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BMO spaces associated with semigroups of operators

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Abstract We study BMO spaces associated with semigroup of operators on noncommutative function spaces (i.e. von Neumann algebras) and apply the results to boundedness of Fourier multipliers on non-abelian discrete groups. We prove an interpolation theorem for BMO spaces and prove the boundedness of a class of Fourier multipliers on noncommutative L_p spaces for all $1 < p < \infty$, with optimal constants in p.

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0 Introduction

The theory of semigroups provides a good framework of studying classical questions from harmonic analysis in a more abstract setting. Our research is particularly motivated by E. Steins' results on Fourier multipliers on L_p spaces and Littlewood-Paley theory for the Laplace-Beltrami operators on compact groups. Our aim is to study BMO spaces which are intrinsically defined by a (some kind of heat-) semigroup and prove fundamental interpolation results. In particular, we want to give a positive answer to the following

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Problem 0.1 *Let (T_t) be a standard semigroup of selfadjoint positive operators on an (abstract) functions space $L_\infty(\Omega)$. Let A be its infinitesimal generator. Is there a BMO space such that*

(a) *BMO serves as an endpoint of interpolation, i.e. $[BMO, L_1(\Omega)]_{\frac{1}{p}} = L_p(\Omega)$;*
(b) *The imaginary powers A^{is}, $s \in \mathbb{R}$ extend to bounded operators from $L_\infty(\Omega)$ to BMO.*
(c) *The estimates in (b) are universal. In particular, the constants involved are dimension free for all the classical heat semigroups on $\Omega = \mathbb{R}^n$.*

We should expect much more singular integral operators for abstract semigroups instead of the imaginary powers mentioned in (b). However, it seems that even in the commutative theory such a BMO space has not yet been identified. An advantage of such a theory is that it provides a natural framework for good, or even optimal dimension free estimates, for Fourier multipliers. Our results apply not only in the commutative, but also in the noncommutative setting (i.e. replacing $L_\infty(\Omega)$ by a von Neumann algebra).

Indeed, BMO spaces, once they can be appropriately defined, provide a very efficient tool in proving results on Fourier-multipliers. BMO spaces associated with semigroups on commutative functions spaces have been studied in [44,48] and very recently in [6,7]. Here 'commutative function space' means that the semigroups of operators under investigation are defined on some $L_\infty(\Omega)$. Note that $L_\infty(\Omega)$ is the prototype of a commutative von Neumann algebra. Even in this commutative setting a general theory of BMO spaces defined intrinsically by the semigroup is far from established.

On the other hand, BMO spaces have been extended to noncommutative function spaces (i.e. von Neumann algebras) in various cases. Let us refer to the seminal work on martingales in [13,21,30,37] and [36], and to [26,32] and [3] for work on operator- or matrix-valued functions. In [27] a first approach towards a $H_1 - BMO$ duality associated with semigroups of operators on von Neumann algebras has been obtained, whereas a duality theory for Averson's subdiagonal algebras is studied in [31].

As in the commutative case, BMO boundedness and interpolation usually gives optimal or at least very good estimates for singular integral operators on L_p. The use of BMO spaces also turns out to be crucial when reducing results on group von Neumann algebras to the semicommutative setting, see [15]. Let us describe one of our main results. Let $\mathcal{N}$ be a von Neumann algebra with a normal trace τ satisfying $\tau(1) = 1$, i.e. $(\mathcal{N}, \tau)$ is a noncommuative probability space. Let (T_t) be semigroup of completely positive maps on $\mathcal{N}$ such that $\tau(T_t(x)) = \tau(x)$ and $T_t(1) = 1$. Then we define the BMO^c column norm by

$$\|x\|_{BMO^c(\mathcal{T})} = \sup_t \|T_t|x - T_t x|^2\|^{1/2}$$

and $\|x\|_{BMO(\mathcal{T})} = \max\{\|x\|_{BMO^c(\mathcal{T})}, \|x^*\|_{BMO^c(\mathcal{T})}\}$. The norm $\|x\|_{BMO_r(\mathcal{T})} = \|x^*\|_{BMO^c(\mathcal{T})}$ is called the row BMO norm and the need of both such norms is well-known from martingale theory.

Theorem 0.2 *Assume that T_t is a standard semigroup of completely positive maps on $\mathcal{N}$ and (T_t) admits a Markov dilation. Then*

$$[BMO(\mathcal{T}), L_1(\mathcal{N})]_{\frac{1}{p}} \;=\; L_p(\mathcal{N})$$

for $1 < p < \infty$.

We investigate other possible intrinsic choices for BMO-norms and compare them. These results are applied to BMO-boundedness of Fourier multipliers on non-abelian discrete groups. We obtain their corresponding L_p-boundedness with optimal constants. Basic examples of Fourier multipliers in this article are noncommutative analogues of E. Stein's imaginary power $(-\triangle)^{i\gamma}$ (see Theorem 3.7 in Example 3.5) and noncommutative analogues of P. A. Meyer's generalized Riesz transforms (see Theorem 4.8). A further application of our results gives optimal constants in Junge/Xu's noncommutative maximal ergodic inequality (see [22]). Many of our results are new even in the commutative setting. In particular, our constants of the L_p bounds of Stein's universal Fourier multipliers are better than those obtained by Stein [40] and Cowling [5] (see Remark 5.6).

1 Preliminaries and notation

1.1 Noncommutative L_p spaces

Let $\mathcal{N}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $\mathcal{S}_+$ be the set of all positive $f \in \mathcal{N}$ such that $\tau(\mathrm{supp}(f)) < \infty$, where $\mathrm{supp}(x)$ denotes the support of f, i.e. the least projection $e \in \mathcal{N}$ such that $ef = f$. Let $\mathcal{S}_{\mathcal{N}}$ be the linear span of S_+. Note that $\mathcal{S}_{\mathcal{N}}$ is an involutive strongly dense ideal of $\mathcal{N}$. For $0 < p < \infty$ define

$$\|f\|_p = \left(\tau\left(|f|^p\right)\right)^{1/p}, \quad x \in \mathcal{S}_{\mathcal{N}},$$

where $|f| = (f^*f)^{1/2}$, the modulus of x. One can check that $\|\cdot\|_p$ is a norm or p-norm on $\mathcal{S}_{\mathcal{N}}$ according to $p \geq 1$ or $p < 1$. The corresponding completion is the noncommutative L_p-space associated with $(\mathcal{N}, \tau)$ and is denoted by $L_p(\mathcal{N})$. By convention, we set $L^\infty(\mathcal{N}) = \mathcal{N}$ equipped with the operator norm $\|\cdot\|$. The elements of $L_p(\mathcal{N})$ can be also described as measurable operators with respect to $(\mathcal{N}, \tau)$. We refer to [38] for more information and for more historical references on noncommutative L_p-spaces. In the sequel, unless explicitly stated otherwise, $\mathcal{N}$ will denote a semifinite von Neumann algebra and τ a normal semifinite faithful trace on $\mathcal{N}$. We will simplify $L_p(\mathcal{N})$ as L_p and the corresponding norms as $\|\cdot\|_p$.

We say an operator T on $\mathcal{N}$ is completely contractive if $T \otimes I_n$ is contractive on $\mathcal{N} \otimes M_n$ for each n. Here, M_n is the algebra of n by n matrices and I_n is the identity operator on M_n. We say an operator T on $\mathcal{M}$ is completely positive if $T \otimes I_n$ is positive on $\mathcal{N} \otimes M_n$ for each n. We will need the following Kadison–Schwarz inequality for unital completely positive contraction T on $L_p(\mathcal{N})$,

$$|T(f)|^2 \le T\left(|f|^2\right), \quad \forall f \in L_p(\mathcal{N}). \tag{1.1}$$

1.2 Standard noncommutative semigroups

Throughout this article we will assume that (T_t) is a semigroup of completely positive maps on a semifinite von Neumann algebra $\mathcal{N}$ satisfying the following *standard assumptions*

(i) Every T_t is a normal completely positive maps on $\mathcal{N}$ such that $T_t(1) = 1$;
(ii) Every T_t is selfadjoint with respect to the trace τ, i.e. $\tau(T_t(f)g) = \tau(fT_t(g))$;
(iii) The family (T_t) is strongly continuous, i.e. $\lim_{t\to 0} T_t f = f$ with respect to the strong topology in $\mathcal{N}$ for any $f \in \mathcal{N}$.

Let us note that (i) and (ii) imply that $\tau(T_t x) = \tau(x)$ for all x, so T_t's are faithful and are contractive on $L_1(\mathcal{N})$. By interpolation, T_t's extend to contractions on $L_p(\mathcal{N})$, $1 \le p < \infty$ and satisfy $\lim_{t\to 0} T_t x = x$ in $L_p(\mathcal{N})$ for all $x \in L_p(\mathcal{N})$. (see [22] for details). Some of these conditions can be weakened, but this is beyond the scope of this article.

Let us recall that such a semigroup admits an infinitesimal (negative) generator A given as $Af = \lim_{t\to 0} t^{-1}(f - T_t(f))$ defined on $\mathrm{dom}(A) = \cup_{1\le p\le\infty} \mathrm{dom}_p(A)$, where

$$\mathrm{dom}_p(A) = \left\{ f \in L_p(\mathcal{N});\ \lim_{t\to 0} t^{-1}(T_t(f) - f) \text{ converges in } L_p(\mathcal{N}) \right\}.$$

It is easy to see that $\frac{1}{s}\int_0^s T_t(f)dt \in \mathrm{dom}_p(A)$ for any $s > 0$, $f \in L_p(\mathcal{N})$, so $\mathrm{dom}_p(A)$ is dense in $L_p(\mathcal{N})$. Denote by A_p the restriction of A on $\mathrm{dom}_p(A)$. Under our assumptions (i)–(iii), A_2 is a positive (unbounded) operator. $A_p T_t = T_t A_p = -\frac{\partial T_t}{\partial t}$ extend to a (same) bounded operator on $L_p(\mathcal{N})$ for all $t > 0, 1 \le p \le \infty$. Therefore, $T_s(f) \in \mathrm{dom}_p(A)$ for any $f \in L_p(\mathcal{N}), 1 \le p \le \infty$.

For a standard semigroup T_s (generated by A), we may consider the *subordinated Poisson semigroup* $\mathcal{P} = (P_t)_{t\ge 0}$ defined by $P_t = \exp(-tA^{\frac{1}{2}})$. (P_t) is again a semigroup of operators satisfying (i)-(iii) above. Note that P_t satisfies $(\partial_t^2 - A)P_t = 0$. By functional calculus and an elementary identity, each P_t can be written as (see e.g. [42]),

$$P_t = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u du. \tag{1.2}$$

The integral on the right hand side of the identity converges with respect to the operator norm on $L_p(N)$ for $1 \le p \le \infty$. Let us define the gradient form Γ associated with T_t,

$$2\Gamma(f, g) = \left(A\left(f^*\right)g\right) + f^*(A(g)) - A\left(f^*g\right),$$

for f, g with $f^*, g, f^*g \in \mathrm{dom}(A)$. For convenience, we assume that there exists a*-algebra $\mathcal{A}$ which is weak* dense in $\mathcal{N}$ such that $T_s(\mathcal{A}) \subset \mathcal{A} \subset \mathrm{dom}(A)$. This assumption is to guarantee that $\Gamma(T_s f, T_s g)$ make senses for $f, g \in \mathcal{A}$, which is not easy to verify in general, although the other form $T_t\Gamma(T_s f, T_s g)$ is what we need essentially in this article and can be read as $T_t((AT_s f^*)T_s g) + T_t(T_s f^*(AT_s g)) - AT_t(T_s f^* T_s g)$ for any $f, g \in L_p(\mathcal{N})$, $1 \le p \le \infty$, $s, t > 0$. The semigroup (T_t) generated by A is said to satisfy the $\Gamma^2 \ge 0$ if

$$\Gamma(T_v f, T_v f) \le T_v\Gamma(f, f)$$

for all $v > 0$, $f \in \mathcal{A}$. It is easy to see $\Gamma^2 \ge 0$ also implies $\Gamma(P_v f, P_v f) \le P_v\Gamma(f, f)$ for any $v > 0$. Denote by the gradient form associated with $(P_t)_t$ by $\Gamma_{A^{\frac{1}{2}}}$.

We will need the following Lemma proved in [14,19]. We add a short proof for the convenience of the reader.

Lemma 1.1 (i) *For any $f \in L_p(\mathcal{N})$, $1 \le p \le \infty$, $s > 0$, we have*

$$T_s|f|^2 - |T_s f|^2 = 2\int_0^s T_{s-t}\Gamma(T_t f, T_t f)dt.$$

(ii) *For any $f \in \mathcal{A}$, we have*

$$\Gamma_{A^{\frac{1}{2}}}(f, f) = \int_0^\infty P_v\Gamma(P_v f, P_v f)dv + \int_0^\infty P_v|P_v' f|^2 dv$$

For any $f \in L_p(\mathcal{N})$, $s, t > 0$, we have

$$P_t\Gamma_{A^{\frac{1}{2}}}(P_s f, P_s f) = \int_0^\infty P_{t+v}\Gamma(P_{s+v} f, P_{s+v} f)dv + \int_0^\infty P_{t+v}|P_{s+v}' f|^2 dv.$$

Here and in the rest of the article, f_t' means $\frac{df_t}{dt}$.

Proof (i): For s fixed, let

$$F_t = T_{s-t}(|T_t f|^2).$$

Then

$$\begin{aligned}\frac{\partial T_{s-t}\left(|T_t f|^2\right)}{\partial t} &= \frac{\partial T_{s-t}}{\partial t}\left(|T_t f|^2\right) + T_{s-t}\left[\left(\frac{\partial T_t}{\partial t} f^*\right) f\right] \\ &\quad + T_{s-t}\left[f\left(\frac{\partial T_t}{\partial t} f^*\right)\right] \\ &= -T_{s-t}\Gamma\left(T_t f, T_t f\right)\end{aligned}$$

Therefore

$$T_s|f|^2 - |T_s f|^2 = -F_s + F_0 = \int_0^s T_{s-t}\Gamma(T_t f, T_t f)dt.$$

(ii): Let

$$F_t = \frac{\partial P_t}{\partial t}(|P_t f|^2) - P_t\left(\frac{\partial P_t f^*}{\partial t} P_t f\right) - P_t\left(P_t f^* \frac{\partial P_t f}{\partial t}\right).$$

Then

$$\begin{aligned}\frac{\partial F_t}{\partial t} &= \frac{\partial^2 P_t}{\partial t^2}\left(|P_t f|^2\right)\\ &\quad - P_t\left(\frac{\partial^2 P_t f^*}{\partial t^2} P_t f\right) - P_t\left(P_t f^* \frac{\partial^2 P_t f}{\partial t^2}\right) - 2P_t\left(\left|\frac{\partial P_t f}{\partial t}\right|^2\right)\\ &= -AP_t(|P_t f|^2) - P_t\left[\left(-AP_t f^*\right) P_t f\right]\\ &\quad - P_t\left[P_t f^*\left(-AP_t f\right)\right] - 2P_t\left(\left|\frac{\partial P_t f}{\partial t}\right|^2\right)\\ &= -P_t\Gamma\left(P_t f, P_t f\right) - 2P_t\left(\left|P_t' f\right|^2\right).\end{aligned}$$

Note that $F_0 = \Gamma_{A^{\frac{1}{2}}}(f, f)$ and $F_t \to 0$ in $\mathcal{N}$ as $t \to \infty$ because of Proposition 1.1. We get

$$\Gamma_{A^{\frac{1}{2}}}(f, f) = \int_0^\infty -\frac{\partial F_t}{\partial t}dt = \int_0^\infty P_t\Gamma\left(P_t f, P_t f\right)dt + 2\int_0^\infty P_t\left(\left|P_t' f\right|^2\right)dt.$$

□

We will use the following inequality from [27].

Proposition 1.2 *Let $f \in \mathcal{N}$ be positive and $0 < t < s$. Then*

$$P_s f \le \frac{s}{t} P_t f.$$

Proof We use (1.2) and $e^{-\frac{s^2}{4u}} \le e^{-\frac{t^2}{4u}}$ for all u. This yields the assertion

$$\frac{P_s f}{s} = \frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u(f)du \le \frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u(f)du = \frac{P_t f}{t}.$$

□

2 BMO norms associated with semigroups of operators

In this part we study several natural BMO-norms associated with a semigroup T_t of completely positive maps. The situation is particularly nice for subordinated semigroups so that the original semigroup satisfies the $\Gamma^2 \geq 0$.

Given a standard semigroup of operators T_t on $\mathcal{N}$ and $f \in \mathcal{N} \cup L_2(\mathcal{N})$, we define

$$\|f\|_{bmo^c(\mathcal{T})} = \sup_t \|T_t|f|^2 - |T_t f|^2\|^{\frac{1}{2}}, \tag{2.1}$$

$$\|f\|_{BMO^c(\mathcal{T})} = \sup_t \|T_t|f - T_t f|^2\|^{\frac{1}{2}}. \tag{2.2}$$

Here and in what follows $\|f\|$ always denote the operator norm of f. The notations $\|\cdot\|_{bmo^c(\mathcal{P})}$, $\|\cdot\|_{BMO^c(\mathcal{P})}$ are used when $\mathcal{T}$ is replaced by the subordinated semigroup (P_t) above. The definitions steam from Garsia's norm for the Poisson semigroup on the circle (see [23]). The $BMO^c(\mathcal{T})$-norm has been studied in [27], motivated by the expression

$$\|f\|_{BMO_1} = \sup_z P_z(|f - f(z)|).$$

This definition appeared in the commutative case in particular in [7,44,48]. Using the $\| \ \|_{BMO_1}$-norm, it is easy to show that the conjugation operator is bounded from L_∞ to BMO_1. Here $f(z)$ gives the value of the harmonic extension in the interior of the circle (see [8]). In some sense $f - P_t f$ is similar to $f - f(z)$, despite the fact that the Poisson integral $P_t f$ still is a function, while $f(z)$ is considered as a constant function in $f - f(z)$.

Proposition 2.1 *Let (T_t) be a standard semigroup of operators. Then $bmo^c(\mathcal{T})$ and $BMO^c(\mathcal{T})$ are semi-norms on $\mathcal{N}$.*

Proof Fix $t > 0$. Let $\mathcal{L}(\mathcal{N} \otimes_{T_t} \mathcal{N})$ be the Hilbert C^*-module over $\mathcal{N}$ with $\mathcal{N}$-valued inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* T_t(a^* c) d.$$

This Hilbert C^*-module is well-known from the GNS-construction for T_t, see [25]. Since T_t is unital, we have a *-homomorphism $\pi : \mathcal{N} \to \mathcal{L}(\mathcal{N} \otimes_{T_t} \mathcal{N})$ such that

$$T_t(f) = e_{11}\pi(f)e_{11}.$$

We then get

$$\begin{aligned} T_t\left(f^* f\right) - T_t\left(f^*\right) T_t(f) &= e_{11}\pi(f)^*\pi(f)e_{11} - e_{11}\pi(f)^* e_{11} e_{11}\pi(f)e_{11} \\ &= e_{11}\pi(f)^*(1 - e_{11})\pi(f)e_{11}. \end{aligned}$$

Therefore,

$$\left\| T_t|f|^2 - |T_t f|^2 \right\|^{\frac{1}{2}} = \left\| (1-e_{11})^{\frac{1}{2}} \pi(f) e_{11} \right\|_{\mathcal{L}(\mathcal{N} \otimes_{T_t} \mathcal{N})},$$
$$\left\| T_t|f - T_t f|^2 \right\|^{\frac{1}{2}} = \| \pi\left(f - e_{11}\pi(f)e_{11}\right) e_{11} \|_{\mathcal{L}(\mathcal{N} \otimes_{T_t} \mathcal{N})}.$$

This shows that $\|\cdot\|_{bmo^c(\mathcal{T})}$ and $\|\cdot\|_{BMO^c(\mathcal{T})}$ are semi-norms. □

Remark 2.2 An alternative proof for $bmo^c(\mathcal{T})$ being a semi-norm can be derived from the identity of Lemma 1.1 (i). Using the GNS construction for the positive form $T_{t-s}\Gamma$ we can find linear maps $u_{ts} : \mathcal{N} \to C(\mathcal{N})$ such that

$$T_t|f|^2 - |T_t f|^2 = \int_0^t |u_{ts}(f)|^2 ds.$$

This provides an embedding in $L_2^c([0,t]) \otimes_{\min} C(\mathcal{N})$.

Proposition 2.3 *Let (T_t) be a standard semigroup and $f \in \mathcal{N} \cup L_2(\mathcal{N})$. Then the following conditions are equivalent:*

(i) $\|f\|_{bmo^c(\mathcal{T})} = 0$.
(ii) $\|f\|_{BMO^c(\mathcal{T})} = 0$.
(iii) $f \in ker(A_\infty) \cup ker(A_2) = \{f \in \mathrm{dom}_\infty(A) \cup \mathrm{dom}_2(A), Af = 0\}$.

Proof Note that (ii) and (iii) both equals to $T_t f = f$ for any t since T_t is faithful. Hence (ii) is equivalent to (iii). Assume (iii), then $\tau(T_t|f|^2 - |T_t f|^2) = 0$ since $T_t f = f$ for all t and T_t is trace preserving. Note $T_t|f|^2 - |T_t f|^2 \geq 0$ by (1.1), so $T_t|f|^2 - |T_t f|^2 = 0$ for any $t > 0$. We get (i). Assume (i), we have $\tau(T_t|f|^2 - |T_t f|^2) = 0$, so $\tau(|f|^2 - |T_t f|^2) = 0$ for any $t > 0$. So $\tau(|f - T_t f|^2) = \tau(|f|^2 - 2|T_t f|^2 + |T_{2t} f|^2) = 0$ for any $t > 0$. So $f = T_t f$ for any $t > 0$. This implies (iii). □

Proposition 2.4 *Let (T_t) be a standard semigroup and $f \in \mathcal{N} \cup L_2(\mathcal{N})$. Then*

(i) $\|T_s f\|_{bmo^c(\mathcal{T})} \leq \|f\|_{bmo^c(\mathcal{T})}$ *for all $s > 0$;*
(ii) $\|f\|_{BMO^c(\mathcal{T})} \leq 2\, \|f\|_{bmo^c(\mathcal{T})} + \sup_t \|T_t f - T_{2t} f\|$.
(iii) *If in addition $\Gamma^2 \geq 0$, then*

$$\|f\|_{BMO^c(\mathcal{T})} \simeq \|f\|_{bmo^c(\mathcal{T})} + \sup_t \|T_t f - T_{2t} f\|.$$

Proof Let us start with (i) and the pointwise estimate

$$0 \leq T_t|T_s f|^2 - |T_{t+s} f|^2 \leq T_{t+s}|f|^2 - |T_{t+s} f|^2.$$

By definition of the $bmo^c(\mathcal{T})$ seminorm this implies

$$\begin{aligned} \|T_s f\|_{bmo^c(\mathcal{T})} &= \sup_t \|T_t|T_s f|^2 - |T_{t+s} f|^2\|^{\frac{1}{2}} \leq \sup_t \|T_{t+s}|f|^2 - |T_{t+s} f|^2\|^{\frac{1}{2}} \\ &\leq \|f\|_{bmo^c(\mathcal{T})}. \end{aligned}$$

For the proof of (ii), we fix $t > 0$ and use the triangle inequality (see Lemma 4.3):

$$\begin{aligned}\left\|T_t\,|f - T_t f|^2\right\|^{\frac{1}{2}} &\le \left\|T_t\,|f - T_t f|^2 - |T_t\,(f - T_t f)|^2\right\|^{\frac{1}{2}} + \left\|\,|T_t\,(f - T_t f)|^2\right\|^{\frac{1}{2}}\\ &\le \|f - T_t f\|_{bmo^c(\mathcal{T})} + \left\||T_t\,(f - T_t f)|^2\right\|^{\frac{1}{2}}\\ &\le \|f\|_{bmo^c(\mathcal{T})} + \|T_t f\|_{bmo^c(\mathcal{T})} + \|T_t\,(f - T_t f)\|\end{aligned}$$

We apply (i) and obtain

$$\|T_t|f - T_t f|^2\|^{\frac{1}{2}} \le 2\|f\|_{bmo^c(\mathcal{T})} + \|T_t(f - T_t f)\|.$$

Taking supremum over t yields the assertion. To prove (iii), we apply Lemma 1.1 (i) and the triangle inequality,

$$\begin{aligned}\left(T_t|f|^2 - |T_t f|^2\right)^{\frac{1}{2}} &= \left(\int_0^t T_{t-s}\Gamma\,(T_s f, T_s f)\,ds\right)^{\frac{1}{2}}\\ &\le \left(\int_0^t T_{t-s}\Gamma(T_s(f - T_t f), T_s(f - T_t f))ds\right)^{\frac{1}{2}}\\ &\quad + \left(\int_0^t T_{t-s}\Gamma(T_{s+t} f, T_{s+t} f)ds\right)^{\frac{1}{2}}\\ (\text{Lemma 1.1 (i)}) &= \left(T_t\,|f - T_t f|^2 - |T_t f - T_{2t} f|^2\right)^{\frac{1}{2}}\\ &\quad + \left(\int_0^t T_{t-s}\Gamma(T_{s+t} f, T_{s+t} f)ds\right)^{\frac{1}{2}}\\ (\Gamma^2 \ge 0) &\le \left(T_t|f - T_t f|^2\right)^{\frac{1}{2}} + \left(\int_0^t T_{2t-2s}\Gamma(T_{2s} f, T_{2s} f)ds\right)^{\frac{1}{2}}\\ (v = 2s) &\le \left(T_t|f - T_t f|^2\right)^{\frac{1}{2}} + \left(\frac{1}{2}\int_0^{2t} T_{2t-v}\Gamma(T_v f, T_v f)dv\right)^{\frac{1}{2}}\\ (\text{Lemma 1.1 (i)}) &= \left(T_t|f - T_t f|^2\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left(T_{2t}|f|^2 - |T_{2t} f|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the norm and the supremum over t on both sides, we get

$$\|f\|_{bmo^c(\mathcal{T})} \le (\sqrt{2} + 2)\|f\|_{BMO^c(\mathcal{T})}.$$

By Choi's inequality (see [4]) we find

$$|T_t f - T_{2t} f|^2 \ \le\ T_t|f - T_t f|^2.$$

Together with (ii), we obtain (iii). □

We now consider BMO-norms associated with the subordinated semigroup $(P_t)_t$.

Proposition 2.5 *Let (T_t) be a standard semigroup and (P_t) be the associated Poisson semigroup. Let $f \in \mathcal{N} \cup L_2(\mathcal{N})$. Then*

(i) $P_b|f|^2 - |P_b f|^2 = 2\int_0^\infty \int_{\max\{0,v-b\}}^v P_{b-v+2t}\hat{\Gamma}(P_v f, P_v f)dtdv;$
(ii) $\sup_b \left\| \int_0^\infty P_{b+s}|P_s' f|^2 \min(s,b)ds \right\| \ \le\ 4\, \|f\|^2_{bmo^c(\mathcal{P})};$
(iii) *If in addition $\Gamma^2 \ \ge\ 0$, then*

$$\frac{1}{4}\int_0^\infty P_{b+s}\hat{\Gamma}(P_s f, P_s f)\min(s,b)ds \ \le\ P_b|f|^2 - |P_b f|^2$$

$$\le 180\int_0^\infty P_{\frac{b}{3}+s}\hat{\Gamma}(P_s f, P_s f)\ \min\left(\frac{b}{3}, s\right) ds.$$

Here $\hat{\Gamma}(f_s, f_s) = \Gamma(f_s, f_s) + |f_s'|^2$.

Proof For the proof of (i) we apply Lemma 1.1 (i) to P_t and get

$$P_b|f|^2 - |P_b f|^2 \ =\ 2\int_0^b P_{b-s}\Gamma_{A^{\frac{1}{2}}}(P_s f, P_s f)ds.$$

Using the formula for $\Gamma_{A^{\frac{1}{2}}}(P_s f, P_s f)$ from Lemma 1.1 (ii), we obtain with the change of variable $(v = s + t)$ that

$$\begin{aligned} P_b|f|^2 - |P_b f|^2 \ &=\ 2\int_0^b\int_0^\infty P_{b-s+t}\hat{\Gamma}\left(P_{s+t} f, P_{s+t} f\right) dtds \\ &=\ 2\int_0^\infty\int_t^{b+t} P_{b-v+2t}\hat{\Gamma}\left(P_v f, P_v f\right) dvdt \\ &=\ 2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b-v+2t}\hat{\Gamma}\left(P_v f, P_v f\right) dtdv. \end{aligned} \tag{2.3}$$

This is (i). Note $\Gamma^2 \geq 0$ implies $\hat{\Gamma}^2 \geq 0$. We apply monotonicity, Proposition 1.2 and split the integral, and get

$$
\begin{aligned}
&2\int_0^\infty \int_{\max\{0,v-b\}}^{v} P_{b-v+2u}\hat{\Gamma}(P_v f, P_v f)du dv \\
&\quad \geq 2\int_0^\infty \int_{\max\{0,v-b\}}^{v} P_{b+2u}\hat{\Gamma}(P_{2v} f, P_{2v} f)du dv \\
&\quad \geq 2\int_0^\infty \left(\int_{\max\{0,v-b\}}^{v} \frac{b+2u}{b+2v} du \right) P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)dv \\
&\quad = \int_0^\infty \frac{2bv + 4v^2 - 2b\max\{0, v-b\} - 4\max\{0, v-b\}^2}{2(b+2v)} P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)dv \\
&\quad \geq \int_0^b P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)v dv + \int_b^\infty \frac{4bv}{2(b+2v)} P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)dv \\
&\quad \geq \frac{1}{2}\int_0^b P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)2v dv + \frac{1}{2}b\int_b^\infty P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)dv \\
&\quad \geq \frac{1}{2}\int_0^\infty P_{b+2v}\hat{\Gamma}(P_{2v} f, P_{2v} f)\min(2v, b)dv.
\end{aligned}
$$

Without $\Gamma^2 \geq 0$ we only obtain

$$
\begin{aligned}
P_b|f|^2 - |P_b f|^2 &\geq \frac{1}{2}\int_0^\infty P_{b+2v}|P'_{2v} f|^2 \min(2v, b)dv \\
&= \frac{1}{4}\int_0^\infty P_{b+v}\left|P'_v f\right|^2 \min(v, b)dv.
\end{aligned}
$$

This is (ii) and the first inequality of (iii). To complete the proof of (iii) we start with (2.3) and $\Gamma^2 \geq 0$:

$$
P_b|f|^2 - |P_b f|^2 = 2\int_0^\infty \int_{\max\{0,v-b\}}^{v} P_{b-v+2u}\hat{\Gamma}(P_v f, P_v f)du dv
$$

$$\begin{aligned}
&\le 2\int_0^\infty \int_{\max\{0,v-b\}}^{v} P_{b-\frac{v}{3}+2u}\hat{\Gamma}(P_{\frac{v}{3}}f, P_{\frac{v}{3}}f)dudv \\
&= 2\int_0^b \int_0^v P_{b-v+2u}\hat{\Gamma}(P_v f, P_v f)dudv \\
&\quad + 2\int_b^\infty \int_{v-b}^{v} P_{b-v+2u}\hat{\Gamma}(P_v f, P_v f)dudv \\
&= I + II.
\end{aligned}$$

For $v \ge b$ we have

$$\frac{b+v}{3} \le b - \frac{v}{3} + 2u \le \frac{5}{3}(b+v).$$

Thus monotonicity implies

$$\begin{aligned}
II &\le 2\int_0^\infty \int_{v-b}^{v} P_{b-\frac{v}{3}+2u}\hat{\Gamma}\left(P_{\frac{v}{3}}f, P_{\frac{v}{3}}f\right)dudv \le 10b\int_b^\infty P_{\frac{b+v}{3}}\hat{\Gamma}\left(P_{\frac{v}{3}}f, P_{\frac{v}{3}}f\right)dv \\
&= 90\int_{\frac{b}{3}}^\infty P_{\frac{b}{3}+s}\hat{\Gamma}\left(P_s f, P_s f\right)\min\left(s, \frac{b}{3}\right)\, ds.
\end{aligned}$$

In the range $v \le b$ and $0 \le u \le v$ we also have

$$\frac{b+v}{3} \le b + 2u - \frac{v}{3} \le \frac{5}{3}(b+v).$$

Again by monotonicity and with $s = \frac{v}{3}$ we obtain

$$I \le 10\int_0^b P_{\frac{b+v}{3}}\hat{\Gamma}\left(P_{\frac{v}{3}}f, P_{\frac{v}{3}}f\right)\, vdv = 90\int_0^{\frac{b}{3}} P_{\frac{b}{3}+s}\hat{\Gamma}(P_s f, P_s f)\, sds.$$

This yields

$$P_b|f|^2 - |P_b f|^2 \le 180\int_0^\infty P_{\frac{b}{3}+s}\hat{\Gamma}\left(P_s f, P_s f\right)\, \min\left(\frac{b}{3}, s\right)ds.$$

□

In view of the classical Carleson-measure-characterization of BMO, we define, for $f \in \mathcal{N} \cup L_2(\mathcal{N})$,

$$\|f\|_{BMO^c(\partial)} = \left\| \sup_t P_t \int_0^t |P_s' f|^2 s ds \right\|^{\frac{1}{2}}, \tag{2.4}$$

$$\|f\|_{BMO^c(\Gamma)} = \sup_t \left\| P_t \int_0^t \Gamma(P_s f, P_s f) s ds \right\|^{\frac{1}{2}}, \tag{2.5}$$

$$\|f\|_{BMO^c(\hat{\Gamma})} = \sup_t \left\| P_t \int_0^t \hat{\Gamma}(P_s f, P_s f) s ds \right\|^{\frac{1}{2}}. \tag{2.6}$$

Theorem 2.6 *Let (T_t) be a standard semigroup. Then*

$$\|f\|_{BMO^c(\partial)} \leq c\|f\|_{BMO^c(\mathcal{P})} \leq c\|f\|_{BMO^c(\mathcal{T})} \tag{2.7}$$

Proof To prove the first inequality, recall that Theorem 3.2 of [27] states that

$$\sup_t \left\| P_t \int_0^t \left| \frac{\partial P_s}{\partial s}(f - P_s f) \right|^2 s ds \right\|^{\frac{1}{2}} \leq \|f\|_{BMO^c(\mathcal{P})}.$$

Then

$$\begin{aligned} \left\| P_t \int_0^t \left| \frac{\partial P_s}{\partial s} f \right|^2 s ds \right\|^{\frac{1}{2}} &\leq \left\| P_t \int_0^t \left| \frac{\partial P_s}{\partial s}(f - P_s f) \right|^2 s ds \right\|^{\frac{1}{2}} \\ &\quad + \left\| P_t \int_0^t \left| \frac{\partial P_s}{\partial s} P_s f \right|^2 s ds \right\|^{\frac{1}{2}} \\ &\leq \|f\|_{BMO^c(\mathcal{P})} + \left\| P_t \int_0^{\frac{t}{2}} \left| \frac{\partial P_s}{\partial s} P_s f \right|^2 s ds \right. \\ &\quad \left. + P_t \int_{\frac{t}{2}}^t \left| \frac{\partial P_s}{\partial s} P_s f \right|^2 s ds \right\|^{\frac{1}{2}} \end{aligned}$$

$$(v=2s) \le \|f\|_{BMO^c(\mathcal{P})} + \left\| \frac{1}{4} P_t \int_0^{\frac{t}{2}} \left| \frac{\partial P_v}{\partial v} f \right|^2 v dv \right.$$

$$\left. + P_t \int_{\frac{t}{2}}^{t} P_{\frac{t}{2}} \left| \frac{\partial P_s}{\partial s} P_{s-\frac{t}{2}} f \right|^2 s ds \right\|^{\frac{1}{2}}$$

$$\left(u = 2s - \frac{t}{2}\right) \le \|f\|_{BMO^c(\mathcal{P})} + \left\| \frac{1}{4} P_t \int_0^{\frac{t}{2}} |P'_v f|^2 v dv \right.$$

$$\left. + \frac{1}{2} P_{\frac{3t}{2}} \int_{\frac{t}{2}}^{\frac{3t}{2}} |P'_u f|^2 u du \right\|^{\frac{1}{2}}$$

$$\le \|f\|_{BMO^c(\mathcal{P})} + \frac{\sqrt{3}}{2} \sup_t \left\| P_t \int_0^t |P'_s f|^2 s ds \right\|^{\frac{1}{2}}.$$

Taking supremum on both sides, we have,

$$\|f\|_{BMO(\partial)} = \sup_t \left\| P_t \int_0^t \left| \frac{\partial P_s}{\partial s} f \right|^2 s ds \right\|^{\frac{1}{2}} \le \frac{1}{1 - \frac{\sqrt{3}}{2}} \|f\|_{BMO^c(\mathcal{P})}.$$

To prove the second inequality, we apply (1.2) and (1.1),

$$P_s|f - P_s f|^2 = \int_0^\infty \phi_s(u) T_u \left| f - \int_0^\infty \phi_s(v) T_v f dv \right|^2 du$$

$$= \int_0^\infty \phi_s(u) T_u \left| \int_0^\infty \phi_s(v)(f - T_v f) dv \right|^2 du$$

$$\le \int_0^\infty \int_0^\infty \phi_s(u)\phi_s(v) T_u |f - T_v f|^2 dv du$$

with $\phi_s(v) = s e^{\frac{-s^2}{4v}} v^{\frac{-3}{2}}$. For $v \le u$, we have

$$\|T_u|f - T_v f|^2\| = \|T_{u-v} T_v |f - T_v f|^2\| \le \|f\|^2_{BMO^c(\mathcal{T})}.$$

For $v > u$, let k be the biggest integer smaller than $\log_2^{\frac{v}{u}}$, we have

$$\begin{aligned}\|T_u|f - T_v f|^2\|^{\frac{1}{2}} &\leq \|T_u|f - T_u f|^2\|^{\frac{1}{2}} + \ \|T_u|T_u f - T_{2u} f|^2\|^{\frac{1}{2}} \\ &\quad + \|T_u|T_{2u} f - T_{4u} f|^2\|^{\frac{1}{2}} + \cdots + \|T_u|T_{2^k u} f - T_v f|^2\|^{\frac{1}{2}} \\ &\leq c\left(\ln \frac{v}{u} + 1\right) \|f\|_{BMO^c(\mathcal{T})}.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\|P_s|f - P_s f|^2\|^{\frac{1}{2}} &\leq \left(\int_0^\infty \int_u^\infty \ln \frac{v}{u} \phi_s(u)\phi_s(v) dv du \right. \\ &\quad \left. + \int_0^\infty \int_0^\infty \phi_s(u)\phi_s(v) dv du \right) \|f\|_{BMO^c(\mathcal{T})} \\ &\leq c\|f\|_{BMO^c(\mathcal{T})}.\end{aligned}$$

Taking supremum over t yields the second inequality. □

Lemma 2.7 *Let (T_t) be a standard semigroup. Then*

$$\|f\|_{BMO^c(\partial)} \simeq \sup_t \left\| \int_0^\infty P_{s+t}|P_s' f|^2 \min(s,t) ds \right\|^{\frac{1}{2}}. \tag{2.8}$$

If in addition $\Gamma^2 \geq 0$,

$$\|f\|_{BMO^c(\Gamma)} \simeq \left\| \sup_t \int_0^\infty P_{s+t}\Gamma(P_s f, P_s f) \min(s,t) ds \right\|^{\frac{1}{2}}. \tag{2.9}$$

$$\|f\|_{BMO^c(\hat{\Gamma})} \simeq \left\| \sup_t \int_0^\infty P_{s+t}\hat{\Gamma}(P_s f, P_s f) \min(s,t) ds \right\|. \tag{2.10}$$

Proof Let $\partial_t = \frac{\partial}{\partial t}$ and $\Gamma_{\partial_t^2}(f, f) = \left|\frac{\partial f_t}{\partial t}\right|^2$ the gradient forms associated with $T_t = e^{t\partial_t^2}$ which satisfies $\Gamma^2_{\partial_t^2} \geq 0$. Then (2.8) follows from (2.9). Moreover, (2.10) follows from $\hat{\Gamma}(f_t, f_t) = \Gamma(f_t, f_t) + |f_t'|^2$. To prove (2.9), we apply the condition $\Gamma^2 \geq 0$ and find

$$\begin{aligned}\left\|\int_0^t P_t\Gamma(P_v f, P_v f)\, v dv\right\| &\le \left\|\int_0^t P_{\frac{v}{2}+t}\Gamma\left(P_{\frac{v}{2}} f, P_{\frac{v}{2}} f\right) v dv\right\| \\ &= 4\left\|\int_0^{\frac{t}{2}} P_{s+t}\Gamma(P_s f, P_s f)\, s ds\right\| \\ &\le 4\left\|\int_0^{\infty} P_{s+t}\Gamma(P_s f, P_s f)\min(s,t) ds\right\|.\end{aligned}$$

For the reversed relation, we use a dyadic decomposition. Indeed, according to Proposition 1.2, we have

$$\frac{2^n t P_{s+t}}{s+t}\Gamma(P_s f, P_s f) \le P_{2^n t}\Gamma(P_s f, P_s f)$$

for $s \ge 2^n t$. This implies

$$\begin{aligned}&\frac{1}{2}\int_0^{\infty} P_{s+t}\Gamma(P_s f, P_s f)\min(s,t) ds \\ &\le \int_0^{\infty} P_{s+t}\Gamma(P_s f, P_s f)\frac{st}{s+t} ds \\ &= \int_0^{2t} P_t\Gamma(P_s f, P_s f)\frac{st}{s+t} ds + \sum_{n=1}^{\infty}\frac{1}{2^n}\int_{2^n t}^{2^{n+1}t}\frac{2^n t P_{s+t}}{s+t}\Gamma(P_s f, P_s f) s ds \\ &\le \int_0^{2t} P_t\Gamma(P_s f, P_s f) s ds + \sum_{n=1}^{\infty}\frac{1}{2^n}\int_{2^n t}^{2^{n+1}t} P_{2^n t}\Gamma(P_s f, P_s f) s ds \\ &\le \int_0^{2t} P_t\Gamma(P_s f, P_s f) s ds + \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^{2^{n+1}t} P_{2^n t}\Gamma(P_s f, P_s f) s ds.\end{aligned}$$

However, we can replace $2t$ by t using $\Gamma^2 \ge 0$ and Lemma 1.2:

$$\begin{aligned}\int_0^{2t} P_t\Gamma(P_s f, P_s f) s ds &\le \int_0^{2t} P_{t+\frac{s}{2}}\Gamma(P_{\frac{s}{2}} f, P_{\frac{s}{2}} f) s ds \\ &= 4\int_0^t P_{t+v}\Gamma(P_v f, P_v f) v dv\end{aligned}$$

$$\leq \ 8 \int_0^t P_t \Gamma(P_v f, P_v f) v dv.$$

Applying this argument for every $2^{n+1} t$, we deduce the assertion. □

Lemma 2.8 *Let* $a > 1$. *Then*

$$\sup_t \|P_t f - P_{at} f\| \ \leq \ \sqrt{2} \left(1 + \log_{\frac{3}{2}} a\right) \|f\|_{BMO^c(\partial)}.$$

Proof For t fixed, we have the

$$\begin{aligned} |P_{3t} f - P_{2t} f|^2 \ &\leq \ P_{\frac{3t}{2}} \left(\left| P_{\frac{3t}{2}} f - P_{\frac{t}{2}} f \right|^2 \right) \ = \ P_{\frac{3t}{2}} \left(\left| \int_{\frac{t}{2}}^{\frac{3t}{2}} P_s' f ds \right|^2 \right) \\ &\leq \ P_{\frac{3t}{2}} \left(t \int_{\frac{t}{2}}^{\frac{3t}{2}} |P_s' f|^2 ds \right) \ \leq \ 2 P_{\frac{3t}{2}} \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} |P_s' f|^2 s ds \right) \\ &\leq \ 2 P_{\frac{3t}{2}} \left(\int_0^{\frac{3t}{2}} |P_s' f|^2 s ds \right). \end{aligned}$$

This implies in particular that

$$\sup_t \|P_t f - P_{\frac{3t}{2}} f\| \ \leq \ \sqrt{2} \|f\|_{BMO^c(\partial)}.$$

For $1 < a \leq \frac{3}{2}$, choose $b \geq 0$ such that $\frac{a-b}{1-b} = \frac{3}{2}$. Then we obtain

$$\begin{aligned} \left\| \, |P_t f - P_{at} f|^2 \right\| &\leq \left\| P_{bt} \left| P_{(1-b)t}(f) - P_{\frac{3}{2}(1-b)t}(f) \right|^2 \right\| \\ &\leq \left\| \left| P_{(1-b)t}(f) - P_{\frac{3}{2}(1-b)t}(f) \right|^2 \right\| \ \leq \ 2 \, \|f\|^2_{BMO^c(\partial)}. \end{aligned} \tag{2.11}$$

We deduce

$$\|P_t(f) - P_{at}(f)\| \ \leq \ \sqrt{2} \, \|f\|_{BMO^c(\partial)} \tag{2.12}$$

for any $1 < a \le \frac{3}{2}$. Consider now $a > \frac{3}{2}$. Let n be the integer part of $\log_{\frac{3}{2}} a$. We may use a telescopic sum

$$P_t f - P_{at} f \;=\; \left(P_t f - P_{\frac{3t}{2}} f\right) + \left(P_{\frac{3t}{2}} f - P_{\frac{3}{2}\frac{3t}{2}} f\right) + \cdots \left(P_{\left(\frac{3}{2}\right)^{nt}} f - P_{at} f\right).$$

We apply (2.12) for every summand. Then the triangle inequality implies the assertion. □

Theorem 2.9 *Let (T_t) be a standard semigroup satisfying $\Gamma^2 \ge 0$. Then $\| \ \|_{BMO^c(\mathcal{P})}$, $\| \ \|_{bmo^c(\mathcal{P})}$ and $\| \ \|_{BMO^c(\hat{\Gamma})}$ are all equivalent on $\mathcal{N} \cup L_2(\mathcal{N})$.*

Proof According to Proposition 2.5 we know that

$$\sup_t \left\| \int_0^\infty P_{s+t}\hat{\Gamma}(P_s f, P_s f) \min(s,t) ds \right\|^{\frac{1}{2}} \sim_{180} \|f\|_{bmo^c(\mathcal{P})}.$$

Then Lemma 2.7 implies that $\| \ \|_{bmo^c(\mathcal{P})}$ and $\| \ \|_{BMO^c(\hat{\Gamma})}$ are equivalent. Proposition 2.4 (iii) provides the upper estimate of $\| \ \|_{bmo^c(\mathcal{P})}$ against $\| \ \|_{BMO^c(\mathcal{P})}$. Conversely, we deduce from Proposition 2.4 (ii), Lemma 2.8, Lemma 2.7 and Proposition 2.5 (i) that

$$\begin{aligned} \|f\|_{BMO^c(\mathcal{P})} &\le 2\|f\|_{bmo^c(\mathcal{P})} + \sup_t \|P_t f - P_{2t} f\| \\ &\le 2\|f\|_{bmo^c(\mathcal{P})} + \sqrt{2}(1 + \log_{\frac{3}{2}} 2)\|f\|_{BMO^c(\hat{\Gamma})} \\ &\le 2\|f\|_{bmo^c(\mathcal{P})} + 2\sqrt{2}\, 2\sqrt{6}\, \|f\|_{bmo^c(\mathcal{P})} \;=\; (2 + 8\sqrt{3})\|f\|_{bmo^c(\mathcal{P})}. \end{aligned}$$

Thus all the norms are equivalent on $\mathcal{N} \cup L_2(\mathcal{N})$. □

3 Bounded Fourier multipliers on BMO

In this section we prove the BMO boundedness for certain singular integrals obtained as a function of the generator for arbitrary semigroups. The ideas for the proof can be traced back to E. Stein's universal L_p-bounded for Fourier multipliers.

Lemma 3.1 *Let Γ be the gradient form associated with a standard semigroup S_t satisfying $\Gamma^2 \ge 0$. Then*

$$\Gamma\left(\int_\Omega f_t d\mu(t), \int_\Omega f_t d\mu(t)\right) \le \int_\Omega |d\mu(t)| \int_\Omega \Gamma(f_t, f_t)|d\mu(t)|, \qquad (3.1)$$

for $\mathcal{N}$-valued function f on a measure space $\{\Omega, \mu\}$ such that $\Gamma(f_t, f_t)$ is weakly measurable. In particular, Let P_t be a Poisson semigroup subordinated to a standard *semigroup T_t. Then,*

$$\Gamma(v\partial P_v f, v\partial P_v f) \le c P_{\frac{v}{2}} \Gamma(f, f),$$

Proof Let us recall the standard module construction for Γ. We consider $\mathcal{A} \otimes_\Gamma \mathcal{N}$ with inner product

$$\left\langle \sum_j a_j \otimes b_j, \sum_k \tilde{a}_k \otimes \tilde{b}_k \right\rangle = \sum_{j,k} b_j^* \Gamma(a_j, \tilde{a}_k) \tilde{b}_k.$$

It is easy to see that $\langle z, z\rangle \ge 0$. Indeed, by inequality (1.1), we see $S_s(|S_{t-s} f|^2)$ is increasing with respect to s for any $s < t$. On the other hand, taking the derivative in s, we find

$$\frac{\partial S_s(|S_{t-s} f|^2)}{\partial s} = S_s \Gamma(S_{t-s} f, S_{t-s} f).$$

This implies $\Gamma(f, f) \ge 0$ for any f. Taking matrices we find similarly $\langle z, z\rangle \ge 0$. Therefore $\mathcal{A} \otimes_\Gamma \mathcal{N}$ is a (non-complete) Hilbert C^*-module over $\mathcal{N}$, and as such isomorphic to $H^c \otimes_{\min} \mathcal{N}$ for some Hilbert space H ([25]). Let $w : \mathcal{A} \otimes_\Gamma \mathcal{N} \to H^c \otimes_{\min} \mathcal{N}$ be the isomorphism. Then we can define $u(f) = w(f \otimes 1)$ and deduce

$$u(f)^* u(g) = \langle f \otimes 1, g \otimes 1\rangle = \Gamma(f, g).$$

With the help of this map it is easy to conclude using the convexity of $|\cdot|^2$

$$\begin{aligned}
\Gamma\left(\int_\Omega f_t d\mu(t), \int_\Omega f_t d\mu(t)\right) &= \left|\int_\Omega u(f_t) d\mu(t)\right|^2 \\
&\le \int_\Omega |d\mu(t)| \int_\Omega |u(f_t)|^2 |d\mu(t)| \\
&= \int_\Omega |d\mu(t)| \int_\Omega \Gamma(f_t, f_t) |d\mu(t)|.
\end{aligned}$$

The convergence of the integral here is in the weak sense, (i.e. considering the real-valued function $(h, \Gamma(f_t, f_t)h)$). By (1.2), we may write $v\partial P_v$ as $\int_0^\infty T_\tau d\mu(\tau)$ with

$\int_0^\infty |d\mu(\tau)| \le c$ and $\int_0^\infty T_\tau |d\mu(\tau)| \le c P_{\frac{v}{2}}$. We deduce from (3.1) and $\Gamma^2 \ \ge\ 0$ that

$$\Gamma(v\partial P_v f, v\partial P_v f) \le c \int_0^\infty \Gamma(T_\tau f, T_\tau f)|d\mu(\tau)|$$
$$\le c \int_0^\infty T_\tau \Gamma(f, f)|d\mu(\tau)| \le c P_{\frac{v}{2}} \Gamma(f, f).$$

□

Recall that $\Gamma_{\partial_t^2}(f, g) = (\partial_t f^*)(\partial_t g)$ is the gradient forms associated with $T_t = e^{t\partial_t^2}$ and satisfies $\Gamma^2_{\partial_t^2} \ge 0$. According to Lemma 3.1, we know that

$$|\partial_v (v\partial_v P_v f)|^2 \le c P_{\frac{v}{2}} |v\partial_v P_v f|^2. \tag{3.2}$$

Since $\widehat{\Gamma} = \Gamma + \Gamma_{\partial_t^2}$, we obtain

$$\widehat{\Gamma}(v P_v' f, v P_v' f) \le c P_{\frac{v}{2}} \widehat{\Gamma}(P_v f, P_v f). \tag{3.3}$$

We now want to define singular integrals of the form $F(A)$ where F is a nice function. We follows Stein's idea and assume that F is given by a Laplace transform. Let a be a scalar valued function such that

$$s \int_s^\infty \frac{|a(v)|^2}{v^2} dv \le c_a^2,$$

for all $s > 0$ and some constant positive c_a. Define M_a as

$$M_a(f) = \int_0^\infty a(t) \frac{\partial P_t f}{\partial t} dt.$$

Lemma 3.2 *Assume T_t be a standard semigroup satisfying $\Gamma^2 \ge 0$. We have*

$$\|M_a(f)\|_{BMO^c(\Gamma)} \le c c_a \|f\|_{BMO^c(\Gamma)}.$$

Proof Let

$$S_t(f) = P_t \int_0^t s\Gamma(P_s f, P_s f) ds.$$

We simplify the notation by using $\Gamma[f] = \Gamma(f, f)$. Let us compute $\|S_t(M_a(f))\|$:

$$\begin{aligned}
\|S_t(M_a(f))\| &= \left\| P_t \int_0^t s\Gamma[P_s M_a(f)]ds \right\| \\
&= \left\| \int_0^t sP_t\Gamma\left[P_s \int_0^\infty a(v) \frac{\partial P_v}{\partial v} f dv \right] ds \right\| \\
&= \left\| \int_0^t sP_t\Gamma\left[\int_0^\infty a(v) \frac{\partial P_{v+s}}{\partial v} f dv \right] ds \right\| \\
&= \left\| \int_0^t sP_t\Gamma\left[\int_s^\infty a(v-s) \frac{1}{v} v \frac{\partial P_v}{\partial v} f dv \right] ds \right\| \\
\text{(first inequality of Lemma 3.1)} &\le \left\| \int_0^t sP_t \left(\int_s^\infty \frac{|a|^2}{v^2} dv \int_s^\infty \Gamma\left[v\frac{\partial P_v}{\partial v} f \right] dv \right) ds \right\| \\
&\le c_a \left\| \int_0^t P_t \left(\int_s^\infty \Gamma\left[v\frac{\partial P_v}{\partial v} f \right] dv \right) ds \right\| \\
\text{(change of variables)} &= 8c_a \left\| \int_0^t P_t \left(\int_{\frac{s}{2}}^\infty \Gamma\left[v\frac{\partial P_v}{\partial v} P_v f \right] dv \right) ds \right\| \\
\text{(inequality (3.1))} &\le cc_a \left\| \int_0^t P_t \left(\int_{\frac{s}{2}}^\infty P_{\frac{v}{2}} \Gamma[P_v f] dv \right) ds \right\| \\
&\le cc_a \left\| \int_0^t \int_{\frac{s}{2}}^\infty P_{\frac{v}{2}+t} \Gamma[P_v f] dv ds \right\| \\
\text{(Integrate } ds \text{ first)} &= cc_a \left\| \int_0^\infty P_{\frac{v}{2}+t} \Gamma[P_v f] \min(t, 2v) dv \right\| \\
\text{(Lemma 2.7)} &\le cc_a \|f\|^2_{BMO^c(\Gamma)}
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|M_a(f)\|_{BMO^c(\Gamma)} = \sup_t \|S_t(M_a(f))\|^{\frac{1}{2}} \le cc_a \|f\|_{BMO^c(\Gamma)}.$$

□

Using (3.2), exactly the same proof shows that, without assuming $\Gamma^2 \geq 0$,

$$\|M_a : BMO^c(\partial) \to BMO^c(\partial)\| \leq cc_a \tag{3.4}$$

Our main tool, the formula $\Gamma(v\partial_v P_v f, v\partial_v P_v f) \leq c\Gamma(P_v f, P_v f)$ from Lemma 3.1 also holds for higher order derivatives that $\Gamma(v^n \partial_v^n P_v f, v^n \partial_v^n P_v f) \leq c_n \Gamma(P_v f, P_v f)$. Therefore the same technique allows us to obtain estimates for operators of the form

$$M_{a,n} = \int_0^\infty a(t)t^{n-1}\partial_t^n P(t)dt.$$

Here $n \in \mathbb{N} \cup \{0\}$ and $\partial_t^n P(t)$ is the n-th derivative of $P(t)$ with respect to t. Let us state this explicitly.

Theorem 3.3 *Let T_t be a standard semigroup. Then*

$$\|M_{a,n}(f)\|_{BMO^c(\partial)} \leq c_n c_a \|f\|_{BMO^c(\partial)}. \tag{3.5}$$

If in addition, T_t satisfies $\Gamma^2 \geq 0$, then

$$\|M_{a,n}(f)\|_{BMO^c(\Gamma)} \leq c_n c_a \|f\|_{BMO^c(\Gamma)}. \tag{3.6}$$

Corollary 3.4 *Let T_t be a standard semigroup satisfying $\Gamma^2 \geq 0$. Then*

$$\|M_a(f)\|_{BMO^c(\mathcal{P})} \leq cc_a \|f\|_{BMO^c(\mathcal{P})}.$$

Proof By Theorem 2.9, we know

$$\|f\|_{BMO^c(\mathcal{P})} \simeq \|f\|_{BMO^c(\hat{\Gamma})}.$$

By the definition of $\hat{\Gamma}$, we see that

$$\|f\|_{BMO^c(\hat{\Gamma})} \simeq \max\{\|f\|_{BMO^c(\Gamma)}, \|f\|_{BMO^c(\partial)}\}.$$

Therefore, Corollary 3.4 follows from Theorem 3.3. □

Example 3.5 Let $-\phi$ be a real valued, symmetric, conditionally negative function on a discrete group G satisfying $\phi(1) = 0$. Let A be the unbounded operator defined on $\mathbb{C}[G]$ as

$$A(\lambda(g)) = \phi(g)\lambda(g).$$

Let $T_t = \exp(-tA)$, i.e.

$$T_t(\lambda(g)) = \exp(-t\phi(g))\lambda(g).$$

$(T_t)_t$ extends to a standard semigroup of operators with generator $-A$ on the group von Neumann algebra $\mathcal{N} = VN(G)$ following Schoenberg's theorem. T_t satisfies $\Gamma^2 \geq 0$ too. Therefore,Theorem 3.3 and Corollary 3.4 applies in this setting. Here we note that M_a is indeed a Fourier multiplier. Indeed, assume that m is a complex valued function of the form

$$m(g) = c\int_0^\infty \phi^{\frac{1}{2}}(g)e^{-t\phi^{\frac{1}{2}}(g)}a(t)dt.$$

Then $M_a(\lambda(g)) = m(g)\lambda(g)$. For example we may consider $a(t) = t^{-2is}$ with s a real number. Then we deduce that $m(g) = \Gamma(1-is)[\phi(g)]^{is}$ is a Fourier multiplier. Note the subordinated semigroup in this case is given by

$$P_t(\lambda(g)) = e^{-t\sqrt{\psi}(g)}\lambda(g).$$

Therefore Corollary 3.4 imply that

$$\|M_a(f)\|_{BMO^c(\mathcal{P})} \leq cc_a\|f\|_{BMO^c(\mathcal{P})}. \tag{3.7}$$

for all $f \in L_2(VN(G))$.

In the remaining part of this article, we will use probabilistic methods to prove an interpolation theorem for semigroup BMO spaces. This in turn allows us to obtain L_p bounds for Fourier multipliers of the form above.

4 Probabilistic models for semigroup of operators

In the section, we introduce BMO spaces for noncommutative martingales and P. A. Meyer's probabilistic model for semigroup of operators. We will apply them in the next section to an interpolation theorem for BMO associated with semigroups.

4.1 Noncommutative martingales

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra equipped with a semifinite normal faithful trace τ. We will say that an increasing family $(\mathcal{M}_t)_{t\geq 0}$ is an *increasing filtration* if if $s < t$ implies $\mathcal{M}_s \subset \mathcal{M}_t$, $\bigcup_t \mathcal{M}_t$ is weakly dense, and the restriction of the trace is semifinite and faithful for every $\mathcal{M}_t$. We refer to [45] for the fact that this implies the existence of a uniquely determined trace preserving conditional expectations $E_t : \mathcal{M} \to \mathcal{M}_t$. By uniqueness we have $E_sE_t = E_{\min(s,t)}$. Right continuity, i.e. $\bigcap_{s>t} \mathcal{M}_s = \mathcal{M}_t$ for all $t \geq 0$, will be part of the assumption when we talk about increasing filtrations. Similarly, we will say that $(\mathcal{M}_t)_{t\geq 0}$ is a *decreasing filtration* if $s < t$ implies $\mathcal{M}_s \supset \mathcal{M}_t$, $\mathcal{M}$ is the weak closure of $\bigcup_t \mathcal{M}_t$, and we have left continuity. Again we have a family of conditional expectations $E_s : \mathcal{M} \to \mathcal{M}_s$ such that $E_sE_t = E_{\max(s,t)}$. We have $\mathcal{M}_0 = \mathcal{M}$, $E_0 = id$ for decreasing filtration and set $\mathcal{M}_\infty = \wedge_t \mathcal{M}_t$ as a convention. Set $\mathcal{M}_\infty = \mathcal{M}, E_\infty = id, \mathcal{M}_0 = \wedge_t \mathcal{M}_t$ for increasing filtration. A (reversed) martingale adapted to $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a family $(x_t) \in L^1(\mathcal{M}) + L^\infty(\mathcal{M})$ such that $E_t(x_s - x_t) = 0$ for any $s > t \geq 0$ for increasing filtration (for $t > s \geq 0$ for decreasing filtration).

For $x \in L_p(\mathcal{M})$, $1 \leq p \leq \infty$, the family (x_t) given by $x_t = E_t x$ is a martingale with respect to $\mathcal{M}_t$. For $2 < p \leq \infty$, we define

$$\|x\|_{L_p^c mo(\mathcal{M})} = \left\| \sup_t{}^+ E_t\left(|x - E_t x|^2\right) \right\|_{\frac{p}{2}}^{\frac{1}{2}},$$

where $\| \sup^+ \cdot \|_{\frac{p}{2}}$ should be understood in the sense of vector-valued noncommutative L_p spaces, see [20,22,35]). Let

$$\|x\|_{L_p mo(\mathcal{M})} = \max\left\{ \|x\|_{L_p^c mo(\mathcal{M})}, \|x^*\|_{L_p^c mo(\mathcal{M})} \right\}.$$

By Doob's inequality, we know that

$$\|x\|_{L_p mo(\mathcal{M})} \leq c_p \|x\|_{L_p(\mathcal{M})}. \tag{4.1}$$

Let $L_p^0(\mathcal{M})$, $1 \leq p \leq \infty$, be the quotient space of $L_p(\mathcal{M})$ by $\{x, x = \mathcal{E}x\}$. Here $\mathcal{E}$ is the projection from $\mathcal{M}$ onto $\wedge_t \mathcal{M}_t$, which equals to E_0 in the case of increasing filtration and equals to E_∞ in the case of decreasing filtration. For $2 < p < \infty$, let $L_p^0 mo(\mathcal{M})$ ($L_p^{c,0} mo(\mathcal{M})$) be the completion of $\mathcal{M}^0 = L_\infty^0(\mathcal{M})$ by $\| \cdot \|_{L_p mo(\mathcal{M})}$ ($\| \cdot \|_{L_p^c mo(\mathcal{M})}$)-norm. For $p = \infty$, we have to consider a weak* completion and denote the completed spaces by $bmo^c(\mathcal{M})$ (resp. $bmo(\mathcal{M})$). We refer the interested readers to [11,16] for more information on noncommutative martingales with continuous filtrations.

We now introduce martingale h_q-space, which are preduals of $L_p mo'$s. Let $\sigma = \{0 = s_0 < s_1, \ldots, s_{n-1} < s_n = \infty\}$ be a finite partition of $[0, \infty]$. For $x \in L^1(\mathcal{M}) + L^\infty(\mathcal{M})$, define the conditioned bracket $\langle x, x\rangle(\sigma)$ ($k \leq n$) as

$$\langle x, x\rangle(\sigma) = \sum_{j=1}^n E_{s_{j-1}} \left| E_{s_j} x - E_{s_{j-1}} x \right|^2.$$

The $h_p^c(\sigma)$, $1 \leq p < \infty$,-norm of x is defined as

$$\|x\|_{h_p^c} = \left\| (\langle x, x\rangle(\sigma))^{\frac{1}{2}} \right\|_{L_p}.$$

Let $\mathcal{U}$ be an ultrafilter refining the natural order given by inclusion on the set of all partitions of $[0, \infty]$. The $h_p^c(\mathcal{U})$ and $h_p^r(\mathcal{U})$-norms of x are defined as

$$\|x\|_{h_p^c} = \lim_{\sigma, \mathcal{U}} \left\| (\langle x, x\rangle(\sigma))^{\frac{1}{2}} \right\|_{L_p}, \quad \|x\|_{h_p^r} = \|x^*\|_{h_p^c}.$$

The $h_p^d(\mathcal{U})$-norm of x is defined as

$$\|x\|_{h_p^d} = \lim_{\sigma,\mathcal{U}} \left(\sum_{s_j \in \sigma} \left\| E_{s_j} x - E_{s_{j-1}} x \right\|_{L_p}^p \right)^{\frac{1}{p}}.$$

It is proved in [16] that these norms do not depend on the choice of $\mathcal{U}$ whenever $\mathcal{U}$ is containing the filter base of tails. Let $h_p^c(\mathcal{M})$ ($h_p^r(\mathcal{M}), h_p^d(\mathcal{M})$) be the collection of all x with finite $h_p^c(\mathcal{U})$ ($h_p^r(\mathcal{M}), h_p^d(\mathcal{U})$)-norm. It is proved in [16] that

$$\left(h_p^c(\mathcal{M})\right)^* = L_q^c mo(\mathcal{M}) = h_q^c(\mathcal{M}), 1 \leq p < 2, \frac{1}{p} + \frac{1}{q} = 1$$
$$h_p^c(\mathcal{M}) + h_p^r(\mathcal{M}) + h_p^d(\mathcal{M}) = L_p(\mathcal{M}), 1 < p < 2.$$

Denote by $h_p(\mathcal{M}) = h_p^c(\mathcal{M}) + h_p^r(\mathcal{M}), H_p(\mathcal{M}) = h_p(\mathcal{M}) + h_p^d(\mathcal{M}), 1 \leq p < 2, h_p(\mathcal{M}) = h_p^c(\mathcal{M}) \cap h_p^r(\mathcal{M})$ for $2 \leq p < \infty$, and $BMO(\mathcal{M}) = (H_1(\mathcal{M}))^* = bmo(\mathcal{M}) \cap (h_1^d(\mathcal{M}))^*$, we have

$$[BMO(\mathcal{M}), L_1(\mathcal{M})]_{\frac{1}{q}} = L_q(\mathcal{M}),$$

for all $1 < q < \infty$.

Recall that a martingale $x = (x_t)_t$ has *a.u.continuous* path provided that, for every $T > 0, \varepsilon > 0$ there exists a projection e with $\tau(1-e) < \varepsilon$ such that the function $f_e : [0, T] \to \mathcal{M}$ given by

$$f_e(t) = x_t e \in \mathcal{M}$$

is continuous. The following observation will be crucial for us.

Lemma 4.1 *Let x^λ be a net of martingales in $\mathcal{M} \cap L_2(\mathcal{M})$ with a. u. continuous path. Suppose x^λ weakly converges in $L_2(\mathcal{M})$ and the limit x is in bmo. Then $x \in BMO$ and*

$$\|x\|_{BMO} \leq c\|x\|_{bmo}.$$

Moreover, let $p > 2$ and $x \in L_p(\mathcal{M})$ with a.u. continuous path. Then $\|x\|_{h_p} \simeq_{c_p} \|x\|_{L_p(\mathcal{M})}$.

Proof We first prove that for martingales $x \in L_2(\mathcal{M}) \cap \mathcal{M}$ with a. u.continuous path, we have

$$\|x\|_{(h_1^d(\mathcal{M}))^*} = 0. \tag{4.2}$$

By Doob's inequality for noncommutative martingales, one can show that a. u. continuity and $x \in L_2(\mathcal{M}) \cap \mathcal{M}$ imply that $x_t = f(t)a$ for some $a \in L_q(\mathcal{M})$ and a continuous function $f : [0, T] \to \mathcal{M}$ for any $q > 2$, $T < \infty$. This implies that

$$\lim_{\sigma,\mathcal{U}} \|d_{t_j}(x)\|_{L_q(\ell_\infty^c)} = \lim_{\sigma,\mathcal{U}} \left\| \sup\nolimits^+_{t_j\in\sigma} d_{t_j}(x)^* d_{t_j}(x) \right\|_{q/2} = 0,$$

for any ultrafilter $\mathcal{U}$ of $[0, T]$ containing the filter base of tails. Note that

$$\left\|d_{t_j}((x)^*)\right\|_{L_q(\ell_\infty^c)} \leq \left\|d_{t_j}(x)\right\|_{L_q(\ell_q)} \leq \left\|d_{t_j}(x)\right\|^{1-\theta}_{L_q(\ell_\infty^c)} \|x\|^{\theta}_{H_q^c(\sigma)} \tag{4.3}$$

for $\theta = \frac{2}{q}$. Thus we also find that

$$\lim_{\sigma,\mathcal{U}} \|d_{t_j}(x)\|_{L_q(\ell_\infty^r)} = 0.$$

We recall from [16] that $\bigcup_{p>1} B_{h_p^{1c}+h_p^{1r}} \subset h_1^d$ are dense in the unit ball of h_1^d. Here the $h_p^{1_c}$ is defined such that the norm of $x \in L_2(\mathcal{M}) \cap (h_p^{1_c})^*$ is given by $\lim_{\sigma,\mathcal{U}} \|d_{t_j}(x)\|_{L_q(\ell_\infty^c)}$. Therefore, x satisfies (4.2) if x is in $\mathcal{M} \cap L_2(\mathcal{M})$ and has a. u. continuous path. Now, let x^λ be a net of weakly L_2-converging martingales in $\mathcal{M} \cap L_2(\mathcal{M})$ with a. u. continuous path. Suppose its weak L_2-limit x is in bmo. Recall from [16] that, for any $y \in H_1^c \cap L_2(\mathcal{M})$ we may find a decomposition such that $y = y_1 + y_2$ with $y_1 \in h_1^c \cap L_2(\mathcal{M})$, $y_2 \in h_1^d \cap L_2(\mathcal{M})$ and $\|y_1\|_{h_1^c} + \|y_2\|_{h_1^d} \leq 2\|y\|_{H_1^c}$. Then

$$\begin{aligned} |\tau(y^*x)| \leq |\tau(y_1^*x)| + |\tau(y_2^*x)| &= |\tau(y_1^*x)| + |\lim_\lambda \tau(y_2^* x^\lambda)| \\ &= |\tau(y_1^*x)| \leq c\|x\|_{bmo}\|y_1\|_{h_1^c}. \end{aligned}$$

Since the unit ball of $H_1^c(\mathcal{M}) \cap L_2(\mathcal{M})$ in dense in the unit ball of $H_1^c(\mathcal{M})$, we get

$$\|x\|_{bmo_c} \leq c\|x\|_{BMO_c}.$$

From (4.3) we have already seen that for martingales $x \in L_q(\mathcal{M})$ with continuous path we have $\lim_{\sigma,\mathcal{U}} \|x\|_{h_q^d(\sigma)} = 0$ because $\|x\|_{h_q^d(\sigma)} = \|d_{t_j}(x)\|_{L_q(\ell_q)}$. Hence we have

$$\|x\|_{H_q^c} \leq C\|x\|_{h_q^c}$$

for $q > 2$ because $H_q^c = h_q^c \cap h_q^d$ for $q > 2$. □

In the previous argument we learned for continuous martingales with a.u. continuous path we have $\|x\|_{h_p^d} = 0$ (see also [11]). In fact, in this paper we might simply take this as a definition. We will need some more results in this direction and state them in the following lemma.

Lemma 4.2 *Let $1 < p < \infty$. We have*

$$(BMO(\mathcal{M}), L_1(\mathcal{M}))_{\frac{1}{p}} = L_p(\mathcal{M}), \tag{4.4}$$

$$(bmo^c(\mathcal{M}), L_2(\mathcal{M}))_{\frac{2}{p}} = h_p^c(\mathcal{M}) \tag{4.5}$$

with equivalence constants $\simeq p$. Suppose that $x \in L_p(\mathcal{M})$, $2 < p < \infty$ and $(E_t x)_t$ is a.u. continuous. We have

$$\|x\|_{L_p mo(\mathcal{M})} + \|\mathcal{E}x\|_{L_p(\mathcal{M})} \simeq \|x\|_{L_p(\mathcal{M})}, \tag{4.6}$$

with equivalence constants $\simeq p$ for $p > 4$.

We say that a standard semigroup (T_t) on a semifinite von Neumann algebra $\mathcal{N}$ admits *a standard Markov dilation* if there exists a larger semifinite von Neumann algebra $\mathcal{M}$, an increasing filtration $(\mathcal{M}_{s]})_{s\,\geq\,0}$ and trace preserving *-homomorphism π_s such that

$$E_{s]}(\pi_t(x)) \;=\; \pi_s(T_{t-s}x) \quad s < t, \;\; x \in \mathcal{N}.$$

We say that (T_t) admits *a reversed Markov dilation* if there exists a larger von Neumann algebra $\mathcal{M}$, a decreasing filtration $(\mathcal{M}_{[s})_{s\,\geq\,0}$, and trace preserving *-homomorphisms $\pi_s : \mathcal{N} \to \mathcal{M}_{[s}$ such that

$$E_{[s}(\pi_t(x)) \;=\; \pi_s(T_{s-t}x) \quad t < s, \;\; x \in \mathcal{N}.$$

We say that (T_t) admits a *Markov dilation* if it admits either a standard dilation or a reversed dilation. We refer to [24] for related questions. A glance at (1.2) shows that a Markov dilation for (T_t) implies that the P_t's are factorable (in the sense of [1]). According to [1], we know that a Markov dilation for T_t (standard or reversed) yields a Markov dilation (standard and reversed) for P_t.

In the noncommutative setting the existence of a Markov dilation is no longer for free, as it is in the commutative case. We refer the reader to [39] for its existence for group von Neumann algebra and to [17] for its existence for finite von Neumann algebra. However, the existence of a Markov dilation allows us to use probabilistic tools for semigroups of operators. In particular, given a a reversed Markov dilation we know that $m(x) = (m_s(x))_{s \geq 0}$ with

$$m_s(x) = \pi_s(T_s x), \tag{4.7}$$

is a martingale with respect to the reversed filtration $(\mathcal{M}_{[s})$. A standard Markov dilation implies that, for any $v > 0$, $m(x) = (m_s(x))_{v \geq s \geq 0}$ with

$$m_s(x) = \pi_s(T_{v-s}x) \tag{4.8}$$

is a martingale with respect to the standard filtration $(\mathcal{M}_{s]})$.

Proposition 4.3 *Let (T_t) be a standard semigroup of operators on $\mathcal{N}$ with reversed Markov dilation $(\pi_t, \mathcal{M}_t)$. Let $x \in L_p(\mathcal{N})$. Then $E_{[s}(\pi_0 x) = \pi_s T_s x$ and*

$$\|\pi_0(x)\|_{L_p^c mo(\mathcal{M})} = \left\| \sup_t{}^+ \pi_t \left(T_t|x|^2 - |T_t x|^2 \right) \right\|_{\frac{p}{2}}^{\frac{1}{2}}, \tag{4.9}$$

for $2 < p \leq \infty$.In particular,

$$\|\pi_0(x)\|_{bmo^c(\mathcal{M})} = \|x\|_{bmo^c(\mathcal{T})}. \tag{4.10}$$

Proof To prove (4.9), we apply the reversed dilation condition and get $E_{[t}\pi_0(x) = \pi_t T_t x$. Then

$$E_{[t} |\pi_0(x)|^2 - \left|E_{[t}\pi_0(x)\right|^2 = \pi_t \left(T_t|x|^2 - |T_t x|^2 \right).$$

Taking the supremum over all t, we obtain the assertion. The Eq. (4.10) now follows immediately from the definition. □

4.2 Meyer's probabilistic model for semigroup of operators

Meyer's probabilistic model provides another way to connect semigroups of operators with martingales. Let us start with an observation due to Meyer [28].

Proposition 4.4 *Let T_t be a semigroup with a standard (resp. reversed) Markov dilation $(\pi_t, \mathcal{M}_t)$. For $x \in \mathrm{dom}(A)$, let $n(x) = (n_s(x))_{s \geq 0}$ with*

$$n_s(x) = \pi_s(x) + \int_0^s \pi_v(A(x))dv, \tag{4.11}$$

for standard Markov dilation and

$$n_s(x) = \pi_s(x) + \int_s^\infty \pi_v(A(x))dv, \tag{4.12}$$

for reversed Markov dilation. Then $n(x)$ is a (reversed) martingale with respect to the filtration $\mathcal{M}_s$.

Proof Let us verify that $E_t n_s(x) = n_t(x)$ for $t > s$ in the reversed dilation case. The verification for $t < s$ in the standard dilation case is similar. Due to the dilation property we have

$$
\begin{aligned}
E_t\left(\pi_s(x)+\int_s^\infty \pi_v(A(x))dv\right) &= \pi_t(T_{t-s}(x))+\int_s^t \pi_t\left(T_{t-v}A(x)\right)dv \\
&\quad+\int_t^\infty \pi_v(A(x))dv \\
&= \pi_t(T_{t-s}(x))+\int_s^t \pi_t\left(\partial_v T_{t-v}(x)\right)dv \\
&\quad+\int_t^\infty \pi_v(A(x))dv \\
&= \pi_t(x)+\int_t^\infty \pi_v(A(x))dv.
\end{aligned}
$$

This means $n(x)$ satisfies the martingale property $E_t n_s(x) = n_t(x)$ for $t > s$. □

The main ingredient in Meyer's model is to use Lévy's stopping time argument for the Brownian motion (see however [9,10] for more compact notation). Given a standard semigroup T_t with generator A, assume (T_t) admits a standard Markov dilation $(\pi_s, \mathcal{M}_s)$. We consider a new generator

$$
\hat{A} = -\frac{d^2}{dt^2} + A.
$$

defined densely on

$$
L^2(\mathbb{R}) \otimes L^2(\mathcal{N}).
$$

This leads to a new semigroup of operators $\hat{T}_t = \exp(-t\hat{A})$ such that

$$
\hat{\Gamma}(f(t), g(t)) = \Gamma(f(t), g(t)) + \frac{df^*(t)}{dt}\frac{dg(t)}{dt}.
$$

Let (B_t) be a classical one dimensional Brownian motion associated with dt (instead of the usual $\frac{1}{2}dt$ in the stochastic differential equation) such that $B_0 = a$ holds with probability 1. Let $\hat{M}_s = M_s^B \otimes \mathcal{M}_s$ with M_s^B the von Neumann algebra of the Brownian motion observed until time s. The Markov dilation for the new semigroup $\hat{T}_t$ is given by $\hat{M}_t$ and $\hat{\pi}_t(f(\cdot)) = \pi_t f(B_t(\cdot))$.

For $x \in L_p(\mathcal{N})$, $1 \le p \le \infty$, denote by Px for the $L_p(\mathcal{N})$-valued function on $[0, \infty)$

$$
Px(t) = P_t(x).
$$

Recall that we write $P'x$ for the functions $\frac{d}{dt}P_t x$. Fix a real number $a > 0$. Let $\mathbf{t}_a$ be the stopping time of B_t first hit 0, i.e.

$$\mathbf{t}_a = \inf\{t : B_t(\omega) = 0\}.$$

The following observation due to P. A. Meyer.

Proposition 4.5 *For any $x \in L_p(\mathcal{N})$, $1 \le p \le \infty$,*

$$\hat{n}_a(x) = (\hat{\pi}_{\mathbf{t}_a \wedge t} P x)_t$$

is a martingale with respect to the filtration

$$\hat{\mathcal{M}}_{t,a} = \bigvee_{v \le t} \hat{\pi}_{\mathbf{t}_a \wedge v}(\mathcal{N} \otimes L^\infty(\mathbb{R})).$$

Proof Apply Proposition 4.4 to $\hat{A}$, $\hat{\pi}_t$, we get $\hat{\pi}_t(Px)$ is a martingale because $\hat{A}$(Px)=0. Therefore, $\hat{\pi}_{\mathbf{t}_a \wedge t}(Px)$ is a martingale too since $\mathbf{t}_a$ is a stopping time. □

Let

$$\hat{\mathcal{M}}_a = \overline{\vee_{t \ge 0} \hat{\mathcal{M}}_{a,t}}^{wot} = \overline{\vee_{\mathbf{t}_a \ge t \ge 0} \mathcal{M}_t}^{wot}.$$

Let $\hat{E}_t$ be the conditional expectation from $\hat{\mathcal{M}}_a$ onto $\hat{\mathcal{M}}_{a,t}$. Proposition 4.5 implies that

$$\hat{E}_t(\pi_{\mathbf{t}_a} x) = \hat{\pi}_{t_a \wedge t} P x = \pi_{\mathbf{t}_a \wedge t} P_{B_{\mathbf{t}_a \wedge t}} x, \tag{4.13}$$

for any $x \in \mathcal{N}$.

Meyer's model allows us to consider martingale spaces with respect to the time and space filtrations simultaneously.Let$L_p^{c,0}mo(\hat{\mathcal{M}}_a)$, $L_p^0 mo(\hat{\mathcal{M}}_a)$, $2 < p \le \infty$ be the martingale spaces with respect to the filtration $\hat{\mathcal{M}}_{a,t}$. Recall that we have an orthogonal projection P_{br} on the subspace consisting of martingales $x = (x_t)$ with the form

$$x_t = \int_0^{t \wedge \mathbf{t}_a} y_s dB_s \tag{4.14}$$

with y_s adapted to $\mathcal{M}_s$. By approximation, we see that $(x_t)_t$ has continuous path, i.e. x_t is continuous on t with respect to the L_p-norm, provided $\sup_t \|y_t\|_{L_p(\hat{\mathcal{M}}_a)} < \infty$.

Denote $P_\Gamma = I - P_{br}$. Recall that it is our convention to write bmo instead of $L_\infty mo$ and $BMO = bmo \cap (h_1^d)^*$.

Lemma 4.6 *Let (T_t) be a standard semigroup admitting a Markov dilation, $(P_t)_t$ the semigroup subordinated to $(T_t)_t$ and $f \in \mathcal{N} \cup L_2(\mathcal{N})$. Then*

(i) $\|f\|_{bmo^c(\mathcal{P})} = \|\hat{n}_a(f)\|_{bmo^c(\hat{\mathcal{M}}_a)}$.

(ii) $\|P_{br}\hat{n}_a(f)\|_{BMO^c(\hat{\mathcal{M}}_a)} \simeq \|P_{br}\hat{n}_a(f)\|_{bmo^c(\hat{\mathcal{M}}_a)} \simeq \sup_b \| \int_0^\infty P_{b+s}|\frac{\partial P_s f}{\partial s}|^2 \min(s, b)ds\|^{\frac{1}{2}}$.

(iii) *If in addition* $\Gamma^2 \geq 0$, *then*

$$\|P_\Gamma \hat{n}_a(f)\|_{bmo^c(\hat{\mathcal{M}}_a)} \simeq \sup_b \left\| \int_0^\infty P_{b+s}\Gamma(P_s f, P_s f) \min(s, b)ds \right\|^{\frac{1}{2}}.$$

Proof We recall that $(\hat{n}_a(f))_t = \hat{\pi}_{\mathbf{t}_a\wedge t}(Pf) = \pi_{\mathbf{t}_a\wedge t}(P_{B_{\mathbf{t}_a\wedge t}}(f))$. So the end element $\hat{n}_a(f) = \pi_{\mathbf{t}_a}(f)$, $(\hat{n}_a(f))_0 = \pi_0(P_{B_0}(f)) = \mathbf{1}(\omega) \otimes \pi_0 P_a f$. Hence we get

$$\hat{E}_t\left(|\hat{n}_a(f)|^2\right) - \left|\hat{E}_t(\hat{n}_a(f))\right|^2 = \pi_{\mathbf{t}_a\wedge t}\left(P_{B_{\mathbf{t}_a\wedge t}}|f|^2 - |P_{B_{\mathbf{t}_a\wedge t}}f|^2\right),$$

for $\mathbf{t}_a(\omega) > t$. Thus in any case we have

$$\begin{aligned} \operatorname{ess\,sup}_{\omega} \left\| \hat{E}_t\left(\left|\hat{n}_a(f)\right|^2\right) - \left|\hat{E}_t\left(\hat{n}_a(f)\right)\right|^2 \right\| &\leq \sup_s \left\|\pi_{\mathbf{t}_a\wedge t}\left(P_s|f|^2 - |P_s f|^2\right)\right\| \\ &\leq \|f\|^2_{bmo^c(\mathcal{P})}. \end{aligned}$$

However, recall that $B_0(\omega) = a$ almost everywhere. This means $B_t = a + \tilde{B}_t$ where $\tilde{B}_t$ is a centered Brownian motion. Since $\limsup_t |\tilde{B}_t|/\sqrt{2t \log\log t} = 1$, we know that with probability 1 the process $|\tilde{B}_t|$ exceeds a. Thus with probability 1 the process B_t hits 0 or $2a$. Hence with probability $\frac{1}{2}$ the process hits $2a$ before it hits 0. Let us assume that $B_{t(\omega)}(\omega) = 2a$ and $B_s(\omega) > 0$ for $0 < s < t(\omega)$. By starting a new Brownian motion at $t(\omega)$, we see with conditional probability $\frac{1}{2}$ we have $B_{t'(\omega)} = 4a$ for some $t(\omega) < t'(\omega)$ and $B_s(\omega) > 0$ for all $t(\omega) < s < t'(\omega)$. By induction we deduce that with probability 2^{-n} the process B_t hits $2^n a$ before it hits 0. Thus given any $b > 0$, we may choose n such that $2^n a > b$. We see that with positive probability there exists $t_n(\omega)$ such that $B_{t_n(\omega)} = 2^n a$ and $B_s(\omega) > 0$ on $[0, t_n(\omega)]$ and B_s is continuous. By continuity there exists $t(\omega) \in [t_n(\omega), \mathbf{t}_a(\omega)]$ such that $B_{t_\omega} = b$. In particular,

$$\begin{aligned} \left\| \hat{E}_{t(\omega)}(\left|\hat{n}_a(f)\right|^2) - \left|\hat{E}_{t(\omega)}\hat{n}_a(f)\right|^2 \right\| &= \left\|\pi_{t(\omega)}\left(P_{B_{t(\omega)}}|f|^2 - \left|P_{B_{t(\omega)}}f\right|^2\right)\right\| \\ &= \left\|P_b|f|^2 - |P_b f|^2\right\|. \end{aligned}$$

Taking the supremum over all b yields (i). For the proof of (iii) we first apply Lemma 2.5.5 and Lemma 2.5.10 (ii) of [14] (note there, ρ_a denotes for $\hat{n}_a$). This immediately yields the first inequality (after a concise review of the involved constant for $\beta = \frac{2}{3}$). For the upper estimate of this term, we recall that with positive probability every value b is hit. Then we start in equality (3.20) for a fixed $b = B_t(\omega)$. We use the monotonicity

$\frac{P_{b+s}(z)}{b+s} \leq \frac{P_t(z)}{t}$ and find

$$\begin{aligned}\mathbb{E}\int_0^{\mathbf{t}_b} T_s(\Gamma(P_{\tilde{B}_s}x, P_{\tilde{B}_s}x))ds &= \frac{1}{2}\int_0^\infty\int_{|b-s|}^{b+s} P_t\Gamma(P_sx, P_sx)dtds\\ &\geq \frac{1}{2}\int_0^\infty \frac{P_{b+s}\Gamma(P_sx, P_sx)}{b+s}\left(\int_{|b-s|}^{b+s} tdt\right)ds\\ &= \int_0^\infty \frac{P_{b+s}\Gamma(P_sx, P_sx)}{b+s}bs\ ds\\ &\geq \frac{1}{2}\int_0^\infty P_{b+s}\Gamma(P_sx, P_sx)\min(b, s)ds.\end{aligned}$$

The proof of the second equivalence of (ii) uses Lemma 2.5.10 (i) of [14] and is similar to (iii) but we only need $|P_tz|^2 \leq P_t|z|^2$ instead of $\Gamma^2 \geq 0$. The first equivalence of (ii) follows from Lemma 4.1 and the fact that $P_b r\hat{n}_a(f)$ has continuous path. □

Lemma 4.7 *For any $y \in L_p(\mathcal{N})$, $2 < p < \infty$, we have*

$$\left\| P_{br}\pi_{\tau_a}y\right\|_{L_pmo(\hat{\mathcal{M}}_a)} + \|P_a y\|_{L_p(\hat{\mathcal{M}}_a)} \simeq \|y\|_{L_p(\mathcal{N})}, \tag{4.15}$$

with equivalent constants $\simeq p$ for $p > 4$. Assume that $(\hat{E}_t\pi_{\tau_a}y)_t$ is a.u. continuous, then

$$\|P_\Gamma\pi_{\tau_a}y\|_{L_pmo(\hat{\mathcal{M}}_a)} + \|P_a y\|_{L_p(\hat{\mathcal{M}}_a)} \simeq \|y\|_{L_p(\mathcal{N})} \tag{4.16}$$

with equivalence constants $\simeq p$ for $p > 4$.

Proof This follows from the fact that $P_{br}\hat{E}_t\pi_{\mathbf{t}_a}y$ has continuous path, $\hat{E}_0\pi_{\tau_a}y = \pi_0 P_a$, Lemma 4.2 of this article, and Lemma 2.5.11 of [14] (note π_{τ_a} is denoted by ρ_a there). □

4.3 Noncommutative Riesz transforms

We will prove a L^∞-BMO boundedness for the noncommutative Riesz transforms studied in [14] in the first subsection.

Recall that the classical Riesz transforms on $\mathbb{R}^n$ can be viewed as $\partial \cdot (-\triangle)^{-\frac{1}{2}}$. Given a standard semigroup of operators $T_t = e^{-tA}$, it is P. A. Meyer's idea to view the generator A as an analogue of $-\triangle$ and the associated bilinear form $\Gamma(f, f)$ as $|\partial f|^2$. The generalized Riesz transform of a function f is $[\Gamma(A^{\frac{1}{2}}f, A^{\frac{1}{2}}f)]^{\frac{1}{2}}$.

As a noncommutative extension of Meyer's result, Junge/Mei proved in [14] that

$$\left\| A^{\frac{1}{2}} f \right\|_{L_p(\mathcal{N})} \leq c_p \left\| [\Gamma(f, f)]^{\frac{1}{2}} \right\|_{L_p(\mathcal{N})},$$

for $2 < p < \infty$ and self adjoint elements $f \in L_p(\mathcal{N})$ with additional assumptions on T_t. We will extend this L_p-boundedness to $L^\infty - BMO$ boundedness.

Theorem 4.8 *Assume T_t admits a Markov dilation and satisfies $\Gamma^2 \geq 0$, we have*

$$\left\| A^{\frac{1}{2}} g \right\|_{BMO^c(\Gamma)} \leq c \max \left\{ \left\| [\Gamma(g, g)]^{\frac{1}{2}} \right\|, \left\| \left[\Gamma(g^*, g^*)\right]^{\frac{1}{2}} \right\| \right\},$$

for $g \in \mathcal{A}$.

Proof By the assumption of a Markov dilation (π_t, E_t) of a standard semigroup $T_t = e^{-tA}$, we have

$$E_t \pi_u f - \pi_t f = \pi_t (T_{u-t} f - f) = \pi_t \int_t^u \frac{\partial T_{r-t}}{\partial r} f dr = -E_t \int_t^u \pi_r A f dr,$$

for $f \in \mathrm{dom}(A)$. Apply to the Markov dilation $(\widehat{\pi}_t, \widehat{E}_t)$ of the new semigroup $\widehat{T}_t = e^{-t\widehat{A}}$, we get

$$\widehat{E}_t \widehat{\pi}_u f - \widehat{\pi}_t f = -\widehat{E}_t \int_t^u \widehat{\pi}_r \widehat{A} f dr.$$

Passing to the stopping time $\mathbf{t}_a$, we get

$$\widehat{E}_t \widehat{\pi}_{\mathbf{t}_a} f - \widehat{\pi}_{\mathbf{t}_a \wedge t} f = -\widehat{E}_t \int_{\mathbf{t}_a \wedge t}^{\mathbf{t}_a} \widehat{\pi}_r \widehat{A} f dr.$$

For a given self adjoint $g \in \mathcal{A}$, let

$$f(s) = \Gamma(P_s g, P_s g),$$

It is an easy calculation by definition of Γ^2 that

$$-\widehat{A} f = 2\Gamma^2(P_s g, P_s g) + 2\Gamma(P_s' g, P_s' g).$$

By $\Gamma^2 \geq 0$, we have

$$-\widehat{A} f \geq 2\Gamma(P_s' g, P_s' g) \geq 0.$$

By Lemma 2.5.10 (ii) of [14] (note ρ_a denotes the same martingale of $\hat{n}_a$),

$$\|P_\Gamma \hat{n}_a(g)\|_{bmo^c} = \sup_t \left\| \widehat{E}_t \int_{\mathbf{t}_a \wedge t}^{\mathbf{t}_a} \widehat{\pi}_r \Gamma(P_s g, P_s g) dr \right\|.$$

Therefore, by Proposition 4.6,

$$\begin{aligned}
\left\| A^{\frac{1}{2}} g \right\|^2_{BMO^c(\Gamma)} &\approx \left\| P_\Gamma \hat{n}_a(A^{\frac{1}{2}} g) \right\|^2_{bmo^c} = \sup_t \left\| \widehat{E}_t \int_{\mathbf{t}_a \wedge t}^{\mathbf{t}_a} \widehat{\pi}_r \Gamma(P'_s g, P'_s g) dr \right\| \\
&\leq \sup_t \left\| -\widehat{E}_t \int_{\mathbf{t}_a \wedge t}^{\mathbf{t}_a} \widehat{\pi}_r \widehat{A} f dr \right\| = \sup_t \left\| \widehat{E}_t \widehat{\pi}_{\mathbf{t}_a} f - \widehat{\pi}_{\mathbf{t}_a \wedge t} f \right\| \\
&\leq \left\| \widehat{\pi}_{\mathbf{t}_a} f \right\| + \sup_t \left\| \widehat{\pi}_{\mathbf{t}_a \wedge t} f \right\| \\
&= \left\| \widehat{\pi}_{\mathbf{t}_a} \Gamma(Pg, Pg) \right\| + \sup_t \left\| \widehat{\pi}_{\mathbf{t}_a \wedge t} \Gamma(Pg, Pg) \right\| \\
&\leq \left\| \widehat{\pi}_{\mathbf{t}_a} P\Gamma(g, g) \right\| + \sup_t \left\| \widehat{\pi}_{\mathbf{t}_a \wedge t} P\Gamma(g, g) \right\| \\
&= \left\| \widehat{\pi}_{\mathbf{t}_a} P\Gamma(g, g) \right\| + \sup_t \left\| \widehat{E}_t \widehat{\pi}_{\mathbf{t}_a} P\Gamma(g, g) \right\| \\
&\leq 2 \left\| \Gamma(g, g) \right\|.
\end{aligned}$$

For non-self adjoint g, we obtain the desired inequality by splitting $g = \frac{g^*+g}{2} + i\frac{ig^*-ig}{2}$. □

Corollary 4.9 *Let T_t be a standard semigroup satisfying $\Gamma^2 \geq 0$ and admitting an a.u. continuous Markov dilation (see definition at the beginning of the next section). We have*

$$\left\| A^{\frac{1}{2}} g \right\|_{L_p(\mathcal{N})} \leq cp \max\{\|\Gamma(g, g)\|_{L_p(\mathcal{N})}, \|\Gamma(g^*, g^*)\|_{L_p(\mathcal{N})}\},$$

for $2 < p < \infty$.

Proof By the same argument used in the proof of Theorem 4.8, we have

$$\left\| P_\Gamma \hat{n}_a \left(A^{\frac{1}{2}} g \right) \right\|_{L_p mo(\hat{\mathcal{M}}_a)} \leq c \max \left\{ \|\Gamma(g, g)\|_{L_p(\mathcal{N})}, \left\| \Gamma\left(g^*, g^*\right) \right\|_{L_p(\mathcal{N})} \right\}.$$

We then obtain

$$\left\| A^{\frac{1}{2}} g \right\|_{L_p(\mathcal{N})} \leq cp \max \left\{ \|\Gamma(g, g)\|_{L_p(\mathcal{N})}, \left\| \Gamma\left(g^*, g^*\right) \right\|_{L_p(\mathcal{N})} \right\}.$$

for $p > 4$ by Lemma 4.7 and Proposition 5.1. The same inequality is proved in [14], Theorem 2.5.13 with constant cp for $2 < p < 4$. □

Remark 4.10 Let $\mathcal{N}$ be a commutative von Neumann algebra, for example $\mathcal{N} = L_\infty(\mathbb{R}^n)$. Let $A = \triangle$ on $\mathbb{R}^n$ and R be the classical Riesz transform, i.e. $R(f) = (\ldots, \partial_i(-\triangle)^{-\frac{1}{2}} f, \ldots)$. It is well known that R is L_p-bounded uniformly on the dimension n for $1 < p < \infty$ (see [29,34,43]). For $p = 1$, a dimension free weak $(1, 1)$ estimate is due to Varopoulos (see [47]). It is desirable to have some results for $p = \infty$ which implies the estimate in the range $1 < p < \infty$ by interpolation. Note, in this case,

$$\|f\|_{BMO(\Gamma)} = \|R(f)\|_{BMO(\partial)}.$$

By Proposition 2.5, we have the dimension free estimate

$$\|R(f)\|_{BMO(\partial)} \leq c\|f\|_\infty.$$

5 Interpolation

We will prove an interpolation theorem for BMO spaces associated with semigroups of operators. Our BMO spaces are then good endpoints for noncommutative L_p spaces.

Let (T_t) be a standard semigroup on $\mathcal{N}$ admitting a (reversed) standard Markov dilation $(\mathcal{M}_t, \pi_t, E_t)$. We say the dilation has *a.u. continuous path* if there exist weakly dense subsets B_p of $L_p(\mathcal{N})$ such that both $m(f)$ and $n(f)$ have a.u. continuous path for all $2 \leq p < \infty$. Here $m(f)$ and $n(f)$ are martingales given as in (4.7) and Proposition 4.4.

Proposition 5.1 *Suppose a standard semigroup T_t satisfies $\Gamma^2 \geq 0$ and admits an a.u. continuous standard (reversed) Markov dilation $(\pi_t, \mathcal{M}_{t]})$. Then the martingale $\hat{n}_a(f) = \hat{\pi}_{\mathbf{t}_a \wedge s}(Pf)$ in Meyer's model is a.u. continuous for all $f \in L_p(\mathcal{N})$, $p > 2$.*

Proof This is the second part of Lemma 2.5.3 (ii) of [14]. □

We use the notation $L_p^0(\mathcal{N})$, $1 \leq p \leq \infty$ for the complemented subspace of $L_p(N)$ which is orthogonal to

$$ker(A_p) = \left\{f \in \mathrm{dom}_p(A), Af = f\right\} = \left\{f \in L_p(\mathcal{N}), \lim_t T_t f = f\right\}.$$

Equivalently, $L_p^0(\mathcal{N}) = \{f \in L_p(\mathcal{N}), \lim_{t\to\infty} T_t f = 0\}$ and hence we may also view $L_p^0(\mathcal{N})$ as a quotient space. The limit is taken with respect to the $\|\cdot\|_{L_p(\mathcal{N})}$-norm for $1 < p < \infty$ and is with respect to the weak* topology for $p = 1, \infty$. Recall from Proposition 2.3 we know that $\|\cdot\|_{bmo^c(\mathcal{T})}$ and $\|\cdot\|_{BMO^c(\mathcal{T})}$ are norms on the quotient space $\mathcal{N}^0 \cup L_2^0(\mathcal{N})$. Note $Af = 0$ implies $T_t f = f$ and $P_t f = f$ for all t, we get $\ker\left(A_\infty^{\frac{1}{2}}\right) = \ker(A_\infty) = \{f, \lim_t P_t f = f\}$. So $\|\cdot\|_{BMO_c(\mathcal{P})}$ and $\|\cdot\|_{bmo^c(\mathcal{P})}$ are norms on $\mathcal{N}^0 \cup L_2^0(\mathcal{N})$ too. The same is true for $\|\cdot\|_{BMO^c(\Gamma)}$, $\|\cdot\|_{BMO^c(\hat{\Gamma})}$, and for $\|\cdot\|_{BMO^c(\partial)}$.

5.1 Interpolation in the finite case

We assume that the underling von Neumann algebra $\mathcal{N}$ is with a finite trace τ in this subsection. In this case, all the BMO-norms associated with semigroups are bigger than the $L_2(\mathcal{N})$-norm up to a constant. Let $BMO(\mathcal{T})$, $BMO(\mathcal{P})$, $BMO(\hat{\Gamma})$, $BMO(\Gamma)$, $BMO(\partial)$ and $bmo(\mathcal{T})$, $bmo(\mathcal{P})$ be the spaces of $f \in L_2^0(\mathcal{N})$ with finite corresponding BMO-norms. We consider the complex interpolation couples $[X, L_p^0(\mathcal{N})]_{\frac{1}{q}}$ with X any of these BMO spaces. See [2] for basic properties of complex interpolation method.

Theorem 5.2 *Let (T_t) be a standard semigroup of operators. We have*

(i) *Assume (T_t) admits a standard Markov dilation. Then*

$$L_{pq}^0(\mathcal{N}) = \left[X, L_p^0(\mathcal{N})\right]_{\frac{1}{q}},$$

with equivalence constant $\simeq pq$ for all $p \geq 1, q > 1$ and X being semigroup-BMO spaces $BMO(\mathcal{T})$, $BMO(\mathcal{P})$, $BMO(\hat{\Gamma})$, $BMO(\partial)$, and $bmo(\mathcal{P})$.

(ii) *Assume (T_t) admits a reversed Markov dilation with a.u. continuous path. Then*

$$L_{pq}^0(\mathcal{N}) = \left[bmo(\mathcal{T}), L_p^0(\mathcal{N})\right]_{\frac{1}{q}},$$

equivalence constant $\simeq pq$ for all $p \geq 1, q > 1$. If, in addition, T_t satisfies $\Gamma^2 \geq 0$, we have

$$L_{pq}^0(\mathcal{N}) = \left[BMO(\Gamma), L_p^0(\mathcal{N})\right]_{\frac{1}{q}},$$

equivalence constant $\simeq pq$ for all $p \geq 1, q > 1$.

Proof For any choice of X, note that the trivial inclusion $\mathcal{N}^0 \subset X$ implies

$$L_{pq}^0(\mathcal{N}) \subset \left[X, L_p^0(\mathcal{N})\right]_{\frac{1}{q}}.$$

Assume a Markov dilation exists, for $X = BMO(\partial)$, we consider Meyer's model in Sect. 4. Note $(\hat{n}_a(x))_0 = \hat{E}_0\pi_{\mathbf{t}_a}(x) = \pi_0 P_a x$ for all $x \in L_2(\mathcal{N}) \supseteq X$. According to Lemma 4.6 (ii), we get that $P_{br}\pi_{\tau_a}$ embeds $BMO(\partial)$ into $BMO(\hat{\mathcal{M}}_a)$. Thus $P_{br}\pi_{\tau_a}$ embeds $[BMO(\partial), L_p^0(\mathcal{N})]_{\frac{1}{q}}$ into $[BMO(\hat{\mathcal{M}}_a), L_p(\hat{\mathcal{M}}_a)]_{\frac{1}{q}}$ because it embeds $L_p(\mathcal{N})$ into $L_p(\hat{\mathcal{M}}_a)$ too. Note

$$\left[BMO\left(\hat{\mathcal{M}}_a\right), L_p\left(\hat{\mathcal{M}}_a\right)\right]_{\frac{1}{q}} = L_{pq}\left(\hat{\mathcal{M}}_a\right)$$

with equivalence constant $\simeq pq$ by Lemma 4.2. We deduce that,

$$\|P_{br}\pi_{\tau_a}x\|_{pq} \leq cpq\|x\|_{[BMO(\partial), L_p^0(\mathcal{N})]_{\frac{1}{q}}}$$

holds for all $x \in [BMO(\partial), L_p^0(\mathcal{N})]_{\frac{1}{q}}$. By Lemma 2.5.11 of [14], we have (note ρ_a there denotes π_{τ_a} and Pr is the projection to $L_p^0(\mathcal{N})$)

$$\|x\|_{pq} \leq 2 \lim_{a\to\infty} \|P_{br}\pi_{\tau_a}x\|_{pq} \leq cpq\|x\|_{[BMO(\partial),L_p^0(\mathcal{N})]_{\frac{1}{q}}}.$$

We obtain the desired result for $X = BMO(\partial)$. For $X = BMO(\mathcal{P}), bmo(\mathcal{P})$, $BMO(\hat{\Gamma}), BMO(\mathcal{T})$, the interpolation result follows from the relation that $\mathcal{N}^0 \subset X \subset BMO(\partial)$ because of Theorem 2.6 and Proposition 2.5 (ii).

We now prove (ii). Assume the admitted Markov dilation has a.u.continuous path. By Proposition 4.3, we see that π_0 embeds $bmo(\mathcal{T})$ into $bmo(\mathcal{M})$. Now, for any $x \in bmo(\mathcal{T}) \subset L_2(\mathcal{N})$, we can find a net $x_\lambda \in L_2(\mathcal{N}) \cap \mathcal{N}$ converging to x in $L_2(\mathcal{N})$. So $\pi_0(x_\lambda) \in L_2(\mathcal{M}) \cap \mathcal{M}$ converging to $\pi_0(x)$ in $L_2(\mathcal{M})$. By Lemma 4.1, $\pi_0(x) \in BMO(\mathcal{M})$ and $\|\pi_0(x)\|_{BMO(\mathcal{M})} \leq c\|x\|_{bmo(\mathcal{M})}$. Therefore, π_0 embeds $bmo(\mathcal{T})$ into $BMO(\mathcal{M})$. By the same argument used for the proof of (i), we obtain the desired result.

We now turn to $BMO(\Gamma)$, Lemma 4.6 (iii) implies that $P_\Gamma\pi_{\tau_a}$ embeds $BMO(\Gamma)$ into $bmo(\hat{\mathcal{M}}_a)$. Note that Proposition 5.1 implies the a.u. continuity of $\hat{n}_a(x) = (\hat{E}_t\pi_{\tau_a}x)_t$ for all $x \in L_2(\mathcal{N}) \cap \mathcal{N}$ assuming $\Gamma^2 \geq 0$. Then $P_\Gamma\pi_{\tau_a}$ embeds $BMO(\Gamma)$ into $BMO(\hat{\mathcal{M}}_a)$ by Lemma 4.1 and the argument used for (ii). Repeat the argument used for the proof of (i), we obtain (iii). □

Remark 5.3 According to [17] we have a Markov dilation for finite von Neumann algebras. Hence $BMO(\partial)$ solves problem (0.1) in this case.

As a consequence, we obtain the boundedness of Fourier multiplier M_a discussed in Sect. 3.

Corollary 5.4 *Let (T_t) be a standard semigroup admitting a Markov dilation. Let M_a be as in Sect. 2. Then*

$$\|M_a f\|_{L_p(\mathcal{N})} \leq c_p\|f\|_{L_p(\mathcal{N})}, \tag{5.1}$$

with c_p in order of $\simeq \max\{p, \frac{1}{p-1}\}$. In particular, for $M_a = L^{is}$, we have

$$\|L^{is} f\|_{L_p(\mathcal{N})} \leq c_{s,p}\|f\|_{L_p(\mathcal{N})}, \tag{5.2}$$

with

$$c_{s,p} \simeq \max\left\{p, \frac{1}{p-1}\right\} |s|^{-\left|\frac{1}{2}-\frac{1}{p}\right|} \exp\left(\left|\frac{\pi s}{2} - \frac{\pi s}{p}\right|\right).$$

Proof Apply Theorem 3.3 and Theorem 5.2 to M_a and their adjoint operators, we have for $f \in L_p^0(\mathcal{N})$,

$$\|M_a f\|_{L_p(\mathcal{N})} \leq c \max\left\{p, \frac{1}{p-1}\right\} \|f\|_{L_p(\mathcal{N})}$$

for all $1 < p < \infty$. Since M_a's vanish on f with $\lim_t T_t f = f$, they are bounded on the whole $L_p(\mathcal{N})$. For $M_a = L^{is}$, we have

$$\|L^{is} f\|_{L_2(\mathcal{N})} \leq \|f\|_{L_p(\mathcal{N})},$$
$$\|L^{is} f\|_{BMO(\partial)} \leq c\Gamma(1-is)^{-1}\|f\|_{BMO(\partial)}.$$

By interpolation, we have, for all $1 < p < \infty$,

$$\|L^{is} f\|_{L_p(\mathcal{N})} \leq c \max\left\{p, \frac{1}{p-1}\right\} \Gamma(1-is)^{-\left|1-\frac{2}{p}\right|}\|f\|_{L_p(\mathcal{N})}.$$

It is well known that, e.g. see page 151 of [46],

$$|\Gamma(1-is)| \simeq |s|^{\frac{1}{2}} e^{-\frac{\pi|s|}{2}}. \tag{5.3}$$

Therefore, we conclude,

$$\|L^{is} f\|_{L_p(\mathcal{N})} \leq c \max\left\{p, \frac{1}{p-1}\right\} |s|^{-\left|\frac{1}{2}-\frac{1}{p}\right|} e^{\left|\frac{\pi s}{2}-\frac{\pi s}{p}\right|}\|f\|_{L_p(\mathcal{N})}.$$

□

Remark 5.5 It is known that standard semigroups (T_t) on von Neumann algebras $VN(G)$ of a discrete group always admit a Markov dilation (see [39]). Moreover, a recent result of Junge/Ricard /Shlyakhtenko (see [17]) shows that standard semigroups (T_t) on any finite von Neumann algebras admits a Markov dilation and for the bounded generators $A_t = t^{-1}(I - T_t)$ the Markov dilations also has almost uniformly continuous path.

Remark 5.6 The L_p-boundedness of Fourier multipliers M_a could be proved directly following E. Stein's Littlewood-paley g-function technique (see [41]) by the noncommutative H_p theory developed in with worse constants. It could be also obtained following a classical argument of M. Cowling (see [5]) through 'transference technique' in the noncommutative setting, which could become available after [17]. However, 'transference technique' does not seem to work for BMO. Cowling did obtain optimal L_p-boundedness constants for the imaginary powers L^{is} on abelian groups, although our method provides a slightly better estimate on s as $s \to \infty$ (see 5.2). But Cowling did not have optimal L_p-boundedness constants for general multipliers M_a's (he had $\simeq \max\{p^{\frac{5}{2}}, (p-1)^{-\frac{5}{2}}\}$, see Theorem 3 of [5]).

As another application, we obtain optimal constants for the noncommutative maximal ergodic inequality proved by Junge and Xu (see Theorem 5.1 and Corollary 5.11 of [22]).

Corollary 5.7 *Suppose (T_t) is a standard semigroup admitting a Markov dilation, then*

$$\|\sup_t T_t f\|_{L_p(\mathcal{N})} \leq c \max\left\{1, \frac{1}{(p-1)^2}\right\} \|f\|_{L_p(\mathcal{N})}. \tag{5.4}$$

Proof The proof is to write $T_t - \frac{1}{t}\int_0^t T_v dv$ as an weighted average of L^{is} for each t as Cowling did (see [5]) and use the uniform estimate of $L_p(\mathcal{N})$-boundedness of L^{is}. From the elementary identity

$$\frac{1}{\pi}\int_0^{+\infty} \lambda^{is}\Gamma(1-is)(1+is)^{-1}ds = e^{-\lambda} - \int_0^1 e^{-u\lambda}du,$$

we deduce by functional calculus that

$$\begin{aligned}\frac{1}{\pi}\int_0^{+\infty} (tL)^{is}\Gamma(1-is)(1+is)^{-1}ds &= e^{-tL} - \int_0^1 e^{-utL}du \\ &= T_t - \frac{1}{t}\int_0^t T_v dv.\end{aligned} \tag{5.5}$$

Theorem 4.1 and Theorem 4.5 of [22] imply that

$$\left\|\sup_t \frac{1}{t}\int_0^t T_v f dv\right\|_{L_p(\mathcal{N})} \leq c \max\left\{p, \frac{1}{(p-1)^2}\right\} \|f\|_{L_p(\mathcal{N})}. \tag{5.6}$$

On the other hand, for any $a \in L_q(\mathcal{N}_+, L_1(0,\infty))$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}&\left|\tau\int_0^\infty a(t)\int_0^{+\infty}(tL)^{is}(f)\Gamma(1-is)(1+is)^{-1}dsdt\right| \\ &= \left|\tau\int_0^{+\infty}\int_0^\infty a(t)t^{is}dtL^{is}(f)\Gamma(1-is)(1+is)^{-1}ds\right| \\ &\leq \sup_s\left\|\int_0^\infty a(t)t^{is}dt\right\|_{L_q(\mathcal{N})}\int_0^{+\infty}\|L^{is}(f)\|_{L_p(\mathcal{N})}|\Gamma(1-is)(1+is)^{-1}|ds.\end{aligned} \tag{5.7}$$

A combination of (5.2), (5.3), (5.5), and (5.7) implies that

$$\left|\tau\int_0^\infty a(t)\left(T_t-\frac{1}{t}\int_0^t T_v dv\right)dt\right| \\ \leq c\max\left\{p,\frac{1}{p-1}\right\}\left\|\int_0^\infty a(t)dt\|_{L_q(\mathcal{N})}\right\| f\|_{L_p(\mathcal{N})}. \tag{5.8}$$

Without loss of generality, assume $f\geq 0$. We deduce by duality (see Proposition 2.1 (iii) of [22]) that,

$$\begin{aligned}
&\|\sup_t T_t f\|_{L_p(\mathcal{N})}\\
&\leq \sup_{a\in L_q(\mathcal{N}_+,L_1(0,\infty)),\|a\|\leq 1}\tau\int_0^\infty a(t)T_t f dt,\\
&= \sup_{a\in L_q(\mathcal{N}_+,L_1(0,\infty)),\|a\|\leq 1}\tau\int_0^\infty a(t)\left(T_t f-\frac{1}{t}\int_0^t T_v f dv+\frac{1}{t}\int_0^t T_v f dv\right)dt,\\
&\leq \sup_{a\in L_q(\mathcal{N}_+,L_1(0,\infty)),\|a\|\leq 1}\tau\int_0^\infty a(t)\left(T_t f-\frac{1}{t}\int_0^t T_v f dv\right)dt\\
&\quad+\left\|\sup_t\frac{1}{t}\int_0^t T_v f dv dt\right\|_{L_p(\mathcal{N})}.
\end{aligned}$$

By (5.6) and (5.8) we obtain,

$$\|\sup_t T_t f\|_{L_p(\mathcal{N})}\leq c\max\left\{p^2,\frac{1}{(p-1)^2}\right\}\|f\|_{L_p(\mathcal{N})}.$$

Note

$$\|\sup_t T_t f\|_{L_\infty(\mathcal{N})}\leq \|f\|_{L_\infty(\mathcal{N})}.$$

Apply the interpolation result of Theorem 3.1 of [22], we obtain

$$\|\sup_t T_t f\|_{L_p(\mathcal{N})}\leq c\max\left\{1,\frac{1}{(p-1)^2}\right\}\|f\|_{L_p(\mathcal{N})},$$

for all $1<p<\infty$. □

5.2 Interpolation in the semifinite case

We will extend Theorem 5.2 to the case that the underling von Neumann algebras $\mathcal{N}$ is semifinite. In this case, BMO is no longer a subspace of L_2. To study the interpolation result, we first have to obtain a larger space that the interpolation couple BMO, L_p belongs to.

5.2.1 L_p-Hilbert module

We will need the following definition and lemma of L_p-Hilbert module due to Junge/Sherman (see [18]). For $p = \infty$ these spaces are well-known through the GNS construction for a completely positive map (see [25,33], Corollary 6.3).

Definition 5.8 Let $\mathcal{M}$ be a semifinite von Neumann algebra. Let E be an $\mathcal{M}$ right module with an $L_{\frac{p}{2}}(\mathcal{M})$-valued inner product $\langle\cdot,\cdot\rangle$. A (right) Hilbert $L_p(\mathcal{M})$ ($1 \leq p < \infty$) module, denoted by $L_p^c(E)$, is the completion of E with respect to the norm $||\cdot|| = \|\langle\cdot,\cdot\rangle\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\mathcal{M})}$. A (right) Hilbert $L_\infty(\mathcal{M})$ module, denoted by $L_\infty^c(E)$ is the completion of E with respect to the strong operator topology, briefly STOP topology. The STOP topology is induced by the family of seminorms $\|x\|_\xi = \tau(\xi\langle x, x\rangle)]^{\frac{1}{2}}$.

Here is an easy proposition which we will use frequently.

Proposition 5.9 *Suppose $(L_\infty^c(E), \langle\cdot,\cdot\rangle)$ is a Hilbert $L_\infty(\mathcal{M})$-module. Suppose a net $x_\lambda \in \mathcal{M}$ converges to $x \in L_\infty(E)$ in the STOP topology. Then $\langle x_\lambda, x_\lambda\rangle$ weak* converges in $\mathcal{M}$. We denote the limit by $\langle x, x\rangle$.*

Given a Hilbert space H, denote by $B(H)$ the space of all bounded operators on H. Choose a norm one element $e \in H$, let P_e be the rank one projection onto Span$\{e\}$. For $0 < p \leq \infty$, let

$$L^p(\mathcal{M}, H_c) = L_p(B(H) \otimes \mathcal{M}))(1 \otimes P_e).$$

Namely, $L^p(\mathcal{M}, H^c)$ is the column subspace of $L^p(B(H) \otimes \mathcal{M}))$ consisting of all elements with the form $x(1 \otimes P_e)$ for $x \in L^p(B(H) \otimes \mathcal{M}))$. The definition of $L^p(\mathcal{M}, H^c)$ does not depend on the choice of e. $L^p(\mathcal{M}, H^c)$ can be identified as the predual of $L^q(\mathcal{M}, H_c)$ with $q = \frac{p}{p-1}$ for $1 \leq p < \infty$. The reader can find more information on $L^p(\mathcal{M}, H_c)$ in Chapter 2 of [12].

Lemma 5.10 *$L_p^c(E)$ is isomorphic to a complemented subspace of $L^p(\mathcal{M}, H_c)$ for some Hilbert space H. Moreover, the isomorphism does not depends on p and*

$$\left(L_p^c(E)\right)^* = L_q^c(E), \tag{5.9}$$

for all $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Here the anti-linear duality bracket $(w, z) = tr(\langle w, z\rangle)$ is used.

5.2.2 Interpolation for $BMO(\partial)$, $BMO(\mathcal{T})$, $BMO(\mathcal{P})$, $BMO(\hat{\Gamma})$ and $bmo(\mathcal{P})$

We use Meyer's model to prove an interpolation result for the BMO space corresponding to the $\|\cdot\|_{bmo(\partial)}$-norm. For $x \in \mathcal{N}^0$ we recall the definition

$$\|x\|_{bmo^c(\partial)} = \sup_t \left\| \int_0^t P_t |P_s'(x)|^2 s ds \right\|^{\frac{1}{2}} \simeq \sup_t \left\| \int_0^\infty P_{s+t} |P_s'(x)|^2 \min(t,s) ds \right\|^{\frac{1}{2}}.$$

Define the $L^\infty(\mathbb{R}_+) \otimes \mathcal{N}$-valued inner product on $\mathcal{N} \otimes \mathcal{N}$ by

$$\langle x \otimes a, y \otimes b \rangle_\partial = a^* \left(\int_0^\infty P_{s+t}(P_s'\left(x^*\right) P_s'(y)) \min(t,s) ds \right) b.$$

Let V be the Hilbert L^∞-module corresponding to this inner product. Let $BMO^c(\partial)$ be the strong operator closure of $\mathcal{N}^0$ in V via the embedding

$$\Phi : x \to x \otimes 1.$$

Let $BMO^r(\partial)$ be the strong operator closure of $\mathcal{N}^0$ in V via the embedding $x \to x^* \otimes 1$.

We define the column and row space of $BMO(\mathcal{T})$, $BMO(\mathcal{P})$, $BMO(\hat{\Gamma})$ and $bmo(\mathcal{P})$ similarly by using Hilbert L_∞-modules corresponding to respective BMO-norms given in Sect. 2.

To understand the intersection of BMO^c and BMO^r, we need the following observation.

Lemma 5.11 *Let $x \in X$ with $X \in \{BMO^c(\partial), BMO^c(\mathcal{T}), BMO^c(\mathcal{P}), BMO^c(\hat{\Gamma}), bmo^c(\mathcal{P})\}$. Then $P_t' x$ exists in $\mathcal{N}$ and*

$$\|P_t' x\|_{\in fty} \leq Ct^{-1} \|x\|_X. \tag{5.10}$$

Proof Fix $t > 0$. Let $x_\lambda \in \mathcal{N}^0 \subset BMO^c(\partial)$ be a net such that $\Phi(x_\lambda) = x_\lambda \otimes 1$ converges in V with respect to the STOP topology. We will show that $P_t' x_\lambda$ weakly converges in $\mathcal{N}$ and the limit (denoted by $P_t' x$) has norm smaller than $ct^{-1}\|x\|_{BMO^c(\partial)}$. This is what we mean by $P_t' x$ exists in $\mathcal{N}$.

We first deduce from Proposition 1.2 that, for $t > 0$,

$$\begin{aligned} \frac{t^2}{2}\left|\frac{\partial P_{2t}}{2t}x_\lambda\right|^2 &\leq \int_0^t \left|\frac{\partial P_{2t}}{2t}x_\lambda\right|^2 sds = \int_0^t |P_{2t-s}P'_s x_\lambda|^2 sds \\ &\leq \int_0^t P_{2t-s}(|P'_s x_\lambda|^2)sds \leq \int_0^t \frac{2t-s}{t+s}P_{t+s}(|P'_s x_\lambda|^2)sds \\ &\leq 2\int_0^\infty P_{t+s}(|P'_s x_\lambda|^2)\min(s,t)ds \qquad (5.11) \\ &= 2\langle \Phi(x_\lambda), \Phi(x_\lambda)\rangle_\partial. \qquad (5.12) \end{aligned}$$

By Proposition 5.9, we know that $P'_t x_\lambda$ converges with respect to the strong operator topology of $\mathcal{N}$ and the limit exists in $\mathcal{N}$ with a norm bounded by $\frac{c}{t}\|x\|_{BMO^c(\partial)}$, since $\Phi(x_\lambda)$ converges in the STOP topology. Note the $\|\cdot\|_{BMO^c(\partial)}$-norm is smaller than any of the other X-norms by Lemma 2.5 (ii) and Theorem 2.6. We obtain (5.10) for all X. □

We say that $x \in BMO^c(\partial)$ belongs to $BMO^r(\partial)$ if $P'_t x = P'_t y$ for some $y \in BMO^r(\partial)$ for all $t > 0$. This y is unique in $BMO^r(\partial)$. In fact, assume there are two weak* convergent nets $y_\lambda, \tilde{y}_\lambda$ in $BMO^r(\partial)$ such that $P'_t x = P'_t y = P'_t \tilde{y}$ holds for the limit elements $y, \tilde{y} \in BMO^r(\partial)$ and any $t > 0$. Then $P'_t(y_\lambda - \tilde{y}_\lambda)$ converges to 0 for any t with respect to the weak* topology of $\mathcal{N}$. Hence $\int_0^\infty P_{b+s}|P'_s(y_\lambda - \tilde{y}_\lambda)^*|^2 sds$ weak * converges to 0 for any b by the dominated convergence theorem. This means $y - \tilde{y} = 0$ in $BMO^r(\partial)$. Set $BMO(\partial)$ to be the space consisting of all such x's equipped with the maximum norm

$$\|x\|_{BMO(\partial)} = \max\{\|x\|_{BMO^c(\partial)}, \|y\|_{BMO^r(\partial)}\},$$

Here y is the unique $y \in BMO^r(\partial)$ such that $P'_t x = P'_t y$ for all $t > 0$ as we explained above. Define $BMO(\mathcal{T})$, $BMO(\mathcal{P})$, $BMO(\hat{\Gamma})$ and $bmo(\mathcal{P})$ to be the intersection of the corresponding row, column spaces similarly.

Once we have these definitions, the same proof of Theorem 5.2 implies

Theorem 5.12 *Let $1 \leq p < \infty$. Assume a standard semigroup T_t admits a standard Markov dilation. Then*

$$\left[X, L_1^0(\mathcal{N})\right]_{\frac{1}{p}} = L_p^0(\mathcal{N}),$$

with equivalence constant in order p for $X = BMO(\partial), BMO(\mathcal{T}), BMO(\mathcal{P}), BMO(\hat{\Gamma})$ or $bmo(\mathcal{P})$.

5.2.3 Interpolation for $bmo(\mathcal{T})$

For the interpolation for $bmo(\mathcal{T})$, besides an appropriate definition of the interpolation couple $bmo(\mathcal{T})$, L_1, we also need to show that $L_p^0(\mathcal{N})$ is dense in $[bmo(\mathcal{T}), L_1^0(\mathcal{N})]_{\frac{1}{p}}$ because we only assume that the special martingales $m(x)$'s have a.u. continuous path and general martingales may not, while in the case of $BMO(\partial)$ we have automatically that all Brownian martingales have continuous path. This difficulty already appeared in the finite case (see the end of the proof of Theorem 5.2). We will go around it by defining an abstract predual of $bmo(\mathcal{T})$.

For a standard semigroup $\mathcal{T} = (T_t)$ on $\mathcal{N}$. We consider the $L_{\frac{p}{2}}(\ell_\infty(\mathbb{R}_+) \otimes \mathcal{N})$-valued inner products on $E = \ell_\infty(\mathbb{R}_+) \otimes (\mathcal{N} \otimes \mathcal{N})$,

$$\langle a \otimes b, c \otimes d\rangle_T^c \;=\; b_t^* T_t \left(a_t^* c_t\right) d_t, \langle a \otimes b, c \otimes d\rangle_T^r \;=\; b_t T_t \left(a_t c_t^*\right) d_t^*,$$

for $a \otimes b \in \ell_\infty(\mathbb{R}_+) \otimes (\mathcal{N} \otimes \mathcal{N})$. Denote by V_p^c (resp. V_p^r) the $L_p(L_\infty(\mathbb{R}_+) \otimes \mathcal{N})$-Hilbert module corresponding to $E, \langle\cdot,\cdot\rangle_{\mathcal{T}}^c$ (resp. $E, \langle\cdot,\cdot\rangle_{\mathcal{T}}^r$).

Let us denote by $w : \mathcal{N} \to E$ the embedding map $w(x)_t = x \otimes 1 - 1 \otimes T_t x$. Then

$$\langle w(x), w(x)\rangle_{\mathcal{T}}^c = T_t|x|^2 - |T_t x|^2.$$
$$\langle w(x), w(x)\rangle_{\mathcal{T}}^r = T_t|x^*|^2 - |T_t x^*|^2.$$

Denote by w_c^* (resp. w_r^*) the adjoint of w with respect to $\mathcal{N}$, $\tau(x^*, y)$; $E, \langle\cdot,\cdot\rangle_{\mathcal{T}}^c$ (resp. $\mathcal{N}$, $\tau(xy^*)$; $E, \langle\cdot,\cdot\rangle_{\mathcal{T}}^r$). We have

$$\begin{aligned} w_c^*(a \otimes b) \;&=\; \sum_t a_t T_t(b_t) - T_t(T_t(a_t)b_t), w_r^*\left(a^* \otimes b^*\right) \\ &=\; \sum_t T_t\left(b_t^*\right) a_t^* - T_t\left(b_t^* T_t\left(a_t^*\right)\right), \end{aligned} \tag{5.13}$$

for $a \otimes b \in \ell_1(\mathbb{R}_+) \otimes (\mathcal{N} \otimes \mathcal{N})$. Indeed, for $x \in \mathcal{N}$ and $z = a \otimes b = (a_t \otimes b_t)_t$,

$$\begin{aligned} \tau(x^* w_c^*(z)) \;&=\; \tau \sum_t \left(\langle x \otimes 1 - 1 \otimes T_t x, a_t \otimes b_t\rangle_{\mathcal{T}}^c\right) \\ &=\; \tau \sum_t \left(T_t\left(x^* a_t\right) b_t\right) - tr\left(T_t\left(x^*\right) T_t(a_t) b_t\right) \\ &=\; \tau \sum_t (x^*(a_t T_t(b_t) - T_t(T_t(a_t)b_t))). \end{aligned}$$

Definition 5.13 (i) The space $bmo^c(\mathcal{T})$ (resp. $bmo^r(\mathcal{T})$) is defined as the weak*-closure of $\mathcal{N}^0$ in V_∞^c (resp. V_∞^r) via the embedding w.

(ii) $h_1^c(\mathcal{T})$ (resp. $h_1^r(\mathcal{T})$) is defined as the quotient of V_1^c (resp. V_1^r) by the kernel of w_c^* (resp. w_r^*). The Hardy space $h_1(\mathcal{T})$ is defined as $h_1^c(\mathcal{T}) + h_1^r(\mathcal{T}) \subset L_1(\mathcal{N})$. More precisely, for $f \in L_1(\mathcal{N})$,

$$\|f\|_{h_1^c(\mathcal{T})} = \inf\{\|v\|_{V_1^c}, w_c^*(v) = f\}.$$

In the following Lemma we report some elementary properties.

Lemma 5.14 (i) $x \in h_1^c(\mathcal{T})$ *iff* $x^* \in h_1^r(\mathcal{T})$.

(ii) $h_1^c(\mathcal{T}) \cap h_1^r(\mathcal{T}) \cap L_p^0(\mathcal{N})$ *is dense in* $L_p^0(\mathcal{N})$ *for* $1 \le p < \infty$.

(iii) $h_1(\mathcal{T}) \cap L_p$ *is dense in* $h_1(\mathcal{T})$ *for all* $1 \le p \le \infty$.

(iv) $(h_1^c(\mathcal{T}))^* = bmo^c(\mathcal{T})$, $(h_1^r(\mathcal{T}))^* = bmo^r(\mathcal{T})$. *Assume* $h_1^c(\mathcal{T}) \cap h_1^r(\mathcal{T})$ *is dense in both* $h_1^c(\mathcal{T})$ *and* h_1^r. *Then* $(h_1(\mathcal{T}))^* = bmo(\mathcal{T}) = bmo^c(\mathcal{T}) \cap bmo^r(\mathcal{T})$.

(v) *Assume that* (T_t) *admits a reversed Markov dilation* $\mathcal{M}_t, \pi_t$. *Then the homomorphism* $\pi_0 : \mathcal{N}^0 \to bmo^c(\mathcal{M})$ *extends to a weakly continuous map on* $bmo^c(\mathcal{T})$ *and* $h_1^c(\mathcal{T}) \cap h_1^r(\mathcal{T})$ *is dense in both* $h_1^c(\mathcal{T})$ *and* $h_1^r(\mathcal{T})$.

Proof (i) is obvious because $a \otimes b \in V_1^c$ iff $a^* \otimes b^* \in V_1^r$ and w is bounded and injective from $\mathcal{N}^0$ to $V_\infty^c \cap V_\infty^r$.For the proof of (ii), we first show that

$$\{aT_t(b) - T_t(T_t(a)b) : t > 0, a, b \in L_2 \cap L_\infty\} \subset L_p^0(\mathcal{N})$$

is dense in $L_p^0(\mathcal{N})$. Indeed, let $y \in L_{p'}(\mathcal{N})$ such that

$$tr(aT_t(b)y) = tr(T_t(T_t(a)b)y) \tag{5.14}$$

holds for all a, b as above. By approximation with support projections and the weak continuity of T_t, we deduce from (5.14) and the self adjointness property of T_t that

$$tr(\tilde{a}y) = \lim_{\lambda,\mu} tr(\tilde{a}e_\mu T_t(e_\lambda)y) = \lim_{\lambda,\mu} tr(T_t(\tilde{a}e_\mu)e_\lambda T_t y) = tr(\tilde{a}T_{2t}y).$$

This shows $T_{2t}(y) = y$ and hence $y \in L_{p'}(\mathcal{N}_0) = \overline{(\ker A)^\perp}^{\|\ \|_p}$. Hence $L_p^0(\mathcal{N}) \cap h_1^c$ is dense in $L_p^0(\mathcal{N})$. Similarly,$L_p^0(\mathcal{N}) \cap h_1^r$ is dense in $L_p^0(\mathcal{N})$.

For (iii), Let A be the set of $a \otimes b = a(t) \otimes b(t)$ with $a(t), b(t) \in L_1(\mathcal{N}) \cap \mathcal{N}$ for all t and $a(t) = b(t) = 0$ except finite many t's. Then $w_c^*(A), w_r^*(A) \in L_p(\mathcal{N})$ and A is dense in V_1^c and is dense in V_1^r. We conclude that $h_1^c(\mathcal{T}) \cap L_p(\mathcal{N})$ is dense in $h_1^c(\mathcal{T})$ and $h_1^r(\mathcal{T}) \cap L_p(\mathcal{N})$ is dense in $h_1^r(\mathcal{T})$. So $h_1(\mathcal{T}) \bigcap L_p(\mathcal{N})$ is dense in $h_1(\mathcal{T})$.

For the proof of (iv) we see that the inclusion map $\iota : h_1^c \to L_1(\mathcal{N})$ is injective. By the Hahn Banach theorem, we deduce that $\iota^*(\mathcal{N}) \subset (h_1^c)^*$ is weakly dense. However, by definition h_1^c is a quotient of V_1^c. Hence $(h_1^c)^*$ is a subspace of V_∞^c. We then deduce from (5.13) that, when restricted to $\mathcal{N}$, the map ι^* is given by $\iota^*(x)(t) = x \otimes 1 - 1 \otimes T_t x$. Thus we have

$$(h_1^c)^* = \iota(\mathcal{N}) = bmo^c.$$

Taking adjoints we get $(h_1^r)^* = bmo^r$. Since $X = h_1^c \cap h_1^r$ is dense in both spaces, we may then embed bmo^c and bmo^r in X^*. We see that the inclusion map $\iota_X : X \to L_1(\mathcal{N})$ is injective and factors through the inclusion map $\iota_{h_1} : h_1 \to L_1(\mathcal{N})$. Since $X \subset h_1$ is dense, we deduce that h_1^* is the weak*-closure of

$$h_1^* = \overline{\iota^*(\mathcal{N})}^{\sigma(h_1^*, h_1)} \subset X^*.$$

Note that the last inclusion is injective and certainly $h_1^* \subset bmo^c \cap bmo^r$ because elements in $\mathcal{N}$ give rise to functionals which coincide on the intersection. For the converse inclusion $bmo^c \cap bmo^r \subset h_1^*$, it suffices to recall that a bounded functional extends uniquely from a dense subspace.

We now prove (v). Recall that a net $x_\lambda \in \mathcal{N}^0$ weakly converges in $bmo^c(\mathcal{T})$, if the inner product $\langle w(x_\lambda), w(x_\lambda)\rangle_{\mathcal{T}}^c = T_t|x|^2 - |T_t x|^2$ weakly converges in $\ell_\infty \otimes \mathcal{N}$. This is equivalent to the weak convergence of $\langle \pi_0 x_\lambda), \pi_0 x_\lambda)\rangle_{\mathcal{E}}^c = \pi_t(T_t|x|^2 - |T_t x|^2)$ in $\ell_\infty \otimes \mathcal{M}$, which is the meaning of weak* convergence of $(\pi_0 x_\lambda)$ in $bmo^c(\mathcal{M})$ (see [16]). Therefore, $\pi_0(bmo^c(\mathcal{T})) \subset bmo^c(\mathcal{M})$ is a weak* closed subspace and $\pi_0^*(h_1^c(\mathcal{M})) = h_1^c(\mathcal{T})$. We obtain the density of $h_1^r(\mathcal{T}) \cap h_1^r(\mathcal{T}) \subset h_1^c(\mathcal{T})$ by the corresponding result on martingale Hardy spaces. □

Lemma 5.15 *Assume that a standard semigroup (T_t) has a reversed Markov dilation with a.u. continuous path. Then*

$$\pi_0^*(H_1^c(\mathcal{M})) \subset h_1^c(\mathcal{T}) \subset L_1(\mathcal{N}).$$

Proof We have seen that $\pi_0^*(h_1^c(\mathcal{M})) = h_1^c(\mathcal{T})$ and $\pi_0^*(H_1^c(\mathcal{M})) \subset \pi_0^*(L_1(\mathcal{M})) = L_1(\mathcal{N})$.Let us recall that $H_1^c(\mathcal{M}) = h_1^c(\mathcal{M}) + h_1^d(\mathcal{M})$. We are going to show that $\pi_0^*(h_1^d(\mathcal{M}))$ vanishes in $L_1(\mathcal{N})$. By density it suffices to consider $\xi \in h_1^d(\mathcal{M}) \cap h_p^d(\mathcal{M})$ for some $1 < p < 2$. Recall that there are weakly dense subsets B_q of $L_q(\mathcal{N})$ such that the martingale $m(f) = (E_{[t}(\pi_0 f))_t$ has a. u. continuous path if T_t admits a reversed Markov dilation with a.u. continuous path. (see the definition at the beginning of this section). Let $y \in B_q$. By Lemma 4.1,

$$\|\pi_0(y)\|_{h_q^d} = 0.$$

This implies

$$|tr(\pi_0^*(\xi^*)y)| = |tr(\xi^* \pi_0(y))| \le \lim_\sigma \|\xi\|_{h_p^d(\sigma)} \|\pi_0(y))\|_{h_q^d(\sigma)} = 0.$$

Hence $tr(\pi_0^*(\xi)\cdot)$ vanishes on a weakly dense set of $L_q(\mathcal{N})$ and is 0 in $L_p(\mathcal{N})$. So it is 0 in $L_1(\mathcal{N})$. Thus π_0^* is 0 on $h_1^d \cap h_p^d$ and hence identically 0. Therefore we have indeed $\pi_0^*(H_1^c(\mathcal{M})) \subset h_1^c(\mathcal{T})$. □

Theorem 5.16 *Let $1 < p < \infty$ and (T_t) be a standard semigroup admitting a reversed Markov dilation with a. u. continuous path. Then*

$$[bmo^0(\mathcal{T}), L_1^0(\mathcal{N})]_{\frac{1}{p}} = [bmo^0(\mathcal{T}), h_1(\mathcal{T})]_{\frac{1}{p}} = [\mathcal{N}^0, h_1(\mathcal{T})]_{\frac{1}{p}} = L_p^0(\mathcal{N}).$$

Proof By Lemma 5.15, we have, for $1 < p \le 2$ and $\frac{1}{p} = \frac{1+\theta}{2}$,

$$\begin{aligned} L_p^0(\mathcal{N}) &= \pi_0^*(\pi_0 L_p^0(\mathcal{N})) \subset \pi_0^*(L_p^0(\mathcal{M})) \\ &\subset \pi_0^*[L_2^0(\mathcal{M}), H_1(\mathcal{M})]_\theta \subset [L_2^0(\mathcal{N}), h_1(\mathcal{T})]_\theta. \end{aligned}$$

Combining this with the trivial inclusion

$$\left[L_2^0(\mathcal{N}), h_1(\mathcal{T})\right]_\theta \subset \left[L_2^0(\mathcal{N}), L_1^0(N)\right]_\theta = L_p^0(\mathcal{N})$$

we equality in this range. Theorem 5.16 follows by duality and Wolffs' theorem (see [30] for a similar argument). □

5.2.4 Interpolation for $BMO(\Gamma)$

Our last concern in this section are interpolation result for $BMO(\Gamma)$ spaces. We first need some definitions. We define a $\ell^\infty(\mathbb{R}_+) \otimes \mathcal{N}$-valued inner product on $\mathcal{N} \otimes \mathcal{N}$ by

$$\langle x \otimes a, y \otimes b\rangle_\Gamma = a^* \int_0^\infty P_{s+t}\Gamma(P_s x, P_s y) \min(s, b) ds b.$$

Let $\mathcal{L}$ be the Hilbert $\ell^\infty(\mathbb{R}_+) \otimes \mathcal{N}$-module corresponding to this inner product. Recall that we denote by P_Γ the projection on the spatial part in Meyer's model.

Definition 5.17 Let $BMO^c(\Gamma)$ be the weak*-closure of $\mathcal{N}^0$ in $\mathcal{L}$ via the embedding

$$\Phi : x \to x \otimes 1.$$

Let $BMO^r(\Gamma)$ be the weak*-closure of $\mathcal{N}^0$ in $\mathcal{L}$ via the embedding $x \to x^* \otimes 1$.

Let $x_\lambda \in \mathcal{N}^0$ be a bounded net in $BMO^c(\Gamma)$ which weak* converges to $x \in BMO^c(\Gamma)$. Recall that $P_b x$ exists in $BMO^c(\Gamma)$ for any $b > 0$ and

$$S(t) = \int_0^\infty P_{t+s}\Gamma\left[P_s(x)\right] \min\{t, s\} ds.$$

exists in $\ell_\infty(\mathbb{R}_+) \otimes \mathcal{N}$ as the weak* limit of

$$S_\lambda(t) = \int_0^\infty P_{t+s}\Gamma\left[P_s(x_\lambda)\right] \min\{t, s\} ds.$$

Here and in the following, $\Gamma[x]$ denotes $\Gamma(x, x)$ for simplification. We need the following lemma to understand the intersection of $BMO^c(\Gamma)$ and $BMO^r(\Gamma)$.

Lemma 5.18 *Let (T_t) be a standard semigroup satisfying $\Gamma^2 \geq 0$. Then, for any $x \in BMO^c(\Gamma)$*

(i) *$P_{2b}\Gamma(P_b x, P_b x)$ exists in $\mathcal{N}$ for any $b > 0$, and*

$$\|P_{2b}\Gamma(P_b x, P_b x)\| \leq \frac{6}{b^2}\|x\|^2_{BMO^c(\Gamma)}.$$

(ii) $T_t|P_b x|^2 - |T_t P_b x|^2$ *exists in* $\mathcal{N}$ *for any* $t, b > 0$ *and*

$$\|T_t|P_b x|^2 - |T_t P_b x|^2\| \leq \frac{6t}{b^2}\|x\|^2_{BMO^c(\Gamma)}.$$

(iii) $P_b x$ *weak* * *converges to* x *in* $BMO^c(\Gamma)$ *as* $b \to 0$.

(iv) $x = 0$ *in* $BMO^c(\Gamma)$ *iff* $P_b x = 0$ *in* $bmo^c(\mathcal{T})$ *for any* $b > 0$. *The similar properties hold for* $y \in BMO^r(\Gamma)$.

Proof Let us fix $b > 0$ and a net $x_\lambda \in \mathcal{N}^0$ such that $\Phi(x_\lambda)$ converges with respect to the STOP topology in $\mathcal{L}$. By (i), we mean that $P_{2b}\Gamma(P_b x_\lambda, P_b x_\lambda)$ weak* converges in $\mathcal{N}$ and the limit is with a norm smaller than $\frac{b^2}{6}\|x\|^2_{BMO^c(\Gamma)}$. We first deduce from $\Gamma^2 \geq 0$ and Proposition 1.2 that

$$\begin{aligned}
\frac{b^2}{2}P_{2b}\Gamma\left(P_b x_\lambda, P_b x_\lambda\right) &= \int_0^b P_{2b}\Gamma\left(P_b x_\lambda, P_b x_\lambda\right) sds\\
&= \int_0^b P_{2b}\Gamma\left(P_{b-s}P_s x_\lambda, P_{b-s}P_s x_\lambda\right) sds\\
&\leq \int_0^b P_{3b-s}\Gamma\left(P_s x_\lambda, P_s x_\lambda\right) sds\\
&\leq 3\int_0^b P_{b+s}\Gamma\left(P_s x_\lambda, P_s x_\lambda\right) sds. \qquad (5.15)
\end{aligned}$$

By Proposition 5.9, $\Phi(x_\lambda)$ converges in the STOP topology implies that the last term in inequality (5.15) weak * converges in $\mathcal{N}$. Thus $\Gamma(P_b x_\lambda, P_b x_\lambda)$ weak * converges in $\mathcal{N}$ and the limit exists in $\mathcal{N}$ with a norm bounded by $\frac{54}{b^2}\|x\|^2_{BMO^c(\Gamma)}$. For (ii), we apply lemma 1.1 (i) and $\Gamma^2 \geq 0$ and get

$$\begin{aligned}
T_t|P_b x_\lambda|^2 - |T_t P_b x_\lambda|^2 &= \int_0^t T_{t-s}\Gamma(T_s P_b x_\lambda, T_s P_b x_\lambda)ds\\
&\leq \int_0^t T_t P_{\frac{2b}{3}}\Gamma\left(P_{\frac{b}{3}}x_\lambda, P_{\frac{b}{3}}x_\lambda\right)ds = tT_t P_{\frac{2b}{3}}\Gamma\left(P_{\frac{b}{3}}x_\lambda, P_{\frac{b}{3}}x_\lambda\right).
\end{aligned}$$

Applying (5.15), we have

$$T_t|P_b x_\lambda|^2 - |T_t P_b x_\lambda|^2 \leq \frac{54t}{b^2}T_t\int_0^b P_{b+s}\Gamma(P_s x_\lambda, P_s x_\lambda)sds.$$

Thus $T_t|P_b x_\lambda|^2 - |T_t P_b x_\lambda|^2$ weak * converges in $\mathcal{N}$ and the limit exists in $\mathcal{N}$ with a norm bounded by $\frac{54t}{b^2}\|x\|^2_{BMO^c(\Gamma)}$ for any $t > 0$.

To prove (iii), we use the same idea in the proof of Lemma 3.2. For any $t > 0$, $0 < b < \min\{t^2, 1\}$. Let $Q_b x = (I - P_b)x = \int_0^b \frac{\partial P_s x}{\partial s} ds$. Then

$$\begin{aligned}
\int_0^\infty P_{t+s}\Gamma[P_s Q_b(x)]\min\{t,s\}ds &= \int_0^\infty P_{t+s}\min\{t,s\}\Gamma\left[\int_s^{b+s}\frac{\partial P_v}{\partial v}x dv\right]ds\\
\text{(first ineq. of Lemma 3.1)} &\leq \int_0^\infty \min\{t,s\}\frac{1}{s}P_{t+s}\left(\int_s^{b+s}\Gamma\left[v\frac{\partial P_v}{\partial v}x\right]dv\right)ds\\
\text{(Prop. 1.2)} &\leq 2\int_0^\infty P_t\left(\int_s^{b+s}\Gamma\left[v\frac{\partial P_v}{\partial v}x\right]dv\right)ds\\
\text{(change of variables)} &= 8\int_0^\infty P_t\left(\int_{\frac{s}{2}}^{\frac{b+s}{2}}\Gamma\left[v\frac{\partial P_v}{\partial v}P_v x\right]dv\right)ds\\
&= 8\int_0^\infty P_t\left(\int_{\frac{s}{2}}^{\frac{b+s}{2}}P_v\Gamma[P_v x]dv\right)ds\\
\text{(Integrate on } ds \text{ first)} &= 8\int_0^\infty P_t P_v\Gamma[P_v x]\min\{2v, b\}dv\\
&\leq 8\int_{\sqrt{b}}^\infty P_t P_v\Gamma[P_v x]b dv + 8\int_0^{\sqrt{b}} P_t P_v\Gamma[P_v x]v dv\\
&\leq 8\sqrt{b}\int_{\sqrt{b}}^\infty P_{t+v}\Gamma[P_v x]\min\{t,v\}dv\\
&\quad + 8\int_0^{\sqrt{b}} P_{t+v}\Gamma[P_v x]\min\{t,v\}dv.
\end{aligned}$$

Thus, for any $t > 0$, $g \in L^1_+(\mathcal{N})$,

$$\tau\left(g\int_0^\infty P_{t+s}\Gamma\left[P_s Q_b(x)\right]\min\{t,s\}ds\right)$$

$$\leq 8\sqrt{b}\tau\left(g\int_{\sqrt{b}}^{\infty}P_{t+v}\Gamma\left[P_v x\right]\min\{t,v\}dv\right)$$
$$+\;8\tau\left(g\int_{0}^{\sqrt{b}}P_{t+v}\Gamma[P_v x]\min\{t,v\}dv\right).$$

This means

$$\lim_{\lambda}\tau\left(g\int_{0}^{\infty}P_{t+s}\Gamma[P_s Q_b(x_\lambda)]\min\{t,s\}ds\right)$$
$$\leq 8\sqrt{b}\lim_{\lambda}\tau\left(g\int_{\sqrt{b}}^{\infty}P_{t+v}\Gamma[P_v x_\lambda]\min\{t,v\}dv\right)$$
$$+\;8\lim_{\lambda}\tau\left(g\int_{0}^{\sqrt{b}}P_{t+v}\Gamma[P_v x_\lambda]\min\{t,v\}dv\right).$$

The first term in the right hand side converges to 0 as $b\to 0$. We claim the second term converges to 0 too. If not, there exists $\epsilon>0$ such that $\tau(g\lim_\lambda\int_0^{\sqrt{b}}P_{t+v}\Gamma[P_v x_\lambda]\min\{t,v\}dv)>\epsilon$ for all b. We reach a contradiction with the absolute continuity of integrals by choosing x_{λ_0} such that $\tau(g\lim_\lambda\int_0^1 P_{t+v}\Gamma[P_v(x_\lambda - x_{\lambda_0})]\min\{t,v\}dv)<\frac{\epsilon}{2}$. The assertion (iii) is proved.

We now prove (iv). Let $x_\lambda\in\mathcal{N}^0$ be a net weak* converges to x in $BMO^c(\Gamma)$. Suppose $x=0$ in $BMO^c(\Gamma)$. The proof of (ii) implies that $w(P_b x_\lambda)$ weakly converges to 0 in V_∞^c. Here w and V_∞^c are the embedding and Hilbert L_∞-module defined for the study of $bmo^c(\mathcal{T})$ in Sect. 5.2.3. Therefore, $P_b x_\lambda$ weakly converges to 0 in $bmo^c(\mathcal{T})$ for every $b>0$. To prove the reverse, recall that $P_b x$ weakly converges to 0 in $bmo^c(\mathcal{T})$ means that $T_t|P_b(x_\lambda)|^2-|T_tP_b(x_\lambda)|^2=\int_0^t T_{t-s}\Gamma(T_sP_bx_\lambda,T_sP_bx_\lambda)ds$ weak * converges to 0 in $\mathcal{N}$ for any $b>0$. Use the same idea as the proof of (ii), we have $tP_bT_t(x_\lambda)$ weakly converges to 0 in $BMO^c(\Gamma)$ for any $b,t>0$. Then $P_{2b}(x_\lambda)$ weakly converges to 0 in $BMO^c(\Gamma)$ for any $b>0$ since b^2P_b is an average of tT_t. This means $P_{2b}x=0$ in $BMO^c(\Gamma)$ for any b. By (iii), we conclude that $x=0$ in $BMO^c(\Gamma)$. The same argument works for $BMO^r(\Gamma)$. □

For $x\in BMO^c(\Gamma)$, $y\in BMO^r(\Gamma)$, we say $x=y$ if $P_b(x-y)=0$ in $bmo^c(\mathcal{T})\cap bmo^r(\mathcal{T})$ for any $b>0$. For $x\in BMO^c(\Gamma)$ given, such a y is unique in $BMO^r(\Gamma)$ because of Lemma 5.18 (iv).

Definition 5.19 Let $BMO(\Gamma)$ be the space of all $x\in BMO^c(\Gamma)$ which belongs to $BMO^r(\Gamma)$ too. Define

$$\|x\|_{BMO(\Gamma)}=\max\left\{\|x\|_{BMO^c(\Gamma)},\|y\|_{BMO^r(\Gamma)}\right\}.$$

Here y is the unique element in $BMO^r(\Gamma)$ such that $P_b(x-y)=0$ in $bmo^c(\mathcal{T})\cap bmo^r(\mathcal{T})$ for all $b>0$.

Theorem 5.20 *Let (T_t) be a standard semigroup satisfying $\Gamma^2\geq 0$ and admitting a reversed Markov dilation with a. u. continuous path. Then*

$$\Big[BMO(\Gamma),L_q^0(\mathcal{N})\Big]_{\frac{q}{p}} \;=\; L_p^0(\mathcal{N}).$$

Proof Let $x\in BMO(\Gamma)\cap L_q^0(\mathcal{N})$. By Proposition 5.1, we know that $\pi_{\mathbf{t}_a}(x)$ has continuous path with respect to the filtration $\hat{\mathcal{M}}_{a,t}$. Note that

$$\pi_{\mathbf{t}_a}(x) \;=\; P_\Gamma\big(\pi_{\mathbf{t}_a}(x)\big)+P_{br}\big(\pi_{\mathbf{t}_a}(x)\big).$$

However, $P_{br}(\pi_{\mathbf{t}_a}(x)) = \int_0^{\mathbf{t}_a}\pi_r(\partial P_{B_r}(x))dr$ is a stochastic integral against the Brownian motion and hence has continuous path. Taking the difference, we know that $P_\Gamma(\pi_{\mathbf{t}_a}(x))$ has a.u. continuous path. We can now copy the proof for $bmo(\mathcal{T})$. More precisely, let $h_1(\Gamma)$ be an abstract predual of $BMO(\Gamma)$. Similar to the proof of Lemma 5.15 we have $\pi_0^*(P_\Gamma H_1^c(\mathcal{M}))\subset h_1^c(\Gamma)$ since $\pi_0^*(h_1^d(\mathcal{M}))=\{0\}$. Then, by the same argument used in the proof of Theorem 5.16, we have

$$L_p^0(\mathcal{N})\subset\Big[L_2^0(\mathcal{N}),h_1(\Gamma)\Big]_{\frac{2-p}{p}},$$

for $1<p<2$. By duality and Wolff's theorem, we obtain the result. □

Open problems. At the end of this article we want to mention some open problems.

(i) *H^1-BMO duality for semigroup of operators.* Fefferman's H^1-BMO duality theory has been studied in the context of semigroups by many researchers. In particular, Varopoulos established an H^1-BMO duality theory for a "good" semigroups by a probabilistic approach. Duong/Yan studied this topic for operators with heat kernel bounds (see [6]). In their proofs, the geometric structure of Euclidean spaces is essential. Mei (see [27]) provides a first approach of this problem in the context of von Neumann algebras with two additional assumptions on the semigroups. The authors expect a more general H^1-BMO duality in the context of semigroups.

(ii) *Comparison of different semigroup BMO-norms*. There are several natural semigroup BMO norms as introduced in this article. A complete comparison of them is in order. In particular, it will be interesting to investigate the conditions on the semigroups so that we have the estimates,
   - (a) $\|\cdot\|_{BMO^c(\mathcal{P})}\simeq\|\cdot\|_{BMO^c(\partial)}\simeq\|\cdot\|_{BMO^c(\Gamma)}$.
   - (b) $\|\cdot\|_{bmo^c(\mathcal{P})}\simeq\|\cdot\|_{BMO^c(\mathcal{P})}\simeq\|\cdot\|_{BMO^c(\hat{\Gamma})}$.
   - (c) $\|\cdot\|_{bmo^c(\mathcal{T})}\simeq\|\cdot\|_{BMO^c(\mathcal{T})}$.
   - (c′) $\sup_t\|T_tx-T_{2t}x\|\leq c\|x\|_{bmo^c(\mathcal{T})}$.
   - (d) $\sup_t\|T_t\int_0^t\left|\frac{\partial T_sx}{\partial s}\right|^2 sds\|\leq c\|x\|^2$.

(iii) The classical BMO functions φ on $\mathbb{R}$ is integrable with respect to $\frac{1}{1+t^2}dt$. *What is a noncommutative analogue of this property ?* A more precise question is, does there exist a normal faithful state τ on $\mathcal{N}$ such that $\tau|x| \leq c\|x\|_{BMO(\mathcal{T})}$ for $x \in \mathcal{N}$.

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