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Infinitely many universally tight torsion free contact structures with vanishing Ozsváth–Szabó contact invariants

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Abstract Ozsváth–Szabó contact invariants are a powerful way to prove tightness of contact structures but they are known to vanish in the presence of Giroux torsion. In this paper we construct, on infinitely many manifolds, infinitely many isotopy classes of universally tight torsion free contact structures whose Ozsváth–Szabó invariant vanishes. We also discuss the relation between these invariants and an invariant on T^3 and construct other examples of new phenomena in Heegaard–Floer theory. Along the way, we prove two conjectures of K. Honda, W. Kazez and G. Matić about their contact topological quantum field theory. Almost all the proofs in this paper rely on their gluing theorem for sutured contact invariants.

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1 Introduction

Contact topology studies isotopy classes of contact structures.[1] These classes come in two main flavors: overtwisted and tight, the latter being further divided into universally tight and virtually overtwisted. Up to now, besides homotopical data, there are only two algebraic objects which have been successfully used to classify such isotopy classes on a general 3-manifold. The first one is Giroux torsion introduced in [6], its definition is recalled in Sect. 2. It is either a non-negative integer or infinite and always infinite for overtwisted classes. It is invariant under isomorphisms, not only isotopies. It shares the monotonicity property of symplectic capacities [14] on one hand and the finiteness property of 3-manifolds complexity [22] on the other hand.

[1] All manifolds in this paper are oriented and all contact structures are positive.

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Indeed, if $(M, \xi) \subset (M', \xi')$ then $\mathrm{Tor}(\xi) \leq \mathrm{Tor}(\xi')$ and, for fixed M and n, there are only finitely many isomorphism classes of contact structures on M whose torsion is at most n. Another way to put it is to say that finite torsion determines contact structures up to isomorphism and a finite ambiguity. More generally, it plays an important role in the coarse classification of tight contact structures [1]. The second object, based on open book decompositions [8], is Ozsváth–Szabó contact invariants introduced in [28] which live in the Heegaard–Floer homology of the ambient manifold. They come in various flavors depending on a choice of coefficients. These invariants are a powerful tool to detect tightness and obstructions to fillability by symplectic or complex manifolds. Its main properties are listed in Theorem 4 below.

It is natural to investigate relations between these two invariants. In [5], Ghiggini, Honda and Van Horn Morris proved that, whenever Giroux torsion is non zero, the contact invariant over $\mathbb{Z}$ coefficients vanishes (we give a new proof of this result in Sect. 6). Here we prove that the converse does not hold.

Main theorem (Section 6) *Every Seifert manifold whose base has genus at least three supports infinitely many (explicit) isotopy classes of universally tight torsion free contact structures whose Ozsváth–Szabó invariant over* $\mathbb{Z}$ *coefficients vanishes.*

In the above theorem, the genus hypothesis cannot be completely dropped because, for instance, on the sphere S^3 and the torus T^3, all torsion free contact structures have non vanishing Ozsváth–Szabó invariants. However, it may hold for genus two bases. Note that the class of Seifert manifolds is the only one where isotopy classes of contact structures are pretty well understood. So the theorem says that examples of universally tight torsion free contact structures with vanishing Ozsváth–Szabó invariant exist on all manifolds we understand, provided there is enough topology (the base should have genus at least three). In this statement, isotopy classes cannot be replaced by conjugacy classes because of the finiteness property explained above. Along the way we prove Conjecture 7.13 of [11].

It is interesting to compare the above theorem (and its proof) with the results in [33] which appeared shortly after the first version of the present paper. In [33], Wendl works in the theory of embedded contact homology, which is conjecturally isomorphic to Heegaard–Floer theory. There he gets examples of universally tight torsion free contact structures with vanishing ECH invariants (and even some examples with vanishing twisted ECH invariants). It is intriguing to compare his list of examples with ours since, while the intersection is non empty, neither is contained in another. Also both papers seem far from explaining clearly when Ozsváth–Szabó invariant vanish. We now have a lot of seemingly harmless contact structures with vanishing invariants but the global picture is unclear. This constrasts with the situation after [5] where one could have naively hoped that torsion explained all vanishings. Note however that, thanks to sutured Heegaard–Floer homology, vanishing still comes from localized parts of the manifolds: we have examples of contact manifolds with boundary such that any contact manifold containing these have vanishing invariant.

Our examples also provide a corollary in the world of Legendrian knots. Ozsváth–Szabó theory provides invariants for Legendrian or transverse knots in different (related) ways, see [30] and references therein. In the standard contact 3-sphere there are still two seemingly distinct ways to define such invariants but, in general contact manifolds, the known invariants all come from the sutured contact invariant of the complement of the knot according to the main theorem proved by Vértesi and Stipsicz in [30]. In this paper they call strongly non loose those Legendrian knots in overtwisted contact manifolds whose complement is tight and torsion free. Corollary 1.2 of that papers states that a Legendrian knot has vanishing invariant when it is not strongly non loose. We prove that the converse does not hold.

Theorem 1 (see the discussion after Proposition 5) *There exists a specific example of overtwisted contact manifold containing a null-homologous strongly non loose Legendrian knot whose sutured invariant vanishes.*

After studying the relationship between Ozsváth–Szabó invariants and Giroux torsion, we now turn to a more specific relation between these invariants and an invariant defined only on the 3-torus. E. Giroux proved that any two incompressible prelagrangian tori of a tight contact structure ξ on T^3 are isotopic. We can then define the Giroux invariant $G(\xi) \in H_2(\mathrm{T}^3)/\pm 1$ to be the homology class of its prelagrangian incompressible tori. Note that there is a "sign ambiguity" because these tori are not naturally oriented. Translated into this language, Giroux proved that two tight contact structures on T^3 are isotopic if and only if they have the same Giroux invariant and the same Giroux torsion, see [6]. This invariant is clearly $\mathrm{Diff}(\mathrm{T}^3)$-equivariant. Since this group acts transitively on primitive elements of $H_2(\mathrm{T}^3)$, we see that all these elements are attained by G. This also proves that all tight contact structures on T^3 which have the same torsion are isomorphic. This classification of tight contact structures on T^3 and a result by Y. Eliashberg shows that torsion free contact structures on T^3 are exactly the Stein fillable ones.

Theorem 2 (Section 4) *There is a unique up to sign $H_1(\mathrm{T}^3)$-equivariant isomorphism between $\widehat{HF}(\mathrm{T}^3)$ and $H^1(\mathrm{T}^3) \oplus H^2(\mathrm{T}^3)$ (on the ordinary cohomology side, H_1 sends H^1 to zero and H^2 to H^1 by slant product). Under this isomorphism, the Ozsváth–Szabó invariant of a torsion free contact structure on T^3 is sent to the Poincaré dual of its Giroux invariant.*

Note that, on $\mathrm{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, cohomology classes can be represented by constant differential forms and 1-dimensional homology classes by constant vector fields. The slant product of the above theorem is then identified with the interior product of vector fields with 2-forms.

The statement about torsion free contact structures is based on the interaction between the action of the mapping class group and first homology group of T^3 on its Ozsváth–Szabó homology and ordinary cohomology. It sheds some light on the sign ambiguity of the contact invariant since the sign ambiguity of the Giroux invariant is very easy to understand.

Corollary 1 *There are infinitely many isomorphic contact structures whose isotopy classes are pairwise distinguished by the Ozsváth–Szabó invariant.*

Theorem 2 proves, via gluing, a conjecture of Honda, Kazez and Matić about the sutured invariants of S^1-invariants contact structures on toric annuli. This conjecture is stated in [11, top of page 35] and will be discussed in Sect. 4 and Proposition 4.

Theorem 2 also have some consequence for the hierarchy of coefficients because $\mathbb{Z}_2$ coefficients can distinguish only finitely many isotopy classes of contact structures (since $\widehat{HF}(Y; \mathbb{Z}_2)$ is always finite).

Corollary 2 *There exists a manifold on which the Ozsváth–Szabó invariant over integer coefficients distinguishes infinitely many more isotopy classes of contact structures than the invariant over $\mathbb{Z}_2$ coefficients.*

In the same spirit, we prove that twisted coefficients are more powerful than $\mathbb{Z}$ coefficients even when the latter give non vanishing invariants.

Proposition 1 (see Proposition 4) *There exist a sutured manifold with two contact structures having the same non vanishing Ozsváth–Szabó invariant over $\mathbb{Z}$ coefficients but which are distinguished by their invariants over twisted coefficients.*

In Sect. 2 we review the work of Giroux on certain contact structures on circle bundles, the easy extension of this work to Seifert manifolds and torsion calculations. In Sect. 3 we review Ozsváth–Szabó contact invariants. In Sect. 4 we prove Theorem 2. In Sect. 5 we review the work of Honda, Kazez and Matić on their contact TQFT and upgrade their SFH groups calculations to twisted coefficients. In Sect. 6, by far the longest, we prove [11, Conjecture 7.13] and the main theorem above.

2 Partitioned contact structures on Seifert manifolds

This section contains preliminary results in contact topology. We first recall the crucial definition of Giroux torsion. The $k\pi$-torsion of a contact manifold (V, ξ) was defined in [6, Definition 1.2] to be the supremum of all integers $n \geq 1$ such that there exist a contact embedding of

$$(T^2 \times [0, 1], \ker(\cos(nk\pi z)dx - \sin(nk\pi z)\, dy)), \quad (x, y, z) \in T^2 \times [0, 1]$$

into the interior of (V, ξ) or zero if no such integer n exists. Of course all $k\pi$-torsions can be recovered from the π-torsion. However when we do not specify k we mean 2π-torsion. This is due to the fact that only 2π-torsion is known to interact with symplectic fillings and Ozsváth–Szabó theory.

A multi-curve in an orbifold surface B is a 1-dimensional submanifold properly embedded in the regular part of B. When B is closed, we will say that a multicurve is essential in B if none of its components bounds a disk containing at most one exceptional point.

Since we want to extend results from circle bundles to Seifert manifolds and most surface orbifolds are covered (in the orbifold sense) by smooth surfaces, the following characterization will be useful.

Lemma 1 *Let Γ be a multicurve in a closed orbifold surface B whose (orbifold) universal cover is smooth. The following statements are equivalent:*

1. *Γ is essential;*
2. *Γ lifts to an essential multicurve in all smooth finite covers of B.*
3. *Γ lifts to an essential multicurve in some smooth finite cover of B.*

Proof We first prove (the contrapositive of) (1) $\implies$ (2). Let Γ be an essential multicurve in B and π be an orbifold covering map from a smooth surface $\hat{B}$ to B. Suppose that a component of the inverse image of Γ bounds an embedded disk $\hat{D}$ in $\hat{B}$. Its image in B is a topological disk D and we only need to prove that this disk contains at most one exceptional point. Using multiplicativity of the orbifold Euler characteristic under the orbifold covering map from $\hat{D}$ to D, we get $\chi(D) > 0$. This proves that D contains at most one exceptional points because its Euler characteristic is $1-s+\sum_{i=1}^{s} 1/\alpha_i$ with $\alpha_i \geq 2$ if it has s exceptional points so $\chi(D) \leq 1-s/2$. So (1) implies (2). Since (2) obviously imply (3), we are left with proving (the contrapositive of) (3) implies (1).

Assume that Γ is not essential and let D be a connected component of the complement of Γ in B which is a disk with at most one exceptional point. In any finite cover $\hat{B}$ of B, this disk lifts to a collection of disks bounded by components of the lift of Γ and containing at most one exceptional point. So Γ is non essential in all finite covers of B. □

The following is the essential definition of this section.

Definition 1 (obvious extension of [7]) A contact structure is partitioned by a multicurve Γ in B if it transverse to the fibers over $B\backslash\Gamma$ and if the surface $\pi^{-1}(\Gamma)$ is transverse to ξ and its characteristics are fibers.

Example 1 ([17,20], see also [23]) Let $V \to B$ be a Seifert manifold and let Γ be a non empty multi-curve in B whose class in $H_2(B, \partial B; \mathbb{Z}_2)$ is trivial. There is a S^1-invariant contact structure on V which is partitioned by Γ. This contact structure is unique up to isotopy among S^1-invariant contact structures.

The following theorem relies on [21, Theorem A] and on easy extensions or consequences of the fourth part of [7]. Of course it also uses a lot the results of [6]. The two papers by Giroux can also be replaced by the Honda versions [12,13]. This theorem could be easily improved to say things about Seifert manifolds with non empty boundary but we will not need such improvements. Recall that a closed Seifert manifold is small if it has at most three exceptional fibers and its base has genus zero. Otherwise it is called large. In particular the bases of large Seifert manifolds admit essential multi-curves. We denote by $e(V)$ the rational Euler number of a Seifert manifold V. See [21] for the conventions used here for Seifert invariants and Euler numbers. In the statement we exclude for convenience the (finitely many) Seifert manifolds which are torus bundles over the circle (see for instance [9] to get the list).

Theorem 3 *Let V be a closed oriented Seifert manifold over a closed oriented orbifold surface.*

1. *A contact structure on V partitioned by a multi-curve Γ is universally tight if and only if one of the following holds:*

   (a) *Γ is empty*
   (b) *V is large and Γ is essential*
   (c) *V is a Lens space (including* S^3 *and* $S^2 \times S^1$*), $e(V) \geq 0$, Γ is connected and each component of its complement contains at most one exceptional point.*
2. *Any universally tight contact structure on V is isotopic to a partitioned contact structure.*
3. *Suppose V is not a torus bundle over the circle. Let ξ be a contact structure on V partitioned by an essential multi-curve Γ. Let n be the greatest integer such that there exist n components of Γ in the same isotopy class of curves. The Giroux torsion of ξ is zero if Γ is empty and at most $\lfloor \frac{n}{2} \rfloor$ otherwise.*
4. *Let ξ_0 and ξ_1 be contact structures on V partitioned by non empty multi-curves denoted by Γ_0 and Γ_1 respectively. If Γ_0 and Γ_1 are isotopic then ξ_0 and ξ_1 are so. If ξ_0 and ξ_1 are isotopic and universally tight then Γ_0 and Γ_1 are isotopic.*

We first comment on some consequences of this theorem which have not much to do with the main stream of the present paper. We can deduce from it and [18] (or [21]) the list (given in corollary 3 below) of Seifert manifolds which carry universally tight contact structures. This list did not appear in the literature while the (much subtler) list of Seifert manifolds which carry tight contact structures (maybe virtually overtwisted) was obtained (with much more work) by Lisca and Stipsicz [19]. In addition, the road taken in that paper to prove existence on large Seifert manifold is much heavier than using the above theorem (but the point of that paper is small manifolds).

Corollary 3 *A closed Seifert manifold V admits a universally tight contact structure if and only if one of the following holds:*

1. *V is large*
2. *V is a Lens space (including* S^3 *and* $S^2 \times S^1$*)*
3. *V has three exceptional fibers which can be numbered such that its Seifert invariants are* $(0, -2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ *with*

$$\frac{\beta_1}{\alpha_1} > \frac{m-a}{m}, \quad \frac{\beta_2}{\alpha_2} > \frac{a}{m}, \quad \text{and} \quad \frac{\beta_3}{\alpha_3} > \frac{m-1}{m}$$

   for some relatively prime integers $0 < a < m$.

The above theorem also proves that all universally tight contact structures on Seifert manifolds interact nicely with the Seifert structure.

Corollary 4 *If ξ is a universally tight contact structure on a closed Seifert manifold V then there exist a locally free* S^1 *action on V such that ξ is either transverse to the orbits or invariant.*

Note that the alternative in the above corollary is not exclusive. A contact structure which is both invariant and transverse to the orbits of a locally free S^1 action exists exactly when $e(V) < 0$, this was proved by Kamishima and Tsuboi [17]. There is only one isomorphism class of contact structure of this type when they exist. This class is of Sasaki type and sometimes called the canonical isomorphism class of contact structures on V.

Proof of Theorem 3 We now outline the main differences between Theorem 3 and the parts which are already written in [7]. First it should be noted that, when V is a Lens space, everything is well understood thanks to the classification theorem of [6] (see also [12]). So we do not consider these Seifert manifolds in the following.

(1) Let ξ be a contact structure on a closed V partitioned by Γ. If Γ is empty then ξ is transverse to the fibers hence universally tight according to [21, Theorem A] (this direction follows rather directly from Bennequin's theorem). If V is large and Γ is essential then the base B of V is covered (in the orbifold sense) by a smooth surface Σ and there is a corresponding circle bundle $\hat{V} \to \Sigma$ covering (honestly) V. The pulled back contact structure is partitioned by the inverse image of Γ which is essential according to Lemma 1 so ξ is universally tight according to [7] (first line of page 252).

Conversely, assume that ξ is universally tight and partitioned by a non empty multi-curve Γ. Assume first the base of V is covered by a smooth surface of genus at least one (for instance if V is large). The manifold V then is covered by a circle bundle over that surface as above. We get from [7, Theorem 4.4] that the lifted contact structure is partitioned by a multi-curve, unique up to isotopy, which is essential. Since the lift of Γ is such a curve, it is essential and Lemma 1 implies that Γ is also essential. In particular V is large.

If no such cover of the base exists (and V is not a Lens space) then its base B is a sphere with exceptional points of order $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ (see [31, Theorem 13.3.6]). In each case B is covered by S^2 and all curves in the regular locus of B bounds a disk whose pre-image in S^2 is disconnected so ξ is virtually overtwisted according to [7, Proposition 4.1 and Lemma 4.7].

(2) Recall that a contact structure ξ on a Seifert manifold is said to have non-negative maximal twisting number[2] if it is isotopic to a contact structure for which there exists a Legendrian regular fiber whose contact framing coincides with the fibration framing. If this property is not satisfied and ξ is universally tight, then [21, Theorem A] ensures that ξ is isotopic to a contact structure partitioned by the empty multi-curve (i.e. transverse to the fibers). We now assume that ξ has non negative maximal twisting number and has been isotoped so that it admits a Legendrian fiber L as above. Let K be a wedge of circles based at L in the smooth part of B (seen as the space of all fibers) let R be a small regular neighborhood of K. We can choose K an R such that the complement R' of R in B is made of disks containing exactly one exceptional point. The techniques of [7] prove that ξ is isotopic to a contact structure which, over R is partitioned by a multicurve Γ_R which intersects all boundary components of R. We now assume this property. Let V' denote the (non necessarily connected) Seifert manifold over R' and ξ' the restricted contact structure. Since Γ_R intersects all components of ∂R, each component T of the boundary of V' contains a Legendrian regular fiber which is either a closed leaf or a circle of singularities of the characteristic foliation $\xi' T$. If ξ' is universally tight then the classification of universally tight contact structures on solid tori directly implies that ξ' is ∂-isotopic to a contact structure partitioned by some $\Gamma_{R'}$ extending Γ_R and we are done. More precisely,

[2] Some texts say zero twisting number in this case.

for each component W of V', this classification guaranties the existence of exactly one isotopy class of universally tight contact structure coinciding with ξ' on $\partial W'$ when W contains no exceptional fiber and two otherwise. In the latter case, the two classes correspond to the two isotopy classes of arcs extending Γ_R inside the base of W (which is a disk with one exceptional point).

So it remains to prove that if ξ has non negative maximal twisting number and is universally tight then each solid torus W isotopic to a fibered one has a universally tight induced contact structure. This is obvious if the universal cover $\tilde{W}$ of W naturally embeds into the universal cover $\tilde{V}$ of V. This $\tilde{V}$ can be built in two stages: first one takes the (orbifold) universal cover of the base B and pulls back the Seifert fibration and then one unwraps the fibers as much as possible. The sought embedding of $\tilde{W}$ obviously exist when the fibers can be completely unwrapped. Due to the classification of orbifolds surfaces the only problematic case if one excludes Lens spaces is when $\tilde{V}$ is S^3 with its (smooth) Hopf fibration. But, by definition of tightness, any tight contact structure on S^3 has negative twisting number with respect to the Hopf fibration so this case does not happen here (the property of having non negative twisting number is obviously inherited by finite covers using lifts of isotopies).

(3) Since we assume that V is not a torus bundle over the circle, all incompressible tori are isotopic to fibered ones (see e.g. [9]).

Suppose first that ξ is partitioned by the empty multicurve (i.e. is transverse to all fibers). It was proved in [21, Theorem A] that such a contact structure has negative maximal twisting number. Suppose by contradiction that it has non vanishing π-torsion. Up to isotopy of ξ there is an annulus in the base which is foliated by circles $(C_t)_{t\in[0,1]}$ such that,

- For all t, the torus T_t above C_t in V is prelagrangian.
- The directions of the Legendrian foliations of the T_t go all over the projective line.

During this full turn around the projective line, the Legendrian direction meets the fiber direction and there are Legendrian curve whose contact framing coincides with the fibration framing so we get a contradiction with the maximal twisting number estimate.

We now assume that ξ is partitioned by a non empty multicurve Γ and that at most n components of Γ are isotopic. Incompressible fibered tori correspond to essential curves in the base orbifold B. To any such curve C correspond an orbifold covering of B by an open annulus $\hat{B}$ and the Seifert fibration lifts to a trivial (smooth) circle fibration $\hat{V}$. The lifted contact structure is partitioned by the inverse image of Γ which is made of as many essential circles as there were components of Γ isotopic to C (at most n) and lines properly embedded in $\hat{B}$. If there exist a contact embedding of a toric annulus with its standard torsion contact structure in V then it lifts to $\hat{V}$ inside some $K \times S^1$ with $K \subset \hat{B}$ compact. The classification of tight contact structures on toric annuli forbids torsion higher than $\lfloor \frac{n}{2} \rfloor$ knowing the partition we have over K. This argument is not new, it was explained to me by E. Giroux around 2005 and was certainly known to him much earlier.

(4) The first part is a straightforward extension of [7, Lemma 4.7]. Suppose now that ξ_0 and ξ_1 are isotopic. If V is not large then we are in case (c) of the first point so that Γ_0 and Γ_1 are trivially isotopic. So we now assume that V is large. In particular

Γ_0 and Γ_1 are essential. By definition, they are isotopic in B if and only if they are isotopic in the smooth surface R obtained from B by removing a small open disk around each exceptional point. By definition of essential curves, no component of Γ_0 or Γ_1 is parallel to the boundary of R. According to W. Thurston, Γ_0 and Γ_1 are isotopic if and only if they have the same geometric intersection number with all closed curves in B [32, Proposition page 421]. These geometric intersections number have a contact topology interpretation explained in [7, Section 4.E] which proves they are invariant under contact structures isotopy exactly as in the circle bundle case. □

3 Contact invariants in sutured Floer homology

In this section we review sutured Heegaard–Floer homology and the contact invariants which live in it.

Heegaard–Floer homology was introduced by Ozsváth and Szabó [27] and extended to sutured manifold by Juház [15]. In the following we will often silently identify a closed manifold M with the sutured manifolds $(M \setminus B^3, S^1)$ and use sutured Floer theory (SFH) also in this case.

We denote the universally twisted $\underline{\mathrm{SFH}}(-M, \Gamma; \mathbb{Z}[H_2(M;\mathbb{Z})])$ by $\underline{\mathrm{SFH}}(-M, \Gamma)$ and, whenever there is no ambiguity on the manifold M we are considering, we denote $\mathbb{Z}[H_2(M;\mathbb{Z})]$ by $\mathbb{L}$.

According to [3, Lemma 10], if a contact invariant vanishes in $\underline{\mathrm{SFH}}$ then it vanishes for all coefficients rings.

Theorem 4 (Ozsváth–Szabó, Honda–Kazez–Matić, Ghiggini–Honda–Van Horn Morris) *Let (M, Γ) be a balanced sutured manifold. To each contact structure ξ on (M, Γ), one can associate a contact invariant $c(\xi)$ in* $\mathrm{SFH}(-M, -\Gamma)/\pm 1$ *and a twisted contact invariant $\underline{c}(\xi)$ in $\underline{\mathrm{SFH}}(-M, -\Gamma)/\mathbb{L}^\times$ satisfying the following properties:*

1. *the set $\underline{c}(\xi)$ is invariant under ∂-isotopy of ξ*
2. *if ξ is overtwisted then $\underline{c}(\xi) = 0$*
3. *if ξ has non zero torsion then $c(\xi) = 0$*
4. *if M is closed and ξ is weakly fillable then $\underline{c}(\xi) \neq 0$*
5. *if M is closed and ξ is strongly fillable then $c(\xi) \neq 0$*
6. *if (M', Γ') is a sutured submanifold of (M, Γ) and ξ is a contact structure on $(M \setminus M', \Gamma \cup \Gamma')$ then there exists a linear map*

$$\underline{\Phi}_\xi : \underline{\mathrm{SFH}}(-M', -\Gamma') \to \underline{\mathrm{SFH}}(-M, -\Gamma)$$

such that, for any contact structure ξ' on (M', Γ'), one has

$$\underline{c}(\xi \cup \xi') = \underline{\Phi}_\xi(\underline{c}(\xi')).$$

If every connected component of $M \setminus int(M')$ intersect ∂M then there are analogous maps over $\mathbb{Z}$ coefficients. They are denoted without underlines.

7. *if (M', ξ') is a contact submanifold of $(M, \xi' \cup \xi)$ then $\underline{c}(\xi') = 0$ implies that $\underline{c}(\xi \cup \xi') = 0$ and analogously over $\mathbb{Z}$ coefficients.*

The construction of the contact invariants (and the isotopy invariance) can be found in [28] for the closed case and [10] in general. The fact that it vanishes for overtwisted contact structures was first proved for the closed case and untwisted coefficients in [28] and follow in general from the last property and the explicit calculation of the twisted contact invariant of a neighborhood of an overtwisted disk found in [10]. The assertion about torsion was proved in [5]. Both assertions about fillings are consequences of [26, Theorem 4.2], using the fact that, for strong fillings, the coefficient ring in this theorem reduces to $\mathbb{Z}$ (see also [4, Theorem 2.13] for an alternative proof of the strong filling property). The gluing properties are proved in [11] for untwisted coefficients and extended to twisted coefficients in [3]. The gluing maps are unique up to multiplication by an invertible element of the relevant coefficients ring. Such maps will be called HKM gluing maps. Over twisted coefficients at least, the last point follows from the previous one but we want to emphasize this last point since it will be crucial in the current paper.

There is one piece of structure of Heegaard–Floer theory which does not seem to have been explicitly discussed[3] in our context up to now: the mapping class group action. Any diffeomorphism of a 3-manifold M acts on any variant of $HF(M)$. Here we need to be precise about what depends on the way a Heegaard diagram is embedded inside a manifold and what does not depend on it. The usual way to do that is to consider embedded Heegaard diagrams as pairs made of a self-indexing Morse function with unique minima and maxima and one of its Morse–Smale pseudo-gradients. Given such a pair (f, X), the Heegaard surface is $f^{-1}(3/2)$ and the Heegaard circles are the intersections of the stable or unstable disks of the index 1 and 2 critical points. We denote the group associated to (f, X) by $HF(f, X)$ (we can use here $\widehat{HF}$, $HF^+, \dots$). Let φ be a diffeomorphism of M. Then [29, Theorem 2.1] gives an isomorphism

$$\Psi : HF(f, X) \to HF(f \circ \varphi, \varphi^* X)$$

which is well defined up to sign. But of course the diffeomorphism φ also gives an isomorphism between the corresponding abstract Heegaard diagrams which then gives an isomorphism Φ between Heegaard–Floer groups. The action of φ on $HF(f, X)$ is defined to be $\Phi^{-1} \circ \Psi$. It is obvious from the construction that the contact invariant is equivariant under this action. What is not obvious is that isotopic diffeomorphisms have the same action so that we get an action of the mapping class group. This has been checked by P. Ozsváth and A. Stipsicz in the context of knot Floer homology in [24]. In this paper we do not use this invariance but use specific diffeomorphisms. Actually this invariance should never be needed in contact geometry since we already know that the contact invariant is a contact structure isotopy invariant so that diffeomorphism isotopy invariance is automatic on the subgroup spanned by contact invariants in any $\widehat{HF}$ or HF^+.

[3] We do not claim to do anything new in this paragraph, but we can not find a reference for it.

4 Contact structures on the three torus

In this section we prove Theorem 2 from the introduction. The following easy lemma is the key algebraic trick.

Lemma 2 *If an isomorphism* $\Phi : \widehat{HF}(\mathrm{T}^3) \to H^1(\mathrm{T}^3) \oplus H^2(\mathrm{T}^3)$ *is* $H_1(\mathrm{T}^3)$*-equivariant then it conjugates the* SL_3 *actions of both sides.*

Proof In this proof we drop T^3 from the notations. We denote by ρ the canonical action of SL_3 on H_1. Let ρ_1 and ρ_2 be two representations of SL_3 on $H^1 \oplus H^2$ which are compatible with the H_1 action, that is:

$$\forall g \in \mathrm{SL}_3, \gamma \in H_1, m \in H^1 \oplus H^2, \quad (\rho(g)\gamma)\rho_i(g)m = \rho_i(g)(\gamma m).$$

We want to prove that $\rho_1 = \rho_2$ since this, applied to the standard action and to the action transported by Φ, will prove the proposition.

We first prove that, for all $g \in \mathrm{SL}_3$, $\rho_1(g)$ and $\rho_2(g)$ agree on H^2. The key property of the H_1 action is that it separates all elements of H^2: for all $m \neq m' \in H^2$, there exists γ in H_1 such that $\gamma m = 0$ and $\gamma m' \neq 0$.

Suppose by contradiction that there exists $g \in \mathrm{SL}_3$ and $m \in H^2$ such that $\rho_1(g)m \neq \rho_2(g)m$. According to the separation property, there exists γ' in H_1 such that $\gamma'\rho_1(g)m = 0$ and $\gamma'\rho_2(g)m \neq 0$. Setting $\gamma = \rho(g)^{-1}(\gamma')$, we get $\rho(g)\gamma\rho_1(g)m = 0$ and $\rho(g)\gamma\rho_2(g)m \neq 0$, so $\rho_1(g)(\gamma m) = 0$ and $\rho_2(g)(\gamma m) \neq 0$, which is absurd since $\rho_1(g)$ and $\rho_2(g)$ are both isomorphisms.

We now prove that the representations agree on H^1. For all $m' \in H^1$, there exists $m \in H^2$ and $\gamma \in H_1$ such that $m' = \gamma m$. So for any $g \in SL_3$ and $i = 1, 2$, we get $\rho_i(g)m' = \rho_i(g)(\gamma m) = \rho(g)\gamma\rho_i(g)m$ and we know that $\rho_1(g)m = \rho_2(g)m$ thanks to the first part so $\rho_1(g)m' = \rho_2(g)m'$. □

Proof of Theorem 2 The existence of such an isomorphism is Proposition 8.4 of [25]. The above lemma proves that, for any Φ as in the statement and any $x \in \widehat{HF}$, x and $\Phi(x)$ have the same stabilizer under the action of SL_3. The uniqueness of Φ follows since primitive elements of $H^1 \oplus H^2$ are characterized up to sign by their stabilizers. Indeed, suppose Φ_1 and Φ_2 are both isomorphisms as in the statement of the proposition. Then $\Phi_{12} := \Phi_1 \circ \Phi_2^{-1}$ is an automorphism such that, for any primitive x, there exists $\varepsilon_x \in \{\pm 1\}$ such that $\Phi_{12}(x) = \varepsilon_x x$. We now consider a $\mathbb{Z}$-basis $e_1, \ldots, e_6$ of $H^1 \oplus H^2$ and compute

$$\sum \varepsilon_{\sum e_i} e_j = \varepsilon_{\sum e_i} \sum e_j = \Phi_{12}\left(\sum e_j\right) = \sum \Phi_{12}(e_j) = \sum \varepsilon_{e_j} e_j$$

so we get that all ε_{e_j} agree with $\varepsilon_{\sum e_i}$ and $\Phi_{12} = \varepsilon_{\sum e_i}\mathrm{Id}$. So Φ_1 and Φ_2 agree up to a global sign.

We now prove that the Poincaré dual of the Giroux invariant and the image of the Ozsváth–Szabó invariant coincide on torsion free contact structures. First remark that the Ozsváth–Szabó invariant belongs to $\widehat{HF}_{-1/2} \simeq H^1$ because the Hopf invariant of tight contact structures on T^3 is $1/2$. So both invariants are primitive elements of H^1.

We prove that the stabilizer of $G(\xi)$ is contained in that of $c(\xi)$ using equivariance of both invariants and the fact that G is a total invariant. For any g in SL_3 and ξ a torsion free contact structure, we have

$$\begin{aligned} gG(\xi) = G(\xi) &\iff G(g\xi) = G(\xi) \\ &\iff g\xi \sim \xi \\ &\implies c(g\xi) = c(\xi) \\ &\iff gc(\xi) = c(\xi) \end{aligned}$$

so we have the announced inclusion of stabilizers and this gives $c(\xi) = G(\xi)$. □

5 The contact TQFT

We now review the contact TQFT of Honda–Kazez–Matić. Let Σ be a non necessarily connected compact oriented surface with all components having non empty boundary. Let F be a finite subset of $\partial\Sigma$ whose intersection with each component of $\partial\Sigma$ is non empty and consists of an even number of points. We assume that the components of $\partial\Sigma \backslash F$ are labelled alternatively by $+$ and $-$. This labelling will always be implicit in the notation (Σ, F). The contact TQFT associates to each (Σ, F) the graded group

$$V(\Sigma, F) = SFH(-(\Sigma \times \mathrm{S}^1), -(F \times \mathrm{S}^1))$$

(strictly speaking, one should replace F by a small translate of F along $\partial\Sigma$ in this formula).

In this construction one can use coefficients in $\mathbb{Z}_2$ or twisted coefficients (including the trivial twisting which leads to $\mathbb{Z}$ coefficients). We denote by $\underline{V}(\Sigma, F)$ the version twisted by $\mathbb{Z}[H_2(\Sigma \times \mathrm{S}^1)]$.

Proposition 2 *Let (Σ, F) be a surface with marked boundary points as above and $\mathbb{M}$ be any coefficient module for the sutured manifold $(\Sigma \times \mathrm{S}^1, F \times \mathrm{S}^1)$. We have, for any coherent orientations system:*

$$\underline{V}(\Sigma, F; \mathbb{M}) \simeq (\mathbb{M}_{(-1)} \oplus \mathbb{M}_{(1)})^{\otimes(\#F/2-\chi(\Sigma))}.$$

The subscripts (-1) and (1) refer to the grading.

Proof The analogous statement over $\mathbb{Z}$ coefficients was proved in [11] using product annuli decomposition, [2, Proposition 7.13]. This technology is not yet available over twisted coefficients but one can actually draw explicit admissible sutured Heegaard diagrams with vanishing differential for these sutured manifolds. We will sketch how to construct them and draw pictures for the three cases where we actually use this computation below.

We first recall what is an (embedded) Heegaard diagram for a (balanced connected) sutured manifold (V, Γ). It consists of a surface S properly embedded in V and circles $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ in S such that:

Fig. 1 (Almost) Heegaard surface for $V = (\mathrm{T}^2 \backslash D^2) \times \mathrm{S}^1$ with two vertical sutures ($n = 1$). Everything lives inside a transparent (*light grey*) cube minus a neighborhood of its vertical edges. The cube's faces are pairwise glued to get $(\mathrm{T}^2 \setminus D^2) \times \mathrm{S}^1$. The picture shows the (almost) Heegaard surface S' of the proof in *medium grey*. The black annulus in the back left is the negative part R_- of the boundary of V. The β compression disks are inside the front, back, left and right faces of the cube (which get glued to two annuli in V). The alpha curves sitting on the cylinders P_1 and P_2 are shown in *black*, each one been divided in two in the representation (there is a front/back α curve and a left/right one)

- $\partial S = \Gamma$
- if we denote by V_+ the connected component of $V \setminus S$ containing R_+, there exist open disks properly embedded in V_+, called compression disks, bounded by the α circles and such that the complement of the compression disks in $V_+ \cup R_+$ retracts by deformation on R_+
- the analogous statement holds for V_- and R_- with the β circles.

We now return to the proposition. Let g be the genus of Σ, r the number of boundary components and $n = \#F/2$. The sutured manifold we study will be denoted by (V, Γ) for concision. We rule out the trivial $(g = 0, r = 1, n = 1)$ case from this discussion as it needs (easy) special treatment. Assume first that $r = 1$ and $n = 1$. Let $a_1, \ldots, a_{2g}$ be a system of disjoints arcs properly embedded in Σ which cuts Σ to a disk. Let $P_1, \ldots, P_{2g}$ be cylinders around the arcs $a_i \times \{\theta_0\}$ for some fixed $\theta_0 \in \mathrm{S}^1$. We can assume that each P_i meets the boundary of V in its positive part R_+. Let S' be the union of R_+ and the cylinders P_i. The surface S obtained by pushing S' to make it properly embedded in V is a Heegaard surface for (V, Γ).

Each cylinder P_i naturally bounds a regular neighborhood $D \times [-1, 1]$ of the arc a_i. Let α_i be the boundary of $D \times \{0\}$ in each P_i. Let β_i be the union of $\{\pm 1\} \times [-1, 1]$ and two arcs in R_+ so that β_i and half of P_i becomes isotopic to a fibered annulus in V. In the case $(g = 1, r = 1, n = 1)$ one can see the embedded surface in Fig. 1 and the Heegaard diagram in Fig. 2.

We then have a Heegaard diagram (S, α, β) for (V, Γ). We now explain what happens when we add some extra boundary components (i.e. $r > 1$). For each extra component T_j we add two cylinders P_{2g+j} and P'_{2g+j} around horizontal arcs $a_{2g+j} \times \{\theta_0\}$ and $a'_{2g+j} \times \{\theta_0\}$. We choose these arcs so that they can be completed by arcs in the positive part of $\partial \Sigma$ to get a circle isotopic to the new boundary component. See Fig. 3

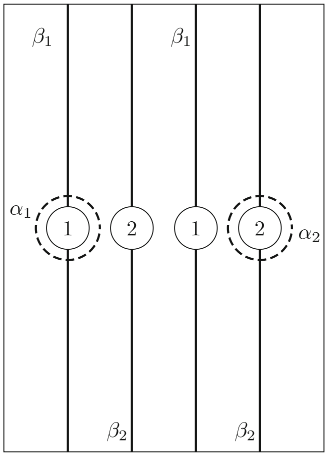


Fig. 2 Heegaard diagram corresponding to Fig. 1. *Top and bottom of the rectangle* are glued to get an annulus. Then disks in the middle are joined by two cylinders according to their labels 1 and 2 to get P_1 and P_2

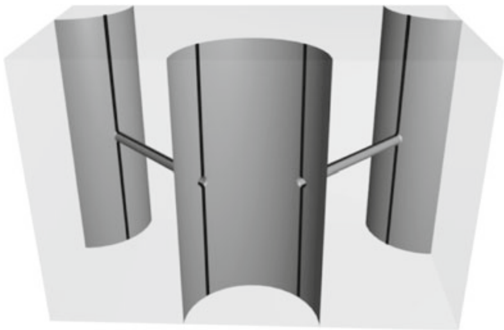


Fig. 3 Heegaard surface for $V = (I \times S^1) \times S^1$ with two vertical sutures ($n = 1$). Everything is inside a transparent *light grey box* which is glued top/bottom and left/right to give a thicken torus. The Heegaard surface in *medium grey* is the union of two vertical annuli and two horizontal cylinders. Note that the two parts in the back are connected in the glued manifold. The β curves are shown in *black*

for the case ($g = 0, r = 2, n = 2$) where the extra boundary component is the back one. We add circles α_{2g+j}, α'_{2g+j}, β_{2g+j} and β'_{2g+j} to the diagram as above (See Fig. 4). When there are extra marked points on the boundary (i.e. $n > r$), we add one cylinder $P_{2g+r-1+k}$ between two positive parts of the relevant boundary component. We add the corresponding circles to the diagram. In the case ($g = 0, r = 1, n = 3$), Fig. 5 shows the Heegaard surface (the extra sutures are the back ones) and Fig. 6 shows the corresponding diagram (Figs. 4, 6).

In this paragraph, whenever we started from the trivial case ($g = 0, r = 1, n = 1$) which was ruled out above, we can use as a starting point the degenerate diagram with Heegaard surface R_+ and no circle.

The constructed diagrams have $2g + 2(r - 1) + (n - r)$ circles of each type and $\#\alpha_i \cap \beta_j = 2\delta_{ij}$. Hence the chain complex has rank $2^{n-\chi(\Sigma)}$. So the proposition follows from the admissibility of these diagrams and the vanishing of the associated differentials.

Each arc a_i, $1 \leq i \leq 2g$ can be extend to a loop $\bar{a}_i$ and each pair of arcs corresponding to extra boundary components can be extended to a loop l_j, $1 \leq j \leq r-1$ such that the collection of tori $\bar{a}_i \times S^1$ and $l_j \times S^1$ gives a basis of $H_2(V, \mathbb{Z})$. This

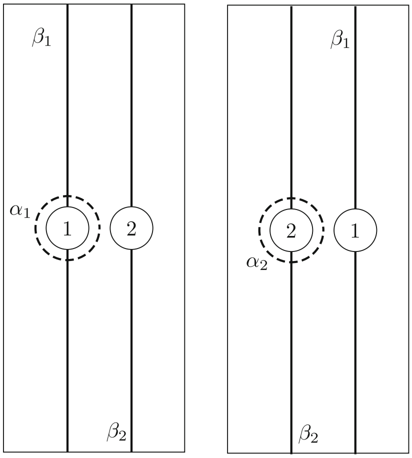


Fig. 4 Heegaard diagram corresponding to Fig. 3. *Top and bottom of each rectangle* are glued to get two annuli. Then disks in the middle are joined by two cylinders according to their labels *1* and *2* to get P_1 and P_2

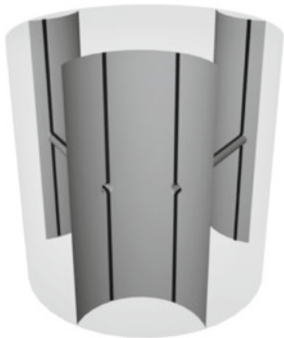


Fig. 5 Heegaard surface for $V = D^2 \times S^1$ with six vertical sutures ($n = 3$). *Top and bottom* are glued. The boundary of V is the transparent torus (drawn as an annulus). The Heegaard surface is the union of three vertical annuli and two horizontal cylinders. The β curves are shown in *black*

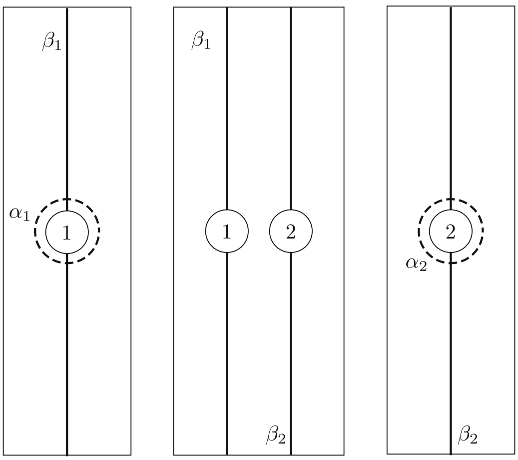


Fig. 6 Heegaard diagram corresponding to Fig. 5. *Top and bottom* of each rectangle are glued to get three annuli. Then disks in the middle are joined by two cylinders according to their labels *1* and *2* to get P_1 and P_2

basis can be realized by periodic domains using the α and β circles associated to the corresponding arcs. So we have a basis of $H_2(V, \mathbb{Z})$ associated to disjoint periodic domains, each having both positive and negative coefficients. Since they have disjoint support, any linear combination of these domains will be admissible and the diagram is admissible.

To compute the differential we note that each region of the complement of the circles in S which is not the base region is either a rectangle or an annulus. In addition each rectangle is adjacent to either a rectangle using the same circles or to the base region or to an annulus. One can then use Lipshitz's formula to prove that the Heegaard–Floer differential vanishes. □

A dividing set for (Σ, F) is a multi-curve K in Σ (see Definition 1). The complement of a dividing set in Σ splits into two (non connected) surfaces $R_\pm$ according to the sign of their intersection with $\partial\Sigma$. The grading of a dividing set is defined to be the difference of Euler characteristics $\chi(R_+) - \chi(R_-)$.

The following definition due to Honda Kazez and Matić is crucial to understand contact invariants of partitioned contact structures.

Definition 2 A dividing set K is said to be isolating if there is a connected component of the complement of K which does not intersect the boundary of Σ.

To each dividing set K for (Σ, F) is associated the contact invariant of the contact structures partitioned by K. All such contact structures are either isotopic according to Theorem 3 or overtwisted so they have the same invariant. These invariants belong to the graded part given by the grading of K.

Theorem 5 [11] *Over $\mathbb{Z}_2$ coefficients, the following are equivalent:*

1. $c(K) \neq 0$
2. $c(K)$ *is primitive*
3. K *is non isolating*

Over $\mathbb{Z}$ coefficients, (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

Conjecture 7.13 of [11] states that the assertions in this theorem are equivalent over $\mathbb{Z}$ coefficients. What remains to be proved is that isolating dividing sets have vanishing invariant. This (and more) will be proved in Sect. 6.

6 Vanishing results

In this section we prove the main theorem from the introduction and the following theorem which finishes off the proof of Conjecture 7.13 of [11]. We use the definitions and notations of the previous section.

Theorem 6 *If K is isolating then $c(K) = 0$ over $\mathbb{Z}$-coefficients.*

Note that the analogous statement over twisted coefficients is known to be false. For instance if we consider on T^3 a contact structure partitioned by four essential circles and remove a small disk meeting one of these circles along an arc then we get an isolating dividing set on a punctured torus whose twisted invariant is sent to a non vanishing invariant according to Theorem 4 since the corresponding contact structures on T^3 are weakly fillable.

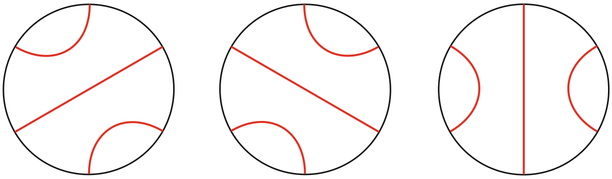
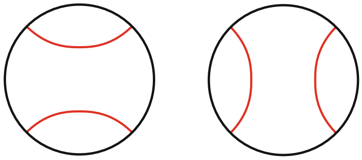


Fig. 7 Bypass relation

Fig. 8 Dividing sets used to prove Lemma 3



Definition 3 We say that dividing sets K_0, K_1 and K_2 are bypass-related if they coincide outside a disk D where they consists of the dividing sets of Fig. 7.

The following lemma is essentially proved in [11] in the combination of proofs of Lemma 7.4 and Theorem 7.6. We write a proof here to explain why twisted coefficients come for free.

Lemma 3 *If K_0, K_1 and K_2 are bypass-related then, for any representatives $\underline{\tilde{c}}_i \in \underline{c}(K_i)$, there exist $a, b \in \mathbb{L}^\times$ such that $\underline{\tilde{c}}_0 = a\underline{\tilde{c}}_1 + b\underline{\tilde{c}}_2$. The same holds over $\mathbb{Z}$ coefficients.*

Proof The first part of the proof concentrates on the disk where the dividing sets differ. Let $\tilde{c}_i^D$ be representatives of the contact invariants of the three dividing sets on a disk D involved in Definition 3. Note that $H_2(D \times S^1)$ is trivial so we now work over $\mathbb{Z}$ coefficients and suppress the underlines.

Because the c_i^D's all belong to the same rank 2 summand of $V(D, F_D)$ there are integers λ, μ and ν not all zero such that

$$\lambda \tilde{c}_0^D = \mu \tilde{c}_1^D + \nu \tilde{c}_2^D. \tag{1}$$

We denote by $K_\pm$ the dividing sets of Fig. 8 and by $c_\pm$ their contact invariants.

Label the points of F_D clockwise by $1, \dots, 6$ starting with the upper right point. Let Φ_j, $j = 1, 2, 3$, denote a HKM gluing map obtained by attaching a boundary parallel arc between points j and $j + 1$. The gluing maps have the following effects:

$$\Phi_1 : c_0^D \mapsto c_+, \quad c_1^D \mapsto c_+, \quad c_2^D \mapsto 0 \tag{2}$$

$$\Phi_2 : c_0^D \mapsto 0, \quad c_1^D \mapsto c_-, \quad c_2^D \mapsto c_- \tag{3}$$

$$\Phi_3 : c_0^D \mapsto c_+, \quad c_1^D \mapsto 0, \quad c_2^D \mapsto c_+ \tag{4}$$

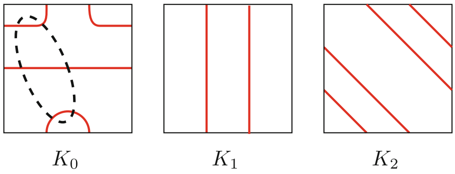


Fig. 9 Dividing sets for Propositions 3 and 4. *Left and right sides of each squares* should be glued to get annuli

Using these equations and the facts that $c_\pm$ are non zero in a torsion free group (see Proposition 2), we get

$$(2) \implies \lambda = \pm\mu$$
$$(3) \implies \mu = \pm\nu$$
$$(4) \implies \lambda = \pm\nu$$

and they are all non zero so we can divide Eq. (1) by λ to get

$$\tilde{c}_0^D = \varepsilon_1 \tilde{c}_1^D + \varepsilon_2 \tilde{c}_2^D. \quad (5)$$

with $\varepsilon_1 = \mu/\lambda$ and $\varepsilon_2 = \nu/\lambda$.

We now return to our full dividing sets. Let D be the disk where the K_i's differ. Denote by F_D the (common) intersection of the K_i's with ∂D. Let ξ_0, ξ_1 and ξ_2 be contact structures partitioned by K_0, K_1 and K_2 respectively and coinciding with some ξ_b outside $D \times \mathrm{S}^1$.

Let $\underline{\Phi} : V(D, F_D) \to \underline{V}(\Sigma, F)$ be a HKM gluing map associated to ξ_b. According to Theorem 4, there exist invertible elements a_i of $\mathbb{L}$ such that $\underline{\Phi}(\tilde{c}_i^D) = a_i \underline{\tilde{c}}_i$ for all i. We now apply $\underline{\Phi}$ to Eq. (5) and put $a = \varepsilon_1 a_1 a_0^{-1}$ and $b = \varepsilon_2 a_2 a_0^{-1}$ □

Using this Lemma, we can reprove the main result of [5].

Proposition 3 [5] *Contact structures with positive Giroux torsion have vanishing contact invariant over $\mathbb{Z}$ coefficients.*

Proof Let (A, F_A) be an annulus with two marked points on each boundary component and consider the dividing sets of Fig. 9. We will denote by ξ_0, ξ_1 and ξ_2 contact structures partitioned by the corresponding K_i. Using the disk whose boundary is dashed, one sees that K_0 is bypass-related to K_1 and K_2. We denote $(A \times \mathrm{S}^1, F_A \times \mathrm{S}^1)$ by (N, Γ).

Let ξ_b be a basic slice on a toric annulus (N', Γ'). We glue (N, Γ) and (N', Γ') to get a new toric annulus. Using the obvious decomposition of $H_1(N)$ and the corresponding one for $H_1(N \cup N')$, we want the dividing slopes to be ∞ (this is the slope of the S^1 factor) and 1 respectively. By changing the sign of the basic slice, we can assume that $\xi_0 \cup \xi_b$ is universally tight. It follows from the classification of tight contact structures on toric annuli that a contact manifold has positive Giroux torsion if and only if it contains a copy of $\xi_0 \cup \xi_b$. Therefore we only need to prove that $c(\xi_0 \cup \xi_b)$ vanishes.

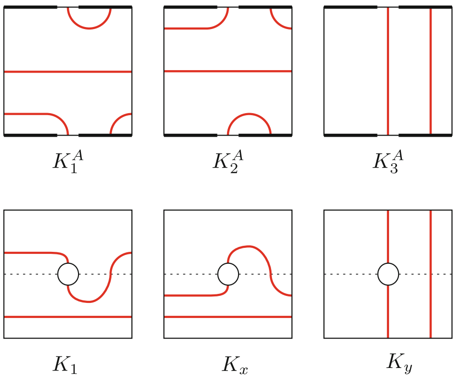


Fig. 10 Dividing sets for Propositions 4 and 5. *On the top row*, *left and right sides of the squares* are glued to make the annulus A. Then the thick parts of ∂A can be glued by translation to make the punctured torus of the *bottom row* where the sides of the squares are glued by translation and the glued part of ∂A is *dashed*

Let $\Phi = \Phi_{\xi_b}$ be a corresponding HKM gluing map. The structures $\xi_1 \cup \xi_b$ and $\xi_2 \cup \xi_b$ are ∂-isotopic and they are basic slices. Using invariance under isotopy, we get $c(\xi_1 \cup \xi_b) = c(\xi_2 \cup \xi_b)$. Let $\tilde{c}_b$ be a representative of this common contact invariant. Let $\tilde{c}_1$ and $\tilde{c}_2$ be representatives of $c(K_1)$ and $c(K_2)$ such that $\tilde{c}_b = \Phi(\tilde{c}_1) = \Phi(\tilde{c}_2)$. Such representatives exist according to the gluing property. We also take any representative $\tilde{c}(K_0) \in c(K_0)$ and denote by $\tilde{c}(\xi_0 \cup \xi_b)$ its image under Φ. This image belong to $c(\xi_0 \cup \xi_b)$ according to the gluing property.

Lemma 3 gives $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$\tilde{c}(K_0) = \varepsilon_1 \tilde{c}_1 + \varepsilon_2 \tilde{c}_2.$$

We then apply Φ to this equation to get:

$$\tilde{c}(\xi_0 \cup \xi_b) = (\varepsilon_1 + \varepsilon_2)\tilde{c}_b. \tag{6}$$

Let (W, ξ_W) be a standard neighborhood of a Legendrian knot (W is a solid torus). We now glue (W, ξ_W) along the boundary component of $N \cup N'$ which is in ∂N so that meridian curves have slope 0. The structure $\xi_W \cup \xi_0 \cup \xi_b$ is overtwisted whereas $\xi_W \cup \xi_1 \cup \xi_b$ (and $\xi_W \cup \xi_2 \cup \xi_b$ which is isotopic to it) is a standard neighborhood of a Legendrian curve so can be embedded into Stein fillable closed contact manifolds. Let Φ_W be a gluing map associated to ξ_W. Applying Φ_W to Eq. (6) and using the vanishing property of overtwisted contact structures, we get

$$0 = (\varepsilon_1 + \varepsilon_2)\Phi_W(\tilde{c}_b).$$

Using that $\Phi_W(\tilde{c}_b)$ is non zero and the fact that the relevant SFH group has no torsion (see [16, Proposition 9.1]) we get $\varepsilon_1 + \varepsilon_2 = 0$. Returning to Eq. (6), we then get $c(\xi_0 \cup \xi_b) = 0$. □

Proposition 4 *Let (A, F_A) be an annulus with two points on each boundary component. Let T be one of the components of $\partial A \times S^1$ and $t = e^{[T]} \in \mathbb{Z}[H_2(A \times S^1)]$. Let K_1^A, K_2^A and K_3^A be the dividing sets of Fig. 10 and let $\tilde{\underline{c}}_1^A$, $\tilde{\underline{c}}_2^A$ and $\tilde{\underline{c}}_3^A$ be any representatives of their contact invariants in $\underline{V}(A, F_A)$.*

1. *There exist invertible elements a and b in* $\mathbb{L}$ *such that:*

$$\underline{\tilde{c}}_1^A = a\underline{\tilde{c}}_2^A + b(t-1)\underline{\tilde{c}}_3^A. \tag{7}$$

2. *Twisted invariants distinguish* K_1^A, K_2^A *and* K_3^A. *Over* $\mathbb{Z}$ *coefficients,* $c(K_2^A)$ *and* $c(K_3^A)$ *are independent but* $c(K_1^A) = c(K_2^A)$.
3. *Let* τ *be the right handed Dehn twist along the core of A. There exist* $\tilde{c}_2 \in c(K_2^A)$ *and* $\tilde{c}_3 \in c(K_3^A)$ *such that for any* $n \in \mathbb{Z}$, $\tilde{c}_3 + n\tilde{c}_2 \in c(\tau^n K_3^A)$.

The second part of this proposition was proved over $\mathbb{Z}$ coefficients in Section 7.5 of [11]. The last part was conjectured in [11, top of page 35].

Proof Because all K_i^A's have zero Euler class, the twisted invariants $\underline{\tilde{c}}_1$, $\underline{\tilde{c}}_2$ and $\underline{\tilde{c}}_3$ all live in the same rank two summand of $\underline{V}(A, F_A)$ so there exist $\lambda, \mu, \nu \in \mathbb{L}$, not all zero, such that

$$\lambda\underline{\tilde{c}}_1^A = \mu\underline{\tilde{c}}_2^A + \nu\underline{\tilde{c}}_3^A. \tag{8}$$

We now use two HKM gluing maps: $\underline{\Phi}_1$ (resp. $\underline{\Phi}_2$) corresponding to gluing the dividing set K_1^A (resp. K_2^A) from the bottom in Fig. 10. We will denote loosely by $K_1^A \cup K_2^A$ for instance the result of gluing K_1^A on the bottom of K_2^A. For any ξ partitioned by K_1^A we can perform a generalized Lutz twist on the unique torus which is foliated by Legendrian fibers and the result is partitioned by $K_1^A \cup K_1^A$ so the main result of [3] gives $\underline{\Phi}_1(\underline{\tilde{c}}_1^A) = d(t-1)\underline{\tilde{c}}_1^A$ for some invertible element d. Since contact structures partitioned by $K_1^A \cup K_2^A$ are overtwisted, we get $\underline{\Phi}_1(\underline{\tilde{c}}_2^A) = 0$. And $K_1^A \cup K_3^A$ is isotopic to K_1^A so there is some invertible e such that $\underline{\Phi}_1(\underline{\tilde{c}}_3^A) = e\underline{\tilde{c}}_1^A$. So when we apply $\underline{\Phi}_1$ to Eq. (8) we get: $\lambda d(t-1)\underline{\tilde{c}}_1^A = \nu e\underline{\tilde{c}}_1^A$.

A similar argument for $\underline{\Phi}_2$ gives invertible elements f and g such that:

$$\mu f(t-1)\underline{\tilde{c}}_2^A + \nu g\underline{\tilde{c}}_2^A = 0.$$

Since $\underline{\text{SFH}}(A, F_A)$ is a free module over the integral domain $\mathbb{L}$ and $\underline{\tilde{c}}_1^A$ and $\underline{\tilde{c}}_2^A$ are non zero (the corresponding contact structures embed into Stein fillable contact manifolds), we get

$$\lambda d(t-1) = \nu e$$
$$\mu f(t-1) + \nu g = 0.$$

so that $\nu = \lambda e^{-1}d(t-1)$ and $\mu = -f^{-1}ge^{-1}d\lambda$. Since λ, μ and ν are not all zero, we get that λ is non zero. Setting $a = -f^{-1}ge^{-1}d$ and $b = e^{-1}d$, Eq. (8) gives the announced relation.

We now prove the second point. We have already met morphisms sending $\underline{\tilde{c}}_1^A$, $\underline{\tilde{c}}_2^A$ and $\underline{\tilde{c}}_3^A$ to elements not related to each other by invertible elements of $\mathbb{L}$. So the invariants $\underline{c}^A(K_i^A)$ are pairwise distinct. Going to $\mathbb{Z}$ coefficients sends $t-1$ to zero so the formula of the first point proves that Ozsváth–Szabó invariants over $\mathbb{Z}$ coefficients

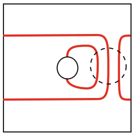


Fig. 11 Dividing set for Proposition 5. Sides of the square are glued to make a punctured torus. The *dashed circle* bounds a disk used to apply Lemma 3

do not distinguish K_1^A and K_2^A. But they distinguish K_1^A and K_3^A as can be seen for instance by using the $\mathbb{Z}$ coefficients version of $\underline{\Phi}_1$.

In order to prove the third point we will use the results of Sect. 4.

Figure 9 shows that K_2^A, K_3^A and $\tau^{-1}K_3^A$ are bypass related. We start with any representatives for the relevant invariants and Lemma 3 gives us instructions to change signs so that we get $\tilde{c}_2^A \in c(K_2^A)$ and $\tilde{c}_3^A \in c(K_3^A)$ with $\tilde{c}_3^A - \tilde{c}_2^A \in c(\tau^{-1}K_3^A)$.

We now stick to these representatives. Using the image of Fig. 9 under τ, we see that K_2^A, K_3^A and τK_3^A are bypass related. So Lemma 3 gives signs ε and ε' such that $\varepsilon\tilde{c}_2^A + \varepsilon'\tilde{c}_3^A$ is in $c(\tau K_3^A)$. We set $\varepsilon_1 = \varepsilon\varepsilon'$ so that $\tilde{c}_2^A + \varepsilon_1\tilde{c}_3^A$ is in $c(\tau K_3^A)$. We want to prove that $\varepsilon_1 = 1$. The only other possibility, $\varepsilon_1 = -1$ would give $c(\tau^{-1}K_3^A) = c(\tau K_3^A)$ but this is forbidden by Theorem 2 since the corresponding contact structures are sent by gluing the two components of A to contact structures on T^3 which are distinguished by Ozsváth–Szabó invariants. So $\tilde{c}_2^A + \tilde{c}_3^A$ is in $c(\tau K_3^A)$. The general case follows from an inductive process using the same arguments. □

Proposition 5 *Let Σ_0 be a punctured torus, F_0 a set of two points on $\partial\Sigma_0$ and K_0 a dividing set on Σ_0 consisting of a circle and an arc, both boundary-parallel (see Fig. 11). Let (x, y) be the image in $\mathbb{L} = \mathbb{Z}[H_2(\Sigma_0 \times \mathrm{S}^1)]$ of a basis of $H_2(\Sigma_0 \times \mathrm{S}^1)$. Let K_x and K_y be dividing sets on Σ_0 made of a boundary parallel arc and one closed curve whose lift in $H_2(\Sigma_0 \times \mathrm{S}^1)$ has homology class $x^{\pm 1}$ and $y^{\pm 1}$ respectively, see Fig. 10. Let $\tilde{\underline{c}}_0$, $\tilde{\underline{c}}_x$ and $\tilde{\underline{c}}_y$ be any representatives of $\underline{c}(K_0)$, $\underline{c}(K_x)$ and $\underline{c}(K_y)$ respectively. In $V(\Sigma_0, F_0)$,*

$$c(K_0) = 0$$

whereas there exist invertible elements λ, μ in $\mathbb{L}$ such that, in $\underline{V}(\Sigma_0, F_0)$,

$$\tilde{\underline{c}}_0 = \lambda(y - 1)\tilde{\underline{c}}_x + \mu(x - 1)\tilde{\underline{c}}_y.$$

Before proving this proposition we discuss its application to Theorem 1 from the introduction. Let T be a torus obtained by filling the boundary of Σ_0 with a disk D. Let a be an arc in D with boundary F_0 which extends smoothly K_0 to a closed multicurve $\bar{K}_0$ in T. Let V be a circle bundle over T with Euler number ± 1. There is an overtwisted contact structure ξ on V partitioned by $\bar{K}_0$ and the fiber over any point of a is a null homologous Legendrian knot. The restriction of ξ to the solid torus over D is a standard Legendrian neighborhood of L according to the easiest case of the classification of tight contact structures on solid tori. So $c(K_0)$ can be seen as the sutured invariant of the Legendrian knot L and we proved Theorem 1. Many more examples of this situation can be constructed using Theorem 6 above.

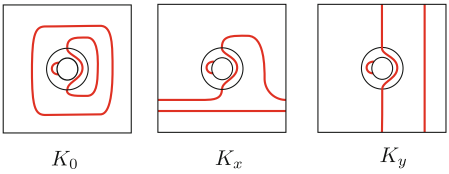


Fig. 12 A gluing for Proposition 5. Opposite edges of each square are glued to get punctured tori

In the above Proposition 5, the formula for the twisted invariant clearly implies vanishing of the untwisted invariant but, for the benefit of readers which are not interested in twisted coefficients, we will explain how to get directly the vanishing result.

Proof Using the disk whose boundary is dashed on Fig. 11, one sees that K_0 is bypass-related to K_1 and K_x from Fig. 10.

The dividing sets K_1 and K_x are obtained from the dividing sets of Proposition 4 as explained in Fig. 10. Let Φ_A be a HKM gluing map associated to the thick annuli of this figure, glued by translation. Let $\underline{\tilde{c}}_i^A$, $i = 1, 2, 3$ be representatives of the $\underline{c}(K_i^A)$. We know there are invertible elements f, g, h in $\mathbb{L}$ such that $\Phi_A(\underline{\tilde{c}}_1^A) = f^{-1}\underline{\tilde{c}}_1$, $\Phi_A(\underline{\tilde{c}}_2^A) = g\underline{\tilde{c}}_x$ and $\Phi_A(\underline{\tilde{c}}_3^A) = h\underline{\tilde{c}}_y$.

Lemma 3 gives $d, e \in \mathbb{L}^\times$ such that

$$\underline{\tilde{c}}_0 = d\underline{\tilde{c}}_1 + e\underline{\tilde{c}}_x. \tag{9}$$

We then apply Φ_A to Eq. (7) from Proposition 4 to get

$$\underline{\tilde{c}}_0 = (r + e)\underline{\tilde{c}}_x + \mu(x - 1)\underline{\tilde{c}}_y. \tag{10}$$

where $r = dfag$ and $\mu = dfhb$ are invertible.

We first prove quickly vanishing of the untwisted invariant and then we'll turn again to twisted coefficients. Over $\mathbb{Z}$ coefficients, the preceding equation reduces to

$$\tilde{c}_0 = (\varepsilon_1 + \varepsilon_2)\tilde{c}_x \tag{11}$$

where ε_1 and ε_2 are ± 1 (invertible elements in the base ring $\mathbb{Z}$).

Let D be a disk divided by an arc K_D and ξ_D a contact structure on $D \times \mathrm{S}^1$ partitioned by K_D. We now glue (Σ_0, K_0) to (D, K_D) and consider a HKM gluing map $\Phi : V(\Sigma_0, F_0) \to \widehat{HF}(\mathrm{T}^3)$ given by ξ_D.

According to Giroux's criterion (contained in the first part of Theorem 3), $\xi_0 \cup \xi_D$ is overtwisted. Since overtwisted contact structures have vanishing invariant, we get $\Phi(\tilde{c}_0) \in c(\xi_0 \cup \xi_D) = 0$. So Eq. (11) gives $0 = (\varepsilon_1 + \varepsilon_2)\Phi(\tilde{c}_x)$. In addition $\Phi(\tilde{c}_x) \in c(\xi_1 \cup \xi_D)$ is non zero because $\xi_1 \cup \xi_D$ is Stein fillable. Since $\widehat{HF}(\mathrm{T}^3)$ has no torsion (see Sect. 4), we get that $\varepsilon_1 + \varepsilon_2 = 0$ and $\tilde{c}_0 = 0$ so $c(K_0) = 0$.

We now return to twisted coefficients. We glue in an annulus divided by two boundary parallel arcs, see Fig. 12. When glued to K_0 we get an overtwisted contact structure while K_x and K_y lead to generalized Lutz modifications on the same dividing set K. Let $\underline{\tilde{c}}$ be a representative of $c(K)$ (which is not zero since its $\mathbb{Z}$ coefficient projection

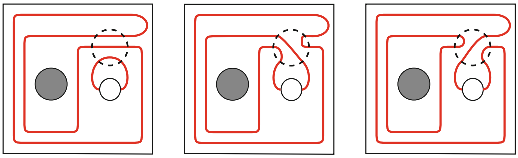


Fig. 13 First inductive step in the proof of Theorem 6. The sides of the squares are glued by translation and the *shaded* disk hides more genus

does not vanish). Using the Ghiggini–Honda formula [3], we get invertible elements u and v in $\mathbb{L}$ such that

$$0 = (x-1)u(r+e)\underline{\tilde{c}} + (y-1)v\mu(x-1)\underline{\tilde{c}}.$$

We can now use that $\underline{V}$ is a free $\mathbb{L}$ module (Proposition 2) and $\mathbb{L}$ is an integral domain to get $r + e = -u^{-1}v\mu(y-1)$ so that Eq. (10) gives the expected formula with $\lambda = -u^{-1}v\mu$. □

Now that we have Proposition 5, the following proof is almost identical to that of [11, Proposition 7.12]

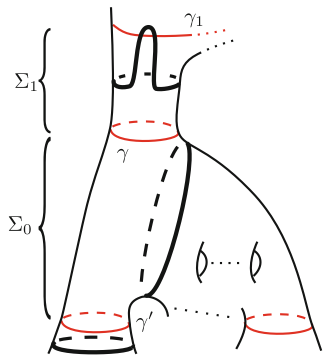
Proof of Theorem 6 First remark that $c(K) = 0$ if K has an isolated annulus because the corresponding contact structures have non zero Giroux torsion. Then we will use two nested inductive proofs to get the general result.

We now start an induction on the number of boundary components of isolated regions.

First assume that K has an isolated region Σ_0 whose boundary is connected. We prove the theorem by induction on the genus of Σ_0. If this genus is zero then any contact structure partitioned by K is overtwisted hence $c(K) = 0$. If this genus is one then Σ_0 is a punctured torus and Σ contains a sub-surface satisfying the assumptions of Proposition 5 so, by this proposition and Theorem 4, $c(K) = 0$. Assume now that the theorem is proved when K has an isolated region with connected boundary and genus at most $g - 1 \geq 1$. If K has an isolated region with genus $g > 1$ then (Σ, K) has a subsurface (Σ_1, K_1) drawn on the left-hand side of Fig. 13 where the sides of the square are glued pairwise and the shaded disk hides a subsurface having genus $g - 1$ and not intersecting K. The dashed circle shows that K_1 is bypass-related to K_1 and K_2. Since K_2 has an isolated punctured torus and K_3 has an isolated region with genus $g - 1$, the inductive hypothesis gives $c(K_2) = c(K_3) = 0$. Lemma 3 combines these two vanishings to give $c(K_1) = 0$. This implies $c(K) = 0$ thanks to Theorem 4. Hence the inductive step is completed and any K having an isolated region with connected boundary has vanishing invariant.

We now prove the induction step for our original inductive proof. We assume the theorem is proved for any dividing set having an isolated region with at most $r - 1 \geq 1$ boundary components. Suppose K has an isolated region Σ_0 with $r > 1$ boundary components. We can assume that Σ_0 is not an annulus since this case is already known. Also, at least one boundary component γ of Σ_0 is adjacent to another region Σ_1 whose closure meets a component γ_1 of $K \setminus \gamma$. Let Σ_0' be a pair of pants in Σ_0 containing γ and another component γ' of $\partial\Sigma_0$ but otherwise does not intersect K. Let Σ' be a

Fig. 14 Construction of Σ' in the second inductive step. The boundary of Σ' is the thickest curve. *Dotted lines in the right hand side* of the picture indicate that Σ_0 can have more or less genus and boundary components. *Dotted lines on top* suggests that only a small portion of Σ_1 and γ_1 is drawn



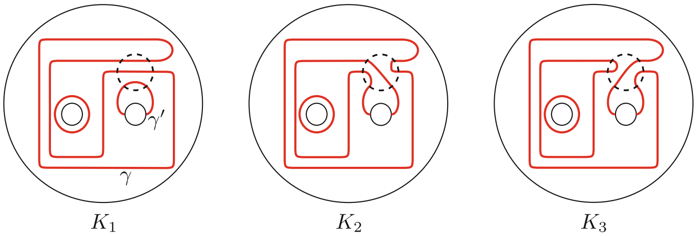


Fig. 15 Second inductive step in the proof of Theorem 6

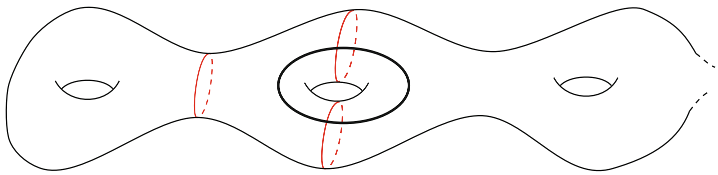


Fig. 16 Universally tight torsion free contact structures with vanishing contact invariants. The partitioning curve is *thin*

small regular neighborhood of the union of Σ_0' and an arc from γ to γ_1 in the complement of K, see Fig. 14. The subsurface Σ' is a pair of pants whose intersection with K is K_1 shown on Fig. 15. This figure also shows the intersections with Σ' of dividing sets K_2 and K_3 which are bypass-related to K. The dividing set K_2 has an isolated region with $r - 1$ boundary components. One of them is the outermost thick circle of Fig. 15, the other ones are not in Σ'. So $c(K_2) = 0$ by inductive assumption. The dividing set K_3 has an annular isolated region so $c(K_3) = 0$. Lemma 3 combines these two vanishings to give $c(K) = 0$. □

Proof of the main theorem Let V be a Seifert manifold over an orbifold B whose base has genus at least three. Let K_0 be the multi-curve of Fig. 16 where B continues to the right and all exceptional points of B are in the right hand side of the picture. Let τ be the right-handed Dehn twist around the thick (black) curve of Fig. 16. Theorem 3 associates to the $\tau^n(K_0)$'s infinitely many isotopy classes of universally tight torsion free contact structures. Note that the genus hypothesis is used here to ensure that our contact structures are torsion free. Proposition 5 and Theorem 4 ensure that they all have vanishing contact invariant over $\mathbb{Z}$ coefficients since there dividing sets all contain a copy the dividing set of Proposition 5. □

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