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Discrete groups and holomorphic functions

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Abstract We determine conditions under which the quotient of a Kähler-Hadamard manifold by a torsion-free, discrete group of isometries admits a big set of nontrivial holomorphic functions. We also generalize a theorem of G. A. Margulis on centerless of cocompact discrete groups of automorphisms of a Carathéodory hyperbolic manifold by a different approach based on the Carathéodory pseudodistance.

Mathematics Subject Classification 32E10 · 32M15 · 32Q05

1 Introduction

The general theory of discrete groups which was initiated by Poincaré and Klein in the late nineteenth century, provides an essential tool of investigating the function theory, geometry and topology of manifolds through their universal coverings (cf. [4,18,32,33]). In this paper, we are interested in the following general problem

> (Q_1) *Under which condition the quotient of a Stein manifold by a torsion-free, discrete group of holomorphic automorphisms is a Stein manifold, or at least admits nontrivial holomorphic functions?*

It is natural to look at manifolds which have a nice geometry and a big group of automorphisms, especially at Hermitian symmetric spaces. The case of the flat complex Euclidean space $\mathbb{C}^n$ is well-understood, in particular, every quotient of $\mathbb{C}^n$ by a

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toroidal group admits no nonconstant holomorphic functions (compare [1]). For the case of the complex ball, Burns-Shnider [8] discovered firstly some Stein ball quotients, Fabritiis-Iannuzzi [11] showed that every ball quotient by an infinite cyclic group is Stein.

In this paper, we deal with general Hadamard manifolds, i.e., complete, simply-connected manifolds of nonpositive sectional curvature. In a popular paper [10], Eberlein-O'Neill extended some of the basic features of the discrete group theory from the Poincaré unit disc to Hadamard manifolds. They introduced a suitable topology, the cone topology, such that every Hadamard manifold X can be compactified by adjoining an ideal sphere $X(\infty)$ to X such that $\overline{X} := X \cup X(\infty)$ is homeomorphic to a closed ball, and every isometry g of X extends to a homeomorphism of $\overline{X}$. Let Isom X denote the group of all orientation-preserving isometrics of X. The elements of Isom X are divided into three types: elliptic, hyperbolic and parabolic isometries. The quotient of X by an infinite cyclic group of hyperbolic isometries is called an axial manifold.

Theorem 1.1 *Let X be a Kähler-Hadamard manifold and let G be a discrete Abelian subgroup of hyperbolic isometries of X. Then X/G is a Stein manifold if one of the following conditions is verified*

(1) *X/G is axial.*
(2) *X is a Hermitian symmetric space of noncompact type.*

If a Hadamard manifold X has pinched negative curvature $-b^2 \leq K \leq -a^2 < 0$, then every discrete group $G \subset \text{Isom}\, X$ acts properly discontinuously on $X \cup \Omega(G)$ where $\Omega(G) := X(\infty) - \Lambda(G)$ and $\Lambda(G) \subset X(\infty)$ is the set of accumulation points of any orbit $G(p)$, $p \in X$. Thus $M = X/G$ can be added a boundary defined as $M(\infty) := \Omega(G)/G$. In this special case, a more precise problem is the following

(Q_2) *Let M be a complete Kähler manifold of curvature $-b^2 \leq K \leq -a^2 < 0$ and q is a point in $M(\infty)$. Does there exist a holomorphic function on $M = X/G$ which approaches ∞ at q?*

It is convenient to use the following terminology

Definition 1.2 Let M be a complex manifold with an ideal boundary ∂M. Let $E \subset \partial M$. We say that M is *holomorphically convex modulo E* if for any discrete $\{x_\nu\}_{\nu=1}^{\infty}$ which has no accumulation points on E, there is a holomorphic function which is unbounded on this sequence. Furthermore, if holomorphic functions on M separate points and give local coordinate systems, then M is said to be *Stein modulo E*.

A discrete group G of isometries of a Hadamard manifold X of pinched negative curvature is called a parabolic group if all nontrivial elements of G are parabolic isometries with a common fixed point $q \in X(\infty)$. If G is parabolic, then $M = X/G$ is called a parabolic manifold. It is known that every parabolic manifold $M = X/G$ is homeomorphic to a cone over $M(\infty)$ with the parabolic vertex q (cf. [10]).

Proposition 1.3 *Every Kähler parabolic manifold is Stein modulo the parabolic vertex.*

It is well-known that there are various non-Stein parabolic manifolds with dimension ≥ 2, e.g., the case when $M(\infty)$ is compact.

The geometry of a complete Riemannian manifold M of curvature $-b^2 \leq K \leq -a^2 < 0$ is dominated by the thin-thick decomposition of Margulis: roughly speaking, there is a constant $\varepsilon_{a,b} > 0$ such that for any $\varepsilon < \varepsilon_{a,b}$ the thin part of M, i.e., the set of points whose injectivity radius is less than $\varepsilon/2$, is a disjoint union of parabolic and axial ends. It induces a natural concept of geometrical finiteness, introduced originally by Marden [23], Beardon-Maskit [5], Thurston [33] for Kleinian groups, and generalized by Bowditch [7] to the case of pinched negative curvature. Geometrical finiteness of M means that $\overline{M} = M \cup M(\infty)$ has finitely many ends, each a *cusp* end.

Theorem 1.4 *Every Kähler geometrically finite manifold is holomorphically convex modulo the parabolic vertices corresponding to the cusp ends.*

We also have a Hartogs type extension theorem

Theorem 1.5 *Let M be a Kähler geometrically finite manifold of dimension ≥ 2 and let V be a neighborhood of the union of compact connected components of $M(\infty)$ in $\overline{M}$. Then every holomorphic function on $M \cap V$ has a holomorphic extension to M. In particular, $M(\infty)$ has at most one compact connected component.*

The complex ball $\mathbb{B}^n_{\mathbb{C}}$ with the Bergman metric distinguishes itself in Kähler-Hadamard manifolds of pinched negative curvature since it has plenty of complete, totally geodesic submanifolds, for instance, the real ball $\mathbb{B}^n_{\mathbb{R}}$ may be isometrically embedded as a totally real, totally geodesic submanifold of $\mathbb{B}^n_{\mathbb{C}}$, i.e., the fixed-point-set of the antiholomorphic involution $z \to \bar{z}$. The Bergman metric of $\mathbb{B}^n_{\mathbb{C}}$ induces a Riemannian metric on $\mathbb{B}^n_{\mathbb{R}}$ which has constant negative sectional curvature $-1/4$. The stabilizer of $\mathbb{B}^n_{\mathbb{R}}$ is the image of the embedding $PO(n,1) \hookrightarrow PU(n,1)$, and every torsion-free, discrete subgroup $G \subset PO(n,1)$ acts on $\mathbb{B}^n_{\mathbb{C}}$ such that the quotient $\mathbb{B}^n_{\mathbb{C}}/G$ is a fiber bundle over a totally real, totally geodesic submanifold $\mathbb{B}^n_{\mathbb{R}}/G$ (cf. [8]).

Proposition 1.6 *For every torsion-free, discrete subgroup G of $PO(n,1)$, $\mathbb{B}^n_{\mathbb{C}}/G$ is Stein modulo $\Lambda(G)$. Furthermore, $\mathbb{B}^n_{\mathbb{C}}/G$ is a Stein manifold if its injectivity radius is positive.*

Burns-Shnider [8] has shown that $\mathbb{B}^n_{\mathbb{C}}/G$ is Stein when $\mathbb{B}^n_{\mathbb{R}}/G$ is compact. On the other hand, $\mathbb{B}^n_{\mathbb{R}}/G$ always admits a Stein neighborhood basis in $\mathbb{B}^n_{\mathbb{C}}/G$ according to a famous result of Grauert [13]. Thus it is natural to ask the following problem

(Q_3) *Is $\mathbb{B}^n_{\mathbb{C}}/G$ always Stein for every torsion-free, discrete subgroup G of $PO(n,1)$?*

Parabolic groups for general Hadamard manifolds (e.g., high-rank Hermitian symmetric spaces of noncompact type) are much more complicated. Nevertheless, every Hermitian symmetric spaces of noncompact type is biholomorphic to an affinely homogeneous Siegel domain of the second kind (cf. [34]). Notice that for the Siegel domain model $\mathbb{H}^n_{\mathbb{C}}$ of $\mathbb{B}^n_{\mathbb{C}}$, parabolic groups fixing ∞ are precisely those discrete groups of unimodular affine automorphisms, i.e., whose complex Jacobian has unit modular.

A particular important subgroup $\mathfrak{P}(D)$ of the group $\mathfrak{A}_1(D)$ of all unimodular affine transformations is the group of *parallel translations*. There is a natural isomorphism $\mathcal{I}$ from $\mathfrak{P}(D)$ to the group of translations of $\mathbb{R}^n \times \mathbb{C}^m$. A subgroup G of $\mathfrak{P}(D)$ is said to be totally real if $\mathcal{I}(G)$ is a subgroup of translations of $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 1.7 *Let D be a Siegel domain of the second kind and let G be a torsion-free, discrete subgroup of $\mathfrak{A}_1(D)$. Then D/G is Stein modulo the ideal boundary point ∞. Suppose furthermore that G is a totally real, torsion-free, discrete subgroup of $\mathfrak{P}(D)$, then D/G is a Stein manifold.*

Corollary 1.8 *If G is a discrete Abelian group of Heisenberg translations, then $\mathbb{H}^n_{\mathbb{C}}/G$ is a Stein manifold.*

The theory of isometries of a Hadamard manifold X relies heavily on the fact that the displacement functions $d(x, g(x))$ is a convex function on X for every $g \in \operatorname{Isom} X$ (compare [4]). For general manifolds, such a nice property will lose. Nevertheless, we still have

Proposition 1.9 *Let X be a complex manifold and let g_0 be a holomorphic automorphism of X. Suppose that there is a point $x_0 \in X$ and a bounded holomorphic function f_0 on X such that $f_0(g_0(x_0)) \neq f_0(x_0)$. Then $c_X^2(x, g_0(x))$ is a* nonconstant *plurisubharmonic function on X. Here c_X denotes the Carathéodory pseudodistance of X.*

As an application, we generalize a theorem of G. A. Margulis as follows

Theorem 1.10 *Let X be a complex manifold and let G be a discrete group of automorphisms of X. Suppose that there exist a central element $g_0 \in G$, a point $x_0 \in X$ and a bounded holomorphic function f_0 on X such that $f_0(g_0(x_0)) \neq f_0(x_0)$. Then $M = X/G$ cannot be compact or admits a compactification M' such that M' is a complex space and $M' - M$ is a complex-analytic variety of codimension ≥ 2 in M'.*

Corollary 1.11 *If a Carathéodory hyperbolic manifold (e.g., bounded domain in $\mathbb{C}^n$) covers a compact complex space or an open set in a compact complex space whose complement is a complex-analytic variety of codimension ≥ 2, then the group of deck transformations of the covering has trivial center.*

The compact case of the above theorem and corollary is due to Margulis (cf. [22], Theorem 3.10, Corollary 3.11). Classical examples satisfying the hypothesis in the corollary are noncompact irredicible arithmetic quotients of bounded symmetric domains (cf. [3]).

The main tool involved is the L^2-theory for $\bar{\partial}-$operator on complete Kähler manifolds. Applications of this method to Complex Differential Geometry were initiated by Siu-Yau, Greene-Wu and Mok (cf. [15,24,30,31]).

The paper is organized as follows. In Sect. 2, we give some backgrounds of Hadamard manifolds. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we prove Propositions 1.3, 1.6. Theorems 1.4, 1.5 are proved in Sects. 5, 6 respectively. In Sect. 7, we investigate quotients of a Siegel domain of the second kind. Finally, we prove Theorem 1.10 and Proposition 1.9.

2 Preliminaries

In this section, we collect some basic materials on the geometry of Hadamard manifolds, following the classical papers [6,7,10] and the monograph [4].

Let X be a Hadamard manifold and let d denote the distance function. For a point $p \in X$, let T_pX denote the tangent space at p. The Cartan-Hadamard theorem states that the exponential map $\exp_p : T_pX \to X$ is a diffeomorphism. Let S_p denote the unit sphere in T_pX. For every $v \in S_p$, there is a unique geodesic $\alpha_v : \mathbb{R} \to X$ such that $\alpha_v'(0) = v$. For two distinct points x_1, x_2 in X, let $\alpha_{x_1x_2}$ denote the unique geodesic segment between x_1 and x_2, i.e., $\alpha_{x_1x_2}(0) = x_1$ and $\alpha_{x_1x_2}(t) = x_2$ where $t = d(x_1, x_2)$. If x_1, x_2 are distinct from p, the angle $\angle_p(x_1, x_2)$ subtended by x_1, x_2 at p is defined as the angle between $\alpha'_{px_1}(0)$ and $\alpha'_{px_2}(0)$, i.e., the vectors in S_p determined by α_{px_1} and α_{px_2} respectively.

Two geodesics α, β are called *asymptotic* if $d(\alpha(t), \beta(t)) \leq \text{const.}$ for all $t \geq 0$. The set $X(\infty)$ of points at infinity for X is defined as the set of all asymptotic classes of X. A truncated cone $T(v, \varepsilon, r)$ with vertex p, axis v, angle ε and radius r is defined as

$$T(v, \varepsilon, r) := \left\{x \in X : \angle_p(\alpha_v(\infty), x) < \varepsilon\right\} - \{x \in X : d(p, x) \leq r\}.$$

The collection of truncated cones at p form a basis for the cone topology. Under this topology, $\overline{X} = X \cup X(\infty)$ is homeomorphic to the closed unit ball and $X(\infty)$ is homeomorphic to the unit sphere. Furthermore, every isometry of X extends to a homeomorphism of $\overline{X} \to \overline{X}$.

For a unit speed geodesic α_v, $v \in S_p$, the Busemann function associated to α_v is defined as

$$f_{\alpha_v}(x) := \lim_{t \to +\infty} d(x, \alpha_v(t)) - t.$$

It enjoys the following properties:

- (B_1) Lipschitz continuity: $|f_{\alpha_v}(x_1) - f_{\alpha_v}(x_2)| \leq d(x_1, x_2), \forall x_1, x_2 \in X$.
- (B_2) f_{α_v} is a C^2-smooth, convex function on X.
- (B_3) If α_v, β_v lie in the same asymptotic class, then $f_{\alpha_v} - f_{\beta_v} = \text{const}$.

From (B_3) we may define the Busemann function f_q at a point $q \in X(\infty)$ as the function f_{α_v} where α_v is a geodesic lying in the asymptotic class of q. The *horosphere* at $q \in X(\infty)$ through a point $x_1 \in X$ is the set

$$S(q, x_1) := \left\{x \in X : f_q(x) = f_q(x_1)\right\}.$$

The *horoball* at q determined by x_1 is defined as

$$B(q, x_1) := \left\{x \in X : f_q(x) < f_q(x_1)\right\}.$$

We also mention the following property:

(B_4) If x_1, x_2 are two distinct points in X, then

$$\begin{aligned}|f_q(x_1) - f_q(x_2)| &= \operatorname{dist}(x_1, S(q, x_2)) = \operatorname{dist}(x_2, S(q, x_1)) \\ &= \operatorname{dist}(S(q, x_1), S(q, x_2)).\end{aligned}$$

If W is a closed totally convex subset of X and $x \in X$, then there is a unique point $\pi_W(x) \in W$ of minimal distance to x, which is called the projection of x to W. The distance $d(x, W) = d(x, \pi_W(x))$ to W is a C^2 convex function on X. This function will play a fundamental role in this paper. If g is an isometry of X such that $g(W) = W$, then we have $g(\pi_W(x)) = \pi_W(g(x))$ for all $x \in X$ (cf. [4], Lemma 6.4). Thus $d(\cdot, W)$ is $g-$invariant. Indeed,

$$\begin{aligned}d(g(x), W) &= d(g(x), \pi_W(g(x))) \\ &= d(g(x), g(\pi_W(x))) \\ &= d(x, \pi_W(x)) \\ &= d(x, W).\end{aligned}$$

If X has negative curvature, then $d(\cdot, W)$ is even strictly convex on $X - W$.

Every isometry $1 \neq g \in \operatorname{Isom} X$ has to belong to one of the following three classes: g is said to be *elliptic* if the displacement function $d_g(x) = d(x, g(x))$ has minimum zero, *hyperbolic* if d_g has positive minimum and *parabolic* if d_g has non minimum. A subgroup G of $\operatorname{Isom} X$ is called properly discontinuous if each compact set in X meets only finitely many images of itself under G. A subgroup G of $\operatorname{Isom} X$ is discrete as a subgroup if and only if it is properly discontinuous. For a discrete group $G \subset \operatorname{Isom} X$, the *limit set* $\Lambda(G)$ of G is defined as the set of accumulation points of some (and hence any) orbit $G(x_0)$, $x_0 \in X$. It is the smallest $G-$invariant closed subset in $X(\infty)$. The *convex hull* $\operatorname{ch}(\Lambda(G))$ of $\Lambda(G)$ is defined as the convex hull of the orbit $G(x_0)$, which is a $G-$invariant closed totally convex subset of X.

Now suppose that X has pinched negative curvature $-b^2 \leq K \leq -a^2 < 0$. Then an isometry g is elliptic, hyperbolic, parabolic iff g has a fixed point in X, g has exactly two fixed points on $X(\infty)$, g has exactly one fixed point on $X(\infty)$. If G is a discrete group of isometries, then it acts properly discontinuously on $X \cup \Omega(G)$ where $\Omega(G) = X(\infty) - \Lambda(G)$. Thus one can define the set $M(\infty)$ of points at infinity of $M = X/G$ by $M(\infty) = \Omega(G)/G$. We write $\overline{M} = M \cup M(\infty)$ and call $M(\infty)$ the boundary of M. An infinite discrete group $G \subset \operatorname{Isom} X$ is called a *parabolic group* if $\Lambda(G)$ consists of a single point q. A particular important property for a parabolic group G is that the Busemann function f_q at q is $G-$invariant. Thus $M = X/G$ such that is topologically a product $F \times \mathbb{R}$ whose horizonal fibers are the level hypersurfaces of this function. If $\Lambda(G)$ consists of two points, then we call G a *axial group*. A axial group G is an infinite cyclic group generated by some hyperbolic isometry. Parabolic and axial groups are called *elementary groups*.

Definition 2.1 Let Γ be a subgroup of a discrete subgroup G of $\operatorname{Isom} X$. A subset Y in $\overline{X}$ is called *precisely invariant* under Γ in G if $g(Y) = Y$ for all $g \in \Gamma$ and $g(Y) \cap Y = \emptyset$ for all $g \in G \backslash \Gamma$.

Let $\varepsilon > 0$. For a given discrete group $G \subset \text{Isom}\, X$ and its orbifold $M = X/G$, the $\varepsilon-$*thin* part $\text{thin}_\varepsilon(M)$ of M is defined as

$$\text{thin}_\varepsilon(M) = \{x \in X : G_\varepsilon(x) = \langle g \in G : d(x, g(x)) < \varepsilon\rangle \text{ is infinite }\}/G.$$

The $\varepsilon-$*thick* part $\text{thick}_\varepsilon(M)$ of M is defined as the closure of $M\backslash\text{thin}_\varepsilon(M)$ in M. According to the Margulis lemma, there is a positive constant $\varepsilon_{a,b}$ (the Margulis constant) such that for any $0 < \varepsilon < \varepsilon_{a,b}$, $\text{thin}_\varepsilon(M)$ is a disjoint union of its connected components, and each component has the form $X_\varepsilon(\Gamma)/\Gamma$ where Γ is a maximal infinite elementary subgroup of G. Here

$$X_\varepsilon(\Gamma) = \{x \in X : \Gamma_\varepsilon = \langle g \in \Gamma : d(x, g(x)) < \varepsilon\rangle \text{ is infinite }\}$$

is precisely invariant under Γ in G. If Γ is parabolic, then $X_\varepsilon(\Gamma)/\Gamma$ is a *Margulis cusp*. If Γ is axial, then $X_\varepsilon(\Gamma)/\Gamma$ is a *Margulis tube*.

The Margulis thin-thick decomposition naturally induces the important concept of geometrical finiteness. Let G be a discrete subgroup of Isom X and let $M = X/G$. Let E be a closed, connected, non-compact set in $\overline{M}$ and let Y be a connected component of the lift of E to $X \cup \Omega(G)$ and let $\Gamma = \text{stab}_G(Y) = \{g \in G : g(Y) = Y\}$. If Γ is parabolic with fixed point $q \in X(\infty)$ and Y is precisely invariant under Γ in G, then E is said to be a *standard cusp region* and q is called a cusp point. We may identify E as a (closed) subset Y/Γ of $(\overline{X} - \{q\})/\Gamma$. Every standard cusp region E determines an unique topological end e of $\overline{M}$ which is called a *cusp* end of $\overline{M}$. The group G is said to be *geometrically finite* if $\overline{M}$ has finitely many ends, each a cusp end. We remark that a parabolic end may not be a cusp end. An equivalent definition of geometrical finiteness is that the intersection of the core $\text{ch}\,(\Lambda(G))/G$ with $\text{think}_\epsilon(M)$ is compact. Classical examples of geometrically finite groups may be constructed from elementary groups via the Klein-Maskit Combination Theorems. A geometrically finite group is always finitely generated and the converse is also true for Fuchsian groups. However, there exist examples of finitely generated geometrically infinite groups when dimension ≥ 3.

3 Proof of Theorem 1.1

We need the following

Definition 3.1 Let X be Hadamard manifold and let G be a discrete subgroup of Isom X. Let X' be a closed submanifold of X. We say that X' is $G-$*periodic* if it is $G-$invariant and has a compact quotient X'/G.

The proof of Theorem 1.1 is based on the following

Proposition 3.2 *Let X be a Kähler-Hadamard manifold and let G be a torsion-free, discrete subgroup of* Isom X. *If there is a $G-$periodic, complete, totally real and totally geodesic submanifold X' of X, then X/G is a Stein manifold.*

Proof Let $d_{X'}$ be the distance to X'. Since X' is totally geodesic, $d^2_{X'}$ is a C^2-convex function on X. We claim that $d^2_{X'}$ is *strictly psh* on X. Fix an arbitrary point $x \in X$.

Let $x' = \pi_{X'}(x)$ be the projection of x to X'. Since at x', $d_{X'}(y)^2$ differs from $d(y, T_{x'}X')^2$ by $o(d(y,x')^2)$, and since $T_{x'}X'$ contains no complex lines, the complex Hessian of $d^2_{X'}$ is positive definite in a small neighborhood of x'. Thus we may assume $x \in X - X'$.

By Lemma 1.13 of [15], it suffices to show

$$D^2 d^2_{X'}(v, v) + D^2 d^2_{X'}(Jv, Jv) > 0, \quad \text{for all } 0 \neq v \in T_x X \tag{1}$$

where J is the complex structure tensor of X. Let $\gamma : [0, 1] \to X$ be a geodesic satisfying $\gamma(0) = x'$ and $\gamma(1) = x$ and let $\beta : (-\varepsilon, \varepsilon) \to X$ be a geodesic such that $\beta'(0) = v$. Consider the following variation of γ: for every $s \in (-\varepsilon, \varepsilon)$, $\gamma_s : [0, 1] \to X$ is the unique geodesic segment connecting $\pi_\alpha(\beta(s))$ with $\beta(s)$. Let $E(\gamma_s)$ be the *energy* of γ_s, i.e.,

$$E(\gamma_s) = \frac{1}{2}\int_0^1 \left|\frac{d\gamma_s}{dt}(t)\right|^2 dt \quad \left(= \frac{1}{2}L(\gamma_s)^2\right)$$

where $L(\gamma_s)$ denotes the length of γ_s. Let W be the transversal vector field of $\{\gamma_s\}$. By the second variation formula of the energy (cf. [17], p. 167), we have

$$D^2 d^2_{X'}(v, v) = 2\frac{d^2E(\gamma_s)}{ds^2}(0) = 2\left(\int_0^1 (|D_{\gamma'}W|^2 - \langle R(W, \gamma')W, \gamma'\rangle)dt\right) > 0 \tag{2}$$

provided $D_{\gamma'}W$ not identically zero, i.e., W is not parallel along γ. Since parallel transports along γ preserve J, we conclude that the transversal vector field corresponding to Jv can not be parallel along γ as X' is totally real. Thus claim follows immediately from (1), (2).

Since X' is G−invariant and totally geodesic (hence totally convex), $d^2_{X'}$ is also G−invariant. Since X/G is diffeomorphic to the normal bundle of the *compact* manifold X'/G in X/G (compare Lemma 3.1 of [6]), we conclude that X/G admits a strictly psh exhaustion function $d^2_{X'}$, hence is a Stein manifold by Grauert's solution of the Levi problem. □

Proof of Theorem 1.1 (1) Let $G = \langle g\rangle$ where g is a hyperbolic isometry. It is well-known that g translates a geodesic α, that is $g(\alpha(t)) = \alpha(t + a)$ for all $t \in \mathbb{R}$ and $a > 0$ is the minimum value of the displacement function d_g (The geodesic α is called an *axis* of g). Clearly α is totally real, thus Proposition 3.2 applies.

(2) It is known that every discrete Abelian group of a Hadamard manifold operates as a lattice on some isometrically embedded flat Euclidean space (as a complete, totally geodesic submanifold) (cf. [4], p. 86). In the special case of Hermitian symmetric spaces of noncompact type, such an embedded Euclidean space contains no complex lines, thus Proposition 3.2 applies. □

4 Proofs of Propositions 1.3, 1.6

We make first of all the following useful observation

Proposition 4.1 *Let* (M, ω) *be complete Kähler manifold and let* ρ *be a* C^2 *strictly psh function on* M *satisfying* $\partial\bar{\partial}\rho + \mathrm{Ric}\,\omega \geq 0$ *where* $\mathrm{Ric}\,\omega$ *denotes the Ricci curvature of* ω*. Then the holomorphic functions on* M *separate points and give local coordinate systems. Furthermore, for any discrete sequence of points* $\{x_\nu\}$ *with* $\rho(x_\nu) \to +\infty$ *as* $\nu \to \infty$ *there is a holomorphic function on* M *which is unbounded on* $\{x_\nu\}$*.*

The proof of this result is based on a standard application of the following L^2 estimate for $\bar{\partial}-$operator (cf. [2,9,16]):

Theorem 4.2 *Let* (M, ω) *be a complete Kähler manifold,* $\dim M = n$*. Let* $\varphi : M \to [-\infty, \infty)$ *be a function which is of* C^2 *outside a discrete subset* $\{x_\nu\}_{\nu=1}^{\infty} \subset M$ *and, near each point* x_ν*,* $\varphi(z^\nu) = C_\nu \log|z^\nu|^2$ *where* C_ν *is a positive constant and* $z^\nu = (z_1^\nu, \cdots, z_n^\nu)$ *are local holomorphic coordinates centered at* x_ν*. Assume that* $\partial\bar{\partial}\varphi + \mathrm{Ric}\,\omega \geq c\omega$ *on* $M \backslash \{x_\nu\}_{\nu=1}^{\infty}$ *for some positive continuous function* c *on* M*. Then for any* $C^\infty\,\bar{\partial}-$*closed* $(0, 1)-$*form* η *on* M *with* $\int_M |\eta|^2 e^{-\varphi} dV < \infty$*, there is a* C^∞ *function* u *on* M *such that* $\bar{\partial}u = \eta$ *and*

$$\int\limits_M |u|^2 e^{-\varphi} dV \leq \int\limits_M \frac{|\eta|^2}{c} e^{-\varphi} dV.$$

Proof of Proposition 4.1 Let $x_1 \neq x_2$ be arbitrary two points in M and let $\{x_\nu\} \subset M$ be a discrete sequence of points such that $\rho(x_\nu) \to +\infty$ as $\nu \to \infty$. By passing to a subsequence if necessary, we may assume that $\rho(x_{\nu+1}) > \rho(x_\nu) + 3$ for all $\nu \geq 2$. For each $\nu \geq 1$, let $z^\nu = (z_1^\nu, \cdots, z_n^\nu)$ be a local holomorphic coordinate centered at x_ν. Take $r_\nu > 0$ such that the holomorphic coordinate balls $B_\nu = \{z^\nu : |z^\nu| < r_\nu\}$, $\nu = 1, 2, \cdots$, are mutually disjoint, and $B_\nu \subset \{\rho(x_\nu) - 1 < \rho < \rho(x_\nu) + 1\}$ for all $\nu \geq 3$.

Let $\chi : \mathbb{R} \to [0, 1]$ be a C^∞ function such that $\chi = 1$ on $(-\infty, 1/2)$ and $\chi = 0$ on $(1, \infty)$. Then one can choose a convex, rapidly increasing function $\lambda \geq 0$ on $\mathbb{R}$ such that

(1) $\partial\bar{\partial}\hat{\varphi} > 0$ on M where

$$\begin{aligned}\hat{\varphi} = \lambda \circ \rho &+ (n+1)\chi(2|z^1|^2/r_1^2)\log 2|z^1|^2/r_1^2 \\ &+ n\sum_{\nu=2}^{\infty}\chi(2|z^\nu|^2/r_\nu^2)\log 2|z^\nu|^2/r_\nu^2.\end{aligned}$$

(2)

$$\sum_{\nu=1}^{\infty} \nu^2 \int\limits_M \frac{|\bar{\partial}\chi(2|z^\nu|^2/r_\nu^2)|^2}{c} e^{-\lambda\circ\rho - 2\rho} dV < \infty$$

where c is the minimal eigenvalue of $\partial\bar{\partial}\rho$ w.r.t. ω. Define $\varphi = \hat{\varphi} + 2\rho$. Clearly, φ is C^2 outside $\{x_\nu\}_{\nu=1}^\infty$ and

$$\partial\bar{\partial}\varphi + \mathrm{Ric}\,\omega \geq \partial\bar{\partial}\rho \geq c\,\omega.$$

Let

$$\eta_0 = \bar{\partial}\chi(2|z^1|^2/r_1^2), \qquad \eta_j = z_j^1\bar{\partial}\chi(2|z^1|^2/r_1^2), \quad 1 \leq j \leq n$$

and

$$\eta_{n+1} = \sum_{\nu=1}^{\infty} \nu\bar{\partial}\chi(2|z^\nu|^2/r_\nu^2).$$

By the above theorem, there are C^∞ functions u_j, $j = 0, 1, \cdots, n+1$, satisfying $\bar{\partial}u_j = \eta_j$ on M and

$$\begin{aligned}\int_M |u_j|^2 e^{-\varphi} dV &\leq \int_M \frac{|\eta_j|^2}{c} e^{-\varphi} dV \\ &\leq \text{const.} \sum_{\nu=1}^{\infty} \nu^2 \int_M \frac{|\bar{\partial}\chi(2|z^\nu|^2/r_\nu^2)|^2}{c} e^{-\lambda\circ\rho-2\rho} dV < \infty.\end{aligned}$$

Let

$$f_0 = \chi(2|z^1|^2/r_1^2) - u_0, \qquad f_j = z_j^1\chi(2|z^1|^2/r_1^2) - u_j, \quad 1 \leq j \leq n$$

and

$$f_{n+1} = \sum_{\nu=1}^{\infty} \nu\chi(2|z^\nu|^2/r_\nu^2) - u_{n+1}.$$

It is easy to see that all f_j are holomorphic functions on M satisfying $f_0(x_1) \neq f_0(x_2)$, the Jacobian of the holomorphic map $x \in M \to (f_1(x), \cdots, f_n(x)) \in \mathbb{C}^n$ is non-vanishing at x_1, and $f_{n+1}(x_\nu) = \nu$ for every $\nu \geq 1$. □

Proof of Proposition 1.3 Let f_q denote the Busemann function at $q \in \Lambda(G)$. Then it is $G-$invariant and satisfies $f_q(x) \to +\infty$ as $x \to X(\infty) - \{q\}$ (compare (B_4)), and the Hessian Comparison Theorem implies $\partial\bar{\partial} f_q \geq a\omega$ (cf. [15,31]). Notice also that $\mathrm{Ric}\,\omega \geq -(2n-1)b^2\omega$. Thus it suffices to apply Proposition 4.1 with $\rho = \frac{(2n-1)b^2}{a} f_q$. □

Proof of Proposition 1.6 Let $d_{\mathbb{B}_{\mathbb{R}}^n}$ be the distance to the totally geodesic submanifold $\mathbb{B}_{\mathbb{R}}^n$ of $\mathbb{B}_{\mathbb{C}}^n$. We are going to verify

$$\partial\bar{\partial} d_{\mathbb{B}_{\mathbb{R}}^n}^2 \geq \text{const.}\,\omega_{\mathrm{Berg}} \tag{3}$$

where ω_{Berg} denotes the Bergman metric of $\mathbb{B}^n_{\mathbb{C}}$. Let $z \in \mathbb{B}^n_{\mathbb{C}}$ and let $\pi_{\mathbb{B}^n_{\mathbb{R}}}(z)$ be the projection of z on $\mathbb{B}^n_{\mathbb{R}}$. Notice that $d_{\mathbb{B}^n_{\mathbb{R}}}$ is $PO(n,1)$-invariant, thus we may assume $\pi_{\mathbb{B}^n_{\mathbb{R}}}(z) = 0$. Since $\mathbb{B}^n_{\mathbb{R}}$ is totally real in $\mathbb{B}^n_{\mathbb{C}}$, the above inequality holds when $d_{\mathbb{B}^n_{\mathbb{R}}}(z) \leq$ const. $\ll 1$. Thus it suffices to consider the case when $d_{\mathbb{B}^n_{\mathbb{R}}}(z) \geq$ const..

Let $\gamma : [0,l] \to \mathbb{B}^n_{\mathbb{C}}$ be a *unit speed* geodesic from 0 to z. Let $T_z(\mathbb{B}^n_{\mathbb{R}})$ be the parallel transport of $T_0(\mathbb{B}^n_{\mathbb{R}})$ to z along γ. Since the complex structure tensor J of $\mathbb{B}^n_{\mathbb{C}}$ is preserved under parallel transports, we conclude that $J(T_z(\mathbb{B}^n_{\mathbb{R}}))$ is orthogonal to $T_z(\mathbb{B}^n_{\mathbb{R}})$. Thus we may write every holomorphic tangent vector at z as $\frac{1}{2}(v - iJv)$ where $v \in T_z(\mathbb{B}^n_{\mathbb{R}})$. Without loss of generality, we may assume $|v| = 1$. Now the parallel translation $v(t)$ of v to $\gamma(t)$ along γ induces a variation of γ, thus by the second variation formula

$$D^2 d^2_{\mathbb{B}^n_{\mathbb{R}}}(v,v) = 2l \left(\int_0^l (|D_{\gamma'} v(t)|^2 - \langle R(v(t),\gamma')v(t),\gamma'\rangle)dt \right) \geq \frac{l^2}{2} \geq \text{const.}$$

because the sectional curvature of the Bergman metric is pinched between -1 and $-1/4$. By Lemma 1.13 of [15], (3) is verified.

To verify the first assertion of the proposition, it suffices to apply Proposition 4.1 with the $G-$invariant function

$$\rho = d^2_{\mathrm{ch}(\Lambda(G))} + C d^2_{\mathbb{B}^n_{\mathbb{R}}}$$

where C is a sufficiently large constant and $d_{\mathrm{ch}(\Lambda(G))}$ is the distance to the closed totally convex set $\mathrm{ch}(\Lambda(G)) \subset \mathbb{B}^n_{\mathbb{C}}$.

If $\mathbb{B}^n_{\mathbb{C}}/G$ has a positive injectivity radius, i.e., it has bounded geometry, then it is not difficult to construct a C^∞ exhaustion function ψ on $\mathbb{B}^n_{\mathbb{C}}/G$ such that $\partial\bar{\partial}\psi \geq -\text{const.}\,\omega_{\mathrm{Berg}}$ (compare [26]). Thus $\psi + C d^2_{\mathbb{B}^n_{\mathbb{R}}}$, $C \gg 1$, gives a strictly psh exhaustion function of $\mathbb{B}^n_{\mathbb{C}}/G$. □

5 Proof of Theorem 1.4

Let us review some basic facts on the L^2-cohomology of $\bar{\partial}-$operator. Let (M,ω) be a complete Kähler manifold, $\dim M = n$. Let (L,h) be a holomorphic Hermitian line bundle on M. Let $C^{r,s}(M,L)$ (resp. $C^{r,s}_0(M,L)$) denote the space of $L-$valued C^∞ (resp. compactly supported C^∞) $(r,s)-$forms on M. Let $L^{r,s}_{(2)}(M,L) = L^{r,s}_{(2)}(M,L;\omega,h)$ be the completion of $C^{r,s}_0(M,L)$ with respect to the norm

$$\|u\| = \left(\int_M |u|^2 dV \right)^{1/2}$$

where $|\cdot| = |\cdot|_{\omega,h}$ is the point-wise norm with respect to the metrics ω, h, and $dV = dV_\omega$ is the volume form with respect to ω. The $\bar{\partial}-$operator is extended naturally to a densely definite, closed operator on $L^{r,s}_{(2)}(M,L)$. Let $\bar\partial^*$ be the adjoint of $\bar\partial$ and let Dom $\bar\partial$, Dom $\bar\partial^*$ denote the domains of $\bar\partial$, $\bar\partial^*$. According to a theorem of Andreotti-Vesentini [2], $C^{r,s}_0(M,L)$ is dense in Dom $\bar\partial$ $\cap$ Dom $\bar\partial^*$ with respect to the graph norm

$$\left(\|u\|^2+\|\bar\partial u\|^2+\|\bar\partial^* u\|^2\right)^{1/2}.$$

The (irreduced) L^2-cohomology group of bi-degree (r,s) is defined as

$$H^{r,s}_{(2)}(M,L)=\frac{L^{r,s}_{(2)}(M,L)\cap \operatorname{Ker}\bar\partial_{r,s}}{L^{r,s}_{(2)}(M,L)\cap \operatorname{Im}\bar\partial_{r,s-1}}.$$

Definition 5.1 We say that the basic estimate holds at bi-degree (r,s) if there exist a compact set $N\subset M$ such that

$$\|u\|^2\le \text{const.}\left(\|\bar\partial u\|^2+\|\bar\partial^* u\|^2+\int_N |u|^2 dV\right)$$

for all $u\in L^{r,s}_{(2)}(M,L)\cap \operatorname{Dom}\bar\partial\cap\operatorname{Dom}\bar\partial^*$.

The following result is well-known (see e.g., [28], Proposition 1.2):

Proposition 5.2 *If the basic estimate holds at bi-degree (r,s), then* $\operatorname{Im}\bar\partial_{r,s-1}$, $\operatorname{Im}\bar\partial^*_{r,s+1}$ *are closed and* $\dim H^{r,s}_{(2)}(M,L)<\infty$.

As an application of Proposition 5.2, we have

Theorem 5.3 *Let (M,ω) be a complete Kähler manifold with* $\operatorname{Ric}\omega\ge -\text{const}.\omega$. *Suppose that there exists a C^2 strictly psh function ρ outside a compact subset $N\subset M$ such that* $\partial\bar\partial\rho\ge \text{const}.\omega$ *holds on $M-N$. Then for any discrete sequence $\{x_\nu\}$ of points in M with $\rho(x_\nu)\to+\infty$ as $\nu\to\infty$, there exists a holomorphic function on M which is unbounded on $\{x_\nu\}$.*

Proof By a multiplier of some cut-off function, we may assume that ρ is a C^2 real function on M. By passing to a subsequence if necessary, we may assume that $\{x_\nu\}\subset M-N$ and $\rho(x_{\nu+1})>\rho(x_\nu)+3$ for all ν. Let $B_\nu=\{z^\nu:|z^\nu|<r_\nu\}\subset M-N$, $\nu=1,2,\ldots,$ be mutually disjoint coordinate patches centered at x_ν such that $B_\nu\subset\{\rho(x_\nu)-1<\rho<\rho(x_\nu)+1\}$ for all ν. Let χ be the cut-off function as in the proof of Proposition 4.1. Choose a convex, rapidly increasing function $\hat\lambda\ge 0$ on $\mathbb{R}$ and a sufficiently large constant C such that

$$\hat\omega:=\partial\bar\partial\left[\hat\lambda\circ\rho+\sum_{\nu=1}^{\infty}\chi(2|z^\nu|^2/r_\nu^2)(-\log(-\log|z^\nu|/2r_\nu))\right]+C\omega\ge\omega$$

holds on M. Thus $\hat{\omega}$ gives a complete Kähler metric on the punctured manifold $\hat{M} = M\backslash\{x_\nu\}_{\nu=1}^{\infty}$. Since $\mathrm{Ric}\,\omega \geq -\mathrm{const}.\omega$ and $\partial\bar{\partial}\rho \geq \mathrm{const}.\omega$ on $M - N$, one may choose a convex, rapidly increasing function $\lambda \geq 0$ on $\mathbb{R}$ such that

(i) $\partial\bar{\partial}\varphi + \mathrm{Ric}\,\omega \geq \hat{\omega}$ holds on $M - N$ where

$$\varphi = \lambda \circ \rho + n \sum_{\nu=1}^{\infty} \chi(2|z^\nu|^2/r_\nu^2) \log 2|z^\nu|^2/r_\nu^2 + \sum_{\nu=1}^{\infty} \chi(2|z^\nu|^2/r_\nu^2)(-\log(-\log|z^\nu|/2r_\nu)).$$

(ii)

$$\sum_{\nu=1}^{\infty} 3^{\nu^2} \int_{\hat{M}} |\bar{\partial}\chi(2|z^\nu|^2/r_\nu^2)|_{\hat{\omega}}^2 e^{-\lambda\circ\rho} dV_\omega < \infty.$$

Let $K_{\hat{M}}^*$ denote the anti-canonical line bundle of $\hat{M}$. Define a Hermitian metric h on $K_{\hat{M}}^*$ by

$$h = e^{-\varphi} \det(\omega_{j\bar{k}})$$

where $\omega = \sum_{j,k} \omega_{j\bar{k}} dz_j d\bar{z}_k$, in local holomorphic coordinates.

The Bochner-Kodaira-Nakano inequality together with Andreotti-Vesentini's approximation theorem imply

$$\|\bar{\partial}u\|_{\hat{\omega},h}^2 + \|\bar{\partial}^*u\|_{\hat{\omega},h}^2 \geq \int_{\hat{M}} \langle[\sqrt{-1}\partial\bar{\partial}\varphi + \mathrm{Ric}\,\omega, \Lambda_{\hat{\omega}}]u, u\rangle_{\hat{\omega},h} dV_{\hat{\omega}} \tag{4}$$

for all $u \in L_{(2)}^{n,s}(\hat{M}, \varphi) \cap \mathrm{Dom}\,\bar{\partial} \cap \mathrm{Dom}\,\bar{\partial}^*$. Let $L_{(2)}^{0,s}(\hat{M}, \varphi)$ be the space of square-integrable $(0, s)-$forms on $\hat{M}$ with respect to the norm

$$\|u\| = \left(\int_{\hat{M}} |u|_{\hat{\omega}}^2 e^{-\varphi} dV_\omega\right)^{1/2}.$$

Clearly we have

$$L_{(2)}^{0,s}(\hat{M}, \varphi) \cong L_{(2)}^{n,s}(\hat{M}, K_{\hat{M}}^*; \hat{\omega}, h).$$

Let R be a sufficiently large number so that $\hat{M}\backslash\hat{B}(p, R) \subset M - N$, where $\hat{B}(p, R) = \{x \in \hat{M} : \hat{d}(p, x) < R\}$, $p \in \hat{M}$ being fixed, and $\hat{d}$ denotes the distance with respect to $\hat{\omega}$. If $u \in L^{0,1}_{(2)}(\hat{M},\varphi) \cap \mathrm{Dom}\,\bar{\partial} \cap \mathrm{Dom}\,\bar{\partial}^*$, let $\hat{u} = [1-\chi(\hat{d}(p_0,\cdot)/2R)]u$. By (4) and hypothesis i), we have

$$\|\bar{\partial}\hat{u}\|^2 + \|\bar{\partial}^*\hat{u}\|^2 \geq \|\hat{u}\|^2.$$

But

$$\bar{\partial}\hat{u} = [1-\chi(\hat{d}(p,\cdot)/2R)]\bar{\partial}u - \bar{\partial}\chi(\hat{d}(p,\cdot)/2R) \wedge u$$
$$\bar{\partial}^*\hat{u} = [1-\chi(\hat{d}(p,\cdot)/2R)]\bar{\partial}^*u + \bar{\partial}\chi(\hat{d}(p,\cdot)/2R) \lrcorner\, u$$

where $''\lrcorner''$ is the contraction operator, thus by the Schwarz inequality

$$\|u\|^2 \leq 2(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) + \mathrm{const.}_R \int\limits_{\hat{B}(p,2R)} |u|^2_{\hat{\omega}} e^{-\varphi} dV_\omega.$$

It follows from Proposition 5.2 that $l := \dim H^{0,1}_{(2)}(\hat{M},\varphi) < \infty$.

Now consider linear independent $(0,1)-$forms

$$\eta_j = \sum_{\nu=1}^{\infty} (2 + z_1^\nu/r_\nu)^{j\nu} \bar{\partial}\chi(2|z^\nu|^2/r_\nu^2), \quad 0 \leq j \leq l.$$

By hypothesis ii), each $\eta_j \in L^{0,1}(\hat{M},\varphi) \cap \mathrm{Ker}\,\bar{\partial}$. Thus there are non-all zero complex numbers $c_0, c_1, \ldots, c_l$ such that the equation $\bar{\partial}u = \sum_{j=0}^{l} c_j\eta_j$ has a solution $u \in L^{0,0}_{(2)}(\hat{M},\varphi)$. Without loss of generality, we may assume that $c_l = 1$. Thus the function

$$f = \sum_{j=0}^{l}\sum_{\nu=1}^{\infty} c_j (2 + z_1^\nu/r_\nu)^{j\nu} \chi(2|z^\nu|^2/r_\nu^2) - u$$

is holomorphic on $\hat{M}$. Since f is square-integrable on the holomorphic coordinate patches $\{z^\nu : |z^\nu| < r_\nu\}$, $\nu \geq 1$, it extends holomorphically across $\{x_\nu\}$ and we have

$$|f(x_\nu)| = \left|\sum_{j=0}^{l} c_j 2^{j\nu}\right| \geq \mathrm{const.} 2^{l\nu}, \quad \nu \gg 1.$$

□

Proof of Theorem 1.4 Since $\overline{M} = M \cup M(\infty)$ is the union of a compact set and a finite number of mutually disjoint standard cusp regions E_k, $1 \leq k \leq m$, there are subsets $Y_k \subset \overline{X}$ and parabolic subgroups $\Gamma_k \subset G$ such that Y_k is precisely invariant under Γ_k in G and $E_k = Y_k/\Gamma_k$. Let f_{q_k} be a Busemann function at $q_k \in \mathrm{Fix}(\Gamma_k)$.

Clearly, f_{q_k} descends to a C^2 function on $\mathrm{int}(E_k)$, where $\mathrm{int}(E_k) = E_k \cap M$. Let E_k' be a closed subset of E_k satisfying

(1) $E_k \backslash E_k'$ is relatively compact in $\overline{M}$,
(2) the closure of $\overline{M} \backslash E_k$ in $\overline{M}$ does not intersect E_k'.

Thus by a multiplier of certain cut-off function we get C^2 real functions ψ_k on M such that $\psi_k = f_{q_k}$ on $\mathrm{int}(E_k')$, $1 \le k \le m$. Let $d_{\mathrm{ch}(\Lambda(G))}$ be the distance to the convex hull $\mathrm{ch}(\Lambda(G))$ of the limit set $\Lambda(G)$. Then $d^2_{\mathrm{ch}(\Lambda(G))}$ is a $G-$invariant, C^2 convex (hence psh) function on X and is strictly convex (hence strictly psh) outside $\mathrm{ch}(\Lambda(G))$. Furthermore, $d^2_{\mathrm{ch}(\Lambda(G))}(x) \to +\infty$ as $x \to \Omega(G)$. To complete the proof, it suffices to apply Theorem 5.3 with

$$\rho = \lambda \circ d^2_{\mathrm{ch}(\Lambda(G))} + \sum_k \psi_k$$

where $\lambda \ge 0$ is a convex, rapidly increasing function. □

6 Proof of Theorem 1.5

The idea of the proof is based on Kohn-Rossi [20]. Without loss of generality, we assume that V does not intersect noncompact components of $M(\infty)$. Let $\psi = d^2_{\mathrm{ch}(\Lambda(G))}$ and let ρ be as in the proof of Theorem 1.4. Choose a sufficiently large number R such that the set $\{\psi < R\} \cap V$ contains a compact strong pseudoconvex boundary. Let $M_R = M - (\{\psi \ge R\} \cap V)$ and let $\phi = -\log(R - \psi)$, $\varphi = \rho + \phi$, and $\omega_R = \omega + \partial\bar\partial\phi$. Then ω_R is a complete Kähler metric on M_R and $\partial\bar\partial\varphi \ge \mathrm{const.}\omega_R$ holds outside of a compact set of M_R.

Let $L^{n,s}_{(2)}(M_R, \varphi) = L^{n,s}_{(2)}(M_R, M_R \times \mathbb{C}; \omega_R, e^{-\varphi})$ and let $\|\cdot\|_\varphi$ denote the norm. Let $H^{n,s}_{(2)}(M_R, \varphi)$ denote the corresponding L^2 cohomology group for (n, s) forms on M_R. A similar argument as § 5 shows that there is a compact set N_R of M_R such that

$$\|u\|^2_\varphi \le 2(\|\bar\partial u\|^2_\varphi + \|\bar\partial^* u\|^2_\varphi) + \mathrm{const.}_R \int_{N_R} |u|^2_{\omega_R} e^{-\varphi} dV_{\omega_R}$$

for all $u \in L^{n,s}_{(2)}(M_R, \varphi) \cap \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^*$ and $s > 0$, hence by Proposition 5.2 we have $\dim H^{n,s}_{(2)}(M_R, \varphi) < \infty$ and the Hodge decomposition

$$L^{n,s}_{(2)}(M_R, \varphi) = H^{n,s}_{(2)}(M_R, \varphi) \oplus \bar\partial\, L^{n,s-1}_{(2)}(M_R, \varphi) \oplus \bar\partial^* L^{n,s+1}_{(2)}(M_R, \varphi).$$

Let $L^{0,s}_{(2)}(M_R, -\varphi) = L^{0,s}_{(2)}(M_R, M_R \times \mathbb{C}; \omega_R, e^{\varphi})$ and let $L^{0,s}_0(M_R, -\varphi)$ denote the space of $(0, s)$ forms $\eta \in L^{0,s}_{(2)}(M_R, -\varphi)$ satisfying $\mathrm{supp}\,\eta \cap \partial M_R \cap V = \emptyset$. Let

$$H^{0,s}_0(M_R, -\varphi) = \frac{\left\{\eta \in L^{0,s}_0(M_R, -\varphi) : \eta \text{ is } C^\infty \text{ and } \bar\partial\eta = 0\right\}}{\left\{\eta \in L^{0,s}_0(M_R, -\varphi) : \eta = \bar\partial\xi,\ \xi \text{ is } C^\infty \text{ and lies in } L^{0,s-1}_0(M_R, -\varphi)\right\}}.$$

Proposition 6.1 $\dim H_0^{0,1}(M_R, -\varphi) \le \dim H_{(2)}^{n,n-1}(M_R, \varphi)$.

Proof Let $u \in L_{(2)}^{n,n-1}(M_R, \varphi)$, $\bar\partial u = 0$, and $\eta \in L_0^{0,1}(M_R, -\varphi)$, $\bar\partial\eta = 0$. Put

$$L_\eta(u) = \int_{M_R} u \wedge \eta = (u, e^\varphi * \bar\eta)_\varphi,$$

where $*$ is the Hodge star operator with respect to ω_R. This depends only on the cohomology classes of u and η for, if $u = \bar\partial w$, then

$$L_\eta(u) = \int_{M_R} \bar\partial(w \wedge \eta) = \int_{M_R} d(w \wedge \eta) = 0$$

by the Gaffney L_1–lemma. If $\eta = \bar\partial\xi$, where $\xi \in L_0^{0,0}(M_R, -\varphi)$, then similar as above,

$$L_\eta(u) = \int_{M_R} d(u \wedge \xi) = 0.$$

Thus the correspondence $\eta \to L_\eta$ induces a homomorphism from $H_0^{0,1}(M_R, -\varphi)$ to the dual space $\left(H_{(2)}^{n,n-1}(M_R, \varphi)\right)^*$ of $H_{(2)}^{n,n-1}(M_R, \varphi)$. It suffices to show that this homomorphism is injective. Suppose that $\eta \in L_0^{0,1}(M_R, -\varphi)$, $\bar\partial\eta = 0$, and $L_\eta = 0$. Then $e^\varphi * \bar\eta \in L_{(2)}^{n,n-1}(M_R, \varphi)$ and $e^\varphi * \bar\eta \perp H_{(2)}^{n,n-1}(M_R, \varphi)$. Since $\bar\partial^* = -e^\varphi * \partial(* e^{-\varphi})$, we have

$$\bar\partial^* \left(e^\varphi * \bar\eta\right) = -e^\varphi * \partial * * \bar\eta = e^\varphi * \partial\bar\eta = 0.$$

Hence by the Hodge decomposition, there are $\xi \in L_{(2)}^{n,n}(M_R, \varphi)$ and $\varsigma \in L_{(2)}^{n,n-2}(M_R, \varphi)$ such that $e^\varphi * \bar\eta = \bar\partial^*\xi + \bar\partial\varsigma$. But

$$\|\bar\partial\varsigma\|_\varphi^2 = (\bar\partial^*\bar\partial\varsigma, \varsigma)_\varphi = 0 \Rightarrow \bar\partial\varsigma = 0,$$

hence $e^\varphi * \bar\eta = \bar\partial^*\xi$. Furthermore, ξ is C^∞ if η is, by the elliptic operator theory. Then $\eta = \bar\partial\zeta$ where $\zeta = e^{-\varphi}\overline{*\xi}$. Since $\eta = 0$ in a neighborhood V_1 of $\partial M_R \cap V$, ζ is holomorphic on $V_1 \cap M_R$. Thanks to the lemma below, $\zeta = 0$ on V_1. Thus $\zeta \in L_0^{0,0}(M_R, -\varphi)$ and the cohomology class of η in $H_0^{0,1}(M_R, -\varphi)$ is zero, completing the proof. □

Lemma 6.2 *If f lies in $L_{(2)}^{0,0}(M_R, -\varphi)$ and is holomorphic in $V_1 \cap M_R$, then $f = 0$ on $V_1 \cap M_R$.*

Proof Notice first the ω_R is equivalent to the Bergman metric of M_R near $\partial M_R \cap V$. Thus for any sufficiently small ε, there is a number r_ε with $r_\varepsilon \to \infty$ as $\varepsilon \to 0$, such that for every $x \in \partial M_{R-\varepsilon}$, the geodesic ball $B(x, r_\varepsilon)$ with respect to the Bergman metric is contained in $V_1 \cap M_R$. Notice that the Bergman metric on M_R and its curvature form approaches near $\partial M_R \cap V$ to those of the Bergman metric on the unit ball (cf. [12,19]). Thus the mean value inequality of Li-Schoen (cf. [21], Theorem 1.2) applies to the psh function $|f|^2 e^\varphi$ on $B(x, r_\varepsilon)$ showing that

$$|f(x)|^2 e^{\varphi(x)} \leq \frac{1}{\mathrm{vol}\,(B(x, r_\varepsilon))} \int\limits_{B(x,r_\varepsilon)} |f|^2 e^\varphi dV \leq \text{const.}\, \|f\|^2_{-\varphi} e^{-cr_\varepsilon}$$

for some positive number c. Since $\varphi(x) \geq$ const.,

$$|f(x)|^2 \leq \text{const.}\, \|f\|^2_{-\varphi} e^{-cr_\varepsilon}.$$

Fix any point $y_0 \in \partial M_R \cap V$. We may take a local holomorphic coordinate centered at y_0 such that ∂M_R is strongly convex near y_0. Then a sufficiently small part of M_R near y_0 can be foliated by complex hyperplanes which intersects $\partial M_R \cap V$ transversely. By the maximal principle, $f = 0$ on every such hyperplance. Hence $f = 0$ on $V_1 \cap M_R$ by the Identity Theorem for holomorphic functions. □

Proof of Theorem 1.5 We follow the standard argument in [20]. Let f be a nonconstant holomorphic function on $V \cap M$. Choose a C^∞ function $\kappa : M \to [0, 1]$ which equals to 1 on $M \backslash V$ and $\operatorname{supp}\kappa \cap (\partial M_R \cap V) = \emptyset$. Let $u_0, \ldots, u_{d_0}$ be a basis for $H^{n,n-1}_{(2)}(M_R, \varphi)$, and let

$$c_{ij} = \int\limits_{M_R} f^j \bar{\partial}\kappa \wedge u_i, \qquad 0 \leq j \leq d_0,\ 1 \leq i \leq d_0.$$

Let $a_j, 0 \leq j \leq d_0$ be non-all zero complex numbers such that $\sum_{j=0}^{d_0} a_j c_{i,j} = 0$ for all $1 \leq i \leq d_0$. Thus $-\left(\sum_{j=0}^{d_0} a_j f^j\right) \bar{\partial}\kappa$ is a $\bar{\partial}-$closed C^∞ form in $L^{0,1}_0(M_R, -\varphi)$ which is orthogonal to $H^{n,n-1}_{(2)}(M_R, \varphi)$, and similar as in the proof of Proposition 6.1 there is a function $u \in L^{0,0}_0(M_R, -\varphi)$ satisfying $\bar{\partial}u = -\left(\sum_{j=0}^{d_0} a_j f^j\right) \bar{\partial}\kappa$. Thus we get a holomorphic function given by $\Phi = \left(\sum_{j=0}^{d_0} a_j f^j\right)(1-\kappa) - u$ on M_R which coincides with $\sum_j a_j f^j$ at $\partial M_R \cap V$.

On the other hand, by Theorem 1.4, there is a non-constant holomorphic function h on M. Since $\dim H^{0,1}_0(M_R, -\varphi) \leq d_0$ by Proposition 6.1, there exist non-all zero complex numbers $c_0, c_1, \cdots, c_{d_0}$, such that there is a function $u_1 \in L^{0,0}_0(M_R, -\varphi)$ satisfying $\bar{\partial}u_1 = -\left(\sum_{j=0}^{d_0} c_j h^j\right) f \bar{\partial}\kappa$. Then $F = \left(\sum_{j=0}^{d_0} c_j h^j\right)(1-\kappa) f - u_1$ is holomorphic on M_R. Let $H = \sum_{j=0}^{d_0} c_j h^j$. Since h is non-constant, it takes infinitely many values, so H is not identically zero. The function F/H is holomorphic on $M_R - \{H = 0\}$ and equals to f on $(M_R \cap V) - \{H = 0\}$.

Since $\sum_{j=0}^{d_0} a_j (F/H)^j = \Phi$ in $(\partial M_R \cap V) - \{H = 0\}$, the identity also holds on the whole $M_R - \{H = 0\}$, thanks to the Identity Theorem for holomorphic functions. Thus F/H is locally bounded on M_R and thus holomorphic on M_R by the Riemann extension theorem. Thus F/H is the desired extension of f.

Suppose that $M(\infty)$ has m connected compact components $N_1, N_2, \ldots, N_m$ with $m \geq 2$. Let $V_1, V_2, \ldots, V_m$ be disjoint neighborhoods of $N_1, N_2, \ldots, N_m$ in $\overline{M}$. Let f be a function which equals 1 on V_1 and equals 0 on V_j for $j \geq 2$. Then f may be extended holomorphically to M, contradicts with the Identity Theorem for holomorphic functions. □

7 Quotients of Siegel domains

Let us recall the following

Definition 7.1 Let $V \subset \mathbb{R}^n$ be an open convex cone contains no straight lines. A map $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ is said to be an $V-$hermitian form if it satisfies the following conditions

(i) For each $z' \in \mathbb{C}^m$, the map $F_{z'} : \mathbb{C}^m \to \mathbb{C}^n$ defined by $F_{z'}(z) = F(z, z')$ is complex linear.
(ii) $\overline{F(z, z')} = F(z', z)$.
(iii) $F(z, z) \in \overline{V}$: the closure of V, for all $z \in \mathbb{C}^m$.
(iv) $F(z, z) = 0$ if and only if $z = 0$.

Definition 7.2 Let $V \subset \mathbb{R}^n$ be an open convex cone contains no straight lines and $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ be an $V-$hermitian form. Then the set

$$D = D(V, F) = \{(z, w) \in \mathbb{C}^m \times \mathbb{C}^n : \mathrm{Im}\, w - F(z, z) \in V\}$$

is called a *Siegel domain* (of the second kind) associated to the pair (V, F).

Lemma 7.3 *Let K_D be the Bergman kernel of D. Then*

(1) $\log K_D$ *approaches* ∞ *at the boundary of* D.
(2) $\mathrm{Ric}\, \partial\bar{\partial} \log K_D \geq -\mathrm{const.}\, \partial\bar{\partial} \log K_D$.

Proof Since every Siegel domain is a convex domain containing no complex lines, thus according to Nikolov–Pflug–Zwonek [27] there is, for each $p \in D$, an embedded polydisc P centered at p such that K_D is uniformly comparable to K_P inside a slightly smaller polydisc. In particular, (1) holds. It is not difficult to see that the technique in [27] is also valid for the Bergman metric and its curvature tensors, from which (2) immediately follows (the case when D is affinely homogeneous is trivial since the Bergman metric is Kähler–Einstein). □

A Siegel domain D has an affine automorphism group $\mathfrak{A}(D)$ which consists of the following linear transformations (cf. [29], p. 25):

$$(z, w) \mapsto (Bz + \eta, Aw + \xi + 2iF(Bz, \eta) + iF(\eta, \eta))$$

where $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{C}^m$, A is a linear transformation of V onto itself, and B is a complex linear transformation satisfying $AF(z, z') = F(Bz, Bz')$ for all $z, z' \in \mathbb{C}^m$. Let $\mathfrak{A}_1(D) = \{g \in \mathfrak{A}(D) : |j_g| \equiv 1\}$ where j_g denotes the complex Jacobian of g, and let $\mathfrak{P}(D)$ be the subgroup of $\mathfrak{A}_1(D)$ which consists of the following parallel translations

$$g_{\xi,\eta}(z, w) = (z + \eta, w + \xi + 2iF(z, \eta) + iF(\eta, \eta)), \quad \xi \in \mathbb{R}^n, \ \eta \in \mathbb{C}^m.$$

There is a natural isomorphism $\mathcal{I}$ from $\mathfrak{P}(D)$ to Isom $(\mathbb{R}^n \times \mathbb{C}^m)$ defined as

$$\mathcal{I} : g_{\xi,\eta} \mapsto \text{identity} + (\xi, \eta).$$

Proof of Theorem 1.7 It is well-known that the Bergman metric is complete on D, thus D/G is a complete Kähler manifold. Since

$$K_D(p) = K_D(g(p))|j_g(p)|^2, \quad g \in \mathrm{Aut}(D),$$

we conclude that $\log K_D$ is $\mathfrak{A}_1(D)-$invariant. Thus by virtue of Lemma 7.3, in order to get the first assertion it suffices to apply Proposition 4.1 with $\rho = C \log K_D$ where C is a sufficiently large constant.

Now suppose that G is a totally real, torsion-free, discrete group. By the celebrated Bieberbach theorem, $\mathcal{I}(G)$ is an Abelian group generated by a finite number of $\mathbb{R}-$linear independent real vectors $(\xi_k, \eta_k) \in \mathbb{R}^n \times \mathbb{R}^m$, $1 \le k \le k_0$. It turns out that G is generated by a finite number of parallel translations $\{g_{\xi_k,\eta_k}\}$ Without loss of generality, we assume that the cone V is contained in the positive octant $\mathbb{R}^n_+$. Thus we have $\mathrm{Im}\, w_j > 0$ for any $1 \le j \le n$ and for all $(z, w) \in D$. Put

$$\Psi(z, w) = \mathrm{Im}\,(w - iF(z, \bar{z})).$$

Since

$$\begin{aligned}\Psi \circ g_{\xi_k,\eta_k}(z, w) &= \mathrm{Im}\,(w + \xi_k + 2iF(z, \eta_k) + iF(\eta_k, \eta_k) - iF(z + \eta_k, \bar{z} + \eta_k)) \\ &= \Psi(z, w)\end{aligned}$$

for all k, we conclude that

$$\psi(z, w) = \sum_{j=1}^{n} \mathrm{Im}\,(w_j - iF_j(z, \bar{z}))$$

where $F = (F_1, \ldots, F_n)$, is a $G-$invariant psh function on D.

Let E_1 be the complex affine subspace of $\mathbb{C}^n$ spanned by $\xi_1, \ldots, \xi_{k_0}$ and $J\xi_1, \ldots, J\xi_{k_0}$, and let E_2 be the real affine subspace of $\mathbb{R}^m$ spanned by $\eta_1, \ldots, \eta_{k_0}$. Let $d_{\mathrm{eucl}}(\cdot, E_1)$ and $d_{\mathrm{eucl}}(\cdot, E_2)$ denote the Euclidean distance to $E_1 \subset \mathbb{C}^n$ and to $E_2 \subset \mathbb{C}^m$

respectively. Since E_2 is $\langle \eta_1, \ldots, \eta_{k_0} \rangle-$period, we may choose a convex, rapidly increasing function $\lambda \geq 0$ such that the $G-$invariant function

$$K_D((z, w)) + d^2_{\text{eucl}}(w, E_1) + \psi(z, w) + \lambda \circ d^2_{\text{eucl}}(z, E_2)$$

gives a strictly psh exhaustion function of D/G. □

Remark 7.4 The construction of ψ is somewhat inspired by [11].

The group $\mathfrak{A}(D)$ also contains *dilations*

$$g_t(z, w) = (\sqrt{t}z, tw), \quad t > 0.$$

They correspond to hyperbolic isometries for the case of Kähler-Hadamard manifolds.

Proposition 7.5 *For every dilation g_t, $t \neq 1$, $D/\langle g_t \rangle$ is a Stein manifold.*

Proof Without loss of generality, we assume that $t > 1$. Let

$$\psi = \log K_D(z) + (m + 2n) \log |w_1|.$$

Clearly, $\psi \circ g_t = \psi$ because $K_D = (K_D \circ g_t)t^{m+2n}$. Notice that there is a fundamental domain of G on which we have $1 \leq |w_j| \leq t^{1/2}$, $1 \leq j \leq n$, thus ψ is a strictly psh exhaustion function on the quotient $D/\langle g_t \rangle$. □

Proof of Corollary 1.8. Following Mok [25], p. 9, we consider two Heisenberg translations

$$g_j(z, w) = \left(z + \eta_j, w + \xi_j + 2iz \cdot \bar{\eta}_j + i|\eta_j|^2\right), \quad j = 1, 2.$$

From

$$\begin{aligned} &g_k \circ g_j(z, w) \\ &\quad = \left(z + \eta_k + \eta_j, w + (\xi_k + \xi_j) + 2iz \cdot (\bar{\eta}_k + \bar{\eta}_j) + 2i\eta_j \cdot \bar{\eta}_k + i(|\eta_k|^2 + |\eta_j|^2)\right) \end{aligned}$$

we learn that g_1 commutes with g_2 if and only if $\eta_1 \cdot \bar{\eta}_2 = \eta_2 \cdot \bar{\eta}_1$, in other words, $\eta_2 = r\eta_1 + \zeta_2$ for some real number r and for some ζ_2 orthogonal to η_1. Now G is Abelian, thus we may make a unitary transformation in z such that every $g \in G$ is of the form

$$g(z, w) = \left(z + \eta, w + \xi + 2iz \cdot \eta + i|\eta|^2\right), \quad \xi \in \mathbb{R}, \ \eta \in \mathbb{R}^{n-1}.$$

Thus the assertion follows immediately from Theorem 1.7. □

Remark 7.6 It remains open whether $\mathbb{H}^n_{\mathbb{C}}/G$ is always non-Stein for every *non-Abelian* discrete group of Heisenberg translations.

8 Proofs of Proposition 1.9, Theorem 1.10

Let Δ be the unit disc in $\mathbb{C}$ and let ds^2_{Poin} denote the Poincaré metric of constant negative curvature -1. The Poincaré distance is defined as

$$d_{\text{Poin}}(w_1, w_2) = \frac{1}{2}\log\frac{1+\left|\frac{w_1-w_2}{1-\bar{w}_1 w_1}\right|}{1-\left|\frac{w_1-w_2}{1-\bar{w}_2 w_1}\right|}, \qquad w_1, w_2 \in \Delta.$$

Let X be a complex manifold and let $\mathcal{O}(X, \Delta)$ denote the set of holomorphic mappings from X to Δ. The *Carathéodory pseudodistance* between two points $x, y \in X$ is defined as

$$c_X(x, y) = \sup\left\{d_{\text{Poin}}(f(x), f(y)) : f \in \mathcal{O}(X, \Delta)\right\}.$$

A manifold X is said to be *Carathéodory hyperbolic* if bounded functions on X separate points and give local coordinates.

Lemma 8.1 *$d^2_{\text{Poin}}(w_1, w_2)$ is convex on $\Delta \times \Delta$ and strictly convex on $\Delta \times \Delta \backslash \{(w_1, w_2) : w_1 = w_2\}$.*

Proof The argument is standard (compare [17], p. 205). For the sake of completeness, we still include the proof here. Since every geodesic $\alpha(t)$ in $\Delta \times \Delta$ is given as $(\alpha_1(t), \alpha_2(t))$ where α_1, α_2 are geodesics in Δ, it suffices to show that $\lambda(t) := d^2_{\text{Poin}}(\alpha_1(t), \alpha_2(t))$ is a convex function of t and strictly convex when $\alpha_1(t) \neq \alpha_2(t)$. For each t with $\alpha_1(t) \neq \alpha_2(t)$, let $\gamma(\cdot, t) : [0, 1] \to \Delta$ denote the geodesic from $\alpha_1(t)$ to $\alpha_2(t)$. Then

$$\lambda(t) = 2\int_0^1 \left|\frac{\partial\gamma}{\partial s}\right|^2 ds$$

and the second variation theorem (cf. [17], p. 167) implies

$$\lambda''(t) = 2\left(\int_0^1 \left|D_{\frac{\partial}{\partial s}}\frac{\partial\gamma}{\partial t}\right|^2 ds - \int_0^1 \left\langle R\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right)\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right\rangle ds\right) > 0$$

since the Poincaré metric has negative curvature. $\square$

Proof of Proposition 1.9 Since pull-backs of psh functions by holomorphic mappings are still psh, we conclude from the above lemma that $d^2_{\text{Poin}}(f(x), f(g_0(x)))$ is psh on X for any $f \in \mathcal{O}(X, \Delta)$. Thus

$$c^2_X(x, g_0(x)) = \sup\left\{d^2_{\text{Poin}}(f(x), f(g_0(x))) : f \in \mathcal{O}(X, \Delta)\right\}$$

is also psh on X.

Now we show that $c_X^2(x, g_0(x))$ is nonconstant. By assumption we have $c_X(x_0, g_0(x_0)) > 0$. Thus there is a nonconstant $f \in \mathscr{O}(X, \Delta)$ such that $c_X(x_0, g_0(x_0)) = d_{\text{Poin}}(f(x_0), f(g_0(x_0)))$. Fix a point x_1 in a sufficiently small coordinate neighborhood of x_0 such that $\partial f(x_1) \neq 0$. We assume $\partial f/\partial z_1(x_1) \neq 0$ for the sake of simplicity. If $c_X^2(x, g_0(x))$ is a constant function, then the psh function $\phi(x) := d_{\text{Poin}}^2(f(x), f(g_0(x)))$ has to be constant since it attains the maximum at a point $x_0 \in X$. On the other hand, by the above lemma, we may take a holomorphic coordinate (w_1, w_2) at $(f(x_1), f(g_0(x_1))) \in \Delta \times \Delta$ such that

$$\frac{\partial^2}{\partial w_j \partial \bar{w}_k} d_{\text{Poin}}^2 = \delta_{jk} \quad \text{at} \quad (f(x_1), f(g_0(x_1)))$$

where δ_{jk} is the Kronecker delta. Thus

$$\frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1} = \left|\frac{\partial f}{\partial z_1}\right|^2 + \left|\frac{\partial f \circ g_0}{\partial z_1}\right|^2 > 0 \quad \text{at} \quad x_1,$$

contradictory. □

Proof of Theorem 1.10 By Proposition 1.9, $\psi(x) := c_X^2(x, g_0(x))$ is a nonconstant psh function on X. Since $c_X^2(g(x), g_0(g(x))) = c_X^2(g(x), g(g_0(x))) = c_X^2(x, g_0(x))$ for any $g \in G$, ψ induces a nonconstant psh function on $M = X/G$. If M is compact, then ψ has to be constant by the maximum principle of psh functions, contradictory. If M admits a compactification M' such that M' is a complex space and $M' - M$ is a complex-analytic variety of codimension ≥ 2 in M', then ψ is also constant by an extension theorem of Grauert-Remmert [14], contradictory. □

Remark 8.2 It seems that the assumption of codim $(M' - M) \geq 2$ is superfluous. Lin [22] has proved that this is true under the additional hypothesis that G is amenable.

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