Math. Ann. (2012) 353:1037–1068
DOI 10.1007/s00208-011-0709-5

Mathematische Annalen

Techniques for the study of singularities with applications to resolution of 2-dimensional schemes

Angélica Benito · Orlando E. Villamayor U.

Received: 27 December 2010 / Revised: 23 May 2011 / Published online: 14 August 2011
© Springer-Verlag 2011

Abstract We give an overview of invariants of algebraic singularities over perfect fields. We then show how they lead to a synthetic proof of embedded resolution of singularities of 2-dimensional schemes.

Mathematics Subject Classification (2000) 14E15

1 Introduction

This paper includes an exposition of recent progress concerning singularities over perfect fields, it is also shown how these results lead to the resolution of singularities of 2-dimensional schemes.

The ultimate motivation of the results reported in this work is the open problem of resolution of singularities in any dimension. We focus here on resolution in the sense of Hironaka, which is a step by step procedure: namely, given a reduced scheme X over a perfect field k, the question is to construct a sequence of blow-ups along smooth centers, each center included in the Hilbert-Samuel stratum of the successive strict transforms of X, so as to define a desingularization.

A. Benito and O. Villamayor are partially supported by MTM2009-07291.

A. Benito (✉) · O. E. Villamayor U.
Universidad Autónoma de Madrid
and ICMAT-UAM, Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain
e-mail: angelica.benito@uam.es

O. E. Villamayor U.
e-mail: villamayor@uam.es

Sections 2 and 3 include an overview of invariants that have been introduced recently for singularities over perfect fields in arbitrary dimension, and finally in Sect. 4 we apply them to give a synthetic proof of resolution of 2-dimensional schemes.

In the case of characteristic zero, given X included in a smooth scheme of dimension d, say $V^{(d)}$, the existence of smooth hypersurfaces of *maximal contact* was used by Hironaka to attach to X inductive invariants in dimension $d-1$. This is done in a way that the problem of resolution, formulated in dimension d, is reformulated as a problem in dimension $d-1$ by *restriction* to smooth hypersurfaces of maximal contact. A drawback of this approach is that such nice hypersurfaces are not unique, and, in addition to that, these hypersurfaces are defined only locally whereas the problem of resolution is global. So a significant difficulty in using this form of induction is that of *patching* local information.

This significant difficulty does not show up in Hironaka's theorem in [31,32], which is existential, but it does arise in the *constructive* proofs of Hironaka's theorem. Namely in proofs which establish an algorithm that indicate which is the smooth center to be blown up in Hironaka's step by step procedure of resolution (see [10,44,45]).

A second drawback in using hypersurfaces of maximal contact for inductive arguments appears when trying to work over fields of positive characteristic. In fact, in this context these hypersurfaces do not always exist [41]. Further information about resolution of singularities in characteristic zero and the previous drawback can be found in [6,11,12,22–24,27,40,49].

Section 2 is an exposition of an approach to resolution, developed in recent years, which avoids the use of maximal contact. In fact, smooth hypersurfaces of maximal contact can be replaced by local projections on smooth schemes; and the traditional notion of *restriction to hypersurfaces of maximal contact* can be replaced by a generalized form of the discriminant, defined in terms of elimination theory. When considering resolution of singularities over fields of characteristic zero this alternative procedure provides the same information as that obtained by using maximal contact. But it has an important advantage over the original constructive proofs of resolution as it trivializes the globalization of local invariants. This clarifies, in particular, the globalization of local data extracted from the Hilbert Samuel function. All this has led to a significant conceptual simplification of constructive resolution of singularities in characteristic zero [13].

A second advantage of this approach is the fact that projections are definable over perfect fields; a feature which has opened the way to the definition of new inductive invariants in positive characteristic. This study is addressed in Sect. 3, which is devoted to the discussion of inductive invariants of singularities over perfect fields, introduced in [8,9]. These are natural extensions of the inductive invariants used by Hironaka in characteristic zero.

As was already mentioned, Hironaka uses inductive invariants in dimension $d-1$, in his step by step procedure, to obtain a resolution of singularities by successive monoidal transformations. These invariants enable him to construct a sequence of blow-ups over $X \subset V^{(d)}$, so as to come to a so called "$d-1$-simplification". This $d-1$-simplification is also known as a reduction to the *monomial case*. He then shows that it is easy to achieve desingularization once X has been transformed into a scheme with singularities in the monomial case.

The inductive invariants, used by Hironaka (in dimension $d-1$) make use of hypersurfaces of maximal contact, and the argument works exclusively in characteristic zero. The alternative approach to induction, using projections, has led to the construction of a sequence of blow-ups over $X \subset V^{(d)}$, so as to come to a "$d-1$-simplification" now over perfect fields. This parallels the $d-1$-simplification in characteristic zero, or say the *reduction to the monomial case*, but the outcome obtained is weaker in positive characteristic. In fact, over fields of positive characteristic the resolution of singularities which are in the monomial case is not straightforward, as it is in the case of characteristic zero. Despite this fact, the reduction to the monomial case is expected to be a simplification of the singularities. The main outcome of Sect. 3 is to show that, if some additional numerical conditions are fulfilled, singularities which are in the monomial case can be resolved.

As an application, in Sect. 4, we show how this alternative approach to induction leads to resolution in the case in which X ($\subset V^{(d)}$) is a 2-dimensional embedded scheme. The task, for future research, would be to show that the said numerical conditions, discussed in Sect. 3, can be attained in higher dimensions.

The reduction to the monomial case in positive characteristic is related with other forms of simplification that appear in other works. For example, a procedure is introduced in the work of Kawanoue and Matsuki [35,36], which also parallels this reduction. See also [34,50].

The general and unifying strategy along this paper is the use of higher differential operators as a tool for the study of singularities over perfect field. Embedded resolution leads to the study of ideals on smooth schemes, and more generally to Rees algebras of ideals on smooth schemes. This reformulation of resolution problems appears already in Hironaka's work. Rees algebras on smooth schemes can be enriched by the action of higher order differential operators, and these enriched algebras are called differential algebras. These are algebras which encode very subtle information of the singularities, and they are the main tool in this alternative form of induction which avoids the use of maximal contact.

Differential Rees algebras allow us to reduce the problem of the resolution of singularities embedded in a smooth d-dimensional scheme, say $V^{(d)}$, to that of hypersurfaces of multiplicity p^e, where p denotes the characteristic of the underlying field. More precisely to hypersurfaces defined by equations of the form

$$f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z].$$

We may assume, in addition, that a sequence of monomial transformation has been defined so that the singularities are in the monomial case. Under these assumptions, a prominent role is played by the constant term a_{p^e} in the study of invariants in Sect. 3, whereas the information of the other intermediate coefficients is somehow encoded by other $d-1$-dimensional invariants. The protagonist role of the constant term, that appears for example in formula (14), seems to resemble a reduction to the study of equations of the form

$$f_{p^e}(z) = z^{p^e} + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z],$$

also known as equations in the purely inseparable case.

These particular features will appear in Sect. 4, were it is shown how the techniques discussed in the previous sections lead to resolution of two dimensional schemes.

The first proof of resolution in dimension 2, is due to Abhyankar (see [2]). It gives a non-embedded proof of resolution of surfaces in positive characteristic. A synthetic and detailed presentation of this proof appears also in [21]. Resolution of arithmetical surfaces (non-equicharacteristic surfaces) was proved by Lipman [37] making use of techniques of duality theory.

Hironaka proves embedded resolution of 2-dimensional schemes over algebraically closed fields in [33]. Moreover, such resolution is attained, as his general proof in characteristic zero, by successive blow-ups along centers included in the Hilbert-Samuel stratum. More recently, in [17], embedded resolution, in the sense of Hironaka, has been proved for 2-dimensional schemes in the non-equicharacteristic case, namely for arithmetical schemes.

So the outcome of Sect. 4, namely that of embedded resolution of two dimensional schemes, is to be taken simply as an application of the alternative form of induction discussed here. In this last section, the resolution of surfaces is divided in five subcases: A), B), C), D1), D2). Each case is defined in accordance to the values attained by the $d-1$-dimensional invariants discussed in Sect. 3.

In studying the effect of quadratic transformations within case A) we make use of an invariant which appears in Hironaka's proof in [33], expressed there in terms of Newton polygons, and known as the β-invariant of the singularity. A similar notion appears also in Abhyankar's work in [2] (see Abhyankar's trick in Lemma 4.12). This invariant is also considered in [15,30].

The task, for future research, would be to apply these techniques to the open problem of embedded resolution of three dimensional schemes. Abhyankar proves resolution in dimension 3, over algebraically closed fields of characteristic $p > 5$ in [4]. This is a non-embedded proof, that make use of geometric arguments (see also [20]). More recently Cossart and Piltant [18,19] prove a theorem of resolution in dimension 3, which holds in any characteristic. The outcome of their procedure is very strong, and it is close to that of embedded resolution, as their resolution only modifies singular points. This last result suggests that embedded resolution of singularities should hold, at least for 3-dimensional schemes over perfect fields. More results concerning resolution of singularities in positive characteristic can be found in [1,3,7,16,25,26,28–30,38,39,42,43,48].

2 Rees algebras, elimination and monoidal transformations

2.1

In problems concerning resolution of singularities it is natural to consider data with two ingredients. The first ingredient is given by a hypersurface, say X, embedded in a d-dimensional smooth scheme, say $V^{(d)}$, and the second is a positive integer, say b.

A first motivation for this approach appears already when we fix b as the highest multiplicity of X. In such case one considers the data $(I(X), b)$, where $I(X) \subset \mathcal{O}_{V^{(d)}}$ is the ideal of definition of X. This, in turn, defines a closed set:

$$\{x \in V^{(d)} \mid \nu_x(I(X)) \geq b\},$$

where $\nu_x(I(X))$ is the order of $I(X)$ at the regular local ring $\mathcal{O}_{V^{(d)},x}$. Namely, the set of points where X takes the highest multiplicity b (i.e., the set of b-fold points of X).

If Y is a smooth closed center included in the set of b-fold points, it defines a "transform" of the data $(I(X), b)$. In this case the transform is given by the strict transform of the hypersurface, which makes use of the integer coordinate b. In fact, if $V^{(d)} \longleftarrow V_1^{(d)}$ is the blow-up at Y, and if X_1 is the strict transform of X, then

$$I(X)\mathcal{O}_{V_1^{(d)}} = I(H)^b I(X_1),$$

where $H \subset V_1^{(d)}$ denotes the exceptional hypersurface. The new data $(I(X_1), b)$ will be called the "transform" of $(I(X), b)$. If the closed set attached to the new data is empty, X_1 has no b-fold points and we shall say that the transformation defines a *resolution of* $(I(X), b)$. In this case we have come closer to the embedded desingularization of the hypersurface.

Hironaka reduces the problem of desingularization to a simultaneous treatment of data of the form, say $(I(X_1), b_1), \ldots, (I(X_s), b_s)$. More precisely, to a simultaneous resolution of these previous data by means of monoidal transformations.

In this work we encode these previous data in algebraic terms. This will lead us to a reformulation of resolution in terms of Rees algebras. The Rees algebra attached to the data $(I(X_1), b_1), \ldots, (I(X_s), b_s)$ will be the $\mathcal{O}_{V^{(d)}}$-algebra of the form $\mathcal{O}_{V^{(d)}}[I(X_1)W^{b_1}, \ldots, I(X_s)W^{b_s}]$, as we explain below.

2.2

A *Rees algebra* over $V^{(d)}$ is an algebra of the form $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$, where $I_0 = \mathcal{O}_{V^{(d)}}$ and each I_n is a coherent sheaf of ideals. Here W denotes a dummy variable introduced to keep track of the degree, so $\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$ is an inclusion of graded algebras. It is always assumed that, locally at any point of $V^{(d)}$, $\mathcal{G}$ is a finitely generated $\mathcal{O}_{V^{(d)}}$-algebra. Namely, that the restriction of $\mathcal{G}$ to an affine set $U \subset V^{(d)}$ is of the form

$$\mathcal{G} = \mathcal{O}_{V^{(d)}}(U)[f_{b_1}W^{b_1}, \ldots, f_{b_s}W^{b_s}](\subset \mathcal{O}_{V^{(d)}}[W])$$

for some *local generators* $\{f_{b_1}, \ldots, f_{b_s}\}$, each $f_{b_i} \in \mathcal{O}_{V^{(d)}}$.

We now set the *singular locus of* $\mathcal{G} = \bigoplus I_n W^n$ to be the closed set:

$$\mathrm{Sing}(\mathcal{G}) := \{x \in V^{(d)} \mid \nu_x(I_n) \geq n \text{ for each } n \in \mathbb{N}\}.$$

It can be checked that for an arbitrary affine set U as before, one have that $\mathrm{Sing}(\mathcal{G}) \cap U = \cap\{x \in V^{(d)} \mid \nu_x(\langle f_{b_i}\rangle) \geq b_i\}$.

Fix a monoidal transformation $V^{(d)} \xleftarrow{\pi_C} V_1^{(d)}$ with center $C \subset \mathrm{Sing}(\mathcal{G})$. For all $n \in \mathbb{N}$, $I_n\mathcal{O}_{V_1^{(d)}}$ admits a factorization of the form

$$I_n\mathcal{O}_{V_1^{(d)}} = I(H_1)^n \cdot I_n^{(1)},$$

where $H_1 = \pi_C^{-1}(C)$ denotes the exceptional hypersurface, and $I(H_1)$ the ideal defining H_1. This defines a Rees algebra over $V_1^{(d)}$, namely $\mathcal{G}_1 = \bigoplus_{n\in\mathbb{N}} I_n^{(1)} W^n$, called the *transform of* $\mathcal{G}$, denoted by

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_1 \\ V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)}. \end{array} \tag{1}$$

A sequence of transformations will be denoted by:

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)} & \xleftarrow{\pi_{C_1}} & \cdots & \xleftarrow{\pi_{C_{r-1}}} & V_r^{(d)} \end{array} \tag{2}$$

and herein we always assume that the exceptional locus of the composite morphism $V^{(d)} \longleftarrow V_r^{(d)}$, say $\{H_1, \ldots, H_r\}$, is a union of hypersurfaces with only normal crossings in $V_r^{(d)}$.

A sequence (2) is a *resolution of* $\mathcal{G}$ if, in addition, $\mathrm{Sing}(\mathcal{G}_r) = \emptyset$.

2.3 Hironaka's main invariants

We shall circumvent the precise definition of Hironaka's notion of *invariant* (see [9, 2.8], and Sect. 3.5 below). But let us mention that his notion of invariant at a point $x \in \mathrm{Sing}(\mathcal{G})$ relates to the local codimension of $\mathrm{Sing}(\mathcal{G})$ at x, and also to the codimension of $\mathrm{Sing}(\mathcal{G}_r)$ at points lying over x, for sequences as (2).

There are two main invariants, introduced by Hironaka, which play a crucial role in his theorem of resolution of singularities [31]. Both are characteristic free, and we shall formulate them within the context of Rees algebras.

(1) *Hironaka's d-dimensional function* Fix a Rees algebra $\mathcal{G} = \oplus_n I_n W^n$, Hironaka's d-dimensional *order* function, say

$$\mathrm{ord} : \mathrm{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}_{>0}$$

is defined by setting

$$\mathrm{ord}(\mathcal{G})(x) = \min_n \left\{ \frac{\nu_x(I_n)}{n} \right\},$$

where $\nu_x(I_n)$ denotes the order of I_n at the regular local ring $\mathcal{O}_{V^{(d)},x}$. Here the invariant at $x \in \mathrm{Sing}(\mathcal{G})$, in Hironaka's sense, is the value $\mathrm{ord}(\mathcal{G})(x)$.

(2) *Hironaka's τ-invariant* This is a positive integer attached here to every closed point $x \in \mathrm{Sing}(\mathcal{G})$. Recall that the tangent space at x is $\mathbb{T}_{V^{(d)},x} = \mathrm{Spec}(gr_M(\mathcal{O}_{V^{(d)},x}))$, where $gr_M(\mathcal{O}_{V^{(d)},x})$ is the graded ring of the regular local ring $\mathcal{O}_{V^{(d)},x}$. An homogeneous ideal $\mathrm{In}_x(\mathcal{G})$ in $gr_M(\mathcal{O}_{V^{(d)},x})$ is defined by $\mathcal{G} = \oplus_n I_n W^n$ at any $x \in \mathrm{Sing}(\mathcal{G}) \subset V^{(d)}$. $\mathrm{In}_x(\mathcal{G})$ is the ideal spanned by the class of I_n in M^n/M^{n+1}, for all $n \geq 1$.

The *tangent cone of* $\mathcal{G}$, say $\mathcal{C}_{\mathcal{G},x} \subset \mathbb{T}_{V^{(d)},x}$, is the homogeneous subscheme defined by $\mathrm{In}_x(\mathcal{G})$ in $gr_M(\mathcal{O}_{V^{(d)},x})$.

Here we view the tangent space as a vector space. A subspace, say $S \subset \mathbb{T}_{V^{(d)},x}$, acts by translation defining, in this way, an additive group scheme over $k(x)$. More precisely, set $tr_v(u) = u + v$ with $v \in S$ and $u \in \mathbb{T}_{V^{(d)},x}$. Define $Ł_{\mathcal{G},x} \subset \mathcal{C}_{\mathcal{G},x}$ to be the biggest additive subscheme so that $\mathcal{C}_{\mathcal{G},x} + Ł_{\mathcal{G},x} = \mathcal{C}_{\mathcal{G},x}$ (the biggest subspace acting on the tangent cone of $\mathcal{G}$). This is called the *subscheme of vertices* of $\mathcal{C}_{\mathcal{G},x}$.

Finally, define the τ*-invariant at* x, say $\tau_{\mathcal{G},x}$, as the codimension of $Ł_{\mathcal{G},x}$ in $\mathbb{T}_{V^{(d)},x}$.

2.4 Rees algebras and differential structure

There is a curious *compatibility* of differential operators on smooth schemes and Hironaka's notion of invariants. Let $V^{(d)}$ denotes a smooth scheme over k.

A Rees algebra $\mathcal{G} = \bigoplus_n I_n W^n$ over $V^{(d)}$ is said to be a *differential Rees algebra over* k if taking restrictions of $\mathcal{G}$ over every open affine set, $D_r(I_n) \subset I_{n-r}$, for any index n and for any k-differential operator D_r of order $r < n$. When a smooth morphism of k-schemes, say $V^{(d)} \xrightarrow{\beta} V^{(d')}$, is fixed, and the previous property holds for differential operators which are $\mathcal{O}_{V^{(d')}}$-linear, or say, β-relative operators, then $\mathcal{G}$ is said to be a β*-relative differential Rees algebra*, or simply β*-differential*.

Proposition 1 [46, Theorems 3.2 and 4.1] *Every Rees algebra* $\mathcal{G}$ *over* $V^{(d)}$ *admits an extension to a new Rees algebra, say* $\mathcal{G} \subset Diff(\mathcal{G})$*, so that* $Diff(\mathcal{G})$ *is a differential Rees algebra. Moreover, this differential algebra has the following properties:*

1. $Diff(\mathcal{G})$ *is the smallest differential Rees algebra containing* $\mathcal{G}$.
2. $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(Diff(\mathcal{G}))$.
3. *The equality in* (2) *is preserved by transformations. In particular, any resolution of* $\mathcal{G}$ *defines a resolution of* $Diff(\mathcal{G})$*, and the converse also holds.*

The property in (3) says that, for the sake of defining a resolution of $\mathcal{G}$, we may always assume that it is a differential Rees algebra. This is an important reduction because, as we shall indicate, differential Rees algebras have very handy properties. In fact, they turn out to be very useful when defining projections and other structures introduced by these projections, as we discuss below.

2.5 Transversal projections and elimination

Once we fix a closed point $x \in V^{(d)}$ it is very simple to construct, for any positive integer $d' \leq d$, a smooth scheme $V^{(d')}$ together with a smooth morphism $\beta : V^{(d)} \longrightarrow V^{(d')}$ (a projection); at least after restriction of $V^{(d)}$ to an étale neighborhood of x. This claim follows, essentially, from the fact that $(V^{(d)}, x)$ is an étale neighborhood of the affine space $\mathbb{A}^{(d)}$ at the origin, say $(\mathbb{A}^{(d)}, \mathbb{O})$ (see [5]). Plenty of smooth morphisms between affine spaces can be constructed (in fact, plenty of surjective linear maps), say $(\mathbb{A}^{(d)}, \mathbb{O}) \longrightarrow (\mathbb{A}^{(d')}, \mathbb{O})$, for $d' \leq d$. In fact, the smoothness of $V^{(d)}$ over the perfect field k ensures that if $\{x_1, \ldots, x_d\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d)},x}$, then $(V^{(d)}, x)$ is an étale neighborhood of $\mathbb{A}^{(d)}_k = \mathrm{Spec}(k[x_1, \ldots, x_d])$ at the origin.

Furthermore, whenever we fix a subspace S of dimension $d-d'$ in $\mathbb{T}_{V^{(d)},x}$, a smooth scheme $V^{(d')}$, together with a smooth morphism, say $\beta : V^{(d)} \longrightarrow V^{(d')}$, which can be constructed so that $ker(d(\beta)_x) = S$ (here $d(\beta)_x : \mathbb{T}_{V^{(d)},x} \longrightarrow \mathbb{T}_{V^{(d')},\beta(x)}$ is a surjective linear transformation).

Fix now a differential Rees algebra over $V^{(d)}$ and a closed point $x \in \mathrm{Sing}(\mathcal{G})$. Set $\tau_{\mathcal{G},x} = e$ (the codimension of $\mathcal{L}_{\mathcal{G},x}(\subset \mathcal{C}_{\mathcal{G},x})$ in the tangent space $\mathbb{T}_{V^{(d)},x}$).

Fix d' so that $d \geq d' \geq d-e$. We say that a smooth morphism $\beta : V^{(d)} \longrightarrow V^{(d')}$ is *transversal* to $\mathcal{G}$ at x if

$$ker(d\beta)_x \cap \mathcal{L}_{\mathcal{G},x} = \mathbb{O}.$$

Here, $ker(d\beta)_x$ has dimension $= d - d' \leq e$ and the previous condition says that both spaces are in general position. Moreover, this is an open condition which holds at points in an open neighborhood of x (see [13, Remark 8.5]).

Definition 1 A smooth morphism $\beta : V^{(d)} \longrightarrow V^{(d')}$ is said to be *transversal* to $\mathcal{G}$, if

1. $\tau_{\mathcal{G},x} \geq d - d'$, and
2. $ker(d\beta)_x \cap \mathcal{L}_{\mathcal{G},x} = \mathbb{O}$,

at any closed point $x \in \mathrm{Sing}(\mathcal{G})$.

Notice that if $\mathcal{G}$ is a differential Rees algebra, then it is, in particular, a β-differential Rees algebra for *any* transversal morphism β. The usefulness of differential Rees algebras relies on this particular fact, as we shall see in the following proposition.

Proposition 2 *Assume that $\beta : V^{(d)} \longrightarrow V^{(d')}$ is transversal to $\mathcal{G}$, and that $\mathcal{G}$ is β-differential. Then a Rees algebra, say $\mathcal{R}_{\mathcal{G},\beta}$, is defined over $V^{(d')}$,*

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{R}_{\mathcal{G},\beta} \\ V^{(d)} & \xrightarrow{\beta} & V^{(d')}, \end{array} \tag{3}$$

(i.e., $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d')}}[W]$). This $\mathcal{R}_{\mathcal{G},\beta}$ is called the elimination algebra of $\mathcal{G}$ defined by β, and has the following properties:

1. *The natural lifting of $\mathcal{R}_{\mathcal{G},\beta}$ to $\mathcal{O}_{V^{(d)}}$, say $\beta^*(\mathcal{R}_{\mathcal{G},\beta}) \subset \mathcal{O}_{V^{(d)}}[W]$, is a subalgebra of $\mathcal{G}$* [47, Theorem 4.13].
2. *$\beta(\mathrm{Sing}(\mathcal{G})) \subset \mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta})$, and $\beta|_{\mathrm{Sing}(\mathcal{G})} : \mathrm{Sing}(\mathcal{G}) \longrightarrow \mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta})$ defines a set theoretical bijection of $\mathrm{Sing}(\mathcal{G})$ with its image. Moreover, corresponding points of $\mathrm{Sing}(\mathcal{G})(\subset V^{(d)})$ and $\mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta})(\subset V^{(d')})$ have the same residue field* [47, 1.15 and Theorem 4.11], *or* [13, 7.1].
3. *Given a smooth sub-scheme $Y \subset \mathrm{Sing}(\mathcal{G})$, then $\beta(Y)(\subset \mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta}))$ is isomorphic to Y. In particular Y defines a transformation of $\mathcal{G}$ and also of $\mathcal{R}_{\mathcal{G},\beta}$* [13, Theorem 9.1 (i)].

4. [13, Theorems 10.1 and 9.1] *A smooth center* $Y \subset \mathrm{Sing}(\mathcal{G})$ *defines a commutative diagram*

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_1 \\ V^{(d)} & \xleftarrow{\pi_Y} & V_1^{(d)} \\ \downarrow \beta & & \downarrow \beta_1 \\ V^{(d')} & \xleftarrow{\pi_{\beta(Y)}} & V_1^{(d')} \\ \mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 \end{array} \tag{4}$$

where $\mathcal{G}_1$ *and* $(\mathcal{R}_{\mathcal{G},\beta})_1$ *denote the transforms of* $\mathcal{G}$ *and* $\mathcal{R}_{\mathcal{G},\beta}$*, respectively. Here* β_1 *is uniquely determined, and defined in the restriction of* $V_1^{(d)}$ *to a neighborhood of* $\mathrm{Sing}(\mathcal{G}_1)$*. This diagram has the following additional properties:*
(4a) $V_1^{(d)} \xrightarrow{\beta_1} V_1^{(d')}$ *is transversal to* $\mathcal{G}_1$ *and* $\mathcal{G}_1$ *is* β_1*-differential. In particular, we get:*

$$\begin{array}{ccc} \mathcal{G}_1 & & \mathcal{R}_{\mathcal{G}_1,\beta_1} \\ V_1^{(d)} & \xrightarrow{\beta_1} & V_1^{(d')} \end{array}$$

(4b) $(\mathcal{R}_{\mathcal{G},\beta})_1$ *coincides with* $\mathcal{R}_{\mathcal{G}_1,\beta_1}$*, the elimination algebra of* $\mathcal{G}_1$ *defined by* β_1.

5. [13, Theorem 10.1] *If a different transversal morphism, say* $\beta' : V^{(d)} \longrightarrow \widetilde{V}^{(d')}$*, defining*

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{R}_{\mathcal{G},\beta'} \\ V^{(d)} & \xrightarrow{\beta'} & \widetilde{V}^{(d')}, \end{array}$$

is considered, then analogous properties to (1), (2), (3) *and* (4) *hold. Moreover, the order of both elimination algebras coincide at any point* $x \in \mathrm{Sing}(\mathcal{G})$*, i.e.,*

$$\mathrm{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x)) = \mathrm{ord}(\mathcal{R}_{\mathcal{G},\beta'})(\beta'(x)).$$

This says a function

$$\mathrm{Ord}^{(d')}(\mathcal{G}) : \mathrm{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}_{>0}$$

can be defined by setting

$$\mathrm{Ord}^{(d')}(\mathcal{G})(x) = \mathrm{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x)).$$

2.6

The previous proposition states that given $\beta : V^{(d)} \longrightarrow V^{(d')}$ transversal to $\mathcal{G}$, and assuming that $\mathcal{G}$ is a β-differential Rees algebra (e.g., $\mathcal{G}$ is an absolute differential

Rees algebra), then a sequence of transformations defined with the conditions in (2), say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_Y} & V_1^{(d)} & \xleftarrow{\pi_{Y_1}} & \cdots & \xleftarrow{\pi_{Y_{r-1}}} & V_r^{(d)} \end{array} \tag{5}$$

gives rise to a diagram, say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_Y} & V_1^{(d)} & \xleftarrow{\pi_{Y_1}} & \cdots & \xleftarrow{\pi_{Y_{r-1}}} & V_r^{(d)} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} & & & & \downarrow{\scriptstyle\beta_r} \\ V^{(d')} & \xleftarrow{\pi_{\beta(Y)}} & V_1^{(d')} & \xleftarrow{\pi_{\beta_1(Y_1)}} & \cdots & \xleftarrow{\pi_{\beta_{r-1}(Y_{r-1})}} & V_r^{(d')} \\ \mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r \end{array} \tag{6}$$

where:

1. For any index i, there is an inclusion $(\mathcal{R}_{\mathcal{G},\beta})_i \subset \mathcal{G}_i$.
2. $\beta_i(\mathrm{Sing}(\mathcal{G}_i)) \subset \mathrm{Sing}((\mathcal{R}_{\mathcal{G},\beta})_i)$, and $\beta_i|_{\mathrm{Sing}(\mathcal{G}_i)} : \mathrm{Sing}(\mathcal{G}_i) \longrightarrow \beta_i(\mathrm{Sing}(\mathcal{G}_i))$ is an identification.
3. Every β_i is transversal to $\mathcal{G}_i$ and $\mathcal{G}_i$ is β_i-differential.
4. $V_i^{(d')} \xleftarrow{\pi_{\beta(Y_i)}} V_{i+1}^{(d')}$ denotes the transformation with center $\beta_i(Y_i)$ (isomorphic to Y_i).
5. $(\mathcal{R}_{\mathcal{G},\beta})_i = \mathcal{R}_{\mathcal{G}_i,\beta_i}$ where the later denotes the elimination algebra of $\mathcal{G}_i$ with respect to β_i.

In the characteristic zero case, the inclusions in (2) are equalities, but in positive characteristic, in general, only the inclusions are ensured.

Remark 1 Given a Rees algebra $\mathcal{G}$ over $V^{(d)}$, the aim is to construct a sequence as (5) which defines a resolution. For this purpose, we can take $\mathcal{G}$ to be a differential Rees algebras. This additional condition ensures that, whenever we construct a transversal morphism $\beta : V^{(d)} \longrightarrow V^{(d')}$, $\mathcal{G}$ will be a β-differential Rees algebra. In fact an absolute differential Rees algebra is always relative differential.

Recall that, passing from $\mathcal{G}$ to $Diff(\mathcal{G})$ does not affect the singular locus, i.e., $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(Diff(\mathcal{G}))$, and moreover the τ-invariant does not change, i.e., $\tau_{\mathcal{G},x} = \tau_{Diff(\mathcal{G}),x}$ at any closed point $x \in \mathrm{Sing}(\mathcal{G})$.

The τ-invariant has very subtle implications in resolution problems. For one thing, $\tau_{\mathcal{G},x}$ is an upper bound of the local codimension of the closed set $\mathrm{Sing}(\mathcal{G}) \subset V^{(d)}$ at the point x. Moreover, if equality holds, i.e., $\tau_{\mathcal{G},x} = \mathrm{codim}_x(\mathrm{Sing}(\mathcal{G})$, then $\mathrm{Sing}(\mathcal{G})$ is smooth in a neighborhood of x, and the resolution can be achieved by blowing-up at $\mathrm{Sing}(\mathcal{G})$. In particular, if $\tau_{\mathcal{G},x} = d$, then $\mathrm{Sing}(\mathcal{G}) = \{x\}$ (an isolated point) and the quadratic transformation at x defines a resolution of $\mathcal{G}$. Therefore, the strategy is to define resolution of Rees algebras on $V^{(d)}$ by decreasing induction on the highest value of τ.

The following theorem is stated under the inductive assumption of existence of resolution of Rees algebras $\mathcal{G}'$ which fulfills the condition $\tau_{\mathcal{G}',x} \geq d - d' + 1$ at any closed point $x \in \mathrm{Sing}(\mathcal{G}')$.

Theorem 1 [13, 10.4] *Let $\mathcal{G}$ be a differential Rees algebra. Assume that $\tau_{\mathcal{G},x} \geq d - d'$ at any closed point $x \in \mathrm{Sing}(\mathcal{G})$. There is a sequence of transformations (5), so that for any local transversal projection, say $\beta : V^{(d)} \longrightarrow V^{(d')}$, the induced sequence in the lower row of (6) is either a resolution, or is such that $(\mathcal{R}_{\mathcal{G},\beta})_r$ is a monomial algebra supported on the exceptional locus. Furthermore, in the latter case the monomial algebra $\beta_r^*((\mathcal{R}_{\mathcal{G},\beta})_r)$ is independent of β.*

2.7

In the case of characteristic zero, it is easy to extend the sequence in Theorem 1 to a resolution. In fact, in such a case, this extension is obtain by blowing-up centers prescribed by the monomial algebra $(\mathcal{R}_{\mathcal{G},\beta})_r$. However, this latter simple construction, that grows from the fact that $\beta_i(\mathrm{Sing}(\mathcal{G}_i)) = \mathrm{Sing}((\mathcal{R}_{\mathcal{G},\beta})_i)$ in (6), (2), does not apply in positive characteristic.

In the forthcoming Sect. 3, we study numerical conditions under which a sequence (5) in the conditions of Theorem 1 does extend to a resolution (Theorem 2). In Sect. 4 we show that these numerical conditions can be easily achieved in low dimension. This fact enables us to prove resolution of singularities of 2-dimensional schemes over perfect fields.

3 H-functions, tight monomial algebras and presentations

3.1

Fix a Rees algebra $\mathcal{G}$ over a d-dimensional smooth scheme $V^{(d)}$. At any closed point $x \in \mathrm{Sing}(\mathcal{G})$, the invariant $\tau_{\mathcal{G},x}$ was defined as the codimension of the subspace $\mathcal{L}_{\mathcal{G},x} \subset \mathbb{T}_{V^{(d)},x}$. The requirement that $\tau_{\mathcal{G},x} \geq d - d'$ at every closed point $x \in \mathrm{Sing}(\mathcal{G})$, ensures that the conditions in Definition 1 holds for a generic smooth morphism $\beta : V^{(d)} \longrightarrow V^{(d')}$.

Under the previous conditions, a so called $d' - dimensional H - function$, say

$$\text{H-ord}^{(d')}(\mathcal{G}) : \mathrm{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}_{>0},$$

was defined in [9]. The definition of this function involves a transversal morphism β, and the induced elimination algebra $\mathcal{R}_{\mathcal{G},\beta}$, among other things. Further details will be outlined below (see 3.2). When specializes to the case of characteristic zero, this function coincides with the *inductive function* introduced by Hironaka, which turned out to be extremely useful in resolution theorems.

Set now a sequence of transformations of $\mathcal{G}$, say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_Y} & V_1^{(d)} & \xleftarrow{\pi_{Y_1}} & \cdots & \xleftarrow{\pi_{Y_{r-1}}} & V_r^{(d)}. \end{array} \tag{7}$$

At any step of the sequence, $i = 1, \ldots, r$, functions

$$\text{H-ord}_i^{(d')}(\mathcal{G}_i) : \text{Sing}(\mathcal{G}_i) \longrightarrow \mathbb{Q}_{>0} \tag{8}$$

are defined.

Moreover, once we fix a transversal projection $\beta : V^{(d)} \longrightarrow V^{(d-1)}$ as above, the previous sequence gives rise to a diagram of the form

$$\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_Y} & V_1^{(d)} & \xleftarrow{\pi_{Y_1}} & \cdots & \xleftarrow{\pi_{Y_{r-1}}} & V_r^{(d)} \\
\beta \downarrow & & \beta_1 \downarrow & & & & \beta_r \downarrow \\
V^{(d')} & \xleftarrow{\pi_{\beta(Y)}} & V_1^{(d')} & \xleftarrow{\pi_{\beta_1(Y_1)}} & \cdots & \xleftarrow{\pi_{\beta_{r-1}(Y_{r-1})}} & V_r^{(d')} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array} \tag{9}$$

with the five properties of (6). In this setting, given $x_i \in \text{Sing}(\mathcal{G}_i)$ the value $\text{H-ord}_i(d')(\mathcal{G}_i)(x_i)$ can be computed from β, and from the liftings β_i (in a explicit manner). Moreover, the explicit calculation of the function will lead to the inequalities:

$$\text{H-ord}_i^{(d')}(\mathcal{G}_i)(x_i) \leq \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_i)(\beta_i(x_i)).$$

A particular feature of the H-functions is their unpredictable behavior under blow-ups. Another aspect is that they are not upper semi-continuous. The interest of the previous inequalities is that the functions in the right hand side, say $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_i)$, are upper semi-continuous (i.e., H-functions are upper bounded by upper semi-continuous functions).

Before we formulate some further applications of the H-functions in Theorem 2, let us indicate how they lead to the definition of a monomial algebra attached to an arbitrary sequence (7).

Definition 2 Fix a differential Rees algebra $\mathcal{G}$, a transversal projection $\beta : V^{(d)} \longrightarrow V^{(d')}$, and a sequence of transformations (7). The *tight monomial algebra* attached to the sequence (7) is a monomial algebra supported on the exceptional locus, say

$$\mathcal{M}_r W^s = \mathcal{O}_{V_r^{(d')}}[I(H_1)^{h_1} \ldots I(H_r)^{h_r} W^s], \tag{10}$$

where the exponents h_i are such that

$$\frac{h_i}{s} = \text{H-ord}_{i-1}^{(d-1)}(\mathcal{G}_{i-1})(\xi_{Y_{i-1}}) - 1.$$

Here $\xi_{Y_{i-1}}$ denotes the generic point of Y_{i-1}, the center of the blow-up.

The real strength of the tight monomial algebra appears in the statement of the following theorem.

Theorem 2 [8, Theorem 6.6] *Fix a differential Rees algebra $\mathcal{G}$ and a transversal projection $\beta : V^{(d)} \longrightarrow V^{(d')}$. Then the following inequalities hold for any sequence*

as (7)*:*

$$\mathrm{ord}(\mathcal{M}_r W^s)(\beta_r(x)) \leq \text{H-ord}_r^{(d')}(\mathcal{G}_r)(x) \leq \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\beta_r(x))$$

and all $x \in \mathrm{Sing}(\mathcal{G}_r)$*. In other words, the upper semi-continous functions* $\mathrm{ord}(\mathcal{M}_r W^s)$ *and* $\mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)$ *are lower and upper bounds, respectively, of the function* $\text{H-ord}_r^{(d')}(\mathcal{G}_r)$*.*

Moreover, if equality holds in the left-hand side of the previous inequalities, then the sequence (7) *can be extended to a resolution of* $\mathcal{G}$*.*

The second half of the previous theorem indicates that the resolution of the Rees algebra $\mathcal{G}$ can be achieved if some suitable numerical conditions hold.

In Sect. 4 it is shown that such numerical conditions can be attained if $\mathcal{G}$ is a Rees algebra attached to the resolution of a 2-dimensional scheme, leading to a resolution of 2-dimensional schemes *a la Hironaka* (by successively blowing-up along smooth centers included in the highest Hilbert-Samuel stratum).

3.2 Explicit computation of the H-functions

In our previous discussion we have treated some properties of H-functions. Here we indicate how to compute explicitly $\text{H-ord}^{(d')}(\mathcal{G})(x)$ at points $x \in \mathrm{Sing}(\mathcal{G})$. Further details about this computation can be found in [8,9].

We first address the case $d' = d - 1$. So assume $\tau_{\mathcal{G},x} \geq 1$ for any $x \in \mathrm{Sing}(\mathcal{G})$. To pave the way to the definition of the H-function in this context, we define, locally at a point $x \in \mathrm{Sing}(\mathcal{G})$ a concept of *p-presentation of* $\mathcal{G}$, say

$$p\mathcal{P}(\beta, z, f_{p^e}(z)). \tag{11}$$

These data consist of:

1. a transversal projection $\beta : V^{(d)} \longrightarrow V^{(d-1)}$ (see Definition 1),
2. a global function on $V^{(d)}$, say z, so that $\{z = 0\}$ defines a section of β (i.e., $\{dz\}$ is a basis of the module of β-differentials, say Ω^1_β), and
3. a monic polynomial of order p^e, say

$$f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z],$$

   where each a_i is a global function on $V^{(d-1)}$, and p is the characteristic of the underlying perfect field k.

In [8, Proposition 2.11] it is proved that such data can always be locally defined, when $\mathcal{G}$ is β-differential. This last requirement imposed no serious conditions as we may assume that $\mathcal{G}$ is a differential algebra. Moreover, it is shown that $\mathcal{G}$ and the graded algebra

$$\mathcal{O}_{V^{(d)}}[f_{p^e}(z)W^{p^e}, \Delta_z^j(f_{p^e}(z))W^{p^e-j}]_{1\leq j\leq p^e-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}) \tag{12}$$

have the same integral closure.

The previous Δ_z^j are β-differential operators defined in terms of the Taylor morphism in the following manner:

Consider the morphism of S-algebras $Tay : S[Z] \longrightarrow S[Z, T]$, defined by setting $Tay(Z) = Z + T$ (Taylor expansion). Here

$$Tay(f(Z)) = f(Z + T) = \sum \Delta^r(f(Z))T^r, \tag{13}$$

and these operators $\Delta^r : S[Z] \longrightarrow S[Z]$ are defined by this morphism. It is well known that $\{\Delta^0, \Delta^1, \ldots, \Delta^r\}$ is a basis of the free module of S-differential operators of order r. The same applies here for $\mathcal{O}_{V^{(d-1)}}[z]$: the set $\{\Delta_z^0, \Delta_z^1, \ldots, \Delta_z^r\}$ consists of differential operators of order r over $V^{(d-1)} \times \mathbb{A}^1$. Moreover, as the smoothness of β ensures that $V^{(d)}$ is étale over $V^{(d-1)} \times \mathbb{A}^1$, the previous set also generates $Diff_\beta^r$, the β-linear differential operators of order r (for $\beta : V^{(d)} \longrightarrow V^{(d-1)}$).

Remark 2 Suppose given a p-presentation of $\mathcal{G}$ as in (11), together with a diagram (9) (with $d' = d - 1$). Then, it is proved in [8, 6.2] that there is natural lifting of $p\mathcal{P}$ to p-presentations of $\mathcal{G}_i$, say $p\mathcal{P}_i(\beta_i, z_i, f_{p^e}^{(i)}(z_i))$, for $i = 1, \ldots, r$. Here $\beta_i : V_i^{(d)} \longrightarrow V_i^{(d-1)}$ are the smooth morphisms in (9), and $z_i = 0$ define a β_i-transversal section.

3.3

We now give a first step towards the definition of the H-functions in (8). Fix a p-presentation of $\mathcal{G}$, say $\mathcal{P}(\beta, z, f_{p^e}(z))$, where here $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e}$. Define the *slope of $p\mathcal{P}$ at $y \in V^{(d-1)}$* as

$$Sl(p\mathcal{P})(y) = \min\left\{\frac{\nu_y(a_{p^e})}{p^e}, \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)\right\} \in \mathbb{Q}_{\geq 0}. \tag{14}$$

We now present a definition that will lead us to the precise value of the H-function in Theorem 3: A p-presentation is *well-adapted to $\mathcal{G}$ at $y \in V^{(d-1)}$* when

(i) $Sl(p\mathcal{P})(y) = \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$, or

(ii) $Sl(p\mathcal{P})(y) = \frac{\nu_y(a_{p^e})}{p^e}$ and $In_y(a_{p^e}) \in gr(\mathcal{O}_{V^{(d-1)},y})$ is not a p^eth power.

Remark 3 Fix a p-presentation $p\mathcal{P}(\beta, z, f_{p^e})$ locally at a point $x \in \operatorname{Sing}(\mathcal{G})$. One can easily modify z and $f_{p^e}(z)$ so as to obtain a new p-presentation, say $p\mathcal{P}'(\beta, z', f'_{p^e}(z'))$, which is well-adapted at $\beta(x)$ (see [8, Section 5]).

Theorem 3 *Fix a Rees algebra $\mathcal{G}$. The H-function, say*

$$\text{H-ord}^{(d-1)}(\mathcal{G}) : \operatorname{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}_{>0},$$

is defined by

$$\text{H-ord}^{(d-1)}(\mathcal{G})(x) = Sl(p\mathcal{P})(\beta(x)) \tag{15}$$

where $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$ *is any* p*-presentation of* $\mathcal{G}$*, which is well-adapted at* $\beta(x)$*. This rational value is independent of any choice as long as* $p\mathcal{P}$ *is well-adapted at* $\beta(x)$*.*

Remark 4 The value H-ord$^{(d-1)}(\mathcal{G})(x)$ in (15) is given by the slope of a well adapted p-presentation $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$. Note that the value of the slope in (14) relies only in the constant coefficient of $f_{p^e}(z)$, say a_{p^e}, and the order of the elimination algebra. This indicates that the contribution of the intermediate coefficients of $f_{p^e}(z)$, say $a_i (1 \leq i \leq p^e - 1)$, is somehow encoded by the constant coefficient a_{p^e} and the elimination algebra.

This resembles an expected (but never proved) behavior of resolution problems in positive characteristic: the reduction to the so called purely inseparable polynomials, namely those of the form $f_{p^e}(z) = z^{p^e} + a_{p^e}$.

In the previous discussion it is assumed that if $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$ is a presentation, then $\mathcal{G}$ is a relative β-differential algebra. This last condition is automatically guaranteed as we assume that $\mathcal{G}$ is an absolute differential algebra, or a transform of an absolute differential algebra.

Remark 5 Fix a differential Rees algebra $\mathcal{G}$, a sequence of transformations (5) and a p-presentation $p\mathcal{P}$ of $\mathcal{G}$. Remark 2 enables us to define p-presentations of $\mathcal{G}_i$, say $p\mathcal{P}_i = p\mathcal{P}_i(\beta_i, z_i, f^{(i)}_{p^e}(z_i))$. Remark 3 indicates how to make use of $p\mathcal{P}_i$ to define a function

$$\text{H-ord}_i^{(d-1)}(\mathcal{G}_i) : \text{Sing}(\mathcal{G}_i) \longrightarrow \mathbb{Q}_{>0},$$

for each index $i = 0, \ldots, r$.

3.4

Now we address the definition of the H-functions in (8) for the general case of arbitrary d' $(1 \leq d' \leq d)$. Set ℓ so that $\tau_{\mathcal{G},x} \geq d - d' = \ell$ at any closed point $x \in \text{Sing}(\mathcal{G})$. Let $\beta : V^{(d)} \longrightarrow V^{(d')}$ be a transversal projection. The problem, in this general case, is to define a notion of p-presentations, with similar properties as that in (11) for de case $d' = d - 1$ (i.e., $\ell = 1$). The extension to this general case is not straightforward, and has been treated in [9, Theorem 5.12]. There it is proved that, if $\tau_{\mathcal{G},x} \geq \ell$ at each closed point $x \in \text{Sing}(\mathcal{G})$, and $\mathcal{G}$ is a differential algebra or a transformation of a differential algebra, then there is a *p-presentation*, say

$$p\mathcal{P} = p\mathcal{P}(\beta, z_1, \ldots, z_\ell, f_{p^{e_1}}(z_1), \ldots, f_{p^{e_\ell}}(z_\ell)),$$

for which $\mathcal{G}$ has the same integral closure as

$$\mathcal{O}_{V^{(d)}}[f_{p^{e_i}}(z_i)W^{p^{e_i}}, \Delta^{j_i}_{z_i}(f_{p^{e_i}}(z_i))W^{p^{e_i}-j_i}]_{1\leq j_i\leq p^{e_i}-1,\ 1\leq i\leq \ell} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}).$$

Moreover the p-presentation is defined so that

- $\{z_1 = \cdots = z_\ell = 0\}$ is a section of the smooth morphism β (i.e., $\{dz_1, \ldots, dz_\ell\}$ is a basis of Ω^1_β),

– $f_{n_i}(z_i)W^{n_i} \in \mathcal{G}_r$ for each index $i = 1, \ldots, \ell$, and the polynomials are of the form:

$$\begin{aligned} f_{p^{e_1}}(z_1) &= z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-\ell)}}[z_1], \\ &\vdots \\ f_{p^{e_\ell}}(z_\ell) &= z_\ell^{p^{e_\ell}} + a_1^{(\ell)} z_\ell^{p^{e_\ell}-1} + \cdots + a_{p^{e_\ell}}^{(\ell)} \in \mathcal{O}_{V^{(d-\ell)}}][z_\ell], \end{aligned} \tag{16}$$

Let us stress here that all coefficients are in dimension $d - \ell$, i.e., $a_{j_i}^{(i)} \in \mathcal{O}_{V^{(d-\ell)}}$. This fact enables us to extend the previous results in Sect. 3.2 (the definitions of slope and of a presentation well-adapted at a point). This will enable us to compute the values of the H-functions in the general setting. Namely to the extension of Theorem 3.

Firstly fix a p-presentation $p\mathcal{P}(\beta, z_1, \ldots, z_\ell, f_{p^{e_1}}(z_1), \ldots, f_{p^{e_\ell}}(z_\ell))$, with $f_{p^{e_i}}(z_i) \in \mathcal{O}_{V^{(d')}}[z_i]$, as in (14). Let the *slope of* $p\mathcal{P}$ *at* $y \in V^{(d')}$ be

$$Sl(p\mathcal{P})(y) = \min_{1 \le i \le \ell} \left\{ \frac{\nu_y(a_{p^{e_i}}^{(i)})}{p^{e_i}}, \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \right\} \in \mathbb{Q}_{\ge 0}.$$

Now, we define a $p\mathcal{P}$ presentation to be *well-adapted to* $\mathcal{G}$ *at* $y \in V^{(d')}$ when

(i) $Sl(p\mathcal{P})(y) = \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$, or

(ii) $Sl(p\mathcal{P})(y) = \frac{\nu_y(a_{p^{e_i}})}{p^{e_i}} < \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ and $In_y(a_{p^{e_i}})$ is not a p^{e_i}th power, for some $i \in \{1, \ldots, \ell\}$.

Also Remark 3 extends to this setting: Once we fix $p\mathcal{P}$ locally at a point $x \in \operatorname{Sing}(\mathcal{G})$, then it can be modified into a new one which is well-adapted at $\beta(x)$ (see Sect. 3.3).

Theorem 4 *Fix a Rees algebra* $\mathcal{G}$ *and assume that* $\tau_{\mathcal{G},x} \ge d - d' = \ell$ *for any* $x \in \operatorname{Sing}(\mathcal{G})$. *The H-function*

$$\text{H-ord}^{(d')} : \operatorname{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}_{>0},$$

is defined by

$$\text{H-ord}^{(d')}(\mathcal{G})(x) = Sl(p\mathcal{P})(\beta(x)) = \min_{1 \le i \le \ell} \left\{ \frac{\nu_{\beta(x)}(a_{p^{e_i}}^{(i)})}{p^{e_i}}, \operatorname{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x)) \right\},$$

where now $p\mathcal{P} = p\mathcal{P}(\beta : V^{(d)} \longrightarrow V^{(d')}, z_1, \ldots, z_\ell, f_{p^{e_1}}, \ldots, f_{p^{e_\ell}})$ *is a p-presentation which is assumed to be well-adapted at* $\beta(x)$. *The value* $\text{H-ord}^{(d')}(\mathcal{G})(x)$ *is independent of any choice (i.e., independent of the p-presentation as long it is well-adapted at* $\beta(x)$*).*

Proof See [9, Theorem 5.12 and Definition 5.13]. □

3.5

The previous theorem indicates how to compute the values of the H-function along $\mathrm{Sing}(\mathcal{G})$. In fact, given a p-presentation of $\mathcal{G}$, it is not difficult to modify it so that at a given point $x \in \mathrm{Sing}(\mathcal{G})$, it is well-adapted at $\beta(x)$. The theorem says, of course, that the value $\text{H-ord}^{(d')}(\mathcal{G})(x)$ is independent of the choice of the p-presentation.

In [9] further properties of these functions are studied, which are related to Hironaka's notion of invariant. This notion has a very precise meaning in the context of resolution of singularities. Roughly speaking, once $V^{(d)}$ is fixed, Hironaka defines an equivalent relation in the class of Rees algebras over $V^{(d)}$. This relation is defined so that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent, then $\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$, and given $x \in \mathrm{Sing}(\mathcal{G}_1)$, also $\tau_{\mathcal{G}_1,x} = \tau_{\mathcal{G}_2,x}$. In [9] it is proved that $\text{H-ord}^{(d')}(\mathcal{G}_1)(x) = \text{H-ord}^{(d')}(\mathcal{G}_2)(x)$ for any point $x \in \mathrm{Sing}(\mathcal{G}_1)$. Namely, that the value $\text{H-ord}^{(d')}(\mathcal{G}_1)(x)$ is an *invariant*.

This last fact will ultimately guarantee that the functions $\text{H-ord}^{(d')}(\mathcal{G})$ will be useful for the problem of resolution of singularities. In fact, resolution of singularities reduces to resolution of Rees algebras, but only if the latter is defined in a way that two equivalent Rees algebras undergo the same resolutions.

4 Embedded resolution of 2-dimensional schemes

Here we address the proof of embedded resolution of 2-dimensional schemes. We show here how our invariants lead to the resolution of a hypersurface embedded in a 3-dimensional smooth scheme. The extension of the resolution of the hypersurface case, treated here, to that of arbitrary 2-dimensional schemes is not straightforward. It is a particular feature of the invariants introduced in the previous sections, studied in [9], which relies essentially on Sect. 3.4 and Theorem 4.

4.1 Stratification of the exceptional locus

We shall take as starting point a diagram (6), in the setting of Theorem 1, for $d = 3$. In which case $V_r^{(d-1)} = V_r^{(2)}$ is a 2-dimensional smooth scheme. Recall that Theorem 1 enables us to assume that the elimination algebra over $V_r^{(2)}$ is monomial. Namely, that

$$(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s,$$

for some integers $\alpha_i \geq 0$. Assume, in addition, that the tight monomial algebra, defined in (10), is of the form

$$\mathcal{M}_r W^s = I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s \quad \text{with } 0 \leq h_i < s. \tag{17}$$

We shall indicate below that the condition $0 \leq h_i < s$ can be achieved by blowing-up permissible centers of codimension 2 in the smooth 3-dimensional scheme $V_r^{(3)}$. In this case, the tight monomial algebra $\mathcal{M}_r W^s$ subject to this condition is said to be *reduced*.

This last assumption guarantees that $\mathrm{Sing}(\mathcal{M}_r W^s) \subset V_r^{(2)}$ has no components of codimension 1, and we claim that neither does $\beta_r(\mathrm{Sing}(\mathcal{G}_r))$, which is, therefore, a finite set of closed points.

Throughout this section, we will always assume that the tight monomial algebra is reduced; that is, every quadratic transformation will be followed by a finite sequence of blow-ups at centers of codimension 2, so as to guarantee that the new tight monomial algebra is reduced.

Locally at $x \in \mathrm{Sing}(\mathcal{G}_r)$ one can fix a p-presentation $p\mathcal{P}(\beta_r, z, f_{p^e}(z))$ well-adapted to $\mathcal{G}_r$ at $\beta_r(x)$, here, $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \in \mathcal{O}_{V_r^{(2)}}[z]$.

The p-presentation $p\mathcal{P}$ can be chosen so as to have the following two properties (see [8, Proposition 5.9 and Definition 3.10]):

- $(\mathcal{R}_{\mathcal{G},\beta})_r \subset \mathcal{M}_r W^s$, and
- $a_j W^j \in \mathcal{M}_r W^s$ for any $j = 1, \ldots, p^e$.

This already proves the claim: if $h_i \geq s$ for some H_i containing x, then $\langle z, I(H_i)\rangle$ defines a smooth component of $\mathrm{Sing}(\mathcal{G}_r)$ of codimension 2 in $V_r^{(3)}$. This monoidal transformation induced the identity map over $V_r^{(2)}$. However h_i drops to $h_i - s$ after blowing up at such component.

So we will assume here that $\mathrm{Sing}(\mathcal{G}_r)$ is a finite set of closed points, and we fix the notation $\mathbf{x} = \beta_r(x)$ along this section.

Recall that

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \min\left\{\frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e}, \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})\right\}.$$

The previous two properties ensures that

$$\mathrm{ord}(\mathcal{M}_r W^s)(\mathbf{x}) \leq \text{H-ord}^{(d-1)}(\mathcal{G}_r)(x). \tag{18}$$

As $(\mathcal{R}_{\mathcal{G},\beta})_r$ is, by assumption, monomial, Proposition 2 (2) ensures that the singular locus of $\mathcal{G}_r$ is entirely included in a union of the exceptional hypersurfaces. We begin by fixing a suitable stratification of the union of the exceptional hypersurfaces in $V_r^{(2)}$, and then we shall construct a stratification on the successive quadratic transformations defined over $V_r^{(2)}$. Here the quadratic transformations are defined canonically, with centers on the finitely many points of the singular locus. This lead to the following definition:

Definition 3 An isolated closed point $x \in \mathrm{Sing}(\mathcal{G}_r)$ is said to be *good* (or green) if

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}).$$

The point x is said to be *bad* (or purple) if

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e} < \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}).$$

The previous definition for a point $x \in \mathrm{Sing}(\mathcal{G}_r)$ leads to a coloring of singular points of $\beta_r(\mathrm{Sing}(\mathcal{G}_r))$ which is a finite set of points in $V_r^{(2)}$. We will say that $\mathbf{x} = \beta_r(x)$ is good (bad) if x is good (bad). In the same manner, exceptional hypersurfaces will be distinguished in terms of the exponents that arise in $\mathcal{M}_r W^s$ and $(\mathcal{R}_{\mathcal{G},\beta})_r$:

Definition 4 We will say that the exceptional hypersurface H_i is *good* (or green) if $h_i = \alpha_i$ and H_i is *bad* (or purple) if $h_i < \alpha_i$.

New quadratic transformations over $V_r^{(2)}$ will be defined, this will introduce new exceptional hypersurfaces, each of which will be either good or bad.

Remark 6 Note from Definition 2 that the closed point $\mathbf{x} \in \beta_r(\mathrm{Sing}(\mathcal{G}_r))$ is good (bad) if and only if the exceptional hypersurface introduced by the quadratic transformation at $\mathbf{x}$ is good (bad).

Our stratification will be defined only along the union of bad hypersurfaces. Singularities lying entirely on the good locus are, to some extend, friendly singularities. In fact, if a closed point x is only included in good hypersurfaces, then $\mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}) = \mathrm{ord}(\mathcal{M}_r W^s)(\mathbf{x})$ and we claim that the transversal parameter z of the p-presentation $p\mathcal{P}$ can be chosen so as to be a hypersurface of maximal contact. To this end, in [8, Theorem 8.5], using (18) and the two properties of $\mathcal{M}_r W^s$ mentioned above, it is proved that

$$\mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}) = \mathrm{ord}(\mathcal{M}_r W^s)(\mathbf{x}) \implies (\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{M}_r W^s$$

in a neighborhood of $\mathbf{x}$. In such case, as $a_j W^j \in \mathcal{M}_r W^s$, we conclude that $a_j W^j \in (\mathcal{R}_{\mathcal{G},\beta})_r \subset \mathcal{G}_r$ for $j = 1, \ldots, p^e$. Check now that zW fulfills the integral condition $\lambda^{p^e} + (a_1 W^1)\lambda^{p^e-1} + \cdots + (a_{p^e} - f_{p^e}(z))W^{p^e} = 0$, which guarantees $zW \in \mathcal{G}_r$. This ensures that $z = 0$ defines a hypersurface of maximal contact.

4.2 Stratification at level r

The image of $\mathrm{Sing}(\mathcal{G}_r)$ is a finite set of points. We will first define a stratification only locally around points of this image. In Sects. 4.3 and 4.4 we shall indicate how to define the stratification after blowing-up these points, and moreover after applying a sequence of quadratic transformations.

According to the previous discussion, we only draw attention to those points of $\mathrm{Sing}(\mathcal{G}_r)$ with images in the union of bad lines of $V_r^{(2)}$. As these points are isolated, we may assume, after restriction, that the point is unique. The following two situations can arise:

- There exists a unique hypersurface H_1 so that $\mathbf{x} \in H_1$ (in which case H_1 must be bad).
- $\mathbf{x}$ is an intersection point of two hypersurfaces, say H_1 and H_2 (at least one of them bad).

The stratification at $V_r^{(2)}$ will be defined as follows: In the first case, we define a unique stratum, which is an affine line containing $\mathbf{x}$, say $\mathbb{A}^1 \subset \mathbb{P}^1 = H_1$. In the second case, we stratify $H_1 \cup H_2$ in two strata: one affine line $\mathbb{A}^1$ so that $\mathbf{x} \in \mathbb{A}^1 \subset H_1$ and the affine line $H_2 \setminus \{\mathbf{x}\}$.

As we are going to blow-up along the singular locus, further quadratic transformations will be defined over $V_r^{(2)}$. This will introduce new exceptional components of

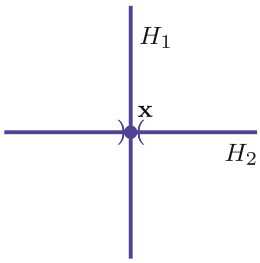
the form $H_j = \mathbb{P}^1$. We indicate in Sects. 4.3 and 4.4 how to stratify the union of new components which are bad (Definition 4). Each stratum, will be either

* an affine line $\mathbb{A}^1$, or
* a point.

A zero-dimensional stratum, a point, will be called an *infinitesimal stratum*. These zero-dimensional strata will always arise as intersection of two exceptional hypersurfaces: one good and the other bad. However, the intersection point of a bad and a good hypersurface is not necessarily a zero-dimensional stratum. These strata will appear and treated in detail in Sect. 4.3 Case C).

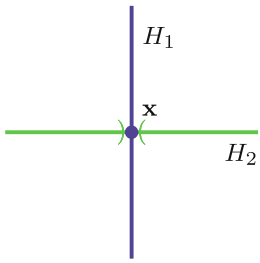
4.3 Stratification and quadratic transformations

Once a stratification is fixed, we blow-up at a singular point, and then a new stratification will be defined. In doing that, we will follow a rule: *the strict transform of an already defined stratum is a stratum*. We therefore need to establish a criterion to stratify points along the exceptional hypersurface, every time we blow up a point.
Case A) In this case we assume that a quadratic transformation is defined at a point **x**, which is bad, and is not a zero-dimensional stratum. This situation can occur within the following sub-cases:



Picture A1

Picture A2

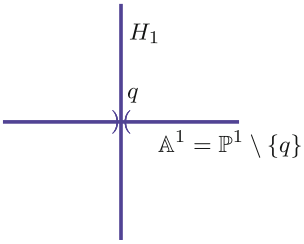


Picture A3

Here, the stratification of the quadratic transformation at **x** will be defined in a way that will not give rise to a zero-dimensional stratum.

Along this section we agree that every time we blow-up a point, the horizontal line will denote the new exceptional component.

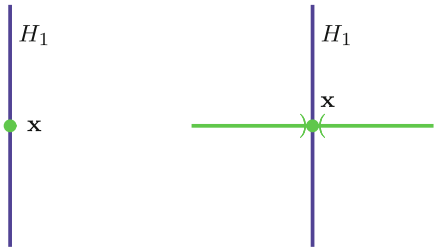
Let H_1 denote again the strict transform of H_1. The new stratification along points of the exceptional component is defined as follows:



New A-stratification after the quadratic transformation

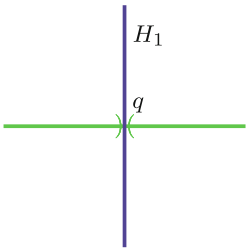
In this case, a unique 1-dimensional stratum $\mathbb{A}^1$ is introduced after the quadratic transformation. This new affine stratum is defined as $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{q\}$, where here $q = H_1 \cap \mathbb{P}^1$. The stratum $H_2 \backslash \mathbf{x}$, in the pictures A1) (or in A3)), defines a new stratum after the quadratic transformation simply by taking its strict transform. Recall that we require that the strict transform of an already defined stratum to be a stratum.

Case B) Here, we study the case of a quadratic transformation at a good point **x** which is only in one bad hypersurface H_1, and is not a zero-dimensional stratum. This can occur in the following sub-cases:



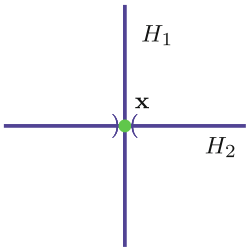
Picture B1 Picture B2

After the quadratic transformation at **x**, the new exceptional hypersurface is good. We define the new stratification by taking the strict transform of the previous stratum defined over H_1:



New B-stratification after the quadratic transformation

Case C) In this case, the quadratic transformation is defined at a good point **x** which is an intersection of two bad lines H_1 and H_2. The point **x** is in the stratum defined by H_1, at least locally at this point. In particular, the point is not a zero-dimensional stratum.



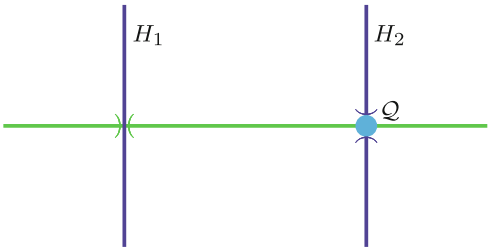
Picture C

The new (horizontal) exceptional line is good and the stratification is defined by:

– The strict transform of the previous strata.

– A new zero-dimensional stratum defined as the intersection of H_2 and the new exceptional good line.

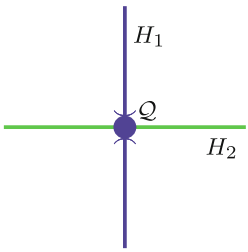
Note that in this case, depicted below, we introduce a zero-dimensional stratum $\mathcal{Q}$:



New C-stratification after the quadratic transformation

4.4 Quadratic transformations at an infinitesimal statum

Case D1) Here we study a quadratic transformations at a bad point $\mathcal{Q}$, which is in addition an zero-dimensional stratum:

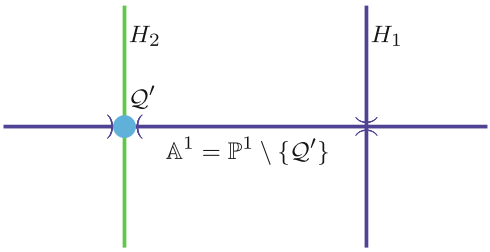


Picture D1

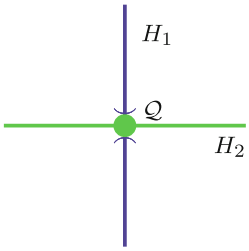
The new stratification after the quadratic transformation is defined by

(a) The strict transforms of the previous strata.
(b) The stratification of the new exceptional line $\mathbb{P}^1$ as the union of the infinitesimal stratum $\mathcal{Q}' = \mathbb{P}^1 \cap H_2$ and the 1-dimensional stratum $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\mathcal{Q}'\}$.

This new stratification is depicted as follows:

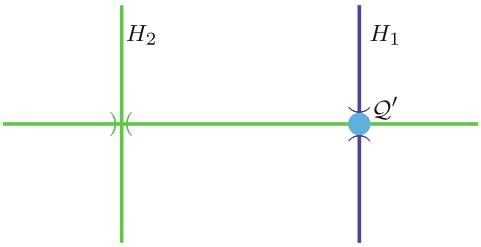


Case D2) This is the case in which the center of the quadratic transformation is a good point $\mathcal{Q}$, which is also an infinitesimal stratum. Namely,



Picture D2

The new stratification is represented by



New D2-stratification after the quadratic transformation

So only a new zero-dimensional stratum $\mathcal{Q}'$ is introduced at the bad locus.

4.5 Definition of the local data and local invariants

Once the stratification has been fixed, notions of *local data* and *local invariants* will be introduced at every isolated point $x \in \mathrm{Sing}(\mathcal{G}_{r'})$ (or equivalently, to $\mathbf{x} \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$. Here, the local data will assign to each 1-dimensional stratum $\mathbb{A}^1 = \mathrm{Spec}(k[y])$ a polynomial $g(y)$ in $k[y]$, and to each zero-dimensional stratum $\mathcal{Q}$ an element of $\mathcal{O}_{V^{(2)}_{r'},\mathcal{Q}}$. Finally, local invariants will be defined in terms of these local data.

• **Local data in case $\mathbf{x} \in \mathbb{A}^1$** Here we take an isolated point $\mathbf{x} \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ in a 1-dimensional stratum $\mathbb{A}^1$ (included in the union of the bad hypersurfaces, and in particular located in a bad hypersurface, say H_1). Let $p\mathcal{P} = p\mathcal{P}(\beta_{r'}, z, f_{p^e})$ be a well-adapted p-presentation at $\mathbf{x}$, where

$$f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \in \mathcal{O}_{V^{(2)}_{r'},\mathbf{x}}[z]. \tag{19}$$

Fix a regular system of parameters $\{x, y\}$ at $\mathcal{O}_{V^{(2)}_{r'},\mathbf{x}}$, so that $\{x = 0\}$ defines $\mathbb{A}^1$ locally at $\mathbf{x}$.

As H_1 is a bad hypersurface, it follows that $\text{H-ord}^{(2)}(\mathcal{G}_{r'})(\xi_{H_1}) = \frac{\nu_{\xi_{H_1}}(a_{p^e})}{p^e} < \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_{r'})(\xi_{H_1})$, where ξ_{H_1} is the generic point of H_1. Note that, in particular, $p^e \cdot \text{H-ord}^{(2)}(\mathcal{G}_{r'})(\xi_{H_1}) \in \mathbb{Z}_{\geq 0}$. Let us denote this integer by $\ell = p^e \cdot \text{H-ord}^{(2)}(\mathcal{G}_{r'})(\xi_{H_1})$. There is a factorization of the form

$$a_{p^e} = x^{\ell}(g(y) + x\Omega(x, y)), \tag{20}$$

where $\Omega(x, y) \in \mathcal{O}_{V^{(2)}_{r'},\mathbf{x}}$, and the exponent $\ell < p^e$ by (17). Note here that H_1 is an exceptional hypersurface introduced by previous quadratic transformation, it can be checked that local coordinates x and y can be chosen so that $g(y)$ is indeed a polynomial in $k[y]$.

In this case, the *local data* at $\mathbf{x}$ will be defined as the pair $(\frac{\ell}{p^e}, g(y)W^{p^e})$.

Definition 5 Fix, with the setting as above, a point $\mathbf{x} \in \mathbb{A}^1$, and the local data $\left(\frac{\ell}{p^e}, g(y)W^{p^e}\right)$. The *order of* g *at* $\mathbf{x}$, say $\mathrm{ord}_{\mathbf{x}}(g)$, is defined as follows:

- If $\ell \neq 0$, $\mathrm{ord}_{\mathbf{x}}(g)$ is the usual order of g at $\mathcal{O}_{\mathbb{A}^1,\mathbf{x}}$.
- If $\ell = 0$, $\mathrm{ord}_{\mathbf{x}}(g)$ is the smallest power of y, which appears in the Taylor expansion of $g(y)$ at the point, that is not a p^eth power.

• **Local data in case $x = \mathcal{Q}$ is an infinitesimal statum** Set local coordinates so that $(\mathcal{R}_{\mathcal{G},\beta})_{r'} = x^a y^b W^s$, assuming that $H_1 = \{x = 0\}$ denotes the good hypersurface through the point. Now, $\text{H-ord}^{(2)}(\mathcal{G}_{r'})(\xi_{H_1}) = \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{H_1}) = \frac{a}{s}$. Define the *local data* as the pair $(\frac{a}{s}, g(y)W^s)$, where $g(y) = y^b$.

Definition 6 (*Local invariants*). Fix a point $\mathbf{x} \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ with local data, say $(\frac{m}{n}, g(y)W^t)$, for a suitable integer t. The *local invariant* assign to $\mathbf{x}$ is defined as:

- If $\mathbf{x} \in \mathbb{A}^1$, then $\mathrm{inv}(\mathbf{x}) = \frac{\mathrm{ord}_{\mathbf{x}}(g)}{p^e}$ (in this case $t = p^e$).
- If $\mathbf{x} = \mathcal{Q}$, then $\mathrm{inv}(\mathbf{x}) = \frac{\nu_{\mathbf{x}}(g)}{s}$ (here $t = s$, and $g(y)W^t = y^b W^s$).

4.6 Invariants and transformations

We now study the behavior of the previous invariants under quadratic transformations, taking into account the distinction in the cases presented in Sect. 4.3.

Case A) Local coordinates $\{x, y\}$ are chosen locally at $\mathcal{O}_{V^{(2)}_{r'},\mathbf{x}}$ so that locally $\mathbb{A}^1 = \{x = 0\}$. In this case, as $\mathbf{x}$ is assumed to be bad, and a well-adapted p-presentation can be chosen so as

$$\text{H-ord}^{(2)}(\mathcal{G}_{r'})(x) = \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e} < \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_{r'})(\mathbf{x}).$$

In addition, the initial form of a_{p^e} in $\mathrm{Gr}_{\mathbf{x}}(\mathcal{O}_{V^{(2)}_{r'}})$ is not a p^eth power.

The objective is to define local invariants after the quadratic transformation at $\mathbf{x}$. This leads to:

1. The definition of local data and invariants at the strict transform of H_1.
2. The definition of local data and invariants at points in the new stratum $\mathbb{A}^1$ (included in the exceptional component).

(1) *Local invariants at the strict transform of* $\mathbb{A}^1 \subset H_1$.

Local factorization at $\mathbf{x}$ is $a_{p^e} = x^\ell(g(y) + x\Omega(x, y))$. At $\mathbf{x}_1$, the origin of the U_y-chart, (with local coordinates $x_1 = \frac{x}{y}$, $y_1 = y$), the factorization is given by

$$a^{(1)}_{p^e} = x_1^\ell y_1^{\ell - p^e}(g(y_1) + x_1 y_1 \Omega').$$

In this case, the new local data, say $\left(\frac{\ell_1}{p^e}, g_1(y_1)W^{p^e}\right)$, is defined by setting $g_1(y_1) = y_1^{\ell-p^e} \cdot g(y_1)$ and $\ell_1 = \ell$.

As we assume that the tight monomial algebra $\mathcal{M}_{r'}W^s$ is reduced (see Sect. 4.1), then $\ell < p^e$, and hence

$$\mathrm{ord}_{\mathbf{x}_1}(g_1) < \mathrm{ord}_{\mathbf{x}}(g). \tag{21}$$

(2) *Invariants along the new stratum* $\mathbb{A}^1$.

Set $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\mathbf{x}_1\} = \mathrm{Spec}(k[Y])$ to be the 1-dimensional stratum defined as in Case A) in Sect. 4.3. We first assign a polynomial $g_1(Y)$ to $\mathbb{A}^1$.

Locally at $\mathbf{x}$, the 1-dimensional stratum containing $\mathbf{x}$ was defined by $\{x = 0\}$. The new stratum along the exceptional hypersurface, the affine line $\mathbb{A}^1$, is the intersection of the exceptional hypersurface with the open chart U_{x_1} (with coordinates $x_1 = x$ and $y_1 = \frac{y}{x}$).

Set $\mathrm{Gr}_{\mathbf{x}}(\mathcal{O}_{V_{r'}^{(2)}}) = k'[X, Y]$ with $X = \mathrm{In}_{\mathbf{x}}(x)$ and $Y = \mathrm{In}_{\mathbf{x}}(y)$, and set $\mathrm{In}_{\mathbf{x}}(a_{p^e}) = \sum_{i+j=d} \lambda_{i,j} X^i Y^j$, the initial form of a_{p^e} at $\mathbf{x}$, where here $d = \nu_{\mathbf{x}}(a_{p^e})$. Finally, define $\widetilde{g}(Y) = \mathrm{In}_{\mathbf{x}}(a_{p^e})|_{X=1}$.

Fix a point in $\mathbb{A}^1$, and after suitable change of coordinates $y_1 = Y + \alpha$, assume that y_1 vanishes at such point. Now set $g_1(y_1) = \widetilde{g}(Y + \alpha)$. Then, the local invariant is $(\frac{\ell_1}{p^e}, g_1(y_1))$, with $\frac{\ell_1}{p^e} = \text{H-ord}^{(2)}(\mathcal{G}_{r'+1})(\xi_{H'})$, where $\xi_{H'}$ is the generic point of the new exceptional hypersurface, say H'. The previous change of variables does not affect the degree of the polynomial g_1. Namely $\deg(\widetilde{g}(Y)) = \deg(g_1(y_1))$.

Lemma 1 (Abhyankar's trick) *Fix a point* $x \in \mathrm{Sing}(\mathcal{G}_{r'})$ *(or equivalently* $\mathbf{x} \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$*) and assume that the setting is as above. For any point* $\mathbf{x}' \in \mathbb{A}^1 \subset H'$,

$$\mathrm{ord}_{\mathbf{x}'}(g_1) \leq \mathrm{ord}_{\mathbf{x}}(g). \tag{22}$$

Proof Consider a_{p^e} as a formal power serie in the variables x and y, (20) indicates that it is expressed as a sum of monomials of the form $x^t y^r$, with $t \geq \ell$. Note, in addition, that a monomial of the form $x^\ell y^M$, where $M = \mathrm{ord}_{\mathbf{x}}(g(y))$, appears in such formal expression.

This leads to the following conclusions:

1. $\mathrm{In}_{\mathbf{x}}(a_{p^e}) = \sum_{i \geq \ell} \lambda_{i,d-i} X^i Y^{d-i}$.
2. $d \leq \ell + M = \deg(x^\ell y^M)$.

From where it is inferred that if $\lambda_{i,d-i} \neq 0$, then $d - i \leq M$. In particular, $\sum_{i \geq \ell} \lambda_{i,d-i} Y^{d-i}$ is a polynomial of degree M_1 with $M_1 \leq \mathrm{ord}_{\mathbf{x}}(g) = M$. We **claim** that $\mathrm{ord}_{\mathbf{x}'}(g_1) \leq M_1$, and this would ensure that the inequality (22). The proof of this claim will be addressed in 4.7, it will make use of the following Lemma. □

Lemma 2 *Fix a polynomial* $g(y) \in k[y]$. *Then at any closed point* $\mathbf{x} \in \mathbb{A}^1_k = \mathrm{Spec}(k[y])$*:*

1. $\mathrm{ord}_{\mathbf{x}}(g) \leq \deg(g)$, *where* $\mathrm{ord}_{\mathbf{x}}(g)$ *is the usual order of* g *at* $\mathcal{O}_{\mathbb{A}^1_k,\mathbf{x}}$.

2. *If $g(y) \notin k[y^{p^e}]$, then p^e-$\mathrm{ord}_{\mathbf{x}}(g) \leq \deg(g)$, where p^e-$\mathrm{ord}_{\mathbf{x}}(g)$ denotes the smallest power of y, that is not a p^eth power, which appears in the Taylor expansion of $g(y)$ at the point.*

Proof Let M denote the degree of $g(y)$.

(1) Fix a change of variables $y_1 = y - \alpha$ so that y_1 vanishes at $\mathbf{x}$, then $g(y) = g_1(y_1)$ is also a polynomial of degree M in y_1. Hence, $\mathrm{ord}_{\mathbf{x}}(g) \leq \deg(g)$.
(2) Let $M' \leq M$ be the biggest integer so that $M' \not\equiv 0 \mod p^e$ and that $y^{M'}$ appears in $g(y)$. Consider the Taylor expansion of $g(y)$ as in (13) applied here for $S[Z] = k[y]$. As $\Delta^{M'}(g) \in k \setminus \{0\}$, then the term $y_1^{M'}$ appears at the Taylor development at a fixed point $\mathbf{x}$. Here $y_1 = y - \alpha$ is defined so as to vanish at $\mathbf{x}$. □

4.7

We address now the proof of the claim stated in Lemma 1. Fix notation as in this lemma, where $\mathbf{x} \in \mathrm{Sing}(\mathcal{G}_{r'})$ and H' is the exceptional hypersurface introduced by blowing-up $\mathbf{x}$. The stratum $\mathbb{A}^1 \subset H'$ is defined by $\mathbb{A}^1 = \mathrm{Spec}(k[y_1])$, where $y_1 = \frac{y}{x}$. We consider $g_1(y_1) \in k[y_1]$ to be naturally identified with $\mathrm{In}_{\mathbf{x}}(a_{p^e})|_{x=1} = \sum_{i \geq \ell} \lambda_{i,d-i} y_1^{d-i}$, so $g_1(y_1)$ is a polynomial of degree $M_1 \leq M$. Recall now the notion of invariant attached to a singular point $\mathbf{x}' \in \mathbb{A}^1$ in Definition 5.

Set as before $d = \nu_{\mathbf{x}}(a_{p^e})$, locally at $\mathbf{x}$. We distinguish two cases:

(a) If $d \not\equiv 0 \mod p^e$, then the local data at $\mathbf{x}'$ is $(\ell_1, g_1(y_1))$ with $\ell_1 \neq 0$, and hence $\mathrm{ord}_{\mathbf{x}'}(g_1)$ is the usual order at $\mathcal{O}_{\mathbb{A}^1,\mathbf{x}'}$. Inequality (22) follows from Lemma 2 (1).
(b) If $d \equiv 0 \mod p^e$, then the local data at $\mathbf{x}'$ is $(\ell_1, g_1(y_1))$ with $\ell_1 = 0$. Here $\mathrm{ord}_{x'}(g_1)$ is provided by $p^e - \mathrm{ord}_{\mathbf{x}'}(g_1)$ as in Lemma 2 (2), and the condition $g_1(y_1) \in k[y_1]$ is guaranteed by the fact that $\mathrm{In}_{\mathbf{x}}(a_{p^e})$ is not a p^e-power.

Lemma 3 *If there is a sequence of quadratic transformations at points $\mathbf{x}_0, \mathbf{x}_1 \ldots$, where each $\mathbf{x}_i$ maps to $\mathbf{x}_{i-1}$ and such that:*

1. *Case* A) *occurs at each point $\mathbf{x}_i$, and*
2. $\mathrm{ord}_{\mathbf{x}_i}(g_i) = \mathrm{ord}_{\mathbf{x}_{i-1}}(g_{i-1})$,

then the sequence must be finite.

Proof The existence of an infinite sequence with the previous conditions would contradict the assumption that $\beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ has no 1-codimensional component. In fact, assume that such a sequence does exist. We claim that at the completion at the closed point $\mathbf{x}_0$, there is a smooth curve whose successive strict transforms passes through the sequence $\mathbf{x}_i$. This follows from the assumption that (21) does not occur. This ensures that there is a smooth curve passing through these points, and a smooth curve with this property would be a 1-dimensional component of the singular locus. This contradicts the hypothesis. □

We assume that $\beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ is a finite set of closed points. Every time we fix one such point, there is a unique procedure of quadratic transformations over it. In fact, after applying a quadratic transformation, we reduce to the case $\beta_{r'+1}(\mathrm{Sing}(\mathcal{G}_{r'+1}))$ is

a finite set of closed points by blowing-up the new exceptional component. Lemma 3 ensures that over $\mathbf{x}_0 \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ only finitely many singular points $\mathbf{x}_0, \mathbf{x}_1 \ldots$ can arise, where each $\mathbf{x}_i$ maps to $\mathbf{x}_{i-1}$, and with the condition that Case A) is preserved and equality holds in (22).

Lemma 4 *Fix* $\mathbf{x}_0 \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$*, there is a uniform bound for the length of all possible sequences of quadratic transformations over* $\mathbf{x}_0$ *in the setting of Lemma* 3.

Proof Fix a point $\mathbf{x} \in \beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$ which we may assume to be isolated and within case A). We claim that after a finite sequence of quadratic transformations over this point, the invariants drops at *any exceptional point* $\mathbf{x}_i$ mapping to $\mathbf{x}$.

To this end we first show that after finitely many blow ups any singular point mapping to $\mathbf{x}$, for which condition A) holds and equality occurs at (22), must be in case A2).

To check this claim note first that under the assumption of the equality in (22), case A2) is stable. Namely, if case A2) holds at a point, and case A) holds at a point after the quadratic transformation, and equality holds at (22), then this exceptional point must also be within case A2). On the other hand, assuming that $\mathbf{x}$ is in condition A1) or A3), for which $\mathbf{x}$ is contained in the hypersurface H_2, Lemma 3 ensures that after finitely many quadratic transformations, every point within case A) and for which equality holds, must be in case A2). In fact, otherwise the exceptional hypersurface H_2 would be a component of $\beta_{r'}(\mathrm{Sing}(\mathcal{G}_{r'}))$, in contradiction with our hypothesis. This last assertion follows using same arguments as before.

The previous finite sequence of quadratic transformations over the point $\mathbf{x}$, constructed so as to be in case A2), introduces finitely many new exceptional components, say $H_{n_1}, \ldots, H_{n_t}$. Let the elimination algebra be of the form $(\mathcal{R}_{\mathcal{G},\beta})_{n_t} = I(H_1)^{\alpha_1} \ldots I(H_{n_t})^{\alpha_{n_t}} W^s$, for some integers $\alpha_i \geq 0$. We claim now that after at most $\alpha_1 + \cdots + \alpha_{n_t}$ quadratic transformations, the inequality (22) will be strict at any point mapping to $\mathbf{x}$ which fulfills condition A).

To check this, note first that locally at any point within condition A2), there is a regular system of parameters $\{x, y\}$ so that $(\mathcal{R}_{\mathcal{G},\beta})_{n_t} = x^a W^s$ with $a = \alpha_i$ for some $i \in \{1, \ldots, n_t\}$. Finally, note that a quadratic transformation at each point introduces a new hypersurface, that any exceptional singular point is in case A2), and that $(\mathcal{R}_{\mathcal{G},\beta})_{n_t+1} = x_1^{a-s} W^s$. This can occur only finitely many times. The claim follows now from the inclusion $\beta_j(\mathrm{Sing}(\mathcal{G}_j)) \subset \mathrm{Sing}((\mathcal{R}_{\mathcal{G},\beta})_j)$. □

Case B) In this case, the stratification is defined by the strict transform of the previous stratum. So attention should be drawn only at the unique point q of the strict transform of H_1.

This parallels the situation of case A) (1) and the invariant strictly drops as in (21).
Case C) If a point $\mathbf{x}$ is within case C), then $\mathbf{x}$ is an intersection of two bad exceptional hypersurfaces H_1 and H_2. Moreover, the point $\mathbf{x}$ is good and belongs to a 1-dimensional stratum $\mathbb{A}^1$ included in H_1.

Therefore a quadratic transformation at such point $\mathbf{x}$, introduces a good exceptional hypersurface $\mathbb{P}^1$. Locally over $\mathbf{x}$ the new stratification is defined by:

- The strict transform of the previous strata.
- A zero-dimensional stratum $\mathcal{Q} = \mathbb{P}^1 \cap H_2$.

The invariants at the strict transform of H_1 are to be dealt with exactly as in Sect. 4.6 Case A) (1). We therefore restrict attention to the data and invariants to be defined at $\mathcal{Q}$.

Let us fix locally at $\mathbf{x}$ coordinates x, y so that $H_1 = \{x = 0\}$ and $H_2 = \{y = 0\}$. Assume that a local presentation is given so that a_{p^e} is as in (20). Therefore, the local invariant at $\mathbf{x}$ is $\left(\frac{\ell}{p^e}, g(y)W^{p^e}\right)$. Set $(\mathcal{R}_{\mathcal{G},\beta})_{r'} = x^a y^b W^s$. As we assume that the point $\mathbf{x}$ is good, then $\frac{a+b}{s} \leq \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e}$.

The point $\mathcal{Q}$ is the origin at the U_x-chart (with coordinates $x_1 = x$, $y_1 = \frac{y}{x}$). Consider the quadratic transformation at $\mathbf{x}$. The new exceptional line is good, and hence the exponents of the tight monomial algebra and the elimination algebra along this hypersurface coincide. So, after reduction, we may assume that at $\mathcal{Q}$, $(\mathcal{R}_{\mathcal{G},\beta})_{r'+1} = x_1^{a+b-sm} y_1^b W^s$, for a suitable integer $m \geq 0$ so that $h_{r'+1} = a + b - sm < s$ (here $h_{r'+1}$ is also the exponent in the tight monomial algebra of the new exceptional hypersurface).

Set $g_1(y_1)W^s = y_1^b W^s$. According to Sect. 4.5, case $x = \mathcal{Q}$, the local data we assign to $\mathcal{Q}$ is $(\frac{a+b-sm}{s}, y_1^b W^s)$.

Lemma 5 *Assume that the conditions in the previous setting hold. Then* $\frac{b}{s} = \mathrm{inv}(\mathcal{Q}) < \mathrm{inv}(\mathbf{x})$.

Proof Set $M = \mathrm{ord}_{\mathbf{x}}(g)$ and recall that $\mathrm{inv}(\mathbf{x}) = \frac{M}{p^e}$. Then

$$\frac{\ell + M}{p^e} \geq \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e} \geq \frac{a+b}{s} > \frac{\ell}{p^e} + \frac{b}{s}.$$

The first inequality follows from the fact that $x^\ell y^M$ is a monomial that appears in the formal expansion of a_{p^e}. The second inequality is due to the fact that $\mathbf{x}$ is a good point. Finally, the last inequality is a consequence of the fact that H_1 is bad and hence $\frac{a}{s} > \frac{\ell}{p^e}$. These inequalities imply that $\frac{M}{p^e} > \frac{b}{s}$. □

Case D1) In this case, we blow-up a bad point $\mathcal{Q}$, which is an infinitesimal stratum, and hence $\mathcal{Q}$ is the intersection of a bad hypersurface H_1 and a good hypersurface H_2. As $\mathcal{Q}$ is bad, the quadratic transformation at $\mathcal{Q}$ will introduce a exceptional line $\mathbb{P}^1$, which is bad. It will give rise to two strata: and infinitesimal stratum $\mathcal{Q}'$ and an affine line $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\mathcal{Q}'\}$. Fix local coordinates $\{x, y\}$ at $\mathcal{Q}$ so that $\{x = 0\}$ defines the bad line H_1 and $\{y = 0\}$ defines the good line H_2. Set $(\mathcal{R}_{\mathcal{G},\beta})_{r'} = x^a y^b W^s$.

Lemma 6 *Fix a point* $\mathbf{x}' \in \mathbb{A}^1$. *The invariant strictly drops, i.e.,*

$$\mathrm{inv}(\mathbf{x}') < \mathrm{inv}(\mathcal{Q}).$$

Proof Locally at $\mathcal{Q}$, the second coordinate of the local data is given by $g(x)W^s = x^a W^s$. So the local invariant at $\mathcal{Q}$ is $\mathrm{inv}(\mathcal{Q}) = \frac{\nu_{\mathcal{Q}}(g)}{s} = \frac{a}{s}$.

As we assume that $\mathcal{Q}$ is bad, $\frac{d}{p^e} := \frac{\nu_{\mathcal{Q}}(a_{p^e})}{p^e} < \mathrm{ord}((\mathcal{R}_{\mathcal{G},\beta})_{r'})(\mathcal{Q}) = \frac{a+b}{s}$, where now a_{p^e} is as in (19). On the other hand, $y = 0$ defines the good line, so we claim that

$\frac{b}{s} < 1$. In fact, good hypersurfaces are, by definition, those for which the corresponding exponents at the elimination algebra, and at the tight monomial algebra, coincide. Since we assume that the tight monomial algebra is reduced (see Sect. 4.1), $\frac{b}{s} < 1$.

Let $\mathrm{In}_{\mathcal{Q}}(a_{p^e}) = \sum_{i+j=d} \lambda_{i,j} X^i Y^j$ denote the initial form of a_{p^e} at $\mathcal{Q}$. Note that $\frac{j}{p^e} \geq \frac{b}{s}$. From the previous inequalities, we obtain

$$\frac{a+b}{s} > \frac{d}{p^e} = \frac{i+j}{p^e} \geq \frac{i}{p^e} + \frac{b}{s},$$

and hence, $\frac{a}{s} > \frac{i}{p^e}$.

At the U_y-chart (with coordinates $y_1 = y, x_1 = \frac{x}{y}$),

$$a_{p^e}^{(1)} = y_1^{d-p^e}(g_1(x_1) + y_1 \Omega'),$$

where $g_1(x_1)$ is obtained by the global polynomial $\mathrm{In}_{\mathcal{Q}}(a_{p^e})|_{Y=1}$. So the previous discussion shows that if $\lambda_{i,j} \neq 0$, then $\frac{a}{s} > \frac{i}{p^e}$. This, in turn, suffices to check that $\mathrm{inv}(\mathbf{x}') = \frac{\mathrm{ord}_{\mathbf{x}'}(g_1)}{p^e} < \frac{a}{s} = \mathrm{inv}(\mathcal{Q})$. □

Now, we study the invariant at $\mathcal{Q}'$. Note that local coordinates at $\mathcal{Q}'$ are given by $x_1 = x, y_1 = \frac{y}{x}$. So the elimination algebra is $(\mathcal{R}_{\mathcal{G},\beta})_{r'+1} = x_1^{a+b-s} y_1^b W^s$ and the strict transform of the good hypersurface H_2 is given by $\{y_1 = 0\}$. Therefore, the second coordinate of the local data at $\mathcal{Q}'$ is $g_1(x) W^s = x_1^{a+b-s} W^s$.

Lemma 7 *With the previous setting,* $\mathrm{inv}(\mathcal{Q}') < \mathrm{inv}(\mathcal{Q})$.

Proof Recall that the second coordinate of the local data at $\mathcal{Q}$ is given by $g(x) W^s = x^a W^s$. By definition $\mathrm{inv}(\mathcal{Q}) = \frac{\nu_{\mathcal{Q}}(g)}{s} = \frac{a}{s}$ and $\mathrm{inv}(\mathcal{Q}') = \frac{\nu_{\mathcal{Q}'}(g_1)}{s} = \frac{a+b-s}{s}$. Since H_2 is a good hypersurface, then $\frac{b}{s} < 1$, from which the strict inequality is clear. □

Case D2) This is, as in D1), the case of a quadratic transformation at a point $\mathcal{Q}$ which is an infinitesimal stratum. It is the intersection of a bad hypersurface H_1 and a good hypersurface H_2. We assume now, in addition, that $\mathcal{Q}$ is a good point. Since the new exceptional hypersurface is good, there is a unique stratum $\mathcal{Q}'$ which will be infinitesimal.

Fix local coordinates $\{x, y\}$ at $\mathcal{Q}$, so that $\{x = 0\}$ defines the bad line H_1 and $\{y = 0\}$ defines the good line H_2. Set $(\mathcal{R}_{\mathcal{G},\beta})_{r'} = x^a y^b W^s$. Recall that the second coordinate of the local data is $g(x) W^s = x^a W^s$.

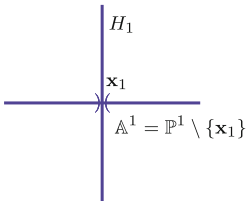
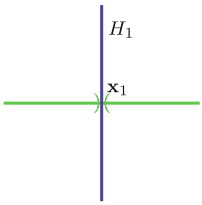
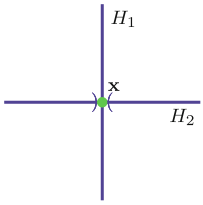
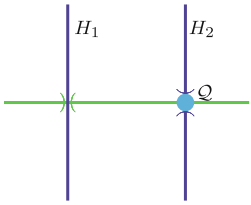
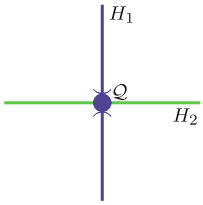
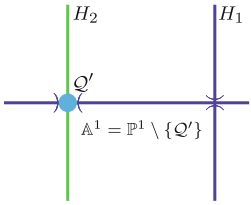
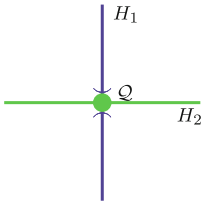
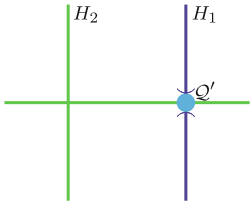
Note that $x_1 = \frac{x}{y}, y_1 = y$ are local coordinates at $\mathcal{Q}'$. The elimination algebra is $(\mathcal{R}_{\mathcal{G},\beta})_{r'+1} = x_1^a y_1^{a+b-s} W^s$. The new good exceptional hypersurface is defined by $y_1 = 0$, so the second coordinate of the local data is $g_1(x_1) W^s = x_1^a W^s$.

Remark 7 If case D2) holds, then

- $\nu_{\mathcal{Q}'}(g_1) = \nu_{\mathcal{Q}}(g)$ and hence $\mathrm{inv}(\mathcal{Q}') = \mathrm{inv}(\mathcal{Q})$.
- On the other hand, $\frac{a+b-s}{s} = \frac{a}{s} + (\frac{b}{s} - 1) < \frac{a}{s}$.

We conclude that case D2) cannot occur in a successive manner more than finitely many times. So, in particular, after finitely many quadratic transformations at infinitesimal strata, case D1) holds.

In the following table we indicate, in a synthetic manner, why resolution is achieved.

Invariants	**Before blow-up**	**After blow-up**
CASE A) • After finitely many quadratic transformations, either the invariant improves (strictly drops), or it remains equal. In this last case, the singular point must be either in case B) or C). (Lemma 1, Lemma 3, and Lemma 4).		
CASE B) • This case adds no new stratum. • The invariant strictly drops, i.e., $\mathrm{inv}(\mathbf{x}) < \mathrm{inv}(\mathbf{x}_1)$.		
CASE C) • Introduces a unique infinitesimal stratum $\mathcal{Q}$. • Invariant strictly drops.		
CASE D1) • Adds two new stratum $\mathcal{Q}'$ and $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\mathcal{Q}'\}$. • Invariants strictly drop at any exceptional point.		
CASE D2) • Only adds a new infinitesimal stratum $\mathcal{Q}'$. • The invariant remains equal, $\mathrm{inv}(\mathcal{Q}') = \mathrm{inv}(\mathcal{Q})$. • Leads to resolution or case D1) after finitely many quadratic transformations.		

Acknowledgments We have profited from discussions with A. Bravo and V. Cossart. We thank the referee for the comments and corrections that helped us to improve the presentation of this paper.

References

1. Abhyankar, S.: Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$. Ann. Math. **63**(2), 256–491 (1956)
2. Abhyankar, S.: Ramification theoretic methods in algebraic geometry. In: Annals of Mathematics Studies, vol. 43. Princeton University Press, Princeton, pp. ix+96 (1959)
3. Abhyankar, S.: Uniformization in p-cyclic extensions of algebraic surfaces over ground fields of characteristic p. Math. Ann. **153**, 81–96 (1964)
4. Abhyankar, S.: Resolution of singularities of embedded algebraic surfaces. In: Pure Appl. Math., vol. 24. Academic Press, New York (1966)
5. Altman, A., Kleiman, S.: Introduction to Grothendieck Duality Theory. In: Lecture Notes in Mathematics, vol. 146. Springer, Berlin (1970)
6. Aroca, J.M., Hironaka, H., Vicente, J.L.: The theory of maximal contact. Mem. Mat. Ins. Jorge Juan (Madrid) **29** (1975)
7. Benito, A.: The τ-invariant and elimination. J. Algebra **324**, 1903–1920 (2010)
8. Benito, A., Villamayor U., O.E.: Monoidal transforms and invariants of singularities in positive characteristic. arxiv.1004.1803v2
9. Benito, A., Villamayor U., O.E.: On elimination of variables in the study of singularities in positive characteristic. arxiv.1103.3462
10. Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. **128**(2), 207–302 (1997)
11. Bravo, A., Encinas, S., Villamayor, O.: A simplified proof of desingularization and applications. Rev. Mat. Iberoamericana **21**, 349–458 (2005)
12. Bravo, A., Garcia-Escamilla, M.L., Villamayor U., O.: On Rees algebras and the globalization of local invariants for resolution of singularities (2011) arxiv:1107.1797
13. Bravo, A., Villamayor U., O.E.: Singularities in positive characteristic, stratification and simplification of the singular locus. Adv. Math. **224**(4), 1349–1418 (2010)
14. Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities with an appendix by H. Hironaka. In: Lecture Notes in Mathematics, vol. 1101. Springer, Berlin (1984)
15. Cossart, V.: Desingularization of embedded excellent surfaces. Tohoku Math. J. (2) **33**(1), 25–33 (1981)
16. Cossart,V.: Sur le polyèdre caractéristique. Thèse d'État. Univ. Paris-Sud, Orsay (1987)
17. Cossart, V., Jannsen, U., Saito, S.: Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes (2009). Preprint. arxiv.0905.2191v1. Accessed 13 May 2009
18. Cossart, V., Piltant, O.: Resolution of singularities of threefolds in positive characteristic. I–II. J. Algebra **320**(3), 1051–1082 (2008)
19. Cossart, V., Piltant, O.: Resolution of singularities of threefolds in positive characteristic. I–II. J. Algebra **321**(7), 1836–1976 (2009)
20. Cutkosky, S.D.: Resolution of singularities for 3-folds in positive characteristic. Am. J. Math. **131**(1), 59–127 (2009)
21. Cutkosky, S.D.: A skeleton key to Abhyankar's proof of embedded resolution of characteristic p surfaces (2010) (Preprint)
22. Encinas, S., Hauser, H.: Strong Resolution of Singularities. Comment. Math. Helv. **77**(4), 821–845 (2002)
23. Encinas, S., Villamayor, O.: Good points and constructive resolution of singularities. Acta Math. **181**(1), 109–158 (1998)
24. Encinas, S., Villamayor U., O.E.: Rees algebras and resolution of singularities. Actas del "XVI Coloquio Latinoamericano de Algebra" (Colonia del Sacramento, Uruguay, 2005). In: Santos, W.F., González-Springer, G., Rittatore, A., Solotar, A. (eds.) Library of the Revista Matemática Iberoamericana, pp. 1–24 (2007)
25. Giraud, J.: Contact maximal en caractéristique positive. Ann. Sci. École Norm. Sup. (4) **8**(2), 201–234 (1975)
26. Giraud, J.: Forme normale d'une fonction sur une surface de caractéristique positive. Bull. Soc. Math. France **111**(2), 109–124 (1983)

27. Giraud, J.: Condition de Jung pour les revêtements radiciels de hauteur 1. Algebraic geometry (Tokyo/Kyoto, 1982). In: Lecture Notes in Mathematics, vol. 1016, pp. 313–333. Springer, Berlin (1983)
28. Hauser, H., Wild singularities and kangaroo points for the resolution in positive characteristic (2009) (Preprint)
29. Hauser, H.: On the problem of resolution of singularities in positive characteristic (Or: A proof we are still waiting for). Bull. Am. Math. Soc. (N.S.) **47**(1), 1–30 (2010)
30. Hauser, H., Wagner, D.: Two new invariants for the resolution of surfaces in positive characteristic (in preparation)
31. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I–II, Ann. of Math. **79**(2), 109–203 (1964)
32. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I–II, Ann. Math. **79**(2), 205–326 (1964)
33. Hironaka, H: Desingularization of excellent surfaces. Advanced Science Seminar in Algebraic Geometry (summer 1967 at Bowdoin College). Mimeographed notes by B. Bennet, Appendix to [14], pp. 99–132
34. Hironaka, H.: Program for resolution of singularities in characteristics $p > 0$. In: Notes from lectures at the Clay Mathematics Institute, September 2008
35. Kawanoue, H.: Toward resolution of singularities over a field of positive characteristic. I. Foundation; the language of the idealistic filtration. Publ. Res. Inst. Math. Sci. **43**(3), 819–909 (2007)
36. Kawanoue, H., Matsuki, K.: Toward resolution of singularities over a field of positive characteristic. (The idealistic filtration program). Publ. Res. Inst. Math. Sci. **46**(2), 359–422 (2010)
37. Lipman, J.: Desingularization of two-dimensional schemes. Ann. Math. (2) **107**(1), 151–207 (1978)
38. Moh, T.T.: On a stability theorem for local uniformization in characteristic p. Publ. Res. Inst. Math. Sci. **23**(6), 965–973 (1987)
39. Moh, T.T.: On a Newton polygon approach to the uniformization of singularities in characteristic p. In: Campillo, A., Narváez, L. (eds.) Algebraic Geometry and Singularities. Proceeding of the Conference on Singularities La Rábida. Birkhäuser, Basel (1996)
40. Narasimhan, R.: Monomial equimultiple curves in positive characteristic. Proc. Am. Math. Soc. **89**, 402–413 (1983)
41. Narasimhan, R.: Hyperplanarity of the equimultiple locus. Proc. Am. Math. Soc. **87**, 403–406 (1983)
42. Oda, T.: Infinitely very near singular points. In: Complex analytic Singularities. Adv. Studies in Pure Math., vol. 8, pp. 363–404. North-Holland, Amsterdam (1987)
43. Piltant, O.: On the Jung method in positive characteristic. Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) **53**(4), 1237–1258 (2003)
44. Villamayor, O.E.: Constructiveness of Hironaka's resolution. Ann. Scient. Ec. Norm. Sup. (4E) **22**, 1–32 (1988)
45. Villamayor, O.E.: Patching local uniformizations. Ann. Scient. Ec. Norm. Sup. 4e **25**, 629–677 (1992)
46. Villamayor U., O.E.: Rees algebras on smooth schemes: integral closure and higher differential operators. Rev. Mat. Iberoamericana **24**(1), 213–242 (2008)
47. Villamayor U., O.E.: Hypersurface singularities in positive characteristic. Adv. Math. **213**(2), 687–733 (2007)
48. Villamayor U., O.E.: Elimination with applications to singularities in positive characteristic. Publ. Res. Inst. Math. Sci. **44**(2), 661–697 (2008)
49. Włodarczyk, J.: Simple Hironaka resolution in characteristic zero. J. Am. Math. Soc. **18**(4), 779–822 (2005)
50. Włodarczyk, J.: Program on resolution of singularities in characteristic p. Notes from lectures at RIMS, Kyoto, December 2008