any $h \leq h_0$, we have*

$$\|w\|_{C^{2,\alpha}(\overline{B_c\cap U_h})} \leq c_0^{-1} \quad \textit{and} \quad c_0 I_n \leq D^2 w \leq c_0^{-1} I_n \textit{ in } B_c \cap U_h \tag{2.22}$$

for some $c_0 > 0$ depending only on $n, \lambda, \Lambda, \alpha$ and ρ.

Now, let us assume in addition that $\partial\Omega$ and $\phi|_{\partial\Omega}$ are $C^{2,\alpha}$ at the origin for some $\alpha \in (0,1)$, that is, we assume that for $x = (x', x_n) \in \partial\Omega \cap B_\rho$, we have

$$\left|x_n - q(x')\right| \leq M\left|x'\right|^{2+\alpha} \quad \text{and} \quad \left|\phi - p(x')\right| \leq M\left|x'\right|^{2+\alpha},$$

where $p(x')$ and $q(x')$ are homogeneous quadratic polynomials.

If h is sufficiently small, then the corresponding rescaling ϕ_h satisfies the hypotheses of ϕ in which the constant M is replaced by an arbitrary small constant σ.

Lemma 2.9 ([27, Lemma 7.4]) *Given any $\sigma > 0$, there exists a small positive constant $h = h_0(M, \sigma, \alpha, n, \lambda, \Lambda, \rho)$ such that on $\partial\Omega_h \cap B_{k^{-1}}$, we have*

$$\left|x_n - q_h(x')\right| \le \sigma \left|x'\right|^{2+\alpha}, \quad \left|q_h(x')\right| \le \sigma \quad \text{and} \quad \left|\phi_h - p(x')\right| \le \sigma \left|x'\right|^{2+\alpha},$$

where $q_h(x') := h^{1/2} q(x')$ is a homogeneous quadratic polynomial.

Remark 2.10 By inspecting the proof of Lemma 7.4 in [27], we see that the following more precise statement holds true: There exists $C = C(M, n, \lambda, \Lambda, \rho) > 0$ such that for any $h \le c$, on $\partial\Omega_h \cap B_{k^{-1}}$ we have

$$\left|x_n - q_h(x')\right| \le Ch^{\frac{1+\alpha}{2}} \left|x'\right|^{2+\alpha}, \quad \left|q_h(x')\right| \le Ch^{\frac{1}{2}} \quad \text{and} \quad \left|\phi_h - p(x')\right| \le Ch^{\frac{\alpha}{2}} \left|x'\right|^{2+\alpha}.$$

Proof of Proposition 2.8. Let $M := max\left\{\|\partial\Omega \cap B_\rho\|_{C^{2,\alpha}}, \|\phi\|_{C^{2,\alpha}(\partial\Omega\cap B_\rho)}\right\}$ and let h_0 be the small constant in Lemma 2.9 corresponding to M and $\sigma = 1$. Then by our assumptions, Lemma 2.9, Remark 2.10 and Lemma 2.7, we can apply [27, Corollary 7.2] to conclude that there exist $C, \delta > 0$ depending on $n, \lambda, \Lambda, \alpha$ and ρ such that

$$\|w\|_{C^{2,\alpha}(\mathcal{C}_0 \cap B_\delta(0))} \le C,$$

where $\mathcal{C}_0 := \{x \in \mathbb{R}^n_+ : |x'| \le x_n\}$ is the cone at the origin with opening $\theta = \pi/4$.

By varying the point under consideration, we then conclude in the similar fashion that

$$\|w\|_{C^{2,\alpha}(\mathcal{C}_{x_0} \cap B_\delta(x_0))} \le C \quad \forall x_0 \in \partial\Omega_h \cap \overline{B_c}. \tag{2.23}$$

Here $\mathcal{C}_{x_0} := \{x \in \mathbb{R}^n_+ : |x - x_0|^2 \le 2|(x - x_0) \cdot \nu_{x_0}|^2\}$ is the cone at x_0 with opening $\theta = \pi/4$ and in the direction of ν_{x_0}, the unit inner normal to $\partial\Omega_h$ at x_0. As a consequence of (2.23) and Caffarelli's interior $C^{2,\alpha}$ estimates [2], we obtain the first estimate in (2.22) from which the second estimate in (2.22) follows. □

2.3 The classes $\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha}$ and $\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$

Fix $n, \rho, \lambda, \Lambda, \kappa$ and α. We define the classes $\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha}$ and $\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$ consisting of the triples (Ω, ϕ, U) satisfying the following sets of conditions (i)–(vii) and (i)–(vi), respectively:

(i) $0 \in \partial\Omega$, $U \subset \Omega \subset \mathbb{R}^n$ are bounded convex domains such that

$$B_k^+ \cap \overline{\Omega} \subset \overline{U} \subset B_{k^{-1}}^+ \cap \overline{\Omega}.$$

(ii) $\phi : \overline{\Omega} \to \mathbb{R}^+$ is convex satisfying $\phi = 1$ on $\partial U \cap \Omega$ and

$$\phi(0) = 0, \quad \nabla\phi(0) = 0, \quad \lambda \le \det D^2\phi \le \Lambda \text{ in } \Omega,$$
$$\partial\Omega \cap \{\phi < 1\} = \partial U \cap \{\phi < 1\}.$$

(iii) (quadratic separation)

$$\frac{\rho}{4}|x - x_0|^2 \le \phi(x) - \phi(x_0) - \nabla\phi(x_0)\cdot(x - x_0)$$
$$\le \frac{4}{\rho}|x - x_0|^2 \qquad \forall x, x_0 \in \partial\Omega \cap B_{\frac{2}{k}}.$$

(iv) (flatness)

$$\partial\Omega \cap \{\phi < 1\} \subset G \subset \{x_n \le \kappa\}$$

where $G \subset B_{2/k}$ is a graph in the e_n direction and its $C^{1,1}$ norm is bounded by κ.

(v) (localization and gradient estimates) ϕ satisfies in U the hypotheses of the Localization Theorem 2.2 at all points on $\partial U \cap B_c$ and

$$|\nabla\phi| \le C_0 \quad \text{in } U \cap B_c.$$

(vi) (Maximal sections around the origin) If $y \in U \cap B_{c^2}$ then the maximal interior section of ϕ in U satisfies:

$$c \ge \bar{h}(y) \ge k_0^2 \operatorname{dist}^2(y, \partial U) \quad \text{and} \quad S_{\phi_h}(y, \bar{h}(y)) \subset U \cap B_c.$$

(vii) (Pogorelov estimates)

$$\|\partial U \cap B_c\|_{C^{2,\alpha}} \le c_0^{-1}$$

and if w is the convex solution to

$$\begin{cases} \det D^2 w = 1 & \text{in } U \\ w = \phi & \text{on } \partial U, \end{cases} \tag{2.24}$$

then

$$\|w\|_{C^{2,\alpha}(\overline{B_c \cap U})} \le c_0^{-1} \quad \text{and} \quad c_0 I_n \le D^2 w \le c_0^{-1} I_n \text{ in } B_c \cap U.$$

The constants k, k_0, c, C_0 above depend only on $n, \rho, \lambda, \Lambda$ and c_0 depends also on α.

Remark 2.11 If $(\Omega, \phi, U) \in \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$ then the Pogorelov estimates in (vii) might not hold. However, ϕ satisfies in U the hypotheses of the Localization Theorem 2.2 at all points on $\partial U \cap B_c$. Thus, if w is the convex solution to (2.24), then by inspecting the proof of Lemma 2.7, we see that w separates quadratically from its tangent planes at any point $x_0 \in \partial U \cap B_c$, that is,

$$\theta\,|x-x_0|^2 \le w(x)-w(x_0)-\nabla w(x_0)\cdot(x-x_0) \le \theta^{-1}\,|x-x_0|^2 \quad \text{for all}\, x\in\partial U.$$

We summarize the discussion at the end of Sect. 2.2, Lemmas 2.5, 2.9 and Proposition 2.8 in the following proposition.

Proposition 2.12 *Let Ω and ϕ satisfy the hypotheses of Theorem 2.2 at the origin. Assume in addition that $\partial\Omega\cap B_\rho$ is $C^{2,\alpha}$ and $\phi\in C^{2,\alpha}(\partial\Omega\cap B_\rho)$ for some $\alpha\in(0,1)$. Then there exists $h_0>0$ depending only on $n,\lambda,\Lambda,\rho,\alpha$, $\|\partial\Omega\cap B_\rho\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega\cap B_\rho)}$ such that for $h\le h_0$ we have*

$$\big(\Omega_h,\phi_h,S_{\phi_h}(0,1)\big)\in\mathcal{P}_{\lambda,\Lambda,\rho,Ch^{1/2},\alpha}\quad \textit{and}\quad \|\partial\Omega_h\cap B_{1/k}\|_{C^{2,\alpha}}\le C'\,h^{1/2}.$$

Here C depends only on n,λ,Λ and ρ; C' depends only on n,λ,Λ,ρ, $\|\partial\Omega\cap B_\rho\|_{C^{2,\alpha}}$, and $\|\phi\|_{C^{2,\alpha}(\partial\Omega\cap B_\rho)}$.

2.4 Geometric properties of boundary sections of solutions to Monge–Ampère equation

In this subsection, we recall some important properties of boundary sections of solutions to the Monge–Ampère equations established in [22]: the engulfing and dichotomic properties, volume estimates, a covering theorem and strong type $p-p$ estimates for the maximal functions corresponding to small sections including boundary ones.

The engulfing property and volume estimates are summarized in the following theorem.

Theorem 2.13 *Assume that Ω and ϕ satisfy (2.1)–(2.3). Then,*

a. (Engulfing property) There exists $\theta_*>0$ depending only on ρ,λ,Λ and n such that if $y\in S_\phi(x,t)$ with $x\in\overline{\Omega}$ and $t>0$, then $S_\phi(x,t)\subset S_\phi(y,\theta_* t)$.

b. (Volume estimates) There exist constants c_*,C_1,C_2 depending only on ρ,λ,Λ and n such that for any section $S_\phi(x,t)$ with $x\in\overline{\Omega}$ and $t\le c_*$, we have

$$C_1 t^{n/2}\le |S_\phi(x,t)|\le C_2 t^{n/2}.$$

Our next property is a dichotomy for sections of solutions to the Monge–Ampère equations: any section is either an interior section or included in a boundary section with a comparable height.

Proposition 2.14 (*Dichotomy*) *Assume that Ω and ϕ satisfy (2.1)–(2.3). Let $S_\phi(x,t)$ be a section of ϕ with $x\in\overline{\Omega}$ and $t>0$. Then one of the following is true:*

(i) $S_\phi(x,2t)$ is an interior section, that is, $S_\phi(x,2t)\subset\Omega$;

(ii) $S_\phi(x, 2t)$ *is included in a boundary section with comparable height, that is, there exists* $z \in \partial\Omega$ *such that* $S_\phi(x, 2t) \subset S_\phi(z, \bar{c}t)$ *for some constant* $\bar{c} = \bar{c}(\rho, \lambda, \Lambda, n) > 0$.

Our covering theorem states as follows.

Theorem 2.15 (Covering theorem) *Assume* Ω *and* ϕ *satisfy* (2.1)–(2.3). *Let* $\mathcal{O} \subset \overline{\Omega}$ *be a Lebesgue measurable set and* $\epsilon > 0$ *small. Suppose that for each* $x \in \mathcal{O}$ *a section* $S_\phi(x, t_x)$ *is given with*

$$\frac{|S_\phi(x, t_x) \cap \mathcal{O}|}{|S_\phi(x, t_x)|} = \epsilon.$$

Then if $\sup\{t_x \ : \ x \ \in \ \mathcal{O}\} < \infty$, *there exists a countable subfamily of sections* $\{S_\phi(x_k, t_k)\}_{k=1}^{\infty}$ *satisfying*

$$\mathcal{O} \subset \bigcup_{k=1}^{\infty} S_\phi(x_k, t_k) \quad \textit{and} \quad |\mathcal{O}| \leq \sqrt{\epsilon}\,\Big|\bigcup_{k=1}^{\infty} S_\phi(x_k, t_k)\Big|.$$

Finally, we have the following global strong-type $p - p$ estimates for the maximal function corresponding to small sections.

Theorem 2.16 (Strong-type p-p estimates) *Assume that* Ω *and* ϕ *satisfy* (2.1)–(2.3). *For* $f \in L^1(\Omega)$, *define*

$$\mathcal{M}(f)(x) = \sup_{t \leq c} \frac{1}{|S_\phi(x, t)|} \int_{S_\phi(x,t)} |f(y)|\, dy \quad \forall x \in \Omega.$$

Then, for any $1 < p < \infty$, *there exists* $C_p > 0$ *depending on* $p, \ \rho, \ \lambda, \ \Lambda$ *and* n *such that*

$$\|\mathcal{M}(f)\|_{L^p(\Omega)} \leq C_p \, \|f\|_{L^p(\Omega)}.$$

3 Global power decay and $W^{2,\delta}$ estimates

In this section, we establish preliminary power decay estimates for the distribution function of the second derivatives of solutions to the linearized Monge–Ampère equations and also their global $W^{2,\delta}$ estimates. We also show under suitable geometric conditions, the cofactor matrices of the Hessian matrices of two convex functions defined on the same domain are close if their Monge–Ampère measures and boundary values are close in the L^∞ norm.

We begin this section by recalling the definitions, introduced in [17], of the quasi distance $d(x, x_0)$ generated by a convex function ϕ and the set $G_M(u, \Omega)$ where the function u is touched from above and below by "quasi paraboloids" generated by this quasi distance.

Definition 3.1 Let Ω be a bounded convex set in $\mathbb{R}^n$ and let $\phi \in C^1(\Omega)$ be a convex function. For any $x \in \Omega$ and $x_0 \in \Omega$, we define the quasi distance $d(x, x_0)$ by

$$d(x, x_0)^2 := \phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0).$$

Definition 3.2 Let Ω and ϕ be as in Definition 3.1. For $u \in C(\Omega)$ and $M > 0$, we define

$$G_M(u, \Omega) = \left\{ \bar{x} \in \Omega : u \text{ is differentiable at } \bar{x} \text{ and } |u(x) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x})| \right.$$
$$\left. \leq \frac{M}{2} d(x, \bar{x})^2 \; \forall x \in \Omega \right\}.$$

We call $\frac{M}{2} d(x, \bar{x})^2$ and $-\frac{M}{2} d(x, \bar{x})^2$ quasi paraboloids of opening M generated by ϕ. When we would like to emphasize the dependence of $d(x, x_0)$ on ϕ, we write $d_\phi(x, x_0)$. Likewise, we write $G_M(u, \Omega, \phi)$ to indicate the dependence on ϕ of the set $G_M(u, \Omega)$. Notice that for $\phi(x) = |x|^2$, we have $d(x, \bar{x}) = |x - \bar{x}|$ is the Euclidean distance.

In the next lemma, we show that if the quasi distance $d(x, x_0)$ is bounded from below by the Euclidean distance $|x - x_0|$ around x_0 then it is also bounded from above by a multiple of this Euclidean distance around x_0. This lemma is a slight modification of [15, Lemma 6.2.1].

Lemma 3.3 *Assume Ω satisfies (2.1) and let $\phi \in C(\overline{\Omega})$ be a convex function satisfying $\lambda \leq \det D^2\phi \leq \Lambda$ in Ω and $\phi = 0$ on $\partial\Omega$. There exists $c = c(n, \lambda, \Lambda, \rho) > 0$ such that if $x_0 \in \Omega$ and*

$$d(x, x_0)^2 \geq \sigma\, |x - x_0|^2 \;\; \textit{in} \;\; B_r(x_0) \subset \Omega \;\; \textit{for some } r > 0,$$

then for all x in a small neighborhood of x_0, we have

$$d(x, x_0)^2 \leq \frac{1}{c^2 \sigma^{n-1}}\, |x - x_0|^2.$$

Proof Let $\varphi(x) := \phi(x) - \phi(x_0) - \nabla\phi(x_0) \cdot (x - x_0)$. Then the strict convexity of ϕ implies that there exists $\delta > 0$ such that $S_\phi(x_0, \delta) := \{x \in \Omega : \varphi(x) < \delta\} \subset B_r(x_0)$. Therefore by the proof of Lemma 6.2.1 in [15], we have $\varphi(x) \leq C(n, \lambda, \Lambda, \rho)\sigma^{-n+1}\, |x - x_0|^2$ for all $x \in \Omega$ satisfying $\varphi(x) \leq \delta$, which gives the conclusion of the lemma. □

The following lemma allows us to estimate the distribution function of D^2u. It is the starting point for our proofs of Theorems 1.1 and 1.2 and the global version of [17, Lemma 2.7].

Lemma 3.4 *Let Ω, ϕ and c be as in Lemma 3.3, and $u \in C^2(\Omega)$. Define*

$$A_\sigma^{loc} := \left\{ x_0 \in \Omega : d(x, x_0)^2 \geq \sigma\, |x - x_0|^2, \textit{for all } x \textit{ in some neighborhood of } x_0 \right\}.$$

Then for any $m > 1$ and $\beta > 0$, we have

$$\left\{x \in \Omega : |D_{ij}u(x)| > \beta^m\right\} \subset \left(\Omega \setminus A^{loc}_{\left(c\beta^{\frac{m-1}{2}}\right)^{\frac{-2}{n-1}}}\right) \cup \left(\Omega \setminus G_\beta(u,\Omega)\right). \quad (3.1)$$

Proof Let $\gamma := \beta^{\frac{m-1}{2}}$. If $\bar{x} \in A^{\text{loc}}_{(c\gamma)^{\frac{-2}{n-1}}} \cap G_\beta(u,\Omega)$, then

$$-\frac{\beta}{2} d(x,\bar{x})^2 \le u(x) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x}) \le \frac{\beta}{2} d(x,\bar{x})^2$$

for each $x \in \Omega$. Since $\bar{x} \in A^{\text{loc}}_{(c\gamma)^{\frac{-2}{n-1}}}$, these together with Lemma 3.3 yield

$$-\frac{\beta\gamma^2}{2}|x-\bar{x}|^2 \le u(x) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x}) \le \frac{\beta\gamma^2}{2}|x-\bar{x}|^2$$

for all x in a small neighborhood of $\bar{x}$, and so $|D_{ij}u(\bar{x})| \le \beta\gamma^2 = \beta^m$. Thus we have proved that

$$A^{\text{loc}}_{(c\gamma)^{\frac{-2}{n-1}}} \cap G_\beta(u,\Omega) \subset \left\{x \in \Omega : |D_{ij}u(x)| \le \beta^m, \text{ for } i,j = 1,\ldots,n\right\}$$

and the lemma follows by taking complements. □

3.1 Power decay estimates

In order to derive global $W^{2,p}$ estimates for solutions u to the linearized Monge–Ampère equation, we will need to estimate the distribution function

$$F(\beta) := \left|\{x \in \Omega : |D_{ij}u(x)| > \beta^m\}\right|$$

for some suitable choice of $m > 1$. It follows from Lemma 3.4 that this can be done if one can get appropriate decay estimates for

$$F_1(\beta) := |\Omega \setminus A^{\text{loc}}_{(c\beta^{\frac{m-1}{2}})^{\frac{-2}{n-1}}}| \quad \text{and} \quad F_2(\beta) := |\Omega \setminus G_\beta(u,\Omega)|.$$

Notice that the function $F_1(\beta)$ involves only the solution ϕ of the Monge–Ampère equation and its power decay is given in the next theorem.

Theorem 3.5 *Assume Ω satisfies (2.1) and $\partial\Omega \in C^{1,1}$. Let $\phi \in C(\overline{\Omega})$ be a convex function such that $1-\epsilon \le \det D^2\phi \le 1+\epsilon$ in Ω and (2.3) holds, where $0 < \epsilon < 1/2$. Then there exists a positive constant M depending only on n and ρ such that*

$$\left|\Omega \setminus A^{loc}_{s^{-2}}\right| \le C'(\epsilon, n, \rho, \|\partial\Omega\|_{C^{1,1}}) \; s^{\frac{\ln \sqrt{C\epsilon}}{\ln M}} \quad \textit{for all} \quad s > 0. \tag{3.2}$$

In particular, for $s = (c\beta^{\frac{m-1}{2}})^{\frac{1}{n-1}}$, *we get*

$$F_1(\beta) \le C'(\epsilon, n, \rho, \|\partial\Omega\|_{C^{1,1}})\, \beta^{-\frac{m-1}{2(n-1)\ln M}\ln\frac{1}{\sqrt{C\epsilon}}} \quad \forall \beta > 0.$$

The small power decay estimates for $F_2(\beta)$ are given in the following proposition. It is the boundary version of Proposition 3.4 in [17].

Proposition 3.6 *Assume that* Ω *and* ϕ *satisfy the assumptions* (2.1)–(2.3). *Assume in addition that* $\partial\Omega \in C^{1,1}$. *Suppose* $u \in C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$, $|u| \le 1$ *in* Ω *and* $\mathcal{L}_\phi u = f$ *in* Ω *with* $\|f\|_{L^n(\Omega)} \le 1$. *Then there exist* $\tau = \tau(n, \lambda, \Lambda, \rho) \in (0, 1/2)$ *and* $C = C(\rho, \lambda, \Lambda, n, \|\partial\Omega\|_{C^{1,1}}) > 0$ *such that*

$$F_2(\beta) = \left|\Omega \setminus G_\beta(u, \Omega)\right| \le \frac{C}{\beta^\tau} \quad \textit{for all} \quad \beta > 0.$$

The next result is a variant of Proposition 3.6 which will be important for the density and improved power decay estimates in Sect. 5.1.

Proposition 3.7 *Let* (Ω, ϕ, U) *be in the class* $\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$. *Suppose* $u \in C(\Omega) \cap C^1(U) \cap W^{2,n}_{loc}(U)$, $|u| \le 1$ *in* Ω *and* $\mathcal{L}_\phi u = f$ *in* U *with* $\|f\|_{L^n(U\cap B_c)} \le 1$. *Then there exist* $\tau = \tau(n, \lambda, \Lambda, \rho) \in (0, 1/2)$ *and* $C = C(n, \lambda, \Lambda, \rho, \kappa) > 0$ *such that*

$$\left|(U \cap B_{c^2}) \setminus G_\beta(u, \Omega)\right| \le \frac{C}{\beta^\tau}\, |U \cap B_{c^2}| \quad \textit{for all} \quad \beta > 0.$$

The above inequality also holds if $U \cap B_{c^2}$ *is replaced by* $S_\phi(0, r)$ *for any universal constant* r *satisfying* $r \le c^6$.

As a consequence of the power decay estimates for $F_1(\beta)$ and $F_2(\beta)$ in Theorem 3.5 and Proposition 3.6, we find that the decay for $F(\beta)$ when $0 < \epsilon < 1/2$ is given by

$$F(\beta) \le C(\epsilon, n, \rho, \|\partial\Omega\|_{C^{1,1}})\, \beta^{-\frac{m-1}{2(n-1)\ln M}\ln\frac{1}{\sqrt{C\epsilon}}} + C\beta^{-\tau}.$$

Since $\frac{m-1}{2(n-1)\ln M}\ln\frac{1}{\sqrt{C\epsilon}} \to \infty$ as $\epsilon \to 0$, we obtain global $W^{2,\delta}$ estimates for all $\delta < \tau/m < 1/2$ for solutions to the linearized Monge–Ampère equation $\mathcal{L}_\phi u = f$ provided that $f \in L^n(\Omega)$ and ϵ is small, that is, $\det D^2\phi$ is close to a constant. However, in the next subsection, we offer a more direct proof of global $W^{2,\delta}$ estimates based on interior estimates without resorting to decay estimates of the distribution function of the second derivatives. Another advantage of this proof is that it works for all Monge–Ampère measures $\det D^2\phi$ bounded away from 0 and ∞.

Remark 3.8 It is now clear that the obstruction to higher integrability of $|D^2u|$ is the small exponent τ in the decay estimates for $\left|\Omega \setminus G_\beta(u, \Omega)\right|$ given by Proposition 3.6.

Most of the paper is devoted to developing tools to improve the decay estimates for $\left|\Omega \backslash G_\beta(u, \Omega)\right|$. In particular, the global stability of cofactor matrices and an approximation lemma in the next two sections will be employed for this purpose.

3.2 Global $W^{2,\delta}$ estimates

In this subsection, we obtain global $W^{2,\delta}(\Omega)$ estimates ($\delta > 0$ small) for solutions to the linearized Monge–Ampère equation $\mathcal{L}_\phi u = f$ when $\det D^2\phi$ is only bounded away from 0 and ∞ and under natural assumptions on the domain Ω and the boundary data of ϕ.

Our main theorem in this subsection is the following.

Theorem 3.9 *Assume Ω and ϕ satisfy the assumptions* (2.1)–(2.3). *Assume in addition that $\partial\Omega \in C^{1,1}$. Let $u \in C(\overline{\Omega}) \cap C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$ be a solution of*

$$\begin{cases} \mathcal{L}_\phi u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exist $p = p(\rho, \lambda, \Lambda, n) > 0$ and $C = C(\rho, \lambda, \Lambda, n, \|\partial\Omega\|_{C^{1,1}}) > 0$ such that

$$\|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^n(\Omega)}.$$

The rest of this subsection is devoted to proving this theorem. The idea is to cover Ω by maximal interior sections whose shapes are under control by Proposition 2.3 and then apply the interior $W^{2,\delta}$ estimates of Gutiérrez and Tournier [18] in these sections. Furthermore, since we can control the number of these sections within certain height due to the $C^{1,1}$ regularity of the boundary $\partial\Omega$, the global estimates follow by adding interior ones.

For reader's convenience, we recall Gutiérrez-Tournier's $W^{2,\delta}$ estimates.

Theorem 3.10 ([18, Theorem 6.3]) *Let Ω be a convex domain such that $B_{k_0} \subset \Omega \subset B_{k_0^{-1}}$. Let $\phi \in C^2(\Omega)$ be a convex function satisfying $\lambda \leq \det D^2\phi \leq \Lambda$ in Ω and $\phi = 0$ on $\partial\Omega$. Let $u \in C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$ be a solution of $\mathcal{L}_\phi u = f$ in Ω. Then, given $\alpha_0 \in (0, 1)$, there exist positive constants δ and C depending only on $\alpha_0, k_0, \lambda, \Lambda$ and n such that*

$$\|D^2u\|_{L^\delta\left(S_\phi(x_0, -\alpha_0\phi(x_0))\right)} \leq C\Big(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}\Big),$$

where $x_0 \in \Omega$ is such that $\min_\Omega \phi = \phi(x_0)$.

Let $0 < p < \min\{\delta, \frac{1}{2}\}$ where $\delta = \delta(\rho, \lambda, \Lambda, n) > 0$ is a small number appearing in Theorem 3.10 corresponding to $\alpha_0 = 1/2$ and $k_0 = k_0(\rho, n, \lambda, \Lambda)$ given by Proposition 2.3.

We will show that the conclusion of Theorem 3.9 holds for the above choice of p. To achieve this, we first estimate the L^p norm of D^2u in the interior of each maximal interior section.

Lemma 3.11 *Assume Ω and ϕ satisfy the assumptions (2.1)–(2.3). Let $u \in C(\overline{\Omega}) \cap C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$ be a solution of*

$$\mathcal{L}_\phi u = f \text{ in } \Omega, \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

Then, there exists a constant $C > 0$ depending only on $p, \rho, \lambda, \Lambda$ and n such that

$$\begin{aligned}
&\|D^2 u\|_{L^p\left(S_\phi\left(y,\frac{\bar{h}(y)}{2}\right)\right)} \\
&\quad \le C\bar{h}(y)^{\frac{n}{2p}-1} |\log \bar{h}(y)|^2 \left(\|u\|_{L^\infty\left(S_\phi(y,\bar{h}(y))\right)} + \bar{h}(y)^{1/2}\|f\|_{L^n\left(S_\phi(y,\bar{h}(y))\right)}\right)
\end{aligned}$$

for all $y \in \Omega$ satisfying $\bar{h}(y) \le c$.

Proof Let $h := \bar{h}(y)$ with $\bar{h}(y) \le c$. We now define the rescaled domain $\tilde{\Omega}_h$ and rescaled functions $\tilde{\phi}_h$, $\tilde{u}_h$ and $\tilde{f}_h$ as in Sect. 2.2 that *preserve the L^∞-norm in a section tangent to the boundary*. For simplicity, let us denote $\tilde{S}_t(0) := S_{\tilde{\phi}_h}(0, t)$ for $t > 0$. Then by Theorem 3.10, we have

$$\|D^2 \tilde{u}_h\|_{L^p\left(\tilde{S}_{\frac{1}{2}}(0)\right)} \le C(p, \rho, \lambda, \Lambda, n)\left(\|\tilde{u}_h\|_{L^\infty(\tilde{S}_1(0))} + \|\tilde{f}_h\|_{L^n(\tilde{S}_1(0))}\right). \tag{3.3}$$

Using the fact

$$D^2 u\left(y + h^{1/2} A_h^{-1} x\right) = h^{-1}(A_h)^t \, D^2 \tilde{u}_h(x)\, A_h,$$

we obtain

$$\begin{aligned}
\int_{S_\phi\left(y,\frac{h}{2}\right)} |D^2 u(z)|^p \, dz &= h^{\frac{n}{2}-p} \int_{\tilde{S}_{\frac{1}{2}}(0)} |A_h^t \, D^2 \tilde{u}_h(x)\, A_h|^p \, dx \\
&\le C\, h^{\frac{n}{2}-p} \,|\log h|^{2p} \int_{\tilde{S}_{\frac{1}{2}}(0)} |D^2 \tilde{u}_h(x)|^p \, dx.
\end{aligned}$$

It follows that

$$\|D^2 u\|_{L^p\left(S_\phi\left(y,\frac{h}{2}\right)\right)} \le C h^{\frac{n}{2p}-1} |\log h|^2 \, \|D^2 \tilde{u}_h\|_{L^p\left(\tilde{S}_{\frac{1}{2}}(0)\right)}. \tag{3.4}$$

Moreover, we have

$$\|\tilde{f}_h\|_{L^n(\tilde{S}_1(0))} = h^{\frac{1}{2}} \|f\|_{L^n(S_\phi(y,h))} \quad \text{and} \quad \|\tilde{u}_h\|_{L^\infty(\tilde{S}_1(0))} = \|u\|_{L^\infty(S_\phi(y,h))}. \tag{3.5}$$

Combining (3.3)–(3.5), we obtain the desired estimate stated in our lemma. □

Finally, we will use the following Vitali covering lemma proved by Savin in [28]; see also [22, Lemma 2.5] for a more general covering result.

Lemma 3.12 ([28, Lemma 2.3]) *Assume* Ω *and* ϕ *satisfy the assumptions* (2.1)–(2.3)*. Then there exists a sequence of disjoint sections* $S_\phi(y_i, \delta_0\bar{h}(y_i))$ *with* $\delta_0 = \delta_0(\lambda, \Lambda, n) > 0$ *such that*

$$\Omega \subset \bigcup_{i=1}^{\infty} S_\phi\left(y_i, \frac{\bar{h}(y_i)}{2}\right).$$

Proof of Theorem 3.9. It follows from Proposition 2.3 (see also [28, Lemma 2.2]) that if $y \in \Omega$ with $\bar{h}(y) \le c$ then

$$\begin{aligned} S_\phi(y, \bar{h}(y)) &\subset y + k_0^{-1}E_h \subset D_{C\bar{h}(y)^{1/2}} \\ &:= \left\{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \le C\bar{h}(y)^{1/2}\right\}, \quad C := 2k_0^{-2}. \end{aligned}$$

By Lemma 3.12, we have

$$\int_\Omega |D^2u|^p dx \le \sum_{i=1}^{\infty} \int_{S_\phi\left(y_i, \frac{\bar{h}(y_i)}{2}\right)} |D^2u|^p dx.$$

There is a finite number of sections $S_\phi(y_i, \bar{h}(y_i))$ with $\bar{h}(y_i) \ge c$ and, by Theorem 3.10, we have in each such section

$$\int_{S_\phi\left(y_i, \frac{\bar{h}(y_i)}{2}\right)} |D^2u|^p \le C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}\right)^p.$$

Now, for $d \le c$ we consider the family $\mathcal{F}_d$ of sections $S_\phi(y_i, \bar{h}(y_i)/2)$ such that $d/2 < \bar{h}(y_i) \le d$. Let M_d be the number of sections in $\mathcal{F}_d$. We claim that

$$M_d \le C_b d^{\frac{1}{2}-\frac{n}{2}} \tag{3.6}$$

for some constant C_b depending only on $\rho, n, \lambda, \Lambda$ and $\|\partial\Omega\|_{C^{1,1}}$. Indeed, we first note that, by [15, Corollary 3.2.4] (see also Theorem 2.13(b)), there exists a constant $C = C(n, \lambda, \Lambda, \rho) > 0$ such that

$$|S_\phi(y_i, \delta_0\bar{h}(y_i))| \ge C\bar{h}(y_i)^{n/2} \ge Cd^{n/2}.$$

Since $S_\phi(y_i, \delta_0\bar{h}(y_i)) \subset D_{Cd^{1/2}}$ are disjoint, we find that

$$M_d C d^{n/2} \le \sum_{i \in \mathcal{F}_d} |S_\phi(y_i, \delta_0\bar{h}(y_i))| \le |D_{Cd^{1/2}}| \le C_* d^{1/2}$$

for some constant C_* depending only on n and $\|\partial\Omega\|_{C^{1,1}}$. Thus (3.6) holds.

It follows from Lemma 3.11 and (3.6) that

$$\sum_{i\in\mathcal{F}_d}\int_{S_\phi\left(y_i,\frac{\bar{h}(y_i)}{2}\right)}|D^2u|^p \le CM_d d^{\frac{n}{2}-p}|\log d|^{2p}\left(\|u\|_{L^\infty(\Omega)}+\|f\|_{L^n(\Omega)}\right)^p$$
$$\le Cd^{\frac{1}{2}-p}|\log d|^{2p}\left(\|u\|_{L^\infty(\Omega)}+\|f\|_{L^n(\Omega)}\right)^p.$$

Adding these inequalities for the sequence $d = c2^{-k}$, $k = 0, 1, 2, \ldots$, and noting that

$$\|u\|_{L^\infty(\Omega)} \le C(n, \rho, \lambda, \Lambda)\|f\|_{L^n(\Omega)},$$

by the ABP estimate, we obtain the desired global L^p estimate for D^2u. □

3.3 Proofs of the power decay estimates

Proof of Theorem 3.5. Let $\{S_\phi(y_i, \bar{h}(y_i)/2)\}$ be the sequence of sections covering Ω given by Lemma 3.12. In what follows we will use the notations as in the proof of Lemma 3.11. We then have

$$\begin{aligned}\left|\Omega \setminus A^{\text{loc}}_{s-2}\right| &\le \sum_{i=1}^{\infty}\left|S_\phi(y_i, \bar{h}(y_i)/2) \setminus A^{\text{loc}}_{s-2}\right|\\ &\le \sum_{k=0}^{\infty}\sum_{i\in\mathcal{F}_{c2^{-k}}}\left|S_\phi(y_i, \bar{h}(y_i)/2) \setminus A^{\text{loc}}_{s-2}\right|\\ &\quad + \sum_{i:\bar{h}(y_i)>c}\left|S_\phi(y_i, \bar{h}(y_i)/2) \setminus A^{\text{loc}}_{s-2}\right| =: I + II. \end{aligned} \tag{3.7}$$

Let us first estimate the summation I corresponding to sections with $\bar{h}(y_i) \le c$. Consider a prototype section $S_\phi(y, h)$ with $h := \bar{h}(y) \le c$. Proposition 2.3 tells us that $S_\phi(y, h)$ is equivalent to an ellipsoid E_h, i.e.,

$$k_0E_h \subset S_\phi(y, h) - y \subset k_0^{-1}E_h,$$

where

$$E_h := h^{1/2}A_h^{-1}B_1, \quad \text{with} \quad \det A_h = 1, \quad \|A_h\|, \|A_h^{-1}\| \le k^{-1}|\log h|.$$

Here k, k_0 depend only on n and ρ. Let $T(x) := h^{-1/2}A_h(x - y)$. Define $\tilde{U}_h := T(S_\phi(y, h))$ and

$$\tilde{\phi}_h(z) := h^{-1}\Big[\phi(T^{-1}z) - \phi(y) - \nabla\phi(y)\cdot(T^{-1}z - y) - h\Big] \quad \text{for } z \in \tilde{U}_h.$$

Then $B_{k_0} \subset \tilde{U}_h \equiv S_{\tilde{\phi}_h}(0,1) \subset B_{k_0^{-1}}$, $1-\epsilon \le \det D^2\tilde{\phi}_h \le 1+\epsilon$ in $\tilde{U}_h$ and $\tilde{\phi}_h = 0$ on $\partial\tilde{U}_h$. By [15, Theorem 3.3.10], there exists $\eta_0 = \eta_0(n,\rho) > 0$ such that

$$S_{\tilde{\phi}_h}(x,t) \Subset \tilde{U}_h \quad \text{for all } x \in S_{\tilde{\phi}_h}(0,1/2) \quad \text{and} \quad t \le \eta_0.$$

Now, let

$$\tilde{D}_s^{\frac{1}{2}} := \left\{x \in S_{\tilde{\phi}_h}(0,1/2) : S_{\tilde{\phi}_h}(x,t) \subset B(x, s\sqrt{t}),\ \forall t \le \eta_0\right\}.$$

Then, by [17, Theorem 2.8], we obtain

$$|S_{\tilde{\phi}_h}(0,1/2) \setminus \tilde{D}_s^{\frac{1}{2}}| \le \frac{|\tilde{U}_h|}{(C\epsilon)^2} s^{-p_\epsilon},$$

where $p_\epsilon := -\dfrac{\ln\sqrt{C\epsilon}}{\ln M}$ with $C, M > 0$ is a constant depending only on n and ρ. Let

$$\tilde{A}_\sigma := \left\{\bar{z} \in \tilde{U}_h : \tilde{\phi}_h(z) \ge \tilde{\phi}_h(\bar{z}) + \nabla\tilde{\phi}_h(\bar{z}) \cdot (z-\bar{z}) + \sigma\,|z-\bar{z}|^2, \quad \forall z \in \tilde{U}_h\right\}.$$

Since $\tilde{D}_s^{\frac{1}{2}} = S_{\tilde{\phi}_h}(0,1/2) \cap \tilde{A}_{s^{-2}}$ by [15, Theorem 6.2.2], we can rewrite the above inequality as

$$|T\big(S_\phi(y,h/2)\big) \setminus \tilde{A}_{s^{-2}}| \le C(\epsilon,n,\rho)\, s^{-p_\epsilon}. \tag{3.8}$$

Let us relate $\tilde{A}_{s^{-2}}$ to A_σ^{loc}. Since $|x-\bar{x}| \le \|A_h^{-1}\|\,|A_h(x-\bar{x})| \le k^{-1}h^{1/2}|\log h|\,|Tx - T\bar{x}|$, we have

$$\begin{aligned}
\tilde{A}_{s^{-2}} &= T\Big\{\bar{x} \in S_\phi(y,h) : \tilde{\phi}_h(Tx) \ge \tilde{\phi}_h(T\bar{x}) + \nabla\tilde{\phi}_h(T\bar{x}) \cdot (Tx - T\bar{x}) \\
&\qquad + s^{-2}\,|Tx - T\bar{x}|^2,\ \forall x \in S_\phi(y,h)\Big\} \\
&\subset T\Big\{\bar{x} \in S_\phi(y,h) : \phi(x) \ge \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x-\bar{x}) \\
&\qquad + (k^{-1}s|\log h|)^{-2}\,|x-\bar{x}|^2,\ \forall x \in S_\phi(y,h)\Big\} \\
&\subset T\left(S_\phi(y,h) \cap A^{\text{loc}}_{(k^{-1}s|\log h|)^{-2}}\right).
\end{aligned}$$

We infer from this and (3.8) that

$$\begin{aligned}
|S_\phi(y,h/2) \setminus A^{\text{loc}}_{(k^{-1}s|\log h|)^{-2}}| &\le C(\epsilon,n,\rho)|\det T|^{-1}\, s^{-p_\epsilon} \\
&= C(\epsilon,n,\rho)\, h^{n/2}\, s^{-p_\epsilon} \quad \forall s > 0,
\end{aligned}$$

or equivalently,

$$|S_\phi(y, h/2) \setminus A^{\mathrm{loc}}_{s^{-2}}| \le C(\epsilon, n, \rho)\, h^{n/2} |\log h|^{p_\epsilon}\, s^{-p_\epsilon} \quad \forall s > 0.$$

Thus the summation I in (3.7) can be estimated as follows

$$\begin{aligned} I &\le C(\epsilon, n, \rho) s^{-p_\epsilon} \sum_{k=0}^{\infty} \sum_{i \in \mathcal{F}_{c2^{-k}}} \bar{h}(y_i)^{n/2} |\log \bar{h}(y_i)|^{p_\epsilon} \\ &\le C(\epsilon, n, \rho) s^{-p_\epsilon} \sum_{k=0}^{\infty} (c2^{-k})^{n/2} |\log(c2^{-k-1})|^{p_\epsilon} M_{c2^{-k}} \\ &\le C\, s^{-p_\epsilon} \sum_{k=0}^{\infty} (c2^{-k})^{1/2} |\log(c2^{-k-1})|^{p_\epsilon} \le C\, s^{-p_\epsilon}. \end{aligned} \tag{3.9}$$

Note that C depends on ϵ, n, ρ and $\|\partial\Omega\|_{C^{1,1}}$, and we have used the bound (3.6) for M_d to obtain the third inequality.

Next let us estimate the summation II corresponding to sections $S_\phi(y_i, \bar{h}(y_i)/2)$ with $\bar{h}(y_i) > c$. Since the family $\{S_\phi(y_i, \delta_0 \bar{h}(y_i))\}$ is disjoint, we infer from the lower bound on volume of sections and $\Omega \subset B_{1/\rho}$ that

$$\#\{i : \bar{h}(y_i) > c\} \le C(n, \rho).$$

Also, by using the standard normalization for interior sections and [17, Theorem 2.8] we get

$$|S_\phi(y_i, \bar{h}(y_i)/2) \setminus A^{\mathrm{loc}}_{s^{-2}}| \le C(\epsilon, n, \rho)\, s^{-p_\epsilon} \quad \text{for all } i \text{ with } \quad \bar{h}(y_i) > c.$$

Therefore,

$$II \le \#\{i : \bar{h}(y_i) > c\} \left[C(\epsilon, n, \rho)\, s^{-p_\epsilon} \right] \le C(\epsilon, n, \rho)\, s^{-p_\epsilon} \quad \forall s > 0. \tag{3.10}$$

By combining (3.7), (3.9) and (3.10) we obtain

$$\begin{aligned} \left|\Omega \setminus A^{\mathrm{loc}}_{s^{-2}}\right| &\le I + II \le C(\epsilon, n, \rho, \|\partial\Omega\|_{C^{1,1}})\, s^{-p_\epsilon} \\ &= C(\epsilon, n, \rho, \|\partial\Omega\|_{C^{1,1}})\ s^{\frac{\ln \sqrt{C\epsilon}}{\ln M}}. \end{aligned}$$

□

The proof of Theorem 3.5 can also be employed to give the proof of Proposition 3.6.

Proof of Proposition 3.6. Let $\{S_\phi(y_i, \bar{h}(y_i)/2)\}$ be the sequence of sections covering Ω given by Lemma 3.12. Then we have

$$\left|\Omega \setminus G_\beta(u,\Omega)\right| \le \sum_{i:\bar h(y_i)>c} \left|S_\phi\left(y_i, \frac{\bar h(y_i)}{2}\right) \setminus G_\beta(u,\Omega)\right| + \sum_{k=0}^{\infty} \sum_{i\in \mathcal{F}_{c2^{-k}}} \left|S_\phi\left(y_i, \frac{\bar h(y_i)}{2}\right) \setminus G_\beta(u,\Omega)\right|. \tag{3.11}$$

By using [17, Proposition 3.4] and arguing as in estimating the term II in the proof of Theorem 3.5, we see that there exist constants $C, \tau > 0$ depending only on n, λ, Λ and ρ with $\tau < 1/2$ such that

$$\sum_{i:\bar h(y_i)>c} \left|S_\phi\left(y_i, \frac{\bar h(y_i)}{2}\right) \setminus G_\beta(u,\Omega)\right| \le \sum_{i:\bar h(y_i)>c} \frac{C}{\beta^\tau} = \frac{C}{\beta^\tau}\#\{i : \bar h(y_i) > c\} \le \frac{C}{\beta^\tau}. \tag{3.12}$$

To estimate the last expression in (3.11), let us consider a prototype section $S_\phi(y,h)$ with $h := \bar h(y) \le c$. We now define the rescaled domains $\tilde\Omega_h, \tilde U_h$ and rescaled functions $\tilde\phi_h, \tilde u_h$ and $\tilde f_h$ as in Sect. 2.2 that *preserve the L^∞-norm in a section tangent to the boundary*. Then

$$\|\tilde f_h\|_{L^n(\tilde U_h)} = h^{1/2}\|f\|_{L^n(S_\phi(y,h))} \le h^{1/2}\|f\|_{L^n(\Omega)} \le 1. \tag{3.13}$$

Therefore, we can apply [17, Proposition 3.4] to obtain for $T(x) := h^{-1/2}A_h(x-y)$

$$\begin{aligned}\left|T(S_\phi(y,h/2)) \setminus G_\beta(\tilde u_h, \tilde\Omega_h, \tilde\phi_h)\right| &= \left|S_{\tilde\phi_h}(0,1/2) \setminus G_\beta(\tilde u_h, \tilde\Omega_h, \tilde\phi_h)\right| \\ &\le \frac{C}{\beta^\tau} \quad \text{for all} \quad \beta > 0.\end{aligned}$$

But as $\tilde u_h \in C^1(\tilde U_h)$ and $d_{\tilde\phi_h}(Tx, T\bar x)^2 = h^{-1}d(x,\bar x)^2$ for all $x, \bar x \in \Omega$, we get

$$T(S_\phi(y,h/2)) \cap G_\beta(\tilde u_h, \tilde\Omega_h, \tilde\phi_h) = T\Big(S_\phi(y,h/2) \cap G_{\beta h^{-1}}(u,\Omega)\Big).$$

Thus we infer from the above inequality that

$$\left|S_\phi(y,h/2) \setminus G_{\beta h^{-1}}(u,\Omega)\right| \le \frac{C}{\beta^\tau}|\det T|^{-1} = \frac{C}{\beta^\tau}h^{\frac{n}{2}},$$

or equivalently,

$$\left|S_\phi(y,h/2) \setminus G_\beta(u,\Omega)\right| \le \frac{C}{\beta^\tau}h^{\frac{n}{2}-\tau} \quad \text{for all} \quad \beta > 0.$$

This together with the estimate (3.6) for M_d yields

$$\sum_{k=0}^{\infty}\sum_{i\in\mathcal{F}_{c2^{-k}}}\left|S_\phi(y_i,\bar h(y_i)/2)\setminus G_\beta(u,\Omega)\right|$$
$$\le \frac{C}{\beta^\tau}\sum_{k=0}^{\infty}\sum_{i\in\mathcal{F}_{c2^{-k}}}\bar h(y_i)^{\frac{n}{2}-\tau}\le\frac{C}{\beta^\tau}\sum_{k=0}^{\infty}(c2^{-k})^{\frac{n}{2}-\tau}M_{c2^{-k}}\le\frac{C'}{\beta^\tau}\sum_{k=0}^{\infty}(c2^{-k})^{\frac{1}{2}-\tau}\le\frac{C'}{\beta^\tau} \tag{3.14}$$

provided that $\tau < 1/2$. Here C' also depends on $\|\partial\Omega\|_{C^{1,1}}$. The desired estimate is now obtained by combining (3.11), (3.12) and (3.14). □

To prove Proposition 3.7, we use the following localized version at the boundary of Lemma 3.12.

Lemma 3.13 *Assume $(\Omega,\phi,U)\in\mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$ and let w be the solution to (2.24). Let ψ denote one of the functions ϕ and w. Then there exists a sequence of disjoint sections $\{S_\psi(y_i,\delta_0\bar h(y_i))\}_{i=1}^{\infty}$, where $\delta_0=\delta_0(n,\lambda,\Lambda)$, $y_i\in U\cap B_{c^2}$ and $S_\psi(y_i,\bar h(y_i))$ is the maximal interior section of ψ in U, such that*

$$U\cap B_{c^2}\subset\bigcup_{i=1}^{\infty}S_\psi\left(y_i,\frac{\bar h(y_i)}{2}\right). \tag{3.15}$$

Moreover, we have

$$S_\psi(y_i,\bar h(y_i))\subset U\cap B_c,\quad \bar h(y_i)\le c. \tag{3.16}$$

If we let M_d^{loc} denote the number of sections $S_\psi(y_i,\bar h(y_i)/2)$ such that $d/2<\bar h(y_i)\le d\le c$, then

$$M_d^{loc}\le C_b d^{\frac{1}{2}-\frac{n}{2}} \tag{3.17}$$

for some constant C_b depending only on $\rho, n, \lambda, \Lambda$ and $\|\partial\Omega\cap B_\rho\|_{C^{1,1}}$.

Proof By Remark 2.11, we can use Proposition 2.3 to get the same conclusion as in Lemma 2.5(e) for sections of ψ with centers in $U\cap B_{c^2}$. All these sections thus satisfy (3.16) and are equivalent to ellipsoids. In particular, ψ is strictly convex in $U\cap B_c$. Furthermore,

$$S_\psi(y_i,\bar h(y_i))\subset\left\{x\in B_c\cap U:\operatorname{dist}(x,\partial\Omega\cap\partial U)\le 2k_0^{-1}\bar h(y_i)^{1/2}\right\}.$$

With this in mind and assuming that the sequence $\{S_\psi(y_i,\delta_0\bar h(y_i))\}_{i=1}^{\infty}$ is disjoint and satisfies (3.15), we argue similarly as in deriving the estimate (3.6) for M_d to obtain (3.17).

It remains to establish the covering (3.15). The crucial point in the proof of Lemma 3.12 is the engulfing property of interior sections which hold for strictly convex solution to the Monge–Ampère equation with bounded right hand side. By our discussion above, ψ is strictly convex in $U \cap B_c$ and thus we obtain (3.15). For completeness, we include the proof here, taken almost verbatim from [28]. By the engulfing property of interior sections of strictly convex solution to the Monge–Ampère equation with bounded right hand side, we can choose δ_0 depending only on n, λ, Λ with the following property. If $y, z \in B_{c^2} \cap U$ with

$$S_\psi(y, \delta_0 \bar{h}(y)) \cap S_\psi(z, \delta_0 \bar{h}(z)) \neq \emptyset \quad \text{and} \quad 2\bar{h}(y) \geq \bar{h}(z)$$

then

$$S_\psi(z, \delta_0 \bar{h}(z)) \subset S_\psi(y, \bar{h}(y)/2).$$

We choose $S_\psi(y_1, \delta_0 \bar{h}(y_1))$ from all sections $S_\psi(y, \delta_0 \bar{h}(y))$, $y \in U \cap B_{c^2}$ such that

$$\bar{h}(y_1) \geq \frac{1}{2} \sup_y \bar{h}(y)$$

then choose $S_\psi(y_2, \delta_0 \bar{h}(y_2))$ as above but only from the remaining sections $S_\psi(y, \delta_0 \bar{h}(y))$ that are disjoint from $S_\psi(y_1, \delta_0 \bar{h}(y_1))$, then $S_\psi(y_3, \delta_0 \bar{h}(y_3))$, etc. Consequently, we easily obtain

$$U \cap B_{c^2} \subset \bigcup_{y \in U \cap B_{c^2}} S_\psi(y, \delta_0 \bar{h}(y)) \subset \bigcup_{i=1}^{\infty} S_\psi(y_i, \delta_0 \bar{h}(y_i)).$$

□

Proof of Proposition 3.7. Our proof is similar to that of Proposition 3.6 using Lemma 3.13. In the proof of Proposition 3.6, we replace $\Omega \backslash G_\beta(u, \Omega)$ by $(U \cap B_{c^2}) \backslash G_\beta(u, \Omega)$, the covering of Ω using Lemma 3.12 by the covering of $U \cap B_{c^2}$ using Lemma 3.13. By (3.16), the first term of the right hand side of (3.11) disappears. For the second term of the right hand side of (3.11), we estimate as in the rest of the proof of Proposition 3.6. Note that, since all sections in the covering for $U \cap B_{c^2}$ satisfy $S_\phi(y_i, \bar{h}(y_i)) \subset B_c \cap U$, instead of (3.13), we now have

$$\|\tilde{f}\|_{L^n(T(S_\phi(y,h)))} = h^{1/2} \|f\|_{L^n(S_\phi(y,h))} \leq h^{1/2} \|f\|_{L^n(U \cap B_c)} \leq 1.$$

In (3.14), we replace M_d by M_d^{loc} and use (3.17) to estimate it. The conclusion of Proposition 3.7 follows. Note that by (2.7), we have $S_\phi(0, r) \subset U \cap B_{c^2}$ if $r \leq c^6$ and the last remark of the proposition follows. □

3.4 Global stability of cofactor matrices

In this subsection, we prove that, under suitable geometric conditions, the cofactor matrices of the Hessian matrices of two convex functions defined on the same domain are close if their Monge–Ampère measures and boundary values are close in the L^∞ norm.

We first start with a stability result at the boundary for the second derivatives and the cofactor matrices of functions in the class $\mathcal{P}$.

Proposition 3.14 *Assume* $(\Omega, \phi, U) \in \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,\kappa,*}$. *Let* $w \in C(\overline{U})$ *be the convex solution to*

$$\begin{cases} \det D^2 w = 1 & \textit{in } U \\ w = \phi & \textit{on } \partial U. \end{cases}$$

Then the following statements hold.

(i) For any $p > 1$, *there exist* $\epsilon_0 = \epsilon_0(p, n, \rho) > 0$ *and* $C = C(p, n, \rho, \kappa) > 0$ *such that*

$$\|D^2\phi - D^2 w\|_{L^p(B_{c^2}\cap U)} \leq C\epsilon^{\frac{\delta}{n(2p-\delta)}} \quad \textit{for all } \epsilon \leq \epsilon_0.$$

(ii) Assume in addition that $(\Omega, \phi, U) \in \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,\kappa,\alpha}$. *Then for any* $q \geq 1$, *there exist* $\epsilon_0 = \epsilon_0(q, n, \rho) > 0$ *and* $C = C(q, n, \rho, \kappa, \alpha) > 0$ *such that*

$$\|\Phi - \mathrm{W}\|_{L^q(B_{c^2}\cap U)} \leq C\epsilon^{\frac{(n-1)\delta}{n(2nq-\delta)}} \quad \textit{for all } \epsilon \leq \epsilon_0.$$

Here $\delta = \delta(n, \rho) > 0$, *and* Φ, W *are the matrices of cofactors of* $D^2\phi$ *and* $D^2 w$, *respectively.*

Proof (i) Our conclusion follows from the following claims.
Claim 1. There exist $\epsilon_0 = \epsilon_0(p, n, \rho) > 0$ small and $C_0 = C_0(p, n, \rho, \kappa) > 0$ such that

$$\|D^2\phi\|_{L^{2p}(B_{c^2}\cap U)} + \|D^2 w\|_{L^{2p}(B_{c^2}\cap U)} \leq C_0 \quad \text{whenever} \quad \epsilon \leq \epsilon_0.$$

Claim 2. There exist $\delta = \delta(n, \rho) \in (0, 1/2)$ and $C = C(n, \rho, \kappa) > 0$ such that

$$\|D^2\phi - D^2 w\|_{L^\delta(B_{c^2}\cap U)} \leq C\epsilon^{1/n} \quad \text{for all} \quad \epsilon < \frac{1}{2}. \tag{3.18}$$

Indeed, let $\theta \in (0, 1)$ be such that

$$\frac{1}{p} = \frac{\theta}{2p} + \frac{1-\theta}{\delta}.$$

Then $1 - \theta = \delta/(2p - \delta)$ and by the interpolation inequality we get

$$\begin{aligned} \|D^2\phi - D^2 w\|_{L^p(B_{c^2}\cap U)} &\leq \|D^2\phi - D^2 w\|^{\theta}_{L^{2p}(B_{c^2}\cap U)} \|D^2\phi - D^2 w\|^{1-\theta}_{L^\delta(B_{c^2}\cap U)} \\ &\leq C\epsilon^{\frac{1-\theta}{n}} = C\epsilon^{\frac{\delta}{n(2p-\delta)}}. \end{aligned}$$

We now turn to the proofs of the claims.

Claim 1 is essentially Savin's global $W^{2,p}$ estimates for the Monge–Ampère equations [28]. For the proof in our setting, we use Lemma 3.13 and follow his arguments. For completeness, we include the proof here. Let ψ denote one of the functions ϕ and w. Then by Lemma 3.13, there exists a sequence of disjoint sections $\{S_\psi(y_i, \delta_0 \bar{h}(y_i))\}_{i=1}^\infty$, where $y_i \in U \cap B_{c^2}$ and $S_\psi(y_i, \bar{h}(y_i))$ is the maximal interior section of ψ in U, such that

$$U \cap B_{c^2} \subset \bigcup_{i=1}^{\infty} S_\psi\left(y_i, \frac{\bar{h}(y_i)}{2}\right).$$

Moreover, we have

$$S_\psi(y_i, \bar{h}(y_i)) \subset U \cap B_c, \quad \bar{h}(y_i) \le c.$$

We will prove that: There exist $\epsilon_0 = \epsilon_0(p, \rho, n) > 0$ small and $C = C(p, \rho, n) > 0$ such that for $\epsilon \le \epsilon_0$, we have

$$\int_{S_\psi\left(y, \frac{\bar{h}(y)}{2}\right)} \left|D^2\psi\right|^{2p} \le C\, \bar{h}(y)^{\frac{n}{2}} |\log \bar{h}(y)|^{4p} \quad \forall y \in U \cap B_{c^2}. \tag{3.19}$$

Given this, we can complete the proof of **Claim 1** as follows. We have

$$\int_{U \cap B_{c^2}} \left|D^2\psi\right|^p \le \sum_{i=1}^{\infty} \int_{S_\psi\left(y_i, \frac{\bar{h}(y_i)}{2}\right)} \left|D^2\psi\right|^{2p} = \sum_{k=0}^{\infty} \sum_{i \in \mathcal{F}_{c2^{-k}}} \int_{S_\psi\left(y_i, \frac{\bar{h}(y_i)}{2}\right)} \left|D^2\psi\right|^{2p}, \tag{3.20}$$

where $\mathcal{F}_d$ is the family of sections $S_\psi(y_i, \frac{\bar{h}(y_i)}{2})$ such that $d/2 < \bar{h}(y_i) \le d \le c$. By (3.19), we have for each $S_\psi(y_i, \frac{\bar{h}(y_i)}{2}) \in \mathcal{F}_d$,

$$\int_{S_\psi\left(y, \frac{\bar{h}(y)}{2}\right)} \left|D^2\psi\right|^{2p} \le C\, |\log d|^{4p} \left|S_\psi(y_i, \delta_0 \bar{h}(y_i))\right|$$

and since

$$S_\psi(y_i, \delta_0 \bar{h}(y_i)) \subset \{x \in B_c \cap U : \operatorname{dist}(x, \partial\Omega \cap \partial U) \le 2k_0^{-1} d^{1/2}\}$$

are disjoint, we find

$$\sum_{i \in \mathcal{F}_d} \int_{S_\psi\left(y_i, \frac{\bar{h}(y_i)}{2}\right)} \left|D^2\psi\right|^{2p} \le C_1\, |\log d|^{4p}\, d^{1/2}$$

where C_1 now depends also on κ which is the upper bound for $\|\partial\Omega \cap B_c\|_{C^{1,1}}$. Therefore, **Claim 1** easily follows from (3.20) by adding these inequalities for $d = c2^{-k},\ k = 0, 1, \ldots$.

It remains to prove (3.19). Let $h := \bar{h}(y)$. Then $h \leq c$. By applying Proposition 2.3 to $S_\psi(y, h)$, we find that it is equivalent to an ellipsoid E_h, i.e.,

$$k_0 E_h \subset S_\psi(y, h) - y \subset k_0^{-1} E_h,$$

where $E_h := h^{1/2} A_h^{-1} B_1$ with $\det A_h = 1$ and $\|A_h\|,\ \|A_h^{-1}\| \leq C|\log h|$. We use the following rescalings similar to those in Sect. 2.2:

$$\tilde{\Omega}_h := h^{-1/2} A_h(\Omega - y),$$

and for $x \in \tilde{\Omega}_h$

$$\tilde{\psi}_h(x) := h^{-1}\left[\psi\left(y + h^{1/2} A_h^{-1} x\right) - \psi(y) - \nabla\psi(y)\cdot\left(h^{1/2} A_h^{-1} x\right) - h\right].$$

Then

$$B_{k_0} \subset S_{\tilde{\psi}_h}(0, 1) \equiv h^{-1/2} A_h\big(S_\phi(y, h) - y\big) \subset B_{k_0^{-1}}.$$

We have

$$\det D^2\tilde{\psi}_h(x) = \det D^2\psi\left(y + h^{1/2} A_h^{-1} x\right) \quad \text{and} \quad \tilde{\psi}_h = 0 \text{ on } \partial S_{\tilde{\psi}_h}(0, 1).$$

For simplicity, we denote $\tilde{S}_t(0) := S_{\tilde{\psi}_h}(0, t)$ for $t > 0$. If $\psi = \phi$ then by Caffarelli's interior $W^{2,p}$ estimates for the Monge–Ampère equation [2], we have

$$\int\limits_{\tilde{S}_{\frac{1}{2}}(0)} \left|D^2\tilde{\psi}_h\right|^{2p} \leq C$$

if $\epsilon \leq \epsilon_0$ small depending only on p, ρ and n. If $\psi = w$ then as $\det D^2 w = 1$, the above inequality obviously holds. Using the fact

$$D^2\psi\left(y + h^{1/2} A_h^{-1} x\right) = A_h^t\, D^2\tilde{\psi}_h(x)\, A_h,$$

we obtain (3.19) from

$$\begin{aligned}\int\limits_{S_\phi\left(y, \frac{h}{2}\right)} |D^2\psi(z)|^{2p}\, dz &= h^{\frac{n}{2}} \int\limits_{\tilde{S}_{\frac{1}{2}}(0)} |A_h^t\, D^2\tilde{\psi}_h(x)\, A_h|^{2p}\, dx \\ &\leq C\, h^{\frac{n}{2}}\, |\log h|^{4p} \int\limits_{\tilde{S}_{\frac{1}{2}}(0)} |D^2\tilde{\psi}_h(x)|^p \leq C\, h^{\frac{n}{2}}\, |\log h|^{4p}.\end{aligned}$$

Finally, we verify **Claim 2** by proving (3.18). As in [16, Lemma 3.4], we note that the difference $v := \phi - w$ is a subsolution (supersolution) of linearized Monge–Ampère equations with bounded right hand side, corresponding to the potentials w and ϕ respectively. We cover $U \cap B_{c^2}$ by sections of w and ϕ using Lemma 3.13. In each of these sections, we can use the one-sided $W^{2,\delta}$ estimates of Gutiérrez-Tournier [18]. Then, adding these estimates as in the proof of Theorem 3.9, we get (3.18). The details are as follows.

Consider the operator $\mathcal{M}u := (\det D^2u)^{1/n}$ and its linearized operator

$$\hat{\mathcal{L}}_u v := \frac{1}{n}(\det D^2u)^{1/n}\operatorname{trace}\big((D^2u)^{-1}D^2v\big).$$

Notice that $\hat{\mathcal{L}}_u v$ and the operator $\mathcal{L}_u v$ defined in (1.1) are related by

$$\mathcal{L}_u v = n(\det D^2u)^{\frac{n-1}{n}}\hat{\mathcal{L}}_u v.$$

Let $v := \phi - w$ and $g := \det D^2\phi$. Since $\mathcal{M}$ is concave, we obtain

$$g^{1/n} - 1 = \mathcal{M}\phi - \mathcal{M}w \leq \hat{\mathcal{L}}_w v$$

and hence

$$\mathcal{L}_w v = n(\det D^2w)^{\frac{n-1}{n}}\hat{\mathcal{L}}_w v \geq -n|g^{1/n} - 1|. \tag{3.21}$$

We also have $\hat{\mathcal{L}}_\phi v \leq \mathcal{M}\phi - \mathcal{M}w \leq |g^{1/n} - 1|$ and thus

$$\mathcal{L}_\phi v = n(\det D^2\phi)^{\frac{n-1}{n}}\hat{\mathcal{L}}_\phi v \leq n(1+\epsilon)^{\frac{n-1}{n}}|g^{1/n} - 1| \leq 2n|g^{1/n} - 1|. \tag{3.22}$$

On the other hand, it follows from the maximum principle ([19, Lemma 3.1]) that

$$\|v\|_{L^\infty(U)} \leq C_n \operatorname{diam}(U)\|g^{1/n} - 1\|_{L^n(U)}. \tag{3.23}$$

We cover $U \cap B_{c^2}$ by sections of w using Lemma 3.13. From (3.21) and (3.23), we can use Gutiérrez-Tournier's one-sided $W^{2,\delta}$ estimates [18] instead of Theorem 3.10 in each of these sections to estimate the L^δ norm of $(D^2v)^+$. After that, we argue as in the proof of Theorem 3.9, and taking into account Lemma 3.13 again to obtain $\delta_1 = \delta_1(n,\rho) \in (0, 1/2)$ and $C_1 = C_1(n,\rho,\kappa) > 0$ such that

$$\begin{aligned}\|(D^2v)^+\|_{L^{\delta_1}(U\cap B_{c^2})} &\leq C_1\Big(\|v\|_{L^\infty(U\cap B_c)} + \|(\mathcal{L}_w v)^-\|_{L^n(U\cap B_c)}\Big)\\ &\leq C_1\|g^{1/n} - 1\|_{L^n(U)}.\end{aligned} \tag{3.24}$$

Similarly, from (3.22), (3.23) and by covering $U \cap B_{c^2}$ by sections of ϕ, we obtain

$$\begin{aligned}\|(D^2v)^-\|_{L^{\delta_2}(U\cap B_{c^2})} &\leq C_2\Big(\|v\|_{L^\infty(U\cap B_c)} + \|(\mathcal{L}_\phi v)^+\|_{L^n(U\cap B_c)}\Big)\\ &\leq C_2\|g^{1/n} - 1\|_{L^n(U)}.\end{aligned} \tag{3.25}$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then, from (3.24) and (3.25), we obtain (3.18) as desired since

$$\|D^2 v\|_{L^\delta(U\cap B_{c^2})} \leq C\|g^{1/n} - 1\|_{L^n(U)} \leq C\epsilon^{1/n}.$$

(ii) The key to the proof is the following estimate

$$\begin{aligned}&\|\Phi - \mathrm{W}\|_{L^q(U\cap B_{c^2})}\\ &\quad\leq C_n\left(\epsilon + \|D^2 w\|^{n-1}_{L^\infty(U\cap B_{c^2})}\|D^2\phi - D^2 w\|^{n-1}_{L^{qn}(U\cap B_{c^2})}\right)\|D^2\phi\|^{n-1}_{L^{qn}(U\cap B_{c^2})}\end{aligned} \tag{3.26}$$

which can be deduced from the proof of Lemma 3.5 in [16].

As in **Claim 1** in the proof of part (i), we have

$$\|D^2\phi\|_{L^{qn}(U\cap B_{c^2})} \leq C_0 \quad \text{for all } \epsilon \leq \epsilon_0, \tag{3.27}$$

where $\epsilon_0 = \epsilon_0(q, n, \rho) > 0$ small and $C_0 = C_0(q, n, \rho, \kappa) > 0$.
On the other hand, by (vii) in the definition of the class $\mathcal{P}$, we have

$$\|D^2 w\|_{L^\infty(U\cap B_{c^2})} \leq C_1(n, \alpha, \rho). \tag{3.28}$$

Putting (3.26)–(3.28) together, we obtain for $\epsilon \leq \epsilon_0$

$$\|\Phi - \mathrm{W}\|_{L^q(U\cap B_{c^2})} \leq C_n\left(\epsilon + C_1^{n-1}\|D^2\phi - D^2 w\|^{n-1}_{L^{qn}(U\cap B_{c^2})}\right)C_0^{n-1}.$$

By applying part (i) of this proposition to $p = qn$, we then get the desired conclusion. □

We also obtain the following global stability of matrices of cofactors.

Lemma 3.15 (Global stability of cofactor matrices) *Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex domain satisfying (2.1) and $\|\partial\Omega\|_{C^3} \leq 1/\rho$. For any $q \geq 1$, there exist $C, \epsilon_0 > 0$ depending only on q, n and ρ with the following property. If ϕ, $w \in C(\overline{\Omega})$ are convex functions satisfying*

$$\begin{cases} 1-\epsilon \leq \det D^2\phi \leq 1+\epsilon & \text{in } \Omega\\ \phi = 0 & \text{on } \partial\Omega\end{cases} \quad \text{and} \quad \begin{cases} \det D^2 w = 1 & \text{in } \Omega\\ w = 0 & \text{on } \partial\Omega,\end{cases}$$

then for some small constant $\delta > 0$ depending only on n and ρ, we have

$$\|\Phi - \mathrm{W}\|_{L^q(\Omega)} \leq C\epsilon^{\frac{(n-1)\delta}{n(2nq-\delta)}} \quad \textit{for all} \quad \epsilon \leq \epsilon_0.$$

Proof The proof follows the lines of the proof of Proposition 3.14 using Proposition 2.4. Here we choose $U = \Omega$, replace $U \cap B_{c^2}$ by Ω and use the covering Lemma 3.12. The estimate (3.28) is now a classical result of Caffarelli–Nirenberg–Spruck [6]. □

3.5 Global $W^{2,1+\epsilon}$ estimates for convex solutions

In this subsection, we establish the global $W^{2,1+\epsilon}$ estimates for convex solutions to the linearized Monge–Ampère equations. These estimates are simple consequence of the global stability of cofactor matrices in Sect. 3.4.

Theorem 3.16 *Let Ω be a uniformly convex domain satisfying (2.1) with $\partial\Omega \in C^3$. Let $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying*

$$0 < \lambda \le \det D^2\phi \le \Lambda \ \ in \ \ \Omega \ \ and \ \ \phi = 0 \ \ on \ \ \partial\Omega.$$

Let v be the convex solution to

$$\begin{cases} \Phi^{ij} v_{ij} = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$. Then, there exist $\gamma > 1$ and $C > 0$ depending only on λ, Λ, n and Ω such that

$$\|D^2 v\|_{L^\gamma(\Omega)} \le C \|f\|_{L^\infty(\Omega)}. \tag{3.29}$$

Remark 3.17 (i) De Phillipis–Figalli–Savin [9] and Schmidt [29] discovered the interior $W^{2,1+\epsilon}$ estimates for convex solution ϕ to the Monge–Ampère equation

$$\det D^2\phi = g \ \text{ in } \Omega, \quad \text{and} \quad \phi = 0 \ \text{ on } \partial\Omega,$$

with $0 < \lambda \le g \le \Lambda$. In these works, the convexity of ϕ plays a crucial role, especially in giving a bound for $|D^2\phi|$ by $\Delta\phi$. Since $\Phi^{ij}\phi_{ij} = n \det D^2\phi = ng$, our theorem is a natural extension of De Phillipis–Figalli–Savin's and Schmidt's estimates.

(ii) The convexity of v and standard arithmetic-geometric inequality give

$$f = \Phi^{ij} v_{ij} = \text{trace}(\Phi\, D^2 v) \ge n(\det\Phi)^{1/n}(\det D^2 v)^{1/n} \ge 0.$$

(iii) It would be interesting to remove the convexity of v in the statement of Theorem 3.16.

Now, we proceed with the proof of Theorem 3.16. To do this, we first establish the following Sobolev stability result.

Proposition 3.18 (Sobolev stability estimates) *Let Ω be a uniformly convex domain with $\partial\Omega \in C^3$. Let $\phi_k \in C(\overline{\Omega}) \cap C^2(\Omega)$ ($k = 1, 2$) be convex Aleksandrov solutions of*

$$\det D^2\phi_k = g_k \ \ in\, \Omega, \quad and \quad \phi_k = 0 \ \ on\, \partial\Omega,$$

with $0 < \lambda \le g_k \le \Lambda$ in Ω. Then there exist $\gamma > 1$, $\alpha \in (0, 1)$ and $C > 0$ depending only on n, λ, Λ and Ω such that

$$\|D^2\phi_1 - D^2\phi_2\|_{L^\gamma(\Omega)} \leq C\|g_1 - g_2\|^{\frac{\alpha}{n}}_{L^1(\Omega)}. \tag{3.30}$$

Proof The interior counterpart of our proposition was established by De Phillipis–Figalli [8]. Here, we will prove the boundary version with a different method. Our proof relies on the $W^{2,\delta}$ estimates of Gutiérrez-Tournier [18] for solutions to the linearized Monge–Ampère equation.

First, using Proposition 2.4, [19, Lemma 3.1] and arguing as in the proof of (3.18) in Proposition 3.14, we find a small $\delta > 0$ and $C_1 > 0$ depending only on n, λ, Λ and Ω such that

$$\|D^2\phi_1 - D^2\phi_2\|_{L^\delta(\Omega)} \leq C_1\|g_1^{\frac{1}{n}} - g_2^{\frac{1}{n}}\|_{L^n(\Omega)} \leq C_1\|g_1 - g_2\|^{\frac{1}{n}}_{L^1(\Omega)}. \tag{3.31}$$

Second, using De Phillipis–Figalli–Savin's and Schmidt's interior $W^{2,1+\epsilon}$ estimates for solutions to the Monge–Ampère equation [9,29] and arguing as in [28], we obtain the following global $W^{2,1+\epsilon}$ estimates

$$\|D^2\phi_1\|_{L^{\gamma_1}(\Omega)} + \|D^2\phi_2\|_{L^{\gamma_1}(\Omega)} \leq C_2, \tag{3.32}$$

where $\gamma_1 > 1$ and $C_2 > 0$ depend only on n, λ, Λ, and Ω.

We now choose $\alpha \in (0, 1)$ sufficiently close to 0 so that

$$\frac{1}{\gamma} := \frac{\alpha}{\delta} + \frac{1-\alpha}{\gamma_1} < 1,$$

i.e., $\gamma > 1$. Then by the interpolation inequality, we obtain

$$\|D^2\phi_1 - D^2\phi_2\|_{L^\gamma(\Omega)} \leq \|D^2\phi_1 - D^2\phi_2\|^{\alpha}_{L^\delta(\Omega)}\|D^2\phi_1 - D^2\phi_2\|^{1-\alpha}_{L^{\gamma_1}(\Omega)}$$

which together with (3.31) and (3.32) yields the estimate (3.30). □

Proof of Theorem 3.16. For any $t \in (0, \|f\|^{-1}_{L^\infty(\Omega)}]$, we have $\phi = \phi + tv$ on $\partial\Omega$ and, by the convexity of v, $\phi \geq \phi + tv$ in Ω. Thus

$$\lambda \leq \det D^2\phi \leq \det D^2(\phi + tv).$$

Moreover by the concavity of the map $\phi \mapsto \log\det D^2\phi$, we obtain

$$\log\det D^2(\phi + tv) \leq log\det D^2\phi + t\phi^{ij}v_{ij} = log\det D^2\phi + t\frac{f}{det D^2\phi}.$$

Therefore,

$$0 \leq \det D^2(\phi + tv) - \det D^2\phi \leq (\det D^2\phi)\left(e^{\frac{tf}{\det D^2\phi}} - 1\right) \leq \Lambda\left(e^{\frac{1}{\lambda}} - 1\right) \quad \text{in}\,\Omega.$$

Applying the stability result in Proposition 3.18, we can find α, $C > 0$, $\gamma > 1$ depending only on n, λ, Λ and Ω such that

$$\|tD^2v\|_{L^\gamma(\Omega)} \leq C\|\det D^2(\phi + tv) - \det D^2\phi\|^{\frac{\alpha}{n}}_{L^1(\Omega)} \leq C\|\Lambda(e^{\frac{1}{\lambda}} - 1)\|^{\frac{\alpha}{n}}_{L^1(\Omega)}.$$

The estimate (3.29) follows by taking $t = \|f\|^{-1}_{L^\infty(\Omega)}$. □

4 Global Hölder estimates and approximation lemma

In this section, we establish global Hölder continuity estimates for solutions to the linearized Monge–Ampère equation under natural assumptions on the domain, Monge–Ampère measure and Hölder continuous boundary data. We then use these Hölder estimates to prove approximation lemmas allowing us to approximate the solution u to $\mathcal{L}_\phi u = f$ by smooth solutions of linearized Monge–Ampère equations associated with convex functions w whose Monge–Ampère measures are close to that of ϕ.

4.1 Global Hölder estimates

In this subsection, we derive global Hölder estimates for solutions to the linearized Monge–Ampère equation in convex domains when the right hand side is assumed to be in L^n and the boundary data is Hölder continuous. These estimates extend the global Hölder estimates in [21] where the domains are assumed to be uniformly convex.

Our first main theorem is concerned with Hölder estimates in a neighborhood of a boundary point. Its precise statement is as follows.

Theorem 4.1 *Assume Ω and ϕ satisfy* (2.1), (2.2), (2.4) *and if $x_0 \in \partial\Omega \cap B_\rho$ then*

$$\rho\,|x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla\phi(x_0)\cdot(x - x_0) \leq \rho^{-1}\,|x - x_0|^2\,,\ \forall x \in \partial\Omega. \tag{4.1}$$

Let $u \in C\left(B_\rho \cap \overline{\Omega}\right) \cap W^{2,n}_{loc}(B_\rho \cap \Omega)$ be a solution to

$$\begin{cases} \Phi^{ij}u_{ij} = f & \text{in } B_\rho \cap \Omega,\\ u = \varphi & \text{on } \partial\Omega \cap B_\rho, \end{cases}$$

where $\varphi \in C^\alpha(\partial\Omega \cap B_\rho)$ for some $\alpha \in (0, 1)$. Then, there exist constants $\beta, C > 0$ depending only on $\lambda, \Lambda, n, \alpha$ and ρ such that

$$|u(x) - u(y)| \leq C|x - y|^\beta\Big(\|u\|_{L^\infty(\Omega\cap B_\rho)} + \|\varphi\|_{C^\alpha(\partial\Omega\cap B_\rho)} + \|f\|_{L^n(\Omega\cap B_\rho)}\Big),$$
$$\forall x, y \in \Omega \cap B_{\rho/2}.$$

As an immediate consequence of Theorem 4.1, we obtain the following estimates which are the global counterparts of Caffarelli–Gutiérrez's interior Hölder estimates for solutions to the linearized Monge–Ampère equation [4].

Theorem 4.2 *Assume Ω and ϕ satisfy* (2.1)–(2.3). *Let $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ be a solution to*

$$\Phi^{ij} u_{ij} = f \quad in\,\Omega, \quad and \quad u = \varphi \quad on\,\partial\Omega,$$

where $\varphi \in C^{\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Then, there exist constants $\beta, C > 0$ depending only on $\lambda, \Lambda, n, \alpha$ and ρ such that

$$|u(x) - u(y)| \leq C|x-y|^{\beta}\Big(\|u\|_{L^{\infty}(\Omega)} + \|\varphi\|_{C^{\alpha}(\partial\Omega)} + \|f\|_{L^n(\Omega)}\Big), \ \forall x, y \in \Omega.$$

The key to the proof of Theorem 4.1 is the following boundary Hölder estimates.

Proposition 4.3 *Let ϕ and u be as in Theorem 4.1. Then, there exist δ, C depending only on $\lambda, \Lambda, n, \alpha, \rho$ such that, for any $x_0 \in \partial\Omega \cap B_{\rho/2}$, we have*

$$|u(x) - u(x_0)| \leq C|x-x_0|^{\frac{\alpha}{\alpha+3n}}\Big(\|u\|_{L^{\infty}(\Omega\cap B_\rho)} + \|\varphi\|_{C^{\alpha}(\partial\Omega\cap B_\rho)} + \|f\|_{L^n(\Omega\cap B_\rho)}\Big),$$
$$\forall x \in \Omega \cap B_{\delta}(x_0).$$

Proof of Theorem 4.1. The boundary Hölder estimates in Proposition 4.3 combined with the interior Hölder continuity estimates of Caffarelli–Gutiérrez [4] and Savin's Localization Theorem [26–28] gives the global Hölder estimates in Theorem 4.1. The precise arguments are almost the same as the proof of [21, Theorem 1.4]. Since [21, Theorem 1.4] is a global result and our theorem is local, we indicate some differences in the arguments. It suffices to prove the theorem for $x, y \in B_{c^2} \cap \Omega$. We use the quadratic separation (4.1) and Proposition 2.3 to show that if $y \in \Omega \cap B_{c^2}$ then the maximal interior section $S_{\phi}(y, \bar{h}(y))$ is contained in $\Omega \cap B_c$ and so tangent to $\partial\Omega$ at $x_0 \in \partial\Omega \cap B_c$ (see Lemma 2.5(e)). Using this fact, Caffarelli–Gutiérrez's interior Hölder estimates [4] and Proposition 4.3, we obtain as in [21]

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2|^{\beta}\left(\|u\|_{L^{\infty}(\Omega\cap B_\rho)} + \|\varphi\|_{C^{\alpha}(\partial\Omega\cap B_\rho)} + \|f\|_{L^n(\Omega\cap B_\rho)}\right)$$
$$\forall z_1, z_2 \in S_{\phi}\left(y, \frac{\bar{h}(y)}{2}\right).$$

The rest of the argument is the same as in [21]. □

The proof of Proposition 4.3 is based on a construction of suitable barriers. Assume ϕ and Ω satisfy the assumptions in the proposition. We also assume for simplicity that $\phi(0) = 0$ and $\nabla\phi(0) = 0$. We now construct a supersolution as in [23, Lemma 6.2].

Lemma 4.4 (Supersolution) *Given δ universally small ($\delta \leq \rho$), define*

$$\tilde{\delta} := \frac{\delta^3}{2} \quad and \quad M_{\delta} := \frac{2^{n-1}\Lambda^n}{\lambda^{n-1}}\frac{1}{\delta^{3n-3}} \equiv \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}}.$$

Then the function

$$w_\delta(x', x_n) := M_\delta x_n + \phi - \tilde{\delta}|x'|^2 - \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}} x_n^2 \quad \textit{for} \quad (x', x_n) \in \overline{\Omega}$$

satisfies

$$\mathcal{L}_\phi(w_\delta) := \Phi^{ij}(w_\delta)_{ij} \leq -n\Lambda \quad \textit{in}\, \Omega,$$

and

$$w_\delta \geq 0 \ \textit{on} \ \partial(\Omega \cap B_\delta), \quad w_\delta \geq \frac{\delta^3}{2} \ \textit{on} \ \Omega \cap \partial B_\delta.$$

Proof We recall from (2.7) that

$$\overline{\Omega} \cap B^+_{ch^{1/2}/|\log h|} \subset S_\phi(0, h) \subset \overline{\Omega} \cap B^+_{Ch^{1/2}|\log h|}.$$

The first inclusion gives $\phi \leq h$ in $\overline{\Omega} \cap B^+_{ch^{1/2}/|\log h|}$ and hence for x close to the origin,

$$\phi(x) \leq C\,|x|^2\,|\log|x||^2\,.$$

Similarly, the second inclusion gives

$$\phi(x) \geq c\,|x|^2\,|\log|x||^{-2} \geq |x|^3$$

for x close to the origin. In conclusion, we have

$$|x|^3 \leq \phi(x) \leq C\,|x|^2\,|\log|x||^2 \tag{4.2}$$

if $|x| \leq \delta$ for δ universally small. Therefore, the choice of $\tilde{\delta}$ gives

$$\phi(x) - \tilde{\delta}|x'|^2 \geq |x|^3 - \tilde{\delta}\,|x|^2 \geq \frac{1}{2}\,|x|^3 = \tilde{\delta} \quad \text{on} \ \Omega \cap \partial B_\delta.$$

On the other hand, the choice of M_δ implies that

$$M_\delta x_n - \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}} x_n^2 \geq 0 \qquad \text{on}\, \overline{\Omega \cap B_\delta}.$$

Hence, $w_\delta \geq \tilde{\delta}$ on $\Omega \cap \partial B_\delta$ while on $\partial\Omega \cap B_\delta$, the quadratic separation (4.1) and $\delta \leq \rho$ give

$$w_\delta \geq \phi - \tilde{\delta}|x'|^2 \geq 0.$$

As a consequence, we obtain the desired inequalities for w_δ on $\partial(\Omega \cap B_\delta)$.

It remains to prove that $\mathcal{L}_\phi(w_\delta) \leq -n\Lambda$ in Ω. If we denote

$$q(x) := \frac{1}{2}\left(\tilde{\delta}|x'|^2 + \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}}x_n^2\right),$$

then

$$\det D^2 q = \frac{\Lambda^n}{\lambda^{n-1}}, \quad D^2 q \geq \tilde{\delta} I.$$

Using the matrix inequality

$$\mathrm{trace}(AB) \geq n(\det A \det B)^{1/n} \text{ for } A, B \text{ symmetric } \geq 0,$$

we get

$$\mathcal{L}_\phi q = \mathrm{trace}(\Phi\, D^2 q) \geq n(\det(\Phi) \det D^2 q)^{1/n} = n\left((\det D^2\phi)^{n-1}\frac{\Lambda^n}{\lambda^{n-1}}\right)^{1/n} \geq n\Lambda.$$

Since $\mathcal{L}_\phi x_n = 0$, we find

$$\begin{aligned}\mathcal{L}_\phi w_\delta &= L_\phi(M_\delta x_n + \phi - 2q) = \Phi^{ij}\phi_{ij} - 2\mathcal{L}_\phi q \\ &= n \det D^2\phi - 2\mathcal{L}_\phi q \leq -n\Lambda \quad \text{in } \Omega.\end{aligned}$$

□

Proof of Proposition 4.3. Our proof follows closely the proof of Proposition 2.1 in [21]. We can suppose that $K := \|u\|_{L^\infty(\Omega\cap B_\rho)} + \|\varphi\|_{C^\alpha(\partial\Omega\cap B_\rho)} + \|f\|_{L^n(\Omega\cap B_\rho)}$ is finite. By working with the function $v := u/K$ instead of u, we can assume in addition that

$$\|u\|_{L^\infty(\Omega\cap B_\rho)} + \|\varphi\|_{C^\alpha(\partial\Omega\cap B_\rho)} + \|f\|_{L^n(\Omega\cap B_\rho)} \leq 1$$

and need to show that the inequality

$$|u(x) - u(x_0)| \leq C|x - x_0|^{\frac{\alpha}{\alpha+3n}} \quad \forall x \in \Omega \cap B_\delta(x_0) \tag{4.3}$$

holds for all $x_0 \in \Omega \cap B_{\rho/2}$, where δ and C depends only on $\lambda, \Lambda, n, \alpha$ and ρ.

We prove (4.3) for $x_0 = 0$. However, our arguments apply to all points $x_0 \in \Omega \cap B_{\rho/2}$ with obvious modifications. For any $\varepsilon \in (0, 1)$, we consider the functions

$$h_\pm(x) := u(x) - u(0) \pm \epsilon \pm \frac{6}{\delta_2^3}w_{\delta_2}$$

in the region

$$A := \Omega \cap B_{\delta_2}(0),$$

where δ_2 is small to be chosen later and the function w_{δ_2} is as in Lemma 4.4. We remark that $w_{\delta_2} \geq 0$ in A by the maximum principle. Observe that if $x \in \partial\Omega$ with $|x| \leq \delta_1(\epsilon) := \epsilon^{1/\alpha}$ then,

$$|u(x) - u(0)| = |\varphi(x) - \varphi(0)| \leq |x|^\alpha \leq \epsilon. \tag{4.4}$$

On the other hand, if $x \in \Omega \cap \partial B_{\delta_2}$ then from Lemma 4.4, we obtain $\frac{6}{\delta_2^3} w_{\delta_2}(x) \geq 3$. It follows that, if we choose $\delta_2 \leq \delta_1$ then from (4.4) and $|u(x) - u(0) \pm \epsilon| \leq 3$, we get

$$h_- \leq 0,\ h_+ \geq 0 \ \text{ on } \partial A.$$

Also from Lemma 4.4, we have

$$\mathcal{L}_\phi h_+ \leq f,\ \mathcal{L}_\phi h_- \geq f \ \text{ in } A.$$

Hence the ABP estimate applied in A gives

$$h_- \leq C_1(n, \lambda) diam(A) \|f\|_{L^n(A)} \leq C_1(n, \lambda)\delta_2 \ \text{ in } \ A \tag{4.5}$$

and

$$h_+ \geq -C_1(n, \lambda) diam(A) \|f\|_{L^n(A)} \geq -C_1(n, \lambda)\delta_2 \ \text{ in } \ A. \tag{4.6}$$

By restricting $\epsilon \leq C_1(n, \lambda)^{\frac{-\alpha}{1-\alpha}}$, we can assume that

$$\delta_1 = \epsilon^{1/\alpha} \leq \frac{\epsilon}{C_1(n, \lambda)}.$$

Then, for $\delta_2 \leq \delta_1$, we have $C_1(n, \lambda)\delta_2 \leq \epsilon$ and thus, for all $x \in A$, we obtain from (4.5) and (4.6) that

$$|u(x) - u(0)| \leq 2\epsilon + \frac{6}{\delta_2^3} w_{\delta_2}(x).$$

Note that, by construction and the estimate (4.2) for the function ϕ, we have in A

$$w_{\delta_2}(x) \leq M_{\delta_2} x_n + \phi(x) \leq M_{\delta_2} |x| + C |x|^2 |\log |x||^2 \leq 2M_{\delta_2} |x|.$$

Therefore, choosing $\delta_2 = \delta_1$ and recalling the choice of M_{δ_2}, we get

$$|u(x) - u(0)| \leq 2\epsilon + \frac{12 M_{\delta_2}}{\delta_2^3} |x| = 2\epsilon + \frac{C_2(n, \lambda, \Lambda)}{\delta_2^{3n}} |x| = 2\epsilon + C_2 \epsilon^{-3n/\alpha} |x| \tag{4.7}$$

for all x, ϵ satisfying the following conditions

$$|x| \leq \delta_1(\epsilon) := \epsilon^{1/\alpha}, \quad \epsilon \leq C_1(n,\lambda)^{\frac{-\alpha}{1-\alpha}} =: c_1. \tag{4.8}$$

Finally, let us choose $\epsilon = |x|^{\frac{\alpha}{\alpha+3n}}$. It satisfies the conditions in (4.8) if

$$|x| \leq \min\left\{c_1^{\frac{\alpha+3n}{\alpha}}, 1\right\} =: \delta.$$

Then, by (4.7), we have $|u(x) - u(0)| \leq (2 + C_2)|x|^{\frac{\alpha}{\alpha+3n}}$ for all $x \in \Omega \cap B_\delta(0)$. □

4.2 Global approximation lemma

In this subsection, we prove an approximation lemma that allows us to compare the solution u to the linearized Monge–Ampère equation $\mathcal{L}_\phi u = f$ to smooth solutions h of linearized Monge–Ampère equations $\mathcal{L}_w h = 0$ associated with convex functions w satisfying $\det D^2 w = 1$. We will estimate the difference $u - h$ in terms of the L^n-norms of f and $\Phi - \mathrm{W}$ where $\Phi = (\Phi^{ij})$ and $\mathrm{W} = (\mathrm{W}^{ij})$ are the matrices of cofactors of $D^2\phi$ and $D^2 w$, respectively. Therefore, in light of the global stability of cofactor matrices in Sect. 3.4, u is well-approximated by h provided that $\det D^2\phi$ is close to 1. This approximation lemma will play a key role in Sect. 5 where we use it to get power decay estimates for the distribution function of the second derivatives of u that are more refined than those provided by Proposition 3.7.

Our approximation lemma, relevant for data of the type $\big(\Omega_h, \phi_h, S_{\phi_h}(0,1)\big)$, states as follows.

Lemma 4.5 *Assume $(\Omega, \phi, U) \in \mathcal{P}_{\frac{1}{2},\frac{3}{2},\rho,\kappa,\alpha}$. Let $r := c^2/4$ where c is as in Remark 2.6. Suppose that $u \in C(\overline{U}) \cap W^{2,n}_{loc}(U)$ is a solution of $\Phi^{ij} u_{ij} = f$ in $U \cap B_{4r}$ with*

$$\|u\|_{L^\infty(U\cap B_{4r})} + \|u\|_{C^{2,\alpha}(\partial U\cap B_{4r})} \leq 1.$$

Let w be defined as in (vii) of the definition of the class $\mathcal{P}$. Assume h is a solution of

$$\begin{cases} \mathrm{W}^{ij} h_{ij} = 0 & \text{in } B_{2r} \cap U \\ h = u & \text{on } \partial(B_{2r} \cap U). \end{cases}$$

Then, there exist $C > 0$ and $\gamma \in (0,1)$ depending only on n, ρ and α such that

$$\|h\|_{C^{1,1}(\overline{B_r\cap U})} \leq C, \tag{4.9}$$

and if $\|\Phi - \mathrm{W}\|_{L^n(B_{2r}\cap U)} \leq r^4$ then

$$\begin{aligned}&\|u - h\|_{L^\infty(B_r\cap U)} + \|f - \mathit{trace}([\Phi - \mathrm{W}]D^2 h)\|_{L^n(B_r\cap U)} \\ &\quad \leq C\left\{\left(1 + \|u\|_{C^{1/2}(\partial U\cap B_{4r})}\right)\|\Phi - \mathrm{W}\|^\gamma_{L^n(B_{2r}\cap U)} + \|f\|_{L^n(U\cap B_{4r})}\right\}.\end{aligned}$$

Proof Observe first that by (vii) in the definition of the class $\mathcal{P}$, the following $C^{2,\alpha}$ and Pogorelov estimates hold

$$\|\partial U \cap B_{4r}\|_{C^{2,\alpha}} \le c_0^{-1},\ \|w\|_{C^{2,\alpha}(\overline{U\cap B_{4r}})} \le c_0^{-1},\ c_0 I_n \le D^2 w \le c_0^{-1} I_n \ \text{ in }\ B_{4r}\cap U. \tag{4.10}$$

Therefore, $W^{ij}\partial_{ij}$ is a uniformly elliptic differential operator with C^α coefficients. Hence, we can employ the standard boundary $C^{2,\alpha}$-estimates for linear uniformly elliptic equation and obtain (4.9) since

$$\begin{aligned}\|h\|_{C^{1,1}(\overline{B_r\cap U})} \le \|h\|_{C^{2,\alpha}(\overline{B_r\cap U})} &\le C(n,\rho,\alpha)\big(\|u\|_{L^\infty(B_{2r}\cap U)} + \|u\|_{C^{2,\alpha}(\partial U\cap B_{2r})}\big)\\ &\le C(n,\rho,\alpha).\end{aligned}$$

Next, since $(\Omega,\phi,U) \in \mathcal{P}_{\frac{1}{2},\frac{3}{2},\rho,\kappa,\alpha}$, by Remark 2.11, the domain U and function ϕ satisfy (2.1), (2.2) and (2.4) and (4.1). Therefore, it follows from Theorem 4.1 with $C^{1/2}$ boundary data that there exist constants $C>0$ and $\beta \in (0,1)$ depending only on n and ρ such that

$$\|u\|_{C^\beta(\overline{B_{2r}\cap U})} \le C\Big(\|u\|_{L^\infty(B_{4r}\cap U)} + \|u\|_{C^{1/2}(\partial U\cap B_{4r})} + \|f\|_{L^n(B_{4r}\cap U)}\Big) \le C\Theta, \tag{4.11}$$

where

$$\Theta := 1 + \|u\|_{C^{1/2}(\partial U\cap B_{4r})} + \|f\|_{L^n(B_{4r}\cap U)}.$$

In view of (4.10), (4.11), and the standard global Hölder estimates for linear uniformly elliptic equations (see [14, Corollary 9.29], [3, Proposition 4.13] and [35, Theorem 1.10]), we can find constants $C>0$ and $\beta' \in (0,\beta)$ depending only on n, ρ and α such that

$$\|h\|_{C^{\beta'}(\overline{B_{2r}\cap U})} \le C\Big(\|h\|_{L^\infty(B_{2r}\cap U)} + \|u\|_{C^\beta(\partial(U\cap B_{2r}))}\Big) \le C\Theta. \tag{4.12}$$

Now let $0<\delta<r$. Then we claim that

$$\|u-h\|_{L^\infty(\partial(B_{2r-\delta}\cap U))} \le C\delta^{\beta'}\Theta, \tag{4.13}$$

and

$$\|D^2 h\|_{L^\infty(B_{2r-\delta}\cap U)} \le C\delta^{\beta'-2-\alpha}\Theta. \tag{4.14}$$

To prove (4.13), we verify that $|(u-h)(x)| \le C\delta^{\beta'}\Theta$ for all $x\in\partial(B_{2r-\delta}\cap U)$. Indeed, if $x\in\partial(B_{2r-\delta}\cap U)$ then we can find $y\in\partial(B_{2r}\cap U)$ such that $|x-y|\le\delta$. Since $u-h=0$ on $\partial(B_{2r}\cap U)$, we get from (4.11) and (4.12) that

$$|(u-h)(x)| = |(u-h)(x)-(u-h)(y)| \le |u(x)-u(y)| + |h(x)-h(y)| \le C\delta^{\beta'}\Theta.$$

To prove (4.14), let $x_0 \in B_{2r-\delta} \cap U$. If $B_{\delta/2}(x_0) \subset B_{2r} \cap U$, then we can apply interior $C^{1,1}$-estimates to $h - h(x_0)$ in $B_{\delta/2}(x_0)$ and use (4.12) to get

$$\|D^2 h(x_0)\| \leq C\delta^{-2}\|h - h(x_0)\|_{L^\infty\left(B_{\frac{\delta}{2}}(x_0)\right)} \leq C\delta^{\beta'-2}\Theta.$$

In case $B_{\delta/2}(x_0) \not\subset B_{2r} \cap U$, then there exists $z_0 \in B_{2r-\delta} \cap \partial U \subset \partial\Omega$ such that $x_0 \in B_{\delta/2}(z_0)$. Hence since $B_\delta(z_0) \cap U \subset B_{2r} \cap U$ and by applying boundary $C^{2,\alpha}$-estimates to $h - h(x_0)$ in $B_{\delta/2}(z_0) \cap U$ we obtain

$$\begin{aligned}
&\|D^2(h - h(x_0))\|_{C^\alpha(B_{\frac{\delta}{2}}(z_0)\cap U)} \\
&\leq C\delta^{-(2+\alpha)}\Big(\|h - h(x_0)\|_{L^\infty(B_\delta(z_0)\cap U)} + \sum_{k=1}^{2} \delta^{k+\alpha}\|D^k(u - h(x_0))\|_{C^\alpha(\partial U\cap B_\delta(z_0))}\Big) \\
&\leq C\delta^{-(2+\alpha)}\Big(\delta^{\beta'}\Theta + \delta^{1+\alpha}\Big) \leq C\delta^{\beta'-2-\alpha}\Theta.
\end{aligned}$$

It follows that $\|D^2 h(x_0)\| \leq C\delta^{\beta'-2-\alpha}\Theta$, and thus (4.14) is proved.

Having (4.13) and (4.14), we now complete the proof of the lemma. Observe that $u - h \in W^{2,n}_{loc}(B_{2r} \cap U)$ is a solution of

$$\Phi^{ij}(u-h)_{ij} = f - \Phi^{ij}h_{ij} = f - [\Phi^{ij} - \mathrm{W}^{ij}]h_{ij} =: F \quad \text{in } B_{2r} \cap U.$$

The ABP estimate together with (4.13) and (4.14) gives

$$\begin{aligned}
&\|u - h\|_{L^\infty(B_{2r-\delta}\cap U)} + \|F\|_{L^n(B_{2r-\delta}\cap U)} \\
&\quad\leq \|u - h\|_{L^\infty(\partial(B_{2r-\delta}\cap U))} + C_n\|F\|_{L^n(B_{2r-\delta}\cap U)} \\
&\quad\leq \|u - h\|_{L^\infty(\partial(B_{2r-\delta}\cap U))} + C_n\|D^2 h\|_{L^\infty(B_{2r-\delta}\cap U)}\|\Phi - \mathrm{W}\|_{L^n(B_{2r}\cap U)} \\
&\qquad + C_n\|f\|_{L^n(U\cap B_{2r})} \\
&\quad\leq C\Big(\delta^{\beta'} + \delta^{\beta'-2-\alpha}\|\Phi - \mathrm{W}\|_{L^n(B_{2r}\cap U)}\Big)\Theta + C_n\|f\|_{L^n(U\cap B_{2r})}.
\end{aligned}$$

If $\|\Phi - \mathrm{W}\|_{L^n(B_{2r}\cap U)} \leq r^4$ then by taking $\delta = \|\Phi - \mathrm{W}\|^{\frac{1}{2+\alpha}}_{L^n(B_{2r}\cap U)}$, we obtain the desired inequality with $\gamma = \beta'/(2+\alpha)$ since

$$\begin{aligned}
&\|u - h\|_{L^\infty(B_r\cap U)} + \|F\|_{L^n(B_r\cap U)} \leq C\|\Phi - \mathrm{W}\|^{\frac{\beta'}{2+\alpha}}_{L^n(B_{2r}\cap U)}\Theta + C_n\|f\|_{L^n(U\cap B_{2r})} \\
&\leq C\Big\{\big(1 + \|u\|_{C^{1/2}(\partial U\cap B_{4r})}\big)\|\Phi - \mathrm{W}\|^{\frac{\beta'}{2+\alpha}}_{L^n(B_{2r}\cap U)} + \|f\|_{L^n(U\cap B_{4r})}\Big\}.
\end{aligned}$$

□

We end this subsection with a result allowing us to estimate the measure of the set where the quasi distance generated by ϕ is bounded from below by certain multiple of the Euclidean distance. This set, when restricted to sections of ϕ, has almost full measure if the Monge–Ampère measure det $D^2\phi$ is close to a constant. Its precise statement is as follows.

Lemma 4.6 *Assume that* $(\Omega, \phi, U) \in \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,\kappa,\alpha}$ *where* $0 < \epsilon < 1/2$. *Define*

$$A_\sigma := \left\{ \tilde{x} \in U : \phi(x) \geq \phi(\tilde{x}) + \nabla\phi(\tilde{x}) \cdot (x - \tilde{x}) + \frac{\sigma}{2} |x - \tilde{x}|^2, \quad \forall x \in B_{c^2} \cap U \right\}. \tag{4.15}$$

Then there exist $\sigma = \sigma(n, \rho, \alpha) > 0$ *and* $C = C(n, \rho, \alpha, \kappa) > 0$ *such that*

$$\left| S_\phi(0, c^9) \setminus A_\sigma \right| \leq C\epsilon^{1/3n} \, |S_\phi(0, c^9)|.$$

Proof We first note that (2.7) implies

$$|S_\phi(0, c^9)| \geq |B_{c^6} \cap U| \geq C. \tag{4.16}$$

Let w be defined as in (vii) in the definition of the class $\mathcal{P}$. Then the following boundary Pogorelov estimates hold

$$c_0 I_n \leq D^2 w \leq c_0^{-1} I_n \;\text{ in }\; B_{c^2} \cap U. \tag{4.17}$$

Let Γ be the convex envelope of $\phi - \frac{w}{2}$ in $U \cap B_{c^2}$. We claim that there exists $C > 0$ depending only on n, ρ, α and κ such that

$$\left| \left\{ \Gamma = \phi - \frac{w}{2} \right\} \cap S_\phi\big(0, c^9\big) \right| \geq \big(1 - C\epsilon^{1/3n}\big) \left| S_\phi\big(0, c^9\big) \right|. \tag{4.18}$$

Assume this claim for a moment. Then by using (4.17) and arguing as in the proof of [15, Theorem 6.1.1], we obtain the desired conclusion. For completeness, we include the proof.

Let the contact set be

$$\mathcal{C} := \left\{ x \in U \cap B_{c^2} : \Gamma(x) = \phi(x) - \frac{w(x)}{2} \right\}.$$

We assert that for $\sigma := c_0/2$, we have

$$\mathcal{C} \cap S_\phi(0, c^9) \subset A_\sigma \cap S_\phi(0, c^9).$$

It then follows from (4.18) that

$$|S_\phi(0, c^9) \setminus A_\sigma| \leq |S_\phi(0, c^9) \setminus \mathcal{C}| \leq C\epsilon^{1/3n} |S_\phi(0, c^9)|.$$

We now proceed with the proof of the claim. Let $x_0 \in \mathcal{C} \cap S_\phi(0, c^9)$, and let l_{x_0} be a supporting hyperplane to Γ at x_0. Since $x_0 \in \mathcal{C}$, we have $l_{x_0}(x_0) = \phi(x_0) - \frac{1}{2} w(x_0)$ and

$$\phi(x) \geq l_{x_0}(x) + \frac{w(x)}{2} \quad \text{for all } x \in U \cap B_{c^2}. \tag{4.19}$$

On the other hand, if $x \in U \cap B_{c^2}$ then the Taylor formula and the first inequality in the Pogorelov estimates (4.17) give

$$\begin{aligned} &w(x) - w(x_0) - \nabla w(x_0) \cdot (x - x_0) \\ &= \int_0^1 t \int_0^1 \langle D^2 w(x_0 + \theta t(x - x_0)) \cdot (x - x_0), x - x_0 \rangle d\theta dt \\ &\geq \int_0^1 t c_0 \, |x - x_0|^2 \, dt = \frac{c_0}{2} \, |x - x_0|^2 \, . \end{aligned}$$

Combining this with (4.19), we deduce that

$$\phi(x) \geq l(x) + \frac{c_0}{4} \, |x - x_0|^2 \quad \forall x \in U \cap B_{c^2},$$

where $l(x)$ is the supporting hyperplane to ϕ at x_0 in $U \cap B_{c^2}$ given by

$$l(x) := l_{x_0}(x) + \frac{1}{2} w(x_0) + \frac{1}{2} \nabla w(x_0) \cdot (x - x_0).$$

Therefore $x_0 \in A_\sigma$ with $\sigma = c_0/2$, proving the assertion.

It remains to prove (4.18). The idea is to compare the image of the gradient mappings of convex functions which are close in L^∞-norms. This idea goes back to Caffarelli (see [2, Lemma 2] and also [24, Lemma 6.2]). Since our setting near the boundary is a bit different, we sketch the proof.

By (4.16), it suffices to consider the case $\epsilon \ll 1$. By the maximum principle ([19, Lemma 3.1]), we have

$$\|\phi - w\|_{L^\infty(U)} \leq C_n \text{diam}\,(U) \|(\det D^2 \phi)^{1/n} - 1\|_{L^n(U)} \leq C \, \epsilon^{1/n} \equiv \bar{\epsilon}.$$

Therefore,

$$\frac{1}{2} w - \bar{\epsilon} \leq \phi - \frac{w}{2} \leq \frac{1}{2} w + \bar{\epsilon} \quad \text{in } U \cap B_{c^2}$$

and since w is convex, we have

$$\frac{1}{2} w - \bar{\epsilon} \leq \Gamma \leq \frac{1}{2} w + \bar{\epsilon} \quad \text{in } U \cap B_{c^2}.$$

Let

$$V_1 = \Big\{ x \in U \cap B_{c^2} : \, \text{dist}\big(x, \partial(U \cap B_{c^2})\big) > \delta \Big\}.$$

Then, using (4.17), we will show that for $1 \gg \delta > \bar{\epsilon}$ to be chosen later, we have

$$\left|\left\{\Gamma = \phi - \frac{w}{2}\right\} \cap V_1\right| \geq (1 - C\delta)|V_1| \tag{4.20}$$

for some C depends on n, ρ, α and κ. Indeed, let

$$V_2 = \left\{x \in U \cap B_{c^2} : \operatorname{dist}\big(x, \partial(U \cap B_{c^2})\big) > 2\delta\right\}.$$

For $x_0 \in V_2$, consider

$$v^*(x) := \frac{1}{2}w(x) - \bar{\epsilon} + \delta\left(r^2 - |x - x_0|^2\right)$$

where

$$\delta = 2\sqrt[3]{\bar{\epsilon}} \sim \epsilon^{1/3n},\ \delta > r > \sqrt{\frac{2\bar{\epsilon}}{\delta}}.$$

Then $B(x_0, r) \subset V_1$ and

$$v^* \leq \Gamma \ \text{ on } \ \partial B(x_0, r) \quad \text{and} \quad v^* \geq \Gamma \ \text{ in } \ B\left(x_0, r - \frac{2\bar{\epsilon}}{\delta r}\right).$$

It follows that $\nabla v^*\big(B\big(x_0, r - \frac{2\bar{\epsilon}}{\delta r}\big) \subset \nabla\Gamma(B(x_0, r)\big)$. Hence

$$\nabla v^*(V_2) \subset \nabla\Gamma(V_1). \tag{4.21}$$

From the C^2 bound on w in (4.17), we have

$$\begin{aligned} D^2 v^* = \frac{1}{2}D^2 w - 2\delta I_n &= \frac{1}{2}\big(1 - 4c_0^{-1}\delta\big)D^2 w + 2\delta\big(c_0^{-1}D^2 w - I_n\big) \\ &\geq \frac{1}{2}\big(1 - 4c_0^{-1}\delta\big)D^2 w. \end{aligned}$$

Therefore, using $\det D^2 w = 1$, we obtain

$$|\nabla v^*(V_2)| = \int_{V_2} \det D^2 v^* \geq \left(\frac{1}{2^n} - C_1\delta\right)|V_2|. \tag{4.22}$$

Next, as Γ is convex with $\Gamma \in C^{1,1}(U \cap B_{c^2})$ and $\det D^2\Gamma = 0$ a.e. outside $\mathcal{C}$, we have

$$|\nabla\Gamma(V_1)| = |\nabla\Gamma(V_1 \cap \mathcal{C})| = \int_{V_1 \cap \mathcal{C}} \det D^2\Gamma. \tag{4.23}$$

We now estimate det $D^2\Gamma$ from above. For this, observe that for any $x \in \mathcal{C}$, the function $\phi - \frac{1}{2}w - \Gamma$ attains its local minimum value 0 at x. Hence,

$$D^2\Gamma(x) \leq D^2\left(\phi - \frac{1}{2}w\right)(x)$$

at any twice differentiable point of Γ and ϕ. Therefore, this inequality holds for a.e $x \in \mathcal{C}$ by Aleksandrov theorem. Note that for symmetric, nonnegative matrices A and B, we have

$$(\det(A+B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}.$$

Thus, for a.e $x \in \mathcal{C}$, we have

$$\begin{aligned}(\det D^2\Gamma(x))^{1/n} &\leq \left(\det D^2\left(\phi - \frac{1}{2}w\right)(x)\right)^{1/n}\\ &\leq (\det D^2\phi)^{1/n} - \left(\det D^2\left(\frac{1}{2}w\right)(x)\right)^{1/n}\\ &\leq (1+\epsilon)^{1/n} - \frac{1}{2} \leq \frac{1}{2} + \frac{\epsilon}{n}.\end{aligned}$$

Combining with (4.23) gives

$$|\nabla\Gamma(V_1)| \leq \left(\frac{1}{2^n} + C_2\epsilon\right)|V_1 \cap \mathcal{C}|.$$

We infer from this, (4.21) and (4.22) that

$$|V_1 \cap \mathcal{C}| \geq \frac{1 - 2^n C_1\delta}{1 + 2^n C_2\epsilon}\,|V_2| \geq (1 - C_3\delta)\,|V_2| \tag{4.24}$$

for $\epsilon \leq \epsilon_0$ with ϵ_0 is a small universal constant.

By (iv) in the definition of the class $\mathcal{P}$, we have $\|B_{c^2} \cap \partial U\|_{C^{1,1}} \leq \kappa$. Consequently,

$$\left|(U \cap B_{c^2})\backslash V_2\right| \leq C_4\delta \quad \text{and} \quad |V_2| \geq |V_1| - C_4\delta$$

for some $C_4 > 0$ depending only on n, ρ and κ. Combining the above inequalities with (4.24), we easily obtain (4.20).

It follows from (4.20), the inclusion $\{\Gamma < \phi - \frac{w}{2}\} \subset U \cap B_{c^2}$ and (4.16) that

$$\begin{aligned}&\left|\left\{\Gamma < \phi - \frac{w}{2}\right\} \cap S_\phi(0, c^9)\right|\\ &\leq \left|\left\{\Gamma < \phi - \frac{w}{2}\right\} \cap V_1\right| + |(U \cap B_{c^2})\backslash V_1| \leq C\delta|V_1| + C_4\delta\\ &\leq C_5\delta \leq C\delta|S_\phi(0, c^9)| \leq C\epsilon^{1/3n}|S_\phi(0, c^9)|.\end{aligned}$$

This gives the claim (4.18) and the proof is complete. □

5 Density and global $W^{2,p}$ estimates

In this section we will prove global $W^{2,p}$ estimates for solutions to the linearized Monge–Ampère equations as stated in the introduction. The key tools are density estimates and a covering lemma.

5.1 Density estimates

In this subsection, by using the approximation lemma in Sect. 4.2 together with the stability of cofactor matrices established in Sect. 3.4, we improve density estimates obtained in Sect. 3 when the Monge–Ampère measure $\det D^2\phi$ is close to 1.

Our first lemma improves the power decay estimates in Proposition 3.7 which say that for $(\Omega, \phi, U) \in \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,*}$, the quantity $|S_\phi(0,r)\setminus G_N(u,\Omega)|$ decays like $CN^{-\tau}$. Here, we improve C by roughly a factor of $\|\Phi - \mathrm{W}\|_{L^n(U)} + \left(\fint_U |f|^n\, dx\right)^{\frac{1}{n}}$ when $*$ is replaced by α, λ and Λ are close to 1, and W is the matrix of cofactors of $D^2 w$ of the solution to the Monge–Ampère equation $\det D^2 w = 1$ with the same boundary values as ϕ. The precise statement is as follows.

Lemma 5.1 *Assume $(\Omega, \phi, U) \in \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,\kappa,\alpha}$ where $0 < \epsilon < 1/2$. Let $r = c^2/4$. Suppose $u \in C(\Omega)\cap C^1(U)\cap W^{2,n}_{loc}(U)$ is a solution of $\mathcal{L}_\phi u = f$ in U that satisfies*

$$\|u\|_{L^\infty(U)} + \|u\|_{C^{2,\alpha}(\partial U\cap B_{4r})} \le 1,$$

and has at most quadratic growth in the sense that

$$|u(x)| \le C^*\left[1 + d(x,x_0)^2\right] \ \text{ in } \ \Omega\setminus U \ \text{ for some } \ x_0 \in B_{r/2}\cap U. \tag{5.1}$$

Then there exist $\tau = \tau(n,\rho) > 0$ and $N_0 = N_0(C^*, n, \rho, \alpha) > 0$ such that for $N \ge N_0$ we have

$$|G_N(u,\Omega)\cap S_\phi(0,c^9)| \ge \left\{1 - C\left(N^{-\tau}\delta_0^\tau + \epsilon^{1/3n}\right)\right\} |S_\phi(0,c^9)|$$

provided that $\|\Phi - \mathrm{W}\|_{L^n(B_{2r}\cap U)} \le r^4$. Here $C = C(n,\rho,\alpha,\kappa) > 0$, W, γ are from Lemma 4.5, and

$$\delta_0 := \left(1 + \|u\|_{C^{1/2}(\partial U\cap B_{4r})}\right)\|\Phi - \mathrm{W}\|^\gamma_{L^n(B_{2r}\cap U)} + \left(\fint_U |f|^n\, dx\right)^{\frac{1}{n}}.$$

Proof Let h be the solution of

$$\mathrm{W}^{ij} h_{ij} = 0 \ \text{ in } \ B_{2r}\cap U, \quad \text{and} \quad h = u \ \text{ on } \partial(B_{2r}\cap U).$$

By Lemma 4.5 and since $U \subset B_{k^{-1}}$, there exists C_0 depending only on n, ρ and α such that

$$\|h\|_{C^{1,1}(\overline{B_r \cap U})} \leq C_0, \tag{5.2}$$

$$\|u - h\|_{L^\infty(\overline{B_r \cap U})} + \|f - \text{trace}([\Phi - \mathrm{W}]D^2 h)\|_{L^n(\overline{B_r \cap U})} \leq C_0\,\delta_0 =: \delta_0'. \tag{5.3}$$

We now consider $h|_{B_r \cap U}$ and then extend h outside $B_r \cap U$ continuously such that

$$\begin{cases} h(x) = u(x) \quad \forall x \in \Omega \setminus (B_{2r} \cap U), \\ \|u - h\|_{L^\infty(\Omega)} = \|u - h\|_{L^\infty(B_r \cap U)}. \end{cases}$$

The maximum principle gives $\|h\|_{L^\infty(B_r \cap U)} \leq \|u\|_{L^\infty(U)} \leq 1$, and thus

$$u(x) - 2 \leq h(x) \leq u(x) + 2 \quad \text{for all } x \in \Omega. \tag{5.4}$$

We claim that if $N \geq N_0$, then

$$\left(B_{\frac{r}{2}} \cap U\right) \cap A_\sigma \subset G_N(h, \Omega) \tag{5.5}$$

where $\sigma = \sigma(n, \rho, \alpha) > 0$ is the constant given by Lemma 4.6 and the set A_σ is defined by (4.15).

Indeed, let $\bar{x} \in (B_{\frac{r}{2}} \cap U) \cap A_\sigma$. By (5.2) we have

$$|h(x) - [h(\bar{x}) + \nabla h(\bar{x}) \cdot (x - \bar{x})]| \leq C_0 |x - \bar{x}|^2 \quad \text{for all } x \in \overline{B_r \cap U},$$

and since $\bar{x} \in A_\sigma$

$$d(x, \bar{x})^2 = \phi(x) - [\phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x})] \geq \frac{\sigma}{2}|x - \bar{x}|^2 \quad \forall x \in B_{4r} \cap U. \tag{5.6}$$

Therefore,

$$\left|h(x) - [h(\bar{x}) + \nabla h(\bar{x}) \cdot (x - \bar{x})]\right| \leq \frac{2C_0}{\sigma} d(x, \bar{x})^2 \quad \forall x \in \overline{B_r \cap U}. \tag{5.7}$$

We next show that by increasing the constant on the right hand side of (5.7), that the resulting inequality holds for all x in Ω.

To see this, we first observe that by the maximum principle $\max_U \phi = \max_{\partial U} \phi = 1$ and by the gradient estimates (v) in the definition of the class $\mathcal{P}$ and $x_0 \in U \cap B_{r/2}$, we have

$$\begin{aligned} d(x, x_0)^2 &= d(x, \bar{x})^2 + [\phi(\bar{x}) - \phi(x_0) - \langle \nabla\phi(x_0), \bar{x} - x_0 \rangle] \\ &\quad + \langle \nabla\phi(\bar{x}) - \nabla\phi(x_0), x - \bar{x} \rangle \\ &\leq d(x, \bar{x})^2 + C_1(1 + |x - \bar{x}|) \qquad \text{for all } x \in \Omega \end{aligned} \tag{5.8}$$

for some universal C_1 depending only on n and ρ.

Next, we observe that if $c_1 = \sigma r/4$ then

$$d(x,\bar{x})^2 \geq c_1|x-\bar{x}| \quad \forall x \in \overline{\Omega} \setminus \overline{B_r \cap U}. \tag{5.9}$$

Indeed, by (5.6) and the fact that $\bar{x} \in B_{\frac{r}{2}} \cap U$, the above inequality holds for all $x \in U \cap \partial B_r$. Now for $x \in \overline{\Omega} \setminus \overline{B_r \cap U}$ we can choose $\hat{x} \in U \cap \partial B_r$ and $\lambda \in (0,1)$ satisfying $\hat{x} = \lambda x + (1-\lambda)\bar{x}$. Then since $d(\hat{x},\bar{x})^2 \geq c_1|\hat{x}-\bar{x}|$ and the function $z \mapsto d(z,\bar{x})^2$ is convex, we obtain

$$\lambda d(x,\bar{x})^2 + (1-\lambda)d(\bar{x},\bar{x})^2 \geq c_1|\lambda x + (1-\lambda)\bar{x} - \bar{x}| = c_1\lambda|x-\bar{x}|$$

which gives $d(x,\bar{x})^2 \geq c_1|x-\bar{x}|$ and hence (5.9) is proved.

We are ready to show that (5.7) holds for all $x \in \Omega$ but with a bigger constant on the right hand side. Let $x \in \Omega \setminus \overline{B_r \cap U}$. Then, recalling $\bar{x} \in B_{\frac{r}{2}} \cap U$ and by (5.9), we have

$$d(x,\bar{x})^2 \geq c_1 r/2 =: c_2.$$

We can estimate using (5.2) and (5.4),

$$\begin{aligned} |h(x) - [h(\bar{x}) + \nabla h(\bar{x})\cdot(x-\bar{x})]| &\leq |h(x)-h(\bar{x})| + C_0|x-\bar{x}| \\ &\leq |u(x)| + C_0(|x-\bar{x}|+1). \end{aligned} \tag{5.10}$$

Consider the following cases:

Case 1: $x \in \overline{U} \setminus \overline{B_r \cap U}$. Using (5.10) and the above lower bound for $d(x,\bar{x})^2$, we obtain

$$\begin{aligned} |h(x) - [h(\bar{x}) + \nabla h(\bar{x})\cdot(x-\bar{x})]| &\leq 1 + C_0(|x-\bar{x}|+1) \leq 1 + C_0(2k^{-1}+1) \\ &\leq C_2\, d(x,\bar{x})^2. \end{aligned}$$

Case 2: $x \in \Omega \setminus \overline{U}$. Using (5.10), (5.1), (5.8), (5.9) and the bound $d(x,\bar{x})^2 \geq c_2$, we find that

$$\begin{aligned} |h(x) - [h(\bar{x}) + \nabla h(\bar{x})\cdot(x-\bar{x})]| &\leq C^*\,[1 + d(x,x_0)^2] + C_0(|x-\bar{x}|+1) \\ &\leq C^*\, d(x,\bar{x})^2 + C_3(|x-\bar{x}|+1) \leq C_4\, d(x,\bar{x})^2. \end{aligned}$$

Therefore if we choose

$$N_0 := \max\left\{\frac{4C_0}{\sigma}, 2C_2, 2C_4\right\},$$

then it follows from the above considerations and (5.7) that

$$|h(x) - [h(\bar{x}) + \nabla h(\bar{x})\cdot(x-\bar{x})]| \leq \frac{N_0}{2}\, d(x,\bar{x})^2 \quad \text{for all } x \in \Omega.$$

This means $\bar{x} \in G_{N_0}(h,\Omega) \subset G_N(h,\Omega)$ for all $N \geq N_0$. Thus claim (5.5) is proved.

Next let

$$u'(x) := \frac{(u-h)(x)}{\delta_0'}, \quad \text{for } x \in \Omega.$$

We infer from (5.3) and the way h was initially defined and extended that

$$\|u'\|_{L^\infty(\Omega)} = \frac{1}{\delta_0'}\|u-h\|_{L^\infty(B_r\cap U)} \le 1,$$
$$\mathcal{L}_\phi u' = \frac{1}{\delta_0'}[\mathcal{L}_\phi u - \mathcal{L}_\phi h] = \frac{1}{\delta_0'}\Big[f - \text{trace}([\Phi - \mathrm{W}]D^2 h)\Big] =: f'(x) \quad \text{in } B_r \cap U.$$

Notice that $\|f'\|_{L^n(B_r\cap U)} \le 1$ by (5.3). Thus we can apply Proposition 3.7 to get

$$\left|S_\phi(0,c^9) \setminus G_{\frac{N}{\delta_0'}}(u',\Omega)\right| \le C\left(\frac{\delta_0'}{N}\right)^\tau |S_\phi(0,c^9)|.$$

As $G_{\frac{N}{\delta_0'}}(u',\Omega) = G_N(u-h,\Omega)$, we then conclude

$$|S_\phi(0,c^9)| - |G_N(u-h,\Omega)\cap S_\phi(0,c^9)| \le C\left(\frac{\delta_0}{N}\right)^\tau |S_\phi(0,c^9)|$$

yielding

$$\begin{aligned}\left\{1 - C\left(\frac{\delta_0}{N}\right)^\tau\right\} |S_\phi(0,c^9)| &\le |G_N(u-h,\Omega)\cap S_\phi(0,c^9)|\\ &\le |G_N(u-h,\Omega)\cap S_\phi(0,c^9)\cap A_\sigma| + \left|S_\phi(0,c^9)\setminus A_\sigma\right|\\ &\le |G_N(u-h,\Omega)\cap S_\phi(0,c^9)\cap A_\sigma| + C\epsilon^{1/3n}|S_\phi(0,c^9)|,\end{aligned}$$

where the last inequality is by Lemma 4.6. Consequently,

$$|G_N(u-h,\Omega)\cap S_\phi(0,c^9)\cap A_\sigma| \ge \left\{1 - C\left[\left(\frac{\delta_0}{N}\right)^\tau + \epsilon^{1/3n}\right]\right\} |S_\phi(0,c^9)|. \quad (5.11)$$

Next observe that $G_N(u-h,\Omega)\cap S_\phi(0,c^9)\cap A_\sigma \subset G_N(u-h,\Omega)\cap G_N(h,\Omega)$ by (5.5). Therefore,

$$G_N(u-h,\Omega)\cap S_\phi(0,c^9)\cap A_\sigma \subset G_{2N}(u,\Omega)\cap S_\phi(0,c^9)$$

which together with (5.11) gives the conclusion of the lemma. □

Having the improved decay estimates in Lemma 5.1, we can now proceed with density estimates when $\det D^2\phi$ is close to a constant. Our next lemma is concerned with second derivative estimates for solutions to $\mathcal{L}_\phi u = f$. It roughly says that in each section $S_\phi(x,t)$ with small height t, we can find a very large portion (as close to the full measure as we want) where u has second derivatives bounded in a controllable manner. The bound on D^2u is made more precise by using the openings of the quasi paraboloids that touch u from below and above. So far, we have no a priori information on the boundedness of D^2u. However, we can still hope for a bound of order $\frac{1}{t}$ for $|D^2u|$ in $S_\phi(x,t)$ as explained in Sect. 2.2 using an L^∞-norm rescaling of our solution. This heuristic idea explains the factor $\frac{N}{t}$ in the estimate of Lemma 5.2 and the way the solution is rescaled in the proof.

Lemma 5.2 *Assume Ω satisfies* (2.1) *and $\phi \in C^{0,1}(\overline{\Omega})$ is a convex function satisfying* (2.3) *and*

$$1-\epsilon \le \det D^2\phi \le 1+\epsilon \quad in\, \Omega.$$

Assume in addition that $\partial\Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $u \in C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$ be a solution of $\mathcal{L}_\phi u = f$ in Ω with $u = 0$ on $\partial\Omega$ and $\|u\|_{L^\infty(\Omega)} \le 1$. Let $0 < \epsilon_0 < 1$. Then there exists $\epsilon > 0$ depending only on ϵ_0, n, ρ and α such that for any $x \in \overline{\Omega}$ and $t \le c_1$ we have

$$\left|G_{\frac{N}{t}}(u,\Omega) \cap S_\phi(x,t)\right| \ge \left\{1-\epsilon_0 - C\left(\frac{\sqrt{t}}{N}\right)^\tau \|f\|^\tau_{L^n(\Omega)}\right\}\left|S_\phi(x,t)\right| \quad \forall N \ge N_1. \tag{5.12}$$

Here $\tau = \tau(n,\rho)$; C and N_1 depend only on n, ρ and α; $c_1 > 0$ is small depending only on n, ρ, α, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$.

Proof If ϵ is small then by the global $W^{2,p}$ estimates for solutions to the Monge–Ampère equations [28, Theorem 1.2], we have $\phi \in W^{2,2n}(\Omega)$ and hence $\phi \in C^1(\overline{\Omega})$.

Let us first consider the case $x \in \partial\Omega$. We can assume that $x = 0$, $\phi(0) = 0$ and $\nabla\phi(0) = 0$. By the Localization Theorem 2.2, we have

$$kE_t \cap \overline{\Omega} \subset S_\phi(0,t) \subset k^{-1}E_t \cap \overline{\Omega},$$

where $E_t := A_t^{-1}B_{t^{1/2}}$ with $A_t x = x - \tau_t x_n$ and

$$\tau_t \cdot e_n = 0, \qquad \|A_t^{-1}\|,\ \|A_t\| \le k^{-1}|\log t|.$$

We now define the rescaled domains Ω_t, U_t and rescaled functions ϕ_t and u_t as in Sect. 2.2 *that preserve the L^∞-norm of u*. We have

$$\mathcal{L}_{\phi_t} u_t(y) = tf(T^{-1}y) =: f_t(y)$$

where $T := t^{-1/2}A_t$ and

$$\|u_t\|_{L^\infty(\Omega_t)} = \|u\|_{L^\infty(\Omega)} \le 1, \quad u_t = 0 \quad \text{on}\, \partial U_t \cap B_k.$$

Moreover, we have from Proposition 2.12 that

$$(\Omega_t, \phi_t, U_t) \in \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,Ct^{1/2},\alpha} \subset \mathcal{P}_{1-\epsilon,1+\epsilon,\rho,1,\alpha}$$

if $t \le \tilde{c}$, where $\tilde{c} > 0$ is a small constant depending only on n, ρ, α, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$.

Now, applying Lemma 5.1 with $C^* = 1$, we obtain

$$|G_N(u_t, \Omega_t, \phi_t) \cap S_{\phi_t}(0, c^9)| \ge \left\{1 - C\left(N^{-\tau}\delta_0^\tau + \epsilon^{1/3n}\right)\right\} |S_{\phi_t}(0, c^9)|$$

for any $N \ge N_0 = N_0(n, \rho, \alpha)$. Here

$$\delta_0 := \|\Phi_t - \mathrm{W}_t\|^\gamma_{L^n\left(B_{\frac{c^2}{2}} \cap U_t\right)} + \left(\fint_{U_t} |f_t|^n \, dy\right)^{\frac{1}{n}}, \tag{5.13}$$

γ is given by Lemma 4.5, w_t is the function in (vii) in the definition of the class $\mathcal{P}$ associated with the triple (Ω_t, ϕ_t, U_t) and W_t is the cofactor matrix of $D^2 w_t$. This together with the stability of cofactor matrices in Proposition 3.14 implies the existence of $\epsilon = \epsilon(\epsilon_0, n, \rho, \alpha) > 0$ such that for $r := c^9$, we have

$$\begin{aligned}
|G_N(u_t, \Omega_t, \phi_t) \cap S_{\phi_t}(0, r)| &\ge \left\{1 - \epsilon_0\beta - CN^{-\tau}\left(\fint_{U_t} |f_t|^n \, dy\right)^{\frac{\tau}{n}}\right\} |S_{\phi_t}(0, r)| \\
&= \left\{1 - \epsilon_0\beta - C\left(\frac{t}{N}\right)^\tau \left(\fint_{S_\phi(0,t)} |f|^n \, dx\right)^{\frac{\tau}{n}}\right\} |S_{\phi_t}(0, r)|,
\end{aligned}$$

where $\beta = \beta(n, \rho) < 1$ is a universal constant to be chosen later.

As $S_{\phi_t}(0, r) = T(S_\phi(0, rt))$, it is easy to see that for $G_N(u, \Omega, \phi) = G_N(u, \Omega)$,

$$G_N(u_t, \Omega_t, \phi_t) \cap S_{\phi_t}(0, r) = T\left(G_{\frac{N}{t}}(u, \Omega, \phi) \cap S_\phi(0, rt)\right).$$

Therefore we conclude that

$$\left|T\left(G_{\frac{N}{t}}(u, \Omega) \cap S_\phi(0, rt)\right)\right| \ge \left\{1 - \beta\epsilon_0 - C\left(\frac{t}{N}\right)^\tau \left(\fint_{S_\phi(0,t)} |f|^n \, dx\right)^{\frac{\tau}{n}}\right\} \left|T(S_\phi(0, rt))\right| \ \forall t \le \tilde{c}.$$

This is equivalent to

$$\left|G_{\frac{N'}{t}}(u,\Omega)\cap S_\phi(x,t)\right| \geq \left\{1-\epsilon_0\beta - C\left(\frac{t}{N'}\right)^\tau \left(\fint_{S_\phi(x,\frac{t}{r})} |f|^n\,dx\right)^{\frac{\tau}{n}}\right\}\left|S_\phi(x,t)\right| \tag{5.14}$$

giving (5.12) for any $N' \geq N_1 \equiv N_0 r$ and $t \leq r\tilde{c}$.

Next we consider the situation that $x \in \Omega$. We then have the following possibilities:

Case 1: $t \leq h/2$, where $h := \bar{h}(x)$.

If $h \geq c$ where c is defined in Proposition 2.3 then the estimate (5.12) is an easy consequence of the interior density estimates [17, Lemma 4.3] which we now recall.

Lemma 5.3 ([17, Lemma 4.2]) *Let $0 < \alpha_0 < 1$ and Ω be a convex domain in $\mathbb{R}^n$ satisfying $B_{k_0} \subset \Omega \subset B_{k_0^{-1}}$ and $u \in C^1(\Omega)\cap W^{2,n}_{loc}(\Omega)$ be a solution of $\Phi^{ij}u_{ij} = f$ in Ω with $\|u\|_{L^\infty(\Omega)} \leq 1$, where $\phi \in C(\overline{\Omega})$ is a convex function satisfying $\phi = 0$ on $\partial\Omega$. Let $0 < \epsilon_0 < 1$. There exists $\epsilon > 0$ depending only on $\epsilon_0, \alpha_0, k_0$ and n such that if*

$$1-\epsilon \leq \det D^2\phi \leq 1+\epsilon \quad in\,\Omega,$$

then for any section $S_\phi(x_0, \frac{t_0}{\alpha_0}) \subset \Omega_{\frac{\alpha_0+1}{2}} := \{x \in \Omega : \phi(x) < (1-\frac{\alpha_0+1}{2})\min_\Omega \phi\}$, we have

$$|G_{\frac{N}{t_0}}(u,\Omega)\cap S_\phi(x_0,t_0)| \geq \left\{1-\epsilon_0 - C\left(\frac{t_0}{N}\right)^\tau \left(\fint_{S_\phi\left(x_0,\frac{t_0}{\alpha_0}\right)} |f|^n\right)^{\frac{\tau}{n}}\right\}|S_\phi(x_0,t_0)|$$

for every $N \geq N_0$. Here C, τ, N_0 are positive constants depending only on α_0, n and k_0.

Now we consider the remaining situation in **Case 1** when $h \leq c$. We define the rescaled domain $\tilde{\Omega}_h$ and rescaled functions $\tilde{\phi}_h, \tilde{u}_h$ and $\tilde{f}_h$ as in Sect. 2.2 that *preserve the L^∞-norm in a section tangent to the boundary*. Now, we apply Lemma 5.3 to the domain $S_{\tilde{\phi}_h}(0,1)$ with $\alpha_0 = 3/4$, $x_0 = 0$ and $t_0 = t/h \leq 1/2$, noting that $(S_{\tilde{\phi}_h}(0,1))_\alpha = S_{\tilde{\phi}_h}(0,\alpha)$ for all $\alpha > 0$. Thus,

$$\begin{aligned}&\left|G_{\frac{Nh}{t}}\left(\tilde{u}_h, S_{\tilde{\phi}_h}(0,1),\tilde{\phi}_h\right)\cap S_{\tilde{\phi}_h}\left(0,\frac{t}{h}\right)\right|\\ &\quad\geq \left\{1-\epsilon_0 - C\left(\frac{t}{hN}\right)^\tau\left(\fint_{S_{\tilde{\phi}_h}\left(0,\frac{4t}{3h}\right)} \left|\tilde{f}_h\right|^n\right)^{\frac{\tau}{n}}\right\}\left|S_{\tilde{\phi}_h}\left(0,\frac{t}{h}\right)\right|. \end{aligned}\tag{5.15}$$

Let $Ty := h^{-1/2}A_h(y-x)$. Then

$$G_{\frac{Nh}{t}}\left(\tilde{u}_h, S_{\tilde{\phi}_h}(0,1), \tilde{\phi}_h\right) \cap S_{\tilde{\phi}_h}\left(0, \frac{t}{h}\right) = T\left(G_{\frac{N}{t}}(u,\Omega) \cap S_\phi(x,t)\right).$$

Changing variables in (5.15) gives

$$|G_{\frac{N}{t}}(u,\Omega) \cap S_\phi(x,t)| \geq \left\{1 - \epsilon_0 - C\left(\frac{t}{N}\right)^\tau \left(\fint_{S_\phi(x,\frac{4t}{3})} |f|^n\right)^{\tau/n}\right\} |S_\phi(x,t)|$$

and hence (5.12) holds.

Case 2: $h/2 < t \leq r\tilde{c}/\bar{c} \equiv c_1$ where $\bar{c} > 1$ is the constant in Proposition 2.14. Then by Proposition 2.14, we know that $S_\phi(x,2t) \subset S_\phi(z,\bar{c}t)$ for some $z \in \partial\Omega$, and by Theorem 2.13(b),

$$C_1 t^{n/2} \leq |S_\phi(x,t)| \leq C_2 t^{n/2} \quad \forall t \leq c_0.$$

Using these inequalities and the estimate (5.14) in the case of boundary section, we get

$$\begin{aligned}\left|S_\phi(x,t) \setminus G_{\frac{N}{t}}(u,\Omega)\right| &\leq \left|S_\phi(z,\bar{c}t) \setminus G_{\frac{N}{\bar{c}t}}(u,\Omega)\right| \\ &\leq \left\{\epsilon_0\beta + C\left(\frac{\bar{c}t}{N}\right)^\tau \left(\fint_{S_\phi(z,\bar{c}t/r)} |f|^n\, dx\right)^{\frac{\tau}{n}}\right\} |S_\phi(z,\bar{c}t)| \\ &\leq \left\{\epsilon_0\beta + C\left(\frac{\sqrt{t}}{N}\right)^\tau \|f\|^\tau_{L^n(\Omega)}\right\} |S_\phi(x,t)|\, C_1^{-1}C_2\bar{c}^{\frac{n}{2}}.\end{aligned}$$

This implies (5.12) as desired by choosing $\beta = C_1C_2^{-1}\bar{c}^{\frac{-n}{2}}$ and $c_1 = r\tilde{c}/\bar{c} = c^9\tilde{c}/\bar{c}$. □

The next lemma is a key technical ingredient in our global $W^{2,p}$ estimates. It propagates a point in a given section where the solution u of $\mathcal{L}_\phi u = f$ has bounded second derivative to almost all points in that section. More precisely, it says that if in a small section $S_\phi(x,t)$ we can find a point where u is touched from above and below by quasi paraboloids of opening γ generated by ϕ then on a set of nearly full measure of $S_\phi(x,t)$, u is touched from above and below by quasi paraboloids of opening $N\gamma$ for some controllable constant N, provided that det $D^2\phi$ is close to a constant.

Lemma 5.4 *Assume Ω is uniformly convex satisfying (2.1) and $\phi \in C^{0,1}(\overline{\Omega})$ is a convex function satisfying (2.3) and*

$$1 - \epsilon \leq \det D^2\phi \leq 1 + \epsilon \quad in\, \Omega.$$

Assume in addition that $\partial\Omega \in C^{2,\alpha}$ *and* $\phi \in C^{2,\alpha}(\partial\Omega)$ *for some* $\alpha \in (0,1)$. *Let* $u \in C^1(\Omega) \cap W^{2,n}_{loc}(\Omega)$ *be a solution of* $\mathcal{L}_\phi u = f$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$. *Let* $0 < \epsilon_0 < 1$. *Then there exists* $\epsilon > 0$ *depending only on* ϵ_0, n, ρ *and* α *such that for any* $x \in \overline{\Omega}$, $t \le c_2$ *and* $S_\phi(x,t) \cap G_\gamma(u,\Omega) \neq \emptyset$ *we have*

$$\left|G_{N\gamma}(u,\Omega) \cap S_\phi(x,t)\right| \ge \left\{1 - \epsilon_0 - C(N\gamma)^{-\tau}\left(\fint_{S_\phi(\tilde{x},\Theta t)} |f|^n\,dx\right)^{\frac{\tau}{n}}\right\}\left|S_\phi(x,t)\right| \tag{5.16}$$

for all $\tilde{x} \in S_\phi(x,t)$ *and* $N \ge N_2$. *Here* τ *and* Θ *depend only on* n *and* ρ; C, c_2 *and* N_2 *depend only on* n, ρ, α, *the uniform convexity of* Ω, $\|\partial\Omega\|_{C^{2,\alpha}}$ *and* $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$.

Proof As explained in the proof of Lemma 5.2, we have $\phi \in C^1(\Omega) \cap W^{2,2n}(\Omega)$ if ϵ is small.

Let us first consider the case $x \in \partial\Omega$. We can assume that $x = 0$, $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Let $h = \theta t$ where $\theta = \theta(n,\rho) > 1$ will be chosen later. Let A_h be the affine transformation as in the Localization Theorem 2.2. We now define the rescaled domains Ω_h, U_h and rescaled functions ϕ_h, $\tilde{u}_h$ and $\tilde{f}_h$ as in Sect. 2.2 *that almost preserve the* L^∞*-norm of* D^2u. Let $T = h^{-1/2}A_h$.

Let $\bar{x} \in S_\phi(0,t) \cap G_\gamma(u,\Omega)$ and $\bar{y} := T\bar{x}$. Then

$$-\gamma\, d(x,\bar{x})^2 \le u(x) - u(\bar{x}) - \nabla u(\bar{x})\cdot(x-\bar{x}) \le \gamma\, d(x,\bar{x})^2, \quad \forall x \in \Omega.$$

By changing variables and recalling that $\Omega_h = T(\Omega)$, $\tilde{u}_h(y) = h^{-1}u(T^{-1}y)$, we get

$$\begin{aligned} -\gamma\,\frac{d(T^{-1}y,T^{-1}\bar{y})^2}{\theta t} &\le \tilde{u}_h(y) - \tilde{u}_h(\bar{y}) - \nabla\tilde{u}_h(\bar{y})\cdot(y-\bar{y}) \\ &\le \gamma\,\frac{d(T^{-1}y,T^{-1}\bar{y})^2}{\theta t}, \ \forall y \in \Omega_h. \end{aligned} \tag{5.17}$$

Since $\bar{x} \in S_\phi(0,t) \subset S_\phi(0,\theta t)$, we have by the engulfing property of sections in Theorem 2.13(a) $S_\phi(0,\theta t) \subset S_\phi(\bar{x},\theta^2 t)$. It follows that $d(x,\bar{x})^2 \le \theta^2 t$ for $x \in S_\phi(0,\theta t)$ yielding $d(T^{-1}y,T^{-1}\bar{y})^2 \le \theta^2 t$ for all $y \in U_h := T(S_\phi(0,h))$. Consequently, if we define

$$v(y) := \frac{1}{\theta\gamma}\left[\tilde{u}_h(y) - \tilde{u}_h(\bar{y}) - \nabla\tilde{u}_h(\bar{y})\cdot(y-\bar{y})\right], \quad y \in \Omega_h, \tag{5.18}$$

then $|v| \le 1$ in U_h. Thanks to Lemma 5.5 below we get for $t \le c_\alpha$

$$\|v\|_{C^{2,\alpha}(\partial U_h \cap B_k)} \le C_\alpha, \tag{5.19}$$

where c_α, C_α depend only on n, ρ, α, the uniform convexity of Ω, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$. By (5.17) we have

$$|v(y)| \le \frac{1}{\theta^2 t} d(T^{-1}y, T^{-1}\bar{y})^2 \le \frac{1}{\theta} d_{\phi_h}(y, \bar{y})^2 \quad \forall y \in T(\Omega), \tag{5.20}$$

where we recall $\theta t = h$ and

$$d_{\phi_h}(y, \bar{y})^2 := \phi_h(y) - \phi_h(\bar{y}) - \nabla\phi_h(\bar{y}) \cdot (y - \bar{y}) = h^{-1} d(T^{-1}y, T^{-1}\bar{y})^2.$$

Moreover

$$\mathcal{L}_{\phi_h} v = (\theta\gamma)^{-1} \mathcal{L}_{\phi_h} \tilde{u_h} = (\theta\gamma)^{-1} \tilde{f_h} \equiv (\theta\gamma)^{-1} f(T^{-1}y) =: \tilde{f}(y).$$

Because $\bar{x} \in S_\phi(0, t)$, we have $\bar{y} = T\bar{x} \in S_{\tilde{\phi}}(0, \frac{1}{\theta})$. Hence, we can choose $\theta > 1$ depending on n, ρ, k such that $\bar{y} \in B_{\frac{c^2}{8}} \cap \tilde{U}$. With this choice of θ, we have by Proposition 2.12

$$(\Omega_h, \phi_h, U_h) \in \mathcal{P}_{1-\epsilon, 1+\epsilon, \rho, Ch^{1/2}, \alpha} \subset \mathcal{P}_{1-\epsilon, 1+\epsilon, \rho, 1, \alpha}$$

if $t \le \tilde{c}$, where $\tilde{c} > 0$ is a small constant depending only on n, ρ, α, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$. Here we can choose $\tilde{c} \le c_\alpha$, and hence it also depends on the uniform convexity of Ω.

Thus, using (5.19) and (5.20), we can apply Lemma 5.1 to $\bar{v} := v/C_\alpha$ to obtain

$$|G_N(\bar{v}, \Omega_h, \phi_h) \cap S_{\tilde{\phi}}(0, c^9)| \ge \left\{1 - C\left(N^{-\tau}\delta_0^\tau + \epsilon^{1/3n}\right)\right\} |S_{\phi_h}(0, c^9)|$$

for any $N \ge N_0$, where δ_0 is as in (5.13). This together with the stability of cofactor matrices in Proposition 3.14 implies the existence of $\epsilon = \epsilon(\epsilon_0, n, \rho, \alpha) > 0$ such that

$$\begin{aligned}
&|G_N(\bar{v}, \Omega_h, \phi_h) \cap S_{\phi_h}(0, r)| \\
&\quad \ge \left\{1 - \epsilon_0\beta - CN^{-\tau}\left(\fint_{U_h} |\tilde{f}|^n \, dy\right)^{\frac{\tau}{n}}\right\} |S_{\phi_h}(0, r)| \\
&\quad = \left\{1 - \epsilon_0\beta - C\left(\frac{1}{\theta\gamma N}\right)^\tau \left(\fint_{S_\phi(0,\theta t)} |f|^n \, dx\right)^{\frac{\tau}{n}}\right\} |S_{\phi_h}(0, r)|,
\end{aligned}$$

where for simplicity we have denoted

$$r := c^9$$

and $\beta = \beta(n, \rho) < 1$ is a universal constant to be chosen later. It follows that

$$\begin{aligned} &|S_{\phi_h}(0, r) \setminus G_N(\bar{v}, \Omega_h, \phi_h)| \\ &\quad \le \left\{ \epsilon_0 \beta + C \left(\frac{1}{\theta \gamma N} \right)^{\tau} \left(\fint_{S_\phi(0,\theta t)} |f|^n \, dx \right)^{\frac{\tau}{n}} \right\} |S_{\phi_h}(0, r)|. \end{aligned}$$

As $S_{\phi_h}(0, r) = T(S_\phi(0, \theta r t))$ and $\bar{v}(y) = \frac{1}{C_\alpha \theta^2 \gamma t}\big[u(T^{-1}y) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (T^{-1}y - \bar{x})\big]$, it is easy to see that

$$G_N(\bar{v}, \Omega_h, \phi_h) \cap S_{\phi_h}(0, r) = T\Big(G_{C_\alpha N \theta \gamma}(u, \Omega) \cap S_\phi(0, r\theta t)\Big).$$

Therefore, by the volume estimates in Theorem 2.13(b), we conclude that

$$\begin{aligned} &|S_\phi(0, rt) \setminus G_{C_\alpha N \theta \gamma}(u, \Omega)| \\ &\quad \le |S_\phi(0, r\theta t) \setminus G_{C_\alpha N \theta \gamma}(u, \Omega)| \\ &\quad \le \left\{ C_1^{-1} C_2 \theta^{\frac{n}{2}} \epsilon_0 \beta + C \left(\frac{1}{C_\alpha \theta \gamma N} \right)^{\tau} \left(\fint_{S_\phi(0,\theta t)} |f|^n \, dx \right)^{\frac{\tau}{n}} \right\} \big|S_\phi(0, rt)\big|. \end{aligned}$$

By setting $N' = C_\alpha N \theta$, $\beta' = C_1^{-1} C_2 \theta^{n/2} \beta$, we can rewrite this as

$$\begin{aligned} &\big|G_{N'\gamma}(u, \Omega) \cap S_\phi(x, t)\big| \\ &\quad \ge \left\{ 1 - \epsilon_0 \beta' - C \left(\frac{1}{\gamma N'} \right)^{\tau} \left(\fint_{S_\phi(x,\frac{\theta}{r}t)} |f|^n \, dx \right)^{\frac{\tau}{n}} \right\} \big|S_\phi(x, t)\big| \end{aligned} \tag{5.21}$$

for any $N' \ge N_2 \equiv C_\alpha N_0 \theta$ and $t \le r\tilde{c}$. From Theorem 2.13(a) we have $S_\phi(x, \frac{\theta}{r}t) \subset S_\phi(\tilde{x}, \frac{\theta_* \theta}{r}t)$ for any $\tilde{x} \in S_\phi(x, t)$. Therefore, by Theorem 2.13(b), we see that (5.21) yields (5.16).

Next we consider the situation that $x \in \Omega$. We then have the following possibilities:

Case 1: $t \le h/2$, where $h := \bar{h}(x)$. This case can be handled as **Case 1** of Lemma 5.2, using now [17, Lemma 4.5] and affine transformations similar to the ones at the beginning of the proof of this lemma.

Case 2: $h/2 < t \le r\tilde{c}/\bar{c} \equiv c_2$, where $\bar{c} > 1$ is the constant in Proposition 2.14. Then by Proposition 2.14, we know that $S_\phi(x, 2t) \subset S_\phi(z, \bar{c}t)$ for some $z \in \partial\Omega$.

Thus, by the estimate (5.21) in the case of boundary section, we get

$$\left|S_\phi(x,t)\setminus G_{N\gamma}(u,\Omega)\right| \le \left|S_\phi(z,\bar{c}t)\setminus G_{N\gamma}(u,\Omega)\right| \le \left\{\epsilon_0\beta' + C\left(\frac{1}{\gamma N}\right)^{\tau}\left(\fint_{S_\phi\left(z,\frac{\theta\bar{c}}{r}t\right)} |f|^n\,dx\right)^{\frac{\tau}{n}}\right\}\left|S_\phi(z,\bar{c}t)\right|. \tag{5.22}$$

For any $\tilde{x}\in S_\phi(x,t)\subset S_\phi(z,\frac{\theta\bar{c}}{r}t)$, we get $S_\phi(z,\frac{\theta\bar{c}}{r}t)\subset S_\phi(\tilde{x},\frac{\theta_*\theta\bar{c}}{r}t)$ by the engulfing property in Theorem 2.13. Now, using (5.22) and the volume estimates in this theorem, we find that

$$\left|S_\phi(x,t)\setminus G_{N\gamma}(u,\Omega)\right| \le \left\{\epsilon_0\beta' C_1^{-1}C_2\bar{c}^{\frac{n}{2}} + C\left(\frac{1}{\gamma N}\right)^{\tau}\left(\fint_{S_\phi(\tilde{x},\frac{\theta_*\theta\bar{c}}{r}t)} |f|^n\,dx\right)^{\frac{\tau}{n}}\right\}\left|S_\phi(x,t)\right|.$$

This gives (5.16) with $\Theta := \theta_*\theta\bar{c}/r$ if we choose β such that $\beta' C_1^{-1}C_2\bar{c}^{n/2} = \beta C_1^{-2}C_2^2(\theta\bar{c})^{n/2} = 1$. □

In the next lemma we prove that the function v defined as in the proof of Lemma 5.4 has uniform $C^{2,\alpha}$ bound on $\partial U_h\cap B_k$.

Lemma 5.5 *Let v be defined as in (5.18). There exist $C_\alpha, c_\alpha > 0$ depending only on n,ρ,α, the uniform convexity of Ω, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$ such that for $t\le c_\alpha$, we have*

$$\|v\|_{C^{2,\alpha}(\partial U_h\cap B_k^+)} \le C_\alpha. \tag{5.23}$$

Proof Since $\partial\Omega$ is $C^{2,\alpha}$ at the origin and Ω is uniformly convex, we have

$$\left|x_n - q(x')\right| \le M\left|x'\right|^{2+\alpha} \quad \text{for } x=(x',x_n)\in\partial\Omega\cap B_\rho,$$

where $q(x')$ is a homogeneous quadratic polynomial with

$$D^2_{x'}q \ge C^{-1}I_{n-1}. \tag{5.24}$$

Recall $h=\theta t$. Then it follows from the definition of U_h and Proposition 2.12 that

$$\left|x_n - h^{1/2}q(x')\right| \le Ch^{\frac{1+\alpha}{2}}\left|x'\right|^{2+\alpha} \quad \text{on } \partial U_h\cap B_k^+ \tag{5.25}$$

if $h \le h_0$, where h_0, C depend only on n, ρ, α and the $C^{2,\alpha}$ norms of $\partial\Omega$ and $\phi|_{\partial\Omega}$ at the origin. Hence by combining with (5.24), we see that if $h \le h_0$ (h_0 now depends also on the uniform convexity of Ω) then on $\partial U_h \cap B_k^+$,

$$\frac{1}{2}h^{1/2}q(x') \le x_n \le 2h^{1/2}q(x'). \tag{5.26}$$

Let

$$l(y) = \frac{-1}{\theta\gamma}\big[\tilde{u}_h(\bar{y}) + \nabla\tilde{u}_h(\bar{y})\cdot(y-\bar{y})\big].$$

Then $l(y) = v(y)$ for $y \in \partial U_h \cap B_k^+$. Since $|v| \le 1$ in U_h, we find that

$$|l(y) - l(z)| = \frac{1}{\theta\gamma}\,|\nabla\tilde{u}_h(\bar{y})\cdot(y-z)| \le 2 \quad \forall y, z \in \partial U_h \cap B_k^+. \tag{5.27}$$

All constants in this lemma, unless otherwise indicated, depend only on n, ρ, α, the uniform convexity of Ω and the $C^{2,\alpha}$ norms of $\partial\Omega$ and $\phi|_{\partial\Omega}$.

We now divide the proof into three steps.

Step 1. l is uniformly Lipschitz at the origin: there exists $L > 0$ such that

$$|l(z) - l(0)| \le L\,|z| \quad \forall z \in \partial U_h \cap B_{k^2}^+.$$

Take $z \in \partial U_h \cap B_{k^2}^+\backslash\{0\}$. Let $\mathcal{C}$ be the curve which is the intersection of $\partial U_h \cap B_k^+$ and the vertical plane (P) passing through z and the origin. Let p and q be the intersection of $\mathcal{C}$ with ∂B_k^+. We now have a plane curve $\mathcal{C}$ in (P) which can be assumed to be the usual xy-plane. It is easy to see from (5.24)–(5.26) that $\mathcal{C}$ is a graph in the y-direction $\mathcal{C} = \{(x, \varphi(x))\}$ with $C^{1,1}$ norm comparable to $h^{1/2}$, that is

$$C^{-1}h^{1/2} \le \varphi^{''}(x) \le Ch^{1/2}.$$

Note that, this also follows from the proof of [23, Lemma 4.2] for the case of uniformly convex domains Ω.

Since $|p| = |q| = k$, we find that

$$y_p \sim h^{1/2}, y_q \sim h^{1/2}, \ \ |x_p| \sim k, |x_q| \sim k.$$

Without loss of generality, we can assume that $y_p \le y_q$ and $x_p < 0 < x_q$, that is, p is on the left half-plane while q is on the right half-plane. The horizontal line through p intersects $\mathcal{C}$ at another point $q^{'}$. Since $\varphi^{''} \le Ch^{1/2}$ and $y_{q^{'}} = y_p \sim h^{1/2}$, we must have $x_{q^{'}} \sim k$. In particular, z lies on the arc $p0q^{'}$. We can assume that z lies on the arc $0q^{'}$. Now, take a ray emanating from $q^{'}$ and parallel to $0z$. This ray is exactly $q^{'}0$ when $z \equiv q^{'}$ and it is $q^{'}p$ when $z \to 0$. Thus, by continuity, there must be a point m on the arc $0p$ such that $q^{'}m$ is parallel to $0z$. Clearly, $\left|q^{'}-m\right| \geq x_{q^{'}} \sim k$. Using $z = \frac{|z|}{\left|q^{'}-m\right|}(q^{'}-m)$, we find from (5.27) that

$$\begin{aligned}
|l(z)-l(0)| &= \frac{1}{\theta\gamma}\left|\nabla\tilde{u}_h(\bar{y})\cdot z\right| = \frac{|z|}{\left|q^{'}-m\right|}\frac{1}{\theta\gamma}\left|\nabla\tilde{u}_h(\bar{y})\cdot(q^{'}-m)\right| \\
&\leq \frac{|z|}{\left|q^{'}-m\right|} \leq L\,|z|\,.
\end{aligned}$$

Thus, l is Lipschitz at 0.

Step 2. Let $\frac{1}{\theta\gamma}\nabla\tilde{u}_h(\bar{y}) = (a', a_n)$. Then

$$\left|a'\right| \leq 2L \quad \text{and} \quad |a_n|\,h^{1/2} \leq CL.$$

First, we note that the projection of $\partial U_h \cap B_k$ on $\{x_n = 0\}$ contains a ball of radius comparable to k. By rotating coordinates in $\{x_n = 0\}$, we can assume that $a' = (A, 0, \ldots, 0)$. Take a curve $\mathcal{C} = \left\{(x, 0, \ldots, 0, \varphi(x)) \mid -k^2 \leq x \leq k^2\right\}$ in $\partial U_h \cap B_k$ that lies in the x_1x_n plane. Note that $\varphi(x) \sim h^{1/2}x^2$. By the Lipschitz property of l in **Step 1**, we have

$$\frac{1}{\theta\gamma}\left|\nabla\tilde{u}_h(\bar{y})\cdot(x, 0, \ldots, 0, \varphi(x))\right| = |Ax + a_n\varphi(x)| \leq L\sqrt{x^2+(\varphi(x))^2} \leq 2L\,|x|\,.$$

Dividing the above inequalities by x and then letting $x \to 0$, we get the desired bound

$$\left|a'\right| = |A| \leq 2L.$$

As a consequence, we have

$$|a_n\varphi(x)| \leq |Ax| + 2L\,|x| \leq 4L\,|x|\,.$$

Using the lower bound on the growth of φ and evaluating at $|x| \sim k^2$, we obtain

$$|a_n|\,h^{1/2} \leq CL.$$

Step 3. We have

$$\|v\|_{C^{2,\alpha}\left(\partial U_h \cap B_k^+\right)} = \|l\|_{C^{2,\alpha}\left(\partial U_h \cap B_k^+\right)} \leq C.$$

Recall from (5.25) that $\partial U_h \cap B_k$ is a graph in the e_n direction, that is,

$$\partial U_h \cap B_k = \left\{(x', \psi(x')) : |x'| \leq C_k\right\},$$

with the following properties:

$$(a)\,\|\nabla\psi\|_{L^\infty} + \|D^2\psi\|_{L^\infty} \le Ch^{1/2}, \qquad (b)\,\|D^2\psi\|_{C^\alpha} \le Ch^{\frac{1+\alpha}{2}}.$$

For $y \in \partial U_h \cap B_k$, we have $y = (x', \psi(x'))$ and

$$l(y) = l(0) - \frac{1}{\theta\gamma}\nabla\tilde{u}_h(\bar{y})\cdot y = l(0) - a'\cdot x' - a_n\psi(x')$$

where $l(0)$ is a constant bounded by 1. Clearly, the $C^{2,\alpha}$ bound for l on $\partial U_h \cap B_k$ now follows from $(a)-(b)$ and **Step 2**. $\square$

5.2 Global $W^{2,p}$ estimates

In this subsection we will use the density estimates established in Sect. 5.1 to derive global $W^{2,p}$-estimates for solution u of the linearized equation $\mathcal{L}_\phi u = f$ when $f \in L^q(\Omega)$ for some $q > n$ as stated in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The assumptions on Ω and ϕ in the statement of our theorem imply that Ω satisfy (2.1) for some $\rho > 0$ and, by Proposition 2.4, ϕ satisfies (2.3). Thus, Ω and ϕ satisfy the conditions of Lemmas 5.2 and 5.4.

By the ABP estimate, it suffices to establish our $W^{2,p}$ estimates in the form

$$\|D^2u\|_{L^p(\Omega)} \le C\Big(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)}\Big).$$

We first observe that by working with the function $v := \dfrac{\epsilon u}{\epsilon\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)}}$ instead of u, it is enough to show that there exist $\epsilon, C > 0$ depending only on p, q, n and Ω such that if $1-\epsilon \le \det D^2\phi \le 1+\epsilon$ in Ω, $\phi = u = 0$ on $\partial\Omega$, $\mathcal{L}_\phi u = f$ in Ω, $\|u\|_{L^\infty(\Omega)} \le 1$ and $\|f\|_{L^q(\Omega)} \le \epsilon$, then

$$\|D^2u\|_{L^p(\Omega)} \le C. \tag{5.28}$$

Notice that $u \in W^{2,s}_{loc}(\Omega)$ for any $n < s < q$ as a consequence of $W^{2,p}_{loc}$ estimates in [17].

Let $N_* = \max\{N_1, N_2\}$ where N_1 and N_2 are the large constants in Lemmas 5.2 and 5.4 and $\hat{c} = \min\{c_1, c_2\}$ where c_1 and c_2 are the small constants in the above lemmas. Fix $M \ge N_*$ so that $1/M < \hat{c}$. Next select $0 < \epsilon_0 < 1/2$ such that

$$M^q\sqrt{2\epsilon_0} = \frac{1}{2}$$

and $\epsilon = \epsilon(\epsilon_0, n, \Omega) = \epsilon(p, q, n, \Omega)$ be the smallest of the constants in Lemmas 5.2 and 5.4. With this choice of ϵ, we are going to show that (5.28) holds. Applying Lemma 5.2 to the function u and using $\|f\|_{L^q(\Omega)} \le \epsilon$ we obtain

$$\left|S_\phi(x,t)\cap G_{\frac{M}{t}}(u,\Omega)\right| \geq \left(1-\epsilon_0-C\epsilon^\tau\right)\,|S_\phi(x,t)|$$

as long as $x\in\overline{\Omega}$ and $t\leq\hat{c}$. By taking ϵ even smaller if necessary we can assume $C\epsilon^\tau<\epsilon_0$. Then it follows from the above inequality that

$$\left|S_\phi(x,t)\setminus G_{\frac{M}{t}}(u,\Omega)\right| \leq 2\epsilon_0\,|S_\phi(x,t)| \quad \text{for any } x\in\overline{\Omega},\ t\leq\hat{c}. \tag{5.29}$$

Let $1/h\leq\hat{c}$. For $x\in\Omega\setminus G_{hM}(u,\Omega)$, define

$$g(t):=\frac{\left|(\Omega\setminus G_{hM}(u,\Omega))\cap S_\phi(x,t)\right|}{|S_\phi(x,t)|}.$$

We have $\lim_{t\to 0}g(t)=1$. Also, if $1/h\leq t\leq\hat{c}$, then (5.29) gives

$$\begin{aligned}\left|(\Omega\setminus G_{hM}(u,\Omega))\cap S_\phi(x,t)\right| &\leq |S_\phi(x,t)\setminus G_{hM}(u,\Omega)|\\ &\leq |S_\phi(x,t)\setminus G_{M/t}(u,\Omega)|\leq 2\epsilon_0\,|S_\phi(x,t)|.\end{aligned}$$

Therefore $g(t)\leq 2\epsilon_0$ for $t\in[1/h,\hat{c}]$. Then by continuity of g, there exists $t_x\leq 1/h$ such that $g(t_x)=2\epsilon_0$.

Thus for any $x\in\Omega\setminus G_{hM}(u,\Omega)$ there is $t_x\leq 1/h\leq\hat{c}$ satisfying

$$\left|\left(\Omega\setminus G_{hM}(u,\Omega)\right)\cap S_\phi(x,t_x)\right| = 2\epsilon_0\,|S_\phi(x,t_x)|. \tag{5.30}$$

We now claim that (5.30) implies

$$S_\phi(x,t_x)\subset\left(\overline{\Omega}\setminus G_h(u,\Omega)\right)\cup\left\{z\in\overline{\Omega}:\mathcal{M}(f^n)(z)>(c^*Mh)^n\right\}, \tag{5.31}$$

where $c^*:=(\frac{\epsilon_0}{C})^{1/\tau}$, and

$$\mathcal{M}(F)(z):=\sup_{t\leq\hat{c}}\frac{1}{|S_\phi(z,t)|}\int\limits_{S_\phi(z,t)}|F(y)|\,dy \quad \forall z\in\overline{\Omega}.$$

Indeed, since otherwise there exists $\bar{x}\in S_\phi(x,t_x)\cap G_h(u,\Omega)$ such that $\mathcal{M}(f^n)(\bar{x})\leq(c^*Mh)^n$. Note also that $t_x\leq\hat{c}$. Then by Lemma 5.4 applied to u we get

$$\left|S_\phi(x,t_x)\cap G_{hM}(u,\Omega)\right| > (1-2\epsilon_0)\,|S_\phi(x,t_x)|$$

yielding

$$\left|\left(\Omega\setminus G_{hM}(u,\Omega)\right)\cap S_\phi(x,t_x)\right| \le \left|S_\phi(x,t_x)\setminus G_{hM}(u,\Omega)\right| < 2\epsilon_0\,|S_\phi(x,t_x)|.$$

This is a contradiction with (5.30) and so (5.31) is proved. We infer from (5.30), (5.31) and Theorem 2.15 that

$$|\Omega\backslash G_{hM}(u,\Omega)| \le \sqrt{2\epsilon_0}\left[|\Omega\backslash G_h(u,\Omega)| + \left|\{x\in\Omega : \mathcal{M}(f^n)(x) > (c^* M h)^n\}\right|\right], \tag{5.32}$$

as long as $1/h \le \hat{c}$.

For $k = 0, 1, \ldots,$ set

$$a_k := |\Omega\setminus G_{M^k}(u,\Omega)| \quad\text{and}\quad b_k := \left|\{x\in\Omega : \mathcal{M}(f^n)(x) > (c^* M M^k)^n\}\right|.$$

Let $h = M$, then we get from (5.32) that $a_2 \le \sqrt{2\epsilon_0}(a_1+b_1)$. Next let $h = M^2$, then $a_3 \le \sqrt{2\epsilon_0}(a_2+b_2) \le 2\epsilon_0 a_1 + 2\epsilon_0 b_1 + \sqrt{2\epsilon_0}\,b_2$. Continuing in this way we conclude that

$$|\Omega\backslash G_{M^{k+1}}(u,\Omega)| = a_{k+1} \le \left(\sqrt{2\epsilon_0}\right)^k a_1 + \sum_{i=1}^{k}\left(\sqrt{2\epsilon_0}\right)^{(k+1)-i} b_i \quad \text{for } k=1,2,\ldots \tag{5.33}$$

We are now ready to prove (5.28). We have

$$\begin{aligned}
\int_\Omega |D_{ij}u|^p\,dx &= p\int_0^\infty t^{p-1}\left|\{x\in\Omega : |D_{ij}u(x)|>t\}\right|dt\\
&= p\int_0^{M^{\frac{q}{p}}} t^{p-1}\left|\{x\in\Omega : |D_{ij}u(x)|>t\}\right|dt\\
&\quad + p\sum_{k=1}^{\infty}\int_{M^{\frac{qk}{p}}}^{M^{\frac{q(k+1)}{p}}} t^{p-1}\left|\left\{x\in\Omega : |D_{ij}u(x)|>t\right\}\right|dt\\
&\le |\Omega| M^q + \left(M^q-1\right)\sum_{k=1}^{\infty} M^{qk}\left|\left\{x\in\Omega :\ |D_{ij}u(x)| > M^{\frac{qk}{p}}\right\}\right|
\end{aligned}$$

$$\leq |\Omega|M^q + \left(M^q - 1\right)\left[\sum_{k=1}^{\infty} M^{qk}\left|\Omega \setminus A^{\mathrm{loc}}_{\left(cM^{\frac{k(q-p)}{2p}}\right)^{\frac{-2}{n-1}}}\right| + \sum_{k=1}^{\infty} M^{qk}\left|\Omega \setminus G_{M^k}(u,\Omega)\right|\right]$$

$$\leq |\Omega|M^q + \left(M^q - 1\right)\left[C(n,\epsilon,\Omega)\sum_{k=1}^{\infty} M^{k\left(q+\left(\frac{q}{p}-1\right)\frac{\ln\sqrt{C\epsilon}}{C}\right)} + \sum_{k=1}^{\infty} M^{qk}\left|\Omega \setminus G_{M^k}(u,\Omega)\right|\right],$$

where we used (3.1) with $m = q/p > 1$ and $\beta = M^k$ in the second inequality and used (3.2) in the last inequality. Since $\epsilon > 0$ is small, the first summation in the last expression is finite and hence (5.28) will follow if we can show that $\sum_{k=1}^{\infty} M^{kq}|\Omega \setminus G_{M^k}(u,\Omega)| \leq C$. For this, let us employ (5.33) to obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} M^{kq}|\Omega \setminus G_{M^k}(u,\Omega)| \\
&\quad\leq a_1 \sum_{k=1}^{\infty} M^{kq}\left(\sqrt{2\epsilon_0}\right)^{k-1} + \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} M^{kq}\left(\sqrt{2\epsilon_0}\right)^{k-i} b_i \\
&\quad= \frac{a_1}{\sqrt{2\epsilon_0}}\sum_{k=1}^{\infty}\left(M^q\sqrt{2\epsilon_0}\right)^k + \left[\sum_{j=1}^{\infty}\left(M^q\sqrt{2\epsilon_0}\right)^j\right]\left[\sum_{i=0}^{\infty} M^{iq} b_i\right] \\
&\quad= \frac{a_1}{\sqrt{2\epsilon_0}}\sum_{k=1}^{\infty} 2^{-k} + \left[\sum_{j=1}^{\infty} 2^{-j}\right]\left[\sum_{i=0}^{\infty} M^{iq} b_i\right] = \frac{a_1}{\sqrt{2\epsilon_0}} + \sum_{i=0}^{\infty} M^{iq} b_i.
\end{aligned}$$

But as $f^n \in L^{\frac{q}{n}}(\Omega)$ and $q > n$, by the strong-type estimate in Theorem 2.16 we have

$$\begin{aligned}
\int_\Omega \left|\mathcal{M}(f^n)(x)\right|^{\frac{q}{n}} dx &\leq C(n,q,\rho)\int_\Omega \left|f^n(x)\right|^{\frac{q}{n}} dx \\
&\leq C(n,q,\rho)\|f\|^q_{L^q(\Omega)} \leq C(n,q,\rho)
\end{aligned}$$

implying $\sum_{i=0}^{\infty}(M^n)^{i\frac{q}{n}} b_i \leq C$. Thus $\sum_{k=1}^{\infty} M^{kq}|\Omega \setminus G_{M^k}(u,\Omega)| \leq C$ and (5.28) is proved. □

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove the theorem for the case $\varphi = 0$ since $\tilde{u} := u - \varphi \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ is the solution to the linearized Monge–Ampère equation

$$\mathcal{L}_\phi \tilde{u} = \tilde{f} \text{ in } \Omega, \quad \text{and} \quad \tilde{u} = 0 \text{ on } \partial\Omega,$$

where $\tilde{f} := f - \Phi^{ij}\varphi_{ij} \in L^q(\Omega)$. Indeed, since $g \in C(\overline{\Omega})$, we have $\phi \in W^{2,\frac{(n-1)qs}{s-q}}(\Omega)$ by Savin's global $W^{2,p}$ estimates [28]. Thus $\Phi^{ij} \in L^{\frac{qs}{s-q}}(\Omega)$ for all i, j and hence $\tilde{f} := f - \Phi^{ij}\varphi_{ij} \in L^q(\Omega)$.

In view of Theorem 1.1 and the interior $W^{2,p}$ estimates obtained in [17], the theorem follows by localizing boundary sections of ϕ using Theorem 2.2. For completeness, we sketch the proof.

The assumptions on Ω and ϕ imply that Ω satisfies (2.1) for some $\rho > 0$ and ϕ satisfies (2.3). Let ϵ be the small constant given by an analogous version of Theorem 1.1 which will be explained later. In particular, ϵ depends only on n, p, q, λ, Λ, ρ and α. Let c be as in Remark 2.6.

Since $g \in C(\overline{\Omega})$, we can find $m \leq c$ depending only on ϵ, λ and the modulus of continuity of g such that

$$|g(x) - g(y)| \leq \lambda\epsilon \text{ for all } x, y \in \overline{\Omega} \text{ satisfying } |x - y| \leq m.$$

Hence it follows from (2.7) that for $s \leq m^3$ and any boundary point $y \in \partial\Omega$, we have

$$|g(x) - g(y)| \leq \lambda\epsilon \text{ for all } x \in S_\phi(y, s). \tag{5.34}$$

Let us consider a boundary point $y \in \partial\Omega$ and for simplicity we assume that $y = 0$. We can assume further that Ω satisfies (2.5), $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Then by the Localization Theorem, there is a linear map $T_s = s^{-1/2} A_s$ such that

$$\overline{\Omega} \cap B_k \subset T_s(S_\phi(0, s)) \subset \overline{\Omega} \cap B_{k^{-1}}, \tag{5.35}$$

where $\det A_s = 1$ and $\|A_s\|, \|A_s^{-1}\| \leq k^{-1}|\log s|$. By working with the function $g(0)^{\frac{-1}{n}}\phi(x)$ instead of $\phi(x)$ and using (5.34), we can also assume that $g(0) = 1$ and

$$1 - \epsilon \leq g \leq 1 + \epsilon \quad \text{in} \quad S_\phi(0, s).$$

We now define the rescaled domains $U_s := T_s(S_\phi(0, s))$, $\Omega_s := T_s(\Omega)$ and the rescaled functions ϕ_s, $u_s := u \circ T_s^{-1}$, f_s as in Sect. 2.2 *that preserve the L^∞-norm of u*. We claim that

$$\|D^2 u_s\|_{L^p\left(S_{\phi_s}(0, c^9)\right)} \leq C\Big(\|u_s\|_{L^\infty(U_s)} + \|f_s\|_{L^q(U_s)}\Big), \tag{5.36}$$

where $C > 0$ depends only on $p, q, n, \rho, \lambda, \Lambda, \alpha$, the uniform convexity of $\partial\Omega$, $\|\partial\Omega\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial\Omega)}$. Then by rescaling back as in the proof of Lemma 3.11 we obtain

$$\begin{aligned} \|D^2 u\|_{L^p\left(S_\phi(y, c^9 s)\right)} &\leq C s^{\frac{n}{2p}-1}|\log s|^2\|u\|_{L^\infty(\Omega)} + C s^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}|\log s|^2\|f\|_{L^q(\Omega)} \\ &\leq C(s)\ \big(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)}\big) \qquad \forall y \in \partial\Omega. \end{aligned} \tag{5.37}$$

Let $\delta := c^9 s$. By (2.7), we know that $S_\phi(y, \delta) \supset \overline{\Omega} \cap B(y, \delta^{2/3})$. Therefore if we let

$$\Omega_{\delta^{2/3}} := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta^{2/3}\},$$

then we can cover the $\delta^{2/3}$ neighborhood of Ω, that is $\Omega\backslash\Omega_{\delta^{2/3}}$, by a finite number of boundary sections $\{S_\phi(y_j,\delta)\}_{j=1}^N$. Then by adding (5.37) over the family $\{S_\phi(y_j,\delta)\}_{j=1}^N$, we arrive at the $W^{2,p}$ estimate at the boundary

$$\|D^2u\|_{L^p(\Omega\backslash\Omega_{\delta^{2/3}})} \leq C\left(\|u\|_{L^\infty(\Omega)}+\|f\|_{L^q(\Omega)}\right).$$

On the other hand, by the interior estimate in [17, Theorem 1.1], we also have

$$\|D^2u\|_{L^p(\Omega_{\delta^{2/3}})} \leq C(\|u\|_{L^\infty(\Omega)}+\|f\|_{L^q(\Omega)}).$$

Our Theorem 1.2 follows from the above inequalities.

We now indicate how to obtain the claim (5.36). The proof consists of reviewing the proof of Theorem 1.1. By (2.7), we have

$$S_{\phi_s}(0,c^9)\subset U_s\cap B_{c^3}.$$

We use Lemma 3.13 to cover $U_s\cap B_{c^2}$. We restrict our estimates on the distribution function for the second derivatives in Lemma 3.4 to $U_s\cap B_{c^2}$. Lemma 5.2 holds with obvious changes for the data (Ω_s,ϕ_s,U_s). So does Lemma 5.4 provided that we have an analogous version of Lemma 5.5 for our data (Ω_s,ϕ_s,U_s). Precisely, let $S_{\phi_s}(y_0,h)$ be a section of ϕ_s in U_s such that $y_0\in\partial U_s\cap B_{c^3}$ and $S_{\phi_s}(y_0,h)\cap G_\gamma(u_s,U_s,\phi_s)\neq\emptyset$ for some $\gamma>0$ (say, $\bar y\in S_{\phi_s}(y_0,h)\cap G_\gamma(u_s,U_s,\phi_s)$). By Lemma 2.5 and the Localization Theorem 2.2, there exists an affine map $\tilde T_h$ such that

$$\tilde T_h(y_0)=y_0 \quad\text{and}\quad \overline{U_s}\cap B_k(y_0)\subset\tilde U_h:=\tilde T_h(S_{\phi_s}(y_0,\theta h))\subset\overline{U_s}\cap B_{k^{-1}}(y_0).$$

Here $\theta>1$ is the same constant at the beginning of the proof of Lemma 5.4. We need to show that the $C^{2,\alpha}$ norm on the boundary $\partial\tilde U_h\cap B_k(y_0)$ of the following function

$$\tilde v(z):=\frac{1}{\theta\gamma h}\left[u_s(\tilde T_h^{-1}z)-u_s(\bar y)-\nabla u_s(\bar y)\cdot(\tilde T_h^{-1}z-\bar y)\right],\quad z\in\tilde T_h(U_s)$$

is bounded by a constant which is independent of the uniform convexity of U_s. The function $\tilde v$ is defined in a similar way to the definition of the function v in (5.18). We note that the uniform convexity of the boundary $\partial\Omega$ plays a key role in the proof of Lemma 5.5. Thus we can not obtain the desired result by repeating the proof of Lemma 5.5 for our data (Ω_s,ϕ_s,U_s) since the uniform convexity of $\partial\Omega_s$ deteriorates as $s\to 0$. However, we can get away from this as follows.

Let $T:=\tilde T_h\circ T_s$. Then T normalizes the section $S_\phi(T_s^{-1}y_0,\theta hs)$, and

$$\|T\|\leq k^{-2}(\theta hs)^{-1/2}\,|\log(\theta h)|\,|\log s|,\quad \|T^{-1}\|\leq k^{-2}(\theta hs)^{1/2}\,|\log(\theta h)|\,|\log s|.$$

Moreover, $\bar x:=T_s^{-1}(\bar y)\in S_\phi(T_s^{-1}y_0,\theta hs)\cap G_{\gamma s^{-1}}(u,\Omega,\phi)$ and

$$T\left(S_\phi(T_s^{-1}y_0,\theta hs)\right)=\tilde T_h(S_{\phi_s}(y_0,\theta h))=\tilde U_h.$$

Therefore, by reviewing the proof of Lemma 5.5 we see that the function

$$v(y) := \frac{1}{\theta(\gamma s^{-1})hs}\left[u(T^{-1}y) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (T^{-1}y - \bar{x})\right], \quad y \in T(\Omega)$$

satisfies

$$\|v\|_{C^{2,\alpha}\left(\partial \tilde{U}_h \cap B_k(\tilde{T}_h(y_0))\right)} \le C_\alpha \tag{5.38}$$

with C_α depending on the uniform convexity of Ω. But since $\tilde{T}_h(y_0) = y_0$ and $\tilde{v} \equiv v$ on $\tilde{U}_h$ as $u_s(y) = u(T_s^{-1}y)$, we conclude that the $C^{2,\alpha}$ norm of $\tilde{v}$ on $\partial \tilde{U}_h \cap B_k(y_0)$ is bounded by the same constant C_α in (5.38). Hence the claim (5.36) follows as explained above. □

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