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Estimate and monotonicity of the first eigenvalue under the Ricci flow

Xiaodong Cao · Songbo Hou · Jun Ling

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Abstract In this paper, we first derive a monotonicity formula for the first eigenvalue of $-\Delta + aR$ $(0 < a \leq 1/2)$ on a closed surface with nonnegative scalar curvature under the (unnormalized) Ricci flow. We then derive a general evolution formula for the first eigenvalue under the normalized Ricci flow. As an application, we obtain various monotonicity formulae and estimates for the first eigenvalue on closed surfaces.

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1 Introduction

Let M ba a n-dimensional closed Riemannian manifold, for any Riemannian metric g, in a given local coordinates $\{x^1, x^2, \ldots, x^n\}$, the Laplace-Beltrami operator Δ is defined by

$$\Delta = \frac{1}{\sqrt{G}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{G} \cdot g^{ij} \frac{\partial}{\partial x^i} \right),$$

X. Cao (✉)
Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA
e-mail: cao@math.cornell.edu

S. Hou
Department of Applied Mathematics, College of Science, East Campus, China Agricultural University, Beijing 100083, People's Republic of China
e-mail: housb10@163.com

J. Ling
Department of Mathematics, Utah Valley University, Orem, UT 84058, USA
e-mail: lingju@uvu.edu

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $(g^{ij}) = (g_{ij})^{-1}$ and $G = \det(g_{ij})$. Since Δ is a self-adjoint elliptic operator, it is well-known that Δ has discrete eigenvalues:

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_n \le \cdots \to \infty.$$

The study of the eigenvalues of the Laplace-Beltrami on a Riemannian manifold has been an active subject of the last 40 years. In particularly, the estimates for lower bounds and upper bounds of the first eigenvalue λ_1 has been an appealing topic for spectral geometry. We refer the readers to [1,5] and [24, Chapter 3] for more details.

The Ricci flow is first introduced by Hamilton [10] to study the geometry of positive Ricci curvature on 3-manifolds. Given a 1-parameter family of Riemannian metrics $g(t)$ on M, defined on a time interval $[0, T)$, Hamilton's Ricci flow is

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc}, \tag{1.1}$$

where Rc is the Ricci curvature of metric g. The Ricci flow has been a powerful tool in studying the geometry and topology for lower dimensional manifolds. In dimension 2, it is proved that starting with any initial metric, the solution to the Ricci flow converges to a constant curvature metric, only depends on the Euler characteristic class of the surface, by Hamilton [11] and Chow [6]. In dimension 3, Hamilton [10] proves that if the initial metric has positive Ricci curvature, then the solution to the Ricci flow converges to the constant curvature (round) metric on a 3-sphere. In a sequence of papers [20,22] and [21], Perelman follows Hamilton's program using Ricci flow with surgery and proves the Poincaré conjecture and Thurston's geometrization conjecture.

Let $(M, g(t))$, $t \in [0, T)$, be a smooth solution to the Ricci flow on a closed Riemannian manifold M. There has been increasing attentions on the study of eigenvalues of geometric operators under the Ricci flow, which plays an important role in understanding geometry and topology of the manifold itself. Perelman [20] studies the first eigenvalue of the Laplace-Beltrami operator with potential R, here R (scalar curvature of metric g) is trace of the Ricci curvature. More precisely, Perelman proves that the first eigenvalue of $-4\Delta + R$ is nondecreasing along the Ricci flow. As an application, he shows that there is no nontrivial steady or expanding breathers on closed manifolds. The eigenvalues of the operator $-4\Delta + R$ also depend on the curvature, even those of the operator Δ are defined independently from curvature. In general, the eigenvalues λ_k are no longer differentiable in time t (for example, cf. [9]). But for the first eigenvalue and first eigenfunction, following the eigenvalue perturbation theory, one can always assume that the first eigenvalue $\lambda(t)$ and first eigenfunction $u(x, t)$ are smooth (in t) along the Ricci flow (cf. [7,12,23]).

The first author [2] proves that the eigenvalues of $-\Delta + \frac{1}{2}R$ are nondecreasing along the Ricci flow on manifolds with nonnegative curvature operator. In [14], Li uses the same technique to show that the monotonicity of the first eigenvalue of $-\Delta + \frac{1}{2}R$ along the Ricci flow without assuming nonnegative curvature operator. A similar result in the physics literature has been given by [19]. For the Laplace-Beltrami operator, Ma [18] derives monotonicity formula of the first eigenvalue on domains with Dirichlet boundary condition along the Ricci flow. In fact, following a remark in [6, Sec. 3], one can see that the determinant of Laplace-Beltrami operator is nondecreasing along normalized Ricci flow (also see [13]). The third author [17] proves a comparison theorem of Faber-Krahn type and obtains a sharp bound for the first eigenvalue of the Laplace-Beltrami operator under the normalized Ricci flow. In [16], the third author shows that, although the first eigenvalue of the Laplace-Beltrami operator is not monotonic along the normalized Ricci flow, an appropriate multiple is monotonic. In [3], the first author derives the evolution formula of the first eigenvalues of $-\Delta + aR$ ($a \geq \frac{1}{4}$) and shows that they are nondecreasing along Ricci flow. This is also proved by Li [14, Theorem 5.2] using various entropy functionals. Along normalized Ricci flow, the first author [3] proves a monotonicity result for the case $a = \frac{1}{4}$ and non-positive average scalar curvature. Besides the study of the evolution of eigenvalues under the Ricci flow, Chang and Lu [4] investigate the behavior of the Yamabe constant along with the Ricci flow.

Most of the work mentioned above deals with the case of (unnormalized) Ricci flow, while [3,13,15–17] deals with normalized Ricci flow. Notice that the eigenvalues of both Δ and $-4\Delta + R$ change under scaling by $\lambda(cg) = \frac{1}{c}\lambda(g)$, $\forall c > 0$, this motivates us to consider the evolution of the first eigenvalue under the normalized Ricci flow,

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc} + \frac{2r}{n} g, \tag{1.2}$$

where $r = \int_M R d\mu / \int_M d\mu$ is the average of scalar curvature R. Under the normalized Ricci flow, the volume of $(M, g(t))$ is a constant for all $t \in [0, T)$.

The rest of this paper is organized as follows. In Sect. 2, we first derive the monotonicity formula of the first eigenvalue of geometric operator $-\Delta + aR$ ($0 < a \leq \frac{1}{2}$) along the Ricci flow on a closed surface with nonnegative scalar curvature. In Sect. 3, we derive a general evolution formula for the first eigenvalue under the normalized Ricci flow for general dimensions. Then we obtain various monotonicity formulae and bounds on the first eigenvalue on closed surfaces.

2 First eigenvalues on surfaces with nonnegative scalar curvature

In this section, we study the first eigenvalues of operators $-\Delta + aR$ ($0 < a \leq \frac{1}{2}$) on closed surfaces with nonnegative scalar curvature. In [3], the first author proved that for any $a \geq \frac{1}{4}$, the first eigenvalues of $-\Delta + aR$ are nondecreasing. When the dimension $n = 2$, we have $\mathrm{Rc} = \frac{1}{2}Rg$. Then the Ricci flow becomes a conformal flow (see [11] and [6])

$$\frac{\partial}{\partial t} g = -Rg. \tag{2.1}$$

The evolution equation of the scalar curvature R is

$$\frac{\partial}{\partial t} R = \Delta R + R^2,$$

it follows that nonnegative scalar curvature is preserved by the Ricci flow. The volume form $d\mu$ satisfies

$$\frac{\partial}{\partial t} d\mu = -R d\mu.$$

Let $\lambda(t)$ be the first eigenvalue of $-\Delta + aR$ $(0 < a \leq \frac{1}{2})$, and $u = u(x,t)$ satisfies

$$-\Delta u(x,t) + aRu(x,t) = \lambda(t)u(x,t),$$

with $\int_M u^2(x,t)d\mu = 1$. By a similar analysis as in [2] and [3], we have the following theorem.

Theorem 2.1 *Let* $(M^2, g(t)), t \in [0,T)$*, be a solution of the Ricci flow on a closed surface* M^2 *with nonnegative scalar curvature. Let* $\lambda(t)$ *and* $u(x,t)$ *be defined as above,* $0 < a \leq \frac{1}{2}$*, then we have*

$$\frac{d}{dt}\lambda(t) = 2a^2 \int R^2u^2 d\mu + (1-2a)\lambda \int Ru^2 d\mu + 2a \int R|\nabla u|^2 d\mu \geq 0.$$

Proof By a direct calculation as in [2], we have

$$\frac{d\lambda}{dt} = -2\int R_{ij}u_{ij}ud\mu + a\int \frac{\partial R}{\partial t}u^2 d\mu.$$

On a surface, $\frac{\partial R}{\partial t} = \Delta R + R^2$, hence it leads to

$$\frac{d\lambda}{dt} = -2\int R_{ij}u_{ij}ud\mu + a\int \Delta R u^2 d\mu + 2a\int |Rc|^2 u^2 d\mu.$$

Using $\Delta R = 2\,\mathrm{div}(\mathrm{div}\,\mathrm{Rc})$, we obtain the desired equality,

$$\begin{aligned}
\frac{d\lambda}{dt} &= -2\int R_{ij}u_{ij}ud\mu + 4a\int R_{ij}(u_iu_j + u_{ij}u)d\mu + 2a\int |Rc|^2u^2 d\mu \\
&= (4a-2)\int R_{ij}u_{ij}ud\mu + 4a\int R_{ij}u_iu_j d\mu + 2a\int |Rc|^2u^2 d\mu \\
&= (2a-1)\int R\Delta u u d\mu + 4a\int R_{ij}u_iu_j d\mu + 2a\int |Rc|^2u^2 d\mu \\
&= (2a-1)\int R(aRu - \lambda u)ud\mu + 4a\int R_{ij}u_iu_j d\mu + a\int R^2u^2 d\mu \\
&= 2a^2\int R^2u^2 d\mu + (1-2a)\lambda\int Ru^2 d\mu + 2a\int R|\nabla u|^2 d\mu \geq 0.
\end{aligned}$$

□

Remark 2.1 In [3], the first author proved a similar result for general dimension and $a \geq \frac{1}{4}$, without any curvature assumption, also see [14] for a different approach by constructing entropy functionals.

Combining this result with Theorem 1.5 in [3], we obtain the following result,

Theorem 2.2 *On a closed surface with nonnegative scalar curvature, $\forall a > 0$, the first eigenvalue of $-\Delta + aR$ is nondecreasing under the Ricci flow.*

3 Monotonic quantities along the normalized Ricci flow

In this section, we derive some monotonic quantities along the normalized Ricci flow (1.2). When $n = 2$, by Gauss-Bonnet Theorem, the average scalar curvature

$$r = \frac{\int_M R d\mu}{\int_M d\mu}$$

is a constant. The normalized Ricci flow again becomes a conformal flow

$$\frac{\partial}{\partial t} g = -(R - r)g. \tag{3.1}$$

As before, we use Δ to denote the Laplace-Beltrami operator and $d\mu$ to denote the volume element of metric $g(t)$. Let $\lambda = \lambda(t)$ be the first eigenvalue of $-\Delta + aR$ ($a \in (0, 1/2]$) and $u = u(x, t)$ be the eigenfunction of λ, with $\int_M u^2(x, t) d\mu = 1$, i.e., we have

$$-\Delta u + aRu = \lambda u. \tag{3.2}$$

We first derive a general evolution equation for λ under the normalized Ricci flow.

Lemma 3.1 *Let $(M^n, g(t)), t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed Riemannian manifold M^n. Let $\lambda(t)$ and $u = u(x, t)$ be defined as above, $0 < a \leq \frac{1}{2}$, then λ evolves by*

$$\begin{aligned} \frac{d}{dt}\lambda = &\left(\int \left[(1 - 2a)R - \frac{2}{n} r \right] u^2 d\mu \right) \lambda + a(2a - 1) \int R^2 u^2 d\mu \\ &+ (2a - 1) \int R|\nabla u|^2 d\mu + 2a \int |\mathrm{Rc}|^2 u^2 d\mu + \int 2R_{ij} \nabla_i u \nabla_j u. \end{aligned} \tag{3.3}$$

Proof By taking derivative of (3.2), we derive that

$$\begin{aligned} \frac{d}{dt}\lambda &= -\int \left(2uR_{ij} \nabla_i \nabla_j u - \frac{2r}{n} u \Delta u \right) d\mu + \int au^2 \left(\Delta R + 2|\mathrm{Rc}|^2 - \frac{2}{n} rR \right) d\mu \\ &= -\int 2uR_{ij} \nabla_i \nabla_j u d\mu - \int \frac{2r}{n} (\lambda - aR) u^2 d\mu \\ &\quad + \int au^2 \Delta R d\mu + \int 2au^2 |\mathrm{Rc}|^2 d\mu - \int \frac{2}{n} rau^2 R d\mu. \end{aligned}$$

Using integration by parts and contracted Bianchi identity, we have

$$-\int 2uR_{ij}\nabla_i\nabla_j u = \int (2u\nabla_i R_{ij})\nabla_j u + \int 2R_{ij}\nabla_i u\nabla_j u,$$

and

$$\begin{aligned}\int (2\nabla_i R_{ij})u\nabla_j u &= \int u(\nabla_j R)\nabla_j u = -\int Ru\Delta u - \int R|\nabla u|^2\\ &= \int (\lambda - aR)Ru^2 - \int R|\nabla u|^2 = \lambda\int Ru^2 - \int aR^2u^2 - \int R|\nabla u|^2,\end{aligned}$$

we arrive at

$$-\int 2uR_{ij}\nabla_i\nabla_j u = \lambda\int Ru^2 - \int aR^2u^2 - \int R|\nabla u|^2 + \int 2R_{ij}\nabla_i u\nabla_j u.$$

It is easy to see that

$$\begin{aligned}-\int \frac{2r}{n}(\lambda - aR)u^2 d\mu &= -\int \frac{2r}{n}\lambda u^2 + \int \frac{2r}{n}aRu^2 d\mu\\ &= -\frac{2r}{n}\lambda\int u^2 d\mu + \frac{2r}{n}a\int Ru^2 d\mu,\end{aligned}$$

and

$$\begin{aligned}\int au^2\Delta R &= \int aR(2u\Delta u + 2|\nabla u|^2)\\ &= -2a\lambda\int Ru^2 d\mu + \int (2a^2R^2u^2 + 2aR|\nabla u|^2)d\mu.\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{d}{dt}\lambda &= \lambda\int Ru^2 - \int aR^2u^2 - \int R|\nabla u|^2 + \int 2R_{ij}\nabla_i u\nabla_j u - \frac{2r}{n}\lambda\\ &\quad - 2a\lambda\int Ru^2 d\mu + \int (2a^2R^2u^2 + 2aR|\nabla u|^2)d\mu + \int 2au^2|\mathrm{Rc}|^2 d\mu\\ &= \lambda\int\left[(1-2a)R - \frac{2}{n}r\right]u^2 d\mu + a(2a-1)\int R^2u^2 d\mu + (2a-1)\\ &\quad \times\int R|\nabla u|^2 d\mu + 2a\int |\mathrm{Rc}|^2u^2 d\mu + \int 2R_{ij}\nabla_i u\nabla_j u.\end{aligned}$$

□

As a consequence of the above lemma, we have the following result on a closed surface with nonnegative scalar curvature.

Theorem 3.1 *Let $(M^2, g(t))$, $t \in [0, T)$, be a solution to the normalized Ricci flow on a closed surface M^2 with nonnegative scalar curvature. $\lambda(t)$ is the first eigenvalue of $-\Delta + aR$, $(0 < a \leq \frac{1}{2})$. Then $e^{rt}\lambda$ is nondecreasing under the normalized Ricci flow.*

Proof Since $R \geq 0$ is preserved by the normalized Ricci flow, it is easy to see that $\lambda \geq 0$. From (3.3), we have

$$\frac{d}{dt}\lambda = -\lambda r + 2a^2 \int R^2 u^2 d\mu + \lambda(1-2a)\int Ru^2 d\mu + 2a\int R|\nabla u|^2 d\mu.$$

Hence we obtain $\frac{d}{dt}(e^{rt}\lambda) \geq 0$, i.e., $e^{rt}\lambda$ is nondecreasing. □

Remark 3.1 In [3], the first author proved a similar result for $a \geq \frac{1}{4}$, but without any curvature assumption.

In the rest of this paper, we derive some bounds for the first eigenvalue λ on closed surfaces. Since dimension $n = 2$, the average scalar curvature r is a constant, indeed

$$r = 4\pi\chi(M)/A,$$

where $\chi(M)$ and A are the Euler class and area of M. We first deal with the case that the surface has negative Euler characteristic class.

Theorem 3.2 ($r < 0$) *Let $(M^2, g(t))$ be a solution to the normalized Ricci flow (3.1) on a compact surface with negative Euler characteristic class. $\lambda(t)$ is the first eigenvalue of $-\Delta + aR$, $(0 < a \leq \frac{1}{2})$. Let t_0 be any time such that $\sigma_0 = \max_M R|_{t=t_0} < 0$ (hence $R < 0$ for $t \geq t_0$), $\rho_0 = \min_M R|_{t=t_0}$. Then we have*

(1) *For $t \geq t_0$, the quantity*

$$c(t)\lambda(t) - \frac{2a^2 r}{\left(1-\frac{r}{\rho_0}\right)b(t)}\left[\frac{r}{\sigma_0} - 1 + \left(1-\frac{r}{\rho_0}\right)^2 e^{r(t-t_0)}\right]$$

is nonincreasing along the normalized Ricci flow. Moreover

$$\lambda(t) \leq \frac{1}{c(t)}\left\{c(t_0)\lambda(t_0) + \frac{2a^2 r}{\left(1-\frac{r}{\rho_0}\right)b(t)}\left[\frac{r}{\sigma_0} - 1 + \left(1-\frac{r}{\rho_0}\right)^2 e^{-rt_0}e^{rt}\right]\right.$$
$$\left. - \frac{2a^2 r}{\left(1-\frac{r}{\rho_0}\right)b(t_0)}\left[\frac{r}{\sigma_0} - 1 + \left(1-\frac{r}{\rho_0}\right)^2\right]\right\},$$

where $p = (\frac{r}{\rho_0} - 1)e^{-rt_0}$, $q = (\frac{r}{\sigma_0} - 1)e^{-rt_0}$, $0 < b(t) = 1 + pe^{rt} < 1$, and $c(t) = 1 + qe^{rt} > 1$.

(2) *Using the same notation as in part* (1). *Set* $t_1 = 0$ *if* $\rho_0 = \sigma_0$, *or* $t_1 = t_0 - \frac{1}{r}\ln(1 - \frac{2r}{\rho_0} + \frac{r}{\sigma_0})$ *otherwise. Then for* $t \geq \max\{t_0, t_1\}$, *the quantity*

$$b(t)\lambda(t) + \frac{2a^2 r}{\left(\frac{r}{\sigma_0} - 1\right)c(t)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{-rt_0}e^{rt}\right]$$

is nondecreasing along the normalized Ricci flow. Moreover

$$\lambda(t) \geq \frac{1}{b(t)}\Bigg\{b(t_1)\lambda(t_1) - \frac{2a^2 r}{\left(\frac{r}{\sigma_0} - 1\right)c(t)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{r(t-t_0)}\right]$$
$$+ \frac{2a^2 r}{\left(\frac{r}{\sigma_0} - 1\right)c(t_1)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{r(t_1-t_0)}\right]\Bigg\}.$$

Proof Notice that in this case, we have $\lambda < 0$. Recall that R evolves by the equation

$$\frac{\partial}{\partial t}R = \Delta R + R^2 - rR.$$

Compare R with the solution of ODE

$$\frac{d}{dt}y = y^2 - ry,$$

with the initial condition $y|_{t=t_0} = \rho_0$ and the initial condition $y|_{t=t_0} = \sigma_0$ respectively, we get

$$\frac{r}{b(t)} \leq R \leq \frac{r}{c(t)} \quad \text{for} \quad t \geq t_0. \tag{3.4}$$

From (3.2), applying the divergence theorem, we have

$$\int_M |\nabla u|^2 d\mu = \lambda - a\int_M Ru^2 d\mu. \tag{3.5}$$

Substituting this into the evolution equation (3.3), which in dimension 2 takes a simpler form

$$\frac{d}{dt}\lambda = \left\{(1-2a)\int Ru^2 d\mu - r\right\}\lambda + 2a\int R|\nabla u|^2 d\mu + 2a^2\int R^2u^2 d\mu, \tag{3.6}$$

and using (3.4), then for $t \geq t_0$ we have

$$\frac{d}{dt}\lambda \leq \left\{(1-2a)\int Ru^2 d\mu - r + \frac{2ar}{c(t)}\right\}\lambda + 2a^2\int R(R - \frac{r}{c(t)})u^2 d\mu.$$

Using (3.4) again, we have

$$0 \leq R^2 - \frac{r}{c(t)} R \leq r^2 \Big(\frac{1}{b(t)^2} - \frac{1}{b(t)c(t)} \Big).$$

Together with the facts that $\lambda < 0$ and $R \geq \frac{r}{c(t)}$, it follows that

$$\frac{d}{dt}\lambda \leq \Big(\frac{1}{c(t)} - 1 \Big) r\lambda + 2a^2 r^2 \Big(\frac{1}{b(t)^2} - \frac{1}{b(t)c(t)} \Big). \tag{3.7}$$

Multiplying the equation by $c(t) > 0$, we can rewrite the above inequality as

$$\frac{d}{dt}\Big(c(t)\lambda(t) \Big) + 2a^2 r^2 \Big(\frac{1}{b(t)} - \frac{c(t)}{b(t)^2} \Big) \leq 0,$$

or

$$\frac{d}{dt}\Big\{ c(t)\lambda(t) + \frac{2a^2 r}{pb(t)} \Big(q + p^2 e^{rt} \Big) \Big\} \leq 0.$$

This yields that

$$\frac{d}{dt}\left\{ c(t)\lambda(t) - \frac{2a^2 r}{\Big(1 - \frac{r}{\rho_0}\Big) b(t)} \left[\frac{r}{\sigma_0} - 1 + \Big(1 - \frac{r}{\rho_0} \Big)^2 e^{-rt_0} e^{rt} \right] \right\} \leq 0.$$

Hence we have

$$\lambda(t) \leq \frac{1}{c(t)} \left\{ c(t_0)\lambda(t_0) + \frac{2a^2 r}{\Big(1 - \frac{r}{\rho_0}\Big) b(t)} \left[\frac{r}{\sigma_0} - 1 + \Big(1 - \frac{r}{\rho_0} \Big)^2 e^{-rt_0} e^{rt} \right] \right.$$
$$\left. - \frac{2a^2 r}{\Big(1 - \frac{r}{\rho_0}\Big) b(t_0)} \left[\frac{r}{\sigma_0} - 1 + \Big(1 - \frac{r}{\rho_0} \Big)^2 \right] \right\}.$$

This proves (1).

To prove the second part, using (3.4), (3.5) and (3.6), we have

$$\frac{d}{dt}\lambda \geq \Big(\frac{r}{b(t)} - r \Big)\lambda + 2a^2 \int R \left(R - \frac{r}{b(t)} \right) u^2 d\mu.$$

Notice that if $\frac{r}{c(t)} \leq \frac{r}{2b(t)}$, then the minimum of the polynomial $P(R) = R^2 - \frac{r}{b(t)} R$ is achieved at the right endpoint $R = \frac{r}{c(t)}$ on the interval $[\frac{r}{b(t)}, \frac{r}{c(t)}]$.

We claim that it is indeed $\frac{r}{c(t)} \leq \frac{r}{2b(t)}$. The last inequality is reduced to

$$\left(1 - \frac{2r}{\rho_0} + \frac{r}{\sigma_0}\right) e^{r(t-t_0)} \leq 1.$$

In the case of $\rho_0 = \sigma_0$, this is automatically true for all $t \geq t_0$. In the case of $\rho_0 < \sigma_0$, which implies that $1 - \frac{2r}{\rho_0} + \frac{r}{\sigma_0} > 0$, the inequality is true for $t \geq \max\{t_0, t_0 - \frac{1}{r}\ln(1 - \frac{2r}{\rho_0} + \frac{r}{\sigma_0})\}$. So for $t \geq \max\{t_0, t_1\}$, using (3.4), we have

$$R\left(R - \frac{r}{b(t)}\right) \geq \frac{r}{c(t)}\left(\frac{r}{c(t)} - \frac{r}{b(t)}\right).$$

Therefore we arrive at

$$\frac{d}{dt}\lambda \geq \left(\frac{r}{b(t)} - r\right)\lambda + 2a^2r^2\left(\frac{1}{c(t)^2} - \frac{1}{b(t)c(t)}\right). \tag{3.8}$$

Since $b(t) > 0$, we can rewrite the above inequality as

$$\frac{d}{dt}\Big(b(t)\lambda(t)\Big) + 2a^2r^2\left(\frac{1}{c(t)} - \frac{b(t)}{c(t)^2}\right) \geq 0,$$

or

$$\frac{d}{dt}\left\{b(t)\lambda(t) + \frac{2a^2r}{qc(t)}\left(p + q^2e^{rt}\right)\right\} \geq 0.$$

This leads to

$$\frac{d}{dt}\left\{b(t)\lambda(t) + \frac{2a^2r}{\left(\frac{r}{\sigma_0} - 1\right)c(t)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{-rt_0}e^{rt}\right]\right\} \geq 0,$$

and it follows that

$$\begin{aligned}\lambda(t) \geq \frac{1}{b(t)}\Bigg\{& b(t_1)\lambda(t_1) - \frac{2a^2r}{\left(\frac{r}{\sigma_0} - 1\right)c(t)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{-rt_0}e^{rt}\right] \\ &+ \frac{2a^2r}{\left(\frac{r}{\sigma_0} - 1\right)c(t_1)}\left[-\left(1 - \frac{r}{\rho_0}\right) + \left(\frac{r}{\sigma_0} - 1\right)^2 e^{-rt_0}e^{rt_1}\right]\Bigg\}\end{aligned}$$

as desired. □

Next we deal with the case that M^2 has positive Euler characteristic class.

Theorem 3.3 ($r > 0$) *Let $(M^2, g(t))$ be a solution to the normalized Ricci flow (3.1) on a closed surface with positive Euler characteristic class. $\lambda(t)$ is the first eigenvalue of $-\Delta + aR$, $(0 < a \leq \frac{1}{2})$. Let t_0 be any time such that the scalar curvature R is positive and $\rho_0 = \min_M R|_{t=t_0} > 0$. Then the quantity*

$$b(t)\lambda(t)$$

is nondecreasing along the normalized Ricci flow for $t \geq t_0$, where

$$b(t) = 1 - \left(1 - \frac{r}{\rho_0}\right)e^{r(t-t_0)}.$$

Moreover, $\lambda(t)$ has a time dependent lower bound, i.e.,

$$\lambda(t) \geq b(t_0)\lambda(t_0)/b(t).$$

Proof Using (3.4), (3.5) and (3.6), it follows that

$$\frac{d}{dt}\lambda \geq \left(\frac{r}{b(t)} - r\right)\lambda + \frac{2a^2 \int R\left(R - \frac{r}{b(t)}\right)u^2 d\mu}{\int u^2 d\mu} \geq \left(\frac{r}{b(t)} - r\right)\lambda.$$

Hence we have that

$$\frac{d}{dt}\left[b(t)\lambda(t)\right] \geq 0,$$

and

$$\lambda(t) \geq b(t_0)\lambda(t_0)/b(t).$$

□

At last we deal with the case that the surface M^2 has vanishing Euler characteristic class.

Theorem 3.4 ($r = 0$) *Let $(M^2, g(t))$ be a solution to the normalized Ricci flow (3.1) on a closed surface with vanishing Euler characteristic class. $\lambda(t)$ is the first eigenvalue of $-\Delta + aR$, $(0 < a \leq \frac{1}{2})$. Then the quantity*

$$\left(1 - \rho_0 t\right)\lambda - \frac{a^2\rho_0}{2}\ln\left(1 - \rho_0 t\right)$$

is nondecreasing along the normalized Ricci flow. Moreover, $\lambda(t)$ has a time dependent lower bound

$$\lambda(t) \geq \frac{1}{\left(1 - \rho_0 t\right)}\lambda(0) + \frac{a^2\rho_0}{2\left(1 - \rho_0 t\right)}\ln\left(1 - \rho_0 t\right),$$

where $\rho_0 = \min_M R|_{t=0}$.

Proof For dimension 2 and Euler characteristic class $\chi = 0$, the evolution equation (3.3) becomes

$$\frac{d}{dt}\lambda = \left\{(1-2a)\int Ru^2 d\mu\right\}\lambda + 2a\int R|\nabla u|^2 d\mu + 2a^2\int R^2u^2 d\mu. \quad (3.9)$$

Now the scalar curvature R evolves under the equation

$$\frac{\partial}{\partial t}R = \Delta R + R^2.$$

We compare R with the solutions of the following ordinary differential equation

$$\frac{d}{dt}y = y^2,$$

with initial values ρ_0 and σ_0 respectively. By maximum principle (see [8]), we have

$$\frac{\rho_0}{1-\rho_0 t} \leq R \leq \frac{\sigma_0}{1-\sigma_0 t}.$$

Substituting the above estimate into (3.9), we get

$$\begin{aligned}\frac{d}{dt}\lambda &\geq \left\{(1-2a)\int Ru^2 d\mu\right\}\lambda + \frac{2a\rho_0}{1-\rho_0 t}\int |\nabla u|^2 d\mu + 2a^2\int R^2u^2 d\mu \\ &= \left\{(1-2a)\int Ru^2 d\mu\right\}\lambda + \frac{2a\rho_0}{1-\rho_0 t}\lambda - \frac{2a^2\rho_0}{1-\rho_0 t}\int Ru^2 d\mu + 2a^2\int R^2u^2 d\mu \\ &\geq \frac{\rho_0}{1-\rho_0 t}\lambda + 2a^2\int\left(R - \frac{\rho_0}{1-\rho_0 t}\right)Ru^2 d\mu,\end{aligned}$$

where we used Eq. (3.5) in the last identity. Using the fact $t^2 - kt \geq -\frac{1}{4}k^2$, we arrive at

$$\frac{d}{dt}\lambda \geq \frac{\rho_0}{1-\rho_0 t}\lambda - \frac{a^2\rho_0^2}{2\left(1-\rho_0 t\right)^2}.$$

This leads to

$$\frac{d}{dt}\left[\left(1-\rho_0 t\right)\lambda - \frac{a^2\rho_0}{2}\ln\left(1-\rho_0 t\right)\right] \geq 0.$$

Therefore we conclude that

$$\lambda(t) \geq \frac{1}{\left(1-\rho_0 t\right)}\lambda(0) + \frac{a^2\rho_0}{2\left(1-\rho_0 t\right)}\ln\left(1-\rho_0 t\right)$$

as desired. □

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