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Heat equation and ergodic theorems for Riemann surface laminations

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Abstract We study the dynamics of possibly singular foliations by Riemann surfaces. The main examples are holomorphic foliations by Riemann surfaces in projective varieties. We introduce the heat equation relative to a positive $\partial\overline{\partial}$-closed current and apply it to the directed currents associated with Riemann surface laminations possibly with singularities. This permits to construct the heat diffusion with respect to various Laplacians that could be defined almost everywhere with respect to the $\partial\overline{\partial}$-closed current. We prove two kinds of ergodic theorems for such currents: one associated to the heat diffusion and one of geometric nature close to Birkhoff's averaging on orbits of a dynamical system. Here the averaging is on hyperbolic leaves and the time is the hyperbolic time. The heat diffusion theorem with respect to a harmonic measure is also developed for real laminations.

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Equation de la chaleur et théorèmes ergodiques pour les laminations par surfaces de Riemann

Résumé On étudie la dynamique des feuilletages, éventuellement singuliers, par surfaces de Riemann. Les exemples principaux sont les feuilletages dans des variétés projectives. Nous introduisons l'équation de la chaleur relative à un courant positif $\partial\bar{\partial}$ fermé et nous appliquons cette approche aux courants associés à un feuilletage (ou à une lamination) par surfaces de Riemann. Nous démontrons deux types de théorèmes ergodiques. Le premier concerne la diffusion de la chaleur relativement aux courants dirigés par un feuilletage. Le second est analogue au théorème ergodique de Birkhoff. On moyenne sur les feuilles revêtues par le disque de Poincaré, le temps étant le temps hyperbolique. La convergence a lieu pour presque tout point de base (relativement au courant). Nous développons également la théorie de la diffusion de la chaleur dans le cas des feuilletages réels, sans hypothèse de géométrie bornée, mais relativement à une mesure harmonique donnée.

Notation

Throughout the paper, $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$, $r\mathbb{D}$ denotes the disc of center 0 and of radius r and $\mathbb{D}_R \subset \mathbb{D}$ is the disc of center 0 and of radius R with respect to the Poincaré metric on $\mathbb{D}$, i.e. $\mathbb{D}_R = r\mathbb{D}$ with $R := \log[(1+r)/(1-r)]$. Poincaré metric on a Riemann surface, in particular on $\mathbb{D}$ and on the leaves of a lamination, is given by a positive (1, 1)-form that we denote by ω_P. The notation $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ is a flow box which is often identified with an open set of the lamination. Here, $\mathbb{T}$ is a transversal and $\mathbb{B}$ is an open set in $\mathbb{R}^n$ for real laminations or in $\mathbb{C}$ for Riemann surface laminations.

1 Introduction

The main goal of this paper is to prove ergodic theorems for possibly singular compact foliations by Riemann surfaces in complex manifolds. The main examples of such foliations are foliations by Riemann surfaces in the complex projective space $\mathbb{P}^k$ of arbitrary dimension (in which case there are always singularities) or in algebraic manifolds.

The dynamics of foliations in $\mathbb{P}^2$ with the line at infinity invariant is well-understood (see the recent book by Ilyashenko–Yakovenko [24]). The analysis relies on the study of the holonomy group on the line at infinity. However, as shown by Jouanolou [25] in dimension 2 and by Lins Neto–Soares [29] for arbitrary k, generic foliations of degree d in $\mathbb{P}^k$ do not have an invariant algebraic curve. So, it is not clear where to start the analysis.

In order to develop an ergodic theory of foliations, Garnett [21] introduced the notion of harmonic measure for non singular foliations. In [1] it was shown that for a foliation by Riemann surfaces in a compact complex manifold with finitely many singular points, there exists a positive current directed by the foliation which is $\partial\bar{\partial}$-closed. Considering the trace of the current with respect to a Hermitian form gives a harmonic measure with respect to that Hermitian metric (see Sect. 5 below). More general cases are considered in [1] and in [17].

Surprisingly enough, Bonatti et al. [3,4] have shown a unique ergodicity result for non singular foliations obtained by suspensions (Ricatti equation). They use deep results on the geodesic flows.

Fornæss and the third author [19] have shown that foliations in $\mathbb{P}^2$, with hyperbolic singularities and without algebraic leaves are uniquely ergodic in the following sense. There is a unique positive dd^c-closed current T of mass 1 tangent to the foliation. This implies in particular the following. Let $\phi_a : \mathbb{D} \to L_a$ denote a universal covering map of the leaf L_a passing through a with $\phi_a(0) = a$. Then, the sequence of currents

$$T_{a,R} := \frac{1}{M_R}(\phi_a)_*\left(\log^+\frac{r}{|\zeta|}\right) \quad \text{with} \quad R := \log\frac{1+r}{1-r}$$

converges to T as $R \to \infty$ independently of the leaf L_a. Here, M_R is a constant to normalize the mass so that the current $T_{a,R}$ has mass 1. The theorem is based on a cohomological intersection theory for positive dd^c-closed currents. This theory is useful only in dimension two. Observe that the foliations considered in that result are generic among foliations of fixed degree.

In the present paper, we deal with foliations in manifolds of arbitrary dimension. We obtain in particular the following result with the notation as above.

Theorem 1.1 *Let M be a Hermitian complex compact manifold. Let $\mathcal{F}$ be a foliation by Riemann surfaces on M with a finite number of singularities. Assume that all the singularities are linearizable. Let T be an extremal positive dd^c-closed current of mass 1 tangent to the foliation. Assume that T has no mass on the union of parabolic leaves, i.e. the leaves which are not uniformized by the unit disc. Then, $T_{a,R} \to T$ as $R \to \infty$ for T-almost every a. Moreover, if $\phi_a : \mathbb{D} \to L_a$ denotes as above a universal covering map, then*

$$\int_{\mathbb{D}_R} |\phi_a'(\zeta)|^2(r - |\zeta|) \cdot i d\zeta \wedge d\bar{\zeta} \simeq \log\frac{1}{1-r} \quad as \quad R \to \infty$$

for T-almost every a.

If ω denotes a Hermitian metric on M, we can also express the previous result as

$$\frac{1}{M_R}(\phi_a)_*\left(\log^+\frac{r}{|\zeta|}\right) \wedge \omega \to T \wedge \omega \quad \text{as} \quad R \to \infty.$$

In which case $T \wedge \omega$ is identified with a harmonic measure in the usual sense (see Sect. 7 below).

For foliations in $\mathbb{P}^k$ of degree $d > 1$, Glutsyuk [23] and Lins Neto [28] have shown that if all the singular points have non degenerate linear part, then the leaves of the foliation are covered by the unit disc. Hence, the previous theorem applies if we just assume that the singularities are linearizable. This is the generic case for foliations of degree d. Brunella [2] has shown that generic foliations of degree d do not admit a positive closed current tangent to the foliation. So, we have to study the statistical

behavior with respect to dd^c-closed currents. We obtain the above result as a conse-
quence of a Birkhoff type ergodic theorem for abstract laminations, with singularities,
by Riemann surfaces with leaves covered by the unit disc (see Theorem 7.1 below).
The invariant measure in the classical Birkhoff's theorem (see [27]) is replaced by a
dd^c-closed current directed by the laminations, the Birkhoff sums are replaced by the
averaging on leaves given by the current $T_{a,R}$ and the time is replaced by the hyperbolic
time in the unit disc. The theorem also applies to laminations without singularities.
The reader will find a number of examples of laminations by Riemann surfaces in the
survey paper by Ghys [22], in [18] and also in [20]. In the study of the complex Hénon
maps, natural laminations with a finite number of singularities appear [16].

The other objective of this paper is to develop the study of the diffusion of the heat
equation for laminations (possibly with singularities) with respect to a positive dd^c-
closed current tangent to the lamination. This permits to extend the classical theory
of Garnett [21] and Candel [7] to foliations with singularities or to foliations with not
necessarily bounded geometry.

When $\mathscr{F}$ is a smooth foliation of a compact Riemannian manifold (M, g) with
smooth leaves (or more generally a lamination by Riemannian manifolds), Garnett
[21] has studied a diffusion process on the leaves of the foliation. The metric g induces
a Laplace operator Δ along the leaves and the diffusion process is associated to the
heat equation

$$\frac{du}{dt} - \Delta u = 0 \quad \text{and} \quad u(0, \cdot) = u_0.$$

Since the leaves have bounded geometry, the classical theory, based on Malliavin [30]
and McKean [31] analytic estimates, applies and one can study the diffusion pro-
cess associated to that equation. L. Garnett proved also an ergodic theorem for the
semi-group $S(t)$ of diffusion operators associated to the heat equation. Recall that a
positive harmonic measure for $\mathscr{F}$ is a positive measure m such that $\langle \Delta u, m \rangle = 0$
for all smooth functions u. The measure m can be decomposed in a flow box as an
average of measures on plaques which are given by harmonic forms. We refer to
Candel–Conlon [7,8] and Walczak [37] for a more recent treatment of L. Garnett's
theory. Their approach relies on uniform estimates of the heat kernel using that the
leaves have bounded geometry with respect to a smooth Riemannian metric.

Candel gives in [7] a nice treatment of L. Garnett's theory. He studies the regularity
of the heat equation using spaces of differentiable functions. The theory does not apply
to Riemann surface laminations with singularities, e.g. to the study of polynomial vec-
tor fields in $\mathbb{C}^{k+1}$, for which we can associate a foliation in $\mathbb{P}^k$. More precisely, let

$$F(z) := \sum_{j=0}^{k} F_j(z) \frac{\partial}{\partial z_j}$$

with F_j homogeneous polynomials of degree $d \geq 1$ without non-trivial common fac-
tor. It induces a foliation with singularities in $\mathbb{P}^k$. The singularities correspond either to
indeterminacy points of $F = [F_0 : \cdots : F_k]$ or to fixed points of F in $\mathbb{P}^k$. In general,
the leaves are not of bounded geometry nor even complete for the induced metric.
They have bounded geometry with respect to the Poincaré metric on the leaves that

we will consider, but then the metric is not in general transversally smooth. Moreover, the regular part of the foliation is not compact.

Here we construct the heat diffusion in a slightly different context. We consider compact laminations by Riemann surfaces with singularities in a Hermitian manifold or abstract laminations by Riemannian leaves without singularities. In the real case, assume that we have a Laplacian Δ along leaves such that for a test function u, regular enough, Δu is continuous. Then, an application of the Hahn–Banach theorem permits to obtain a harmonic measure m, see e.g. Garnett [21]. In the complex case with singularities, the construction of m is different, see [1]. One has to use the notion of plurisubharmonic functions. They have the property to be subharmonic on every leaf independently of the lamination. One can also use an averaging process as in [17]. We get a positive $\partial\bar{\partial}$-closed current directed by the foliation.

When such a current is given, we can consider other natural Laplacians. They are obtained by considering Hermitian forms on the foliation which vary measurably in the transversal directions. We then get harmonic measures m with respect to these Laplacians and we introduce a heat equation associated to m. We can develop a Hilbert space theory with respect to that equation using Lax–Milgram and Hille–Yosida theorems. More precisely, given u_0 in the domain $\mathrm{Dom}(\Delta)$ of Δ, we solve

$$\frac{du}{dt} = \Delta u \quad \text{and} \quad u(0, \cdot) = u_0$$

with $u(t, \cdot) \in \mathrm{Dom}(\Delta)$. The theory is sufficient to get an ergodic theorem for that diffusion. So, we rather get the heat equation in the space $(M, \mathcal{F}, m)$. The Laplacians are not necessarily symmetric operators in $L^2(m)$ and the natural ones depend on m. The idea is to consider the heat equation on a highly irregular object: the harmonic current. We replace manifolds and foliations by currents. Consequently, we can relax the assumption of smoothness. Our metrics are only transversally measurable and the most interesting one is defined only almost everywhere with respect to the current. It is the one which gives a self adjoint Laplacian and for which we can prove that the heat diffusion is mixing. The question of speed of mixing for appropriate classes of observables is open.

For clarity, we first develop the theory of the heat diffusion with respect to a harmonic measure in the context of real laminations without singularities. So, we give a self-contained proof of the ergodic theorem in the Riemannian case without any use of bounded geometry nor delicate estimates on the heat kernel. This will permit further generalizations. We get also the mixing for the diffusion associated to a natural Laplacian with coefficients defined only m-almost everywhere. We then apply the same ideas to the complex case with singularities. In some sense, we treat harmonic measures and dd^c-closed currents as manifolds and we solve the heat equation relatively to those measures and currents. The case of $\bar{\partial}$-equation induced on a current was studied by Berndtsson and the third author in [1]. The methods here are however quite different.

In Sect. 2 we discuss some general facts about harmonic currents on real or complex laminations.

In Sect. 3 we introduce the Poincaré metric on leaves. The main point there is the introduction of a measurable way to choose "good large discs" on the leaves (Corollary

3.3 will be crucial in our proof of Theorem 7.1). The results are written for locally compact laminations with hyperbolic leaves.

In Sect. 4 we give precise estimates for the Poincaré metric on a "hyperbolic folia-tion" with linearizable singularities. This permits to introduce the trace measure of a positive dd^c-closed current with respect to the Poincaré metric on a foliation.

In Sect. 5 we develop the theory of the heat diffusion with respect to a harmonic measure. This allows us to develop the classical theory of L. Garnett and A. Candel without any smoothness assumption on the transverse regularity of the laminations. We obtain in particular an ergodic theorem and mixing results for the heat diffusion in a quite general context.

In Sect. 6 we study the heat equation with respect to a positive dd^c-closed current T of bidimension $(1, 1)$ with very weak regularity assumptions. The fact that the cur-rent is associated to a lamination is not needed, though the main application we have in mind is singular foliations by Riemann surfaces in complex manifolds. We obtain ergodicity and mixing results for the heat equation with respect to a harmonic measure for these foliations. We also prove L^p ergodic theorems, $1 \leq p < \infty$, for foliations with singularities.

A Birkhoff's type ergodic theorem is given in Sect. 7. The proof of this result uses the preliminary results from Sect. 3 to 6. We need only that Poincaré metric is trans-versally measurable. In some important cases this metric is known to be continuous out of the singularities [6,9,18,28]. It turns out that this continuity would not simplify very much our study.

2 Currents on a lamination

In this section, we will give some basic notions and properties of currents for lamina-tions. We refer to Demailly [13] and Federer [15] for currents on manifolds.

Let X be a locally compact space. Consider an atlas $\mathscr{L}$ of X with charts

$$\Phi_i : \mathbb{U}_i \to \mathbb{B}_i \times \mathbb{T}_i,$$

where $\mathbb{T}_i$ is a locally compact metric space, $\mathbb{B}_i$ is a domain in $\mathbb{R}^n$ and Φ_i is a homeo-morphism defined on an open subset $\mathbb{U}_i$ of X. We say that $(X, \mathscr{L})$ is a *real lamination of dimension n* if all the changes of coordinates $\Phi_i \circ \Phi_j^{-1}$ are of the form

$$(x, t) \mapsto (x', t'), \quad x' = \Psi(x, t), \quad t' = \Lambda(t)$$

where Ψ, Λ are continuous functions, Ψ is smooth with respect to x and its partial derivatives of any order with respect to x are continuous.

The open set $\mathbb{U}_i$ is called a *flow box* and the manifold $\Phi_i^{-1}\{t = c\}$ in $\mathbb{U}_i$ with $c \in \mathbb{T}_i$ is a *plaque*. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. A *leaf L* is a minimal connected subset of X such that if L intersects a plaque, it contains the plaque. So, a leaf L is a connected real manifold of dimension n immersed in X which is a union of plaques. It is not difficult to see that $\overline{L}$ is also a lamination. We will

only consider *oriented laminations*, i.e. the case where the Φ_i preserve the canonical orientation on $\mathbb{R}^n$. So, the leaves of X inherit the orientation given by the one of $\mathbb{R}^n$. A *transversal* in a flow box is a closed set of the box which intersects every plaque in one point. In particular, $\Phi_i^{-1}(\{x\} \times \mathbb{T}_i)$ is a transversal in $\mathbb{U}_i$ for any $x \in \mathbb{B}_i$. In order to simplify the notation, we often identify $\mathbb{T}_i$ with $\Phi_i^{-1}(\{x\} \times \mathbb{T}_i)$ for some $x \in \mathbb{B}_i$ or even identify $\mathbb{U}_i$ with $\mathbb{B}_i \times \mathbb{T}_i$ via the map Φ_i.

From now on, we fix an atlas on X which is locally finite. For simplicity, assume that the associated local coordinates extend to a neighbourhood of the closure of each flow box in the atlas. If $\Phi : \mathbb{U} \to \mathbb{B} \times \mathbb{T}$ is such a flow box, we assume for simplicity that $\mathbb{B}$ is contained in the ball of center 0 and of radius 3 in $\mathbb{R}^n$ and $\mathbb{T}$ is a locally compact metric space of diameter ≤ 1. If the lamination is embedded in a Riemannian manifold, it is natural to consider the metric on that manifold. In the abstract setting, it is useful to introduce a metric for the lamination. First, consider the metric on the flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ which is induced by the ones on $\mathbb{R}^n$ and on $\mathbb{T}$. So, the flow box has diameter ≤ 7 with respect to this metric. Consider two points $a, b \in X$, a sequence $a_0, \ldots, a_m$ with $a_0 = a$, $a_m = b$ and a_i, a_{i+1} in a same flow box $\mathbb{U}_i$. Denote by l_i the distance between a_i and a_{i+1} in $\mathbb{U}_i$. Define the distance between a, b as the infimum of $\sum l_i$ over all choices of a_i and $\mathbb{U}_i$. This distance is locally equivalent to the distance in flow boxes.

We recall now the notion of currents on a manifold. Let M be a real oriented manifold of dimension n. We fix an atlas of M which is locally finite. Up to reducing slightly the charts, we can assume that the local coordinates system associated to each chart is defined on a neighbourhood of the closure of this chart. For $0 \leq p \leq n$ and $l \in \mathbb{N}$, denote by $\mathscr{D}_l^p(M)$ the space of p-forms of class $\mathscr{C}^l$ with compact support in X and $\mathscr{D}^p(X)$ their intersection. If α is a p-form on X, denote by $\|\alpha\|_{\mathscr{C}^l}$ the sum of the $\mathscr{C}^l$-norms of the coefficients of α in the local coordinates. These norms induce a topology on $\mathscr{D}_l^p(M)$ and $\mathscr{D}^p(M)$. In particular, a sequence α_j converges to α in $\mathscr{D}^p(M)$ if these forms are supported in a fixed compact set and if $\|\alpha_j - \alpha\|_{\mathscr{C}^l} \to 0$ for every l.

A current of degree p and of dimension $n - p$ on M (a p-current for short) is a continuous linear form T on $\mathscr{D}^{n-p}(M)$ with values in $\mathbb{C}$. The value of T on a test form α in $\mathscr{D}^{n-p}(M)$ is denoted by $\langle T, \alpha \rangle$ or $T(\alpha)$. The current T is of order $\leq l$ if it can be extended to a continuous linear form on $\mathscr{D}_l^{n-p}(M)$. The order of T is the minimal integer $l \geq 0$ satisfying this condition. It is not difficult to see that the restriction of T to a relatively compact open set of M is always of finite order. Define

$$\|T\|_{-l,K} := \sup \left\{ |\langle T, \alpha \rangle|, \quad \alpha \in \mathscr{D}^{n-p}(M), \quad \|\alpha\|_{\mathscr{C}^l} \leq 1, \quad \mathrm{supp}(\alpha) \subset K \right\}$$

for $l \in \mathbb{N}$ and K a compact subset of M. This quantity may be infinite when the order of T is larger than l.

Consider now a real lamination of dimension n as above. The notion of differential forms on manifolds can be extended to laminations, see Sullivan [35]. A p-form on X can be seen on the flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ as a p-form on $\mathbb{B}$ depending on the parameter $t \in \mathbb{T}$. For $0 \leq p \leq n$, denote by $\mathscr{D}_l^p(X)$ the space of p-forms α with compact support satisfying the following property: α restricted to each flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ is a p-form

of class $\mathscr{C}^l$ on the plaques whose coefficients and all their derivatives up to order l depend continuously on the plaque. The norm $\| \cdot \|_{\mathscr{C}^l}$ on this space is defined as in the case of real manifold using a locally finite atlas of X. We also define $\mathscr{D}^p(X)$ as the intersection of $\mathscr{D}_l^p(X)$ for $l \geq 0$. *A current of bidegree p and of dimension $n - p$ on X* is a continuous linear form on the space $\mathscr{D}^{n-p}(X)$ with values in $\mathbb{C}$. A p-current is of order $\leq l$ if it can be extended to a linear continuous form on $\mathscr{D}_l^{n-p}(X)$. The restriction of a current to a relatively compact open set of X is always of finite order. The norm $\| \cdot \|_{-l,K}$ on currents is defined as in the case of manifolds. The following result gives us the local structure of a current. It shows in particular that we can consider the restriction of a current to a measurable family of plaques.

Proposition 2.1 *Let T be a p-current on a lamination X and let $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ be a flow box as above which is relatively compact in X. Let l be the order of the restriction of T to $\mathbb{U}$. Then, there is a positive Radon measure μ on $\mathbb{T}$ and a measurable family of p-currents T_a of order l on $\mathbb{B}$ for μ-almost every $a \in \mathbb{T}$ such that if K is compact in $\mathbb{B}$, the integral $\langle \mu, \|T_a\|_{-l,K} \rangle$ is finite and*

$$\langle T, \alpha \rangle = \int_{\mathbb{T}} \langle T_a, \alpha(\cdot, a) \rangle d\mu(a) \quad for \ \alpha \in \mathscr{D}_l^{n-p}(\mathbb{U}).$$

Proof Observe that if we have a local disintegration as above for two currents T, T', then it is easy to get such a disintegration for $T + T'$ using the sum $\mu + \mu'$ of the corresponding measures μ, μ'. This property allows us to make several reductions below. Using a partition of unity, we can reduce the problem to the case where T has compact support in $\mathbb{U} \subset \mathbb{R}^n \times \mathbb{T}$. If $(x_1, \ldots, x_n)$ is a coordinate system in $\mathbb{R}^n$ and $1 \leq i_1 < \cdots < i_{n-p} \leq n$, we only have to prove the proposition for the current $T \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{n-p}}$. Therefore, we can assume that T is an n-current, i.e. a distribution.

Now, since T is of order l, it can be seen as a continuous linear form on the derivatives φ_I of order l of a test function $\varphi \in \mathscr{D}_l^0(\mathbb{U})$. By Hahn–Banach theorem, there are distributions T_I of order 0 such that $T = \sum T_I(\varphi_I)$. It is enough to prove the proposition for each T_I instead of T. So, we can assume that T is of order 0, i.e. a Radon measure.

Since T can be written as a difference of two positive measures, we only have to consider the case where T is positive. Define π the canonical projection from U to $\mathbb{T}$ and $\mu := \pi_*(T)$. The disintegration of T along the fibers of π gives the result. $\qquad \square$

The following result shows that the above local decomposition of a current is almost unique.

Proposition 2.2 *With the notation of Proposition 2.1, if μ' and T_a' are associated with another decomposition of T in $\mathbb{U}$, then there is a measurable function $\lambda > 0$ on a measurable set $\mathbb{S} \subset \mathbb{T}$ such that $T_a = 0$ for μ-almost every $a \notin \mathbb{S}$, $T_a' = 0$ for μ'-almost every $a \notin \mathbb{S}$, $\mu' = \lambda\mu$ on $\mathbb{S}$ and $T_a = \lambda(a)T_a'$ for μ and μ'-almost every $a \in \mathbb{S}$.*

Proof Consider first the case where $T = 0$. We show that $T_a = 0$ for μ-almost every a. Let α be a test form in $\mathscr{D}^{n-p}(\mathbb{B})$. Define $\eta(a) := \langle T_a, \alpha \rangle$. If χ is a continuous

function with compact support in $\mathbb{T}$, we have by Proposition 2.1

$$\int \chi \eta d\mu = \langle T, \chi \alpha \rangle = 0.$$

It follows that $\eta(a) = 0$ for μ-almost every a. Applying this property to a dense sequence of test forms $\alpha_j \in \mathscr{D}^{n-p}(\mathbb{B})$ allows us to conclude that $T_a = 0$ for μ-almost every a.

Consider now the general case. Define

$$\mathbb{S} := \{a \in \mathbb{T}, \ T_a \neq 0, T'_a \neq 0\}.$$

Observe that the restriction of T to

$$E := \{a \in \mathbb{T}, \ T_a \neq 0, \ T'_a = 0\} \quad \text{and} \quad E' := \{a \in \mathbb{T}, \ T_a = 0, \ T'_a \neq 0\}$$

vanishes. Then, using the first case, we obtain that μ has no mass on E and μ' has no mass on E'. Therefore, $T_a = 0$ for μ-almost every $a \notin \mathbb{S}$ and $T'_a = 0$ for μ'-almost every $a \notin \mathbb{S}$. Using the first case, we also deduce that a measurable subset of $\mathbb{S}$ has positive μ measure if and only if this is the case for μ'. Therefore, there is a function $\lambda > 0$ such that $\mu' = \lambda \mu$ on $\mathbb{S}$. Observe that $T_a - \lambda T'_a$ and μ define a decomposition of 0 in $\mathbb{U}$. It follows that $T_a = \lambda T'_a$ for μ-almost every $a \in \mathbb{S}$. $\qquad\square$

A 0-current T on a lamination is *positive* if it is of order 0 and if in the local description as in Proposition 2.1, the currents T_a are given by positive functions on $\mathbb{B}$ for μ-almost every $a \in \mathbb{T}$. Consider a Riemannian metric g which is smooth on the leaves of X and such that its restriction to a flow box depends in a measurable way on the plaques. A 0-current T of order 0 is called *g-harmonic* (or simply *harmonic* if there is no confusion) if in the local description as above, the currents T_a are given by g-harmonic functions.

Consider now the complex setting. In the definition of the lamination $(X, \mathscr{L})$, when the $\mathbb{B}_i$ are domains in $\mathbb{C}^n$ and Φ_i are holomorphic with respect to x, we say that X is a *complex lamination of dimension n*. In this case, the complex structure on $\mathbb{B}_i$ induces a complex structure on the leaves of X. Therefore, in the definition of lamination, it is enough to assume that $\Psi_i(x, t)$ depends continuously on t; indeed Cauchy's formula implies that all partial derivatives of this function with respect to x satisfy the same property.

Let X be a complex lamination of dimension n. Denote by $\mathscr{D}_l^{p,q}(X)$ and $\mathscr{D}^{p,q}(X)$ the spaces of forms in $\mathscr{D}_l^{p+q}(X)$ and $\mathscr{D}^{p+q}(X)$ respectively whose restriction to plaques is of bidegree (p, q). A $(p + q)$-current is of bidegree (p, q) if it vanishes on forms of bidegree $(n - p', n - q')$ with $(p', q') \neq (p, q)$. The operators ∂ and $\bar{\partial}$ act on currents as in the case of manifolds. If T is a (p, q)-current, then ∂T and $\bar{\partial}T$ are defined by

$$\langle \partial T, \alpha \rangle := (-1)^{p+q+1} \langle T, \partial \alpha \rangle \quad \text{for all test } (n - p - 1, n - q)\text{-form } \alpha$$

and

$$\langle \overline{\partial} T, \alpha \rangle := (-1)^{p+q+1} \langle T, \overline{\partial} \alpha \rangle \quad \text{for all test } (n-p, n-q-1)\text{-form } \alpha.$$

We call $\partial\overline{\partial}$-*closed current or pluriharmonic current* a $(0, 0)$-current T on X such that $\partial\overline{\partial}T = 0$ (in dimension $n = 1$, we will say simply "harmonic" instead of "pluriharmonic"). The following result shows that in dimension $n = 1$ this notion coincides with the notion of g-harmonic current if we consider Poincaré metric on the hyperbolic leaves of X and standard metrics on the parabolic ones or more generally conformal metrics on the leaves.

Proposition 2.3 *Let T be a pluriharmonic current on a complex lamination X. Let $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ be a flow box as above which is relatively compact in X. Then, T is a normal current, i.e. T and dT are of order 0. Moreover, there is a positive Radon measure μ on $\mathbb{T}$ and for μ-almost every $a \in \mathbb{T}$ there is a pluriharmonic function h_a on $\mathbb{B}$ such that if K is compact in $\mathbb{B}$ the integral $\langle \mu, \|h_a\|_{L^1(K)} \rangle$ is finite and*

$$\langle T, \alpha \rangle = \int_{\mathbb{T}} \left(\int_{\mathbb{B}} h_a(z)\alpha(z, a) \right) d\mu(a) \quad \text{for} \quad \alpha \in \mathscr{D}_0^{n,n}(X).$$

Proof Using Proposition 2.1, we easily deduce that T_a is $\partial\overline{\partial}$-closed for μ-almost every $a \in \mathbb{T}$. It follows that T_a is given by a pluriharmonic function h_a on $\mathbb{B}$. If K, L are compacts in $\mathbb{B}$ with $K \Subset L$, the harmonic property implies that $\|h_a\|_{L^1(K)} \lesssim \|h_a\|_{-l,L}$ for $0 \leq l < \infty$ and $\|dh_a\|_{L^1(K)} \lesssim \|h_a\|_{L^1(L)}$. This implies that T and dT are of order 0 and completes the proof. $\square$

For complex laminations, there is a notion of positivity for currents of bidegree (p, p) which extends the same notion for (p, p)-currents on complex manifolds. We shortly recall the last one that we will use later.

A (p, p)-form on a complex manifold M of dimension n is *positive* if it can be written at every point as a combination with positive coefficients of forms of type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

where the α_j are $(1, 0)$-forms. A (p, p)-current or a (p, p)-form T on M is *weakly positive* if $T \wedge \varphi$ is a positive measure for any smooth positive $(n-p, n-p)$-form φ. A (p, p)-current T is *positive* if $T \wedge \varphi$ is a positive measure for any smooth weakly positive $(n-p, n-p)$-form φ. If M is given with a Hermitian metric ω, $T \wedge \omega^{n-p}$ is a positive measure on M. The mass of $T \wedge \omega^{n-p}$ on a measurable set E is denoted by $\|T\|_E$ and is called *the mass of T on E*. The mass $\|T\|$ of T is the total mass of $T \wedge \omega^{n-p}$. We will use the following local property of positive $\partial\overline{\partial}$-closed currents which is due to Skoda [33]. Recall that $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

Lemma 2.4 *Let B_r denote the ball of center 0 and of radius r in $\mathbb{C}^n$. Let T be a positive $\partial\overline{\partial}$-closed (p, p)-current in a ball B_{r_0}. Define $\beta := dd^c\|z\|^2$ the standard Kähler form where z is the canonical coordinates on $\mathbb{C}^n$. Then the function*

$r \mapsto \pi^{-(n-p)} r^{-2(n-p)} \|T \wedge \beta^{n-p}\|_{B_r}$ *is increasing on* $0 < r \leq r_0$. *In particular, it is bounded on* $]0, r_1]$ *for any* $0 < r_1 < r_0$.

The limit of the above function when $r \to 0$ is called *the Lelong number* of T at 0. The lemma shows that Lelong's number exists and is finite.

Lemma 2.5 *Let T be a positive current of bidimension* $(1, 1)$ *with compact support on a complex manifold M. Assume that $dd^c T$ is a negative measure on $M \backslash E$ where E is a finite set. Then, T is a dd^c-closed current on M.*

Proof Since E is finite, we prove first as in [20] that T has finite mass and that $dd^c T$ is a negative measure on M. On the other hand, we have

$$\langle dd^c T, 1 \rangle = \langle T, dd^c 1 \rangle = 0.$$

It follows that $dd^c T = 0$ on M. □

3 Riemann surface laminations

In this section, we consider a *Riemann surface lamination*, i.e. a complex lamination X as above of dimension $n = 1$. The lamination has no singular points but we do not assume that it is compact. What we have in mind as an example is the regular part of a compact lamination with singularities. Consider also a Hermitian metric on X, i.e. Hermitian metrics on the leaves of $\mathscr{L}$ whose restriction to each flow box defines Hermitian metrics on the plaques that depend continuously on the plaques. It is not difficult to construct such a metric using a partition of unity. Observe that all the Hermitian metrics on X are locally equivalent. So, from now on, fix a Hermitian metric on X. It is given by a strictly positive smooth $(1, 1)$-form ω on X.

We will need some basic properties of these laminations. Let S be a hyperbolic Riemann surface, i.e. a Riemann surface whose universal covering is the unit disc $\mathbb{D}$ in $\mathbb{C}$. Let $\phi : \mathbb{D} \to S$ be a universal covering map which is unique up to an automorphism on $\mathbb{D}$. The fundamental group $\pi_1(S)$ can be identified with a group of automorphisms of $\mathbb{D}$. Since the Poincaré metric on $\mathbb{D}$ is invariant under the automorphism group, it induces via ϕ a metric on S that we also call *the Poincaré metric*. It is smooth and the surface S is complete with respect to that metric. By convention, Poincaré metric (pseudo-metric to be precise) on a parabolic Riemann surface vanishes identically.

Poincaré metric on the leaves of X defines a positive $(1, 1)$-form ω_P, which a priori is not necessarily transversally continuous. The continuity is proved in some important cases, see Candel and Gómez-Mont [9] and [18]. Consider a hyperbolic leaf L_a passing through a point a and a universal covering map $\phi_a : \mathbb{D} \to L_a$ such that $\phi_a(0) = a$. The map ϕ_a is unique up to a rotation on $\mathbb{D}$. Define

$$\vartheta(a) := \|D\phi_a(0)\|^{-2},$$

where $\|D\phi_a(0)\|$ is the norm of the differential of ϕ_a at 0 with respect to the Euclidean metric on $\mathbb{D}$ and the fixed Hermitian metric on L. Recall that at 0 the Poincaré metric

on $\mathbb{D}$ is equal to twice the Euclidean metric. The above definition does not depend on the choice of ϕ_a. We obtain that

$$\omega_P = 4\vartheta\omega.$$

Recall also that ω_P is an extremal metric in the sense that if $\tau : \mathbb{D} \to L_a$ is a holomorphic map such that $\tau(0) = a$, then $\|D\tau(0)\| \leq \vartheta(a)^{-1/2}$. The equality occurs in the last estimate only when τ is a universal covering map of L_a.

Consider an open set $V \subset \mathbb{C}$ and a sequence of holomorphic maps $\tau_n : V \to X$, i.e. holomorphic maps from V to leaves of X. We say that τ_n *converge locally uniformly* to a holomorphic map $\tau : V \to X$ if any point $z_0 \in V$ admits a neighbourhood V_0 such that for n large enough τ_n and τ restricted to V_0 have values in the same flow box and τ_n converge uniformly to τ on V_0. This notion coincides with the local uniform convergence with respect to the metric on X introduced in Sect. 2. A family $\mathscr{F}$ of holomorphic maps from V to X is said to be *normal* if any infinite set $\mathscr{F}' \subset \mathscr{F}$ admits a sequence which converges locally uniformly to a holomorphic map. We have the following proposition.

Proposition 3.1 *Let X be a Riemann surface lamination as above. Then the Poincaré metric ω_P is a measurable $(1, 1)$-form on X. In particular, the union of parabolic leaves is a measurable set. Moreover, the function ϑ associated with ω_P is locally bounded.*

Proof By definition, ϑ is a non-negative function. We first show that it is locally bounded. Consider a small neighbourhood W of a point $a \in X$ and a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ containing W. We can assume that $\mathbb{B}$ is the disc of center 0 and of radius 3 in $\mathbb{C}$ and that W is contained in $\mathbb{D} \times \mathbb{T}$ where $\mathbb{D}$ is the unit disc in $\mathbb{C}$. Consider the family of holomorphic map $\tau : \mathbb{D} \to \mathbb{U}$ such that $\tau(z) = (z + b, t)$ with $(b, t) \in W$. It is clear that $\|D\tau(0)\|$ is bounded from below by a strictly positive constant independent of (b, t). Therefore, the extremality of Poincaré metric implies that ϑ is bounded from above on W. This gives the last assertion in the proposition.

It remains to show that ϑ is a measurable function. Fix a sequence K_n of compact subsets of X such that K_n is contained in the interior of K_{n+1} and that $X = \cup K_n$. We only have to show that ϑ is measurable on K_0. For all positive integer n, denote by $\mathscr{F}_n$ the family of holomorphic maps $\tau : \mathbb{D} \to K_n$ such that $\|D\tau\|_\infty \leq n$. It is not difficult to see using flow boxes that this family is compact. Therefore, the function

$$\xi_n(a) := \sup\left\{\|D\tau(0)\| : \quad \tau \in \mathscr{F}_n \text{ with } \tau(0) = a\right\}$$

is upper semi-continuous on $a \in K_0$. The extremality of Poincaré metric implies that $\xi_n \leq \vartheta^{-1/2}$. It is now enough to show that $\vartheta^{-1/2} = \sup_n \xi_n$. We distinguish two cases.

Let $a \in K_0$ be a point such that L_a is hyperbolic and consider a universal covering map $\phi_a : \mathbb{D} \to L_a$ with $\phi_a(0) = a$. Define $\tau_r(z) := \phi_a(rz)$ for $z \in \mathbb{D}$ and $0 < r < 1$. It is clear that $\tau_r(\mathbb{D})$ is relatively compact in X and $\|D\tau_r\|$ is finite. So, τ_r belongs to $\mathscr{F}_n$ for n large enough. On the other hand, we have

$$\vartheta^{-1/2}(a) = \|D\phi_a(0)\| = \lim_{r \to 1} \|D\tau_r(0)\|.$$

Therefore, $\vartheta^{-1/2} = \sup_n \xi_n$ on hyperbolic leaves.

Let $a \in K_0$ be a point such that L_a is parabolic and consider a map $\phi_a : \mathbb{C} \to L_a$ with $\phi_a(0) = a$. Define $\tau_r(z) := \phi_a(rz)$. It is also clear that τ_r belongs to $\mathscr{F}_n$ for n large enough and we have

$$\vartheta^{-1/2}(a) = +\infty = \lim_{r \to \infty} \|D\tau_r(0)\|.$$

It follows that $\vartheta^{-1/2} = \sup_{n \geq 0} \xi_n$ and this completes the proof. $\square$

Consider now a flow box $\Phi : \mathbb{U} \to \mathbb{B} \times \mathbb{T}$ as above. Recall that for simplicity, we identify $\mathbb{U}$ with $\mathbb{B} \times \mathbb{T}$ and $\mathbb{T}$ with the transversal $\Phi^{-1}(\{z\} \times \mathbb{T})$ for some point $z \in \mathbb{B}$. We have the following result.

Proposition 3.2 *Let ν be a positive Radon measure on $\mathbb{T}$. Let $\mathbb{T}_1 \subset \mathbb{T}$ be a measurable set such that $\nu(\mathbb{T}_1) > 0$ and L_a is hyperbolic for any $a \in \mathbb{T}_1$. Then, for every $\epsilon > 0$ there is a compact set $\mathbb{T}_2 \subset \mathbb{T}_1$ with $\nu(\mathbb{T}_2) > \nu(\mathbb{T}_1) - \epsilon$ and a family of universal covering maps $\phi_a : \mathbb{D} \to L_a$ with $\phi_a(0) = a$ and $a \in \mathbb{T}_2$ that depends continuously on a.*

Proof Recall that the universal covering maps $\phi_a : \mathbb{D} \to L_a$ are obtained from each other by composing with a rotation on $\mathbb{D}$. For the rest of the proof, denote by ϕ_a the universal covering map such that in the coordinates on the flow box $\mathbb{U}$, the derivative of ϕ_a at 0 is a positive real number. We can replace $\mathbb{T}_1$ with some compact set in the support of ν in order to assume that $\mathbb{T}_1$ is compact and contained in the support of ν. We will use the notation in the previous proposition with K_0 larger than $\mathbb{T}_1$. By Lusin's theorem, we can replace $\mathbb{T}_1$ by a suitable compact set in order to assume that ξ_n and ϑ are continuous on $\mathbb{T}_1$. So, the sequence ξ_n is increasing and converge uniformly to $\vartheta^{-1/2}$ on $\mathbb{T}_1$.

Claim Let $0 < r < 1$ and $\delta > 0$ be two constants. Then, there is a compact set $\mathbb{T}_r \subset \mathbb{T}_1$ with $\nu(\mathbb{T}_r) > \nu(\mathbb{T}_1) - \delta$ and an integer N such that $\|D\phi_a\| \leq N$ on $r\mathbb{D}$ and $\phi_a(r\mathbb{D}) \subset K_N$ for all $a \in \mathbb{T}_r$.

We first explain how to deduce the proposition from the claim. Using this property for $r_n = 1 - 1/n$, $\delta_n = 2^{-n}\epsilon$ with $n \geq 2$, define $\mathbb{T}_2 := \cap \mathbb{T}_{r_n}$. It is clear that $\nu(\mathbb{T}_2) > \nu(\mathbb{T}_1) - \epsilon$ and the family $\{\phi_a\}_{a \in \mathbb{T}_2}$ is locally bounded on $\mathbb{D}$. If $a_n \to a$ in $\mathbb{T}_2$, since ϑ is continuous, any limit value ϕ of ϕ_{a_n} satisfies $\|D\phi(0)\| = \vartheta(a)^{-1/2}$. Therefore, ϕ is a universal covering map of L_a. We deduce that ϕ is equal to ϕ_a because the derivatives of ϕ_{a_n} and ϕ_a are real positive. Hence, the family $\{\phi_a\}_{a \in \mathbb{T}_2}$ is continuous.

It remains to prove the above claim. Let $\mathscr{E}_n$ denote the family of $\tau \in \mathscr{F}_n$ such that $a := \tau(0)$ is in $\mathbb{T}_1$ and $\|D\tau(0)\| = \xi_n(a)$. This family is not empty since $\mathscr{F}_n$ is compact. Let $\mathscr{E}_n^+$ be the family of $\tau \in \mathscr{E}_n$ as above such that in the coordinates on the flow box $\mathbb{U}$, the derivative of τ at 0 is a positive real number. We can obtain such a map by composing a map in $\mathscr{E}_n$ with an appropriate rotation on $\mathbb{D}$. The continuity of ξ_n implies that $\mathscr{E}_n$ and $\mathscr{E}_n^+$ are compact.

The map which associates to $\tau \in \mathscr{E}_n^+$ its value at 0 is continuous. We recall that if $f : X_1 \to X_2$ is a continuous surjective map between two compact metric spaces, then

f admits an inverse measurable selection, i.e. there is $g : X_2 \to X_1$ measurable such that $f \circ g$ is identity, [12, p. 82]. So, the map $\tau \mapsto \tau(0)$ on $\mathscr{E}_n^+$ admits a measurable inverse map. More precisely, there is a measurable family $\{\tau_{n,a}\}_{a \in \mathbb{T}_1} \subset \mathscr{E}_n^+$ such that $\tau_{n,a}(0) = a$. Therefore, the measure ν on $\mathbb{T}_1$ induces a measure on $\mathscr{E}_n^+$. We can extract from $\{\tau_{n,a}\}_{a \in \mathbb{T}_1}$ a compact subset of measure almost equal to $\nu(\mathbb{T}_1)$. Hence, there is a compact set $\mathbb{T}_1' \subset \mathbb{T}_1$ such that $\nu(\mathbb{T}_1') > \nu(\mathbb{T}_1) - \delta$ and $\{\tau_{n,a}\}_{a \in \mathbb{T}_1'}$ is compact for every n. In other words, the family $\{\tau_{n,a}\}_{a \in \mathbb{T}_1'}$ depends continuously on $a \in \mathbb{T}_1'$.

For each a fixed in $\mathbb{T}_1'$, the extremal property of Poincaré metric implies that $\tau_{n,a} \to \phi_a$ locally uniformly when $n \to \infty$. Define for all positive integer N, $\mathbb{T}_{1,N}$ the set of $a \in \mathbb{T}_1'$ such that $\|D\tau_{n,a}\| \le N$ on $r\mathbb{D}$ and $\tau_{n,a}(r\mathbb{D}) \subset K_N$ for all n. This is an increasing sequence of compact sets which converges to $\mathbb{T}_1'$. So, for N large enough, $\nu(\mathbb{T}_{1,N}) > \nu(\mathbb{T}_1) - \delta$. We can choose a compact subset $\mathbb{T}_r \subset \mathbb{T}_{1,N}$ such that $\nu(\mathbb{T}_r) > \nu(\mathbb{T}_1) - \delta$. Clearly, $\mathbb{T}_r$ satisfies the claim.   □

Let $\phi_a : \mathbb{D} \to L_a$ be a covering map of L_a with $\phi_a(0) = a$. Denote $L_{a,R} := \phi_a(\mathbb{D}_R)$, where $\mathbb{D}_R \subset \mathbb{D}$ is the disc of center 0 and of radius R. Here, the radius is with respect to the Poincaré metric on $\mathbb{D}$. Since ϕ_a is unique up to a rotation on $\mathbb{D}$, $L_{a,R}$ is independent of the choice of ϕ_a. We will need the following result.

Corollary 3.3 *Let $R > 0$ be a positive constant. Then, under the hypothesis of Proposition 3.2, there is a countable family of compact sets $\mathbb{S}_n \subset \mathbb{T}_1$, $n \ge 1$, with $\nu(\cup_n \mathbb{S}_n) = \nu(\mathbb{T}_1)$ such that $L_{a,R} \cap \mathbb{S}_n = \{a\}$ for every $a \in \mathbb{S}_n$. Moreover, there are universal covering maps $\phi_a : \mathbb{D} \to L_a$ with $\phi_a(0) = a$ which depend continuously on $a \in \mathbb{S}_n$.*

Proof We first show that there is a compact set $\mathbb{S} \subset \mathbb{T}_2$ with $\nu(\mathbb{S}) > 0$ such that $L_{a,R} \cap \mathbb{S} = \{a\}$ for every $a \in \mathbb{S}$. By Proposition 3.2, the last assertion in the corollary holds for $\mathbb{S}$. Consider $\mathbb{T}_2'$ the support of the restriction of ν to $\mathbb{T}_2$ and an open neighbourhood $\mathbb{T}'$ of $\mathbb{T}_2'$ which is relatively compact in $\mathbb{T}$. By Proposition 3.2, the number of points in $\Sigma_a := L_{a,R} \cap \mathbb{T}'$ is bounded independently on $a \in \mathbb{T}_2'$ because the minimal number plaques covering $\overline{L}_{a,R}$ is bounded. Fix an $a_0 \in \mathbb{T}_2'$ such that $\#\Sigma_{a_0}$ is maximal. Also by Proposition 3.2, $\#\Sigma_a$ is lower semi-continuous on $a \in \mathbb{T}_2'$. The maximality of $\#\Sigma_{a_0}$ implies that if V is a neighbourhood of a, small enough, $\#\Sigma_a = \#\Sigma_{a_0}$ for $a \in \mathbb{T}_2' \cap \overline{V}$. It follows that Σ_a depends continuously on $a \in \mathbb{T}_2' \cap \overline{V}$. We then deduce that if V is small enough, $\Sigma_a \cap \overline{V} = \{a\}$ for $a \in \mathbb{T}_2' \cap \overline{V}$. It is enough to take $\mathbb{S} := \mathbb{T}_2' \cap \overline{V}$; we have $\nu(\mathbb{S}) > 0$ by definition of $\mathbb{T}_2'$.

Consider now the family $\mathscr{G}$ of all countable unions G of such compact sets $\mathbb{S}$. Let λ denote the supremum of $\nu(G)$ for $G \in \mathscr{G}$. So, there is a sequence G_n in $\mathscr{G}$ such that $\nu(G_n) \to \lambda$. The union $G_\infty := \cup_n G_n$ is also an element of $\mathscr{G}$. So, we have $\nu(G_\infty) = \lambda$. Now, it is enough to check that $\lambda = \nu(\mathbb{T}_1)$. If not, we have $\nu(\mathbb{T}_1 \backslash G_\infty) > 0$. Hence, we can apply the above construction of $\mathbb{S}$ in $\mathbb{T}_1 \backslash G_\infty$ instead of $\mathbb{T}_1$. We necessarily have $\nu(G_\infty \cup \mathbb{S}) > \lambda$ which is a contradiction. So, we can choose compact sets $\mathbb{S}_n$ satisfying the corollary with $\cup \mathbb{S}_n = G_\infty$.   □

4 Laminations with singularities

We call *Riemann surface lamination with singularities* the data $(X, \mathscr{L}, E)$ where X is a locally compact space, E a closed subset of X and $(X \backslash E, \mathscr{L})$ is a Riemann surface lamination. The set E is *the singularity set* of the lamination. In order to simplify the presentation, we will mostly consider the case where X is a closed subset of a complex manifold M of dimension $k \geq 1$ and E is a locally finite subset of X. We assume that M is endowed with a Hermitian metric ω. We also assume that the complex structures on the leaves of the foliation coincide with the ones induced by M, that is, the leaves of $(X \backslash E, \mathscr{L})$ are Riemann surfaces holomorphically immersed in M. The main example we have in mind is a foliation by Riemann surfaces in the projective space $\mathbb{P}^k$ described in the introduction.

Proposition 4.1 *Let $(X, \mathscr{L}, E)$ be a lamination with isolated singularities in a complex manifold M as above. Let T be a positive harmonic current of $X \backslash E$. Then, the linear form $\alpha \mapsto \langle T, \alpha_{|X \backslash E} \rangle$ for $\alpha \in \mathscr{D}^{1,1}(M \backslash E)$ defines a positive $\partial\bar{\partial}$-closed current on $M \backslash E$. Moreover, it has locally finite mass on M and when X is compact, the extension of T by zero, always denoted by T, is a positive $\partial\bar{\partial}$-closed current of bidimension $(1, 1)$ on M.*

Proof Observe that $\alpha_{|X}$ is smooth with compact support in $X \backslash E$ which is positive (resp. $\partial\bar{\partial}$-exact) if α is positive (resp. $\partial\bar{\partial}$-exact). Therefore, using a partition of unity and the local description of T, we see that T defines a positive $\partial\bar{\partial}$-closed current on $X \backslash E$. Since E is finite, T has locally finite mass on M, see [20]. If moreover X is compact, the extension of T by zero is a positive $\partial\bar{\partial}$-closed current on M, see Lemma 2.5. □

So, if T is a positive harmonic current on $X \backslash E$, its mass with respect to the Hermitian metric on M is locally finite. We call *Poincaré mass* of T the mass of T with respect to Poincaré metric ω_P on $X \backslash E$, i.e. the mass of the positive measure $m_P := T \wedge \omega_P$. A priori, Poincaré mass may be infinite near the singular points. The following proposition gives us a criterion for the finiteness of this mass. It can be applied to generic foliations in $\mathbb{P}^k$.

We say that a vector field F on $\mathbb{C}^k$ is *generic linear* if it can be written as

$$F(z) = \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j}$$

where λ_j are non-zero complex numbers. The integral curves of F define a Riemann surface foliation on $\mathbb{C}^k$. The condition $\lambda_j \neq 0$ implies that the foliation has an isolated singularity at 0. Consider a lamination X with isolated singularities E in a manifold M as above. We say that a singular point a of X is *linearizable* if there is a local holomorphic coordinates system of M near a on which the leaves of X are integral curves of a generic linear vector field.

Proposition 4.2 *Let $(X, \mathscr{L}, E)$ be a compact lamination with isolated singularities in a complex manifold M. If a is a linearizable singularity of X, then any positive harmonic current on X has locally finite Poincaré mass near a.*

Fix a positive harmonic current T on X. So, by Proposition 4.1, we can identify T with a positive $\partial\bar{\partial}$-closed current on M. For the rest of the proof, we don't need the compactness of M. Since the Poincaré metric increases when we replace M with an open subset, it is enough to consider the case where M is the polydisc $(2\mathbb{D})^k$ in $\mathbb{C}^k$ and X is the lamination associated with the vector field

$$F(z) = \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j}$$

where $\lambda_j = s_j + i t_j$ and $s_j, t_j \in \mathbb{R}$. We need the following lemma.

Lemma 4.3 *For every point $a \in \mathbb{D}^k \setminus \{0\}$, there is a holomorphic map $\tau : \mathbb{D} \to L_a \cap (2\mathbb{D})^k$ such that $\tau(0) = a$ and $\|D\tau(0)\| \geq c\|a\|\,|\log\|a\||$ where $c > 0$ is a constant independent of a.*

Proof We only have to consider a very close to 0. Let $\psi_a : \mathbb{C} \to \mathbb{C}^k \setminus \{0\}$ be the holomorphic map defined by

$$\psi_a(\xi) := \left(a_1 e^{\lambda_1 \xi}, \ldots, a_k e^{\lambda_k \xi} \right) \quad \text{for} \quad \xi \in \mathbb{C}$$

where a_j are the coordinates of a. We have $\psi_a(0) = a$ and $\psi_a(\mathbb{C})$ is an integral curve of F. Write $\xi = u + iv$ with $u, v \in \mathbb{R}$. The domain $\psi_a^{-1}(\mathbb{D}^k)$ in $\mathbb{C}$ is defined by the inequalities

$$s_j u - t_j v \leq -\log|a_j| \quad \text{for} \quad j = 1, \ldots, k.$$

So, $\psi_a^{-1}(\mathbb{D}^k)$ is a convex polygon (not necessarily bounded) which contains 0 since $\psi_a(0) = a \in \mathbb{D}^k$. Observe that the distance between 0 and the line $s_j u - t_j v = -\log|a_j|$ is proportional to $-\log|a_j|$. Therefore, $\psi_a^{-1}(\mathbb{D}^k)$ contains a disc of center 0 and of radius

$$c' \min\{-\log|a_1|, \ldots, -\log|a_k|\} \geq -c' \log\|a\|$$

for some constant $c' > 0$ independent of a.

Define the map $\tau : \mathbb{D} \to \mathbb{C}^l$ by $\tau(\xi) := \psi_a(-c' \log\|a\|\xi)$. It is clear that $\tau(0) = a$ and $\tau(\mathbb{D}) \subset L_a \cap \mathbb{D}^k$. We also have

$$\|D\tau(0)\| = -c' \log\|a\| \|D\psi_a(0)\| \geq -c' \log\|a\| \min_j |\lambda_j| \|a\|.$$

The lemma follows.   □

Proof of Proposition 4.2. We use in $\mathbb{C}^k$ the standard Kähler metric $\beta := i\partial\bar{\partial}\|z\|^2$. Recall that $\omega_P = 4\vartheta\beta$. Lemma 4.3 implies that $\vartheta(a) \lesssim \|a\|^{-2}|\log\|a\||^{-2}$. Let T be a positive $\partial\bar{\partial}$-closed current on $(2\mathbb{D})^k$. We only have to show that the integral on the the ball $B_{1/2}$, with respect to the measure $T \wedge \beta$, of the radial function

$\widetilde{\vartheta}(r) := r^{-2}|\log r|^{-2}$ is finite. Let $m(r)$ denote the mass of $T \wedge \beta$ on B_r. By Lemma 2.4, we have $m(r) \lesssim r^2$. Using an integration by parts, the considered integral is equal, up to finite constants, to

$$- \int_0^1 m(r)\widetilde{\vartheta}'(r)dr$$

It is clear that the last integral is finite. The proposition follows. □

It is shown in Candel and Gómez-Mont [9], see also [18] that when a is a hyperbolic singularity, $\vartheta(z) \to \infty$ when $z \to a$. We only consider in this paper Poincaré metric on the regular part of the lamination. It is quite often that some leaves of the lamination can be compactified near singular points by adding these points and sometimes it is natural to consider the Poincaré metric of the extended leaves. Since Poincaré metric decreases when we extend these leaves, several results we obtain also apply to extended leaves as well.

If $\mathbb{T}$ is a transversal in $X \setminus E$ then all positive harmonic currents on $(X, \mathscr{L}, E)$ have finite mass near $\mathbb{T}$ with respect to the Poincaré metric on the leaves of $X \setminus (E \cup \mathbb{T})$. This may give us a technical tool in order to study parabolic leaves. Another situation where we have currents with finite Poincaré mass is the following. Let $\pi : M' \to M$ be a proper finite holomorphic map and $(X', \mathscr{L}', E')$ be a compact Riemann surface lamination on M' with isolated singularities which is the pull-back of a lamination $(X, \mathscr{L}, E)$ with linearizable singularities as above. If T' is a positive harmonic current on M', then its Poincaré mass is bounded by the Poincaré mass of the positive harmonic current $\pi_*(T')$ of X since π contracts the Poincaré metric.

We have the following properties of positive harmonic currents where the existence of positive harmonic currents was obtained in [1,17,21]. We also refer to [8, p. 105] for a version of the proposition in the case without singularities and to [17, Th. 3.14] for a result related to the last assertion.

Proposition 4.4 *Let $(X, \mathscr{L}, E)$ be a compact Riemann surface lamination with isolated singularities in a Hermitian complex manifold (M, ω). Let $\mathscr{G}$ be the family of positive harmonic currents directed by $\mathscr{L}$ of mass 1 on X. Then, $\mathscr{G}$ is a non-empty compact simplex and for any $T \in \mathscr{G}$ there is a unique probability measure ν on the set of extremal elements in $\mathscr{G}$ such that $T = \int T' d\nu(T')$. Moreover, two different extremal elements in $\mathscr{G}$ are mutually singular.*

Proof Recall that *the mass* of T is the mass of the measure $T \wedge \omega$. The fact that $\mathscr{G}$ is compact convex is clear. By Choquet's representation theorem [11], we can decompose T into extremal elements as in the proposition. We show that the decomposition is unique. According to Choquet–Meyer's theorem [11, p. 163], it is enough to check that the cone generated by $\mathscr{G}$ is a lattice. More precisely, there is a natural order in this cone: $T_1 \prec T_2$ if $T_2 - T_1$ is in the cone. We have to show that given two elements T_1, T_2 in the cone there is a minimal element $\max\{T_1, T_2\}$ larger than T_1, T_2 and a maximal element $\min\{T_1, T_2\}$ smaller than T_1, T_2 with respect to the above order.

Define $T := T_1 + T_2$. Using the description of currents in flow boxes, we see that $T_i = \theta_i T$ for some functions θ_i in $L^1(T \wedge \omega)$, $0 \leq \theta_i \leq 1$. Define also $\max\{T_1, T_2\} := \max\{\theta_1, \theta_2\}T$ and $\min\{T_1, T_2\} := \min\{\theta_1, \theta_2\}T$. It is enough to show that these currents belong to the cone generated by $\mathscr{G}$. In a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ as above, if μ is the transversal measure associated to T, then T_i is given by harmonic functions $h_{i,a}$ on the plaques $\mathbb{B} \times \{a\}$ for μ-almost every a. The current $\min\{T_1, T_2\}$ is associated with $\min\{h_{1,a}, h_{2,a}\}$ which is positive superharmonic. Therefore, $\min\{T_1, T_2\}$ is a positive current on M and $dd^c \min\{T_1, T_2\}$ is a negative measure on $M \backslash E$. By Lemma 2.5, $\min\{T_1, T_2\}$ is harmonic and is proportional to an element of $\mathscr{G}$. It follows that $\max\{T_1, T_2\} = T - \min\{T_1, T_2\}$ is also proportional to an element of $\mathscr{G}$. This completes the proof of the first assertion in the proposition.

Let T, T' be two different extremal elements in $\mathscr{G}$. We show that they are mutually singular. Using the local description of currents, we can write $T' = \theta T + T''$ where θ is a positive function in $L^1(T \wedge \omega)$ and $T \wedge \omega$, $T'' \wedge \omega$ are mutually singular. Using also the local description of currents, we see that θT and T'' are necessarily harmonic. There is a union Σ of leaves such that T'' has no mass outside Σ and T has no mass on Σ. If T'' is non-zero, since T' is extremal, it has no mass outside Σ and then T, T' are mutually singular. Assume that $T'' = 0$.

We can find a number $c > 0$ such that $\{\theta \geq c\}$ and $\{\theta \leq c\}$ have positive measure with respect to $T \wedge \omega$. Define $T_c^+ := \max\{\theta - c, 0\}T$, $T_c^- := \max\{c - \theta, 0\}T$. Since $T_c^+ = \max\{T', cT\} - cT$ and $T_c^- = \max\{cT, T'\} - T'$, these currents are harmonic. So, we can choose a set Σ' which is a union of leaves such that T_c^+ has no mass outside Σ' and $\theta > c$ on Σ'. The choice of c implies that T has positive mass outside Σ'. Since T is extremal, we deduce that T has no mass on Σ'. It follows that $T_c^+ = 0$ and then $\theta \leq c$ almost everywhere. Using T_c^-, we prove in the same way that $\theta \geq c$ almost everywhere. Finally, we have $T' = cT$ and since T, T' have the same mass we get $T = T'$. This is a contradiction. $\square$

Recall that a leaf L in $(X, \mathscr{L}, E)$ is *wandering* if it is not closed in $X \backslash E$ and if there is a point $p \in L$ and a flow box $\mathbb{U}$ containing p such that $L \cap \mathbb{U}$ is just one plaque. Note that if L is wandering, the above property is true for every $p \in L$. The set of closed leaves and the set of wandering leaves are measurable. We get as for real smooth foliations [8, p. 113] the following result.

Theorem 4.5 *Let M be a complex manifold and ω a Hermitian form on M. Let $(X, \mathscr{L}, E)$ be a compact lamination with isolated singularities in M and T a positive harmonic current on X. Then the set of wandering leaves of X has zero measure with respect to the measure $T \wedge \omega$.*

Proof Assume that the set Y of wandering leaves has positive $T \wedge \omega$ measure. Choose a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ such that $Y \cap U$ has positive measure. For each $m \geq 1$, cover $\mathbb{T}$ by a finite family of open sets $\mathbb{S}_{m,n}$ of diameter $\leq 1/m$. Denote by $Y_{m,n}$ the set of leaves which intersect $\mathbb{S}_{m,n}$ at only one point. So, the union of $Y_{m,n}$ has positive measure since it contains $Y \cap \mathbb{U}$. We can choose m, n such that $Y_{m,n}$ has positive measure.

By Proposition 2.3, the restriction of T to $Y_{m,n}$ is a harmonic current. So, we can restrict T to $Y_{m,n}$ in order to assume that T has no mass outside $Y_{m,n}$. We deduce from

the definition of $Y_{m,n}$ that T can be decomposed into extremal harmonic currents supported on one single leaf of $Y_{m,n}$. We can assume that T is extremal. So, there is a leaf L in $Y_{m,n}$ and a positive harmonic function h such that $T = h[L]$. We show that L is closed in $X \setminus E$ and this will give us a contradiction. For this purpose, one only has to prove that L has finite area.

Observe that h is integrable on L with respect to the metric ω since it defines a positive current on M. Therefore, it is enough to show that h is constant. Assume that h is not constant. Let $t < s$ be two different values of h. Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth concave increasing function such that $\chi(x) = x$ if $x \leq t$ and $\chi(x) = s$ if $x \geq s + 1$. So, $\chi(h) \leq h$ and then $\chi(h)$ is integrable on L. Moreover, we have since h is harmonic

$$i\partial\overline{\partial}\chi(h) = \chi'(h)i\partial\overline{\partial}h + \chi''(h)i\partial h \wedge \overline{\partial}h = \chi''(h)i\partial h \wedge \overline{\partial}h \leq 0.$$

We deduce using flow boxes that $T' := \chi(h)[L]$ is a positive current with compact support in M such that $i\partial\overline{\partial}T' \leq 0$ on $M \setminus E$. By Lemma 2.5, $i\partial\overline{\partial}T' = 0$ and then $\chi(h)$ is harmonic. Finally, since $\chi(h)$ has a maximum on L, it is constant. This gives a contradiction. □

5 Heat equation on a real lamination

Consider a real lamination $(X, \mathscr{L})$ of dimension n as in Sect. 2. We assume that X is compact and endowed with a continuously smooth Riemannian metric g on the leaves. By continuous smoothness, we mean that in flow boxes the coefficients of g and their derivatives, of any order, along the plaques depend continuously on the plaques. So, we can consider the corresponding Laplacian Δ and the gradient ∇ on leaves. Several results below for Δ can be deduced from Candel–Conlon [8] and Garnett [21] but results on $\widetilde{\Delta}$ are new. Observe that in our context Δ is not symmetric in $L^2(m)$.

A measure m on X is called g-*harmonic* (or simply *harmonic* when there is no confusion) if

$$\int \Delta u \, dm = 0 \quad \text{for} \quad u \in \mathscr{D}^0(X).$$

We will consider the operator Δ in $L^2(m)$ and the above identity holds for u in the domain $\text{Dom}(\Delta)$ of Δ that we will define later. We have the following elementary result.

Proposition 5.1 *Let Ω be the volume form associated with g. Then there is a one to one correspondence between harmonic (resp. positive harmonic) measures m and harmonic (resp. positive harmonic) current T such that $m = T \wedge \Omega$.*

Proof It is clear that $T \wedge \Omega$ is a harmonic (positive) measure if T is a harmonic (positive) 0-current. Consider now a decomposition of m in a flow box as in Proposition 2.1

$$m = \int m_a \, d\mu(a)$$

with m_a a measure on $\mathbb{B} \times \{a\}$. Since m is harmonic, we deduce that m_a is harmonic in the flow box for μ-almost every a. So, by Weyl's lemma, there is a harmonic function h_a such that $m_a = h_a \Omega$. We have $m = T \wedge \Omega$ where T is locally given by

$$T = \int h_a [\mathbb{B} \times \{a\}] d\mu(a),$$

where $[\mathbb{B} \times \{a\}]$ is the current of integration on the plaque $\mathbb{B} \times \{a\}$. When m is positive, it is easy to see that h_a is positive for μ-almost every a. The result follows. $\square$

Consider a harmonic probability measure m on X. Write $m = T \wedge \Omega$ as in Proposition 5.1. In a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$, by Proposition 2.1, the current T can be written as

$$T = \int h_a [\mathbb{B} \times \{a\}] d\mu(a),$$

where h_a is a positive harmonic function on $\mathbb{B}$ and μ is a positive measure on the transversal $\mathbb{T}$. We can restrict μ in order to assume that $h_a \neq 0$ for μ-almost every a. Under this condition, Proposition 2.2 says that up to a multiplicative function, μ and h_a are uniquely determined by T.

In what follows, the differential operators ∇, Δ and $\widetilde{\Delta}$ are considered in $L^2(m)$. We introduce the Hilbert space $H^1(m)$ as the completion of $\mathscr{D}^0(X)$ with respect to the norm

$$\|u\|_{H^1}^2 := \int |u|^2 dm + \int |\nabla u|^2 dm.$$

Recall that the gradient ∇ is defined by

$$\langle \nabla u, \xi \rangle_g = du(\xi)$$

for all tangent vector ξ along a leaf and for $u \in \mathscr{D}^0(X)$. We consider ∇ as an operator in $L^2(m)$ and $H^1(m)$ its domain.

Define in a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ as above the Laplace type operator

$$-\widetilde{\Delta} u = -\Delta u - \langle h_a^{-1} \nabla h_a, \nabla u \rangle_g = -\Delta u - Fu,$$

where F is a vector field. The uniqueness of h_a and μ implies that F does not depend on the choice of the flow box. Therefore, Fu and $\widetilde{\Delta} u$ are defined globally m-almost everywhere when $u \in \mathscr{D}^0(X)$.

We recall some classical results of functional analysis that we will use later. The reader will find an exposition in Brezis [5]. A linear operator A on a Hilbert space L is called *monotone* if $\langle Au, u \rangle \geq 0$ for all u in the domain $\mathrm{Dom}(A)$ of A. Such an operator is *maximal monotone* if moreover for any $f \in L$ there is a $u \in \mathrm{Dom}(A)$ such that $u + Au = f$. In this case, the domain of A is always dense in L and the graph of A is closed.

A family $S(t) : L \to L, t \in \mathbb{R}_+$, is *a semi-group of contractions* if $S(t + t') = S(t) \circ S(t')$ and if $\|S(t)\| \leq 1$ for all $t, t' \geq 0$. We will apply the following theorem to our Laplacian operators and for $L := L^2(m)$. It says that any maximal monotone operator is the infinitesimal generator of a semi-group of contractions.

Theorem 5.2 (Hille–Yosida) *Let A be a maximal monotone operator on a Hilbert space L. Then there is a semi-group of contractions $S(t) : L \to L, t \in \mathbb{R}_+$, such that for $u_0 \in \mathrm{Dom}(A)$, $u(t, \cdot) := S(t)u_0$ is the unique $\mathscr{C}^1$ function from $\mathbb{R}_+$ to L with values in $\mathrm{Dom}(A)$ which satisfies*

$$\frac{\partial u(t, \cdot)}{\partial t} + Au(t, \cdot) = 0 \quad and \quad u(0, \cdot) = u_0.$$

When A is self-adjoint and $u_0 \in L$, the function $u(t, \cdot)$ is still continuous on $\mathbb{R}_+$ and is $\mathscr{C}^1$ on $\mathbb{R}_+^*$ with values in $\mathrm{Dom}(A)$ and we have the estimate

$$\left\| \frac{\partial u}{\partial t} \right\| \leq \frac{1}{t} \|u_0\| \quad for\ t > 0.$$

In order to check that our operators are maximal monotone, we will apply the following result to $H := H^1(m)$.

Theorem 5.3 (Lax–Milgram) *Let $e(u, v)$ be a continuous bilinear form on a Hilbert space H. Assume that $e(u, u) \geq \|u\|_H^2$ for $u \in H$. Then for every f in the dual H^* of H there is a unique $u \in H$ such that $e(u, v) = \langle f, v \rangle$ for $v \in H$.*

Define for $u, v \in \mathscr{D}^0(X)$

$$q(u, v) := - \int (\Delta u)v dm, \qquad e(u, v) := q(u, v) + \int uv dm$$

and

$$\tilde{q}(u, v) := - \int (\tilde{\Delta} u)v dm = q(u, v) - \int (Fu)v dm,$$

$$\tilde{e}(u, v) := \tilde{q}(u, v) + \int uv dm.$$

Note that these identities still hold for $v \in L^2(m)$ and u in the domain of Δ and of $\tilde{\Delta}$ that we will define later.

Lemma 5.4 *We have for $u, v \in \mathscr{D}^0(X)$*

$$\tilde{q}(u, v) = \int \langle \nabla u, \nabla v \rangle_g dm \quad and \quad \int (\tilde{\Delta} u)v dm = \int u(\tilde{\Delta} v)dm.$$

In particular, $\tilde{q}(u, v)$ and $\tilde{e}(u, v)$ are symmetric in u, v and

$$\int \tilde{\Delta} u dm = \int \Delta u dm = \int Fu dm = 0 \quad for\ \ u \in \mathscr{D}^0(X).$$

Proof Using a partition of unity, we can assume that u and v have compact support in a flow box as above. It is then enough to consider the case where T is supported by a plaque $\mathbb{B} \times \{a\}$ and given by a harmonic function h_a. The first identity in the lemma can be deduced from classical identities in Riemannian geometry, see e.g. [10]. Indeed, we have

$$\langle \nabla u, \nabla v \rangle_g + v \Delta u = \text{div}(v \nabla u)$$

and the first identity is equivalent to

$$\int_{\mathbb{B}} \text{div}(v \nabla u) h_a \Omega = - \int_{\mathbb{B}} (Fu) v h_a \Omega$$

which is easily obtained by integration by parts.

It follows from the first identity in the lemma that $\widetilde{q}$ and $\widetilde{e}$ are symmetric. The second assertion in the lemma is a consequence. We then deduce the last identities in the lemma using that $\widetilde{\Delta} 1 = 0$ and that m is harmonic. Note that the lemma still holds for u, v in the domain of Δ and $\widetilde{\Delta}$ that we will define later. $\qquad\square$

Lemma 5.5 *The bilinear forms* $q, \widetilde{q}, e$ *and* $\widetilde{e}$ *extend continuously to* $H^1(m) \times H^1(m)$. *Moreover, we have* $q(u, u) = \widetilde{q}(u, u)$ *and* $e(u, u) = \widetilde{e}(u, u)$ *for* $u \in H^1(m)$.

Proof The first identity in Lemma 5.4 implies that $\widetilde{q}$ and $\widetilde{e}$ extend continuously to $H^1(m)$ and the identity is still valid for the extension of $\widetilde{q}$. In order to prove the same property for q and e, it is enough to show that $q - \widetilde{q}$ is bounded on $H^1(m) \times H^1(m)$.

We use the description of $\widetilde{\Delta}$ in a flow box $\mathbb{U}$ as above. By Harnack's inequality, $h_a^{-1} \nabla h_a$ are locally bounded uniformly on a. Since X is compact, we deduce that the vector field F has bounded coefficients. Therefore, by Cauchy–Schwarz's inequality

$$|q(u, v) - \widetilde{q}(u, v)| = \left| \int (Fu) v \, dm \right| \leq C \left(\int |\nabla u|^2 dm \right)^{1/2} \left(\int v^2 dm \right)^{1/2}.$$

So, $q - \widetilde{q}$ is bounded and hence q, e extend continuously on $H^1(m) \times H^1(m)$.

We prove now the identities in the lemma. We can assume that u is in $\mathscr{D}^0(X)$ since this space is dense in $H^1(m)$. We have

$$\Delta(u^2) = \text{div}(\nabla u^2) = \text{div}(2u \nabla u) = 2u \Delta u + 2|\nabla u|^2.$$

Since m is harmonic, the integral of $\Delta(u^2)$ vanishes. It follows that

$$- \int u \Delta u \, dm = \int |\nabla u|^2 dm.$$

We deduce from the definition of q, e and the first identity in Lemma 5.4 that $q(u, u) = \widetilde{q}(u, u)$ and $e(u, u) = \widetilde{e}(u, u)$. This completes the proof. $\qquad\square$

Define the domain Dom(Δ) of Δ (resp. Dom($\widetilde{\Delta}$) of $\widetilde{\Delta}$) as the space of $u \in H^1(m)$ such that $q(u, \cdot)$ (resp. $\widetilde{q}(u, \cdot)$) extends to a linear continuous form on $L^2(m)$. Since $\Delta - \widetilde{\Delta}$ is given by a vector field with bounded coefficients, we have Dom(Δ) = Dom($\widetilde{\Delta}$). In a flow box, we can show using Federer's version of Lusin's theorem [15, Th. 2.3.5] that a function u in $L^2(m)$ belongs to $H^1(m)$ if the gradient ∇u, defined as a vector field with distribution coefficients on generic plaques, is in $L^2(m)$. An analogous property holds for Δ and $\widetilde{\Delta}$. In fact, Dom(Δ) is the completion of $\mathscr{D}^0(X)$ for the norm $\sqrt{\|u\|_{L^2(m)}^2 + \|\Delta u\|_{L^2(m)}^2}$. It is clear that if $u \in$ Dom(Δ) then Δu in the sense of distributions with respect to $\mathscr{D}^0(X)$ as test functions, is in L^2. This allows us to extend Lemma 5.4 to u, v in Dom(Δ). The converse is true but we don't use it. We have the following proposition.

Proposition 5.6 *Let $(X, \mathscr{L})$ be a compact real lamination of dimension n endowed with a continuously smooth Riemannian metric g on the leaves. Let m be a harmonic probability measure on X. Then the associated operators $-\Delta$ and $-\widetilde{\Delta}$ are maximal monotone on $L^2(m)$. In particular, they are infinitesimal generators of semi-groups of contractions on $L^2(m)$ and their graphs are closed.*

Proof The last assertion is the consequence of the first one, Theorem 5.2 and the properties of maximal monotone operators. So, we only have to prove the first assertion. We deduce from Lemmas 5.4 and 5.5 that for $u \in \mathscr{D}^0(X)$

$$\langle -\Delta u, u \rangle = \langle -\widetilde{\Delta} u, u \rangle = \int \langle \nabla u, \nabla u \rangle_g dm = \|\nabla u\|_{L^2}^2 \geq 0.$$

By continuity, we can extend the inequalities to u in Dom($-\Delta$) = Dom($-\widetilde{\Delta}$). So, $-\Delta$ and $-\widetilde{\Delta}$ are monotone. We also obtain for $u \in H^1(m)$ that

$$e(u, u) = \widetilde{e}(u, u) \geq \|u\|_{H^1}^2.$$

By Theorem 5.3, for any $f \in L^2(m)$, there is $u \in H^1(m)$ such that

$$e(u, v) = \langle f, v \rangle_{L^2(m)} \quad \text{for} \quad v \in H^1(m).$$

So, u is in Dom(Δ) and the last equation is equivalent to $u - \Delta u = f$. Hence, $-\Delta$ is maximal monotone. The case of $-\widetilde{\Delta}$ is treated in the same way. Note that since $-\widetilde{\Delta}$ is symmetric and maximal monotone, it is self-adjoint but $-\Delta$ is not symmetric. □

We have the following theorem.

Theorem 5.7 *Under the hypothesis of Proposition 5.6, let $S(t), t \in \mathbb{R}_+$, denote the semi-group of contractions associated with the operator $-\Delta$ or $-\widetilde{\Delta}$ which is given by the Hille–Yosida theorem. Then the measure m is $S(t)$-invariant and $S(t)$ is a positive contraction in $L^p(m)$ for all $1 \leq p \leq \infty$.*

Proof We prove that m is invariant, that is

$$\langle m, S(t)u_0 \rangle = \langle m, u_0 \rangle \quad \text{for} \quad u_0 \in \mathscr{D}^0(X).$$

We will see later that this identity holds also for $u_0 \in L^1(m)$ because $S(t)$ is a contraction in $L^1(m)$ and $\mathscr{D}^0(X)$ is dense in $L^1(m)$. Define $u := S(t)u_0$ and

$$\eta(t) := \langle m, S(t)u_0 \rangle = \langle m, u(t, \cdot) \rangle.$$

We deduce from Theorem 5.2 that η is of class $\mathscr{C}^1$ on $\mathbb{R}_+$ and that

$$\eta'(t) = \langle m, S'(t)u_0 \rangle = \langle m, Au(t, \cdot) \rangle$$

where A is the operator $-\Delta$ or $-\widetilde{\Delta}$. By Lemma 5.4, the last integral vanishes. So, η is constant and hence m is invariant.

In order to prove the positivity of $S(t)$, it is enough to show the following *maximum principle*: if u_0 is a function in $\mathscr{D}^0(X)$ such that $u_0 \leq K$ for some constant K, then $u(t, x) \leq K$. To show the maximum principle we use a trick due to Stampacchia [5]. Fix a smooth bounded function $G : \mathbb{R} \to \mathbb{R}_+$ with bounded first derivative such that $G(t) = 0$ for $t \leq 0$ and $G'(t) > 0$ for $t > 0$. Put

$$H(s) := \int_0^s G(t)dt.$$

Consider the non-negative function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\xi(t) := \int H(u(t, \cdot) - K)dm.$$

By Theorem 5.2, ξ is of class $\mathscr{C}^1$. We want to show that it is identically zero. Define $v(t, x) := u(t, x) - K$. We have $Av(t, x) = Au(t, x)$. Using in particular that G is bounded, we obtain

$$\xi'(t) = \int G(u(t, \cdot) - K)\frac{\partial u(t, \cdot)}{\partial t}dm$$

$$= -\int G(u(t, \cdot) - K)Au(t, \cdot)dm$$

$$= -\int G(v(t, \cdot))Av(t, \cdot)dm.$$

When $A = -\widetilde{\Delta}$, by Lemma 5.4, the last integral is equal to

$$-\int \langle \nabla G(v), \nabla v \rangle_g dm = -\int G'(v)|\nabla v|^2 dm \leq 0.$$

Thus, $\xi'(t) \leq 0$. We deduce that $\xi = 0$ and hence $u(x, t) \leq K$.

When $A = -\Delta$, since $F = \widetilde{\Delta} - \Delta$ is a vector field with bounded coefficients, the considered integral is equal to

$$-\int G'(v)|\nabla v|^2 dm + \int G(v)Fvdm = -\int G'(v)|\nabla v|^2 dm + \int FH(v)dm.$$

By Lemma 5.4, the last integral vanishes. So, we also obtain that $\xi'(t) \leq 0$. This completes the proof of the maximum principle which implies the positivity of $S(t)$.

The positivity of $S(t)$ together with the invariance of m imply that

$$\|S(t)u_0\|_{L^1(m)} \leq \|u_0\|_{L^1(m)} \quad \text{for} \quad u_0 \in \mathscr{D}^0(X).$$

It follows that $S(t)$ extends continuously to a positive contraction in $L^1(m)$ since $\mathscr{D}^0(X)$ is dense in $L^1(m)$. The uniqueness of the solution in Theorem 5.2 implies that $S(t)1 = 1$. This together with the positivity of $S(t)$ imply that $S(t)$ is a contraction in $L^\infty(m)$. Finally, the classical theory of interpolation between the Banach spaces $L^1(m)$ and $L^\infty(m)$ implies that $S(t)$ is a contraction in $L^p(m)$ for all $1 \leq p \leq \infty$, see Triebel [36]. $\square$

We have the following proposition which can be applied to functions whose derivatives of orders 1 and 2 are in $L^2(m)$.

Proposition 5.8 *Let $(X, \mathscr{L})$, g and m be as in Proposition 5.6. Then every function u_0 in $\mathrm{Dom}(\Delta)$ (which is equal to $\mathrm{Dom}(\widetilde{\Delta})$) such that $\Delta u_0 \geq 0$ (resp. $\widetilde{\Delta} u_0 \geq 0$) is constant on the leaf L_a for m-almost every a. Moreover, if m is an extremal positive harmonic measure, then u_0 is constant m-almost everywhere.*

Proof We know that

$$\int \Delta u_0 dm = \int \widetilde{\Delta} u_0 dm = 0.$$

So, the hypothesis implies that $\Delta u_0 = 0$ (resp. $\widetilde{\Delta} u_0 = 0$). By Lemmas 5.4 and 5.5, we deduce that

$$\int |\nabla u_0|^2 dm = -\int (\widetilde{\Delta} u_0)u_0 dm = -\int (\Delta u_0)u_0 dm = 0.$$

It follows that $\nabla u_0 = 0$ almost everywhere with respect to m. Thus, u_0 is constant on L_a for m-almost every a. When m is extremal, this property implies that u_0 is constant m-almost everywhere, since every measurable set of leaves has zero or full m measure. $\square$

We deduce from the above results the following ergodic theorem.

Corollary 5.9 *Under the hypothesis of Theorem 5.7, for all $u_0 \in L^p(m)$, $1 \leq p < \infty$, the average*

$$\frac{1}{R} \int_0^R S(t)u_0 dt$$

converges pointwise m-almost everywhere and also in $L^p(m)$ to an $S(t)$-invariant function u_0^* when R goes to infinity. Moreover, u_0^* is constant on the leaf L_a for m-almost every a. If m is an extremal harmonic measure, then u_0^* is constant m-almost everywhere.

Proof The first assertion is a consequence of the ergodic theorem as in Dunford–Schwartz [14, Th. VIII.7.5]. We get a function u_0^* which is $S(t)$-invariant. For the rest of the proposition, since $S(t)$ is a contraction in $L^p(m)$, it is enough to consider the case where u_0 is in $\mathscr{D}^0(X)$.

By Proposition 5.8, we only have to check that u_0^* is in the domain of A and $Au_0^* = 0$. Define

$$u_R := \frac{1}{R} \int_0^R S(t)u_0 dt.$$

This function belongs to $\mathrm{Dom}(A)$. Since u_R converges to u_0^* in $L^2(m)$ and the graph of A is closed in $L^2(m) \times L^2(m)$, it is enough to show that $Au_R \to 0$ in $L^2(m)$. We have

$$Au_R = \frac{1}{R} \int_0^R Au(t, \cdot) dt = -\frac{1}{R} \int_0^R \frac{\partial}{\partial t} u(t, \cdot) dt = \frac{1}{R} u_0 - \frac{1}{R} u(R, \cdot).$$

Since $S(t)$ is a contraction in $L^2(m)$, the last expression tends to 0 in $L^2(m)$. The result follows. $\square$

We will need the following lemma.

Lemma 5.10 *Let $\widehat{m} = \theta m$ be a harmonic measure, not necessarily positive, where θ is a function in $L^2(m)$. Let $\widehat{m} = \widehat{m}^+ - \widehat{m}^-$ be the minimal decomposition of $\widehat{m}$ as the difference of two positive measures. Then $\widehat{m}^\pm$ are harmonic.*

Proof Let $S(t)$ be the semi-group of contractions in $L^1(m)$ associated with $-\Delta$ as above. Define the action of $S(t)$ on measures by

$$\langle S(t)\widehat{m}, u_0 \rangle := \langle \widehat{m}, S(t)u_0 \rangle \quad \text{for} \quad u_0 \in L^2(m).$$

Consider a function $u_0 \in \mathscr{D}^0(X)$ and define $\eta(t) := \langle S(t)\widehat{m}, u_0 \rangle$. By Theorem 5.2, this is a $\mathscr{C}^1$ function on $\mathbb{R}_+$. We have since $\widehat{m}$ is harmonic and θ is in $L^2(m)$

$$\eta'(t) = \langle \widehat{m}, S'(t)u_0 \rangle = \langle \widehat{m}, -\Delta(S(t)u_0) \rangle = 0.$$

To see the last equality, we can use a partition of unity and the local description of $\widehat{m}$ on a flow box. So, η is constant. It follows that $S(t)\widehat{m} = \widehat{m}$. Since $S(t)$ is a positive contraction, we deduce that $S(t)\widehat{m}^{\pm} = \widehat{m}^{\pm}$. So, the functions $\eta^{\pm}(t) := \langle \widehat{m}^{\pm}, S(t)u_0 \rangle$ are constant. As above, we have

$$\langle \widehat{m}^{\pm}, \Delta u_0 \rangle = -\langle \widehat{m}^{\pm}, S'(0)u_0 \rangle = (\eta^{\pm})'(0) = 0.$$

Hence, $\widehat{m}^{\pm}$ are harmonic.   □

We also obtain the following result, see Candel–Conlon [8].

Corollary 5.11 *Under the hypothesis of Proposition* 5.6, *the family $\mathscr{H}$ of harmonic probability measures on X is a non-empty compact simplex and for any $m \in \mathscr{H}$ there is a unique probability measure v on the set of extremal elements in $\mathscr{H}$ such that $m = \int m' dv(m')$. Moreover, two different extremal harmonic probability measures are mutually singular.*

Proof Elements in $\mathscr{H}$ are defined as probability measures m such that $\langle m, \Delta u \rangle = 0$ for $u \in \mathscr{D}^0(X)$. It is clear that $\mathscr{H}$ is convex and compact. The fact that $\mathscr{H}$ is non-empty is well-known and is a consequence of the Hahn–Banach theorem, see Garnett [21]. Indeed, by maximum principle for subharmonic functions, the distance in $\mathscr{D}_0^0(X)$ between 1 and the space $\{\Delta\varphi, \ \varphi \in \mathscr{D}^0(X)\}$ is equal to 1. Therefore, Hahn–Banach's theorem implies the existence of a linear form m on $\mathscr{D}_0^0(X)$ of norm ≤ 1 vanishing on $\{\Delta\varphi, \ \varphi \in \mathscr{D}^0(X)\}$ such that $m(1) = 1$. The form defines a harmonic measure of mass ≤ 1 which is necessarily a probability measure because $m(1) = 1$.

By Choquet's representation theorem [11], we can decompose m into extremal measures as in the corollary. We show that the decomposition is unique. Define $m := m_1 + m_2$ and θ_i a function in $L^1(m), 0 \leq \theta_i \leq 1$, such that $m_i = \theta_i m$. Define also $m_1 \vee m_2 := \max\{\theta_1, \theta_2\}m$ and $m_1 \wedge m_2 := \min\{\theta_1, \theta_2\}m$. As in Proposition 4.4, it is enough to show that these measures belong to the cone generated by $\mathscr{H}$. By Lemma 5.10, since $m_1 - m_2$ is harmonic, $m' := \max\{\theta_1 - \theta_2, 0\}m$ is harmonic. It follows that $m_1 \vee m_2 = m' + m_2$ and $m_1 \wedge m_2 = m_1 - m'$ are harmonic. This completes the proof of the first assertion as a consequence of the Choquet–Meyer theorem [11, p. 163].

Consider now two different extremal elements m, m' in $\mathscr{H}$. We show that they are mutually singular. Consider the decomposition of m and m' in a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ as above

$$m = \int h_a[\mathbb{B} \times \{a\}] \wedge \Omega d\mu(a) \quad \text{and} \quad m' = \int h'_a[\mathbb{B} \times \{a\}] \wedge \Omega d\mu'(a).$$

We can restrict μ, μ' in order to assume that $h_a \neq 0$ for μ-almost every a and $h'_a \neq 0$ for μ'-almost every a. Write $\mu' = \varphi\mu + \mu''$ where φ is a positive function in $L^1(\mu)$ and μ, μ'' are mutually singular. Choose a measurable set A on the transversal $\mathbb{T}$ such that $\mu''(\mathbb{T}\backslash A) = 0$ and $\mu(A) = 0$. Let Σ denote the union of the leaves which intersect A. Then, m has no mass on Σ. If μ'' is non-zero, then m' has positive mass on Σ. Since m' is extremal, it has no mass outside Σ. We deduce that m and m' are mutually singular.

Assume now that $\mu'' = 0$. Multiplying h_a by $\varphi(a)$ allows us to assume that $\mu = \mu'$. Therefore, there is a non-negative function $\theta \in L^1(m)$ such that $m' = \theta m$. We can find a number $c > 0$ such that $\{\theta \geq c\}$ and $\{\theta \leq c\}$ have positive m-measure. Define $m_c^+ := \max\{\theta - c, 0\}m$ and $m_c^- := \max\{c - \theta, 0\}m$. By Lemma 5.10 applied to $m + m'$, $m' - cm$ instead of m, m' and a bounded function instead of θ, the measures $m_c^\pm$ are harmonic. So, we can choose a set Σ' which is a union of leaves such that m_c^+ has no mass outside Σ' and $\theta > c$ on Σ'. The choice of c implies that m has positive mass outside Σ'. Since m is extremal, we deduce that m has no mass on Σ'. It follows that $m_c^+ = 0$ and then $\theta \leq c$ almost everywhere. Using m_c^-, we prove in the same way that $\theta \geq c$ almost everywhere. Finally[1], we have $m' = cm$ and since m, m' have the same mass, we get $m = m'$. This is a contradiction and completes the proof. $\qquad\square$

The following result gives a version of the mixing property in our context. The classical case is due to Kaimanovich [26] who uses in particular the smoothness of the Brownian motion, see also Candel [7] who relies on a version of the zero-two law due to Ornstein and Sucheston [32].

Theorem 5.12 *Under the hypothesis of Theorem 5.7, assume moreover that m is extremal. If $S(t)$ is associated to $-\widetilde{\Delta}$, then $S(t)u_0$ converge to $\langle m, u_0 \rangle$ in $L^p(m)$ when $t \to \infty$ for $u_0 \in L^p(m)$ with $1 \leq p < \infty$. In particular, $S(t)$ is mixing, i.e.*

$$\lim_{t \to \infty} \langle S(t)u_0, v_0 \rangle = \langle m, u_0 \rangle \langle m, v_0 \rangle \quad \text{for} \quad u_0, v_0 \in L^2(m).$$

Proof Since $S(t)$ is a contraction in $L^p(m)$ for $1 \leq p \leq \infty$, it is enough to show the above convergence for u_0 in a dense subspace of $L^p(m)$. If u_0 is bounded, we have, $\|S(t)u_0\|_\infty \leq \|u_0\|_\infty$. Therefore, we only have to show the above convergence in $L^1(m)$. Since $S(t)$ preserves constant functions, we can assume without loss of generality that $\langle m, u_0 \rangle = 0$. We can also assume that $u_0 \in H^1(m)$ and we will show that $S(t)u_0 \to 0$ in $H^1(m)$.

Define $u := S(t)u_0$. Using Cauchy–Schwarz's inequality, Lemma 5.5 and the last assertion in Theorem 5.2, we obtain that

$$\|\nabla u\|_{L^2(m)}^2 \leq \|\widetilde{\Delta}u\|_{L^2(m)} \|u\|_{L^2(m)} \lesssim \frac{1}{t} \|u_0\|_{L^2(m)}^2.$$

It follows that $S(t)$ is bounded in $H^1(m)$ uniformly on t. So, we can consider a limit value u_∞ of $S(t)u_0$, $t \to \infty$, in the weak sense in $H^1(m)$. We have

$$\|\nabla u_\infty\|_{L^2(m)} \leq \limsup_{t \to \infty} \|\nabla u\|_{L^2(m)} = 0.$$

Using the description of m and u_∞ in flow boxes, we deduce that u_∞ is constant on almost every leaf with respect to m. Since m is extremal, u_∞ is constant m-almost everywhere. Finally, since $S(t)$ preserves m, we have $\langle m, S(t)u_0 \rangle = 0$ and

---

[1] We didn't find a simpler argument. The fact that $q(u, v)$ is not symmetric is a difficulty.

then $\langle m, u_\infty \rangle = 0$. It follows that $u_\infty = 0$. We deduce that $S(t)u_0 \to 0$ weakly in $H^1(m)$.

Now, recall that $-\widetilde{\Delta}$ is self-adjoint on $L^2(m)$. Hence, $S(t)$ is also self-adjoint since it is obtained as the limit of $(I - tn^{-1}\widetilde{\Delta})^n$ where I is the identity operator. Therefore, we have

$$\|S(t)u_0\|_{L^2}^2 = \langle S_{2t}u_0, u_0 \rangle_{L^2(m)} \to 0.$$

It follows that $S(t)u_0 \to 0$ in $H^1(m)$. Note that using E.M. Stein's theorem in [34] on self-adjoint contractions of L^p, we can prove that $S(t)u_0 \to \langle m, u_0 \rangle$ m-almost everywhere for $u_0 \in L^1(m)$. □

Proposition 5.13 *Let* $m = \int m' d\nu(m')$ *be as in Corollary* 5.11. *Then, the closures of* $\Delta(\mathscr{D}^0(X))$ *and of* $\widetilde{\Delta}(\mathscr{D}^0(X))$ *in* $L^p(m)$, $1 \leq p \leq 2$, *are the space of functions* $u_0 \in L^p(m)$ *such that* $\int u_0 dm' = 0$ *for* ν-*almost every* m'. *In particular, if* m *is an extremal harmonic probability measure, then this space is the hyperplane of* $L^p(m)$ *defined by the equation* $\int u_0 dm = 0$.

Proof We only consider the case of Δ; the case of $\widetilde{\Delta}$ can be treated similarly. It is clear that $\Delta(\mathscr{D}^0(X))$ is a subset of the space of $u_0 \in L^p(m)$ such that $\int u_0 dm' = 0$ for ν-almost every m' and the last space is closed in $L^p(m)$. Consider a function $\theta \in L^q(m)$, with $1/p + 1/q = 1$, which is orthogonal to $\Delta(\mathscr{D}^0(X))$. So, θm is a harmonic measure. Since $p \leq 2$, we have $\theta \in L^2(m)$. We have to show that θ is constant with respect to ν-almost every m'.

Consider the disintegration of m along the fibers of θ. There is a probability measure ν' on $\mathbb{R}$ and probability measures m_c on $\{\theta = c\}$ such that $m = \int m_c d\nu'(c)$. By Lemma 5.10, for any $c \in \mathbb{R}$, the measure $\max\{\theta, c\}m$ is harmonic. Therefore, m_c is harmonic for ν'-almost every c. If ν_c is the probability measure associated with m_c as in Corollary 5.11, we deduce from the uniqueness in this lemma that

$$\nu = \int \nu_c d\nu'(c).$$

Now, since θ is constant m_c-almost everywhere, it is constant with respect to ν_c-almost every m'. We deduce from the above identity that θ is constant for ν-almost every m'. This completes the proof. □

Remark 5.14 We will see in the next section an analogous development for the heat equation on Riemann surface laminations with isolated singularities. It is known that a compact Riemann surface lamination with a tame set of singularities always admits a harmonic probability measure [1]. For Riemannian laminations it would be interesting to find natural hypothesis on singularities which guarantees the existence of such a measure.

6 Case of singular Riemann surface laminations

We will consider in this section the heat equation for positive harmonic currents associated with a compact Riemann surface lamination possibly with singularities. The

main example we have in mind is the case of a current supported on the set of hyper-bolic leaves which are endowed with the Poincaré metric. Since in general, Poincaré metric does not depend continuously on the leaves, it is important that we can relax the strong regularity of the metric which is a necessary condition in the classical setting, see [7,21].

Consider a more general situation. Let T be a positive $\partial\bar{\partial}$-closed current of bidi-mension $(1, 1)$ with compact support in a complex manifold M of dimension k. For simplicity, fix a Hermitian form ω on M. So, $T \wedge \omega$ is a positive measure. We assume that T is regular in the sense of [1], that is, there exists a $(1, 0)$-form τ defined almost everywhere with respect to $T \wedge \omega$ such that

$$\partial T = \tau \wedge T \quad \text{and} \quad \int i\tau \wedge \bar{\tau} \wedge T < \infty.$$

In this context, the Hörmander L^2-estimates are proved in [1] for the $\bar{\partial}$-equation induced on T. Fix also a positive $(1, 1)$-form β which is defined almost everywhere with respect to $T \wedge \omega$ such that $T \wedge \beta$ is of finite mass. We assume that β is *strictly T-positive* in the sense that $T \wedge \omega$ is absolutely continuous with respect to $T \wedge \beta$. This condition[2] does not depend on the choice of ω and allows us to define the operators ∇_β^∂, $\nabla_\beta^{\bar{\partial}}$ and ∇_β on $u \in \mathscr{D}^0(M)$ by

$$(\nabla_\beta^\partial u)T \wedge \beta := i\partial u \wedge \bar{\tau} \wedge T = i\partial(u\bar{\partial}T),$$
$$(\nabla_\beta^{\bar{\partial}} u)T \wedge \beta := -i\bar{\partial}u \wedge \tau \wedge T = -i\bar{\partial}(u\partial T)$$

and

$$\nabla_\beta := \nabla_\beta^\partial + \nabla_\beta^{\bar{\partial}}.$$

Define also the operators Δ_β and $\widetilde{\Delta}_\beta$ on $u \in \mathscr{D}^0(M)$ by

$$(\Delta_\beta u)T \wedge \beta := i\partial\bar{\partial}u \wedge T \quad \text{and} \quad \widetilde{\Delta}_\beta := \Delta_\beta + \frac{1}{2}\nabla_\beta.$$

Let m_β denote the measure $T \wedge \beta$. We introduce the Hilbert space $H_\beta^1(T) \subset L^2(m_\beta)$ associated with T and β as the completion of $\mathscr{D}^0(M)$ with respect to the norm[3]

$$\|u\|_{H_\beta^1}^2 := \int |u|^2 T \wedge \beta + i\int \partial u \wedge \bar{\partial}u \wedge T.$$

Observe that operators ∇_β^∂, $\nabla_\beta^{\bar{\partial}}$ and ∇_β are defined on $H_\beta^1(T)$ with values in $L^1(m_\beta)$.

---

[2] The form β plays the role of a "Hermitian metric" on the current T that can be seen as a "generalized submanifold" of M.

[3] The second integral does not depend on β. This is an important difference in comparison with the analogous notion in the real setting.

Define also for $u, v \in \mathscr{D}^0(M)$ (for simplicity, we only consider real-valued functions)

$$q(u, v) := -\int (\Delta_\beta u) v T \wedge \beta, \quad e(u, v) := q(u, v) + \int u v T \wedge \beta$$

and

$$\widetilde{q}(u, v) := -\int (\widetilde{\Delta}_\beta u) v T \wedge \beta, \quad \widetilde{e}(u, v) := \widetilde{q}(u, v) + \int u v T \wedge \beta.$$

We will define later the domain of Δ and $\widetilde{\Delta}$ which allows us to extend these identities to more general u and v. The following lemma also holds for u, v in the domain of Δ and $\widetilde{\Delta}$.

Lemma 6.1 *We have for* $u, v \in \mathscr{D}^0(M)$

$$\widetilde{q}(u, v) = \operatorname{Re} \int i \partial u \wedge \overline{\partial} v \wedge T \quad and \quad \int (\widetilde{\Delta}_\beta u) v T \wedge \beta = \int u (\widetilde{\Delta}_\beta v) T \wedge \beta.$$

In particular, $\widetilde{q}(u, v)$ *and* $\widetilde{e}(u, v)$ *are symmetric in* u, v *and*

$$\int (\widetilde{\Delta}_\beta u) T \wedge \beta = \int (\Delta_\beta u) T \wedge \beta = \int (\nabla_\beta u) T \wedge \beta = 0 \quad for \ \ u \in \mathscr{D}^0(X).$$

Proof Note that $i \partial \overline{\partial} u$ is a real form. Since T is $\partial \overline{\partial}$-closed, the integral of $i \partial \overline{\partial} u^2 \wedge T$ vanishes. We deduce using Stoke's formula that

$$\begin{aligned}
\widetilde{q}(u, v) &= -\int \left(\Delta_\beta u + \frac{1}{2} \nabla_\beta u\right) v T \wedge \beta \\
&= -\int i \partial \overline{\partial} u \wedge v T - \operatorname{Re} \int i \partial u \wedge \overline{\tau} \wedge v T \\
&= -\operatorname{Re} \int i \partial \overline{\partial} u \wedge v T - \operatorname{Re} \int i \partial u \wedge \overline{\tau} \wedge v T \\
&= \operatorname{Re} \int i \partial u \wedge \left[\overline{\partial}(v T) - v \overline{\partial} T\right] = \operatorname{Re} \int i \partial u \wedge \overline{\partial} v \wedge T.
\end{aligned}$$

This gives the first identity in the lemma.

We also have since T is $\partial \overline{\partial}$-closed

$$\int (\nabla_\beta u) T \wedge \beta = 2\operatorname{Re} \int i \partial u \wedge \overline{\partial} T = 2\operatorname{Re} \int -i u \partial \overline{\partial} T = 0.$$

The other assertions in the lemma are obtained as in Lemma 5.4. $\qquad\square$

We say that T is *β-regular* if $i \tau \wedge \overline{\tau} \wedge T \leq \beta \wedge T$. We will see that this hypothesis is satisfied for foliations with linearizable singularities and for $\beta := \omega_P$. We have the following lemma.

Lemma 6.2 *The bilinear forms $\widetilde{q}$ and $\widetilde{e}$ extend continuously to $H^1_\beta(T) \times H^1_\beta(T)$. Assume that T is β-regular. Then the same property holds for q and e. Moreover, we have $q(u, u) = \widetilde{q}(u, u)$ and $e(u, u) = \widetilde{e}(u, u)$ for $u \in H^1_\beta(T)$.*

Proof The first assertion is deduced from Lemma 6.1. Assume that T is β-regular. Using Cauchy–Schwarz's inequality, we have for $u, v \in \mathscr{D}^0(M)$

$$
\begin{aligned}
|q(u, v) - \widetilde{q}(u, v)|^2 &\le \left| \int \partial u \wedge v \overline{\tau} \wedge T \right|^2 \\
&\le \left(\int i \partial u \wedge \overline{\partial} u \wedge T \right) \left(\int i v^2 \tau \wedge \overline{\tau} \wedge T \right) \\
&\le \left(\int i \partial u \wedge \overline{\partial} u \wedge T \right) \left(\int v^2 \beta \wedge T \right)
\end{aligned}
$$

which implies the second assertion. We also have for $u \in \mathscr{D}^0(M)$

$$
\begin{aligned}
q(u, u) - \widetilde{q}(u, u) &= \operatorname{Re} \int (\nabla^{\partial}_\beta u) u T \wedge \beta = \operatorname{Re} \int i \partial u \wedge u \overline{\tau} \wedge T \\
&= \frac{1}{2} \operatorname{Re} \int i \partial u^2 \wedge \overline{\partial} T = -\frac{1}{2} \operatorname{Re} \int i u^2 \partial \overline{\partial} T = 0.
\end{aligned}
$$

This allows us to obtain the rest of the lemma.   □

Define the domain $\operatorname{Dom}(\widetilde{\Delta}_\beta)$ of $\widetilde{\Delta}_\beta$ (resp. $\operatorname{Dom}(\Delta_\beta)$ of Δ_β when T is β-regular) as the space of $u \in H^1_\beta(T)$ such that $\widetilde{q}(u, \cdot)$ (resp. $q(u, \cdot)$) extends to a linear continuous form on $L^2(m_\beta)$. When T is β-regular, we have seen in the proof of Lemma 6.2 that $q(u, v) - \widetilde{q}(u, v)$ is continuous on $H^1_\beta(T) \times L^2(m_\beta)$. Therefore, $\operatorname{Dom}(\Delta_\beta) = \operatorname{Dom}(\widetilde{\Delta}_\beta)$. We have the following proposition.

Proposition 6.3 *Let T be a positive $\partial \overline{\partial}$-closed current of bidimension $(1, 1)$ on a complex manifold M and β a strictly T-positive form as above. Then the associated operator $-\widetilde{\Delta}_\beta$ (resp. $-\Delta_\beta$ when T is β-regular) is maximal monotone on $L^2(m_\beta)$ where $m_\beta := T \wedge \beta$. In particular, it is the infinitesimal generator of semi-groups of contractions on $L^2(m_\beta)$ and its graph is closed.*

Proof By Lemmas 6.1 and 6.2, we have for $u \in H^1_\beta(T)$

$$
\widetilde{e}(u, u) \ge \|u\|^2_{H^1_\beta} \quad \text{and} \quad e(u, u) \ge \|u\|^2_{H^1_\beta}.
$$

We only have to follow the same lines as in the proof of Proposition 5.6.   □

We have the following theorem.

Theorem 6.4 *Under the hypothesis of Proposition 6.3, let $S(t), t \in \mathbb{R}_+$, denote the semi-group of contractions associated with the operator $-\widetilde{\Delta}$ (resp. $-\Delta$ when T is β-regular) which is given by the Hille–Yosida theorem. Then the measure $m_\beta := T \wedge \beta$ is $S(t)$-invariant and $S(t)$ is a positive contraction in $L^p(m_\beta)$ for all $1 \le p \le \infty$.*

Proof The proof is almost the same as in Theorem 5.7. We only give here some computations which are slightly different. We have with mostly the same notation

$$\xi(t) := \int H(u(t, \cdot) - K)T \wedge \beta$$

and

$$\xi'(t) = -\int G(v(t, \cdot))Av(t, \cdot)T \wedge \beta$$

with $A = -\widetilde{\Delta}$ or $A = -\Delta$. In the first case, by Lemma 6.1, the last integral is equal to

$$-\mathrm{Re} \int i\partial G(v) \wedge \bar{\partial}v \wedge T = -\mathrm{Re} \int iG'(v)\partial v \wedge \bar{\partial}v \wedge T$$

$$= -\int iG'(v)\partial v \wedge \bar{\partial}v \wedge T \leq 0.$$

When $A = -\Delta_\beta$, there is an extra term which is equal to

$$-\frac{1}{2}\int G(v)\nabla_\beta vT \wedge \beta = -\mathrm{Re} \int iG(v)\partial v \wedge \bar{\partial}T$$

$$= -\mathrm{Re} \int i\partial H(v) \wedge \bar{\partial}T = \mathrm{Re} \int iH(v)\partial\bar{\partial}T = 0.$$

We can now follow closely the proof of Theorem 5.7. □

As in the real case, we deduce from the above results the following ergodic theorem.

Corollary 6.5 *Under the hypothesis of Theorem 6.4, for all $u_0 \in L^p(m_\beta)$, $1 \leq p < \infty$, the average*

$$\frac{1}{R}\int_0^R S(t)u_0 dt$$

converges pointwise m_β-almost everywhere and also in $L^p(m_\beta)$ to an $S(t)$-invariant function u_0^* when R goes to infinity.

Consider now a compact Riemann surface lamination $(X, \mathscr{L}, E)$ with isolated singularities in M and T a positive harmonic current on the lamination. We have seen in Proposition 4.1 that T is also a positive $\partial\bar{\partial}$-closed current of bidimension $(1, 1)$ on M. By Proposition 2.3 and Lemma 3.1, we can decompose T into the sum of a positive harmonic current on the union of the parabolic leaves and another on the union of the hyperbolic leaves. We want to study the second part of T. From now on, assume for simplicity that T has no mass on the union of the parabolic leaves.

With the local decomposition of T, we have $\partial T = \tau \wedge T$ with $\tau = h_a^{-1} \partial h_a$ when $h_a \neq 0$ and $\tau = 0$ otherwise. On almost every leaf, when $\tau \neq 0$, $i\tau \wedge \bar{\tau}$ defines a metric with curvature -1. It follows that $i\tau \wedge \bar{\tau}$ is bounded by the Poincaré metric ω_P on the leaves, see [19] for details. Therefore, all the above results can be applied for T with $\beta := \omega_P$ if $T \wedge \omega_P$ is a measure of finite mass, in particular, for laminations with linearizable singularities, e.g. generic foliations in $\mathbb{P}^k$, see Proposition 4.2 (they can be also applied to $i\tau \wedge \bar{\tau}$ and $\widetilde{\Delta}$ when $\tau \neq 0$). In what follows, we will use the index P instead of β, e.g. Δ_P instead of Δ_β. Note that $T \wedge \omega_P$ is a harmonic measure on X with respect to the Poincaré metric on the leaves. We have the following result.

Proposition 6.6 *Let $(X, \mathcal{L}, E)$ be a compact Riemann surface lamination in a complex manifold M with isolated singularities and ω_P the Poincaré metric on the leaves. Let T be a positive harmonic current on the lamination without mass on the set of parabolic leaves. Assume that $m_P := T \wedge \omega_P$ is a measure of finite mass. Then every function u_0 in $\mathrm{Dom}(\Delta_P)$ such that $\Delta_P u_0 \geq 0$ (resp. $\widetilde{\Delta}_P u_0 \geq 0$) is constant on the leaf L_a for m_P-almost every a. If moreover T is an extremal positive harmonic current, then u_0 is constant m_P-almost everywhere.*

Proof We have seen in the above discussion that T is ω_P-regular. So, using Lemmas 6.1 and 6.2 we have the identities

$$\int \Delta_P u_0 T \wedge \omega_P = \int \widetilde{\Delta}_P u_0 T \wedge \omega_P = 0$$

and

$$\int i\partial u_0 \wedge \bar{\partial} u_0 \wedge T = -\int (\widetilde{\Delta}_P u_0) u_0 T \wedge \omega_P = -\int (\Delta_P u_0) u_0 T \wedge \omega_P = 0.$$

So, we can repeat the proof of Proposition 5.8.   □

Note that since the Poincaré metric is not bounded in general, we don't have a priori a one to one correspondence between harmonic currents and harmonic measures. However, we have the following lemma that can be easily obtained using local description of currents.

Lemma 6.7 *Let T be a positive harmonic current with finite Poincaré mass. If m is a positive harmonic measure such that $m \leq m_P := T \wedge \omega_P$, then there is a positive harmonic current $S \leq T$ such that $m = S \wedge \omega_P$. In particular, if T is extremal, then m_P is an extremal positive harmonic measure.*

This allows us to prove the following result as in Corollary 5.9.

Corollary 6.8 *Under the hypothesis of Proposition 6.6 and with the notation of Corollary 6.5, u_0^* is constant on the leaf L_a for m_P-almost every a. Moreover, if T is an extremal positive harmonic current, then u_0^* is constant m_P-almost everywhere.*

The following result gives us the mixing for the operator $-\widetilde{\Delta}_P$. We can also obtain a pointwise convergence using E.M. Stein's theorem.

Theorem 6.9 *Under the hypothesis of Proposition* 6.6, *if $S(t)$ is associated with $-\widetilde{\Delta}_P$ and if T is extremal, $S(t)u_0 \to \langle m_P, u_0 \rangle$ in $L^p(m)$ when $t \to \infty$ for every $u_0 \in L^p(m_P)$ with $1 \le p < \infty$.*

Proof We follow closely the proof of Theorem 5.12 using similar notation. In this case, we use flow boxes away from the singularities of the lamination. We only need to observe that

$$2i \int \partial u \wedge \bar{\partial} u \wedge T = \|\nabla u\|^2_{L^2(T \wedge \omega_P)}$$

where ∇ is the gradient along the leaves with respect to the metric ω_P. □

As in the real case, we have the following result.

Proposition 6.10 *Let $T = \int T' d\nu(T')$ be as in Propositions* 6.6 *and* 4.4. *Then, the closures of $\Delta_P(\mathscr{D}^0(X))$ and of $\widetilde{\Delta}_P(\mathscr{D}^0(X))$ in $L^p(m_P)$, $1 \le p \le 2$, are the space of functions $u_0 \in L^p(m_P)$ such that $\int u_0 T' \wedge \omega_P = 0$ for ν-almost every T'. In particular, if T is an extremal positive harmonic current of finite Poincaré mass, then this space is the hyperplane of $L^p(m_P)$ defined by the equation $\int u_0 T \wedge \omega_P = 0$.*

Proof The proof follows the lines as in Proposition 5.13. With the analogous notation, we obtain that θm restricted to $\{c \le \theta \le c'\}$ is harmonic for all c, c'. By Lemma 6.7, this measure is associated to a harmonic current. This allows us to disintegrate T along the fibers of θ and to conclude using Proposition 4.4. □

Remark 6.11 For all the above results, when the lamination has no singular points, the hypothesis that it is embedded in a complex manifold is unnecessary. In general, it is enough to assume that locally near singular points, the lamination can be holomorphically embedded in a complex manifold on which T is a dd^c-closed current.

7 Birkhoff's type theorem for laminations

In this section, we will give an analogue of Birkhoff's ergodic theorem in the context of a compact lamination $(X, \mathscr{L}, E)$ with isolated singularities on a complex manifold M. In the case where the set of singularities is empty, the property that the lamination is embedded in a complex manifold is unnecessary, see e.g. Remark 7.2 below. Many examples of abstract compact laminations are constructed in [8] and [22]. Suspension of a group action gives already a large class of laminations without singularities.

Let T be a positive harmonic current on $(X, \mathscr{L}, E)$. We have seen that T is also a positive $\partial\bar{\partial}$-closed current of bidimension $(1, 1)$ on M. We assume that T has no mass on the union of parabolic leaves and that $m_P := T \wedge \omega_P$ is a probability measure where ω_P denotes the Poincaré metric on the leaves. So, m_P is a harmonic measure on X with respect to ω_P.

For any point $a \in X \setminus E$ such that the corresponding leaf L_a is hyperbolic, consider a universal covering map $\phi_a : \mathbb{D} \to L_a$ such that $\phi_a(0) = a$. This map is uniquely

defined by a up to a rotation on $\mathbb{D}$. Denote by $r\mathbb{D}$ the disc of center 0 and of radius r with $0 < r < 1$. In the Poincaré metric, this is also the disc of center 0 and of radius

$$R := \log \frac{1+r}{1-r}.$$

So, we will also denote by $\mathbb{D}_R$ this disc. For all $0 < R < \infty$, consider

$$m_{a,R} := \frac{1}{M_R} (\phi_a)_* \left(\log^+ \frac{r}{|\zeta|} \omega_P \right).$$

where $\log^+ := \max\{\log, 0\}$, ω_P denotes also the Poincaré metric on $\mathbb{D}$ and

$$M_R := \int \log^+ \frac{r}{|\zeta|} \omega_P = \int \log^+ \frac{r}{|\zeta|} \frac{2}{(1-|\zeta|^2)^2} i d\zeta \wedge d\bar{\zeta}.$$

So, $m_{a,R}$ is a probability measure which depends on a, R but does not depend on the choice of ϕ_a. Here is the main theorem in this section.

Theorem 7.1 *Let $(X, \mathscr{L}, E)$ be a compact lamination with isolated singularities in a complex manifold M and ω_P the Poincaré metric on the leaves. Let T be an extremal positive harmonic current of Poincaré mass 1 on $(X, \mathscr{L}, E)$ without mass on the union of parabolic leaves. Then for almost every point $a \in X$ with respect to the measure $m_P := T \wedge \omega_P$, the measure $m_{a,R}$ defined above converges to m_P when $R \to \infty$.*

Remark 7.2 Let $(X, \mathscr{L})$ be a compact lamination by Riemann surfaces without singularities. Let T be a positive harmonic current directed by the lamination which is extremal, with full mass on hyperbolic leaves and with Poincaré mass 1. Then, the conclusions of Theorem 7.1 are still valid. The proof is essentially the same but we need Proposition 5.13 instead of Proposition 6.10.

For every $0 < R < \infty$, we introduce the operator B_R by

$$B_R u(a) := \frac{1}{M_R} \int_{|\zeta|<1} \log^+ \frac{r}{|\zeta|} (\phi_a)^* (u\omega_P) = \langle m_{a,R}, u \rangle.$$

Note that for $u \in L^1(m_P)$, the function $B_R u$ is defined m_P-almost everywhere. So, the convergence in Theorem 7.1 is equivalent to the convergence $B_R u(a) \to \langle m_P, u \rangle$ for u continuous and for m_P-almost every a.

Proposition 7.3 *Under the hypothesis of Theorem 7.1, for every $u \in L^1(m_P)$, we have*

$$\int (B_R u) T \wedge \omega_P = \int u T \wedge \omega_P.$$

In particular, B_R is positive and of norm 1 in $L^p(m_P)$ for all $1 \le p \le \infty$.

Proof Fix an $R > 0$. The positivity of B_R is clear. Since B_R preserves constant functions, its norm in $L^p(m_P)$ is at least equal to 1. It is also clear that B_R is an operator of norm 1 on $L^\infty(m_P)$. If B_R is of norm 1 on $L^1(m_P)$, by interpolation [36], its norm on $L^p(m_P)$ is also equal to 1. So, the second assertion is a consequence of the first one. Define another operator B'_R by

$$B'_R u(a) := \frac{1}{M'_R} \int_{\mathbb{D}_R} (\phi_a)^*(u\omega_P) \quad \text{where} \quad M'_R := \int_{\mathbb{D}_R} (\phi_a)^*(\omega_P).$$

Note that M'_R is the Poincaré area of $\mathbb{D}_R$ which is also the Poincaré area of $\phi_a(\mathbb{D}_R)$ counted with multiplicity. The operator B_R can be obtained as an average of B'_t on $t \le R$. So, it is enough to prove the first assertion of the proposition for B'_R instead of B_R.

We can assume that u is positive and using a partition of unity, we can also assume that u has support in a compact set of a flow box $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ as in Sect. 3. Let $\mathbb{T}_1$ be the set of $a \in \mathbb{T}$ such that L_a is hyperbolic. We will use the decomposition of T and the notation as in Proposition 2.3. By hypothesis, we can assume that the measure μ has total mass on $\mathbb{T}_1$. Now, we apply Corollary 3.3 to μ instead of ν and $4\lambda R$ instead of R for a fixed constant λ large enough. Let $\Sigma_n(R)$ denote the union of $L_{a,R}$ for $a \in \mathbb{S}_n$. Define by induction the function u_n as follows: u_1 is the restriction of u to $\Sigma_1(R)$ and u_n is the restriction of $u - u_1 - \cdots - u_{n-1}$ to $\Sigma_n(R)$. We have $u = \sum u_n$. So, it is enough to prove the proposition for each u_n. For simplicity, assume that $u = u_n$.

We use now the properties of $\mathbb{S}_n$ given in Corollary 3.3. The set $\Sigma_n(4\lambda R)$ is a smooth lamination and the restriction T_n of T to $\Sigma_n(4\lambda R)$ is a positive harmonic current. Observe that $B_R u_n$ vanishes outside $\Sigma_n(\lambda R)$ and does not depend on the restriction of T to $X \setminus \Sigma_n(2\lambda R)$. Since there is a natural projection from Σ_n to the transversal $\mathbb{S}_n$, the extremal positive harmonic currents on Σ_n are supported by a leaf and defined by a harmonic function. Therefore, we can reduce the problem to the case where $T = h[L_{a,4\lambda R}]$ with $a \in \mathbb{S}_n$ and h is positive harmonic on $L_{a,4\lambda R}$.

Define $\widehat{u} := u \circ \phi_a$, $\widehat{h} := h \circ \phi_a$ and $\widehat{B'_R u} := (B'_R u) \circ \phi_a$. The function $\widehat{h}$ is harmonic on $\mathbb{D}_{4\lambda R}$. Choose a measurable set $A \subset \mathbb{D}_{2\lambda R}$ such that ϕ_a defines a bijection between A and $L_{a,2\lambda R}$. Denote also by ω_P the Poincaré metric on $\mathbb{D}$ and dist the associated distance. We first observe that

$$\widehat{B'_R u}(0) := \frac{1}{M'_R} \int_{\text{dist}(\zeta,0)<R} \widehat{u}(\zeta)\omega_P(\zeta).$$

If η is a point in $\mathbb{D}$ and $\tau : \mathbb{D} \to \mathbb{D}$ is an automorphism such that $\tau(0) = \eta$, then $\phi_a \circ \tau$ is also a covering map of L_a but it sends 0 to $\phi_a(\eta)$. We apply the above formula to this covering map. Since τ preserves ω_P and dist, we obtain

$$\widehat{B'_R u}(\eta) := \frac{1}{M'_R} \int_{\text{dist}(\zeta,\eta)<R} \widehat{u}(\zeta)\omega_P(\zeta).$$

Hence, we have to show the following identity

$$
\int\limits_{A} \left[\frac{1}{M'_R} \int\limits_{\text{dist}(\zeta,\eta)<R} \widehat{u}(\zeta)\omega_P(\zeta) \right] \widehat{h}(\eta)\omega_P(\eta) = \int\limits_{A} \widehat{u}(\zeta)\widehat{h}(\zeta)\omega_P(\zeta).
$$

Let W denote the set of points $(\zeta, \eta) \in \mathbb{D}^2$ such that $\eta \in A$ and $\text{dist}(\zeta, \eta) < R$. Let W' denote the symmetric of W with respect to the diagonal, i.e. the set of (ζ, η) such that $\zeta \in A$ and $\text{dist}(\zeta, \eta) < R$. Since $\widehat{h}$ is harmonic, we have

$$
\widehat{h}(\zeta) = \frac{1}{M'_R} \int\limits_{\text{dist}(\zeta,\eta)<R} \widehat{h}(\eta)\omega_P(\eta).
$$

Therefore, our problem is to show that the integrals of $\Phi := \widehat{u}(\zeta)\widehat{h}(\eta)\omega_P(\zeta) \wedge \omega_P(\eta)$ on W and W' are equal.

Consider the map $\phi := (\phi_a, \phi_a)$ from $\mathbb{D}^2$ to L_a^2. The fundamental group $\Gamma := \pi_1(L_a)$ can be identified with a group of automorphisms of $\mathbb{D}$. Since, Γ^2 acts on $\mathbb{D}^2$ and preserves the form Φ, our problem is equivalent to showing that each fiber of ϕ has the same number of points in W and in W'. We only have to consider the fibers of points in $L_{a,2\lambda R} \times L_{a,2\lambda R}$ since $B'_R u_n$ is supported on $L_{a,\lambda R}$. Fix a point $(\zeta, \eta) \in A^2$ and consider the fiber F of $\phi(\zeta, \eta)$. By definition of A, the numbers of points in $F \cap W$ and $F \cap W'$ are respectively equal to

$$
\#\{\gamma \in \Gamma, \ \text{dist}(\gamma \cdot \zeta, \eta) < R\} \quad \text{and} \quad \#\{\gamma \in \Gamma, \ \text{dist}(\zeta, \gamma \cdot \eta) < R\}.
$$

Since Γ preserves the Poincaré metric, the first set is equal to

$$
\{\gamma \in \Gamma, \ \text{dist}(\zeta, \gamma^{-1} \cdot \eta) < R\}.
$$

It is now clear that the two numbers are equal. This completes the proof.   □

We have the following ergodic theorem.

Theorem 7.4 *Under the hypothesis of Theorem* 7.1, *if u is a function in $L^p(m_P)$, with $1 \le p < \infty$, then $B_R u$ converge in $L^p(m_P)$ towards a constant function u^* when $R \to \infty$.*

Proof We show that it is enough to consider the case where $p = 1$. By Proposition 7.3, it is enough to consider u in a dense subset of $L^p(m_P)$, e.g. $L^\infty(m_P)$. For u bounded, we have $\|B_R u\|_\infty \le \|u\|_\infty$. Therefore, if $B_R u \to u^*$ in $L^1(m_P)$ we have $B_R u \to u^*$ in $L^p(m_P)$, $1 \le p < \infty$.

Now, assume that $p = 1$. Since B_R preserves constant functions, by Proposition 6.10 applied to $p = 1$, it is enough to consider $u = \Delta_P v$ with $v \in \mathscr{D}^0(M)$. We have to show that $B_R u$ converges to 0. Note that since v is in $\mathscr{D}^0(M)$, the function $\Delta_P v$ is defined at every point outside the parabolic leaves by the formula $(\Delta_P v)\omega_P := i\partial\overline{\partial} v$ on the leaves.

Define

$$T_{a,R} := \frac{1}{M_R}(\phi_a)_* \left(\log^+ \frac{r}{|\zeta|} \right).$$

We have $m_{a,R} = T_{a,R} \wedge \omega_P$ and

$$B_R u(a) = B_R(\Delta_P v)(a) = \langle T_{a,R}, (\Delta_P v)\omega_P \rangle = \langle T_{a,R}, i\partial\bar{\partial}v \rangle = \langle i\partial\bar{\partial}T_{a,R}, v \rangle.$$

Since $M_R \to \infty$, it is easy to see that the mass of $\partial\bar{\partial}T_{a,R}$ tends to 0 uniformly on a. So, the last integral tends to 0 uniformly on a. The result follows.

Let $p(x, y, t)$ be the heat kernel on the disc $\mathbb{D}$ with respect to the Poincaré metric ω_P on $\mathbb{D}$. This is a positive function on $\mathbb{D}^2 \times \mathbb{R}_+^*$ smooth when (x, y) is outside the diagonal of $\mathbb{D}^2$. It satisfies

$$\int_{\mathbb{D}} p(x, y, t)\omega_P(y) = 1 \quad \text{and} \quad \frac{1}{2\pi} \log \frac{1}{|y|} = \int_0^\infty p(0, y, t)dt.$$

The function $p(0, \cdot, t)$ is radial, see e.g. Chavel [10, p. 246]. Define the operator S_t by

$$S_t u(a) := \int_{\mathbb{D}} p(0, \cdot, t)(u \circ \phi_a)\omega_P.$$

Observe that the family S_t with $t > 0$ is a semi-group of operators, i.e. $S_{t+t'}u = S_t \circ S_{t'}u$ for $u \in \mathscr{D}^0(M)$.

Lemma 7.5 *The operator S_t extends continuously to an operator of norm* 1 *on $L^P(m_P)$ for $1 \le p \le \infty$. Moreover, there is a constant $c > 0$ such that for all $\epsilon > 0$ and $u \in L^1(m_P)$, we have*

$$m_P\{\widetilde{S}u > \epsilon\} \le c\epsilon^{-1}\|u\|_{L^1(m_P)},$$

where the operator $\widetilde{S}$ is defined by

$$\widetilde{S}u(a) := \limsup_{R \to \infty} \left| \frac{1}{R} \int_0^R S_t u(a)dt \right|.$$

Proof It is clear that S_t is positive and preserves constant functions. Its norm on $L^\infty(m_P)$ is equal to 1. Since $p(0, \cdot, t)$ is a radial function, as in Proposition 7.3, S_t is an average of B_R'. We then obtain in the same way the first assertion of the lemma. The second one is a direct consequence of Lemma VIII.7.11 in Dunford–Schwartz [14]. This lemma says that if S_t is a semi-group acting on $L^1(m)$ for some probability

measure m such that $\|S_t\|_{L^1} \leq 1$, $\|S_t\|_{L^\infty} \leq 1$ and $t \mapsto S_t u$ is measurable with respect to the Lebesgue measure on t, then

$$m\{\widetilde{S}u > \epsilon\} \leq c\epsilon^{-1}\|u\|_{L^1}$$

where $\widetilde{S}$ is defined as above.   □

Consider also the operator $\widetilde{B}$ given by

$$\widetilde{B}u(a) := \limsup_{R \to \infty} |B_R u(a)|.$$

We have the following lemma.

Lemma 7.6 *There is a constant $c > 0$ such that for all $\epsilon > 0$ and $u \in L^1(m_P)$ we have*

$$m_P\{\widetilde{B}u > \epsilon\} \leq c\epsilon^{-1}\|u\|_{L^1(m_P)}.$$

Proof Since we can write $u = u^+ - u^-$ with $\|u\|_{L^1(m_P)} = \|u^+\|_{L^1(m_P)} + \|u^-\|_{L^1(m_P)}$, it is enough to consider u positive with $\|u\|_{L^1(m_P)} \leq 1$. Write $u = \sum_{i \geq 0} u_i$ with u_i positive and bounded such that $\|u_i\|_{L^1(m_P)} \leq 4^{-i}$. We will show that $\widetilde{S}u_i = \widetilde{B}u_i$. This, together with Lemma 7.5 applied to u_i and to $2^{-i-1}\epsilon$ gives the result.

So, in what follows, assume that $0 \leq u \leq 1$. We show that $\widetilde{S}u = \widetilde{B}u$. This assertion will be an immediate consequence of the following estimate

$$\left| B_R u(a) - \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} S_t u(a)dt \right| \leq cR^{-1/2}\sqrt{\log R}$$

for m_P-almost every a and c a constant independent of u. Observe that the integrals in the left hand side of the last inequality can be computed on $\mathbb{D}$ in term of $\widehat{u} := u \circ \phi_a$ and of the Poincaré metric ω_P on $\mathbb{D}$. So, in order to simplify the notation, we will work on $\mathbb{D}$. We have to show that

$$\left| B_R \widehat{u}(0) - \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} S_t \widehat{u}(0)dt \right| \leq cR^{-1/2}\sqrt{\log R}$$

where

$$B_R \widehat{u}(0) := \frac{1}{M_R} \int_{\mathbb{D}_R} \log \frac{r}{|\zeta|} \widehat{u}\omega_P \quad \text{and} \quad S_t \widehat{u}(0) := \int_{\mathbb{D}} p(0,\cdot,t)\widehat{u}\omega_P.$$

A direct computation shows that $|M_R - 2\pi R|$ is bounded by a constant and the area of $\mathbb{D}_R$ is of order e^R for R large. Define

$$D_R \widehat{u}(0) := \frac{1}{M_R} \int_{\mathbb{D}_R} \log \frac{1}{|\zeta|} \widehat{u} \omega_P = \frac{2\pi}{M_R} \int_{\mathbb{D}_R} \left[\int_0^\infty p(0, \zeta, t) dt \right] \widehat{u} \omega_P.$$

We have

$$|B_R \widehat{u}(0) - D_R \widehat{u}(0)| \leq \frac{1}{M_R} \log r \int_{\mathbb{D}_R} \omega_P \lesssim \frac{1}{M_R} (1 - r) e^R \lesssim \frac{1}{R}.$$

Therefore, we can replace $B_R \widehat{u}(0)$ with $D_R \widehat{u}(0)$. We have

$$\left| \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} S_t \widehat{u}(0) dt - D_R \widehat{u}(0) \right|$$

$$= \left| \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} \left[\int_{\mathbb{D}} p(0, \zeta, t) \widehat{u} \omega_P \right] dt - \frac{2\pi}{M_R} \int_0^\infty \left[\int_{\mathbb{D}_R} p(0, \zeta, t) \widehat{u} \omega_P \right] dt \right|$$

$$\leq \frac{2\pi}{M_R} \int_{\mathbb{D}_R} \left[\int_{\frac{M_R}{2\pi}}^\infty p(0, \zeta, t) dt \right] \omega_P + \frac{2\pi}{M_R} \int_{\mathbb{D} \backslash \mathbb{D}_R} \left[\int_0^{\frac{M_R}{2\pi}} p(0, \zeta, t) dt \right] \omega_P.$$

The two last integrals are equal since using properties of the heat kernel $p(x, y, t)$ summarized above, we have

$$\frac{2\pi}{M_R} \int_{\mathbb{D}} \left[\int_0^{\frac{M_R}{2\pi}} p(0, \zeta, t) dt \right] \omega_P = 1 = \frac{2\pi}{M_R} \int_{\mathbb{D}_R} \left[\int_{t=0}^\infty p(0, \zeta, t) dt \right] \omega_P.$$

Therefore, it remains to check that

$$\frac{2\pi}{M_R} \int_{\mathbb{D}_R} \left[\int_{\frac{M_R}{2\pi}}^\infty p(0, \zeta, t) dt \right] \omega_P \lesssim R^{-1/2} \sqrt{\log R}.$$

To prove this, split the integral over $\mathbb{D}_R$ into two integrals over $\mathbb{D}_R \backslash \mathbb{D}_{R'}$ and over $\mathbb{D}_{R'}$ where $R' := R - R^{1/2} \sqrt{2 \log R}$. Using the properties of the heat kernel and some

direct computation, e.g. $\log|\zeta| \simeq |\zeta| - 1$, we can bound the first part by

$$\frac{2\pi}{M_R} \int_{\mathbb{D}_R \setminus \mathbb{D}_{R'}} \log\frac{1}{|\zeta|}\omega_P \lesssim \frac{R - R'}{M_R} \lesssim \frac{R - R'}{R} \lesssim R^{-1/2}\sqrt{\log R}.$$

For the second part, we claim that

$$\int_{\frac{M_R}{2\pi}}^{\infty} p(0, \zeta, t)dt \le c\frac{R^{1/2}}{\sqrt{\log R}} \log\frac{1}{|\zeta|} \quad \text{for} \quad \zeta \in \mathbb{D}_{R'}$$

where $c > 0$ is a constant independent of R and ζ. Taking this for granted, since the Poincaré area of $\mathbb{D}_R$ is of order e^R, using the definition of M_R, we get

$$\frac{1}{M_R}\int_{\mathbb{D}_R} \log\frac{1}{|\zeta|}\omega_P = 1 - \frac{1}{M_R}\log r \int_{\mathbb{D}_R} \omega_P \lesssim 1 + \frac{1}{R}(1 - r)e^R \lesssim 1.$$

Therefore, our second integral to estimate is bounded by a constant times

$$\frac{1}{\sqrt{\log R}}R^{-1/2}\frac{1}{M_R}\int_{\mathbb{D}_R} \log\frac{1}{|\zeta|}\omega_P \lesssim R^{-1/2}\frac{1}{\sqrt{\log R}} \lesssim R^{-1/2}\sqrt{\log R}.$$

Now, it remains to prove the above claim.

Denote by ρ the Poincaré distance between 0 and ζ. It is given by the formula

$$\rho := \log\frac{1 + |\zeta|}{1 - |\zeta|}.$$

Recall a formula in Chavel [10, p. 246]

$$p(0, \zeta, t) = \frac{\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}}\int_{\rho}^{\infty} \frac{se^{-\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}}ds.$$

So, we have $\sqrt{\cosh s - \cosh \rho} \gtrsim e^{s/2}$ when $s \ge \rho + 1$ and $\sqrt{\cosh s - \cosh \rho} \gtrsim e^{s/2}\sqrt{s - \rho}$ for $\rho < s \le \rho + 1$. Since $|M_R - 2\pi R|$ is bounded by a constant, the

integral in the claim is bounded up to some constants by

$$\int\limits_R^\infty \left[\frac{\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}} \int\limits_\rho^\infty \frac{se^{-\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} ds \right] dt$$

$$\lesssim \int\limits_R^\infty \left[\frac{\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}} \int\limits_\rho^\infty \frac{se^{-\frac{s^2}{4t}}}{e^{s/2}} ds \right] dt + \int\limits_R^\infty \left[\frac{\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}} \int\limits_\rho^{\rho+1} \frac{se^{-\frac{s^2}{4t}}}{e^{s/2}\sqrt{s-\rho}} ds \right] dt$$

$$\lesssim \int\limits_R^\infty \frac{1}{\sqrt{t}} \left[\int\limits_\rho^\infty \frac{s}{2\sqrt{t}} e^{-\left(\frac{s}{2\sqrt{t}}+\frac{\sqrt{t}}{2}\right)^2} d\left(\frac{s}{2\sqrt{t}}\right) \right] dt + \int\limits_R^\infty \left[\frac{\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}} \frac{(\rho+1)e^{-\frac{\rho^2}{4t}}}{e^{\rho/2}} \right] dt$$

$$\lesssim \int\limits_R^\infty \frac{1}{\sqrt{t}} e^{-\left(\frac{\rho}{2\sqrt{t}}+\frac{\sqrt{t}}{2}\right)^2} dt \lesssim e^{-\rho} \int\limits_R^\infty \frac{1}{\sqrt{t}} e^{-\left(\frac{\rho}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} dt.$$

We used here that $\rho \le R \le t$. Observe that the function $(t, \rho) \mapsto \frac{\sqrt{t}}{2} - \frac{\rho}{2\sqrt{t}}$ is increasing in t and decreasing in ρ. Therefore,

$$\frac{\sqrt{t}}{2} - \frac{\rho}{2\sqrt{t}} \ge \frac{R-R'}{2\sqrt{R}} \gtrsim \sqrt{\log R}$$

for $\rho \le R' \le R$ and $t \ge R$. We can bound the last expression in the above sequence of inequalities by a constant times

$$e^{-\rho} \frac{1}{\sqrt{\log R}} \int\limits_R^\infty e^{-\left[\frac{\rho}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right]^2} \left[\frac{\sqrt{t}}{2} - \frac{\rho}{2\sqrt{t}} \right] d\left[\frac{\sqrt{t}}{2} - \frac{\rho}{2\sqrt{t}} \right] \le \frac{e^{-\rho}}{\sqrt{\log R}} e^{-\frac{(R-R')^2}{4R}}.$$

Finally, we obtain the claim using that $R' := R - R^{1/2}\sqrt{2\log R}$ and

$$e^{-\rho} = \frac{1-|\zeta|}{1+|\zeta|} \lesssim \log \frac{1}{|\zeta|}$$

for $\zeta \in \mathbb{D}_{R'}$. This completes the proof. $\qquad\square$

Proof of Theorem 7.1. Let u be a function in $L^1(m_P)$. It is enough to show that $B_R u(a) \to \langle m_P, u \rangle$ for m_P-almost every a. Since this is true when u is constant, we can assume without loss of generality that $\langle m_P, u \rangle = 0$. Fix a constant $\epsilon > 0$ and define $E_\epsilon(u) := \{\widetilde{B}u \ge \epsilon\}$. To prove the theorem it suffices to show that $m_P(E_\epsilon(u)) = 0$.

By Proposition 6.10, $\Delta_P(\mathscr{D}^0(M))$ is dense in the hyperplane of functions with mean 0 in $L^1(m_P)$. Consequently, for every $\delta > 0$ we can choose a smooth function v such that $\|\Delta_P v - u\|_{L^1(m_P)} < \delta$. We have

$$E_\epsilon(u) \subset E_{\epsilon/2}(u - \Delta_P v) \cup E_{\epsilon/2}(\Delta_P v).$$

Therefore,

$$m_P\big(E_\epsilon(u)\big) \le m_P\big(E_{\epsilon/2}(u - \Delta_P v)\big) + m_P\big(E_{\epsilon/2}(\Delta_P v)\big).$$

We have

$$B_R(\Delta_P v)(a) = \langle T_{a,R}, i\partial\overline{\partial} v\rangle = \langle i\partial\overline{\partial}T_{a,R}, v\rangle.$$

The last integral tends to 0 uniformly on a since the mass of $\partial\overline{\partial}T_{a,R}$ satisfies this property. Hence, $m_P\big(E_{\epsilon/2}(\Delta_P v)\big) = 0$.

On the other hand, by Lemma 7.6, we have

$$m_P\big(E_{\epsilon/2}(u - \Delta_P v)\big) = m_P\big(\widetilde{B}(u - \Delta_P v) > \epsilon/2\big)$$
$$\le 2c\epsilon^{-1}\|u - \Delta_P v\|_{L^1(m_P)} \le 2c\epsilon^{-1}\delta.$$

Since δ is arbitrary, we deduce from the last estimate that $m_P\big(E_\epsilon(u)\big) = 0$. This completes the proof of the theorem. □

Remark 7.7 When T is not extremal, for a function u in $L^p(m_P)$, $1 \le p < \infty$, we still have the convergence of $B_R u$ pointwise m_P-almost everywhere and also in $L^p(m_P)$ to a function which is constant on the leaves but not necessarily constant globally. This property can be deduced from Theorems 7.1 and 7.4 using the decomposition of T into extremal currents and that B_R has norm 1 in $L^p(m_P)$.

If $\phi_a : \mathbb{D} \to L_a$ is a universal covering map over a hyperbolic leaf L_a with $\phi_a(0) = a$, define

$$T'_{a,R} := (\phi_a)_* \left(\log^+ \frac{r}{|\zeta|}\right).$$

We have the following result which can be applied to foliations in $\mathbb{P}^k$ with linearizable isolated singularities.

Theorem 7.8 *Under the hypothesis of Theorem 7.1, we have the convergence*

$$\|T'_{a,R}\|^{-1}T'_{a,R} \to \|T\|^{-1}T$$

for almost every point $a \in M$ with respect to $T \wedge \omega$.

Proof Let α be a smooth $(1, 1)$-form on M. We have to show that

$$\|T'_{a,R}\|^{-1}\langle T'_{a,R}, \alpha\rangle \to \|T\|^{-1}\langle T, \alpha\rangle$$

for almost every a with respect to $T \wedge \omega$ or equivalently with respect to m_P since T has no mass on the set of parabolic leaves. This property applied to a dense countable family of α gives the result. The restriction of α to the lamination can be written as

$\alpha = \varphi \omega_P$ where φ is in $L^1(m_P)$ because $\alpha \wedge T$ is a finite measure. Now, we have seen in the proof of Theorem 7.1 that $\langle m_{a,R}, \varphi \rangle$ converges to $\langle m_P, \varphi \rangle = \langle T, \alpha \rangle$. It follows that

$$M_R^{-1} \langle T'_{a,R}, \alpha \rangle \to \langle T, \alpha \rangle.$$

It remains to show that

$$M_R^{-1} \| T'_{a,R} \| \to \| T \|.$$

But this is a consequence of the above convergence applied to ω instead of α. $\qquad\square$

Remark 7.9 We have seen that $\| T'_{a,R} \| \simeq M_R$ for almost every a with respect to $T \wedge \omega$. The estimate implies that

$$\int_{r\mathbb{D}} |\phi'_a(z)|^2 \log \frac{r}{|z|} i dz \wedge d\overline{z} \simeq \log \frac{1}{1-r}$$

which gives a quantitative information on the behavior of ϕ_a near the singularities. When the lamination admits no positive closed invariant current, it was shown in [17, Th. 5.3] that the left hand side integral tends to infinity for every a. We finally get Theorem 1.1 announced in the introduction. It is a consequence of Theorem 7.1, Proposition 4.2 and of the present remark.

Remark 7.10 We have seen in the proof of Theorem 7.8 that for m_P-almost every a the area (counted with multiplicity) of $\phi_a(\mathbb{D}_R)$ with respect to the Poincaré metric is equivalent to the one with respect to the metric ω. As a consequence, in Theorems 7.1 and 7.8, we can replace $\log^+ \frac{r}{|\xi|}$ with the restriction of $\log \frac{1}{|\xi|}$ to $\mathbb{D}_R$ because the integral of $(\log r)\omega_P$ over $\mathbb{D}_R$ is bounded. This will give us ergodic theorems with a little bit more geometric flavor since $\log \frac{1}{|\xi|}$ is independent of R.

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