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On uniqueness of semi-wavefronts

Diekmann–Kaper theory of a nonlinear convolution equation re-visited

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Abstract Motivated by the uniqueness problem for monostable semi-wave -fronts, we propose a revised version of the Diekmann and Kaper theory of a nonlinear convolution equation. Our version of the Diekmann–Kaper theory allows (1) to consider new types of models which include nonlocal KPP type equations (with either symmetric or anisotropic dispersal), nonlocal lattice equations and delayed reaction–diffusion equations; (2) to incorporate the critical case (which corresponds to the slowest wavefronts) into the consideration; (3) to weaken or to remove various restrictions on kernels and nonlinearities. The results are compared with those of Schumacher (J Reine Angew Math 316: 54–70, 1980), Carr and Chmaj (Proc Am Math Soc 132: 2433–2439, 2004), and other more recent studies.

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1 Introduction

The main goal of this paper is to develop a version of the fundamental Diekmann and Kaper theory [10–12] (the DK theory for short) of a nonlinear convolution equation

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for the scalar integral equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R}, \tag{1}$$

in the case of monostable nonlinearity g. Throughout the paper (X, μ) will denote a measure space with finite measure μ, $K(s,\tau) \geq 0$ will be integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} K(s,\tau)ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, will be continuous in φ for every fixed $\tau \in X$. When X is just a single point (i.e. $\#X = 1$), Eq. (1) coincides with the nonlinear convolution equation from [12].

In a biological context, φ is the size of an adult population, so we are interested in non-negative solutions of (1). Following the terminology of [22], we call a bounded continuous non-constant solution $\varphi : \mathbb{R} \to \mathbb{R}_+$ semi-wavefront if either $\varphi(-\infty) = 0$ or $\varphi(+\infty) = 0$. We will always assume φ to satisfy $\varphi(-\infty) = 0$, since the other case can be easily transformed to this one via the change of variables $\zeta(t) = \varphi(-t)$, with Eq. (1) assuming the form

$$\zeta(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau) g(\zeta(t-s),\tau) ds, \quad K_1(s,\tau) := K(-s,\tau).$$

We would like to emphasize that the nonlinearity g and semi-wavefronts are generally non-monotone [19] (nevertheless, typically semi-wavefronts are strictly increasing in some vicinity of $-\infty$ [1,18,37]). The non-monotonicity of waves complicates their analysis. For instance, the wave uniqueness is easier to establish within a subclass of monotone solutions [8,23,39].

Actually the 'largely open uniqueness question' [6] is central in our research where we follow the scheme elaborated in [12]. This means that after assuming the existence of a semi-wavefront to (1), we study its asymptotic behavior at infinity trying then to demonstrate the wave uniqueness (modulo translation). Similarly to other authors, we work mostly with the first positive eigenvalue λ_l of the linearization of (1) at zero. As a consequence, our analysis excludes from the consideration so called "pushed" fronts [13,22,34] associated to the second positive eigenvalue λ_r. Analogously to [12], the existence of semi-wavefronts to (1) is not investigated here.

There are various motivations to study the above equation, mainly from the theory of traveling waves for nonlinear models (e.g. reaction–diffusion equations with delayed response [1,23,36,38,39], equations with non-local dispersal [2,4,7,8,28,33], lattice systems [6,16,26,30]). Only a few of these models take the simplest form with $\#X = 1$ of (1). Therefore our first goal is to show that the basic framework of [12] can be extended to include much broader class of convolution type equations than it was initially intended. Here is a simple step to create such a general direct extension of results in [12]. It would be interesting to consider further generalizations of (1) in order to include more applications (for example, equations with distributed delays considered in [16,17], see also [25,33,39]). However, we do not pursue this direction in our current work. After all, ours is not the first attempt to expand the DK theory.

Schumacher has mentioned, while studying equation

$$c\varphi'(t) = g(\varphi, \mu_c * g(\varphi)),$$

the impossibility of transforming it into the form to which the DK theory could be applied [33, p.54]. Instead, Schumacher has developed an approach which is based on guidelines of the DK theory and, at the same time, which is technically rather different from that in [12]. In particular, in order to extend the DK uniqueness theorem, Schumacher has used a comparison method for differential inequalities combined with Nagumo-point argument. In this respect, his work [33] is very close to the recent contributions [6–8,26].

Similarly to [33], the present studies also follow the mainstream of the DK ideology. In difference with [33] and trying to apply our results to delayed equations (where in general the comparison argument does not work), we preserve the original idea of the DK theory in the proof of uniqueness. Now, from the technical point of view our approach to Eq. (1) differs from the methods used by Diekmann and Kaper, Schumacher and Carr and Chmaj [4] in many key points. Even though the logical sequence of results here basically is the same as in [12], our proofs are essentially different. In particular, we do not use the Titchmarsh theory of Fourier integrals [12,16] nor we use the Ikehara Tauberian theorem [4,8,39] in order to obtain asymptotic expansions of solutions (a necessary key component of each uniqueness proof). We have found more convenient for our purpose the use of a suitable L^2-variant of the bootstrap argument (as it was suggested by Mallet-Paret in [31, p. 9–10]).

As a consequence of the DK strategy, we also present a non-existence result and describe properties of the kernel K which is proved to satisfy exponential convergence estimates (Mollison's condition [8]). Here the fulfillment of the Mollison's condition means that the characteristic function

$$\chi(z) := 1 - \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-zs} ds$$

is well defined for all z from some maximal non-degenerate interval (which can be open, closed, half-closed, finite or infinite). One of the key results of the theory says that, under rather mild assumptions on g, K the presence of a semi-wavefront $\varphi, \varphi(-\infty) = 0$, guarantees the existence of a minimal positive zero λ_l to $\chi(z)$. Let us also mention here a new type of non-existence theorem proposed not long ago by Yagisita [41] for a nonlocal analogue of the KPP equation. Yagisita introduced the concept of a *periodic traveling wave solution with average speed c* and his version of the non-existence result (given in terms of these solutions) is stronger than the standard one. The spreading properties of the nonlocal KPP type equations with 'fat-tailed' kernels were recently considered by Garnier [21].

Next, as it is known the DK and Schumacher uniqueness theorems do not apply to the critical fronts (when $\chi(\lambda_l) = \chi'(\lambda_l) = 0$).[1] As an example, let us consider the

[1] Also, the DK result assumes that fronts are monotone and Schumacher considers only what he calls 'regular solutions' (i.e. those having an appropriate exponential convergence at $-\infty$).

nonlocal KPP equation

$$u_t = J * u - u + g(u),\ x \in \mathbb{R}, \qquad g(0) = g(1) = 0, \quad g > 0 \text{ on } (0, 1) \tag{2}$$

proposed in [28]. Here continuous birth function f is supposed to be differentiable at 0, with $g(s) = g'(0)s + O(s^{1+\alpha}),\ s \to 0+$, for some $\alpha > 0$, and to satisfy the KPP condition [28] $f'(s) \le f'(0),\ s \in (0, 1)$. Measurable kernel $J \ge 0, \int J ds = 1$ is allowed to be asymmetric and non compactly supported. This agrees with the initial idea of Kolmogorov et al. [28] who interpreted $J(x)dx$ as the probability that an individual passes a distance between x and $x + dx$. It is easy to see that the DK theory does not apply to (2). Under the above mentioned assumptions, Schumacher [33, Example 2] has proved uniqueness of all *non-critical* wavefronts for (2). Now, the first examples of nonlocal monostable equations in which the uniqueness problem was completely solved (what includes the case of critical and non-monotone wavefronts) appeared in Chen and Guo [6] and Carr and Chmaj [4]. In particular, Carr and Chmaj [4] achieved an important extension of the DK theory for the special case of Eq. (2). By assuming several additional conditions in [4] that J must be even, compactly supported and

$$|g(s) - g(t)| \le g'(0)|t - s|, \quad s, t \ge 0, \tag{3}$$

they showed that the minimal wavefront $\varphi(x + c_0 t)$ to (2) satisfying $0 \le \varphi(s) \le 1,\ s \in \mathbb{R}$, is unique up to translation. Carr and Chmaj's work has motivated the second goal of our research: to get an improvement of the DK theory that includes the critical semi-wavefronts. Theorem 3 below gives such an extension for general model (1). In the particular case of Eq. (2) our result (stated as Theorem 5) establishes the uniqueness of *critical* wavefronts under the same assumptions on J and f as in [33]. See Sect. 6.1 for more details, further discussion and references.

The necessity of the subtangential Lipschitz condition (3) [4,12,16,36] could be considered as a weak point of the DK uniqueness theorem, cf. [1,6,8,22,26,33]. For instance, as it was established recently by Coville et al. [8], neither (3) nor $g'(s) \le g'(0),\ s \in (0, 1)$, is necessary to prove the uniqueness of non-stationary monotone traveling fronts to (2). Instead of that, it was supposed in [8] that generally asymmetric $J \in C^1(\mathbb{R})$ is compactly supported with $J(a) > 0,\ J(b) > 0$ for some $a < 0 < b$, while $g \in C^1(\mathbb{R})$ has to satisfy $g'(0)g'(1) < 0,\ g(s) \le g'(0)s,\ s \ge 0$, and $g \in C^{1,\alpha}$ near 0. The proof in [8] follows ideas of [7] and is mainly based on the sliding methods proposed by Berestycki and Nirenberg [3] (see [7,8] for a comprehensive state-of-art overview about (2) and [5,30] for the further references). The above discussion explains our third goal in this paper: to weaken various convergence and smoothness conditions of the DK theory, and especially condition (3). It is worthwhile to note that a similar task was also considered in [33]. The related improvements can be found in Theorems 3 and 4. In the latter theorem, we remove condition (3) by assuming a little more smoothness for g and exploiting the absence of zeros for $\chi(z)$ in the vertical strip $\lambda_l < \Re z < \lambda_r$ (see Lemma 2). Incidentally, Theorems 4 justifies the following principle for monostable equations: "fast positive semi-wavefronts are unique (modulo translation)". In the last section, we apply this principle to reaction–diffusion equations with delayed Mackey–Glass type nonlinearities.

The main results of this paper are stated as Theorems 3, 4 below. We apply them to nonlocal integro-differential equations (Sect. 6.1), nonlocal lattice systems (Sect. 6.2), nonlocal (Sect. 6.3) and local (Sect. 6.4) reaction–diffusion equations with discrete delays. In Theorem 1, we give a short proof of the necessity of the Mollison's condition for the existence of semi-wavefronts. Theorem 2 provides a non-existence result.

2 Mollison's condition

In this section, we consider somewhat more general equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s), t-s, \tau) ds, \tag{4}$$

where measurable $g : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R}_+$ is continuous in the first two variables for every fixed $\tau \in X$. We suppose additionally that, for some measurable $p(\tau) \geq 0$ and $\delta > 0,\ \bar{s} \leq 0$, it holds

$$g(v, s, \tau) \geq p(\tau) v,\ \ v \in (0, \delta),\ \ s \leq \bar{s},\ \ \tau \in X. \tag{5}$$

First, we present a simple proof of the necessity of the following Mollison's condition (cf. [8]) for the existence of the semi-wavefronts:

$$\int_{\mathbb{R}} \int_X K(s,\tau) p(\tau) d\mu(\tau) e^{-sz} ds \text{ is finite for some } z \in \mathbb{R} \setminus \{0\}. \tag{6}$$

Theorem 1 *Let continuous* $\varphi : \mathbb{R} \to [0, +\infty)$ *satisfy* (4) *and suppose that* $\varphi(-\infty) = 0$ *and* $\varphi(t) \not\equiv 0,\ t \leq t'$ *for each fixed* t'. *If* (5) *holds and*

$$\int_X \int_{\mathbb{R}} K(s,\tau) p(\tau) ds d\mu(\tau) \in (1, \infty), \tag{7}$$

then $\int_{-\infty}^0 \varphi(s) e^{-s\bar{x}} ds$ *and* $\int_{\mathbb{R}} \int_X K(s,\tau) p(\tau) d\mu(\tau) e^{-s\bar{x}} ds$ *are convergent for an appropriate* $\bar{x} > 0$. *Furthermore, supp* $K \cap (\mathbb{R}_+ \times X) \neq \emptyset$.

Remark 1 Looking for heteroclinic solutions of the simple logistic equation $x' = -\beta x + x(1 + \beta - x)$ with $\beta > 0$, we obtain an example of (1) where supp $K \cap (\mathbb{R}_- \times X) = \emptyset$ under conditions of the above theorem.

Proof Since the support of K generally is unbounded, we will truncate K by choosing integer N such that

$$\kappa := \int_X \int_{-N}^{N} K(s,\tau) p(\tau) ds d\mu(\tau) > 1, \text{ and } 0 \leq \varphi(t) < \delta, \quad t < \bar{s} - N.$$

Integrating Eq. (4) between t' and $t < \bar{s} - 2N$, we find that

$$\begin{aligned}\int_{t'}^{t} \varphi(v)dv &\geq \int_X d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} g(\varphi(v-s), v-s, \tau)dvds \\ &\geq \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} \varphi(v-s)dvds \\ &= \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \left(\int_{t'-s}^{t'} + \int_{t'}^{t} + \int_{t}^{t-s} \right) \varphi(v)dvds,\end{aligned}$$

from which

$$\int_{t'}^{t} \varphi(v)dv \leq \frac{2\delta \int_X \int_{-N}^{N} |s| K(s,\tau)p(\tau)dsd\mu(\tau)}{\int_X \int_{-N}^{N} K(s,\tau)p(\tau)dsd\mu(\tau) - 1}, \quad t' < t < \bar{s} - 2N.$$

Hence, the increasing function

$$\psi(t) = \int_{-\infty}^{t} \varphi(s)ds \tag{8}$$

is well defined for all $t \in \mathbb{R}$ and

$$\psi(t) \geq \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)\psi(t-s)ds \geq \kappa\psi(t-N), \quad t < \bar{s} - 2N.$$

Consider $h(t) = \psi(t)e^{-\gamma t}$ where $\kappa = e^{\gamma N}$, cf. [4]. For all $t < \bar{s} - 2N$ we have

$$h(t-N) = \psi(t-N)e^{-\gamma(t-N)} \leq \frac{1}{\kappa}\psi(t)e^{-\gamma t}e^{\gamma N} = h(t)$$

and $\gamma = N^{-1}\ln\kappa > 0$. Hence $\sup_{t \leq 0} h(t) < \infty$ and $\psi(t) = O(e^{\gamma t})$, $t \to -\infty$. After taking $\bar{x} \in (0, \gamma)$ and integrating by parts, we obtain

$$\int_{-\infty}^{t} \varphi(s)e^{-\bar{x}s}ds = \psi(t)e^{-\bar{x}t} + \bar{x}\int_{-\infty}^{t} \psi(s)e^{-\bar{x}s}ds$$

that proves the first statement of the theorem. Finally,

$$e^{-\bar{x}t}\psi(t) = \int_X d\mu(\tau)\int_{\mathbb{R}} e^{-\bar{x}s}K(s,\tau)e^{-\bar{x}(t-s)}\psi_1(t-s,\tau)ds,$$

where $\psi_1(u,\tau) := \int_{-\infty}^{u} g(\varphi(s),s,\tau)ds \geq p(\tau)\int_{-\infty}^{u}\varphi(s)ds,\ u \leq \bar{s}-N$. The latter yields

$$\begin{aligned}\int_{-\infty}^{\bar{s}-N} e^{-\bar{x}v}\psi(v)dv &= \int_X d\mu(\tau)\int_{\mathbb{R}} e^{-\bar{x}s}K(s,\tau)\int_{-\infty}^{\bar{s}-N} e^{-\bar{x}(v-s)}\psi_1(v-s,\tau)dvds\\ &\geq \int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0} e^{-\bar{x}s}K(s,\tau)ds\int_{-\infty}^{\bar{s}-N} e^{-\bar{x}v}\psi(v)dv,\\ \mathcal{K}_-(\bar{x}) &:= \int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0} e^{-\bar{x}s}K(s,\tau)ds \leq 1,\\ &\qquad \text{(note that } \psi(s) > 0,\ s\in\mathbb{R}\text{)},\end{aligned} \tag{9}$$

so that

$$\mathcal{K}_-(0) = \int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0} K(s,\tau)ds \leq 1 < \int_X p(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)ds,$$

which completes the proof of the theorem.

Remark 2 Suppose that $|g(\varphi(s),s,\tau)| \leq C$ where C does not depend on s,τ. Then

$$|\varphi(t+h)-\varphi(t)| \leq C\int_X d\mu(\tau)\int_{\mathbb{R}} |K(s+h,\tau)-K(s,\tau)|ds.$$

Since the translation is continuous in $L_1(\mathbb{R})$ [14, Example 5.4], we find that $\varphi(t)$ is uniformly continuous on $\mathbb{R}$. It is easy to see that the convergence of the integral $\int_{-\infty}^{0}\varphi(s)ds < \infty$ combined with the uniform continuity of φ gives $\varphi(-\infty) = 0$. In this way, $\int_{-\infty}^{0}\varphi(s)ds < \infty$ implies that $\int_{-\infty}^{0} e^{-xs}\varphi(s)ds < \infty$ for small positive x.

Remark 3 It is easy to see that the global non-negativity of g is not necessary in the case of K having bounded support (uniformly in $\tau \in X$).

Now, let $\varphi, K, g, \bar{x}$ be as in Theorem 1 and $\sup_{s\in\mathbb{R}} \varphi(s) < \infty$. Set

$$\Phi(z) = \int_{\mathbb{R}} e^{-zs}\varphi(s)ds, \ \mathcal{K}(z) = \int_{\mathbb{R}}\int_{X} K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds,$$

and denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\phi < \Re z < \gamma_\phi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Evidently, $\sigma_\phi, \sigma_K \le 0$ and $\gamma_\phi, \gamma_K \ge \bar{x} > 0$. Since φ, K are both non-negative, by [40, Theorem 5b, p. 58], $\gamma_\phi, \ \gamma_K, \sigma_\phi, \ \sigma_K$ are singular points of $\Phi(z), \mathcal{K}(z)$ (whenever they are finite). A simple inspection of the proof of Theorem 1 suggests the following

Lemma 1 *Assume φ, g, K are as in Theorem 1. Then $\sigma_K \le \sigma_\phi < \gamma_\phi \le \gamma_K$. Furthermore, $\mathcal{K}(\gamma_\phi)$ is always a finite number.*

Proof For all $z \in (0, \gamma_\phi), \ t \le 0$, we have

$$\psi(t) = \int_{-\infty}^{t} (\varphi(s)e^{-zs})e^{zs}ds \le e^{zt}\int_{-\infty}^{0} \varphi(s)e^{-zs}ds,$$

so that $\int_{-\infty}^{0} \psi(s)e^{-z's}ds < \infty$ for each $z' \in (0, \gamma_\phi)$ and, due to (9), we get

$$\mathcal{K}_{-}(z) := \int_{X} p(\tau)d\mu(\tau)\int_{-\infty}^{0} e^{-zs}K(s,\tau)ds \le 1$$

for all $z \in (0, \gamma_\phi)$. Hence, using the Beppo Levi monotone convergence theorem, we obtain that $\mathcal{K}_{-}(\gamma_\phi) \le 1$. As a consequence, $\mathcal{K}(\gamma_\phi)$ is finite and $\gamma_K \ge \gamma_\phi$.

Corollary 1 *Assume that*

$$\lim_{z\to\gamma_K-}\int_{\mathbb{R}}\int_{X} K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds = +\infty.$$

Then γ_ϕ is a finite number and $\gamma_\phi < \gamma_K$.

3 Abscissas of convergence

In this section, we investigate the abscissas of convergence for the bilateral Laplace transforms of K and bounded non-negative φ satisfying $\varphi(-\infty) = 0, \varphi(t) \not\equiv 0, \ t \le t'$, for each fixed t', and solving our main Eq. (1). Now we are supposing that the continuous $g(\cdot,\tau) : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable at 0 with $g'(0+,\tau) > 0$ for each fixed τ. Then the non-negative functions

$$\lambda_\delta^+(\tau) := \sup_{u\in(0,\delta)} \frac{g(u,\tau)}{u}, \ \lambda_\delta^-(\tau) := \inf_{u\in(0,\delta)} \frac{g(u,\tau)}{u}, \qquad \delta > 0, \quad \tau \in X,$$

are well defined, measurable, monotone in δ and pointwise converging:

$$\lim_{\delta\to 0+} \lambda_\delta^{\pm}(\tau) = g'(0+, \tau).$$

The *characteristic* function χ associated with the variational equation along the trivial steady state of (1) is defined by

$$\chi(z) := 1 - \int_{\mathbb{R}}\int_X K(s, \tau)g'(0+, \tau)d\mu(\tau)e^{-sz}ds.$$

It is supposed to be negative at $z = 0$: $\chi(0) < 0$. Since condition (5) is obviously satisfied with $p(\tau) = \lambda_\delta^-(\tau)$ and

$$\lim_{\delta\to 0+}\int_{\mathbb{R}}\int_X K(s, \tau)\lambda_\delta^-(\tau)d\mu(\tau)ds = \int_{\mathbb{R}}\int_X K(s, \tau)g'(0+, \tau)d\mu(\tau)ds > 1$$

by the monotone convergence theorem, all results of Sect. 2 hold true for Eq. (1). Furthermore, we have the following

Theorem 2 *Assume* $\chi(0) < 0$. *Let* $\varphi : \mathbb{R} \to [0, +\infty)$ *be a semi-wavefront to Eq. (1). If* $\varphi(-\infty) = 0$ *and* $\varphi(t) \not\equiv 0,\ t \le t'$ *for each fixed* t', *then* $\chi(z)$ *has a zero on* $(0, \gamma_\phi] \subset (0, \gamma_K] \subset \mathbb{R} \cup \{+\infty\}$.

Remark 4 (1) If $\varphi(+\infty) = 0$ then a similar statement can be proved. Namely, in such a case $\chi(z)$ has a zero on $[\sigma_K, 0)$. (2) It should be noted that Theorem 2 also provides a non-existence result: if $\chi(x) < 0$ for all $x \in (0, \gamma_K]$ then Eq. (1) does not have any semi-wavefront vanishing at $-\infty$.

Proof For real positive $z \in (0, \gamma_\phi)$ we consider the integrals

$$\Phi(z) = \int_{\mathbb{R}} e^{-zs}\varphi(s)ds,\ \mathcal{G}(z, \tau) := \int_{\mathbb{R}} e^{-zs}g(\varphi(s), \tau)ds,\ \mathcal{K}(z, \tau) := \int_{\mathbb{R}} e^{-zs}K(s, \tau)ds.$$

Since φ is non-negative and bounded, and since $g'(0+, \tau) > 0$ exists, the convergence of $\mathcal{G}(z, \tau)$ (for positive z) is equivalent to the convergence of $\Phi(z)$. Applying the bilateral Laplace transform to Eq. (1), we obtain that

$$\Phi(z) = \int_X \mathcal{K}(z, \tau)\mathcal{G}(z, \tau)d\mu(\tau). \tag{10}$$

Obviously, $\mathcal{K}, \mathcal{G}, \Phi$ are positive at each real point of the convergence.

Let us prove that $\chi(z)$ has a zero on $(0, \gamma_\phi]$. First, we suppose that $\Phi(\gamma_\phi) = \lim_{z\to\gamma_\phi -}\Phi(z) = \infty$. In such a case, we claim that

$$\lim_{z\to\gamma_\phi -}\frac{\mathcal{G}(z, \tau)}{\Phi(z)} = g'(0, \tau).$$

Indeed, let T_δ be the rightmost non-positive number such that $\varphi(s) \le \delta$ for $s \le T_\delta$. Then

$$\lambda_\delta^- \int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds \le \int_{-\infty}^{T_\delta} e^{-zs} g(\varphi(s), \tau)ds \le \lambda_\delta^+ \int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds,$$

$$\int_{T_\delta}^{+\infty} e^{-zs}(g(\varphi(s), \tau) + \varphi(s))ds \le \frac{\sup_{s\in\mathbb{R}}(g(\varphi(s), \tau) + \varphi(s))}{z} e^{-\gamma_\phi T_\delta}.$$

As a consequence, for each positive $\delta > 0$,

$$\lambda_\delta^- \le \liminf_{z\to\gamma_\phi -} \frac{\mathcal{G}(z, \tau)}{\Phi(z)} \le \limsup_{z\to\gamma_\phi -} \frac{\mathcal{G}(z, \tau)}{\Phi(z)} \le \lambda_\delta^+,$$

that proves our claim.

Now, by using the Fatou lemma as $z \to \gamma_\phi -$ in

$$\int_X \mathcal{K}(z, \tau)\frac{\mathcal{G}(z, \tau)}{\Phi(z)} d\mu(\tau) = 1,$$

we obtain

$$1 - \chi(\gamma_\phi) = \int_X \mathcal{K}(\gamma_\phi, \tau) g'(0, \tau) d\mu(\tau) \le 1.$$

Therefore $\chi(\gamma_\phi) \ge 0$, and since $\chi(0) < 0$ we get the required assertion.

Hence, we may suppose that $\Phi(\gamma_\phi) = \lim_{z\to\gamma_\phi -} \Phi(z) > 0$ is finite. Since $\varphi(t) \not\equiv 0,\ t \le t'$ for each fixed t', in such a case $\gamma_\phi < \infty$. Due to Lemma 1, the value $\mathcal{K}(\gamma_\phi)$ is also finite. Set

$$\zeta(t) := \varphi(t)e^{-\gamma t},\ K_1(s, \tau) := e^{-\gamma s} K(s, \tau),\ \text{where}\ \gamma := \gamma_\phi.$$

Then, for $t < T_\delta - N$, we have from (1) that $\int_{-\infty}^t \zeta(v)dv =$

$$\begin{aligned}\int_{-\infty}^{t} \varphi(v)e^{-\gamma v}dv &\ge \int_X d\mu(\tau) \int_{-N}^{N} K_1(s, \tau) \int_{-\infty}^{t} g(\varphi(v - s), \tau)e^{-\gamma(v-s)}dvds \\ &\ge \int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau)K_1(s, \tau) \int_{-\infty}^{t} \zeta(v - s)dvds \\ &\ge \left(\int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau)K_1(s, \tau)ds\right) \int_{-\infty}^{t-N} \zeta(v)dv.\end{aligned}$$

Suppose now on the contrary that the characteristic equation

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_{X} K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-sz} ds = 0$$

has not real roots on $[0, \gamma_\phi]$. Then $\chi(0) < 0$ implies $\chi(\gamma) < 0$. As a consequence, in virtue of the monotone convergence theorem,

$$\lim_{\delta \to 0+, N \to +\infty} \int_X d\mu(\tau) \int_{-N}^{N} \lambda_\delta^-(\tau) K_1(s,\tau) ds = 1 - \chi(\gamma) > 1.$$

Hence, for some appropriate $\delta, N > 0$, increasing function $\xi(t) = \int_{-\infty}^t \zeta(s) ds$ satisfies $\xi(t) \geq \kappa_\delta \xi(t - N)$, $t < T_\delta - N$ with $\kappa_\delta > 1$. Arguing now as in the proof of Theorem 1 below (8) we conclude that the integral $\int_{-\infty}^t \zeta(s) e^{-zs} ds$ converges for all small positive z, contradicting to the definition of γ_ϕ.

Remark 5 A natural question is whether there exists φ satisfying assumptions of Theorem 2 and such that $\Phi(\gamma_\phi)$ is finite. Actually, it is well known that $\Phi(\gamma_\phi) = \infty$ under some additional conditions on K, g. For example, this happens if $g(u,\tau) \leq g'(0,\tau)u,\ u \geq 0$ (other conditions can be found in Corollary 3). Indeed, due to the above proof, the only case to be examined is when $\chi(\gamma) \geq 0$ and $\Phi(\gamma) < \infty,\ \gamma := \gamma_\phi$. We have

$$\varphi(t) e^{-\gamma t} \leq \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau) g'(0,\tau) \varphi(t-s) e^{-\gamma(t-s)} ds,\ t \in \mathbb{R}, \qquad (11)$$

where both sides of the inequality are continuous[2] functions of t. If either (i) $\chi(\gamma) > 0$ or (ii) inequality (11) is strict at some point t_0, we will integrate (11) over $\mathbb{R}$ to get a contradiction: $\Phi(\gamma) \leq \Phi(\gamma)(1 - \chi(\gamma))$ (case (i)), $\Phi(\gamma) < \Phi(\gamma)(1 - \chi(\gamma))$ (case (ii)). In consequence we are left to assume that

$$\varphi(t) e^{-\gamma t} = \int_{\mathbb{R}} \left[\int_X K_1(s,\tau) g'(0,\tau) d\mu(\tau) \right] \varphi(t-s) e^{-\gamma(t-s)} ds, \quad t \in \mathbb{R},$$

and $\int_{\mathbb{R}} \int_X K_1(s,\tau) g'(0,\tau) d\mu(\tau) ds = 1$. However, after integrating the latter equality over $(t, +\infty)$ and then using Lemma 7 with Remark 7, we get again a contradiction.

It is clear that $\chi(z)$ is concave on (σ_K, γ_K), where $\chi''(z) < 0$. Since $\chi(0)$ is negative, χ can have at most two real zeros, and they must be of the same sign. We will denote them (if they exist) by $\lambda_l \leq \lambda_r$. Under assumption of the existence of a semi-wavefront

[2] This property becomes obvious if we rewrite (11) without exponential factors.

φ vanishing at $-\infty$, χ has at least one positive root λ_l. Finally, it is clear that χ is analytical in the vertical strip $\Re z \in (0, \gamma_K)$.

Notation At this stage, it is convenient to introduce the following notation:

$$\lambda_{rK} = \begin{cases} \lambda_r, & \text{if } \lambda_r \text{ exists}, \\ \gamma_K, & \text{otherwise}. \end{cases}$$

Lemma 2 *Equation $\chi(z) = 0$ does not have roots in the open strip $\Sigma := \Re z \in (\lambda_l, \lambda_{rK})$. Furthermore, the only possible zeros on the boundary Σ are λ_l, λ_r.*

Proof Observe that if $\chi(z_0) = 0$ for some $z_0 \in \Sigma$, then $\chi(\Re z_0) > 0$ since χ is concave, $\chi(\lambda_l) = 0$ and $\Re z_0 \in (\lambda_l, \min\{\lambda_r, \gamma_K\})$. On the other hand,

$$1 = |\int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-sz_0} ds| \le \int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-s\Re z_0} ds$$

and therefore $\chi(\Re z_0) \le 0$, a contradiction. Now, if $\chi(\lambda_l + i\omega) = 0$ for some $\omega \neq 0$ then similarly

$$1 = 1 - \chi(\lambda_l + i\omega) = |1 - \chi(\lambda_l + i\omega)| \le 1 - \chi(\lambda_l) = 1,$$

so that

$$\int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-s\lambda_l} (1 - \cos \omega s) ds = 0.$$

Thus $K(s,\tau)(1 - \cos \omega s) = 0$ for almost all $\tau \in X$, so that $K(s,\tau) = 0$ a.e. on $X \times \mathbb{R}$, a contradiction.

4 A bootstrap argument

The main purpose of this section is to prove several auxiliary statements needed in the studies of the asymptotic behavior of solutions $\varphi(t)$ at $t = -\infty$. Usually proofs of the uniqueness are based on the derivation of appropriate asymptotic formulas with one or two leading terms (at $t = -\infty$ as in [4,12,16,39] or at $t = +\infty$ as in [23]). Our approach is based on an asymptotic integration routine often used in the theory of functional differential equations, e.g. see [27], [31, Proposition 7.1] or [20]. Thus we use neither the Titchmarsh theory of Fourier integrals [35] nor the powerful Ikehara Tauberian theorem [4,12]. First we will apply our methods to get an asymptotic formula for the integral $\psi(t) := \int_{-\infty}^t \varphi(s) ds$. Since $\psi \in C^1(\mathbb{R})$ is increasing and positive, this function is somewhat easier to treat than the solution $\varphi(t)$.

Here and everywhere in the sequel, the functions φ, K, g, χ satisfy all conditions of Sect. 3. In particular, $\chi(0) < 0$. We also will use the following hypotheses (**SB**), (**EC**$_\rho$):

(SB) $\gamma_\phi < \gamma_K$ and, for some measurable $C(\tau) > 0$ and $\alpha, \sigma \in (0, 1]$,

$$|g'(0,\tau) - \frac{g(u,\tau)}{u}| \le C(\tau)u^\alpha, \quad u \in (0,\sigma),$$

$$\zeta(x) := \int_{X\times\mathbb{R}} C(\tau)K(s,\tau)e^{-sx}dsd\mu < +\infty, \ x \in (0,\gamma_K). \tag{12}$$

(EC$_\rho$) For every $x \in (0,\rho)$, $\rho \le \gamma_\phi$, there exists some positive C_x such that

$$0 \le \varphi(t) \le C_x e^{xt}, \quad t \le 0. \tag{13}$$

There are several situations when **(EC$_\rho$)** can be easily checked:

Lemma 3 *Condition* **(EC$_\rho$)** *is satisfied in either of the following two cases:*

(i) $\varphi \in C^1(\mathbb{R})$ *and the integral* $\int_\mathbb{R} e^{-xs}\varphi'(s)ds$ *converges absolutely for all* $x \in (0,\rho)$*;*

(ii) (cf. [12]) $\rho < \gamma_\phi$ *and there exist measurable* $d_1, d_2,\ d_1d_2 \in L^1(X)$*, such that*

$$0 \le K(s,\tau) \le d_1(\tau)e^{\rho s}, \ s \in \mathbb{R}, \ \tau \in X,$$

$$|g(u,\tau)| \le d_2(\tau)u, \ u \ge 0. \tag{14}$$

Proof (i) For each $x \in (0,\rho)$ we have that

$$\varphi(t) = \int_{-\infty}^{t} \varphi'(s)ds = \int_{-\infty}^{t} e^{xs}\varphi'(s)e^{-xs}ds \le e^{xt}\int_\mathbb{R} e^{-xs}|\varphi'(s)|ds =: C_x e^{xt}.$$

(ii) Since $\rho < \gamma_\phi$, the integral $\int_\mathbb{R} e^{-xs}\varphi(s)ds$ converges for all $x \in (0,\rho]$. If $x \in (0,\rho],\ t \le 0$, then

$$\varphi(t)e^{-xt} \le \varphi(t)e^{-\rho t} = \int_X d\mu(\tau)\int_\mathbb{R} K(s,\tau)e^{-\rho s}e^{-\rho(t-s)}g(\varphi(t-s),\tau)ds$$
$$\le C := \int_X d_1(\tau)d_2(\tau)d\mu(\tau)\int_\mathbb{R} e^{-\rho s}\varphi(s)ds.$$

The following simple proposition will be used several times in the sequel:

Lemma 4 *Assume that* $h(s)e^{-sx} \in L^1(\mathbb{R})$ *for all* $x \in [a,b]$*. Then*

$$H(x,y) := \int_\mathbb{R} h(s)e^{-sx-isy}ds, \ y \in \mathbb{R},$$

is uniformly (with respect to $y \in \mathbb{R}$*) continuous on* $[a,b]$.

Proof Take an arbitrary $\varepsilon > 0$ and let $N > 0$ be such that

$$\int_{\mathbb{R}\setminus[-N,N]} |h(s)|e^{-sx}ds < 0.25\varepsilon, \ x \in [a,b].$$

Since e^t is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|x_1 - x_2| \le \delta, \ s \in [-N, N]$ implies $|e^{-x_1 s} - e^{-x_2 s}| < 0.5\varepsilon/|h|_1$. But then

$$|H(x_1, y) - H(x_2, y)| \le 0.5\varepsilon + \int_{-N}^{N} |h(s)||e^{-x_1 s} - e^{-x_2 s}|ds < \varepsilon, \ y \in \mathbb{R}.$$

Corollary 2 *With h as in Lemma 4, we have that* $\lim_{y\to\infty} H(x, y) = 0$ *uniformly on* $x \in [a, b]$.

Proof Due to Lemma 4, for each $\varepsilon > 0$ there exists a finite sequence $a := x_0 < x_1 < x_2 < \cdots < x_m =: b$ possessing the following property: for each x there is x_j such that $|H(x_j, y) - H(x, y)| < 0.5\epsilon$ uniformly on y. Now, by the Riemann–Lebesgue lemma, $\lim_{y\to\infty} H(x_j, y) = 0$ for every j. Therefore, for all j and some $M > 0$, we have that $|H(x_j, y)| < 0.5\epsilon$ if $|y| \ge M$. This implies that

$$|H(x, y)| \le |H(x_j, y) - H(x, y)| + |H(x_j, y)| < \epsilon, \quad |y| \ge M, \ x \in [a, b],$$

and the corollary is proved.

As we know, the property $\varphi(-\infty) = 0$ implies the exponential decay $\psi(t) = O(e^{zt})$ at $-\infty$ for each $z \in (0, \gamma_\phi)$. It is clear also that $\psi(t) = O(t)$ as $t \to +\infty$. Hence, for each fixed $z \in (0, \gamma_\phi)$, we can integrate Eq. (1) twice, to find that $\Psi(z) := \int_{\mathbb{R}} e^{-zv}\psi(v)dv$ satisfies

$$\begin{aligned}
\Psi(z) &= \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} \int_{\mathbb{R}} e^{-z(v-s)} \int_{-\infty}^{v-s} g(\varphi(u),\tau)dudvds \\
&= \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} g(\varphi(u),\tau)dudvds \\
&= \left(\int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)e^{-zs}ds\right) \int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}(z), \quad \text{where}
\end{aligned}$$

$$\mathcal{R}(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))dudv.$$

Therefore $\chi(z)\Psi(z) = \mathcal{R}(z)$. Set now

$$\mathbf{G}(z,\tau) := \int_{\mathbb{R}} e^{-zv} G(v,\tau)dv, \quad G(v,\tau) := \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

Lemma 5 *Assume* (14), (**SB**), (**EC**$_{2\epsilon}$) *for some small* $2\epsilon \in (0, \gamma_K - \gamma_\phi)$. *Then given* $a, b \in (0, \gamma_\phi + \alpha\epsilon)$ *there exists* $\rho > 0$ *depending on* φ, a, b *such that*

$$|\mathbf{G}(z,\tau)| \le \rho(\tau)/|z| := \rho(C(\tau) + d_2(\tau) + g'(0,\tau))/|z|,$$
$$\Re z \in [a,b] \subset (0, \gamma_\phi + \alpha\epsilon).$$

Proof For $x := \Re z \in (0, \gamma_\phi + \alpha\epsilon),\ v \le 0$, we have

$$e^{-xv}|G(v,\tau)| \le e^{-xv}C(\tau)\int_{-\infty}^{v} (\varphi(u))^{1+\alpha}du \le e^{-xv}C_\epsilon^\alpha C(\tau)\psi(v)e^{\alpha\epsilon v},$$

so that $e^{-x\cdot}|G(\cdot,\tau)| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. After integrating by parts, we obtain

$$\int_{-N}^{N} e^{-zv}G(v,\tau)dv = \frac{G(-N,\tau)e^{zN} - G(N,\tau)e^{-zN}}{z}$$
$$+\frac{1}{z}\int_{-N}^{N} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

This yields

$$|\int_{\mathbb{R}} e^{-zv}G(v,\tau)dv| = \frac{1}{|z|}|\int_{\mathbb{R}} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du|$$
$$\le \frac{1}{|z|}\left(C_\epsilon^\alpha C(\tau)\int_{-\infty}^{0} e^{-(\Re z - \alpha\epsilon)u}\varphi(u)du + |\varphi|_\infty(g'(0,\tau) + d_2(\tau))\int_{0}^{+\infty} e^{-\Re z u}du\right).$$

Corollary 3 *In addition, assume that* $\int_{\mathbb{R}\times X} K(s,\tau)\rho(\tau)e^{-sx}d\mu ds$ *converges for all* $x \in (0, \gamma_K)$. *Then* $\chi(\gamma_\phi) = 0$ *and, for appropriate* $\varepsilon_1 > 0,\ a, m \in \mathbb{R}$, $k \in \{0, 1\}$, *and continuous* $r \in L^2(\mathbb{R})$, *it holds that*

$$\psi(t+m) = (a-t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi + \varepsilon_1)t}r(t),\ t \in \mathbb{R}.$$

It should be noted that depending on the geometric properties of g, the value of γ_ϕ can be minimal (the case of a pulled semi-wavefront [13,22,34]) or maximal (the case of a pushed semi-wavefront, ibid.) positive zero of $\chi(z)$. Observe that, due to the monotonicity of ψ, we can also use here the Ikehara Tauberian theorem [4]. However it gives a slightly different result.

Proof Set $z := x + iy$. For a fixed $0 < x < \gamma_\phi + \alpha\epsilon$ we have

$$|\mathcal{R}(z)| = |\int_X \mathbf{G}(z,\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} ds d\mu| \le \frac{1}{|z|} \int_X \rho(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-xs} ds d\mu,$$

so that $\mathcal{R}(z)$ is regular in the strip $0 < \Re z < \gamma_\phi + \alpha\epsilon$. Thus we can deduce from $\Psi(z) = \mathcal{R}(z)/\chi(z)$ that $\gamma_\phi = \gamma_\psi$ (e.g. see [12, Lemma 4.4], the definition of γ_ψ is similar to that of γ_ϕ) must be a positive zero of $\chi(z)$ and $\Psi(\gamma_\phi) = \infty$. It is clear that $\mathcal{R}(x + i\cdot)$ is also bounded and square integrable on $\mathbb{R}$ (for each fixed x). Take now γ', γ'' such that $0 < \gamma' < \gamma_\phi < \gamma'' < \gamma_\phi + \alpha\epsilon$. Then we may shift the path of integration in the inversion formula for the Laplace transform (e.g. see [31, p. 10]) to obtain

$$\psi(t) = \frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\Psi(z)dz = -\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + \frac{e^{\gamma'' t}}{2\pi i}\left\{\int_{-\infty}^{+\infty} e^{ist} a_1(s) ds\right\},$$

where the first term is different from 0 and $a_1(s) = \mathcal{R}(\gamma'' + is)/\chi(\gamma'' + is)$ is square integrable on $\mathbb{R}$. Here we recall that, by Corollary 2, $\lim_{y\to\infty} \chi(x + iy) = 1$ uniformly on $x \in [\gamma', \gamma'']$. Since $\chi''(z) < 0$, $x \in (0, \gamma_K)$, for some $a, m \in \mathbb{R}$ we get $\psi(t + m) = (a - t)^k e^{\gamma_\phi t} + e^{\gamma'' t} r(t)$.

Lemma 6 *Assume all conditions of Lemma 5 except $\gamma_\phi < \gamma_K$. If*

$$1 - \chi_1(x_0) := \int_{\mathbb{R}} \int_X K(s,\tau)d_2(\tau)d\mu(\tau)e^{-sx_0} ds \le 1,$$

for some $x_0 \in (0, \gamma_K)$, then γ_ϕ coincides with the minimal positive zero λ_l of $\chi(z)$.

Proof Since $d_2(\tau) \ge g'(0,\tau)$, we obtain that $x_0 \in [\lambda_l, \lambda_{rK}]$ and $\lambda_l < \gamma_K$.
Case I. $\gamma_\phi < \gamma_K$. Then, by Corollary 3, we have $\chi(\gamma_\phi) = 0$ so that $\gamma_\phi \in \{\lambda_l, \lambda_r\}$. Suppose that $\gamma_\phi > \lambda_l$, this implies $x_0 \le \gamma_\phi = \lambda_r$. We have

$$\Psi(z) = \left(\int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)d_2(\tau)e^{-zs} ds\right) \int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}_1(z), \quad \text{where}$$

$$\mathcal{R}_1(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs} ds \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} (g(\varphi(u),\tau) - d_2(\tau)\varphi(u))du dv,$$

or, in a shorter form,

$$\chi_1(z)\Psi(z) = \mathcal{R}_1(z). \tag{15}$$

It is clear that $x_0 = \gamma_\phi = \lambda_r > \lambda_l$ implies immediately that $g'(0,\tau) = d_2(\tau)$ a.e. on X and that $\chi_1(z) = \chi(z),\ \mathcal{R}(z) = \mathcal{R}_1(z)$. As we have seen in the proof of Corollary 3, this guarantees that $\mathcal{R}_1(x_0)$ is a finite number. Of course, $\mathcal{R}_1(x_0)$ is also well defined if $x_0 < \gamma_\phi$. Now, it is clear that $\mathcal{R}_1(x_0) \le 0$ because of $g(u,\tau) \le d_2(\tau)u,\ u \ge 0$. We claim that, in fact, $\mathcal{R}_1(x_0) < 0$. Indeed, otherwise $g(u,\tau) = d_2(\tau)u,\ u \ge 0$, for almost all $\tau \in X$ that yields $d_2(\tau) = g'(0,\tau)$ and $\mathcal{R}_1(z) \equiv 0$ leading to a contradiction: $\Psi(z) \equiv 0$ and $\psi(t) \equiv 0$.

Now, from $\mathcal{R}_1(x_0) < 0$, $\Psi(x_0) > 0$, $\chi_1(x_0) \ge 0$, we deduce that Ψ must have a pole at $x_0 = \gamma_\phi < \gamma_K$. But then $\chi_1(\gamma_\phi) = \chi(\gamma_\phi)$ implies $\chi_1(z) \equiv \chi(z)$, $\mathcal{R}(z) = \mathcal{R}_1(z)$. Hence, $\lambda_l < \lambda_r = x_0 < \gamma_K$ and $\gamma_\phi = x_0$ is a simple pole of Ψ. Therefore we can proceed as in the proof of Corollary 3 taking $0 < \gamma' < \gamma_\phi = \lambda_r < \gamma'' < \gamma_\phi + \alpha\epsilon$ to obtain

$$\begin{aligned}\psi(t) &= \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\Psi(z)dz = -\mathrm{Res}_{z=\lambda_r}\frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + e^{\gamma'' t} r_1(t) \\ &= Ae^{\gamma_\phi t} + e^{\gamma'' t} r_1(t), \quad \text{where } A := -\frac{\mathcal{R}(\lambda_r)}{\chi'(\lambda_r)} < 0,\ r_1 \in L^2(\mathbb{R}),\end{aligned}$$

contradicting to the positivity of ψ.
Case II. $\gamma_\phi = \gamma_K$. Since $x_0 < \gamma_K = \gamma_\phi$ and $\mathcal{R}_1(x_0) < 0$, we similarly deduce from (15) that x_0 is a singular point of $\Psi(z)$, a contradiction.

5 The uniqueness theorems

To prove our uniqueness results we will need more strong property of φ than the merely convergence of $\int_{\mathbb{R}} e^{-zs}\varphi(s)ds$ for all $\Re z \in (0,\gamma_\phi)$ (even combined, as in Sect. 4, with **(EC$_\epsilon$)** for some small $\epsilon > 0$). This property, assumed everywhere in the sequel, is **(EC$_{\gamma_\phi}$)**. The nonlinearity g is supposed to satisfy the hypothesis **(SB)**.

The following assertion is crucial for extension of the Diekmann–Kaper theory on the critical case $\chi(\lambda_l) = \chi'(\lambda_l) = 0$.

Lemma 7 *There is no continuous v which satisfies $v(-\infty) = 1,\ v(+\infty) = 0$ and*

$$v(t) \le \int_{\mathbb{R}} N(s)v(t-s)ds,$$

where measurable $N(s) \ge 0,\ s \in \mathbb{R}$, is such that

$$\int_{\mathbb{R}} N(s)ds = 1, \quad \int_{\mathbb{R}} sN(s)ds = 0, \quad \int_{\mathbb{R}} |s|N(s)ds < \infty.$$

Remark 6 We are deeply indebted to an anonymous referee for suggesting the present statement and proof of Lemma 7. Comparing with our original result (which can be found at http://www.arXiv.org), the referee's version is stronger and has a simpler proof.

Proof First we observe that, without restricting the generality, we may assume that $v \in C^1(\mathbb{R})$ with the finite norm $|v|_{C^1} := \sup_{s\in\mathbb{R}, j=0,1} |v^{(j)}(s)|$ and $v'(\pm\infty) = 0$. Indeed, if we consider

$$w(t) := \int_t^{t+1} v(s)ds, \quad t \in \mathbb{R},$$

then $w \in C^1(\mathbb{R})$ has the same properties as v, with $|w'(t)| \le 2\sup_{s\in\mathbb{R}} |v(s)|$ and $w'(\pm\infty) = 0$.

Set now

$$f(t) := \int_{\mathbb{R}} N(s)v(t-s)ds - v(t) = \int_{\mathbb{R}} N(s)(v(t-s) - v(t))ds \ge 0.$$

Following [2, p.113], we will integrate the above relation over $[-a, a]$ to obtain, after an application of Fubini's theorem,

$$\int_{-a}^{a} f(t)dt = -\int_{-a}^{a} \left(\int_{\mathbb{R}} sN(s) \int_0^1 v'(t-\theta s)d\theta ds \right) dt$$
$$= \int_{\mathbb{R}} sN(s) \int_0^1 (v(-a-\theta s) - v(a-\theta s))\, d\theta ds =: g(a).$$

Applying Lebesgue's dominated convergence theorem, we find easily that $g(a) \to \int_{\mathbb{R}} sN(s)ds = 0$ as $a \to +\infty$. Therefore $\int_{\mathbb{R}} f(t)dt = 0$ so that non-negative continuous $f(t)$ is identically zero. Then

$$v(t) = \int_{\mathbb{R}} N(s)v(t-s)ds,$$

which yields

$$v'(t) = \int_{\mathbb{R}} N(s)v'(t-s)ds.$$

Since $v'(\pm\infty) = 0,\ v(-\infty) = 1,\ v(+\infty) = 0$, we conclude that $v'(t)$ attains its absolute minimum value $m < 0$ at some leftmost point t_0 as well as at some rightmost point $t_1 \geq t_0$. But then $u(t) := (v'(t+t_0) + v'(t+t_1))/2 - m > 0$ for all $t \neq 0$ and we get a contradiction as follows:

$$0 = u(0) = \int_{\mathbb{R}} N(s)u(-s)ds > 0.$$

Remark 7 The above proof shows that both conditions $\int sN(s)ds = 0$ and $\int |s|N(s)ds < \infty$ of Lemma 7 can be omitted when $v(t) = \int N(s)v(t-s)ds$.

Now we are ready to prove our first uniqueness result:

Theorem 3 *Assume* **(SB)** *except* $\gamma_\phi < \gamma_K$ *as well as* **(EC$_{\gamma_\phi}$)** *and suppose further that* $\chi(0) < 0,\ \chi(\gamma_K -) \neq 0,$

$$|g(u,\tau) - g(v,\tau)| \leq g'(0,\tau)|u-v|,\ u, v \geq 0. \tag{16}$$

Then Eq. (1) *has at most one bounded positive solution* $\varphi,\ \varphi(-\infty) = 0$. *Furthermore,* γ_ϕ *coincides with the minimal positive zero* λ_l *of* $\chi(z)$ *and such a solution (if exists) has the following representation:*

$$\varphi(t+m) = (a-t)^k e^{\lambda_l t} + e^{(\lambda_l+\delta)t} r(t), \quad with\, continuous\; r \in L^2(\mathbb{R}),$$

for some appropriate $a, m \in \mathbb{R}$, $\delta > 0$. *Here* $k = 0$ [*respectively,* $k = 1$] *if* λ_l *is a simple* [*respectively, double*] *root of* $\chi(z) = 0$.

Remark 8 By Lemma 6, $\chi(\gamma_K -) \neq 0$ yields $\gamma_\phi = \lambda_l < \gamma_K$. We assume this stronger assumption instead of $\gamma_\phi < \gamma_K$ since it is more easy to use. In the section of applications, the condition $\chi(\gamma_K -) \neq 0$ is slightly modified in order to take into account the dependence of χ, γ_K on the wave velocity c. Recall that we need $\gamma_\phi < \gamma_K$ to apply the bootstrap argument.

Proof Step I: Asymptotic behavior at $-\infty$. It is clear that Eq. (1) can be written as the linear inhomogeneous equation

$$\varphi(t) = \int_X d\mu \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)\varphi(t-s)ds + \mathcal{D}(t), \quad t \in \mathbb{R}, \tag{17}$$

where all integrals are converging and

$$\mathcal{D}(t) := \int_X d\mu \int_{\mathbb{R}} K(s,\tau)(g(\varphi(t-s),\tau) - g'(0,\tau)\varphi(t-s))ds \leq 0,\ t \in \mathbb{R}.$$

Take $C(\tau), \sigma, \zeta(x)$ as in **(SB)**. Observe that without restricting the generality, we can assume in **(SB)** that $(1+\alpha)\gamma_\phi < \gamma_K$. Since Eq. (1) is translation invariant, we can suppose that $\varphi(t) < \sigma$ for $t \le 0$. Applying the bilateral Laplace transform to (17), we obtain that

$$\chi(z)\Phi(z) = \int_{\mathbb{R}} e^{-zt}\mathcal{D}(t)dt =: \mathbf{D}(z).$$

We claim that, due to conditions **(SB)** and $(\mathbf{EC}_{\gamma_\phi})$, function $\mathbf{D}$ is regular in the strip $\Pi_\alpha = \{z : \Re z \in (0, (1+\alpha)\gamma_\phi)\}$. Indeed, we have

$$\mathbf{D}(x+iy) = \int_{\mathbb{R}} e^{-iyt}[e^{-xt}\mathcal{D}(t)]dt.$$

Given $x := \Re z \in (0, (1+\alpha)\gamma_\phi)$, we choose x' sufficiently close from the left to γ_ϕ to satisfy $-x + (1+\alpha)x' > 0$. Then

$$\begin{aligned}
|e^{-xt}\mathcal{D}(t)| \ &\le e^{-xt}\left[\int_X C(\tau)d\mu \int_t^{+\infty} K(s,\tau)C_{x'}^{1+\alpha}e^{(1+\alpha)x'(t-s)}ds + \right. \\
&\left. +2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] \\
&\le e^{-xt}\left[e^{(1+\alpha)x't}C_{x'}^{1+\alpha}\zeta((1+\alpha)x')\right. \\
&\left. +2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] \\
&=: e^{-xt}\left[e^{(1+\alpha)x't}A_1 + 2|\varphi|_\infty \int_X g'(0,\tau)d\mu\right. \\
&\left. \times \int_{-\infty}^t K(s,\tau)e^{-(1+\alpha)x's}e^{(1+\alpha)x's}ds\right] \\
&\le e^{(-x+(1+\alpha)x')t}\left[A_1 + 2|\varphi|_\infty(1-\chi((1+\alpha)x'))\right] \\
&=: A_2 e^{(-x+(1+\alpha)x')t}, \quad t \in \mathbb{R}.
\end{aligned}$$

Since clearly $\mathcal{D}(t)$ is bounded on $\mathbb{R}$, the above calculation shows that $e^{-xt}\mathcal{D}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1, \infty]$ once $x \in (0, (1+\alpha)\gamma_\phi)$. As a consequence, for each such x the function $\mathbf{d}_x(y) := \mathbf{D}(x + i \cdot y)$ is bounded and square integrable on $\mathbb{R}$.

By our assumptions, $\chi(z)$ is also regular in the domain Π_α, while $\Phi(z) = \mathbf{D}(z)/\chi(z)$ is regular in $\Re z \in (0, \gamma_\phi)$ and meromorphic in Π_α. In virtue of Lemma 2, we can suppose that $\Phi(z)$ has a unique singular point γ_ϕ in Π_α which is either simple or double pole.

Now, for some $x'' \in (0, \gamma_\phi)$, using the inversion theorem for the Fourier transform, we obtain that for an appropriate sequence of integers $N_j \to +\infty$

$$\varphi(t) = \frac{1}{2\pi i} \lim_{j\to+\infty} \int_{x''-iN_j}^{x''+iN_j} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} dz$$

almost everywhere on $\mathbb{R}$, e.g. see [31, p. 9–10]. Next, if $x \in (\gamma_\phi, (1+\alpha)\gamma_\phi)$ then

$$\int_{x''-iN}^{x''+iN} \frac{e^{zt}\mathbf{D}(z)dz}{\chi(z)} = \left(\int_{x-iN}^{x+iN} + \int_{x''-iN}^{x-iN} - \int_{x''+iN}^{x+iN} \right) \frac{e^{zt}\mathbf{D}(z)dz}{\chi(z)} - 2\pi i \mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)}.$$

Since, by Corollary 2,

$$\lim_{j\to+\infty} \max_{z\in[x''\pm iN_j, x\pm iN_j]} (|\mathbf{D}(z)| + |1-\chi(z)|) = 0, \tag{18}$$

we conclude that, for each fixed $t \in \mathbb{R}$

$$\lim_{j\to+\infty} \int_{x''\pm iN_j}^{x\pm iN_j} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} dz = 0.$$

Observe also that due to Lemma 2 and Corollary 2 [cf. (18)], the function $\chi(z)$ does not have zero other than $\lambda_l = \gamma_\phi$ in a small strip centered at $\Re z = \lambda_l$. Therefore

$$\varphi(t) = -\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} + \frac{e^{xt}}{2\pi} \int_{\mathbb{R}} \frac{e^{iyt}\mathbf{d}_x(y)}{\chi(x+iy)} dy.$$

It should be noted here that $\mathbf{D}(\gamma_\phi) < 0$ since otherwise $\mathcal{D}(t) \equiv 0$ implying $\chi(z)\Phi(z) = \mathbf{D}(z) \equiv 0$ so that $\Phi(z) \equiv 0$, a contradiction. Since

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} = \frac{e^{\gamma_\phi t}\mathbf{D}(\gamma_\phi)}{\chi'(\gamma_\phi)}, \quad \text{if } \lambda_l < \lambda_{rK},$$

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathbf{D}(z)}{\chi(z)} = \frac{2e^{\gamma_\phi t}}{\chi''(\gamma_\phi)} \left(t\mathbf{D}(\gamma_\phi) + \mathbf{D}'(\gamma_\phi) - \mathbf{D}(\gamma_\phi)\frac{\chi'''(\gamma_\phi)}{3\chi''(\gamma_\phi)} \right), \quad \text{if } \lambda_l = \lambda_r,$$

we get the desired representation.

Step II: Uniqueness. By the contrary, suppose that φ_1 and φ_2 are different solutions of (1) in the sense that $\varphi_1(t) \notin \{\varphi_2(t+s),\ s \in \mathbb{R}\}$. Due to Step I we may suppose that φ_1, φ_2 have the same main parts of their asymptotic representations:

$$\varphi_j(t) = (a_j - t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} r_j(t),\ r_j \in L^2(\mathbb{R}).$$

Therefore $\omega(t) := \varphi_2(t) - \varphi_1(t) = e^{(\gamma_\phi+\delta)t} r(t),\ t \in \mathbb{R},\ r \in L^2(\mathbb{R})$, in the case of $\lambda_l < \lambda_{rK}$ and $\omega(t) = (a_2 - a_1)e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} r(t),\ t \in \mathbb{R},\ r \in L^2(\mathbb{R})$, in the case of $\lambda_l = \lambda_r$. Set

$$w(t) := \int_{t-1}^{t} |\omega(s)| ds,$$

it is clear that $w \in C^1(\mathbb{R})$ is bounded and has bounded derivative on $\mathbb{R}$, in fact, $0 < |w'|_\infty, |w|_\infty \le \max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}$. Furthermore, if $\lambda_l < \lambda_{rK}$ then

$$w(t) = \int_{t-1}^{t} |e^{(\gamma_\phi+\delta)s} r(s)| ds \le e^{(\gamma_\phi+\delta)t} \int_{t-1}^{t} |r(s)| ds \le e^{(\gamma_\phi+\delta)t} \sqrt{\int_{t-1}^{t} r^2(s) ds},$$

so that $w(t) = e^{(\gamma_\phi+\delta)t} o(1)$ at $t = -\infty$. Now, if $\lambda_l = \lambda_r$, we know that

$$\omega(t) = a e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} r(t),$$

where we can suppose that $a \ge 0$. Therefore

$$-e^{(\gamma_\phi+\delta)t} |r(t)| \le |\omega(t)| - a e^{\gamma_\phi t} \le e^{(\gamma_\phi+\delta)t} |r(t)|,$$

so that, in view of the above estimation of $w(t)$, we get

$$|\omega(t)| = a e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} r_1(t),\ \text{with } |r_1(t)| \le |r(t)|,$$
$$w(t) = \int_{t-1}^{t} |\omega(s)| ds = \frac{a(1 - e^{-\gamma_\phi})}{\gamma_\phi} e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} o(1), \quad t \to -\infty.$$

We have the following:

$$\omega(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)(g(\varphi_2(t-s),\tau) - g(\varphi_1(t-s),\tau)) ds,$$
$$|\omega(t)| \le \int_X g'(0,\tau) d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) |\omega(t-s)| ds,$$

$$\int_{t-1}^{t} |\omega(u)|du \leq \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) \int_{t-1}^{t} |\omega(u-s)|duds,$$

and, finally,

$$w(t) \leq \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)w(t-s)ds. \tag{19}$$

After multiplying the both sides of (19) by $e^{-\gamma_\phi t}$ and setting $v(t) := w(t)e^{-\gamma_\phi t}$, we find that

$$v(t) \leq \int_{\mathbb{R}} N(s)v(t-s)ds,$$

$$\text{where } N(s) := \int_X g'(0,\tau)K(s,\tau)e^{-\gamma_\phi s}d\mu(\tau) \quad \text{satisfies} \quad \int_{\mathbb{R}} N(s)ds = 1.$$

Case I. (*noncritical*) If $\lambda_l < \lambda_{rK}$, then $v(\pm\infty) = 0$. Since $v(t) \geq 0$, there exists a finite t_m such that $v(t_m) = M := \max_{s\in\mathbb{R}} v(s)$. If $M = 0$ then $v(t) \equiv 0$, and the uniqueness follows. If $M > 0$ set $t_1 := \min v^{-1}(M) \leq \max v^{-1}(M) =: t_2$ and consider $v_1(t) := (v(t+t_1) + v(t+t_2))/2$. We have $v_1(t) < M,\ t \in \mathbb{R} \setminus \{0\}$, so that

$$M = v_1(0) < \int_{\mathbb{R}} N(s)dsM = M,$$

a contradiction.
Case II. (*critical*) Now, if $\lambda_l = \lambda_r$, we have that $v(-\infty) = a(1-e^{-\gamma_\phi})/\gamma_\phi$, $v(+\infty) = 0$. Furthermore, $\int_{\mathbb{R}} sN(s)ds = 0$ because of $\chi(\gamma_\phi) = \chi'(\gamma_\phi) = 0$. Since the assumption $a = 0$ was already considered in Case I, we will suppose that $a > 0$. In fact, after normalizing if necessary, we can assume that $v(-\infty) = 1$. Finally, since $\gamma_\phi < \gamma_K$, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}} N(s)e^{xs}ds = 1 - \chi(\gamma_\phi - x) < \infty \quad \text{for all } |x| < \delta.$$

Therefore $\int_{\mathbb{R}} |s|N(s)ds < \infty$ and an application of Lemma 7 yields $v(t) \equiv 0$. This contradiction completes the proof of Theorem 3.

Let us consider now the situation when the subtangential Lipschitz condition of Theorem 3 is not satisfied. In such a case, we prove the uniqueness under somewhat stronger hypotheses (**SB*), (EC***):

(SB*) Either one of the following conditions holds

$$|g(u,\tau)-g(v,\tau)-g'(0,\tau)(u-v)|\le C(\tau)|u-v|(u^\alpha+v^\alpha),\ u,v\in(0,\sigma),$$
$$|g'(u,\tau)-g'(0,\tau)|\le C(\tau)u^\alpha,\ u\in(0,\sigma),$$

for some $\alpha,\sigma\in(0,1]$ and measurable $C(\tau)>0$ satisfying (12). Furthermore, there exist $\hat{\epsilon}\in(0,\gamma_\phi)$ and measurable $d_1(\tau)$ such that

$$0\le K(s,\tau)\le d_1(\tau)e^{\hat{\epsilon}s},\quad s\in\mathbb{R}.$$

(EC*) Either one of the following two assumptions is satisfied:

(i) Each solution of (1) is C^1-smooth and if $\varphi_1,\varphi_2\in C^1(\mathbb{R})$ satisfy (1) and the integral $\int_{\mathbb{R}}e^{-zs}(\varphi_2(s)-\varphi_1(s))ds$ converges absolutely then the integral $\int_{\mathbb{R}}e^{-zs}(\varphi_2'(s)-\varphi_1'(s))ds$ also converges absolutely.

(ii) There exists $\delta_0>0$ such that, for each $x\in(\lambda_{rK}-\delta_0,\lambda_{rK})$, it holds

$$0\le K(s,\tau)\le d_{2x}(\tau)e^{xs},\ s\in\mathbb{R},$$

for some $\mu-$measurable $d_{2x}(\tau)$.

Theorem 4 *Assume* **(SB*)***,* **(EC*)** *and suppose that*

$$|g(u,\tau)-g(v,\tau)|\le\lambda(\tau)|u-v|,\ u,v\ge 0,\tau\in X,\tag{20}$$

for some measurable $\lambda(\tau)$ different from $g'(0,\tau)$ and that the function

$$\chi_1(z)=1-\int_{\mathbb{R}}\int_X K(s,\tau)\lambda(\tau)d\mu(\tau)e^{-sz}ds$$

is well defined on $[0,\lambda_{rK})$. *If, in addition,* $\lambda d_j\in L^1(X),\ j=1,2,\ \chi(0)<0$ *and* $\chi_1(m)\ge 0$ *for some* $m\in(0,\lambda_{rK})$*, then Eq.* (1) *has at most one bounded positive solution* $\varphi,\ \varphi(-\infty)=0$. *Furthermore,* γ_ϕ *coincides with the minimal simple positive zero* λ_l *of* $\chi(z)$ *and, for appropriate* $t_0\in\mathbb{R}$, $\delta>0$,

$$\varphi(t+t_0)=e^{\lambda_l t}+e^{(\lambda_l+\delta)t}r(t),\quad with\,continuous\ r\in L^2(\mathbb{R}).$$

Proof Using Lemma 6 and the above conditions, we find that $\lambda_l=\gamma_\phi<m<\lambda_{rK}\le\gamma_K$. Hence, due to Lemma 3, the assumptions of the theorem guarantee the fulfillment of the hypotheses **(SB)** and **(EC$_{\gamma_\phi}$)**. Therefore all arguments of Step I in the proof of Theorem 3 can be repeated (with a unique change in the estimation of $e^{-xt}\mathcal{D}(t)$ where $g'(0,\tau),\chi$ is replaced with $\lambda(\tau),\chi_1$). Thus each pair φ_1,φ_2 of solutions of (1) can be supposed to have the same main parts of their asymptotic representations: $\varphi_j(t)=e^{\lambda_l t}+e^{(\lambda_l+\delta)t}r_j(t),\ r_j\in L^2(\mathbb{R})$. The further proof is divided in three steps.
Step I. Again, we consider bounded function $\omega(t):=\varphi_2(t)-\varphi_1(t)=e^{(\lambda_l+\delta)t}r(t)$,

$t \in \mathbb{R},\ r \in L^2(\mathbb{R})$. If $\Re z \in (0, \lambda_l + \delta)$, then $\int_{\mathbb{R}} e^{-zs}\omega(s)ds$ converges absolutely and from condition **(EC*)**(i) we have

$$|\omega(t)| = |\int_{-\infty}^{t} \omega'(s)ds| = |\int_{-\infty}^{t} e^{xs}\omega'(s)e^{-xs}ds| \le e^{xt}\int_{\mathbb{R}} e^{-xs}|\omega'(s)|ds =: C_x e^{xt},$$

for all $x \in (0, \lambda_l + \delta)$ and $t \in \mathbb{R}$. Similarly, we obtain from **(SB*)**, **(EC*)**(ii) that

$$\begin{aligned}|\omega(t)| &= |\int_X d\mu \int_{\mathbb{R}} K(s,\tau)\Big(g(\varphi_1(t-s),\tau) - g(\varphi_2(t-s),\tau)\Big)ds| \\ &\le e^{xt}\int_X \lambda(\tau)d\mu \int_{\mathbb{R}} K(s,\tau)e^{-xs}e^{-x(t-s)}|\omega(t-s)|ds \\ &\le e^{xt}\int_X \lambda(\tau)(d_1(\tau) + d_{2,\lambda_l+\delta}(\tau))d\mu \int_{\mathbb{R}} e^{-xs}|\omega(s)|ds,\ x \in (\hat{\epsilon}, \lambda_l + \delta),\ t \in \mathbb{R}.\end{aligned}$$

In each of these two cases, for every $x \in (\hat{\epsilon}, \lambda_l + \delta)$ there exists an appropriate $C_x > 0$ such that $|\omega(t)| \le C_x e^{xt},\ t \in \mathbb{R}$. Set

$$\Gamma = \sup\{x \ge \lambda_l |\ \exists C_x : |\omega(t)| \le C_x e^{xt},\ t \in \mathbb{R}\},$$

we claim that $\Gamma \ge \lambda_{rK}$. Indeed, on the contrary, suppose that $\Gamma < \lambda_{rK}$ and let $x_0 \in (\hat{\epsilon}, \Gamma)$, $\alpha > 0$, $\gamma_0 \in (\hat{\epsilon}, \lambda_l)$ be such that $x_* = x_0 + \alpha\gamma_0 \in (\Gamma, \lambda_{rK})$. We have that

$$\omega(t) = \int_X d\mu \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)\omega(t-s)ds + \mathcal{E}(t),\ t \in \mathbb{R}, \tag{21}$$

with bounded

$$\mathcal{E}(t) := \int_X d\mu \int_{\mathbb{R}} K(s,\tau)\Big(g(\varphi_1(t-s),\tau) - g(\varphi_2(t-s),\tau) - g'(0,\tau)\omega(t-s)\Big)ds.$$

Now, independently on assumptions chosen in **(SB*)**, we have

$$\begin{aligned}|g(\varphi_1(s),\tau) - g(\varphi_2(s),\tau) - g'(0,\tau)\omega(s)| &\le C(\tau)|\omega(s)|(|\varphi_1(s)| + |\varphi_2(s)|)^{\alpha} \\ \le k_2 C(\tau)\min\{C_{x_0}e^{(x_0+\alpha\gamma_0)s}, (|\varphi_1|_\infty + |\varphi_2|_\infty)^{1+\alpha}\} &\le k_3 C(\tau)e^{x_* s},\ s \in \mathbb{R},\end{aligned}$$

where k_i depend on x_0, γ_0 and $|\varphi_j|_\infty$ only. Hence,

$$|\mathcal{E}(t)| \leq 4e^{x_* t}\max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}\int_X \lambda(\tau)d\mu \int_{-\infty}^{t} K(s,\tau)e^{-x_* s}ds$$
$$+ke^{x_* t}\int_X C(\tau)d\mu \int_t^{+\infty} K(s,\tau)e^{-x_* s}ds$$
$$\leq e^{x_* t}\Big(4\max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}(1-\chi_1(x_*)) + k\zeta(x_*)\Big) =: Ae^{x_* t}, \quad t \in \mathbb{R}.$$

Therefore $e^{-xt}\mathcal{E}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1,\infty]$ once $x \in (0, x_*)$. Using Lemma 2, we can repeat now the arguments of Step I of Theorem 3 (below the estimation of $|e^{-xt}\mathcal{D}(t)|$) to conclude that $\omega(t) = e^{xt}r_x(t)\ \ t \in \mathbb{R},\ r_x \in L^2(\mathbb{R})$, for each $x \in (\lambda_l, x_*)$. This implies the absolute convergence of $\int_{\mathbb{R}} e^{-xs}\omega(s)ds$ for every $x \in (\lambda_l, x_*)$. But as we have seen at the beginning of Step I, this yields $|\omega(s)| \leq B_x e^{xs},\ s \in \mathbb{R},\ x \in (\lambda_l, x^*)$ for appropriate B_x. Therefore $\Gamma \geq x_* > \Gamma$, a contradiction. In this way, we have proved that

$$|\omega(s)| \leq B_x e^{xs},\ s \in \mathbb{R},\ x \in (\hat{\varepsilon}, \min\{\lambda_r, \gamma_K\}). \tag{22}$$

*Step II.*Suppose that $\chi_1(m) > 0$ for some $m \in (0, \lambda_{rK})$, it is clear that $m > \lambda_l$ and

$$\kappa := \int_{\mathbb{R}}\int_X K(s,\tau)\lambda(\tau)d\mu(\tau)e^{-sm}ds < 1.$$

We now define $\bar{\omega}(t) := |\omega(t)|e^{-mt} \geq 0,\ t \in \mathbb{R}$. By (22), we obtain that $\bar{\omega}(\pm\infty) = 0$ and $\bar{\omega}(t_m) = \max_{s\in\mathbb{R}} \bar{\omega}(s) \geq 0$ for some $t_m \in \mathbb{R}$. Since

$$\omega(t) = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)(g(\varphi_2(t-s),\tau) - g(\varphi_1(t-s),\tau))ds,$$

we have

$$\bar{\omega}(t_m) = |\omega(t_m)|e^{-mt_m} \leq \int_X \lambda(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-ms}|\omega(t_m-s)|e^{-m(t_m-s)}ds$$
$$\leq \bar{\omega}(t_m)\int_X \lambda(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-ms}ds = \bar{\omega}(t_m)\kappa.$$

Hence, $\bar{\omega}(\tau) = 0$ and the uniqueness follows.

Step III. Suppose now that $\chi_1(m) = \max_{s\in(0,\lambda_{rK})}\chi_1(s) = 0$. Then additionally $\chi_1'(m) = 0$. Since $\lambda(\tau)$ is different from $g'(0,\tau)$, we have also that $\lambda_l < m$. Furthermore, $\bar{\omega}(t) := |\omega(t)|e^{-mt} \geq 0,\ t \in \mathbb{R}$ has the same properties as in Step II: $\bar{\omega}(\pm\infty) = 0,\ \bar{\omega}(t_m) = \max_{s\in\mathbb{R}}\bar{\omega}(s) \geq 0$ for some $t_m \in \mathbb{R}$ and

$$\bar{\omega}(t) \leq \int_{\mathbb{R}} \left(\int_X K(s,\tau)\lambda(\tau)e^{-ms}d\mu(\tau) \right) \bar{\omega}(t-s)ds.$$

Since the assumption $\bar{\omega}(t_m) = 0$ immediately implies the uniqueness, we may assume that $0 \leq \bar{\omega}(t) \leq 1 = \bar{\omega}(t_m) = 1$, $t \in \mathbb{R}$, for some finite rightmost t_m. Then

$$1 \leq \int_{\mathbb{R}} N_\lambda(s)\bar{\omega}(t_m - s)ds \leq \int_{\mathbb{R}} N_\lambda(s)ds = 1,$$

where $N_\lambda(s) := \int_X K(s,\tau)\lambda(\tau)e^{-ms}d\mu(\tau)$. This implies that $N_\lambda(s)\bar{\omega}(t_m - s) = N_\lambda(s)$ a.e. and $\bar{\omega}(t_m - s) = 1$ for all s such that $N_\lambda(s) > 0$. Now, since $\int_{\mathbb{R}} N_\lambda(s)ds = 1$, $\int_{\mathbb{R}} sN_\lambda(s)ds = 0$, there is a subset of $\mathbb{R}_-$ of positive measure where $N_\lambda(s) > 0$. This means that t_m does not possess the property to be the rightmost point where $\bar{\omega}(t_m) = 1$, a contradiction. In consequence, $\bar{\omega}(t) \equiv 0$ that proves the uniqueness.

Remark 9 It is enlightening to compare Theorem 4 and Theorem 2 in [33] where somewhat similar ideas were exploited. Indeed, from pure analytical estimations, without the use of asymptotic representations of solutions and without using the properties of χ indicated in Lemma 2, Schumacher deduced that $\Gamma \geq \lambda_{rK}$ (under assumptions made in [33]). In any case, monotonicity restrictions on the convolution term in [33] do not allow to consider various interesting models (cf. Sects. 6.3–6.4 below).

6 Applications

In this section, Theorems 3 and 4 are applied to several models which can be written as (1). This allows to improve or complement the uniqueness results in [1,4,8,12,16,36]. Everywhere in this section we assume that locally Lipschitzian $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(0) = 0$, is differentiable at 0 with $g'(0) > 0$.

6.1 A nonlocal integro-differential equation [4,8,9,21,28,33]

Consider the equation

$$u_t = J * u - u + g(u), \tag{23}$$

where $J \geq 0$, $\int_{\mathbb{R}} Jds > 0$. Let $\gamma^\#$ denote an extended positive real number such that $\int_{\mathbb{R}} J(s)e^{-zs}ds$ is convergent for $z \in [0, \gamma^\#)$ and is divergent when $z > \gamma^\#$. As it can be easily deduced from Theorem 1, the existence of such $\gamma^\#$ is automatically assured by the existence of positive semi-wavefronts $u(t,x) = \phi(x+ct)$, $\phi(-\infty) = 0$ to (23). Traveling wave profile ϕ solves

$$c\phi' = J * \phi - \phi + g(\phi). \tag{24}$$

In order to replace condition (3) with less restrictive

$$g'(s) \leq g'(0) \text{ a.e. on } \mathbb{R}_+, \tag{25}$$

we use the following trick. Set $g_\beta(s) = g(s) + \beta s$ for some positive β. We claim that β can be chosen in such a way that g_β satisfies the Lipshitz condition with a constant $g'_\beta(0) = \beta + g'(0)$. First observe that our proof of uniqueness compares two different solutions ϕ_1, ϕ_2. Since they are uniformly bounded by some positive $M > 0$, we can restrict our attention to a finite interval $[0, M]$ where g is globally Lipschitzian. But then there exists $\beta > 0$ such that $g'(0) \geq g'(s) \geq -2\beta - g'(0)$ almost everywhere on $[0, M]$. In consequence, we get the necessary estimation

$$-g'(0) - \beta \leq g'_\beta(s) = \beta + g'(s) \leq \beta + g'(0) \text{ a.e. on } [0, M].$$

Hence, instead of (24) we will consider

$$c\phi' = J * \phi - (1 + \beta)\phi + g_\beta(\phi). \tag{26}$$

Let us suppose that $c > 0$ (the case $c < 0$ is similar). Since ϕ is non-negative and bounded, it should satisfy

$$\begin{aligned}\phi(t) &= \frac{1}{c}\int\limits_{-\infty}^{t} e^{-(t-s)(1+\beta)/c}\Big(J * \phi(s) + g_\beta(\phi(s))\Big)ds \\ &= k * (J * \phi)(t) + k * g_\beta(\phi)(t) = (k * J) * \phi(t) + k * g_\beta(\phi)(t),\end{aligned} \tag{27}$$

where $k(s) = c^{-1}e^{-s(1+\beta)/c}$, $s \geq 0$ and $k = 0$ if $s < 0$. Thus, Eq. (27) can be written as (4), with $X = \{\tau_1, \tau_2\}$ and

$$K(s, \tau) = \begin{cases} k * J(s), & \tau = \tau_1, \\ k(s), & \tau = \tau_2, \end{cases} \quad g(s, \tau) = \begin{cases} s, & \tau = \tau_1, \\ g_\beta(s), & \tau = \tau_2. \end{cases}$$

Finally, independently on the sign of c, we find that

$$\begin{aligned}\chi(z, c) &= 1 - \int\limits_{\mathbb{R}} K(s, \tau_1)e^{-zs}ds - (g'(0) + \beta)\int\limits_{\mathbb{R}} K(s, \tau_2)e^{-zs}ds \\ &= 1 - \frac{1}{1 + \beta + cz}\int\limits_{\mathbb{R}} J(s)e^{-zs}ds - \frac{g'(0) + \beta}{1 + \beta + cz} =: \frac{\tilde{\chi}(z, c)}{1 + \beta + cz}.\end{aligned}$$

Let c_* be the minimal value of c for which

$$\tilde{\chi}(z, c) := 1 - g'(0) + cz - \int\limits_{\mathbb{R}} J(s)e^{-sz}ds$$

has at least one positive zero. It is easy to see that

$$c_* = \inf_{z>0} \frac{1}{z} \left\{ -1 + g'(0) + \int_{\mathbb{R}} J(s) e^{-sz} ds \right\}$$

can be positive, negative (in these cases inf can be replaced with min) or zero. By Theorem 2, $c \geq c_*$ for each admissible wave speed c. The next result is a direct consequence of Theorem 3.

Theorem 5 *Suppose (25) together with* $1 - \int_{\mathbb{R}} J(s)ds < g'(0)$ *and*

$$|g(u) - g'(0)u| \leq Cu^{1+\alpha}, \; u \in (0, \sigma) \; \text{ for some } \; \alpha, \sigma \in (0, 1], \tag{28}$$

Then Eq. (24) *has at most one bounded positive solution* φ, $\varphi(-\infty) = 0$, *for each* $c \neq 0$ *(if* $\tilde{\chi}(\gamma^{\#}-, c_*) \neq 0$*) or for each* $c \neq 0, c_*$ *(if* $\tilde{\chi}(\gamma^{\#}-, c_*) = 0$*).*

Proof Suppose that $c > 0$ (the case $c < 0$ is similar). We only have to check the assumptions **(EC**$_{\gamma_\phi}$**)**, **(SB)** except $\gamma_\phi(c) < \gamma_K(c)$, $\chi(0, c) < 0$ and $\chi(\gamma_K-, c) \neq 0$ of Theorem 3.
Step I. It is clear that $g(\cdot, \tau)$ satisfies (16), where $g'(0, \tau_1) = 1, \; g'(0, \tau_2) = g'(0) + \beta$. Moreover, we have $|g(u, \tau) - g'(0, \tau)u| \leq C(\tau)u^{1+\alpha}, \; u \in (0, \sigma)$, where $C(\tau) = 0$ if $\tau = \tau_1$ and $C(\tau) = C$ if $\tau = \tau_2$.
Step II. For each $z > -\frac{1+\beta}{c}$ we have $\int_{\mathbb{R}} k(s) e^{-zs} ds = \frac{1}{1+\beta+cz} < +\infty$ so that $\gamma_K(c) = \gamma^{\#}$ because of $\int_{\mathbb{R}} k * J(s) e^{-zs} ds = \int_{\mathbb{R}} J(s) e^{-zs} ds/(1 + \beta + cz)$. (Observe here that $\gamma_K(c) = \min\{\gamma^{\#}, -(1 + \beta)/c\}$ if $c < 0$. However, if $\gamma_K(c) = -(1 + \beta)/c$ then $\chi(\gamma_K(c), c) = \infty$ so that $\gamma_\phi(c) < \gamma_K(c)$ due to Corollary 1).
Step III. If φ solves (24), then $\varphi \in C^1(\mathbb{R})$ and for each $0 < z < \gamma_\phi$ we obtain

$$\begin{aligned} c \int_{\mathbb{R}} e^{-zs} |\varphi'(s)| ds \; &\leq \int_{\mathbb{R}} e^{-zs} J * \varphi(s) ds + \int_{\mathbb{R}} e^{-zs} \varphi(s) ds + \int_{\mathbb{R}} e^{-zs} g(\varphi(s)) ds \\ &\leq \left(\int_{\mathbb{R}} e^{-zs} J(s) ds + 1 + g'(0) \right) \int_{\mathbb{R}} e^{-zs} \varphi(s) ds < +\infty. \end{aligned}$$

Thus, by Lemma 3, condition **(EC**$_{\gamma_\phi}$**)** is satisfied.
Step IV. We have $\chi(0, c) = (1 - \int_{\mathbb{R}} J(s)ds - g'(0))/(1 + \beta) < 0$. Now, if $\gamma^{\#} < +\infty$, then $\tilde{\chi}(\gamma^{\#}-, c_*) \neq 0$ implies that $\chi(\gamma^{\#}-, c_*) \neq 0$ and $\gamma_\phi(c_*) = \lambda_l(c_*) < \gamma^{\#}$. Since $\chi(z, c)$ is strictly increasing in c for each fixed $z > 0$, function $\lambda_l(c)$ is strictly decreasing. Hence $\gamma_\phi(c) = \lambda_l(c) < \gamma^{\#}$ for each $c \geq c_*$. Similar considerations shows that $\gamma_\phi(c) < \gamma^{\#}$ for each $c > c_*$ if $\chi(\gamma^{\#}-, c_*) = 0$. Finally, in the case $\gamma^{\#} = +\infty$ we have that $\chi(+\infty, c) \in \{1, -\infty\} \not\ni 0$, so that $\chi(\gamma_K-, c) \neq 0$ holds automatically.

Remark 10 Our approach allows to remove several restrictions on J and g assumed in the Carr and Chmaj uniqueness result [4, Theorem 2.1]. In the cited work g is supposed to satisfy (3) and J to be an even compactly supported function with $\int_{\mathbb{R}} J ds = 1$. These properties were essential in the proof of Theorem 2.1 in [4] even though (3) was not mentioned explicitly there. Similarly, conditions $J \in C^1(\mathbb{R})$, $J(a) > 0$, $J(b) > 0$ for some $a < 0 < b$, and of J compactly supported were used by Coville *et al.* It was assumed in [8] that $g'(0)g'(1) < 0$ together with $g(u)/u \le g'(0),\ u > 0$, instead of more restrictive $g'(u) \le g'(0),\ u > 0$. See also [8] for non-uniqueness of stationary traveling fronts ($c = 0$). Next, Schumacher [33], using a comparison method for differential inequalities combined with a Nagumo-point argument, established uniqueness of *regular and non-critical* semi-wavefronts to Eq. (23) for general J and g satisfying (25). The trick allowing to weaken the Lipschitz restriction (3) is due to Thieme and Zhao [36] (as far as we know). Usually it was applied under reversed inequality $f'(s) \ge f'(0)$ to the second (damping) term of equation, e.g. see also [17] and Sect. 6.3 for further generalizations. Here we show that this trick shows to be useful also in the case of birth functions. We would like to note that Theorem 5 remains true if we introduce a small delay $h > 0$ in the term $g(\varphi(t-h))$. Indeed, in such a case it suffices to replace $k(s)$ with a positive fundamental solution $v(s)$ of the scalar delayed equation $cv'(s) = -v(s) - \beta v(s-h)$.

6.2 Nonlocal lattice equations [6,16,26,30,42]

Now we consider semi-wavefronts $w_j(t) = u(j + ct), u(-\infty) = 0$, of the nonlocal lattice equation

$$\frac{dw_j(t)}{dt} = D\sum_{k=-1}^{1}[w_{j+k}(t) - w_j(t)] - dw_j(t) + \sum_{k\in Z}\beta(j-k)g(w_k(t-r)),\ \ j \in Z,$$

where $\beta(k) \ge 0,\ \sum_{k\in Z}\beta(k) = 1$. Let $\gamma^{\#}$ be an extended positive real number such that $\sum_{k\in Z}\beta(k)e^{-zk}$ converges when $z \in [0,\gamma^{\#})$ and is divergent when $z > \gamma^{\#}$. By Cauchy-Hadamard formula, $\gamma^{\#} = -\limsup_{k\to+\infty} k^{-1}\ln\beta(-k)$, where by convention $\ln(0) = -\infty$. The wave profile u satisfies

$$\begin{aligned} cu'(x) &= D[u(x+1) + u(x-1) - 2u(x)] - du(x) \\ &\quad + \sum_{k\in Z}\beta(k)g(u(x-k-cr)). \end{aligned} \tag{29}$$

Again we take $c > 0$ for simplicity. Since u is bounded, we find that

$$\begin{aligned} u(t) &= \frac{1}{c}\int_{-\infty}^{t} e^{-\frac{2D+d}{c}(t-s)}\left[Du(s+1) + Du(s-1) + \sum_{k\in Z}\beta(k)g(u(s-k-cr))\right]ds \\ &= D(H_{-1} + H_1) * u(t) + \sum_{k\in Z}\beta(k)H_{k+cr} * g(u)(t), \end{aligned} \tag{30}$$

where

$$H_\tau(t) = \begin{cases} \frac{1}{c} e^{-\frac{2D+d}{c}(t-\tau)}, & t \geq \tau, \\ 0, & t < \tau. \end{cases}$$

Thus (30) can be written as (1), with $X = \{\tau_1, \tau_2\}$ and

$$K(s,\tau) = \begin{cases} D(H_{-1}(s) + H_1(s)), & \tau = \tau_1, \\ \sum_{k\in Z} \beta(k) H_{k+cr}(s), & \tau = \tau_2, \end{cases} \quad g(s,\tau) = \begin{cases} s, & \tau = \tau_1, \\ g(s), & \tau = \tau_2. \end{cases}$$

Next, $\chi(z,c) = 1 - \int_{\mathbb{R}} K(s,\tau_1)e^{-sz}ds - g'(0)\int_{\mathbb{R}} K(s,\tau_2)e^{-sz}ds =$

$$1 - \frac{2D\cosh(z)}{2D+d+cz} - \frac{g'(0)e^{-crz}}{2D+d+cz}\sum_{k\in Z}\beta(k)e^{-kz} =: \frac{\tilde{\chi}(z,c)}{2D+d+cz}.$$

Let c_* be the minimal value of c for which

$$\tilde{\chi}(z,c) := d + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-crz}\sum_{k\in Z}\beta(k)e^{-kz}$$

has at least one positive zero. It is easily seen that c_* is well defined and is finite. By Theorem 2, $c \geq c_*$ for each admissible wave speed c.

We are ready to apply our uniqueness results to (29).

Theorem 6 *Suppose that g satisfies (3), (28) and $g'(0) > d$. Then Eq. (29) has at most one bounded positive solution u, $u(-\infty) = 0$, for each $c \neq 0$ (if $\tilde{\chi}(\gamma^\#-, c_*) \neq 0$) or for each $c \neq 0, c_*$ (if $\tilde{\chi}(\gamma^\#-, c_*) = 0$).*

Proof *Step I*. Obviously, $g(\cdot,\tau)$ verifies (3) with $g'(0,\tau_1) = 1$ and $g'(0,\tau_2) = g'(0)$. Moreover, we have $|g(u,\tau) - g'(0,\tau)u| \leq C(\tau)u^{1+\alpha}$, $u \in (0,\sigma)$, where $C(\tau_1) = 0$ and $C(\tau_2) = C$.
Step II. If $0 < z < \gamma_\#$, we get

$$\int_{\mathbb{R}\times X} K(s,\tau)e^{-zs}ds\,d\mu = \int_{\mathbb{R}} D(H_{-1}(s) + H_1(s))e^{-zs}ds$$
$$+ \int_{\mathbb{R}} \sum_{k\in Z}\beta(k)H_{k+cr}(s)e^{-zs}ds = \frac{2D\cosh(z)}{2D+d+cz} + \frac{e^{-crz}}{2D+d+cz}\sum_{k\in Z}\beta(k)e^{-kz}.$$

Therefore $\gamma_K = \gamma_\#$ (if $c > 0$) and $\gamma_K = \min\{\gamma_\#, -(2D+d)/c\}$ (if $c < 0$).

Step III. If u solves (29) with $c > 0$, then for each $0 < z < \gamma_\phi$ we obtain

$$c\int_{\mathbb{R}} |u'(s)|e^{-zs}ds \le D\int_{\mathbb{R}} (u(s+1)+u(s-1)+2u(s))e^{-zs}ds$$
$$+d\int_{\mathbb{R}} u(s)e^{-zs}ds + g'(0)\sum_{k\in Z}\beta(k)\int_{\mathbb{R}} u(s-k-cr)e^{-zs}ds$$
$$=\left(2D(\cosh(z)+1)+d+g'(0)e^{-zcr}\sum_{k\in Z}\beta(k)e^{-zk}\right)\int_{\mathbb{R}} u(s)e^{-zs}ds < +\infty.$$

Thus, by Lemma 3, condition $(\mathbf{EC}_{\gamma_\phi})$ is satisfied.
Step IV . We have $\chi(0) = (d - g'(0))/(2D+d) < 0$. The proof of $\gamma_\phi(c) < \gamma^\#$ is the same as in Step IV of the previous section and is omitted.

Remark 11 Our approach allows to improve the uniqueness results of [16, Theorem 3.1], where additional conditions $\beta(k) = \beta(-k)$ and $\chi(\gamma_K-) = -\infty$ are assumed. Moreover, [16, Theorem 3.1] does not establish the uniqueness of the minimal wave. Similarly to Sect. 6.1, condition (3) in Theorem 6 can be replaced with more weak (25) if the nonlinear term is local and non-delayed. See [26], where a local and non-delayed variant of (29) was considered. Similarly to [7,8] and under the same conditions on g as in [8], Guo and Wu prove their uniqueness result [26, Theorem 2] by means of the comparison argument. To establish the uniqueness in the degenerate case $(g'(0)-d)(g'(1)-d) = 0$ (cf. Remark 10), about which is the main concern of [5], Chen *et al.* developed new interesting tools (magnification, compression, blow-up techniques, modified sliding method). Finally, we mention Ma and Zou uniqueness result from [30], where a local version of (29) is investigated. The Lipschitz condition (3) is not required in [30], it is supposed instead that $g'(s) \ge 0,\ g(s)/s \le g'(0),\ s > 0$.

6.3 Nonlocal reaction–diffusion equation [15,17,24,32,36,38]

Here, we consider positive semi-wavefronts $u(t,x) = \phi(x+ct),\ \phi(-\infty) = 0$, for non-local delayed reaction–diffusion equations

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_{\mathbb{R}} k(w)g(u(t-h,x-w))dw, \quad h > 0, \tag{31}$$

where $f \in C^1(\mathbb{R}_+,\mathbb{R}_+),\ f(0) = 0$, is strictly increasing and $k \ge 0,\ \int_{\mathbb{R}} k ds = 1$, can be asymmetric (see [38] for further details concerning wave solutions in the presence of asymmetric non-local interaction). Let $\gamma^\# > 0$ denote an extended positive real number such that $\int_{\mathbb{R}} k(s)e^{-zs}ds$ converges when $z \in [0,\gamma^\#)$ and diverges if $z > \gamma^\#$. It is clear that profile ϕ must satisfy

$$y''(t) - cy'(t) - f(y(t)) + \int_{\mathbb{R}} k(s)g\left(y(t - ch - s)\right) ds = 0, \quad t \in \mathbb{R}. \qquad (32)$$

Equation (32) can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_{\mathbb{R}} k_h(w)g(y(t - w))dw = 0, \quad t \in \mathbb{R},$$

where $k_h(w) = k(w - ch)$ and $f_\beta(s) = \beta s - f(s)$ for some $\beta > 0$.

Again, without restricting the generality, we may suppose that f_β is a Lipschitzian function with $\text{Lip} f_\beta = \beta - \inf_{s\geq 0} f'(s)$. Indeed, our proof of uniqueness compares two solutions ϕ_1, ϕ_2. Since they are uniformly bounded by some positive $M > 0$, we can restrict our attention to a finite interval $[0, M]$. Let $\beta > f'(0)$ be such that $f_\beta(s) = \beta s - f(s) \geq 0$ for all $s \in [0, M]$ and $\max_{s\in[0,M]} f'(s) \leq 2\beta - \inf_{s\geq 0} f'(s)$. But then

$$\left| \frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} \right| \leq \left(\beta - \inf_{s\geq 0} f'(s)\right), \quad s_1, s_2 \in [0, M].$$

Next, it is easy to see that the wave profile ϕ solves the equation

$$\phi(t) = \frac{1}{\sigma(c)} \left(\int_{-\infty}^{t} e^{\nu(t-s)} (\mathcal{G}\phi)(s)ds + \int_{t}^{+\infty} e^{\mu(t-s)} (\mathcal{G}\phi)(s)ds \right),$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and $(\mathcal{G}\phi)(t) := \int_{\mathbb{R}} k_h(s)g(\phi(t - s))ds + f_\beta(\phi(t))$. Equivalently,

$$\phi(t) = (\mathcal{K} * k_h) * g(\phi)(t) + \mathcal{K} * f_\beta(\phi)(t),$$

where

$$\mathcal{K}(s) = \sigma^{-1}(c) \begin{cases} e^{\nu s}, & s \geq 0, \\ e^{\mu s}, & s < 0. \end{cases}$$

We can invoke now Theorems 3, 4 where $X = \{\tau_1, \tau_2\}$ and

$$K(s, \tau) = \begin{cases} (\mathcal{K} * k_h)(s), & \tau = \tau_1, \\ \mathcal{K}, & \tau = \tau_2, \end{cases} \qquad g(s, \tau) = \begin{cases} g(s), & \tau = \tau_1, \\ f_\beta(s), & \tau = \tau_2. \end{cases}$$

Observe that $g(\cdot, \tau)$ meets (20) with $\lambda(\tau_1) = g'(0)$, $\lambda(\tau_2) = \beta - \inf_{s\geq 0} f'(s)$. If $f'(0) \leq f'(v)$ for all $v \geq 0$, as in [36], then $\beta - \inf_{s\geq 0} f'(s) = \beta - f'(0) = f'_\beta(0)$. We have also that

$$\chi_1(z,c) = 1 - g'(0)\int_{\mathbb{R}} K(s,\tau_1)e^{-sz}ds - (\beta - \inf_{s\geq 0} f'(s))\int_{\mathbb{R}} K(s,\tau_2)e^{-sz}ds$$
$$= 1 - \frac{\beta - \inf_{s\geq 0} f'(s)}{\beta + cz - z^2} - \frac{g'(0)e^{-zch}}{\beta + cz - z^2}\int_{\mathbb{R}} k(s)e^{-zs}ds =: \frac{\tilde{\chi}_1(z)}{\beta + cz - z^2}.$$

We see that $\gamma_K = \min\{\mu, \gamma^\#\}$ so that $\gamma_\phi < \mu$. Let $c_\star$ be the minimal value of c for which

$$\tilde{\chi}_1(z,c) := cz - z^2 + \inf_{s\geq 0} f'(s) - g'(0)e^{-zch}\int_{\mathbb{R}} k(s)e^{-zs}ds$$

has at least one positive zero. This value is finite, well defined and does not depend on β. We will write c_* instead of $c_\star$ in the special case when $f'(0) \leq f'(v)$ for all $v \geq 0$. In such a case, we have $f'(0) = \inf_{s\geq 0} f'(s)$ and therefore $\chi_1 = \chi$. By Theorem 2, $c \geq c_*$ for each admissible wave speed c.

Theorem 7 *Suppose g satisfies (3), $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, and $g, f \in C^{1,\alpha}$ in some neighborhood of* 0*, and $g(0) = f(0) = 0,\ g'(0) > f'(0)$. Then Eq. (31) has at most one positive semi-wavefront $u(t,x) = \phi(x+ct),\ \phi(-\infty) = 0$, for each $c \geq c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) \neq 0$) or for each $c > c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) = 0$).*

Proof Observe that $\beta\chi(0) \leq f'(0) - g'(0) < 0$, and $\chi_1(\gamma^\#-, c_\star) \neq 0$ if $\tilde{\chi}_1(\gamma^\#-, c_\star) \neq 0$. *First let $c \geq c_\star > c_*$*, then $\chi_1(x,c) < \chi(x,c)$ so that $\chi_1(m,c) = 0$ for some $m \in (0, \lambda_{rK}]$. It is clear that $m = \lambda_{rK}$ if and only if $m = \gamma^\#$. Since $\chi_1(z,c)$ is strictly increasing in c for each fixed positive z, this implies that $c = c_\star$ and $\chi_1(\gamma^\#-, c_\star) = 0$. Consequently, $m \in (0, \lambda_{rK})$ for each $c \geq c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) \neq 0$) or for each $c > c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) = 0$).

Next, if $c_\star = c_*$ then $\chi_1 = \chi$ and the inequality $\chi(\gamma^\#-, c_*) \neq 0$ guarantees that $\lambda_l(c_*) = \gamma_\phi(c_*) < \gamma^\#$ for $c = c_*$. If $c > c_*$ then we have again $\lambda_l(c) = \gamma_\phi(c) < \lambda_l(c_*) < \gamma^\#$ because $\lambda_l(c)$ is monotone decreasing in c.

Step I. Since $|f'_\beta(0)u - f_\beta(u)| = |f'(0)u - f(u)|$, for an appropriate C, σ, it holds $|g(u,\tau) - g'(0,\tau)u| \leq C(\tau)u^{1+\alpha},\ u \in (0,\sigma)$.

Step II. We claim that for each $x \in (0, \gamma_K)$ and some $d_j(x)$ it holds

$$0 \leq K(s,\tau_j) \leq d_j(x)e^{xs}, s \in \mathbb{R}.$$

Indeed, if $j = 2$, we can even take $x = \mu,\ d_2 = 1/\sigma(c)$. Next, we have

$$K(t,\tau_1) = \frac{1}{\sigma(c)}\left[\int_{t-ch}^{+\infty} e^{\mu(t-ch-v)}k(v)dv + \int_{-\infty}^{t-ch} e^{\nu(t-ch-v)}k(v)dv\right]$$
$$\leq \frac{e^{-xch}}{\sigma(c)}\left[\int_{\mathbb{R}} e^{-xv}k(v)dv\right]e^{xt}.$$

Since $\lambda_{rK} \le \gamma_K = \min\{\gamma^\#, \mu\}$, the exponential estimations of K in **(SB*)**, **(EC*)**(ii) are verified. This observation completes the proof of the theorem.

Remark 12 Theorem 7 improves [36, Theorem 4.3], where the uniqueness was established under assumption that either $f(s) = f'(0)s$ or $g(s) = g'(0)s$ and K is the Gaussian kernel. The latter restrictions were also removed by Fang and Zhao whose recent work [17] contains important improvements over the previous results. In any case, [17, Theorems 4.1, 4.2] and [36, Theorem 4.3] do not consider the uniqueness of the minimal waves. See also [29,38] and references therein about the existence of semi-wavefronts in (31) and its limit form (33) studied below.

6.4 Uniqueness of fast traveling fronts in delayed equations

Here we study positive semi-wavefronts $u(t, x) = \phi(x + ct)$, $\phi(-\infty) = 0$, to

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x)), \ x \in \mathbb{R}, \tag{33}$$

where g is a Lipschitzian function such that $|g'|_{L^\infty} > g'(0)$. Profile ϕ solves the delay differential equation

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t - hc)) = 0, \quad t \in \mathbb{R}. \tag{34}$$

Similarly to Sect. 6.3 (where we take now $\beta = 0$), we find that ϕ satisfies

$$\phi(t) = \mathcal{K} * g(\phi)(t), \ \ \mathcal{K}(s) = \frac{1}{\sigma(c)} \begin{cases} e^{\nu(s-ch)}, & s \ge ch, \\ e^{\mu(s-ch)}, & s < ch, \end{cases}$$

which is exactly the form considered in the DK theory (formally, we set $X = \{\tau\}$, $K(s, \tau) = \mathcal{K}$ and $g(s, \tau) = g(s)$). Nevertheless, since $L > g'(0)$, the Diekmann–Kaper uniqueness theorem does not apply to (34).

In order to use Theorem 4, we first note that

$$\chi_1(z, c) = 1 - L \int_{\mathbb{R}} \mathcal{K}(s)e^{-sz}ds = 1 - \frac{Le^{-zhc}}{1 + cz - z^2}.$$

is well defined on (ν, μ). Thus, $\gamma_K = \mu$ and since $\lim_{x\to\mu-} \int_{\mathbb{R}} \mathcal{K}(s)e^{-sx}ds = +\infty$ we obtain that $\gamma_\phi < \gamma_K$. The exponential estimations of K in **(SB*)**, **(EC*)**(ii) are also obviously verified.

Finally, let $c_\star$ be the minimal value of c for which the equation $z^2 - cz - 1 + Le^{-chz} = 0$ has at least one positive root. This value is well defined and positive. It is easy to see that, for each $c > c_\star$ there exists $m > 0$ close to λ_l from the right and such that $\chi_1(m) > 0$. Hence, we get the following

Theorem 8 *Suppose that* $|g(s) - g(t)| \le L|t - s|$, $s, t \ge 0$, *and that* $g \in C^{1,\alpha}$ *in some neighborhood of* 0 *with* $g'(0+) > 1$. *Then, for every* $c > c_\star$ *Eq. (34) has at most one bounded positive solution* ϕ *vanishing at* $-\infty$.

Theorem 8 gives an alternative proof of the uniqueness result in [1, Theorem 1.1] where it was additionally assumed that $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and that $g''(0+)$ is finite. Moreover, we give here an easily calculable lower bound $c_\star$ for the 'uniqueness' speeds. Observe that if $L = g'(0)$, then $c_\star$ coincides with the minimal speed of propagation c_*. Now, the situation when $L > g'(0)$, $h > 0$, is clearly more complicated: in particular, the existence of the minimal speed c_* with the usual properties is not yet proved in such a case (at least, as far as we know). In any event, all the available evidence supports the following affirmation: 'positive semi-wavefronts of (33) are unique (up to translation) for each admissible wave speed c'.

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