Math. Ann. (2013) 356:1247–1282
DOI 10.1007/s00208-012-0871-4

Mathematische Annalen

Heegner cycles and higher weight specializations of big Heegner points

Francesc Castella

Received: 21 September 2012 / Revised: 23 October 2012 / Published online: 7 December 2012

Abstract Let $\mathbf{f}$ be a p-ordinary Hida family of tame level N, and let K be an imaginary quadratic field satisfying the Heegner hypothesis relative to N. By taking a compatible sequence of twisted Kummer images of CM points over the tower of modular curves of level $\Gamma_0(N) \cap \Gamma_1(p^s)$, Howard has constructed a canonical class $\mathfrak{Z}$ in the cohomology of a self-dual twist of the big Galois representation associated to $\mathbf{f}$. If a p-ordinary eigenform f on $\Gamma_0(N)$ of weight $k > 2$ is the specialization of $\mathbf{f}$ at ν, one thus obtains from $\mathfrak{Z}_\nu$ a higher weight generalization of the Kummer images of Heegner points. In this paper we relate the classes $\mathfrak{Z}_\nu$ to the étale Abel-Jacobi images of Heegner cycles when p splits in K.

Contents

F. Castella (✉)
Department of Mathematics and Statistics, McGill University, Montreal, Canada
e-mail: castella@math.mcgill.ca

1 Introduction

Fix a prime $p > 3$ and an integer $N > 4$ such that $p \nmid N\phi(N)$. Let

$$f_o = \sum_{n>0} a_n q^n \in S_k(X_0(N))$$

be a p-ordinary newform of even weight $k = 2r \geq 2$ and trivial nebentypus. Thus f_o is an eigenvector for all the Hecke operators T_n with associated eigenvalues a_n, and a_p is a p-adic unit for a choice of embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ that will remain fixed throughout this paper. Also let $\mathcal{O}$ denote the ring of integers of a (sufficiently large) finite extension $L/\mathbf{Q}_p$ containing all the a_n.

For $s > 0$, let X_s be the compactified modular curve of level

$$\Gamma_s := \Gamma_0(N) \cap \Gamma_1(p^s),$$

and consider the tower

$$\cdots \to X_s \xrightarrow{\alpha} X_{s-1} \to \cdots$$

with respect to the degeneracy maps described on the non-cuspidal moduli by

$$(E, \alpha_E, \pi_E) \mapsto (E, \alpha_E, p \cdot \pi_E),$$

where α_E denotes a cyclic N-isogeny on the elliptic curve E, and π_E a point of E of exact order p^s. The group $(\mathbf{Z}/p^s\mathbf{Z})^\times$ acts on X_s via the diamond operators

$$\langle d \rangle : (E, \alpha_E, \pi_E) \mapsto (E, \alpha_E, d \cdot \pi_E)$$

compatibly with α under the reduction $(\mathbf{Z}/p^s\mathbf{Z})^\times \to (\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$. Set $\Gamma := 1 + p\mathbf{Z}_p$. Letting J_s be the Jacobian variety of X_s, the inverse limit of the system induced by Albanese functoriality,

$$\cdots \to \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O} \to \mathrm{Ta}_p(J_{s-1}) \otimes_{\mathbf{Z}_p} \mathcal{O} \to \cdots, \tag{1.1}$$

is equipped with an action of the Iwasawa algebras $\widetilde{\Lambda}_{\mathcal{O}} := \mathcal{O}[[\mathbf{Z}_p^\times]]$ and

$$\Lambda_{\mathcal{O}} := \mathcal{O}[[\Gamma]].$$

Let $\mathfrak{h}_s$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_ℓ ($\ell \nmid Np$), $U_\ell := T_\ell$ ($\ell | Np$), and the diamond operators $\langle d \rangle$ ($d \in (\mathbf{Z}/p^s\mathbf{Z})^\times$) acting on the space $S_k(X_s)$ of cusp forms of weight k and level Γ_s. Hida's ordinary projector

$$e^{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}$$

defines an idempotent of $\mathfrak{h}_s$, projecting to the maximal subspace of $\mathfrak{h}_s$ where U_p acts invertibly. We make each $\mathfrak{h}_s$ into a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra by letting the group-like element attached to $z \in \mathbf{Z}_p^\times$ act as $z^{k-2}\langle z\rangle$.

Taking the projective limit with respect to the restriction maps induced by the natural inclusion $S_k(X_{s-1}) \hookrightarrow S_k(X_s)$, we obtain a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra

$$\mathfrak{h}^{\mathrm{ord}} := \varprojlim_s e^{\mathrm{ord}}\mathfrak{h}_s \tag{1.2}$$

which can be seen to be *independent* of the weight $k \geq 2$ used in its construction.

After a highly influential work [16] of Hida, one can associate with f_o a certain local domain $\mathbb{I}$ quotient of $\mathfrak{h}^{\mathrm{ord}}$, finite flat over $\Lambda_{\mathcal{O}}$, with the following properties. For each n, let $\mathbf{a}_n \in \mathbb{I}$ be the image of T_n under the projection $\mathfrak{h}^{\mathrm{ord}} \to \mathbb{I}$, and consider the formal q-expansion

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]].$$

We say that a continuous $\mathcal{O}$-algebra homomorphism $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ is an *arithmetic prime* if there is an integer $k_\nu \geq 2$, called the *weight* of ν, such that the composition $\Gamma \to \mathbb{I}^\times \to \overline{\mathbf{Q}}_p^\times$ agrees with $\gamma \mapsto \gamma^{k_\nu-2}$ on an open subgroup of Γ of index $p^{s_\nu-1} \geq 1$. Denote by $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, which will often be seen as sitting inside $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$. If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, F_ν will denote its residue field. Then:

- for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, there exists an ordinary p-stabilized newform[1]

$$\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$$

  such that $\nu(\mathbf{f}) \in F_\nu[[q]]$ gives the q-expansion of $\mathbf{f}_\nu$;
- if $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$, there exists a normalized newform $\mathbf{f}_\nu^\sharp \in S_{k_\nu}(X_0(N))$ such that

$$\mathbf{f}_\nu(q) = \mathbf{f}_\nu^\sharp(q) - \frac{p^{k_\nu-1}}{\nu(\mathbf{a}_p)}\mathbf{f}_\nu^\sharp(q^p); \tag{1.3}$$

- there exists a unique $\nu_o \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ such that $f_o = \mathbf{f}_{\nu_o}^\sharp$.

In particular, after "p-stabilization" (1.3), the form f_o fits in the p-adic family $\mathbf{f}$.

Similarly for the associated Galois representation V_{f_o}: the continuous $\mathfrak{h}^{\mathrm{ord}}$-linear action of the absolute Galois group $G_{\mathbf{Q}}$ on the module

$$\mathbb{T} := \mathbb{T}^{\mathrm{ord}} \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathbb{I}, \qquad \text{where } \mathbb{T}^{\mathrm{ord}} := \varprojlim_s e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O}), \tag{1.4}$$

gives rise to a "big" Galois representation $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \to \mathrm{Aut}(\mathbb{T})$ such that

$$\nu(\rho_{\mathbf{f}}) \cong \rho_{\mathbf{f}_\nu}^* \qquad \text{for every } \nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}),$$

[1] As defined in [29, (1.3.7)].

where $\rho^*_{\mathbf{f}_\nu}$ is the contragredient of the (cohomological) p-adic Galois representation $\rho_{\mathbf{f}_\nu} : G_{\mathbf{Q}} \to \mathrm{Aut}(V_{\mathbf{f}_\nu})$ attached to $\mathbf{f}_\nu$ by Deligne; in particular, one recovers $\rho^*_{f_o}$ from $\rho_{\mathbf{f}}$ by specialization at ν_o.

Assume from now on that the residual representation $\bar{\rho}_{f_o}$ is irreducible; then $\mathbb{T}$ can be shown to be free of rank 2 over $\mathbb{I}$. (See [23, Théorème 7].) Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ containing an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}, \tag{1.5}$$

and denote by H the Hilbert class field of K. Under this *Heegner hypothesis* relative to N (but with no extra assumptions on the prime p), the work [19] of Howard produces a compatible sequence $U_p^{-s} \cdot \mathfrak{X}_s$ of cohomology classes with values in a certain twist of the ordinary part of (1.1), giving rise to a canonical "big" cohomology class $\mathfrak{X}$, the *big Heegner point* (of conductor 1), in the cohomology of a self-dual twist $\mathbb{T}^\dagger$ of $\mathbb{T}$. Moreover, if every prime factor of N splits in K, it follows from his results that the class

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X})$$

lies in Nekovář's extended Selmer group $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$. In particular, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$ as above, the specialization $\mathfrak{Z}_\nu$ belongs to the Bloch-Kato Selmer group $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(k_\nu/2))$ of the self-dual representation $\mathbb{T}^\dagger \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}^\sharp_\nu}(k_\nu/2)$. The classes $\mathfrak{Z}_\nu$ may thus be regarded as a natural higher weight analogue of the Kummer images of Heegner points on modular Abelian varieties (associated with weight 2 eigenforms).

But for any of the above $\mathbf{f}^\sharp_\nu$, one has an alternate (and completely different!) method of producing such a higher weight analogue. Briefly, if $k_\nu = 2r_\nu > 2$, associated to any elliptic curve A with CM by $\mathcal{O}_K$, there is a null-homologous cycle $\Delta^{\mathrm{heeg}}_{A,r_\nu}$, a so-called *Heegner cycle*, on the $(2r_\nu - 1)$-dimensional Kuga–Sato variety W_{r_ν}, giving rise to an H-rational class in the Chow group $\mathrm{CH}^{r_\nu+1}(W_{r_\nu})_0$ with $\mathbf{Q}$-coefficients. Since the representation $V_{\mathbf{f}^\sharp_\nu}(r_\nu)$ appears in the étale cohomology of W_{r_ν}:

$$H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu}, \mathbf{Q}_p)(r_\nu) \xrightarrow{\pi_{\mathbf{f}^\sharp_\nu}} V_{\mathbf{f}^\sharp_\nu}(r_\nu),$$

by taking the images of the cycles $\Delta^{\mathrm{heeg}}_{A,r_\nu}$ under the p-adic étale Abel-Jacobi map

$$\Phi^{\text{ét}}_H : \mathrm{CH}^{r_\nu+1}(W_{r_\nu})_0(H) \to H^1(H, H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu}, \mathbf{Q}_p)(r_\nu))$$

and composing with the map induced by $\pi_{\mathbf{f}^\sharp_\nu}$ on H^1's, we may consider the classes

$$\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu, K}(\Delta^{\mathrm{heeg}}_{r_\nu}) := \mathrm{Cor}_{H/K}(\pi_{\mathbf{f}^\sharp_\nu}\Phi^{\text{ét}}_H(\Delta^{\mathrm{heeg}}_{A,r_\nu})).$$

By the work [28] of Nekovář, these classes are known to lie in the same Selmer group as $\mathfrak{Z}_\nu$, and the question of their comparison thus naturally arises.

Main Theorem (Thm. 5.11) *Assume that p splits in K and that $\mathfrak{Z}$ is not $\mathbb{I}$-torsion. Then for any $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $k_\nu = 2r_\nu > 2$ with $k_\nu \equiv k \pmod{2(p-1)}$ and trivial character, we have*

$$\langle \mathfrak{Z}_\nu, \mathfrak{Z}_\nu \rangle_K = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu, K}(\Delta^{\text{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu, K}(\Delta^{\text{heeg}}_{r_\nu}) \rangle_K}{u^2 (4D)^{r_\nu - 1}},$$

where $\langle , \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(r_\nu))$, $u := |\mathcal{O}_K^\times|/2$, and $-D < 0$ is the discriminant of K.

Thus assuming the non-degeneracy of the p-adic height pairing, it follows that the étale Abel-Jacobi images of Heegner cycles are p-adically interpolated by $\mathfrak{Z}$. We also note that $\mathfrak{Z}$ is conjecturally *always* not $\mathbb{I}$-torsion ([19, Conj. 3.4.1]), and that by [18, Cor. 5] this conjecture can be verified in any given case by exhibiting the non-vanishing of an appropriate L-value (a derivative, in fact).

This paper is organized as follows. Section 2 is aimed at proving an expression for the formal group logarithms of ordinary CM points on X_s using Coleman's theory of p-adic integration. Our methods here are drawn from [1, Sect. 4], which we extend in weight 2 to the case of level divisible by an arbitrary power of p, but with ramification restricted to a *potentially crystalline* setting. Not quite surprisingly, this restriction turns out to make our computations essentially the same as theirs, and will suffice for our purposes.

In Sect. 3 we recall the generalised Heegner cycles and the formula for their p-adic Abel-Jacobi images from *loc.cit.*, and discuss the relation between these and the more classical Heegner cycles.

In Sect. 4 we deduce from the work [30] of Ochiai a "big" logarithm map that will allow as to move between different weights in the Hida family.

Finally, in Sect. 5 we prove our main results. The key observation is that, when p splits in K, the combination of CM points on X_s taken in Howard's construction appears naturally in the evaluation of the critical twist of a p-adic modular form at a canonical trivialized elliptic curve. The expression from Sect. 2 thus yields, for infinitely many ν of weight 2, a formula for the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$ in terms of certain values of a p-adic modular form of weight 0 associated with $\mathbf{f}_\nu$ (Thm. 5.8). When extended by p-adic continuity to an arithmetic prime ν of higher even weight, this expression is seen to agree with the formula from Sect. 3, and by the interpolation properties of the big logarithm map it corresponds to the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$. Our main results follow easily from this.

Finally, we note that an extension of the results in this paper, and in particular of the Main Theorem above, has a number of arithmetic applications arising from the connection with the theory of p-adic L-functions. (See [5].)

2 Preliminaries

2.1 p-Adic modular forms

To avoid some issues related to the representability of certain moduli problems, in this section we change notations from the Introduction, letting X_s be the compactified modular curve of level $\Gamma_s := \Gamma_1(Np^s)$, viewed as a scheme over $\mathrm{Spec}(\mathbf{Q}_p)$. Let $\pi : \mathcal{E}_s \to X_s$ be the universal generalized elliptic curve over X_s, and let

$$\underline{\omega}_{X_s} := \pi_* \Omega^1_{\mathcal{E}_s/X_s}(\log Z_s)$$

be the pushforward of the invertible sheaf of relative differentials on $\mathcal{E}_s/X_s$ with possible log-poles along the inverse image of the cuspidal subscheme $Z_s \subset X_s$.

Algebraically, $H^0(X_s, \underline{\omega}_{X_s}^{\otimes 2})$ gives the space of modular forms of weight 2 and level Γ_s (defined over $\mathbf{Q}_p$). Consider the complex

$$\Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s) : 0 \longrightarrow \mathcal{O}_{X_s} \xrightarrow{d} \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s) \longrightarrow 0 \tag{2.1}$$

of sheaves on X_s. The algebraic *de Rham cohomology* of X_s

$$H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p) := \mathbb{H}^1(X_s, \Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s))$$

is a finite-dimensional $\mathbf{Q}_p$-vector space equipped with a *Hodge filtration*

$$0 \subset H^0(X_s, \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)) \subset H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p),$$

and by the Kodaira–Spencer isomorphism $\underline{\omega}_{X_s}^{\otimes 2} \cong \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)$, every cusp form $f \in S_2(X_s)$ (in particular) defines a cohomology class $\omega_f \in H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p)$.

Let X be the complete modular curve of level $\Gamma_1(N)$, also viewed over $\mathrm{Spec}(\mathbf{Q}_p)$, and consider the subspaces of the associated rigid analytic space X^{an}:

$$X^{\mathrm{ord}} \subset X_{<1/(p+1)} \subset X_{<p/(p+1)} \subset X^{\mathrm{an}}.$$

To define these, let $\mathcal{X}_{/\mathbf{Z}_p}$ be the canonical integral model of X over $\mathrm{Spec}(\mathbf{Z}_p)$, and let $X_{\mathbf{F}_p} := \mathcal{X} \times_{\mathbf{Z}_p} \mathbf{F}_p$ denote its special fiber. The *supersingular points* $SS \subset X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ is the finite set of points corresponding to the moduli of supersingular elliptic curves (with $\Gamma_1(N)$-level structure) in characteristic p. Let E_{p-1} be the Eisenstein series of weight $p-1$ and level 1, seen as a global section of the sheaf $\underline{\omega}_X^{\otimes(p-1)}$. (Recall that we are assuming $p \geq 5$.) The reduction of E_{p-1} to $X_{\mathbf{F}_p}$ is the *Hasse invariant*, which defines a section of the reduction of $\underline{\omega}_X^{\otimes(p-1)}$ with SS as its locus of (simple) zeroes. If $x \in X(\overline{\mathbf{Q}}_p)$, let $\bar{x} \in X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ denote its reduction. Each point $\bar{x} \in SS$ is smooth in $X_{\mathbf{F}_p}$, and the *ordinary locus* of X.

$$X^{\mathrm{ord}} := X^{\mathrm{an}} \setminus \bigcup_{\bar{x} \in SS} D_{\bar{x}}$$

is defined to be the complement of their residue discs $D_{\bar{x}} \subset X^{\mathrm{an}}$. The function $|E_{p-1}(x)|_p$ defines a local parameter on $D_{\bar{x}}$, and with the normalization $|p|_p = p^{-1}$, $X_{<1/(p+1)}$ (resp. $X_{<p/(p+1)}$) is defined to be complement in X^{an} of the subdiscs of $D_{\bar{x}}$ where $|E_{p-1}(x)|_p \leq p^{-1/(p+1)}$ (resp. $|E_{p-1}(x)|_p \leq p^{-p/(p+1)}$), for all $\bar{x} \in SS$.

Using the *canonical subgroup* H_E (of order p) attached to every elliptic curve E corresponding to a closed point in $X_{<p/(p+1)}$, the *Deligne–Tate map*

$$\phi_0 : X_{<1/(p+1)} \to X_{<p/(p+1)}$$

is defined by sending $E \mapsto E/H_E$ (with the induced action on the level structure) under the moduli interpretation. This map is a finite morphism which by definition lifts to characteristic zero the absolute Frobenius on $X_{\mathbf{F}_p}$. (See [21, Thm. 3.1].)

For every $s > 0$, the Deligne–Tate map ϕ_0 can be iterated $s-1$ times on the open rigid subspace $X_{<p^{2-s}/(p+1)}$ of X^{an} where $|E_{p-1}(x)|_p > p^{-p^{2-s}/(p+1)}$. Letting $\alpha_s : X_s \to X$ be the map forgetting the "$\Gamma_1(p^s)$-part" of the level structure, define

$$\mathcal{W}_1(p^s) \subset X_s^{\mathrm{an}}$$

to be the open rigid subspace of X_s whose closed points correspond to triples (E, α_E, π_E) whose image under α_s lands inside $X_{<p^{2-s}(p+1)}$ and are such that π_E generates the canonical subgroup of E *of order* p^s (as in [4, Def. 3.4]).

Define $\mathcal{W}_2(p^s) \subset X_s^{\mathrm{an}}$ is the same manner, replacing $p^{2-s}/(p+1)$ by $p^{1-s}/(p+1)$ in the definition of $\mathcal{W}_1(p^s)$. Then we obtain a lifting of Frobenius $\phi = \phi_s$ on X_s making the diagram

$$\begin{array}{ccc} \mathcal{W}_2(p^s) & \xrightarrow{\ \phi\ } & \mathcal{W}_1(p^s) \\ \downarrow{\scriptstyle \alpha_s} & & \downarrow{\scriptstyle \alpha_s} \\ X_{<p^{1-s}(p+1)} & \xrightarrow{\ \phi_0\ } & X_{<p^{2-s}(p+1)}. \end{array}$$

commutative by sending a point $x = (E, \alpha_E, \iota_E) \in \mathcal{W}_2(p^s)$, where $\iota_E : \boldsymbol{\mu}_{p^s} \hookrightarrow E[p^s]$ is an embedding giving the $\Gamma_1(p^s)$-level structure on E, to $x' = (\phi_0 E, \phi_0 \alpha_E, \iota'_E)$, where ι'_E is determined by requiring that $\alpha_s(x')$ lands in $X_{<p^{2-s}/(p+1)}$ and for each $\zeta \in \boldsymbol{\mu}_{p^s} - \{1\}$, $\iota'_E(\zeta) = \phi_0 Q$ if $\iota_E(\zeta) = pQ$. (Cf. [11, Sect. B.2].)

Let $k \in \mathbf{Z}$, and denote by $\underline{\omega}_{X_s^{\mathrm{an}}}$ the rigid analytic sheaf on X_s^{an} deduced from $\underline{\omega}_{X_s}$. Let $I_s := \{v \in \mathbf{Q} : 0 \le v < p^{2-s}/(p+1)\}$, and for $p^{-v} \in I_s$ define the affinoid subdomain $X_s(v)$ of X_s^{an} inside $\mathcal{W}_1(p^s)$ whose closed points x satisfy $|E_{p-1}(x)|_p \ge p^{-v}$. Then $X_s(0)$ is the connected component of the ordinary locus of X_s containing the cusp ∞. The space of *p-adic modular forms* of weight k and level Γ_s (defined over $\mathbf{Q}_p$) is the p-adic Banach space

$$M_k^{\mathrm{ord}}(X_s) := H^0(X_s(0), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

and the space of *overconvergent p-adic modular forms* of weight k and level Γ_s is the p-adic Fréchet space

$$M_k^{\mathrm{rig}}(X_s) := \varprojlim_{v} H^0(X_s(v), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

where the limit is with respect to the natural restriction maps as $v \in I_s$ increasingly approaches $p^{2-s}/(p+1)$. By restriction, a classical modular form in $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ defines an (obviously) overconvergent p-adic modular form of the same weight an level. Moreover, the action of the diamond operators on X_s gives rise to an action of $(\mathbf{Z}/p^s\mathbf{Z})^\times$ on the spaces of p-adic modular forms which agrees with the action on $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ under restriction.

We say that a ring R is a *p-adic ring* if the natural map $R \to \varprojlim R/p^n R$ is an isomorphism. For varying $s > 0$, the data of a compatible sequence of embeddings $\boldsymbol{\mu}_{p^s} \hookrightarrow E$ as R-group schemes, amounts to the data of an embedding $\boldsymbol{\mu}_{p^\infty} \hookrightarrow E[p^\infty]$ of p-divisible groups, and also to the given of a *trivialization* of E over R, i.e. an isomorphism

$$\imath_E : \hat{E} \to \hat{\mathbf{G}}_m$$

of the associated formal groups. The space $\mathbf{M}(N)$ of *Katz p-adic modular functions* of tame level N (over $\mathbf{Z}_p$) is the space of functions f on trivialized elliptic curves with $\Gamma_1(N)$-level structure over arbitrary p-adic rings, assigning to the isomorphism class of a triple $(E, \alpha_E, \imath_E)$ over R a value $f(E, \alpha_E, \imath_E) \in R$ whose formation is compatible under base change. If R is a fixed p-adic ring, by only considering p-adic rings which are R-algebras, we obtain the notion of Katz p-adic modular functions defined over R, forming the space $\mathbf{M}(N)\widehat{\otimes}_{\mathbf{Z}_p} R$ which will also be denoted by $\mathbf{M}(N)$ with an abuse of notation.

The action of $z \in \mathbf{Z}_p^\times$ on a trivialization gives rise to an action of $\mathbf{Z}_p^\times$ on $\mathbf{M}(N)$:

$$\langle z \rangle f(E, \alpha_E, \imath_E) := f(E, \alpha_E, z \cdot \imath_E),$$

and given a character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, R^\times)$, we say that $f \in \mathbf{M}(N)$ has *weight-nebentypus* χ if $\langle z \rangle f = \chi(z) f$ for all $z \in \mathbf{Z}_p^\times$. If k is an integer, denoting by z^k the k-th power character on $\mathbf{Z}_p^\times$, the subspace $M_k^{\mathrm{ord}}(Np^s, \varepsilon)$ of $M_k^{\mathrm{ord}}(X_s)$ consisting of p-adic modular forms with nebentypus $\varepsilon : (\mathbf{Z}/p^s\mathbf{Z})^\times \to R^\times$ can be recovered as

$$M_k^{\mathrm{ord}}(Np^s, \varepsilon) \cong \{f \in \mathbf{M}(N) : \langle z \rangle f = z^k \varepsilon(z) f, \quad \text{for all } z \in \mathbf{Z}_p^\times\}. \tag{2.2}$$

Since it will play an important role later, we next recall from [14, Sect. III.6.2] the definition in terms of moduli of the twist of p-adic modular forms by characters of not necessarily finite order. Let R be a p-adic ring, and let $(E, \alpha_E, \imath_E)$ be a trivialized elliptic curve with $\Gamma_1(N)$-level structure over R. For each $s > 0$, consider the quotient $E_0 := E/\imath_E^{-1}(\boldsymbol{\mu}_{p^s})$, and let $\varphi_0 : E \to E_0$ denote the projection. Since $p \nmid N$, φ_0 induces a $\Gamma_1(N)$-level structure α_{E_0} on E_0, and since $\ker(\varphi_0) \cong \boldsymbol{\mu}_{p^s}$, the dual $\check{\varphi}_0 : E_0 \to E$ is étale, inducing an isomorphism of the associated formal groups. Thus (with a slight abuse of notation) $\imath_{E_0} := \imath_E \circ \check{\varphi}_0 : \hat{E}_0 \xrightarrow{\sim} \hat{\mathbf{G}}_m$ is a trivialization of E_0, and since we have an embedding $\jmath : \mathbf{Z}/p^s\mathbf{Z} \cong \ker(\check{\varphi}_0) \hookrightarrow E_0[p^s]$, we deduce an isomorphism

$$E_0[p^s] \cong \boldsymbol{\mu}_{p^s} \oplus \mathbf{Z}/p^s\mathbf{Z}$$

which we use to bijectively attach a p^s-th root of unity ζ_C to every étale subgroup $C \subset E_0[p^s]$ of order p^s, in such a way that 1 is attached to $\ker(\check{\varphi}_0)$.

Now for $f \in \mathbf{M}(N)$ and $a \in \mathbf{Z}_p$, define $f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}$ to be the rule on trivialized elliptic curves given by

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(E, \alpha_E, \iota_E) = \frac{1}{p^s} \sum_C \zeta_C^{-a} \cdot f(E_0/C, \alpha_C, \iota_C) \tag{2.3}$$

where the sum is over the étale subgroups $C \subset E_0[p^s]$ of order p^s, and where α_C (resp. ι_C) denotes the $\Gamma_1(N)$-level structure (resp. trivialization) on the quotient E_0/C naturally induced by α_{E_0} (resp. ι_{E_0}).

Lemma 2.1 *The assignment* $a + p^s\mathbf{Z}_p \rightsquigarrow \left(f \mapsto f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}\right)$ *gives rise to an* $\mathrm{End}_R \mathbf{M}(N)$*-valued measure* μ_{Gou} *on* $\mathbf{Z}_p$*.*

Proof Let $\sum_n a_n q^n$ be the q-expansion of f, i.e. the value that it takes at the triple $(\mathrm{Tate}(q), \alpha_{\mathrm{can}}, \iota_{\mathrm{can}}) = (\mathbf{G}_m/q^{\mathbf{Z}}, \zeta_N, \boldsymbol{\mu}_{p^\infty} \hookrightarrow \mathbf{G}_m/q^{\mathbf{Z}})$ over the p-adic completion of $R((q))$. By the q-expansion principle, the claim follows immediately from the equality

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(q) = \sum_{n \equiv a \bmod p^s} a_n q^n,$$

which is shown by adapting the arguments in [14, p. 102]. □

Definition 2.2 (Gouvêa) Let $f \in \mathbf{M}(N)$ and $\chi : \mathbf{Z}_p \to R$ be any continuous multiplicative function. The *twist* of f by χ is

$$f \otimes \chi := \left(\int_{\mathbf{Z}_p} \chi(x) d\mu_{\mathrm{Gou}}(x) \right) (f) \in \mathbf{M}(N).$$

This operation is compatible with the usual character twist of Hecke eigenforms:

Lemma 2.3 *Let* $\chi : \mathbf{Z}_p^\times \to R^\times$ *be a continuous character extended by zero on* $p\mathbf{Z}_p$*. If* $f \in \mathbf{M}(N)$ *has* q*-expansion* $\sum_n a_n q^n$*, then* $f \otimes \chi$ *has* q*-expansion* $\sum_n \chi(n) a_n q^n$*, and if* f *has weight-nebentypus* $\kappa \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, R^\times)$*, then* $f \otimes \chi$ *has weight-nebentypus* $\chi^2\kappa$*.*

Proof See [14, Cor. III.6.8.i] and [14, Cor. III.6.9]). □

In particular, twisting by the identity function of $\mathbf{Z}_p$ we obtain an operator $d : \mathbf{M}(N) \to \mathbf{M}(N)$ whose effect on q-expansions is $q\frac{d}{dq}$. For every $k \in \mathbf{Z}$, we see from (2.2) and Lemma 2.3, that this restricts to a map

$$d : M_k^{\mathrm{ord}}(X_s) \to M_{k+2}^{\mathrm{ord}}(X_s)$$

which increases the weight by 2 and preserves the nebentypus. Moreover, for $k = 0$, the arguments in [9, Prop. 4.3] can be adapted to show that d restricts to a linear map $M_0^{\mathrm{rig}}(X_s) \to M_2^{\mathrm{rig}}(X_s)$, viewing $M_k^{\mathrm{rig}}(X_s) \hookrightarrow M_k^{\mathrm{ord}}(X_s)$ by restriction.

2.2 Comparison isomorphisms

Let ζ_s be a primitive p^s-th root of unity, and let F be a finite extension of $\mathbf{Q}_p(\zeta_s)$ over which X_s acquires stable reduction, i.e. such that the base extension $X_s \times_{\mathbf{Q}_p} F$ admits a stable model over the ring of integers $\mathcal{O}_F$ of F. For the ease of notation, from now on we will denote $X_s \times_{\mathbf{Q}_p} F$ (as well as the associated rigid analytic space) simply by X_s.

Let $\mathscr{X}_s$ be the minimal regular model of X_s over $\mathcal{O}_F$, and denote by F_0 the maximal unramified subfield of F. The work [17] of Hyodo–Kato endows the F-vector space $H^1_{\mathrm{dR}}(X_s/F)$ with a canonical F_0-structure

$$H^1_{\mathrm{log-cris}}(\mathscr{X}_s) \hookrightarrow H^1_{\mathrm{dR}}(X_s/F) \tag{2.4}$$

equipped with a semi-linear Frobenius operator φ.

After the proof [33] of the Semistable conjecture of Fontaine–Jannsen, these structures are known to agree with those attached by Fontaine's theory to the p-adic G_F-representation

$$V_s := H^1_{\text{ét}}(\overline{X}_s, \mathbf{Q}_p). \tag{2.5}$$

More precisely, since X_s has semistable reduction, V_s is semistable in the sense of Fontaine, and there is a canonical isomorphism $D_{\mathrm{st}}(V_s) \to H^1_{\mathrm{log-cris}}(\mathscr{X}_s)$, inducing an isomorphism

$$D_{\mathrm{dR}}(V_s) \xrightarrow{\sim} H^1_{\mathrm{dR}}(X_s/F) \tag{2.6}$$

as filtered φ-modules after extension of scalars to F.

Consider the étale Abel-Jacobi map $\mathrm{CH}^1(X_s)_0(F) \to H^1(F, V_s(1))$ constructed in [28], which in this case agrees with the usual Kummer map

$$\delta_F : J_s(F) \to H^1(F, \mathbf{Q}_p \otimes \mathrm{Ta}_p(J_s)),$$

where $J_s = \mathrm{Pic}^0(X_s)$ is the connected Picard variety of X_s. (See [*loc.cit.*, Ex. (2.3)]).

Let $g \in S_2(X_s)$ be a newform with primitive nebentypus of p-power conductor, let V_g the p-adic Galois representation attached to g, which is equipped with a Galois-equivariant projection $V_s \to V_g$, and let V_g^* be the representation contragredient to V_g, so that $V_g(1)$ and V_g^* are in Kummer duality. Also let L_g be a finite extension of $\mathbf{Q}_p$ over which the Hecke eigenvalues of g are defined. By [3, Ex. 3.11], the image of the induced composite map:

$$\delta_{g,F} : J_s(F) \xrightarrow{\delta_F} H^1(F, V_s(1)) \to H^1(F, V_g(1)) \tag{2.7}$$

lies in the Bloch-Kato "finite" subspace $H^1_f(F, V_g(1))$, and by our assumption on the nebentypus of g, the Bloch-Kato exponential map gives an isomorphism

$$\exp_{F,V_g(1)} : \frac{D_{\mathrm{dR}}(V_g(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))} \to H^1_f(F, V_g(1)) \tag{2.8}$$

whose inverse will be denoted by $\log_{F,V_g(1)}$.

Our aim in this section is to compute the images of certain degree 0 divisors on X_s under the p-adic Abel-Jacobi map $\delta^{(p)}_{g,F}$, defined as the composition

$$\delta^{(p)}_{g,F} : J_s(F) \xrightarrow{\delta_{g,F}} H^1_f(F, V_g(1)) \xrightarrow{\log_{F,V_g(1)}} \frac{D_{\mathrm{dR}}(V_g(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))} \xrightarrow{\sim} (\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*))^\vee, \tag{2.9}$$

where the last identification arises from the de Rham pairing

$$\langle\ ,\ \rangle : D_{\mathrm{dR}}(V_g(1)) \times D_{\mathrm{dR}}(V_g^*) \to D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L_g \cong L_g \tag{2.10}$$

with respect to which $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*)$ are exact annihilators of each other. A basic ingredient for this computation will be the following alternate description of the logarithm map $\log_{F,V_g(1)}$.

Recall the interpretation of $H^1(F, V_g(1))$ as the space $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}(G_F)}(\mathbf{Q}_p, V_g(1))$ of extensions of $V_g(1)$ by $\mathbf{Q}_p$ in the category of p-adic G_F-representations. Since F contains $\mathbf{Q}_p(\zeta_s)$, V_g is a crystalline G_F-representation in the sense of Fontaine, and under that interpretation the Bloch-Kato "finite" subspace corresponds to those extensions which are crystalline (see [26, Prop. 1.26], for example):

$$H^1_f(F, V_g(1)) \cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathrm{cris}}(G_F)}(\mathbf{Q}_p, V_g(1)). \tag{2.11}$$

Now consider a crystalline extension

$$0 \to V_g(1) \to W \to \mathbf{Q}_p \to 0. \tag{2.12}$$

Since $D_{\mathrm{cris}}(V_g(1))^{\varphi=1} = 0$ by our assumptions, the resulting extension of φ-modules

$$0 \to D_{\mathrm{cris}}(V_g(1)) \to D_{\mathrm{cris}}(W) \to F_0 \to 0 \tag{2.13}$$

admits a unique section $s_W^{\mathrm{frob}} : F_0 \to D_{\mathrm{cris}}(W)$ with $s_W^{\mathrm{frob}}(1) \in D_{\mathrm{cris}}(W)^{\varphi=1}$. Extending scalars from F_0 to F in (2.13) and taking Fil^0-parts, we take an arbitrary section $s_W^{\mathrm{fil}} : F \to \mathrm{Fil}^0 D_{\mathrm{dR}}(W)$ of the resulting exact sequence of F-vector spaces

$$0 \to \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) \to \mathrm{Fil}^0 D_{\mathrm{dR}}(W) \to F \to 0 \tag{2.14}$$

and form the difference

$$t_W := s_W^{\mathrm{fil}}(1) - s_W^{\mathrm{frob}}(1),$$

which can be seen in $D_{\mathrm{dR}}(V_g(1))$, and whose image modulo $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ is well-defined.

Lemma 2.4 *Under the identification* (2.11)*, the above assignment*

$$0 \to V_g(1) \to W \to \mathbf{Q}_p \to 0 \quad \rightsquigarrow \quad t_W \bmod \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$$

defines an isomorphism which agrees with the Bloch-Kato logarithm map

$$\log_{F, V_g(1)} : H^1_f(F, V_g(1)) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V_g(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))}.$$

Proof See [26, Lem. 2.7], for example. □

Let $\Delta \in J_s(F)$ be the class of a degree 0 divisor on X_s with support contained in the finite set of points $S \subset X_s(F)$. The extension class $W = W_\Delta$ (2.12) corresponding to $\delta_{g,F}(\Delta)$ can then be constructed from the étale cohomology of the open curve $Y_s := X_s \smallsetminus S$, as explained in [1, Sect. 4.1]. We describe the associated $s^{\mathrm{fil}}_{W_\Delta}$ and $s^{\mathrm{frob}}_{W_\Delta}$.

By [33] (or also [13]), denoting g-isotypical components by the superscript g, there is a canonical isomorphism of $F_0 \otimes_{\mathbf{Q}_p} L_g$-modules

$$D_{\mathrm{cris}}(V_g) \cong H^1_{\mathrm{log-cris}}(\mathscr{X}_s)^g \tag{2.15}$$

compatible with φ-actions and inducing an $F \otimes_{\mathbf{Q}_p} L_g$-module isomorphism

$$D_{\mathrm{dR}}(V_g) \cong H^1_{\mathrm{dR}}(X_s/F)^g \tag{2.16}$$

after extension of scalars.

Writing $\Delta = \sum_{Q \in S} n_Q.Q$ for some $n_Q \in \mathbf{Z}$, we assume from now on that the reductions of the points $Q \in S$ are smooth and pair-wise distinct. Assume from now on that the reduction of S in the special fiber is stable under the absolute Frobenius. Like $H^1_{\mathrm{dR}}(X_s/F)$, the F-vector space $H^1_{\mathrm{dR}}(Y_s/F)$ is equipped with a canonical F_0-structure

$$H^1_{\mathrm{log-cris}}(\mathscr{Y}_s) \hookrightarrow H^1_{\mathrm{dR}}(Y_s/F), \tag{2.17}$$

a Frobenius operator still denoted by φ, and a Hecke action compatible with that in (2.4). Thus for $W = W_\Delta$ the exact sequence (2.13) is obtained as the pullback

$$\begin{array}{ccccc} D_{\mathrm{cris}}(V_g(1)) & \hookrightarrow & D_{\mathrm{cris}}(W_\Delta) & \xrightarrow{\rho} & F_0 \otimes_{\mathbf{Q}_p} L_g \\ \Vert & & \downarrow & & \downarrow \Delta \\ H^1_{\mathrm{log-cris}}(\mathscr{X}_s)^g(1) & \hookrightarrow & H^1_{\mathrm{log-cris}}(\mathscr{Y}_s)^g(1) & \xrightarrow{\oplus \mathrm{res}_Q} & (F_0 \otimes_{\mathbf{Q}_p} L_g)_0^{\oplus S} \end{array} \tag{2.18}$$

of the bottom extension of φ-modules with respect to the $F_0 \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q \in S}$, where the subscript 0 indicates taking the degree 0 subspace.

On the other hand, after extending scalars from F_0 to F and taking Fil^0-parts, (2.14) is given by the pullback[2]

$$\begin{array}{ccccc} \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) & \hookrightarrow & \mathrm{Fil}^0 D_{\mathrm{dR}}(W_\Delta) & \xrightarrow{\rho} & F \otimes_{\mathbf{Q}_p} L_g \\ \| & & \downarrow & & \downarrow \Delta \\ \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_s/F)^g & \hookrightarrow & \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y_s/F)^g & \xrightarrow{\oplus \mathrm{res}_Q} & (F \otimes_{\mathbf{Q}_p} L_g)_0^{\oplus S} \end{array} \tag{2.19}$$

of the bottom exact sequence of free $F \otimes_{\mathbf{Q}_p} L_g$-modules with respect to the $F \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q \in S}$.

Let $g^* \in S_2(X_s)$ be the form *dual* to g, defined as the newform associated with the twist $g \otimes \varepsilon_g^{-1}$, and let $\omega_{g^*} \in H^0(X_s, \Omega^1_{X_s/F})$ be its associated differential, so that $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*) = \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{g^*}) = (F \otimes_{\mathbf{Q}_p} L_g).\omega_{g^*}$. Thus $\delta^{(p)}_{g,F}(\Delta)$ is determined by the value

$$\delta^{(p)}_{g,F}(\Delta)(\omega_{g^*}) = \langle t_{W_\Delta}, \omega_{g^*} \rangle \tag{2.20}$$

of the pairing (2.10), which corresponds to the Poincaré pairing on $H^1_{\mathrm{dR}}(X_s/F)$ under the identification (2.16). Using rigid analysis, we now give an expression for the latter pairing that will make (2.20) amenable to computations.

Let $\mathcal{X}_s$ be the canonical balanced model of X_s over $\mathbf{Z}_p[\zeta_s]$ constructed by Katz and Mazur (see [22, Ch. 13]). The special fiber $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathbf{F}_p$ is a reduced disjoint union of Igusa curves over $\mathbf{F}_p$ intersecting at the supersingular points. Exactly two of these components are isomorphic to the Igusa curve $\mathrm{Ig}(\Gamma_s)$ representing the moduli problem $([\Gamma_1(N)], [\mathrm{bal}.\Gamma_1(p^s)^{\mathrm{can}}])$ over $\mathbf{F}_p$, and we let I_∞ be the one that contains the reduction of $\mathcal{W}_1(p^s) \times_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s)$, and I_0 be the other. (These two are the two "good" components in the terminology of [24]).

By the universal property of the regular minimal model, there exists a morphism

$$\mathscr{X}_s \to \mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathcal{O}_F \tag{2.21}$$

which reduces to a sequence of blow-ups on the special fiber. Letting κ be the residue field of F, define $\mathcal{W}_\infty \subset X_s$ (resp. $\mathcal{W}_0 \subset X_s$) to be the inverse image under the reduction map via $\mathscr{X}_s$ of the unique irreducible component of $\mathscr{X}_s \times_{\mathcal{O}_F} \kappa$ mapping bijectively onto $I_\infty \times_{\mathbf{F}_p} \kappa$ (resp. $I_0 \times_{\mathbf{F}_p} \kappa$) in $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ via the reduction of (2.21). Similarly, define $\mathcal{U} \subset X_s$ by considering the irreducible components of $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ different from $I_\infty \times_{\mathbf{F}_p} \kappa$ and $I_0 \times_{\mathbf{F}_p} \kappa$. Letting SS denote (the degree of) the supersingular divisor of $\mathrm{Ig}(\Gamma_s)$, it follows that $\mathcal{U}$ intersects $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$) in a union of SS supersingular annuli.

Since they reduce to smooth points, the residue class D_Q of each $Q \in S$ is conformal to the open unit disc $D \subset \mathbf{C}_p$. Fix an isomorphism $h_Q : D_Q \xrightarrow{\sim} D$ that sends Q to 0, and for a real number $r_Q < 1$ in $p^{\mathbf{Q}}$, denote by $\mathcal{V}_Q \subset D_Q$ the annulus consisting of

[2] Notice the effect of the Tate twist on the filtrations.

the points $x \in D_Q$ with $r_Q < |h_Q(x)|_p < 1$. In the same manner, we define annuli $\mathcal{V}_z$ for each z in the cuspidal subscheme $Z_s \subset X_s$.

Attached to any (oriented) annulus $\mathcal{V}$, there is a *p-adic annular residue map*

$$\mathrm{Res}_{\mathcal{V}} : \Omega^1_{\mathcal{V}} \to \mathbf{C}_p$$

defined by expanding $\omega \in \Omega^1_{\mathcal{V}}$ as $\omega = \sum_{n\in\mathbf{Z}} a_n T^n \frac{dT}{T}$ for a fixed uniformizing parameter T on $\mathcal{V}$ (compatible with the orientation), and setting $\mathrm{Res}_{\mathcal{V}}(\omega) = a_0$. This descends to a linear functional on $\Omega^1_{\mathcal{V}}/d\mathcal{O}_{\mathcal{V}}$. (Cf. [6, Lem. 2.1]).

For any *basic wide-open* $\mathcal{W}$ (as in [4, p. 34]), define

$$H^1_{\mathrm{rig}}(\mathcal{W}) := \mathbb{H}^1(\mathcal{W}, \Omega^\bullet(\log Z)) \cong \Omega^1_{\mathcal{W}}/d\mathcal{O}_{\mathcal{W}}, \tag{2.22}$$

where $\Omega^\bullet(\log Z)$ denotes the complex of rigid analytic sheaves on $\mathcal{W}$ deduced from (2.1) by analytification and pullback, and consider the basic wide-opens

$$\widetilde{\mathcal{W}}_\infty := \mathcal{W}_\infty \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q) \quad \text{and} \quad \widetilde{\mathcal{W}}_0 := \mathcal{W}_0 \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q).$$

As follows from the arguments in [2, Lem. 4.4.1], the spaces $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)$ and $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)$ are each equipped with a natural action of the Hecke operators T_ℓ $(\ell \nmid Np)$ compatible with the Hecke action on $H^1_{\mathrm{dR}}(Y_s/F)$ under restriction.

Lemma 2.5 • *The natural restriction maps induce an isomorphism*

$$H^1_{\mathrm{dR}}(Y_s/F)^g \;\to\; H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)^g \oplus H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)^g.$$

- *A class $\omega \in H^1_{\mathrm{dR}}(Y_s/F)^g$ belongs to the natural image of $H^1_{\mathrm{dR}}(X_s/F)^g$ if an only if it can be represented by a pair of differentials $(\omega_\infty, \omega_0) \in \Omega^1_{\widetilde{\mathcal{W}}_\infty} \times \Omega^1_{\widetilde{\mathcal{W}}_0}$ with vanishing p-adic annular residues.*
- *If η and ω are any two classes in $H^1_{\mathrm{dR}}(X_s/F)^g$, their Poincaré pairing can be computed as*

$$\langle \eta, \omega\rangle = \sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_\infty} \mathrm{Res}_{\mathcal{V}}(F_{\omega_\infty|_{\mathcal{V}}} \cdot \eta_\infty|_{\mathcal{V}}) + \sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_0} \mathrm{Res}_{\mathcal{V}}(F_{\omega_0|_{\mathcal{V}}} \cdot \eta_0|_{\mathcal{V}}), \tag{2.23}$$

where for each annulus $\mathcal{V}$, $F_{\omega_{\mathcal{V}}}$ denotes any solution to $dF_{\omega_{\mathcal{V}}} = \omega_{\mathcal{V}}$ on $\mathcal{V}$.

Proof By an excision argument, the first assertion is easily deduced from [10, Thm. 2.1] as in [2, Lem. 4.4.2]; the second and third are shown by adapting the arguments in [9, §5] for each of the two components, as they are proven in [7, Prop. 1.3] for $s = 1$. (See also [10, §3].) □

2.3 Coleman p-adic integration

Coleman's theory provides a coherent choice of local primitives that will allow us to compute (2.20) using the formula (2.23).

Recall the lift of Frobenius $\phi : \mathcal{W}_2(p^s) \to \mathcal{W}_1(p^s)$ described in Sect. 2.1, where $\mathcal{W}_i(p^s)$ are the strict neighborhoods of the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞ described there. Recall also the wide open space $\mathcal{W}_\infty$ described in the preceding section, which also contains $X_s(0)$ by construction.

Proposition 2.6 (Coleman) *Let $g = \sum_{n>0} b_n q^n \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, so that b_p is such that $U_p g = b_p g$. Then there exists a locally analytic function F_{ω_g} on $\mathcal{W}_\infty$ which is unique up to a constant on $\mathcal{W}_\infty$ and such that*

- *$dF_{\omega_g} = \omega_g$ on $\mathcal{W}_\infty$, and*
- *$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} \in M_0^{\mathrm{rig}}(X_s)$.*

Proof This follows from the general result of Coleman [8, Thm. 10.1]. Indeed, a computation on q-expansions shows that the action of the Frobenius lift ϕ on differentials agrees with that of pV, with V the map sending $q \mapsto q^p$, in the sense that $\phi^*\omega_g = p\omega_{Vg}$ on $\mathcal{W}'_\infty := \phi^{-1}(\mathcal{W}_\infty \cap \mathcal{W}_1(p^s))$. Since the differential $\omega_{g^{[p]}} = \omega_g - b_p\omega_{Vg}$ attached to

$$g^{[p]} = \sum_{(n,p)=1} b_n q^n$$

becomes exact upon restriction to $\mathcal{W}'_\infty$, this shows that the polynomial $L(T) = 1 - \frac{b_p}{p}T$ is such that

$$L(\phi^*)\omega_g = 0.$$

Finally, since g has primitive nebentypus, b_p has complex absolute value $p^{1/2}$, and hence [8, Thm. 10.1] can be applied with $L(T)$ as above. □

Attached to a primitive p^s-th root of unity ζ, there is an automorphism w_ζ of X_s which interchanges the components $\mathcal{W}_\infty$ and $\mathcal{W}_0$ (see [2, Lem. 4.4.3]).

Corollary 2.7 *Define $\phi' := w_\zeta \circ \phi \circ w_\zeta$. With hypotheses as in Proposition 2.6, there exists a unique locally analytic function F'_{ω_g} on $\mathcal{W}_0$ which vanishes at 0, satisfies $dF'_{\omega_g} = \omega_g$ on $\mathcal{W}_0$, and $F'_{\omega_g} - \frac{b_p}{p}(\phi')^* F'_{\omega_g}$ is rigid analytic on a wide-open neighborhood $\mathcal{W}'_0$ of $w_\zeta X_s(0)$ in $\mathcal{W}_0$.*

Proof Proposition 2.6 applied to the differential $\omega'_g := w_\zeta^*\omega_g$ gives the existence of a locally analytic function $F_{\omega'_g}$ with $F'_{\omega_g} := w_\zeta^* F_{\omega'_g}$ having the desired properties. The uniqueness of F'_{ω_g} follows immediately from that of $F_{\omega'_g}$. □

We refer to the locally analytic function F_{ω_g} (resp. F'_{ω_g}) appearing in Proposition 2.6 as the *Coleman primitive* of g on $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$). Let $g = \sum_{n>0} b_n q^n$ be as in

Proposition 2.6. The q-expansion $\sum_{(n,p)=1} \frac{b_n}{n} q^n$ corresponds to a p-adic modular form g' vanishing at ∞ and satisfying $dg' = g^{[p]}$, where d is the operator described at the end of Section 2.1, which here corresponds to the differential operator $\mathcal{O}_{\mathcal{W}} \to \Omega^1_{\mathcal{W}}$ for any subspace $\mathcal{W} \subset X_s$. Set $d^{-1}g^{[p]} := g'$.

Corollary 2.8 *If F_{ω_g} is the Coleman primitive of g on $\mathcal{W}_\infty$ which vanishes at ∞, then*

$$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} = d^{-1}g^{[p]}.$$

Proof Since $d^{-1}g^{[p]}$ is an overconvergent rigid analytic primitive of $\omega_{g^{[p]}}$, and the operator $L(\phi^*) = 1 - \frac{b_p}{p}\phi^*$ acting on the space of locally analytic functions on $\mathcal{W}'_\infty$ is invertible, we see that $L(\phi^*)^{-1}(d^{-1}g^{[p]})$ satisfies the defining properties of F_{ω_g}. Since $d^{-1}g^{[p]}$ vanishes at ∞, the result follows. □

Now we can give a closed formula for the p-adic Abel-Jacobi images of certain degree 0 divisors on X_s.

Proposition 2.9 *Assume $s > 1$. Let $g \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, let P be an F-rational point of X_s factoring through $X_s(0) \subset X_s$, and let $\Delta \in J_s(F)$ be the divisor class of $(P) - (\infty)$. Then*

$$\delta^{(p)}_{g,F}(\Delta)(\omega_{g^*}) = F_{\omega_{g^*}}(P), \tag{2.24}$$

where $F_{\omega_{g^*}}$ is the Coleman primitive of ω_{g^*} on $\mathcal{W}_\infty$ which vanishes at ∞.

Proof By (2.20), we must compute $\langle t_{W_\Delta}, \omega_{g^*}\rangle = \langle s^{\text{fil}}_{W_\Delta}, \omega_{g^*}\rangle - \langle s^{\text{frob}}_{W_\Delta}, \omega_{g^*}\rangle$, where

- $s^{\text{fil}}_{W_\Delta} \in \text{Fil}^1 D_{\text{dR}}(W_\Delta)$ is such that $\rho(s^{\text{fil}}_{W_\Delta}) = 1$ in (2.19), and
- $s^{\text{frob}}_{W_\Delta} \in D_{\text{cris}}(W_\Delta)^{\varphi=1}$ is such that $\rho(s^{\text{frob}}_{W_\Delta}) = 1$ in (2.18).

By Lemma 2.5, we see that these can be represented, respectively, by

- η^{fil}_Δ a section of $\Omega^1_{X_s/F}$ over Y_s with simple poles at P and ∞ and with
  - $\text{Res}_P(\eta^{\text{fil}}_\Delta) = 1$, while $\text{Res}_\infty(\eta^{\text{fil}}_\Delta) = 0$ for all $Q \in S - \{P\}$;
  - $\text{Res}_\infty(\eta^{\text{fil}}_\Delta) = -1$, while $\text{Res}_z(\eta^{\text{fil}}_\Delta) = 0$ for all $z \in Z_s - \{\infty\}$,
- $\eta^{\text{frob}}_\Delta = (\eta^{\text{frob}}_\infty, \eta^{\text{frob}}_0) \in \Omega^1_{\widetilde{\mathcal{W}}_\infty} \times \Omega^1_{\widetilde{\mathcal{W}}_0}$ with
  - $(\phi^*\eta^{\text{frob}}_\infty, (\phi')^*\eta^{\text{frob}}_\Delta) = (p \cdot \eta^{\text{frob}}_\infty + dG_\infty, p \cdot \eta^{\text{frob}}_0 + dG_0)$ with G_∞ and G_0 rigid analytic on $\phi^{-1}\widetilde{\mathcal{W}}_\infty$ and $(\phi')^{-1}\widetilde{\mathcal{W}}_0$, respectively;
  - $\text{Res}_{\mathcal{V}}(\eta^{\text{frob}}_\Delta) = 0$ for all supersingular annuli $\mathcal{V}$; and
  - $\text{Res}_{\mathcal{V}_Q}(\eta^{\text{frob}}_\Delta) = \text{Res}_Q(\eta^{\text{fil}}_\Delta)$ $(Q \in S)$, $\text{Res}_{\mathcal{V}_z}(\eta^{\text{frob}}_\Delta) = \text{Res}_z(\eta^{\text{fil}}_\Delta)$ $(z \in Z_s)$.

The arguments in [1, Prop. 3.21] can now be straightforwardly adapted to deduce the result. Indeed, using the defining properties of the Coleman primitives $F_{\omega_{g^*}}$ and $F'_{\omega_{g^*}}$ of ω_{g^*} on $\mathcal{W}_\infty$ and $\mathcal{W}_0$, respectively, one first shows that

$$\sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_\infty} \text{Res}_{\mathcal{V}}(F_{\omega_{g^*}} \cdot \eta^{\text{frob}}_\infty) = 0 \quad \text{and} \quad \sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_0} \text{Res}_{\mathcal{V}}(F'_{\omega_{g^*}} \cdot \eta^{\text{frob}}_0) = 0 \tag{2.25}$$

as in [*loc.cit.*, Lemma 3.20]. On the other hand, using the same primitives, one shows as in [*loc.cit.*, Lemma 3.19] that

$$\sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_\infty} \mathrm{Res}_{\mathcal{V}}(F_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = F_{\omega_{g^*}}(P) \quad \text{and} \quad \sum_{\mathcal{V}\subset\widetilde{\mathcal{W}}_0} \mathrm{Res}_{\mathcal{V}}(F'_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = 0. \qquad (2.26)$$

Substituting (2.26) and (2.25) into the formula (2.23) for the Poincaré pairing (and using that $s > 1$, so that there is no overlap between the supersingular annuli in $\widetilde{\mathcal{W}}_\infty$ and the supersingular annuli in $\widetilde{\mathcal{W}}_0$), the result follows. □

3 Generalised Heegner cycles

Let $X_1(N)$ be the compactified modular curve of level $\Gamma_1(N)$ defined over $\mathbf{Q}$, and let $\mathcal{E}$ be the universal generalized elliptic curve over $X_1(N)$. (Recall that $N > 4$). For $r > 1$, denote by W_r the $(2r-1)$-dimensional Kuga-Sato variety[3], defined as the canonical desingularization of the $(2r-2)$-nd fiber product of $\mathcal{E}$ with itself over $X_1(N)$. By construction, the variety W_r is equipped with a proper morphism

$$\pi_r : W_r \to X_1(N)$$

whose fibers over a noncuspidal closed point of $X_1(N)$ corresponding to an elliptic curve E with $\Gamma_1(N)$-level structure is identified with $2r-2$ copies of E. (For a more detailed description, see [1, Sect. 3.1].)

Let K be an imaginary quadratic field of odd discriminant $-D < 0$. It will be assumed throughout that K satisfies the following hypothesis:

Assumption 3.1 All the prime factors of N split in K.

Denote by $\mathcal{O}_K$ the ring of integers of K, and note that by this assumption we may choose an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$ that we fix once and for all.

Let A be a fixed elliptic curve with CM by $\mathcal{O}_K$. The pair $(A, A[\mathfrak{N}])$ defines a point P_A on $X_0(N)$ rational over H, the Hilbert class field of K. Choose one of the square-roots $\sqrt{-D} \in \mathcal{O}_K$, let $\Gamma_{\sqrt{-D}} \subset A \times A$ be the graph of $\sqrt{-D}$, and define

$$\Upsilon_{A,r}^{\mathrm{heeg}} := \Gamma_{\sqrt{-D}} \times \overset{(r-1)}{\cdots} \times \Gamma_{\sqrt{-D}}$$

viewed inside W_r by the natural inclusion $(A \times A)^{r-1} \to W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting P_A. Let ϵ_W be the projector from [1, (2.1.2)], and set

$$\Delta_{A,r}^{\mathrm{heeg}} := \epsilon_W \Upsilon_{A,r}^{\mathrm{heeg}}, \qquad (3.1)$$

which is an $(r-1)$-dimensional null-homologous cycle on W_r defining an H-rational class in the Chow group $\mathrm{CH}^r(W_r)_0$ (taken with $\mathbf{Q}$-coefficients, as always here).

[3] Perhaps most commonly denoted by W_{2r-2}; cf. [35] and [27], for example.

These cycles (3.1) are usually referred to as *Heegner cycles* (of conductor one, weight $2r$), and they share with classical Heegner points (as in [15]) many of their arithmetic properties (see [25,27,35]).

We next recall a variation of the previous construction introduced in the recent work [1] of Bertolini–Darmon–Prasanna. Let A be the CM elliptic curve fixed above, and consider the variety[4]

$$X_r := W_r \times A^{2r-2}.$$

For each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, represented by an ideal $\mathfrak{a} \subset \mathcal{O}_K$ prime to N, let $A_{\mathfrak{a}} := A/A[\mathfrak{a}]$ and denote by $\varphi_{\mathfrak{a}}$ the degree $\mathrm{N}\mathfrak{a}$-isogeny

$$\varphi_{\mathfrak{a}} : A \to A_{\mathfrak{a}}.$$

The pair $\mathfrak{a} * (A, A[\mathfrak{N}]) := (A_{\mathfrak{a}}, A_{\mathfrak{a}}[\mathfrak{N}])$ defines a point $P_{A_{\mathfrak{a}}}$ in $X_0(N)$ rational over H. Let $\Gamma^t_{\varphi_{\mathfrak{a}}} \subset A_{\mathfrak{a}} \times A$ be the transpose of the graph of $\varphi_{\mathfrak{a}}$, and set

$$\Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \Gamma^t_{\varphi_{\mathfrak{a}}} \times \overset{(2r-2)}{\cdots} \times \Gamma^t_{\varphi_{\mathfrak{a}}} \subset (A_{\mathfrak{a}} \times A)^{2r-2} = A_{\mathfrak{a}}^{2r-2} \times A^{2r-2} \xrightarrow{(\iota_{\mathfrak{a}}, \mathrm{id}_A)} X_r,$$

where $\iota_{\mathfrak{a}}$ is the natural inclusion $A_{\mathfrak{a}}^{2r-2} \to W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting $P_{A_{\mathfrak{a}}}$. Letting ϵ_A be the projector from [1, (1.4.4)], the cycles

$$\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \epsilon_A \epsilon_W \Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} \tag{3.2}$$

define classes in $\mathrm{CH}^{2r-1}(X_r)_0(H)$ and are referred to as *generalised Heegner cycles*.

We will assume for the rest of this paper that K also satisfies the following:

Assumption 3.2 The prime p splits in K.

Let $g \in S_{2r}(X_0(N))$ be a normalized newform, and let V_g be the p-adic Galois representation associated to g by Deligne. By the Künneth formula, there is a map

$$H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1)) \longrightarrow H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \mathrm{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)),$$

which composed with the natural Galois-equivariant projection

$$H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \mathrm{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)) \xrightarrow{\pi_g \otimes \pi_{\mathrm{N}^{r-1}}} V_g(r)$$

induces a map

$$\pi_{g,\mathrm{N}^{r-1}} : H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1))) \longrightarrow H^1(F, V_g(r))$$

over any number field F. In the following we fix a number field F containing H.

[4] Notice that our indices differ from those in [1].

Now consider the étale Abel-Jacobi map

$$\Phi_F^{\text{ét}} : \mathrm{CH}^{2r-1}(X_r)_0(F) \to H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p)(2r-1))$$

constructed in [28]. Let F_p be the completion of $\imath_p(F)$, and denote by loc_p the induced localization map from G_F to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_p)$. Then we may define the *p-adic Abel-Jacobi map* AJ_{F_p} by the commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{CH}^{2r-1}(X_r)_0(F) & \xrightarrow{\pi_{g,\mathrm{N}^{r-1}} \circ \Phi_F^{\text{ét}}} & H^1(F, V_g(r)) & \xrightarrow{\mathrm{loc}_p} & H^1(F_p, V_g(r)) \\ & \searrow^{\cdots} & & & \cup \\ & & & & H^1_f(F_p, V_g(r)) \\ & \searrow_{\mathrm{AJ}_{F_p}} & & & \downarrow {\scriptstyle \log_{F_p, V_g(r)}} \\ & & & & \mathrm{Fil}^1 D_{\mathrm{dR}}(V_g(r-1))^\vee, \end{array} \tag{3.3}$$

where the existence of the dotted arrow follows from [28, Thm.(3.1)(i)], and the vertical map is given by the logarithm map of Bloch-Kato, as it appeared in (2.9) for $r = 1$. Using the comparison isomorphism of Faltings [12], the map AJ_{F_p} may be evaluated at the class $\omega_g \otimes e_\zeta^{\otimes r-1}$, with e_ζ an F_p-basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong F_p$.

The main result of [1] yields the following formula for the p-adic Abel-Jacobi images of the generalised Heegner cycles (3.2) which we will need.

Theorem 3.3 (Bertolini–Darmon–Prasanna) *Let $g = \sum_n b_n q^n \in S_{2r}(X_0(N))$ be a normalized newform of weight $2r \geq 2$ and level N prime to p. Then*

$$(1 - b_p p^{-r} + p^{-1}) \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{1-r} \cdot \mathrm{AJ}_{F_p}(\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}}, r})(\omega_g \otimes e_\zeta^{\otimes r-1})$$
$$= (-1)^{r-1}(r-1)! \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-r} g^{[p]}(\mathfrak{a} * (A, A[\mathfrak{N}])),$$

where $g^{[p]} = \sum_{(n,p)=1} b_n q^n$ is the p-depletion of g.

Proof See the proof of [1, Thm. 5.13]. □

We end this section by relating the images of Heegner cycles and of generalised Heegner cycles under the p-adic height pairing. (Cf. [1, Sect. 3.4]).

Consider $\Pi_r := W_r \times A^{r-1}$ seen as a subvariety of $W_r \times X_r = W_r \times W_r \times (A^2)^{r-1}$ via the map

$$(\mathrm{id}_{W_r}, \mathrm{id}_{W_r}, (\mathrm{id}_A, \sqrt{-D})^{r-1}).$$

Denoting by π_W and π_X the projections onto the first and second factors of $W_r \times X_r$, the rational equivalence class of the cycle Π_r gives rise to a map on Chow groups

$$\Pi_r : \mathrm{CH}^{2r-1}(X_r) \to \mathrm{CH}^{r+1}(W_r)$$

induced by $\Pi_r(\Delta) = \pi_{W,*}(\Pi_r \cdot \pi_X^* \Delta)$.

Lemma 3.4 *We have*

$$\langle \Delta^{\mathrm{heeg}}_{A,r}, \Delta^{\mathrm{heeg}}_{A,r} \rangle_{W_r} = (4D)^{r-1} \cdot \langle \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}, \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r} \rangle_{X_r},$$

where $\langle , \rangle_{W_r}$ *and* $\langle , \rangle_{X_r}$ *are the* p*-adic height pairings of* [26] *on* $\mathrm{CH}^{r+1}(W_r)_0$ *and* $\mathrm{CH}^{2r-1}(X_r)_0$*, respectively.*

Proof The image $\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r})$ remains unchanged if we replace $\Gamma_{\sqrt{-D}}$ by $Z_A := \Gamma_{\sqrt{-D}} - (A \times \{0\}) - D(\{0\} \times A)$ (see [27, §II(3.6)]). Since $Z_A \cdot Z_A = -2D$, we easily see from the construction of Π_r that

$$\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r}) = (-2D)^{r-1} \cdot \Phi^{\text{ét}}_F(\Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r})). \tag{3.4}$$

On the other hand, if $\langle , \rangle_A$ denotes the Poincaré pairing on $H^1_{\mathrm{dR}}(A/F)$, we have

$$\langle (\sqrt{-D})^* \omega, (\sqrt{-D})^* \omega' \rangle_A = D \cdot \langle \omega, \omega' \rangle_A,$$

for all $\omega, \omega' \in H^1_{\mathrm{dR}}(A/F)$. By the definition of the p-adic height pairings $\langle , \rangle_{W_r}$ and $\langle , \rangle_{X_r}$ (factoring through $\Phi^{\text{ét}}_F$), we thus see that

$$\langle \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}, \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r} \rangle_{X_r} = D^{r-1} \cdot \langle \Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}), \Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}) \rangle_{W_r}. \tag{3.5}$$

Combining (3.4) and (3.5), the result follows. □

4 Big logarithm map

Let $\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ be a Hida family passing through (the ordinary p-stabilization of) a p-ordinary newform $f_o \in S_k(X_0(N))$ as described in the Introduction. We begin this section by recalling the definition of a certain twist of $\mathbf{f}$ such that all of its specializations at arithmetic primes of *even weight* correspond to p-adic modular forms with trivial weight-nebentypus.

Decompose the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}$ as the product

$$\varepsilon_{\mathrm{cyc}} = \omega \cdot \epsilon : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times = \boldsymbol{\mu}_{p-1} \times \Gamma.$$

Since k is even, the character ω^{k-2} admits a square root $\omega^{\frac{k-2}{2}} : G_{\mathbf{Q}} \to \boldsymbol{\mu}_{p-1}$, and in fact two different square roots, corresponding to the two different lifts of $k-2 \in$

$\mathbf{Z}/(p-1)\mathbf{Z}$ to $\mathbf{Z}/2(p-1)\mathbf{Z}$. Fix for now a choice of $\omega^{\frac{k-2}{2}}$, and define the *critical character* to be

$$\Theta := \omega^{\frac{k-2}{2}} \cdot [\epsilon^{1/2}] : G_{\mathbf{Q}} \to \Lambda_{\mathcal{O}}^{\times}, \tag{4.1}$$

where $\epsilon^{1/2} : G_{\mathbf{Q}} \to \Gamma$ denotes the unique square root of ϵ taking values in Γ.

Remark 4.1 As noted in [19, Rem. 2.1.4], the above choice of Θ is for most purposes largely indistinguishable from the other choice, namely $\omega^{\frac{p-1}{2}}\Theta$, where

$$\omega^{\frac{p-1}{2}} : \mathrm{Gal}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q}) \xrightarrow{\sim} \{\pm 1\} \qquad (p^* = (-1)^{\frac{p-1}{2}} p).$$

Nonetheless, for a given f_o as above, our main result (Theorem 5.11) will specifically apply *to only one* of the two possible choices for the critical character.

The *critical twist* of $\mathbb{T}$ is then defined to be the module

$$\mathbb{T}^{\dagger} := \mathbb{T} \otimes_{\mathbb{I}} \mathbb{I}^{\dagger} \tag{4.2}$$

equipped with the diagonal $G_{\mathbf{Q}}$-action, where $\mathbb{I}^{\dagger} = \mathbb{I}(\Theta^{-1})$ is $\mathbb{I}$ as a module over itself with $G_{\mathbf{Q}}$ acting via the character $G_{\mathbf{Q}} \xrightarrow{\Theta^{-1}} \Lambda_{\mathcal{O}}^{\times} \to \mathbb{I}^{\times}$.

Lemma 4.2 *Let $\rho_{\mathbb{T}^{\dagger}} : G_{\mathbf{Q}} \to \mathrm{Aut}(\mathbb{T}^{\dagger})$ be the Galois representation carried by $\mathbb{T}^{\dagger}$. Then for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ we have*

$$\nu(\rho_{\mathbb{T}^{\dagger}}) \cong \rho_{\mathbf{f}'_\nu} \otimes \varepsilon_{\mathrm{cyc}}^{r_\nu},$$

where $\mathbf{f}'_\nu$ is a character twist of $\mathbf{f}_\nu$ of the same weight with trivial nebentypus. In other words, defining $\mathbb{V}_\nu^{\dagger} := \mathbb{T}^{\dagger} \otimes_{\mathbb{I}} F_\nu$ and letting $V_{\mathbf{f}'_\nu}$ be the representation space of $\rho_{\mathbf{f}'_\nu}$, we have

$$\mathbb{V}_\nu^{\dagger} \cong V_{\mathbf{f}'_\nu}(r_\nu), \tag{4.3}$$

and in particular $\mathbb{V}_\nu^{\dagger}$ is isomorphic to its Kummer dual.

Proof This follows from a straightforward computation explained in [29, (3.5.2)] for example (where $\mathbb{T}^{\dagger}$ is denoted by T). ☐

Let $\theta : \mathbf{Z}_p^{\times} \to \Lambda_{\mathcal{O}}^{\times}$ be such that $\Theta = \theta \circ \varepsilon_{\mathrm{cyc}}$. It follows from the preceding lemma that the formal q-expansion

$$\mathbf{f}^{\dagger} = \mathbf{f} \otimes \theta^{-1} := \sum_{n>0} \theta^{-1}(n)\mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

(where we put $\theta^{-1}(n) = 0$ whenever $p|n$) is such that, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight, $\mathbb{V}_\nu^{\dagger}$ is the Galois representation attached to the specialization $\mathbf{f}_\nu \otimes \theta_\nu^{-1}$ of $\mathbf{f}^{\dagger}$, which by Lemma 2.3 is a p-adic modular form of weight 0 and trivial nebentypus.

We next recall some of the local properties of the big Galois representation $\mathbb{T}$. Let $I_w \subset D_w \subset G_{\mathbf{Q}}$ be the inertia and decomposition groups at the place $w|p$ induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. In the following we will identify D_w with the absolute Galois group $G_{\mathbf{Q}_p}$. Then by a result of Mazur and Wiles (see [34, Thm. 2.2.2]) there exists a filtration of $\mathbb{I}[D_w]$-modules

$$0 \to \mathscr{F}_w^+\mathbb{T} \to \mathbb{T} \to \mathscr{F}_w^-\mathbb{T} \to 0 \tag{4.4}$$

with $\mathscr{F}_w^{\pm}\mathbb{T}$ free of rank one over $\mathbb{I}$ and with the Galois action on $\mathscr{F}_w^-\mathbb{T}$ unramified, given by the character $\alpha : D_w/I_w \to \mathbb{I}^\times$ sending an arithmetic Frobenius σ_p to $\mathbf{a}_p$. Twisting (4.4) by Θ^{-1} we define $\mathscr{F}_w^{\pm}\mathbb{T}^\dagger$ in the natural manner.

Let $\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}, \mathbb{I})$ be the contragredient[5] of $\mathbb{T}$, and consider the $\mathbb{I}$-module

$$\mathbb{D} := (\mathscr{F}_w^+\mathbb{T}^* \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}, \tag{4.5}$$

where $\mathscr{F}_w^+\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathscr{F}^-\mathbb{T}, \mathbb{I}) \subset \mathbb{T}^*$, and $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$.

Fix once and for all a compatible system $\zeta = \{\zeta_s\}$ of primitive p^s-th roots of unity, and denote by e_ζ the basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$ corresponding to $1 \in \mathbf{Q}_p$ under the resulting identification $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p$.

Lemma 4.3 *The module $\mathbb{D}$ is free of rank one over $\mathbb{I}$, and for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ there is a canonical isomorphism*

$$\mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \cong \frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}. \tag{4.6}$$

Proof Since the action on $\mathscr{F}_w^+\mathbb{T}^*$ is unramified, the first claim follows from [30, Lemma 3.3] in light of the definition (4.5) of $\mathbb{D}$. The second can be deduced from [30, Lemma 3.2] as in the proof of [30, Lemma 3.6]. □

With the same notations as in Lemma 4.3, we denote by $\langle, \rangle_{\mathrm{dR}}$ the pairing

$$\langle, \rangle_{\mathrm{dR}} : \mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \times \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}(r_\nu - 1)) \to F_\nu \tag{4.7}$$

deduced from the usual de Rham pairing

$$\frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}^*(1 - r_\nu)) \to F_\nu$$

via the identification (4.6) and the isomorphism $V_{\mathbf{f}_\nu}^* \cong V_{\mathbf{f}_\nu^*}(k_\nu - 1)$.

Theorem 4.4 (Ochiai) *Assume that the residual representation $\bar{\rho}_{f_o}$ is irreducible, fix an $\mathbb{I}$-basis η of $\mathbb{D}$, and set $\lambda := \mathbf{a}_p - 1$. There exists an $\mathbb{I}$-linear map*

$$\mathrm{Log}_{\mathbb{T}^\dagger}^{(\eta)} : H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \to \mathbb{I}[\lambda^{-1}]$$

[5] So that $\mathbb{T}^* \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}_\nu}$ for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

such that if $\mathfrak{Y} \in H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger)$ *and* $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ *has weight* $k_\nu = 2r_\nu \geq 2$*, then*

$$\nu(\mathrm{Log}^{(\eta)}_{\mathbb{T}^\dagger}(\mathfrak{Y})) = \frac{(-1)^{r_\nu-1}}{(r_\nu-1)!} \times \begin{cases} \left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^{-1}\left(1-\frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right)\langle \log_{V_{\mathbf{f}_\nu}(r_\nu)}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu = \mathbb{1}; \\ \frac{1}{G(\vartheta_\nu^{-1})}\left(\frac{\nu(\mathbf{a}_p)}{p^{r_\nu-1}}\right)^{s_\nu}\langle \log_{s,V_{\mathbf{f}_\nu}(r_\nu)}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu \neq \mathbb{1}, \end{cases} \tag{4.8}$$

where

- $\log_{V_{\mathbf{f}_\nu}(r_\nu)}$ *(resp.* $\log_{s,V_{\mathbf{f}_\nu}(r_\nu)}$*) is the Bloch-Kato logarithm map for* $V_{\mathbf{f}_\nu}(r_\nu)$ *over* $\mathbf{Q}_p$ *(resp.* $\mathbf{Q}_{p,s} := \mathbf{Q}_p(\boldsymbol{\mu}_{p^s})$*),*
- $\eta'_\nu \in \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}^*_\nu}(r_\nu-1))$ *is such that* $\langle \eta_\nu \otimes e_\zeta^{\otimes r_\nu}, \eta'_\nu\rangle_{\mathrm{dR}} = 1$*,*
- $\vartheta_\nu : \mathbf{Z}_p^\times \to F_\nu^\times$ *is the finite order character* $z \mapsto \theta_\nu(z)z^{1-r_\nu}$*,*
- $s > 0$ *is such that the conductor of* ϑ_ν *is* p^s*, and*
- $G(\vartheta_\nu^{-1})$ *is the Gauss sum* $\sum_{x \bmod p^s} \vartheta_\nu^{-1}(x)\zeta_s^x$*.*

Proof Let $\Lambda_{\mathrm{cyc}} = \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]]$ be the cyclotomic Iwasawa algebra, where

$$\Gamma_{\mathrm{cyc}} := \mathrm{Gal}(\mathbf{Q}_{p,\infty}/\mathbf{Q}_p) \cong \mathbf{Z}_p,$$

and consider the $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$-modules $\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$ and $\mathscr{F}_w^+\mathcal{T}^* := \mathscr{F}_w^+\mathbb{T}^*\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}} \otimes \omega^{\frac{k-2}{2}}$, the latter being equipped with the diagonal action of $G_{\mathbf{Q}_p}$. Also let γ_o be a topological generator of Γ_{cyc} and $\mathcal{I} := (\lambda, \gamma_o) \subset \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}} \cong \mathbb{I}[[\Gamma_{\mathrm{cyc}}]]$. Consider the $\mathbb{I}$-algebra isomorphism

$$\mathrm{Tw}_{\theta_1} : \mathbb{I}[[\Gamma_{\mathrm{cyc}}]] \to \mathbb{I}[[\Gamma_{\mathrm{cyc}}]] \tag{4.9}$$

given by $\mathrm{Tw}_{\theta_1}([\sigma]) = \epsilon^{1/2}(\sigma)[\sigma]$ for $\sigma \in \Gamma_{\mathrm{cyc}}$, where $\epsilon^{1/2}$ is the unique square-root of the wild component of the cyclotomic character. By [30, Prop. 5.3] there exists an injective $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$-linear map

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*} : \mathcal{I}\mathcal{D} \to H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*)$$

with cokernel killed by $\mathcal{I}$ which interpolates the Bloch-Kato exponential over the arithmetic primes of $\mathbb{I}$ and of Λ_{cyc}. Notice that letting $\mathscr{F}_w^+\mathcal{T}^\dagger$ be the module $\mathscr{F}_w^+\mathcal{T}^*$ with the $\mathbb{I}[[\Gamma_{\mathrm{cyc}}]]$-action twisted by θ_1, there is a Galois equivariant projection $\mathscr{F}_w^+\mathcal{T}^\dagger \to \mathscr{F}_w^+\mathbb{T}^\dagger$. The composition

$$\mathcal{I}\mathcal{D} \xrightarrow{\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*) \xrightarrow{\mathrm{Tw}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^\dagger) \xrightarrow{\mathrm{Cor}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \tag{4.10}$$

is an $\mathbb{I}$-linear map making for every $\nu \in \mathcal{X}_{\text{arith}}(\mathbb{I})$ as in the statement the diagram

$$\begin{array}{ccc} \mathcal{I}\mathcal{D} & \xrightarrow{\operatorname{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger}} & H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \\ \downarrow{\scriptstyle \operatorname{Sp}_{\nu,\zeta}} & & \downarrow{\scriptstyle \operatorname{Sp}_\nu} \\ \frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))} \otimes \mathbf{Q}_{p,s} & \longrightarrow & H^1(\mathbf{Q}_{p,s}, \mathscr{F}_w^+ V_{\mathbf{f}_\nu}(r_\nu)) \end{array}$$

commutative, where $\operatorname{Sp}_{\nu,\zeta}$ is given by the composition of (4.6) with the map

$$\mathcal{D} = \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\text{cyc}} \to \mathbb{D}_\nu \otimes D_{\text{dR}}(\mathbf{Q}_p(r_\nu)) \otimes \mathbf{Q}_{p,s}$$

induced by specialization at ν on $\mathbb{D}$ and $\sigma \mapsto e_\zeta^{\otimes r_\nu} \otimes \zeta_s^\sigma$ ($\sigma \in \Gamma_{\text{cyc}}$) on Λ_{cyc}, and where the bottom horizontal arrow is given by:

$$(-1)^{r_\nu - 1}(r_\nu - 1)! \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)\left(1 - \frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right)^{-1} \exp_{V_{\mathbf{f}_\nu}(r_\nu)} & \text{if } \vartheta_\nu = \mathbb{1}; \\ G(\vartheta_\nu^{-1})\left(\frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^{s_\nu} \exp_{s, V_{\mathbf{f}_\nu}(r_\nu)} & \text{if } \vartheta_\nu \neq \mathbb{1} \end{cases}$$

with $\exp_{V_{\mathbf{f}_\nu}(r_\nu)}$ (resp. $\exp_{s,V_{\mathbf{f}_\nu}(r_\nu)}$) the Bloch-Kato exponential map for $V_{\mathbf{f}_\nu}(r_\nu)$ over $\mathbf{Q}_p$ (resp. $\mathbf{Q}_{p,s}$). The map $\operatorname{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger}$ factors through an injective $\mathbb{I}$-linear map

$$\operatorname{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger} : \mathbb{D}^\dagger \to H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger),$$

where $\mathbb{D}^\dagger := \mathcal{I}\mathcal{D} \otimes_{\mathbf{Z}_p} \mathbb{I}[[\Gamma_{\text{cyc}}]]/(\gamma_o^2 - \gamma_o')$ with γ_o' a topological generator of Γ. (Recall for the Introduction that Γ acts on $\mathbb{I}$ via the diamond operators.)

Now if $\mathfrak{Y} \in H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger)$, then $\lambda \cdot \mathfrak{Y}$ lands in the image $\operatorname{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger}$ and so

$$\operatorname{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y}) := \lambda^{-1} \cdot (\operatorname{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger})^{-1}(\lambda \cdot \mathfrak{Y}) \in \mathbb{I}[\lambda^{-1}] \otimes_{\mathbb{I}} \mathbb{D}^\dagger$$

is well-defined. Thus defining $\operatorname{Log}_{\mathbb{T}^\dagger}^{(\eta)}(\mathfrak{Y}) \in \mathbb{I}[\lambda^{-1}]$ by the relation

$$\operatorname{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y}) = \operatorname{Log}_{\mathbb{T}^\dagger}^{(\eta)}(\mathfrak{Y}) \cdot \eta \otimes 1$$

the result follows. □

5 The big Heegner point

In this chapter we prove the main results of this paper, relating the étale Abel-Jacobi images of Heegner cycles to the specializations at higher even weights of the big Heegner point $\mathfrak{Z}$ (whose definition is recalled below), from where a deformation of the p-adic Gross-Zagier formula of Nekovář over a Hida family follows at once. There are two key points to the proof: the properties of the big logarithm map deduced from the work of Ochiai as explained in the preceding section, and the local study of (almost all) the weight 2 specializations of $\mathfrak{Z}$ taken up in the following.

5.1 Weight two specializations

Recall form Sect. 3 that K is a fixed imaginary quadratic field in which all prime factors of N split, and that $\mathfrak{N} \subset \mathcal{O}_K$ is a fixed cyclic N-ideal, i.e. such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$. We also assume that p splits in K, and let $\mathfrak{p}$ be the prime of K above p induced by our fixed embedding $\imath_p$, and by $\bar{\mathfrak{p}}$ the other. Finally, A is a fixed elliptic curve with CM by $\mathcal{O}_K$ defined over the Hilbert class field H of K.

Let $R_0 = \widehat{\mathbf{Z}}_p^{\rm nr}$ be the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, which we view as an overfield of H via $\imath_p$. Since p splits in K, A admits a trivialization

$$\imath_A : \hat{A} \to \hat{\mathbf{G}}_m$$

over R_0 with $\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A[\mathfrak{p}^s]$ for every $s > 0$. Letting α_A be the cyclic N-isogeny on A with kernel $A[\mathfrak{N}]$, the triple $(A, \alpha_A, \imath_A)$ thus defines a trivialized elliptic curve with $\Gamma_0(N)$-level structure defined over R_0.

Set $A_0 := A/A[\mathfrak{p}^s]$ and let $(A_0, \alpha_{A_0}, \imath_{A_0})$ be the trivialized elliptic curve deduced from $(A, \alpha_A, \imath_A)$ via the projection $A \to A_0$. Let $C \subset A_0[p^s]$ be any étale subgroup of order p^s, and set $A_s := A_0/C$. Finally, let $(A_s, \alpha_{A_s}, \imath_{A_s})$ be the trivialized elliptic curve with $\Gamma_0(N)$-level structure deduced from $(A_0, \alpha_{A_0}, \imath_{A_0})$ via the projection $A_0 \to A_s$, and consider the triple

$$h_s = (A_s, \alpha_{A_s}, \imath_{A_s}(\zeta_s)), \tag{5.1}$$

which defines an algebraic point on the modular curve X_s.

Write $p^* = (-1)^{\frac{p-1}{2}}p$, and let ϑ be the unique continuous character

$$\vartheta : G_{\mathbf{Q}(\sqrt{p^*})} \to \mathbf{Z}_p^\times/\{\pm 1\} \tag{5.2}$$

such that $\vartheta^2 = \varepsilon_{\rm cyc}$. Notice the inclusion $G_{H_{p^s}} \subset G_{\mathbf{Q}(\sqrt{p^*})}$ for any $s > 0$, where H_{p^s} denotes the ring class field of K of conductor p^s.

Lemma 5.1 *The curve A_s has CM by the order $\mathcal{O}_{p^s}$ of K of conductor p^s, and the point h_s is rational over $L_{p^s} := H_{p^s}(\boldsymbol{\mu}_{p^s})$. In fact we have*

$$h_s^\sigma = \langle \vartheta(\sigma) \rangle \cdot h_s \tag{5.3}$$

for all $\sigma \in \mathrm{Gal}(L_{p^s}/H_{p^s})$.

Proof The first assertion is clear, and immediately from the construction we also see that α_{A_s} is the cyclic N-isogeny on A_s with kernel $A_s[\mathfrak{N} \cap \mathcal{O}_{p^s}]$. It follows that the point (5.1) gives rise to precisely the point $h_s \in X_s(\mathbf{C})$ in [19, Eq. (4)]. The result thus follows from [*loc.cit.*, Cor. 2.2.2]. □

If ν is an arithmetic prime of $\mathbb{I}$, we let ψ_ν denote its *wild character*, defined as the composition of $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ with the structure map $\Gamma = 1 + p\mathbf{Z}_p \to \mathbb{I}^\times$. The nebentypus of $\mathbf{f}_\nu$ is then given by $\varepsilon_{\mathbf{f}_\nu} = \psi_\nu \omega^{k-k_\nu}$, where $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \boldsymbol{\mu}_{p-1} \subset \mathbf{Z}_p^\times$ is the Teichmüller character.

Recall the critical characters Θ and θ from Sect. 4, and for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, consider the $F_\nu^\times$-valued Hecke character of K given by

$$\chi_\nu(x) = \Theta_\nu(\mathrm{art}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))) \tag{5.4}$$

for all $x \in \mathbb{A}_K^\times$. Notice that since χ_ν has finite order, it may alternately be seen as character on G_K via the Artin reciprocity map $\mathrm{art}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$.

Let $\mathcal{O}_{\mathbf{C}_p}$ be the ring of integers of the completion of $\overline{\mathbf{Q}}_p$. For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, after fixing an embedding $F_\nu \to \overline{\mathbf{Q}}_p$, the form $\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$ defines a p-adic modular form $\mathbf{f}_\nu \in \mathbf{M}(N)$. Finally, recall the dual form $\mathbf{f}_\nu^*$ defined as in the paragraph before (2.20).

Lemma 5.2 *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})} \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}), \tag{5.5}$$

where $u = |\mathcal{O}_K^\times|/2$, $G(\theta_\nu^{-1})$ is the Gauss sum $\sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$, and for every $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(L_{p^s}/H)$.

Proof Notice that the expression in the right hand side of (5.5) does not depend on the choice of lifts $\tilde{\sigma}$. Indeed, as explained in [18, p. 808] the character $\chi_{0,\nu} := \chi_\nu|_{\mathbb{A}_{\mathbf{Q}}^\times}$, seen as a Dirichlet character in the usual manner, is such that $\chi_{0,\nu}^{-1} = \theta_\nu^2$. But since the weight of ν is 2, we have $\theta_\nu^2 = \varepsilon_{\mathbf{f}_\nu} = \varepsilon_{\mathbf{f}_\nu^*}^{-1}$ (see [18, p. 806]), and our claim thus follows immediately from (5.3).

To compute the above value of the twist $d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu$ we follow Definition 2.2. The integer $s > 1$ in the statement is such that θ_ν factors through $(\mathbf{Z}/p^s\mathbf{Z})^\times$, therefore

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) &= \sum_{a \bmod p^s} \theta_\nu(a) \left(\int_{a+p^s\mathbf{Z}_p} d\mu_{\mathrm{Gou}}(x) \right) (d^{-1}\mathbf{f}_\nu^{*[p]})(A, \alpha_A, \imath_A) \\ &= \frac{1}{p^s} \sum_{a \bmod p^s} \theta_\nu(a) \sum_C \zeta_C^{-a} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_0/C, \alpha_C, \imath_C), \end{aligned} \tag{5.6}$$

where as before $A_0 := A/\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A/A[\mathfrak{p}^s]$ and the sum is over the étale subgroups $C \subset A_0[p^s]$ of order p^s. Letting γ_s be a generator of $\mathbf{Z}/p^s\mathbf{Z}$, these subgroups correspond bijectively with the cyclic subgroups $C_u = \langle \zeta_s^u . \gamma_s \rangle \subset \boldsymbol{\mu}_{p^s} \times \mathbf{Z}/p^s\mathbf{Z}$, with u running over the integers modulo p^s, and we set $\zeta_{C_u} = \zeta_s^u$.

Since θ_ν does not factor through $(\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$, we have $\sum_{a \bmod p^s} \theta_\nu(a)\zeta_s^{-ua} = 0$ whenever $u \notin (\mathbf{Z}/p^s\mathbf{Z})^\times$. Continuing from (5.6), we thus obtain

$$d^{-1}\mathbf{f}_\nu^{*[p]}\otimes\theta_\nu(A,\alpha_A,\iota_A)=\frac{1}{p^s}\sum_{a \bmod p^s}\theta_\nu(a)\sum_{u \bmod p^s}\zeta_s^{-ua}\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u},\alpha_{C_u},\iota_{C_u})$$
$$=\frac{1}{p^s}\sum_{u\in(\mathbf{Z}/p^s\mathbf{Z})^\times} d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u},\alpha_{C_u},\iota_{C_u})\sum_{a \bmod p^s}\theta_\nu(a)\zeta_s^{-ua}$$
$$=\frac{1}{G(\theta_\nu^{-1})}\sum_{u\in(\mathbf{Z}/p^s\mathbf{Z})^\times}\theta_\nu^{-1}(u)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u},\alpha_{C_u},\iota_{C_u}),$$

with the last equality obtained by a change of variables. The result thus follows from the relation

$$\sum_{u\in(\mathbf{Z}/p^s\mathbf{Z})^\times}\theta_\nu^{-1}(u)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u},\alpha_{C_u},\iota_{C_u})=u\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)}\chi_\nu^{-1}(\tilde\sigma)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde\sigma}),$$

where $u=|\mathcal{O}_K^\times|/2$, and for each $\sigma\in\mathrm{Gal}(H_{p^s}/H)$, $\tilde\sigma\in\mathrm{Gal}(L_{p^s}/H)$ lifts σ. □

Keeping the above notations, let $\Delta_s\in J_s(L_{p^s})$ be the divisor class of $(h_s)-(\infty)$, and consider the element in $J_s(L_{p^s})\otimes_{\mathbf{Z}}F_\nu$ given by

$$\widetilde{Q}_{\chi_\nu}:=\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)}\Delta_s^{\tilde\sigma}\otimes\chi_\nu^{-1}(\tilde\sigma), \tag{5.7}$$

where for every $\sigma\in\mathrm{Gal}(H_{p^s}/H)$, $\tilde\sigma$ is any lift to $\mathrm{Gal}(L_{p^s}/H)$.

Let F_s be the completion of $\iota_p(L_{p^s})$, and consider the p-adic Abel-Jacobi map $\delta^{(p)}_{\mathbf{f}_\nu,F_s}$ defined in (2.9) which we extend by F_ν-linearity to a map

$$\delta^{(p)}_{\mathbf{f}_\nu,F_s}: J_s(L_{p^s})\otimes_{\mathbf{Z}}F_\nu\longrightarrow(\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}))^\vee.$$

Proposition 5.3 *Let $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $s>1$ be as in Lemma 5.2. Then*

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)}\chi_\nu^{-1}(\tilde\sigma)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde\sigma})=\delta^{(p)}_{\mathbf{f}_\nu,F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{5.8}$$

Proof The integer $s>1$ in the statement is so that the nebentypus $\varepsilon_{\mathbf{f}_\nu}$ of $\mathbf{f}_\nu$ is primitive modulo p^s. Moreover, since p splits in K, we see from the construction that the point h_s lies in the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞. Thus Proposition 2.9 applies, giving

$$\delta^{(p)}_{\mathbf{f}_\nu,F_s}(\Delta_s)(\omega_{\mathbf{f}_\nu^*})=F_{\omega_{\mathbf{f}_\nu^*}}(h_s),$$

where $F_{\omega_{\mathbf{f}_\nu^*}}$ is the Coleman primitive of $\omega_{\mathbf{f}_\nu^*}$ from Proposition 2.6 vanishing at ∞, and by linearity

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)}\chi_\nu^{-1}(\tilde\sigma)\cdot F_{\omega_{\mathbf{f}_\nu}}(h_s^{\tilde\sigma})=\delta^{(p)}_{\mathbf{f}_\nu,F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{5.9}$$

Since ϕ lifts the Deligne–Tate map to X_s, we see that ϕh_s is defined over the subfield $H_{p^{s-1}}(\zeta_s) \subset L_{p^s}$. If b_p denotes the U_p-eigenvalue of $\mathbf{f}_\nu^*$, by Corollary 2.8 we obtain

$$\sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}) = \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde{\sigma}}) - \frac{b_p}{p}\sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \cdot F_{\omega_{\mathbf{f}_\nu^*}}(\phi h_s^{\tilde{\sigma}})$$
$$= \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde{\sigma}}),$$

where all the sums are over $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, and the second equality follows immediately from the fact θ_ν is primitive modulo p^s. The result thus follows from (5.9). □

Still with the same notations, recall Hida's ordinary projector (1.2) and set $y_s := e^{\mathrm{ord}}h_s$, which naturally lies in $e^{\mathrm{ord}}J_s(L_{p^s})$ (see [19, p.100]). Equation (5.3) then amounts to the fact that

$$y_s^\sigma = \Theta(\sigma) \cdot y_s \tag{5.10}$$

for all $\sigma \in \mathrm{Gal}(L_{p^s}/H_{p^s})$, where Θ is the critical character (4.1). Denoting by $J_s^{\mathrm{ord}}(L_{p^s})^\dagger$ the module $e^{\mathrm{ord}}J_s(L_{p^s})$ with the Galois action twisted by Θ^{-1}, and by $y_s^\dagger$ the point y_s seen in this new module, (5.10) translates into the statement that

$$y_s^\dagger \in H^0(H_{p^s}, J_s^{\mathrm{ord}}(L_{p^s})^\dagger).$$

Lemma 5.4 (Howard) *The classes*

$$x_s := \mathrm{Cor}_{H_{p^s}/H}(y_s^\dagger) \in H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger) \tag{5.11}$$

are such that

$$\alpha_* x_{s+1} = U_p \cdot x_s, \qquad \text{for all } s > 0$$

under the Albanese maps induced from the degeneracy maps $\alpha : X_{s+1} \to X_s$.

Proof This is shown in the course of the proof of [19, Lemma 2.2.4]. □

Abbreviate by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)$ the module $e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O})$ from the Introduction, and denote by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger$ this same module with the Galois action twisted by Θ^{-1}. By the Galois and Hecke-equivariance of the twisted Kummer map

$$\mathrm{Kum}_s : H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger) \to H^1(H, \mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger)$$

constructed in [19, p. 101], Lemma 5.4 implies that the cohomology classes $\mathfrak{X}_s := \mathrm{Kum}_s(x_s)$ are such that $\alpha_*\mathfrak{X}_{s+1} = U_p \cdot \mathfrak{X}_s$, for all $s > 0$.

Definition 5.5 (Howard) The *big Heegner point* of conductor one is the cohomology class $\mathfrak{X}$ given by the image of $\varprojlim_s U_p^{-s} \cdot \mathfrak{X}_s$ under the natural map induced by the $\mathfrak{h}^{\mathrm{ord}}[G_{\mathbf{Q}}]$-linear projection $\varprojlim_s (\mathrm{Ta}_p^{\mathrm{ord}}(J_s)^{\dagger}) \to \mathbb{T}^{\dagger}$.

Our object of study is in fact

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X}), \tag{5.12}$$

which [19, Conj. 3.4.1] predicts to be not $\mathbb{I}$-torsion. For $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, let $L(s, \mathbf{f}_\nu, \chi_\nu)$ be the Rankin-Selberg convolution L-function of [20, §1]. In the spirit of the classical Gross-Zagier theorem, one has the following criterion.

Theorem 5.6 (Howard) *If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has weight 2 and non-trivial nebentypus, then*

$$\mathfrak{Z}_\nu \neq 0 \iff L'(1, \mathbf{f}_\nu, \chi_\nu) \neq 0, \tag{5.13}$$

and if the non-vanishing holds for at least one such ν, then $\mathfrak{Z}$ is not $\mathbb{I}$-torsion.

Proof See [18, Prop. 3] for the equivalence (5.13), and [*loc.cit*, Cor. 5] for the last implication. We outline the proof for future reference. For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus, consider (with the same notations as above)

$$Q_{\chi_\nu} := \sum_{\tau \in \mathrm{Gal}(L_{p^s}/K)} \Delta_s^{\tau} \otimes \chi_\nu^{-1}(\tau) \in J_s(L_{p^s}) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}. \tag{5.14}$$

If $e_{\mathbf{f}_\nu}$ denotes the idempotent of the Hecke algebra (tensored with $\overline{\mathbf{Q}}$) defined by the eigenform $\mathbf{f}_\nu$, the arguments in [18, pp. 809–810] show that

$$\mathfrak{Z}_\nu \neq 0 \iff e_{\mathbf{f}_\nu} Q_{\chi_\nu} \neq 0, \tag{5.15}$$

and by the “twisted Gross-Zagier theorem” [20, Thm. 4.6.2], one has

$$e_{\mathbf{f}_\nu} Q_{\chi_\nu} \neq 0 \iff L'(1, \mathbf{f}_\nu, \chi_\nu) \neq 0.$$

□

Corollary 5.7 *Assume that there is a $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that $L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$. Then the localization map*

$$\mathrm{loc}_{\mathfrak{p}} : H^1_f(K, \mathbb{V}^{\dagger}_\nu) \to H^1_f(\mathbf{Q}_p, \mathbb{V}^{\dagger}_\nu)$$

is injective at all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Proof By [18, Cor. 5], the assumption implies that $\mathfrak{Z}$ is nontorsion, and by [19, Cor. 3.4.3] that $\widetilde{H}^1_f(K, \mathbb{T}^{\dagger})$ has rank 1 over $\mathbb{I}$. By [19, Lemma 2.1.7], it follows that

$$H^1_f(K, \mathbb{V}^{\dagger}_\nu) = \mathfrak{Z}_\nu . F_\nu,$$

for all but finitely many ν of weight 2 and non-trival nebentypus. On the other hand, since $\dim_{F_\nu} H^1_f(\mathbf{Q}_p, \mathbb{V}^\dagger_\nu)$ for every ν of weight 2 with non-trivial nebentypus, we see that it suffices to show that one has the implication

$$\mathfrak{Z}_\nu \neq 0 \quad \Longrightarrow \quad \mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu) \neq 0 \tag{5.16}$$

for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus. (Indeed, (5.16) will show that $\mathrm{loc}_{\mathfrak{p}}$ is injective at infinitely many ν, and by [19, Lemma 2.1.7] it will follow that the kernel of the localization map

$$\mathrm{loc}_{\mathfrak{p}} : \widetilde{H}^1_f(K, \mathbb{T}^\dagger) \to H^1(\mathbf{Q}_p, \mathbb{T}^\dagger)$$

must be $\mathbb{I}$-torsion, hence contained in only finitely arithmetic primes).

The point Q_{χ_ν} (5.14) defines a K-rational point on a twist J_{χ_ν} of J_s by the character χ_ν^{-1}. Since the localization map

$$J_{\chi_\nu}(K) \otimes_{\mathbf{Z}} \mathbf{Q} \to J_{\chi_\nu}(K_{\mathfrak{p}})$$

is injective, we thus see that (5.16) follows from (5.15), hence the result. □

For any class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, taking a representative $\mathfrak{a} \subset \mathcal{O}_K$ prime to Np, define

$$\mathfrak{a} * (A, \alpha_A, \imath_A) := (A_{\mathfrak{a}}, \alpha_{A_{\mathfrak{a}}}, \imath_{A_{\mathfrak{a}}}),$$

where $A_{\mathfrak{a}} = A/A[\mathfrak{N}]$, $\alpha_{A_{\mathfrak{a}}} = A_{\mathfrak{a}}[\mathfrak{N}]$, and $\imath_{A_{\mathfrak{a}}}$ is the trivialization $\hat{A}_{\mathfrak{a}} \xrightarrow{\hat{\varphi}_{\mathfrak{a}}^{-1}} \hat{A} \xrightarrow{\imath_A} \hat{\mathbf{G}}_m$ induced by the projection $\varphi_{\mathfrak{a}} : A \to A_{\mathfrak{a}}$.

Theorem 5.8 *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character ψ_ν, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)) = u \frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})} \log_{s, V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\omega_{\mathbf{f}_\nu^*}), \tag{5.17}$$

where $u = |\mathcal{O}_K^\times|/2$, and $G(\theta_\nu^{-1})$ is the Gauss sum $\sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$.

Proof Since clearly $d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1} = d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu$, letting F_s be the completion of $\imath_p(L_{p^s})$ it suffices to establish the equality

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = u \frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})} \log_{F_s, V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{X}_\nu))(\omega_{\mathbf{f}_\nu^*}). \tag{5.18}$$

Combining the formulas from Lemma 5.2 and Proposition 5.3, we have

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})} \delta^{(p)}_{\mathbf{f}_\nu, F_s}(\widetilde{Q}_{\chi_\nu}). \tag{5.19}$$

Now the integer $s > 1$ is such that the natural map $\mathbb{T} \to \mathbb{V}_\nu$ can be factored as

$$\mathbb{T} \to \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \to \mathbb{V}_\nu, \tag{5.20}$$

and we have $\mathbb{V}_\nu^\dagger \cong \mathbb{V}_\nu$ as $G_{L_{p^s}}$-modules. Tracing through the construction of $\mathfrak{X}$, we see that the image of $U_p^s \cdot \mathfrak{X}_\nu$ in $H^1(L_{p^s}, \mathbb{V}_\nu^\dagger)$ agrees with the image of $\widetilde{Q}_{\chi_\nu}$ under the composite map (where the unlabelled arrow is induced by (5.20))

$$\begin{aligned} J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu &\xrightarrow{\mathrm{Kum}_s} H^1(L_{p^s}, \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}} F_\nu) \xrightarrow{e^{\mathrm{ord}}} H^1(L_{p^s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_{\mathbf{Z}} F_\nu) \\ &\longrightarrow H^1(L_{p^s}, \mathbb{V}_\nu) \cong H^1(L_{p^s}, \mathbb{V}_\nu^\dagger). \end{aligned} \tag{5.21}$$

Since U_p acts on $\mathbb{V}_\nu^\dagger$ as multiplication by $\nu(\mathbf{a}_p)$, we thus arrive at the equality

$$\mathrm{Kum}_s(e^{\mathrm{ord}}\widetilde{Q}_{\chi_\nu}) = \nu(\mathbf{a}_p)^s \cdot \mathrm{res}_{L_{p^s}/H}(\mathfrak{X}_\nu) \in H^1(L_{p^s}, \mathbb{V}_\nu). \tag{5.22}$$

By [32, Prop. 1.6.8], this shows that the restriction to $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{X}_\nu)$ to G_{F_s} is contained in the Bloch-Kato finite subspace $H^1_f(F_s, \mathbb{V}_\nu) \cong H^1_f(F_s, \mathbb{V}_\nu^\dagger)$. Since the map $\delta^{(p)}_{\mathbf{f}_\nu, F_s}$ is defined by the commutativity of the diagram

$$\begin{array}{ccccc} J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu & \xrightarrow{(4.21)} & H^1(L_{p^s}, \mathbb{V}_\nu) & \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} & H^1(F_s, \mathbb{V}_\nu) \\ & \searrow^{\cdots} & & & \cup \\ & & & & H^1_f(F_s, \mathbb{V}_\nu) \\ & \searrow_{\delta^{(p)}_{\mathbf{f}_\nu, F_s}} & & & \downarrow \log_{F_s, \mathbb{V}_\nu} \\ & & & & (\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathbb{V}_\nu))^\vee, \end{array}$$

we thus see that (5.18) follows from (5.19) and (5.22). □

Remark 5.9 The expression in the left hand side of (5.17) can be interpreted as the value of a certain p-adic Rankin L-series at a point outside the range of classical interpolation, and hence Theorem 5.8 may be seen as a p-adic analogue of the Gross-Zagier formula for the classes $\mathfrak{Z}_\nu$, in the same spirit as the main result of [1]. This interpretation, which does not play a direct role in this paper, is studied further in the companion paper [5].

5.2 Higher weight specializations

Now we can prove our main result. Recall from the Introduction that f_o is a p-ordinary newform of level N prime to p, even weight $k \geq 2$ and trivial nebentypus, that

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

is the Hida family passing through the ordinary p-stabilization of f_o, and that K is an imaginary quadratic field such that every prime factor of pN is split in K.

If $\mathbf{f}_\nu$ is the ordinary p-stabilization of a p-ordinary newform $\mathbf{f}_\nu^\sharp$ of even weight $2r_\nu > 2$ and trivial nebentypus, the Heegner cycle $\Delta_{A,r_\nu}^{\mathrm{heeg}}$ has been defined in Sect. 3, and by [28, Thm. (3.1)(i)] the class

$$\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{heeg}}) := \mathrm{Cor}_{H/K}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,H}(\Delta_{A,r_\nu}^{\mathrm{heeg}})) \tag{5.23}$$

lies in the Bloch-Kato Selmer group $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$.

On the other hand, by [19, Prop. 2.4.5], the big Heegner point $\mathfrak{X}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(H, \mathbb{T}^\dagger)$ (defined in [*loc.cit.*, Def. 2.4.2]), and since $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger) \cong H^1_f(K, \mathbb{V}_\nu^\dagger)$ as explained in [19, p. 114]) and $\mathbb{V}_\nu^\dagger \cong V_{\mathbf{f}_\nu^\sharp}(r_\nu)$ by Lemma 4.2, the class

$$\mathfrak{Z}_\nu = \mathrm{Cor}_{H/K}(\mathfrak{X}_\nu)$$

naturally lies in $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$ as well. Our main result relates these two classes.

Recall that the following hypotheses are being assumed throughout this paper.

Assumption 5.10 The residual representation $\bar{\rho}_{f_o}$ is irreducible, and the semi-simplification of $\overline{\rho}_{f_o}|_{G_{\mathbf{Q}_p}}$ is non-scalar.

Theorem 5.11 *Let ν_o be the arithmetic prime of $\mathbb{I}$ such that $\mathbf{f}_{\nu_o}$ is the ordinary p-stabilization of f_o, and let $\mathbb{T}^\dagger = \mathbb{T} \otimes \Theta^{-1}$ be the critical twist of $\mathbb{T}$ such that ϑ_{ν_o} is the trivial character*[6]*. Assume that there is a $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that*

$$L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0. \tag{5.24}$$

Then for all but finitely many arithmetic primes $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $2r_\nu > 2$ with $2r_\nu \equiv k \pmod{2(p-1)}$, we have

$$\langle \mathfrak{Z}_\nu, \mathfrak{Z}_\nu \rangle_K = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{heeg}}), \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{heeg}})\rangle_K}{u^2(4D)^{r_\nu - 1}}, \tag{5.25}$$

where $\langle , \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$, $u = |\mathcal{O}_K^\times|/2$, and $-D < 0$ is the discriminant of K.

Proof Since $\mathfrak{Z} \in \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$, the localization $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ lies in the kernel of the natural map

$$H^1(\mathbf{Q}_p, \mathbb{T}^\dagger) \to H^1(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger),$$

6 As opposed to $\omega^{\frac{p-1}{2}}$.

and since $H^0(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger) = 0$ by [19, Lemma 2.4.4], the class $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ can be seen as sitting inside $H^1(\mathbf{Q}_p, \mathscr{F}_w^+ \mathbb{T}^\dagger)$. Thus upon taking an $\mathbb{I}$-basis η of $\mathbb{D}$, we can form

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger) := u \cdot \mathrm{Log}_{\mathbb{T}^\dagger}^{(\eta)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})) \in \mathbb{I}[\lambda^{-1}] \qquad (\lambda := \mathbf{a}_p - 1).$$

On the other hand, consider the continuous function on $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$ given by

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger) : \nu \mapsto \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1} \mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \iota_A)).$$

(Its continuity can be checked by staring at the q-expansion of $d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}$ and appealing to the results in [14, § I.3.5], for example.)

By the specialization property (4.8) of the map $\mathrm{Log}_{\mathbb{T}^\dagger}^{(\eta)}$, we see that Theorem 5.8 can be reformulated as follows: For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial wild character, there exists a unit $\Omega_\nu^{(\eta)} \in \mathcal{O}_\nu^\times$ such that

$$\nu\left(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\right) = \Omega_\nu^{(\eta)} \cdot \nu\left(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)\right). \tag{5.26}$$

In fact,

$$\Omega_\nu^{(\eta)} = \langle \eta_\nu \otimes e_\zeta^{\otimes r_\nu}, \omega_{\mathbf{f}_\nu^*} \rangle_{\mathrm{dR}} \tag{5.27}$$

under the pairing (4.7), so that $\omega_{\mathbf{f}_\nu^*} = \Omega_\nu^{(\eta)} \cdot \eta_\nu'$ with η_ν' as defined in Theorem 4.4.[7] Since both $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)$ and $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)$ are continuous functions of ν, (5.26) shows that the map $\nu \mapsto \Omega_\nu^{(\eta)}$ is continuous, and hence (5.27) is valid for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Now let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be as in the statement. Then $\theta_\nu(z) = z^{r_\nu - 1}\vartheta_\nu(z) = z^{r_\nu - 1}$ as characters on $\mathbf{Z}_p^\times$, from where if follows that

$$\begin{aligned} \nu\left(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\right) &= \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \iota_A)) \\ &= \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-r_\nu}\mathbf{f}_\nu^{[p]}(\mathfrak{a} * (A, A[\mathfrak{N}])). \end{aligned}$$

By Theorem 3.3, setting

$$\Delta_{r_\nu}^{\mathrm{bdp}} := \mathrm{Norm}_{H/K}(\Delta_{\varphi_{(1)}, r_\nu}^{\mathrm{bdp}}) = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{1-r} \cdot \Delta_{\varphi_{\mathfrak{a}}, r_\nu}^{\mathrm{bdp}} \in \mathrm{CH}^{2r_\nu - 1}(X_{r_\nu})_0(K), \tag{5.28}$$

[7] That $\Omega_\nu^{(\eta)}$, which a priori just lies in F_ν, is indeed a unit is shown in [31, Prop. 6.4].

this shows that

$$\nu\left(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^{\dagger})\right) = \mathcal{E}_{\nu}(r_{\nu})\mathcal{E}_{\nu}^{*}(r_{\nu})\frac{(-1)^{r_{\nu}-1}}{(r_{\nu}-1)!}\mathrm{AJ}_{\mathbf{Q}_p}(\Delta_{r_{\nu}}^{\mathrm{bdp}})(\omega_{\mathbf{f}_{\nu}^{\sharp}}\otimes e_{\zeta}^{\otimes r_{\nu}-1})$$
$$= \mathcal{E}_{\nu}(r_{\nu})\mathcal{E}_{\nu}^{*}(r_{\nu})\frac{(-1)^{r_{\nu}-1}}{(r_{\nu}-1)!}\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}})))(\omega_{\mathbf{f}_{\nu}^{\sharp}}\otimes e_{\zeta}^{\otimes r_{\nu}-1}), \tag{5.29}$$

where

$$\mathcal{E}_{\nu}(r_{\nu}) := \left(1-\frac{p^{r_{\nu}-1}}{\nu(\mathbf{a}_p)}\right), \qquad \mathcal{E}_{\nu}^{*}(r_{\nu}) := \left(1-\frac{\nu(\mathbf{a}_p)}{p^{r_{\nu}}}\right),$$

and $\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}} := \pi_{\mathbf{f}_{\nu}^{\sharp},\mathrm{N}^{r_{\nu}-1}}\circ\Phi_{K}^{\text{ét}}$ with notations as in the diagram (3.3) defining $\mathrm{AJ}_{\mathbf{Q}_p}$.

On the other hand, by the specialization property of the map $\mathrm{Log}_{\mathbb{T}^{\dagger}}^{(\eta)}$ we have

$$\nu\left(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^{\dagger})\right) = u\frac{(-1)^{r_{\nu}-1}}{(r_{\nu}-1)!}\mathcal{E}_{\nu}(r_{\nu})^{-1}\mathcal{E}_{\nu}^{*}(r_{\nu})\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_{\nu}))(\eta_{\nu}'). \tag{5.30}$$

Comparing (5.30) and (5.29), we thus conclude form (5.26) that

$$\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_{\nu}))(\omega_{\mathbf{f}_{\nu}^{\sharp}}\otimes e_{\zeta}^{\otimes r_{\nu}-1}) = \frac{1}{u}\mathcal{E}_{\nu}(r_{\nu})^{2}\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}})))(\omega_{\mathbf{f}_{\nu}^{\sharp}}\otimes e_{\zeta}^{\otimes r_{\nu}-1}).$$

Since $\mathrm{Fil}^{1}D_{\mathrm{dR}}(V_{\mathbf{f}_{\nu}^{\sharp}}(r_{\nu}-1))$ is spanned by $\omega_{\mathbf{f}_{\nu}^{\sharp}}\otimes e_{\zeta}^{\otimes r_{\nu}-1}$, it follows that

$$\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_{\nu})) = \frac{1}{u}\mathcal{E}_{\nu}(r_{\nu})^{2}\log_{\mathbb{V}_{\nu}^{\dagger}}(\mathrm{loc}_{\mathfrak{p}}(\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}}))),$$

and since $\log_{\mathbb{V}_{\nu}^{\dagger}}$ is an isomorphism, that

$$\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_{\nu}) = \frac{1}{u}\mathcal{E}_{\nu}(r_{\nu})^{2}\mathrm{loc}_{\mathfrak{p}}(\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}})). \tag{5.31}$$

Our nonvanishing assumption (5.24) implies on the one hand, by Theorem 5.6, that $\mathfrak{Z}_{\nu}$ is non-zero for all but finitely many $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, and on the other hand, by Corollary 5.7, that the localization map $\mathrm{loc}_{\mathfrak{p}}$ is injective for all but finitely many $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. In particular, we thus see from (5.31) that we have

$$\langle\mathfrak{Z}_{\nu},\mathfrak{Z}_{\nu}\rangle_{K} = \frac{1}{u^{2}}\mathcal{E}_{r_{\nu}}(r_{\nu})^{4}\langle\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}}),\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{bdp}})\rangle_{K}$$
$$= \frac{1}{u^{2}}\mathcal{E}_{r_{\nu}}(r_{\nu})^{4}\frac{\langle\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{heeg}}),\Phi_{\mathbf{f}_{\nu}^{\sharp},K}^{\text{ét}}(\Delta_{r_{\nu}}^{\mathrm{heeg}})\rangle_{K}}{(4D)^{r_{\nu}-1}}$$

for all but finitely many ν as in the statement, where the last equality follows from Lemma 3.4 in light of the definitions (5.23) and (5.28). The result follows. □

Acknowledgments It is a pleasure to thank my advisor, Prof. Henri Darmon, for suggesting that I work on this problem, and for sharing with me some of his wonderful mathematical insights. I thank both him and Adrian Iovita for critically listening to me while the results in this paper were being developed, and also Jan Nekovář and Victor Rotger for encouragement and helpful correspondence. It is a pleasure to acknowledge the debt that this work owes to Ben Howard, especially for pointing out an error in an early version of this paper, and for providing several helpful comments and corrections. Finally, I am very thankful to an anonymous referee whose valuable comments and suggestions had a considerable impact on the final form of this paper.

References

1. Bertolini, Massimo, Darmon, Henri, Prasanna, Kartik: Generalised Heegner cycles and p-adic Rankin L-series. to appear in Duke Math. Journal.
2. Breuil, Christophe, Emerton, Matthew: Représentations p-adiques ordinaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et compatibilité local-global. Astérisque **331**, 255–315 (2010)
3. Bloch, Spencer, Kato, Kazuya: L-functions and Tamagawa numbers of motives. In: The Grothendieck Festschrift, Vol. I, volume 86 of Progr. Math., pages 333–400. Birkhäuser Boston, Boston, MA, (1990)
4. Buzzard, Kevin: Analytic continuation of overconvergent eigenforms. J. Amer. Math. Soc. 16(1): 29–55 (electronic) (2003)
5. Castella, Francesc: p-adic L-functions and the p-adic variation of Heegner points. preprint, (2012)
6. Coleman, Robert F.: Reciprocity laws on curves. Compositio Math. **72**(2), 205–235 (1989)
7. Coleman, Robert F.: A p-adic inner product on elliptic modular forms. In: Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), volume 15 of Perspect. Math., pages 125–151. Academic Press, San Diego, CA, (1994)
8. Coleman, Robert F.: A p-adic Shimura isomorphism and p-adic periods of modular forms. In: p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), volume 165 of Contemp. Math., pages 21–51. Amer. Math. Soc., Providence, RI, (1994)
9. Coleman, Robert F.: Classical and overconvergent modular forms. Invent. Math. **124**(1–3), 215–241 (1996)
10. Coleman, Robert F.: Classical and overconvergent modular forms of higher level. J. Théor. Nombres Bordeaux **9**(2), 395–403 (1997)
11. Coleman, Robert F.: p-adic Banach spaces and families of modular forms. Invent. Math. **127**(3), 417–479 (1997)
12. Faltings, Gerd: Crystalline cohomology and p-adic Galois-representations. In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, (1989)
13. Faltings, Gerd: Almost étale extensions. Astérisque, (279):185–270, 2002. Cohomologies p-adiques et applications arithmétiques, II
14. Gouvêa, Fernando Q.: Arithmetic of p -adic modular forms, volume 1304 of Lecture Notes in Mathematics. Springer-Verlag, Berlin (1988)
15. Gross, Benedict H.: Heegner points on $X_0(N)$. In: Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 87–105. Horwood, Chichester, (1984)
16. Hida, Haruzo: Iwasawa modules attached to congruences of cusp forms. Ann. Sci. École Norm. Sup. (4) **19**(2), 231–273 (1986)
17. Hyodo, Osamu., Kato, Kazuya.: Semi-stable reduction and crystalline cohomology with logarithmic poles. Astérisque, (223):221–268, 1994. Périodes p-adiques (Bures-sur-Yvette, 1988)
18. Howard, Benjamin: Central derivatives of L-functions in Hida families. Math. Ann. **339**(4), 803–818 (2007)
19. Howard, Benjamin: Variation of Heegner points in Hida families. Invent. Math. **167**(1), 91–128 (2007)
20. Howard, Benjamin: Twisted Gross-Zagier theorems. Canad. J. Math. **61**(4), 828–887 (2009)
21. Katz, Nicholas M.: p-adic properties of modular schemes and modular forms. In: Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, (1973)

22. Katz, Nicholas M., Mazur, Barry: Arithmetic moduli of elliptic curves, volume 108 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ (1985)
23. Mazur, B., Tilouine, J.: Représentations galoisiennes, différentelles de Käller at “conjectures principales”. Publ. Math. Inst. Hautes Études Sci. **71**, 65–103 (1990)
24. Mazur, B., Wiles, A.: On p-adic analytic families of Galois representations. Compositio Math. **59**(2), 231–264 (1986)
25. Nekovář, Jan: Kolyvagin’s method for Chow groups of Kuga-Sato varieties. Invent. Math. **107**(1), 99–125 (1992)
26. Nekovář, Jan.: On p-adic height pairings. In: Séminaire de Théorie des Nombres, Paris, 1990–91, volume 108 of Progr. Math., pages 127–202. Birkhäuser Boston, Boston, MA, (1993)
27. Nekovář, Jan: On the p-adic height of Heegner cycles. Math. Ann. **302**(4), 609–686 (1995)
28. Nekovář, Jan.: p-adic Abel-Jacobi maps and p-adic heights. In The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), volume 24 of CRM Proc. Lecture Notes, pages 367–379. Amer. Math. Soc., Providence, RI, (2000)
29. Nekovář, Jan, Plater, Andrew: On the parity of ranks of Selmer groups. Asian J. Math. **4**(2), 437–497 (2000)
30. Ochiai, Tadashi: A generalization of the Coleman map for Hida deformations. Amer. J. Math. **125**(4), 849–892 (2003)
31. Ochiai, Tadashi: On the two-variable Iwasawa main conjecture. Compos. Math. **142**(5), 1157–1200 (2006)
32. Rubin, Karl.: Euler systems, volume 147 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study
33. Tsuji, Takeshi: p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. Invent. Math. **137**(2), 233–411 (1999)
34. Wiles, A.: On ordinary λ-adic representations associated to modular forms. Invent.Math. **94**(3), 529–573 (1988)
35. Zhang, Shouwu: Heights of Heegner cycles and derivatives of L-series. Invent. Math. **130**(1), 99–152 (1997)