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On the subordinate killed B.M in bounded domains and existence results for nonlinear fractional Dirichlet problems

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Abstract We take up in this paper the existence of positive continuous solutions for some nonlinear boundary value problems with fractional differential equation based on the fractional Laplacian $(-\Delta_{|D})^{\frac{\alpha}{2}}$ associated to the subordinate killed Brownian motion process Z_{α}^{D} in a bounded $C^{1,1}$ domain D. Our arguments are based on potential theory tools on Z_{α}^{D} and properties of an appropriate Kato class of functions $K_{\alpha}(D)$.

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1 Introduction

Let $\chi = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Brownian motion in $\mathbb{R}^d, d \geq 2$ and $\pi = (\Omega, \mathcal{G}, T_t)$ be an $\frac{\alpha}{2}$-stable process subordinator starting at zero, where $0 < \alpha < 2$ and such that χ and π are independent. In this paper, we always assume that D is a bounded $C^{1,1}$- domain in $\mathbb{R}^d$. We are interested in the subordinate killed Brownian motion process which is a symmetric Hunt process that we denote by Z_{α}^{D}. This process is obtained by killing χ at τ_D, the first exit time of χ from D giving the process χ^D and then subordinating this killed Brownian motion using the $\frac{\alpha}{2}$-stable subordinator T_t.

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The infinitesimal generator of Z_α^D is the fractional power $(-\Delta_{|D})^{\frac{\alpha}{2}}$ of the negative Dirichlet Laplacian in D, which is a prototype of non-local operator and a very useful object in analysis and partial differential equations, see, for instance [14,17].

Until recently and despite its importance, the process Z_α^D was not fully developed. This process was first studied in [7], where among other things, an one to one correspondence between the family of positive harmonic functions of the killed Brownian motion χ^D and the family of positive harmonic functions of the subordinate killed Brownian motion Z_α^D was established. This correspondence was improved later in [6]. In particular, it was shown in [6], that there are no non-trivial bounded harmonic functions for Z_α^D. While the classical formulation of the Dirichlet problem becomes impossible, the authors of [6] provide an appropriate reformulated Dirichlet problem associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$ (see Proposition 3 and Remark 2 below). This approach allows us to study two different nonlinear Dirichlet problems associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$ and to transfer existence results about nonlinear equations based on Brownian motion techniques, obtained in [12] and [13], into existence results in the new situation as it is stated in Theorems 3 and 4 below.

On the other hand, a precise description of Z_α^D in terms of the underlying Brownian motion χ and the subordinator π was given in [16]. As a consequence, the authors of [16] established the behavior of the Green function G_α^D of Z_α^D. Later, in [15], new lower bounds for G_α^D were proved giving sharp estimates on G_α^D and also sharp estimates for the density of Z_α^D were performed. These bounds will be useful for our study. In particular, this enables us to introduce a functional class $K_\alpha(D)$ called fractional Kato class, which is characterized by an integral condition involving G_α^D. This class is quite rich (see Proposition 8) and it is a key tool for proving our existence results.

The content of this paper is organized as follows. In Sect. 2, we recapitulate some potential theory tools pertaining to the process Z_α^D developed in particular in [6] and [7]. Then, we present our main results (see Theorems 3 and 4). In Sect. 3, we establish some estimates and properties of G_α^D. We give in Sect. 4 some interesting properties of the class $K_\alpha(D)$ including a careful analysis about continuity of some potential functions. Our main results are proved in Sects. 5 and 6.

2 Notation and setting

2.1 Potential theory associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$

Let $p^D(t,x,y)$ be the transition density of the semi-group $(P_t^D)_{t>0}$ corresponding to the killed Brownian motion χ^D and η_t^α be the density of T_t such that for every $t,s>0$, $\int_0^\infty \eta_t^\alpha(u)\exp(-su)du = \exp(-ts^{\frac{\alpha}{2}})$. Further, we have $\int_0^\infty \eta_s^\alpha(t)ds = \frac{1}{\Gamma(\frac{\alpha}{2})}t^{\frac{\alpha}{2}-1}, t>0$.

Then the semi-group $(Q_t^\alpha)_{t>0}$ generated by the process Z_α^D is given by

$$Q_t^\alpha f(x) := \int_0^\infty P_s^D f(x)\eta_t^\alpha(s)ds = \int_D q^\alpha(t,x,y)f(y)dy, \text{ for } f\in B^+(D),$$

where $q^{\alpha}(t,x,y) := \int_0^{\infty} p^D(s,x,y)\eta_t^{\alpha}(s)ds$ is the density of Q_t^{α} and $B^+(D)$ denotes the set of nonnegative Borel measurable functions defined on D.

It is shown in [15] that for any $T > 0$, we have

$$q^{\alpha}(t,x,y) \approx \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2 + t^{\frac{2}{\alpha}}}, 1\right) t^{\frac{-d}{\alpha}} \left(1 + \frac{|x-y|^2}{t^{\frac{2}{\alpha}}}\right)^{-\frac{d+\alpha}{2}}, t < T \text{ and } x, y \in D. \tag{2.1}$$

Here and throughout the paper $\delta(x)$ denotes the Euclidean distance between x and the boundary ∂D of D and for nonnegative functions f and g defined on a set S, we write $f \approx g$ if there exists $c > 0$ such that $\frac{1}{c} f \leq g \leq cf$ on S and we say that f is comparable to g.

The Green function $G_{\alpha}^D(x,y)$ associated to $(Q_t^{\alpha})_{t>0}$ is a continuous function on $D \times D$ except along the diagonal and is given by

$$G_{\alpha}^D(x,y) = \int_0^{\infty} q^{\alpha}(t,x,y)dt = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^{\infty} p^D(t,x,y)t^{\frac{\alpha}{2}-1}dt. \tag{2.2}$$

We will denote $G^D(x,y)$ the Green function associated to $(P_t^D)_{t>0}$ (i.e. $\alpha = 2$).

The following sharp estimates on $G_{\alpha}^D(x,y)$ are given in [15],

$$G_{\alpha}^D(x,y) \approx \frac{1}{|x-y|^{d-\alpha}} \, min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right), \quad x, y \in D. \tag{2.3}$$

These interesting inequalities extend those for the Green function G^D of the killed Brownian motion χ^D, in the case $d \geq 3$ (see [18]) and consequently it was shown a $3G$-inequality for G^D (see [8]) allowing to introduce and study the Kato class of functions $K(D)$ (see [13], for $d \geq 3$ and [19] for $d = 2$). This class was extensively used in the study of various elliptic differential equations in bounded domains (see [2,13] and [19]).

Analogously, Theorem 1 below provides a fundamental $3G$-inequality for G_{α}^D, as a consequence of the estimates (2.3). For the proof, we refer to [15].

Theorem 1 (3G-Theorem) *There exists a positive constant C_0 such that for all x, y, z in D, we have*

$$\frac{G_{\alpha}^D(x,z)G_{\alpha}^D(z,y)}{G_{\alpha}^D(x,y)} \leq C_0 \left(\frac{\delta(z)}{\delta(x)} G_{\alpha}^D(x,z) + \frac{\delta(z)}{\delta(y)} G_{\alpha}^D(y,z)\right) \tag{2.4}$$

This allows us to introduce a new fractional Kato class of functions in D which will be denoted by $K_{\alpha}(D)$ and defined as follows.

Definition 1 A Borel measurable function q in D belongs to the Kato class $K_\alpha(D)$ if q satisfies the following condition

$$\lim_{r\to 0}\left(\sup_{x\in D}\int\limits_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\,|q(y)|\,dy\right)=0.$$

As a typical example of functions in $K_\alpha(D)$, we cite $q(x)=\delta(x)^{-\lambda}$, $\lambda<\alpha$.

Remark 1 Replacing G_α^D by G^D in Definition 1 above, we find again the Kato class $K(D)$ introduced in [13,19].

Furthermore, since for $x, y \in D$, we have

$$\frac{G^D(x,y)}{G_\alpha^D(x,y)}\approx |x-y|^{2-\alpha}\log\left(2+\frac{\delta(x)\delta(y)}{|x-y|^2}\right)^{(3-d)^+},$$

we deduce that there exists $c>0$ such that for $x, y \in D$, $G^D(x,y)\leq cG_\alpha^D(x,y)$. Consequently, we conclude that

$$K_\alpha(D)\subset K(D).$$

Let us define the potential kernel G_α^D of Z_α^D on $B^+(D)$ by

$$G_\alpha^D\psi(x)=\int\limits_D G_\alpha^D(x,y)\psi(y)dy.$$

By ([7, Proposition 1]), we have $G_\alpha^D\psi\neq\infty$ if and only if $G_\alpha^D\psi\in L^1_{loc}(D)$.

Also by ([7, p. 222]), we have the following interesting relation between the potential kernels G_α^D and G^D : For any $\psi\in B^+(D)$, we get

$$G_\alpha^D(G_{2-\alpha}^D\psi)=G^D\psi. \tag{2.5}$$

Then, using (2.3) and (2.5), it is easy to see, as in the classical case, that the following assertions are equivalent

(i) $G_\alpha^D\psi\neq\infty$.
(ii) $\int_D\delta(y)\psi(y)dy<\infty$.

On the other hand, for any $\psi\in B^+(D)$ such that $\int_D\delta(y)\psi(y)dy<\infty$ and for any $\phi\in C_c^\infty(D)$ we have (see [7, p. 230])

$$\int\limits_D\psi(x)(-\Delta_{|D})^{\frac{\alpha}{2}}\phi(x)dx=-\int\limits_D G_{2-\alpha}^D\psi(x)\Delta\phi(x)dx<\infty,$$

that is

$$(-\Delta_{|D})^{\frac{\alpha}{2}}\psi = -\Delta G^D_{2-\alpha}\psi (in\ the\ distributional\ sense). \tag{2.6}$$

In particular, by (2.5) we obtain that

$$(-\Delta_{|D})^{\frac{\alpha}{2}} G^D_\alpha \psi = \psi \quad in\ D\ (in\ the\ distributional\ sense). \tag{2.7}$$

In what follows we recall the definition of excessive and harmonic functions associated to the process Z^D_α (see [6]).

Definition 2 A nonnegative Borel measurable function h on D is said to be harmonic with respect to Z^D_α if $h \neq \infty$ on D and if for every relatively compact open subset $U \subset \overline{U} \subset D$, we have

$$h(x) = \widetilde{E}^x \left[h\left(Z^D_\alpha(\widetilde{\tau}_U)\right)\right], \quad x \in U,$$

where $\widetilde{E}^x$ stands for the expectation with respect to Z^D_α starting from x and $\widetilde{\tau}_U := \inf \left\{t > 0 : Z^D_\alpha(t) \notin U\right\}$.

Definition 3 A nonnegative Borel measurable function s on D is said to be excessive with respect to Z^D_α if $s \neq \infty$ on D and satisfies

$$Q^\alpha_t s(x) \leq s(x), t > 0, x \in D$$

and

$$\lim_{t\to 0} Q^\alpha_t s(x) = s(x).$$

We are going to use $\mathcal{H}^D_\alpha$ to denote the collection of all nonnegative functions on D which are harmonic with respect to Z^D_α and $\mathcal{S}^D_\alpha$ to denote the collection of all excessive functions on D with respect to Z^D_α.

Also we denote by $\mathcal{H}^D$ and $\mathcal{S}^D$ respectively the collections of the classical nonnegative harmonic functions and excessive functions on D (i.e. with respect to χ^D). Recall that $\mathcal{H}^D_\alpha \subset \mathcal{S}^D_\alpha$ and $\mathcal{H}^D \subset \mathcal{S}^D$. An important connection between $\mathcal{S}^D_\alpha$ and $\mathcal{S}^D$ was established in [7] and improved later in [6]. More precisely, it was shown in [6] that $G^D_{2-\alpha}$ is a bijection from $\mathcal{S}^D_\alpha$ to $\mathcal{S}^D$ and the same from $\mathcal{H}^D_\alpha$ to $\mathcal{H}^D$. We can summarize the result of [6, Theorem 3.1] as follows.

Theorem 2 *If $s \in \mathcal{S}^D$, there exists a function $g \in \mathcal{S}^D_\alpha$, such that $s(x) = G^D_{2-\alpha} g(x)$ on D, given by the formula*

$$g(x) = \frac{1-\frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-2+\frac{\alpha}{2}} \left(s(x) - P^D_t s(x)\right) dt.$$

Conversely, if $g \in \mathcal{S}_\alpha^D$, then $s = G_{2-\alpha}^D g$ is in $\mathcal{S}^D$.

Moreover, $g \in \mathcal{H}_\alpha^D$ if and only if $s = G_{2-\alpha}^D g \in \mathcal{H}^D$.

Using this correspondence between $\mathcal{H}_\alpha^D$ and $\mathcal{H}^D$, the following properties are obtained in [6, theorem 3.2 and proposition 3.8]

Proposition 1 (i) *Every function $h \in \mathcal{H}_\alpha^D$ is continuous.*
(ii) *If $h \in \mathcal{H}_\alpha^D$ is bounded, then $h \equiv 0$.*

Note that we have the following relation between the functions in $\mathcal{H}_\alpha^D$ and the solutions (in the distributional sense) of the equation $(-\Delta_{|D})^{\frac{\alpha}{2}} u = 0$ (see [6, proposition 3.11]).

Proposition 2 *If $h \in \mathcal{H}_\alpha^D$, then*

$$\forall \phi \in C_c^\infty(D), \int_D h(x)(-\Delta_{|D})^{\frac{\alpha}{2}} \phi(x) dx = 0. \tag{2.8}$$

Conversely, suppose that h is a nonnegative continuous function such that $\int_D \delta(y) h(y) dy$ is finite and (2.8) is satisfied. Then $h \in \mathcal{H}_\alpha^D$.

Now, let us introduce the Martin kernel associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$. Fix a point $x_0 \in D$ and let

$$M^D(x, z) := \lim_{D \ni y \longrightarrow z} = \frac{G^D(x, y)}{G^D(x_0, y)}, \quad x \in D, \ z \in \partial D,$$

be the the Martin kernel of χ^D based at x_0. It is well known from the general potential theory that for each fixed $z \in \partial D$, the function $x \longrightarrow M^D(x, z) \in \mathcal{H}^D$ (see [1]).

Since $G_{2-\alpha}^D$ is a bijection from $\mathcal{H}_\alpha^D$ to $\mathcal{H}^D$, we define the function $K_\alpha^D(x, z)$ on $D \times \partial D$ by

$$G_{2-\alpha}^D \left(K_\alpha^D(\cdot, z) \right)(x) = M^D(x, z).$$

Then for each fixed $z \in \partial D$, $K_\alpha^D(\cdot, z) \in \mathcal{H}_\alpha^D$.

Let M_α^D be the function defined on $D \times \partial D$ by

$$M_\alpha^D(x, z) = \frac{K_\alpha^D(x, z)}{K_\alpha^D(x_0, z)}, \quad x \in D, \ z \in \partial D.$$

Then we have for each $z \in \partial D$, $M_\alpha^D(\cdot, z) \in \mathcal{H}_\alpha^D$. Moreover, M_α^D is jointly continuous on $D \times \partial D$ and satisfies for each $x \in D$

$$M_\alpha^D(x, z) = \lim_{y \longrightarrow z \in \partial D} \frac{G_\alpha^D(x, y)}{G_\alpha^D(x_0, y)}.$$

$M_\alpha^D(x,z)$ is called the Martin kernel based at x_0 for Z_α^D (see [6]).

On the other hand, by Martin's representation theorem (see [1]), there exists a finite positive measure σ on ∂D such that

$$1 = \int_{\partial D} M^D(\cdot, z)\sigma(dz).$$

We know (see [4, p. 16]) that for every continuous function f on ∂D, the unique solution h of the Dirichlet problem $\Delta h = 0$, $\lim_{x \longrightarrow z \in \partial D} h(x) = f(z)$ is given by

$$M^D f(x) = \int_{\partial D} M^D(x,z) f(z)\sigma(dz), \quad x \in D.$$

Hence putting for a continuous function f on ∂D

$$M_\alpha^D f(x) = \int_{\partial D} M_\alpha^D(x,z) f(z)\nu(dz), \quad x \in D,$$

where $\nu(dz) = K_\alpha^D(x_0, z)\sigma(dz)$, we obtain that $M_\alpha^D f \in \mathcal{H}_\alpha^D$ and $G_{2-\alpha}^D(M_\alpha^D f) = M^D f$.

Recall that by [6], we have

$$M_\alpha^D f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} E^x \left(f\left(X_{\tau_D}\right) \tau_D^{\frac{\alpha}{2}-1} \right). \tag{2.9}$$

Note that, if f is the constant 1, then $M_\alpha^D 1$ is the function in $\mathcal{H}_\alpha^D$ playing the role of the constant 1 in $\mathcal{H}^D$ i.e.

$$G_{2-\alpha}^D\left(M_\alpha^D 1\right) = 1, \tag{2.10}$$

and by Theorem 2,

$$M_\alpha^D 1(x) = \frac{1-\frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-2+\frac{\alpha}{2}} \left(1 - P_t^D 1(x)\right) dt.$$

We remark that it was shown in ([16, remark 3.3]) that

$$M_\alpha^D 1(x) \approx \delta(x)^{\alpha-2} \ in \ D. \tag{2.11}$$

Moreover, we have the following Proposition due to [6].

Proposition 3 *Let f be a nonnegative continuous function on ∂D. The function $M_\alpha^D f$ is the unique function $h \in \mathcal{H}_\alpha^D$ such that*

$$\lim_{x \longrightarrow z \in \partial D} \frac{h(x)}{M_\alpha^D 1(x)} = f(z).$$

Remark 2 Proposition 3 provides the solvability of the following reformulated Dirichlet problem associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$. Namely, if f is a nonnegative continuous function on ∂D, then $M_\alpha^D f$ is the unique continuous solution of

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = 0 \ \ in \ D \ \ (in \ the \ distributional \ sense) \\ \lim_{x \longrightarrow z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = f(z). \end{cases}$$

2.2 Main results

As it is mentioned above, the main goal of this paper is to prove two existence theorems, stated in Theorems 3 and 4 below, for fractional differential equations with reformulated Dirichlet boundary condition.

Our first purpose is to study the following problem

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = \varphi(\cdot, u) \ \ in \ D \ \ (in \ the \ distributional \ sense) \\ \lim\limits_{x \longrightarrow \partial D} \delta(x)^{2-\alpha} u(x) = 0. \end{cases} \tag{2.12}$$

In view of (2.11), we remark that the boundary condition in (2.12) is equivalent to $\lim_{x \longrightarrow z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = 0$. The nonlinearity φ is required to satisfy the assumptions

($\mathbf{H}_1$) φ is a non-trivial nonnegative measurable function in $D \times (0, \infty)$ which is continuous and nonincreasing with respect to the second variable.
($\mathbf{H}_2$) $\forall c > 0$, $x \to \delta(x)^{2-\alpha} \varphi(x, c\delta(x)^{\alpha-2})$ is in $K_\alpha(D)$.

Note that $x \to \partial D$ means that x tends to a point ξ of ∂D.

As a typical example of functions φ satisfying (H_1) and (H_2) , we quote $\varphi(x, u) = k(x)u^{-\sigma}$, where $\sigma \geq 0$ and k is a nonnegative measurable function in D such that the function

$$x \to k(x)\delta(x)^{(\sigma+1)(2-\alpha)} \in K_\alpha(D).$$

Using a fixed point theorem, we prove in Sect. 5 the following

Theorem 3 *Assume $(H_1) - (H_2)$. Then problem (2.12) has a positive continuous solution u in D satisfying*

$$u(x) = G_\alpha^D(\varphi(\cdot, u))(x), x \in D. \tag{2.13}$$

Note that this result extends a result of [12] in the elliptic case (i.e. $\alpha = 2$).

For our second purpose, we are interested in the following problem

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u + u\varphi(\cdot, u) = 0 \ \ in\ D\ (in\ the\ distributional\ sense) \\ \lim\limits_{\substack{x \longrightarrow z \\ z\in\partial D}} \frac{u(x)}{M_\alpha^D 1(x)} = f(z), \end{cases} \tag{2.14}$$

where f is a non-trivial nonnegative continuous function on ∂D and the nonlinear term is required to satisfy the following assumptions

($\mathbf{H}_3$) φ is a nonnegative measurable function in $D \times (0, \infty)$.
($\mathbf{H}_4$) For all $c > 0$, there exists a nonnegative function $q_c \in K_\alpha(D)$ such that the map $s \to s\left[q_c(x) - \varphi\left(x, s\delta(x)^{\alpha-2}\right)\right]$ is continuous and nondecreasing on $[0, c]$, for all $x \in D$.

To illustrate, let us present an example. Let $p > 0$ and k be a nonnegative measurable function such that the function

$$x \to k(x)\delta(x)^{(\alpha-2)p} \in K_\alpha(D).$$

Then the function $\varphi(x, u) = k(x)u^p$ satisfies (H_3) and (H_4).

Using a potential theory approach, we establish in Sect. 6 the following

Theorem 4 *Assume* $(H_3) - (H_4)$. *Then problem* (2.14) *has a positive continuous solution u in D. Moreover, u satisfies the following*

$$cM_\alpha^D f(x) \le u(x) \le M_\alpha^D f(x), \tag{2.15}$$

where $c \in (0, 1)$.

We end this section by noting that solutions for the nonlinear problems (2.12) and (2.14) associated to $(-\Delta_{|D})^{\frac{\alpha}{2}}$ blow up at the boundary ∂D. On the contrary, for the classical case (i.e. $\alpha = 2$), solutions of elliptic nonlinear problems corresponding to (2.12) and (2.14) are bounded (see [12,13]).

From here on, c denotes a positive constant which may vary from line to line. Also we refer to $C(\overline{D})$ the collection of all continuous functions in $\overline{D}$ and $C_0(D)$ the subclass of $C(\overline{D})$ consisting of functions which vanish continuously on ∂D.

3 Estimates and properties of G_α^D

We provide in this section some estimates on the Green function $G_\alpha^D(x, y)$ and some interesting properties of the potential kernel G_α^D, related to potential theory.

Proposition 4 *For each* $x, y \in D$, *we have*

$$G_\alpha^D(x, y) \approx \frac{\delta(x)\delta(y)}{|x-y|^{d-\alpha}\left(|x-y|^2 + \delta(x)\delta(y)\right)} \tag{3.1}$$

and

$$\delta(x)\delta(y) \le c\, G_\alpha^D(x,y). \tag{3.2}$$

Moreover, if $|x-y| \geqslant r$ *then*

$$G_\alpha^D(x,y) \le c\frac{\delta(x)\delta(y)}{r^{d+2-\alpha}}. \tag{3.3}$$

Proof Since for each $a, b > 0$, we have $\min(a,b) \approx \frac{ab}{a+b}$, then from (2.3) we deduce (3.1). Inequalities (3.2) and (3.3) follow immediately from (3.1). □

Proposition 5 *If f and g are in $B^+(D)$ such that $g \le f$ and the potential function $G_\alpha^D f$ is continuous in D. Then the potential function $G_\alpha^D g$ is also continuous in D.*

Proof Let $\theta \in B^+(D)$ be such that $f = g + \theta$. So, we have $G_\alpha^D f = G_\alpha^D g + G_\alpha^D \theta$. Now, since $G_\alpha^D g$ and $G_\alpha^D \theta$ are two lower semi-continuous functions in D, we deduce the result. □

It is the same as the case $\alpha = 2$, the potential kernel G_α^D satisfies the complete maximum principle, i.e. for each $f \in B^+(D)$ and $v \in \mathcal{S}_\alpha^D$, such that $G_\alpha^D f \le v$ in $\{f > 0\}$, we have $G_\alpha^D f \le v$ in D (see [3, Chap. II, proposition 7.1]). Consequently, we deduce the following

Proposition 6 *Let $h \in B^+(D)$ and $v \in \mathcal{S}_\alpha^D$. Let w be a Borel measurable function in D such that $G_\alpha^D(h|w|) < \infty$ and $v = w + G_D^\alpha(hw)$. Then w satisfies*

$$0 \le w \le v.$$

Proof Since $G_\alpha^D(h|w|) < \infty$, then we have

$$G_\alpha^D\left(hw^+\right) \le v + G_\alpha^D\left(hw^-\right) \text{ in } \{w > 0\} = \left\{w^+ > 0\right\}.$$

Now, since the function $v + G_\alpha^D\left(hw^-\right)$ is in $\mathcal{S}_\alpha^D$, then we deduce by the complete maximum principle that

$$G_\alpha^D\left(hw^+\right) \le v + G_\alpha^D\left(hw^-\right), \text{ in } D.$$

That is

$$G_\alpha^D(hw) \le v = w + G_\alpha^D(hw).$$

Hence, we obtain

$$0 \le w \le w + G_\alpha^D(hw) = v.$$

□

Remark 3 Let $\lambda \in \mathbb{R}$ and q be the function defined on D by

$$q(x) = \frac{1}{(\delta(x))^{\lambda}}.$$

As it is mentioned above, for any $\psi \in B^{+}(D)$, the function $G_{\alpha}^{D}\psi$ is a potential if and only if $\int_D \delta(y)\psi(y)dy < \infty$. Then by ([9, lemma p. 726]), we conclude that $G_{\alpha}^{D}q$ is a potential if and only if $\lambda < 2$. We shall give in Proposition 7 below, estimates on $G_{\alpha}^{D}q$, for $\lambda < 2$.

This will provide us a class of potential functions p defined on D and satisfying

$$p(x) \approx (\delta(x))^{\beta}, \alpha - 2 < \beta \leq 1.$$

To this end, we need the following lemma due to [11].

In what follows, we put for $x \in D$

$$D_1 = \left\{ y \in D, |x-y|^2 \leq \delta(x)\delta(y) \right\}$$

$$D_2 = \left\{ y \in D, |x-y|^2 \geq \delta(x)\delta(y) \right\}.$$

Lemma 1 *Let $x \in D$, then we have*

(i) *If $y \in D_1$, then*

$$\frac{3-\sqrt{5}}{2}\delta(x) \leq \delta(y) \leq \frac{3+\sqrt{5}}{2}\delta(x) \text{ and } |x-y| \leq \frac{1+\sqrt{5}}{2}\min(\delta(x), \delta(y)).$$

(ii) *If $y \in D_2$, then*

$$\max(\delta(x), \delta(y)) \leq \frac{\sqrt{5}+1}{2}|x-y|.$$

In particular,

$$B\left(x, \frac{\sqrt{5}-1}{2}\delta(x)\right) \subset D_1 \subset B\left(x, \frac{\sqrt{5}+1}{2}\delta(x)\right).$$

Proposition 7 *Let $d_0 = diam(D)$ and q be the function defined on D by $q(x) = \delta(x)^{-\lambda}$, $\lambda < 2$. For $x \in D$, we have*

(i) $G_{\alpha}^{D}q(x) \approx \delta(x)^{\alpha-\lambda}, \quad if \alpha - 1 < \lambda < 2$
(ii) $G_{\alpha}^{D}q(x) \approx \delta(x)\log(\frac{2d_0}{\delta(x)}), \quad if \lambda = \alpha - 1$
(iii) $G_{\alpha}^{D}q(x) \approx \delta(x), \quad if \lambda < \alpha - 1$

Proof Let $\lambda < 2$. We obtain from (2.3) that

$$I(x) = \int_D G_\alpha^D(x,y)\frac{1}{(\delta(y))^\lambda}dy \approx I_1(x) + I_2(x),$$

where

$$I_1(x) = \int_{D_1} \frac{1}{|x-y|^{d-\alpha}}\frac{1}{(\delta(y))^\lambda}dy$$

and

$$I_2(x) = \int_{D_2} \frac{\delta(x)\delta(y)^{1-\lambda}}{|x-y|^{d-\alpha+2}}dy.$$

It is clear from Lemma 1 that

$$\frac{1}{c}\frac{1}{\delta(x)^\lambda}\int_{B(x,\frac{\sqrt{5}-1}{2}\delta(x))}\frac{dy}{|x-y|^{d-\alpha}} \le I_1(x) \le \frac{c}{(\delta(x))^\lambda}\int_{B(x,\frac{\sqrt{5}+1}{2}\delta(x))}\frac{dy}{|x-y|^{d-\alpha}},$$

i.e.

$$\frac{1}{c}\frac{1}{(\delta(x))^\lambda}\int_0^{\frac{\sqrt{5}-1}{2}\delta(x)} r^{\alpha-1}dr \le I_1(x) \le \frac{c}{(\delta(x))^\lambda}\int_0^{\frac{\sqrt{5}+1}{2}\delta(x)} r^{\alpha-1}dr.$$

This implies that

$$I_1(x) \approx (\delta(x))^{\alpha-\lambda}, \quad x \in D. \tag{3.4}$$

Now, we shall estimate $I_2(x)$. Let $\alpha - 1 < \lambda < 2$. We derive the estimates by considering two cases.

Case 1: $\alpha - 1 < \lambda < \alpha$. Since for each $x \in D$, $y \in D_2$, we have $\delta(y) \le \frac{\sqrt{5}+1}{2}|x-y|$, then we get

$$I_2(x) \le c(\delta(x))^{\alpha-\lambda}\int_{D_2}\left(\frac{\delta(x)}{\delta(y)}\right)^{\lambda+1-\alpha}\frac{1}{|x-y|^d}dy.$$

Using the fact that $0 < \lambda + 1 - \alpha < 1$, we deduce from [11, corollary 2.8] that

$$I_2(x) \le c(\delta(x))^{\alpha-\lambda}, \quad x \in D.$$

Case 2: $\alpha \le \lambda < 2$. We distinguish two subcases:

If $\lambda \leq 1$, we have by Lemma 1

$$I_2(x) \leq c \int_{D_2} \frac{\delta(x)}{|x-y|^{d+\lambda+1-\alpha}} dy$$
$$\leq c\delta(x) \int_{\frac{\sqrt{5}-1}{2}\delta(x)}^{2d_0} r^{\alpha-\lambda-2} dr$$
$$\leq c(\delta(x))^{\alpha-\lambda}.$$

If $1 < \lambda < 2$, it follows from Lemma 1 that

$$I_2(x) \leq c(\delta(x))^{\alpha-\lambda} \int_{D_2} \left(\frac{\delta(x)}{\delta(y)}\right)^{\lambda-1} \frac{1}{|x-y|^d} dy.$$

Since $0 < \lambda - 1 < 1$, we deduce again from [11, corollary 2.8] that $I_2(x) \leq c(\delta(x))^{\alpha-\lambda}$. This together with (3.4) gives the assertion (i).

Now, let $\lambda < \alpha - 1$. Then $2 - \alpha < 1 - \lambda$ and by Lemma 1, we obtain

$$I_2(x) \leq c\delta(x).$$

Thus, the assertion (iii) follows immediately from (3.2) to (3.4).

Finally, let $\lambda = \alpha - 1$. We remark from (i) that $G^D_{2-\alpha}(\delta(\cdot)^{-1})(x) \approx \delta(x)^{1-\alpha}$. So using (2.5), we deduce that

$$G^D_\alpha \left(\delta(\cdot)^{1-\alpha}\right)(x) \approx G^D_\alpha G^D_{2-\alpha}(\delta(\cdot)^{-1})(x) = G^D(\delta(\cdot)^{-1})(x).$$

Hence (ii) holds by using the following estimates proved in [2, example 6 (ii)]

$$G^D(\delta(\cdot)^{-1})(x) \approx \delta(x)\log(\frac{2d_0}{\delta(x)}), x \in D.$$

□

4 The Kato class $K_\alpha(D)$

We look in this section at some interesting properties of functions belonging to the Kato class $K_\alpha(D)$ (see Definition 1). In particular, we characterize this class by means of the density $q^\alpha(t, x, y)$ of the semigroup $\left(Q^\alpha_t\right)_{t>0}$. Also a careful analysis about equicontinuity of some family of functions is performed. First to illustrate the class $K_\alpha(D)$, let us present the following.

4.1 A subclass in $K_\alpha(D)$

Proposition 8 *Let $p > \frac{d}{\alpha}$ and $q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $d_0 = diam(D)$ and θ be a nonegative continuous function in $(0, 2d_0)$ satisfying for some $\eta > 0$ the following conditions:*

(i) *The function $t \to t^{\alpha - \frac{d}{p}}\theta(t)$ is nondecreasing on $(0, \eta)$ and $\lim_{t\to 0^+} t^{\alpha-\frac{d}{p}}\theta(t) = 0$.*
(ii) *The function $t \to \max(\theta(t), 1)$ is nonincreasing on $(0, \eta)$.*
(iii) *The function $t \to t^{\alpha-1-\frac{d-1}{p}}\theta(t) \in L^q((0, \eta))$.*

Then we have

$$\theta(\delta(\cdot))L^p(D) \subset K_\alpha(D).$$

Proof Let $p > \frac{d}{\alpha}$ and $q \geq 1$ be such that $\frac{1}{q} + \frac{1}{p} = 1$. Let $\varphi \in L^p(D)$ and $\theta : (0, 2d_0) \longrightarrow [0, \infty)$ be a continuous function satisfying (i)–(iii). Let $r > 0$ and $x \in D$ then

$$\begin{aligned}\int_{B(x,r)\cap D} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|\theta(\delta(y))dy &= \int_{B(x,r)\cap D_1} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|\theta(\delta(y))dy \\ &\quad + \int_{B(x,r)\cap D_2} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|\theta(\delta(y))dy \\ &= I_1(x) + I_2(x)\end{aligned}$$

where D_1 and D_2 are the sets define before Lemma 1. We aim to show that $I_1(x)$ and $I_2(x)$ tend to zero as $r \longrightarrow 0$, uniformly on x.

First, we remark by using (2.3) and Lemma 1 that

$$\frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y) \leq \frac{c}{|x - y|^{d-\alpha}}, \text{ if } x \in D \text{ and } y \in D_1 \tag{4.1}$$

and

$$\frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y) \leq c\frac{(\delta(y))^2}{|x - y|^{d+2-\alpha}}, \text{ if } x \in D \text{ and } y \in D_2. \tag{4.2}$$

Now let us estimate $I_1(x)$. For simplicity, we put

$$\beta = \frac{3 - \sqrt{5}}{2}, \sigma = \frac{1 + \sqrt{5}}{2}$$

and $\rho(x) = \min(r, \beta\delta(x))$. So by Lemma 1, we have for $y \in B(x, r) \cap D_1$

$$|x - y| \leq \min(r, \sigma\delta(x)) \leq \frac{\sigma}{\beta}\min(r, \beta\delta(x)) = \frac{\sigma}{\beta}\rho(x).$$

Since θ satisfies (ii), we get by the Hölder inequality and (4.1),

$$\begin{aligned}
I_1(x) &\le c \int\limits_{B(x,r)\cap D_1} \frac{1}{|x-y|^{d-\alpha}} |\varphi(y)| \max(\theta(\delta(y)), 1) dy \\
&\le c \, \|\varphi\|_p \max(\theta(\beta\delta(x)), 1) \left(\int\limits_{B(x,r)\cap D_1} |x-y|^{(\alpha-d)q} \, dy \right)^{\frac{1}{q}} \\
&\le c \, \|\varphi\|_p \max\left(\theta(\beta\delta(x)), 1\right) \left(\int\limits_0^{\frac{\sigma}{\beta}\rho(x)} t^{d-1+(\alpha-d)q} dt \right)^{\frac{1}{q}} \\
&\le c \, \|\varphi\|_p \max(\theta(\rho(x)), 1)(\rho(x))^{\alpha-\frac{d}{p}} \\
&\le c \, \|\varphi\|_p \max\left((\rho(x))^{\alpha-\frac{d}{p}} \theta(\rho(x)), (\rho(x))^{\alpha-\frac{d}{p}} \right).
\end{aligned}$$

Then, since θ satisfies (i), we deduce that

$$I_1(x) \le c \, \|\varphi\|_p \max\left(r^{\alpha-\frac{d}{p}} \theta(r), r^{\alpha-\frac{d}{p}} \right).$$

This implies by (i) that $I_1(x)$ tends to zero as $r \to 0$, uniformly on x.

To control $I_2(x)$, we remark by (i) that the function $t^2\theta(t) = \left(t^{\alpha-\frac{d}{p}}\theta(t)\right) t^{2-\alpha+\frac{d}{p}}$ is nondecreasing. So, using the fact that if $y \in D_2$, $\delta(y) \le \sigma|x-y|$, we obtain

$$\begin{aligned}
I_2(x) &\le c \int\limits_{B(x,r)\cap D_2} \frac{(\delta(y))^2 \, \theta\left(\delta(y)\right)}{|x-y|^{d+2-\alpha}} |\varphi(y)| \, dy \\
&\le c \int\limits_{B(x,r)\cap D_2} \frac{1}{|x-y|^{d-\alpha}} \theta(\sigma|x-y|)|\varphi(y)|dy.
\end{aligned}$$

By the Hölder inequality, we obtain

$$\begin{aligned}
I_2(x) &\le c \, \|\varphi\|_p \left(\int\limits_{B(x,r)\cap D_2} |x-y|^{(\alpha-d)q} \left(\theta(\sigma\,|x-y|)\right)^q dy \right)^{\frac{1}{q}} \\
&\le c \, \|\varphi\|_p \left(\int_0^{\sigma r} t^{\left(\alpha-1-\frac{d-1}{p}\right)q} (\theta(t))^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Now, using condition (iii), we deduce that $I_2(x)$ tends to zero as $r \to 0$, uniformly in x. This ends the proof. □

As application of the above proposition, we quote

Example 1 Let $m \in \mathbb{N}^*$ and a be a sufficiently large positive real number such that the function

$$\theta(t) = t^{-\lambda} \prod_{k=1}^{m} \left(\log_k \frac{a}{t}\right)^{-\mu_k}$$

is defined and positive on $(0, 2d_0)$, where $\log_k(x) = \log \circ \log \circ \cdots \circ \log(x)$ (k times).
Let $p > \frac{d}{\alpha}$, then if one of the following conditions is satisfied

- $\lambda < \alpha - \frac{d}{p}$ and $\mu_k \in \mathbb{R}$ for $k \in \mathbb{N}^*$,
- $\lambda = \alpha - \frac{d}{p}$, $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 1 - \frac{1}{p}$, $\mu_k > 1 - \frac{1}{p}$ and $\mu_j \in \mathbb{R}$ for $j > k$,

we have

$$\theta(\delta(\cdot))L^p(D) \subset K_\alpha(D).$$

4.2 Properties of functions in $K_\alpha(D)$

Lemma 2 *Let φ be a function in $K_\alpha(D)$. Then the function*

$$x \to \delta(x)^2\varphi(x),$$

is in $L^1(D)$. In particular, $K_\alpha(D) \subset L^1_{loc}(D)$.

Proof Let $\varphi \in K_\alpha(D)$, then there exists $r > 0$ such that for each $x \in D$,

$$\int_{D\cap B(x,r)} \frac{\delta(y)}{\delta(x)} G^D_\alpha(x, y)\, |\varphi(y)|\, dy \leq 1.$$

Let $x_1, x_2, \ldots, x_m$ in D be such that $D \subset \bigcup_{i=1}^{m} B(x_i, r)$, then by (3.2) there exists $c > 0$ such that for each $i \in \{1, 2, \ldots, m\}$ and $y \in B(x_i, r) \cap D$, we have

$$\delta(y)^2 \leq c \frac{\delta(y)}{\delta(x_i)} G^D_\alpha(x_i, y).$$

Hence, we have

$$\begin{aligned}\int_D \delta(y)^2\, |\varphi(y)|\, dy &\leq c \sum_{i=1}^{m} \int_{D\cap B(x_i,r)} \frac{\delta(y)}{\delta(x_i)} G^D_\alpha(x_i, y)\, |\varphi(y)|\, dy \\ &\leq cm < \infty.\end{aligned}$$

This completes the proof. □

In the sequel, we use the notation

$$\|\varphi\|_D := \sup_{x\in D} \int_D \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|dy$$

and

$$a_\alpha(\varphi) := \sup_{x,y\in D} \int_D \frac{G_\alpha^D(x, z)G_\alpha^D(z, y)}{G_\alpha^D(x, y)} |\varphi(z)|dz.$$

Proposition 9 *Let φ be a function in $K_\alpha(D)$, then*

$$a_\alpha(\varphi) \le 2C_0\|\varphi\|_D < \infty,$$

where C_0 is the constant given in Theorem 1.

Proof Let $\varphi \in K_\alpha(D)$, then the first inequality follows immediately from Theorem 1. Now to prove that $\|\varphi\|_D$ is finite, we consider $r > 0$ such that for $x \in D$

$$\int_{B(x,r)\cap D} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|dy \le 1.$$

So using (3.3) and Lemma 2, we deduce that for $x \in D$

$$\begin{aligned}\int_D \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|dy &\le \int_{B(x,r)\cap D} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|dy \\ &\quad + \int_{B^c(x,r)\cap D} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y)|\varphi(y)|dy \\ &\le 1 + \frac{C}{r^{d+2-\alpha}} \int_D (\delta(y))^2|\varphi(y)|dy < \infty.\end{aligned}$$

This ends the proof. □

Proposition 10 *Let φ be a function in $K_\alpha(D)$. Then for any function h in $\mathcal{S}_\alpha^D$ and $x \in D$, we have*

$$\int_D G_\alpha^D(x, y)\,|\varphi(y)|\,h(y)dy \le a_\alpha(\varphi)h(x). \tag{4.3}$$

Moreover, we have for $x_0 \in \overline{D}$

$$\lim_{r\to 0}\left(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap B(x_0,r)} G_\alpha^D(x,y)\,|\varphi(y)|\,h(y)dy\right)=0. \tag{4.4}$$

Proof Let h be a function in $\mathcal{S}_\alpha^D$. Then by [3, Chap. II, proposition 3.11], there exists a sequence (f_k) of nonnegative measurable functions in D such that for all $y \in D$

$$h(y)=\sup_k \int_D G_\alpha^D(y,z)f_k(z)dz.$$

Hence, it is enough to prove (4.3) and (4.4) for $h(y) = G_\alpha^D(y,z)$ uniformly in $z \in D$.
Let $\varphi \in K_\alpha(D)$. We have for all $x, z \in D$

$$\int_D G_\alpha^D(x,y)G_\alpha^D(y,z)|\varphi(y)|dy \le a_\alpha(\varphi)G_\alpha^D(x,z).$$

Then (4.3) holds. Now, we shall prove (4.4). Let $\varepsilon > 0$ and $r_0 > 0$ such that

$$\sup_{\xi\in D}\int_{D\cap B(\xi,r_0)}\frac{\delta(y)}{\delta(\xi)}G_\alpha^D(\xi,y)|\varphi(y)|dy \le \varepsilon.$$

Let $r > 0$. We deduce from Theorem 1 and (3.3) that for all $x, z \in D$

$$\begin{aligned}
&\frac{1}{G_\alpha^D(x,z)}\int_{D\cap B(x_0,r)} G_\alpha^D(x,y)G_\alpha^D(y,z)|\varphi(y)|dy\\
&\le C_0\int_{D\cap B(x_0,r)}\left(\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)+\frac{\delta(y)}{\delta(z)}G_\alpha^D(y,z)\right)|\varphi(y)|dy\\
&\le 2C_0\sup_{\xi\in D}\int_{D\cap B(x_0,r)}\frac{\delta(y)}{\delta(\xi)}G_\alpha^D(\xi,y)|\varphi(y)|dy\\
&\le 2C_0\sup_{\xi\in D}\left(\int_{D\cap B(\xi,r_0)}\frac{\delta(y)}{\delta(\xi)}G_\alpha^D(\xi,y)|\varphi(y)|dy\right.\\
&\quad\left.+\int_{D\cap B(x_0,r)\cap B^c(\xi,r_0)}\frac{\delta(y)}{\delta(\xi)}G_\alpha^D(\xi,y)|\varphi(y)|dy\right)
\end{aligned}$$

$$\leq 2C_0\varepsilon + \frac{c}{r_0^{d+2-\alpha}} \int\limits_{D\cap B(x_0,r)} (\delta(y))^2|\varphi(y)|dy.$$

By letting $r \to 0$ and using Lemma 2, we reach (4.4). □

Corollary 1 *Let $\alpha - 2 \leq \beta \leq 1$. Then there exists $c > 0$ such that for any φ in $K_\alpha(D)$*

$$\sup_{x\in D} \int\limits_D \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y)|\varphi(y)|dy \leq c a_\alpha(\varphi) \tag{4.5}$$

and for $x_0 \in \overline{D}$

$$\lim_{r\to 0}\left(\sup_{x\in D} \int\limits_{D\cap B(x_0,r)} \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y)|\varphi(y)|dy\right) = 0. \tag{4.6}$$

Proof By (2.11), the function $x \to \delta(x)^{\alpha-2}$ is comparable to $M_\alpha^D 1$ which is in $\mathcal{H}_\alpha^D$. Also, we know from Proposition 7, that for $\alpha - 2 < \beta \leq 1$, the function $x \to \delta(x)^\beta$ is comparable to $G_\alpha^D\left(\delta(\cdot)^{\alpha-\beta}\right)$ which is in $\mathcal{S}_\alpha^D$. Hence (4.5) and (4.6) are obtained obviously from (4.3) and (4.4). □

Remark 4 Let φ be a function in $K_\alpha(D)$ and putting $\beta = 1$ in (4.5), we obtain that $\|\varphi\|_D \leq c a_\alpha(\varphi)$ and by Proposition 9, we deduce that $\|\varphi\|_D \approx a_\alpha(\varphi)$.

Corollary 2 *Let φ be a function in $K_\alpha(D)$. Then the function $x \to \delta(x)^{\alpha-1}\varphi(x)$ is in $L^1(D)$.*

Proof Let $x_0 \in D$. By (3.2) and (4.5), it follows that

$$\int\limits_D \delta(y)^{\alpha-1}|\varphi(y)|dy \leq c \int\limits_D \left(\frac{\delta(y)}{\delta(x_0)}\right)^{\alpha-2} G_\alpha^D(x_0,y)|\varphi(y)|dy < \infty.$$

□

4.3 Characterization of $K_\alpha(D)$ by means of $q^\alpha(t,x,y)$

Lemma 3 *For each $t > 0$ and x, y in D, we have*

$$\int\limits_0^t q^\alpha(s,x,y)ds \leq G_\alpha^D(x,y). \tag{4.7}$$

Moreover, if $|x-y| \leq t^{1/\alpha}$ then

$$G_\alpha^D(x,y) \leq c\int\limits_0^t q^\alpha(s,x,y)ds. \tag{4.8}$$

Proof Let $t > 0$ and $x, y \in D$. The inequality (4.7) holds obviously from (2.2). Now, we suppose that $|x-y| \le t^{\frac{1}{\alpha}}$. Using (2.1), we have

$$\int_0^t q^\alpha(s,x,y)ds \ge c\int_0^t s^{\frac{-d}{\alpha}} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2 + s^{\frac{2}{\alpha}}}, 1\right)\left(1 + \frac{|x-y|^2}{s^{\frac{2}{\alpha}}}\right)^{-\frac{d+\alpha}{2}} ds$$

Put $r = |x-y|^2 s^{-\frac{2}{\alpha}}$, then we have

$$\begin{aligned}\int_0^t q^\alpha(s,x,y)ds &\ge c|x-y|^{\alpha-d} \int_{|x-y|^2 t^{-\frac{2}{\alpha}}}^{\infty} r^{\frac{d-\alpha}{2}-1}(1+r)^{-\frac{d+\alpha}{2}} \min\left(\frac{\delta(x)\delta(y)}{(1+\frac{1}{r})|x-y|^2}, 1\right)dr \\ &\ge c|x-y|^{\alpha-d}\int_1^\infty r^{-\alpha-1}\min\left(\frac{\delta(x)\delta(y)}{(1+\frac{1}{r})|x-y|^2}, 1\right)dr \\ &\ge c|x-y|^{\alpha-d}\min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}, 1\right).\end{aligned}$$

Now we deduce the inequality (4.8) from (2.3). □

Lemma 4 *Let φ be a nonnegative function in $K_\alpha(D)$, then for each $r > 0$, we have*

$$\sup_{0<t<1}\left(\sup_{x\in D}\int_{(|x-y|\ge r)\cap D} \frac{\delta(y)}{\delta(x)} q^\alpha(t,x,y)\varphi(y)dy\right) := M(r) < \infty. \tag{4.9}$$

Proof Let $0 < t < 1$ and $0 < r \le |x-y|$. Using the fact that for $a, b \in (0,\infty)$ we have $\min(a,b) \approx \frac{ab}{a+b}$, we deduce from (2.1) that

$$\begin{aligned}\frac{\delta(y)}{\delta(x)} q^\alpha(t,x,y) &\le c\frac{t\delta^2(y)}{\delta(y)\delta(x) + |x-y|^2 + t^{\frac{2}{\alpha}}}\left(|x-y|^2 + t^{\frac{2}{\alpha}}\right)^{-\frac{d+\alpha}{2}} \\ &\le c\delta^2(y)\left(|x-y|^2 + t^{\frac{2}{\alpha}}\right)^{-\frac{d+\alpha+2}{2}} \\ &\le c\delta^2(y) r^{-d+\alpha+2}.\end{aligned}$$

Then we conclude from Lemma 2 that

$$M(r) \le c\int_D \delta^2(y)\varphi(y)dy < \infty.$$

This leads to (4.9). □

Proposition 11 *A Borel measurable function φ in D belongs to the class $K_\alpha(D)$ if and only if*

$$\lim_{t\to 0}\left(\sup_{x\in D}\int_D\int_0^t \frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\,|\varphi(y)|\,dsdy\right)=0. \tag{4.10}$$

Proof Suppose that the function φ satisfies (4.10), then using (4.8) for $t=r^\alpha$, we deduce that

$$\int_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\varphi(y)dy\le c\int_0^{r^\alpha}\int_D\frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\varphi(y)dyds.$$

This implies that

$$\lim_{r\longrightarrow 0}\left(\sup_{x\in D}\int_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\varphi(y)dy\right)=0$$

and so $\varphi\in K_\alpha(D)$.

Conversely, suppose that φ is a nonnegative function in $K_\alpha(D)$. Let $\varepsilon>0$ and $r>0$ such that

$$\sup_{x\in D}\int_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\varphi(y)dy\le\varepsilon.$$

Using (4.7) and (4.9), we have for $0<t<1$

$$\begin{aligned}0\le\int_D\int_0^t\frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\varphi(y)dsdy&=\int_{D\cap B(x,r)}\int_0^t\frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\varphi(y)dsdy\\&\quad+\int_{(|x-y|\ge r)\cap D}\int_0^t\frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\varphi(y)dsdy\\&\le\int_{D\cap B(x,r)}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)\varphi(y)dy\\&\quad+\int_0^t\int_{(|x-y|\ge r)\cap D}\frac{\delta(y)}{\delta(x)}q^\alpha(s,x,y)\varphi(y)dyds\\&\le\varepsilon+tM(r).\end{aligned}$$

Then φ satisfies (4.10). □

4.4 Equicontinuity

In order to prove our existence results, we need the following theorem. The idea of the proof follows closely from the properties of functions in $K_\alpha(D)$.

Theorem 5 *Let $\alpha - 2 \leq \beta < 1$. Let φ be a nonnegative function in $K_\alpha(D)$, then the family of functions*

$$\Lambda_\varphi = \left\{ x \longrightarrow T(\theta)(x) = \int_D \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y)\theta(y)dy, \theta \in K_\alpha(D), |\theta| \leq \varphi \right\}$$

is uniformly bounded and equicontinuous in $\overline{D}$. Consequently Λ_φ is relatively compact in $C_0(D)$.

Proof Let φ be a nonnegative function in $K_\alpha(D)$ and $\theta \in K_\alpha(D)$ such that $|\theta| \leq \varphi$ in D. By (4.5), we have

$$\sup_{x\in D} |T\theta(x)| \leq \sup_{x\in D} \int_D \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y)\varphi(y)dy < +\infty.$$

Hence Λ_φ is uniformly bounded.

Let us prove the equicontinuity. Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. By (4.6), there exists $r > 0$ such that

$$\sup_{\zeta\in D} \int_{D\cap B(x_0,2r)} \left(\frac{\delta(y)}{\delta(\varsigma)}\right)^\beta G_\alpha^D(\zeta,y)\varphi(y)dy \leq \varepsilon.$$

If $x_0 \in D$ and $x, x' \in B(x_0, r) \cap D$, then we have

$$\begin{aligned}
\left|T\theta(x) - T\theta(x')\right| &\leq \int_D \left| \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y) - \left(\frac{\delta(y)}{\delta(x')}\right)^\beta G_\alpha^D(x',y) \right| \varphi(y)dy \\
&\leq 2 \sup_{\zeta\in D} \int_{D\cap B(x_0,2r)} \left(\frac{\delta(y)}{\delta(\zeta)}\right)^\beta G_\alpha^D(\zeta,y)\varphi(y)dy \\
&\quad + \int_{D\cap B^c(x_0,2r)} \left| \left(\frac{\delta(y)}{\delta(x)}\right)^\beta G_\alpha^D(x,y) - \left(\frac{\delta(y)}{\delta(x')}\right)^\beta G_\alpha^D(x',y) \right| \varphi(y)dy \\
&\leq 2\varepsilon + I\left(x, x'\right).
\end{aligned}$$

On the other hand, since $|x - x_0| \le r$ and $|x' - x_0| \le r$, then for $y \in B^c(x_0, 2r)$, we have $|x - y| \ge r$ and $|x' - y| \ge r$. So we deduce that

$$\left|\left(\frac{\delta(y)}{\delta(x)}\right)^{\beta} G_{\alpha}^{D}(x,y) - \left(\frac{\delta(y)}{\delta(x')}\right)^{\beta} G_{\alpha}^{D}(x',y)\right| \le c\frac{\delta(y)^{\beta+1}}{r^{d+2-\alpha}} \le c\delta(y)^{\alpha-1}.$$

Now since the function $x \to \frac{G_D^{\alpha}(x,y)}{\delta(x)^{\beta}}$ is continuous off the diagonal, we conclude by Corollary 2 and the dominated convergence theorem that $I(x, x')$ tends to zero as $|x - x'| \to 0$.

If $x_0 \in \partial D$ and $x \in B(x_0, r) \cap D$, then we have

$$\begin{aligned} |T\theta(x)| &\le \sup_{\varsigma\in D} \int_{D\cap B(x_0,2r)} \left(\frac{\delta(y)}{\delta(\zeta)}\right)^{\beta} G_{\alpha}^{D}(\varsigma,y)\varphi(y)dy \\ &\quad + \int_{D\cap B^c(x_0,2r)} \left(\frac{\delta(y)}{\delta(x)}\right)^{\beta} G_{\alpha}^{D}(x,y)\varphi(y)dy \\ &\le \varepsilon + J(x) \end{aligned}$$

Now since $\beta < 1$, we have by (3.1) that $\frac{G_{\alpha}^{D}(x,y)}{\delta(x)^{\beta}} \to 0$ as $|x - x_0| \to 0$, for $y \in B^c(x_0, 2r)$. So by same argument as for $I(x, x')$, we prove that $J(x)$ tends to 0 as $|x - x_0| \to 0$. Consequently, by Ascoli's theorem, we deduce that Λ_{φ} is relatively compact in $C_0(D)$. □

5 Proof of Theorem 3

In this section, we aim at proving the existence of a positive continuous solution for the following boundary value problem

$$(P_{\lambda}) \begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = \varphi(\cdot, u) \;\; in \;\; D \;\; (in\ the\ distributional\ sense) \\ \lim\limits_{x\longrightarrow\partial D} \frac{u(x)}{M_{\alpha}^{D}1(x)} = \lambda, \end{cases}$$

where λ is a nonnegative constant.

Remark 5 (i) For $\lambda > 0$, we shall prove also the uniqueness of the solution of problem (P_{λ}).
(ii) We remark that problem (P_0) is equivalent to problem (2.12).

Lemma 5 *Let w be a nonnegative continuous function in D, satisfying*

$$\lim_{x\longrightarrow\partial D} \frac{w(x)}{M_{\alpha}^{D}1(x)} = \lambda \ge 0. \tag{5.1}$$

Then $G^D_{2-\alpha}w$ is continuous in D and $\lim_{x\longrightarrow\partial D} G^D_{2-\alpha}w(x) = \lambda$.

Proof Since the function $x \to \frac{w(x)}{M^D_\alpha 1(x)}$ is nonnegative and continuous in D and satisfies (5.1), it follows that there exists $c > 0$ such that for $x \in D$, we get

$$0 \leq \frac{w(x)}{M^D_\alpha 1(x)} \leq c.$$

This implies by (2.10) and Proposition 5 that $G^D_{2-\alpha}w$ is continuous in D and consequently we have $\int_D \delta(y)w(y)dy < \infty$.

Now, for $\eta > 0$, we denote by D_η the set defined by

$$D_\eta = \{x \in D; \delta(x) < \eta\}.$$

Let $\varepsilon > 0$, then it follows from (5.1) that there exists $\eta_0 > 0$ such that

$$|w(x) - \lambda M^D_\alpha 1(x)| \leq \varepsilon M^D_\alpha 1(x), \quad x \in D_{\eta_0}.$$

So for $x \in D_{\frac{\eta_0}{2}}$, we deduce from (2.3), (2.10), and (2.11) that

$$\begin{aligned}
|G^D_{2-\alpha}w(x) - \lambda| &\leq \int_D G^D_{2-\alpha}(x, y)\left|w(y) - \lambda M^D_\alpha 1(y)\right| dy \\
&\leq \int_{D_{\eta_0}} G^D_{2-\alpha}(x, y)\left|w(y) - \lambda M^D_\alpha 1(y)\right| dy \\
&\quad + \int_{D^c_{\eta_0}} G^D_{2-\alpha}(x, y)\left|w(y) - \lambda M^D_\alpha 1(y)\right| dy \\
&\leq \varepsilon + c\int_{D^c_{\eta_0}} \frac{\delta(x)\delta(y)}{|x-y|^{d+2-\alpha}}(w(y) + \lambda M^D_\alpha 1(y))dy \\
&\leq \varepsilon + c\delta(x)\left(\int_D \delta(y)w(y)dy + \lambda\int_D (\delta(y))^{\alpha-1}\, dy\right).
\end{aligned}$$

Hence it follows that $G^D_{2-\alpha}w(x) \longrightarrow \lambda$ as $x \longrightarrow \partial D$. This completes the proof. □

Lemma 6 *Let φ be a function satisfying (H_1) and (H_2) and w be a positive continuous function in D such that*

$$\lim_{x\longrightarrow\partial D} \frac{w(x)}{M^D_\alpha 1(x)} = \lambda > 0. \tag{5.2}$$

Then we have the following

(i) $G_\alpha^D(\varphi(\cdot, w)) \in C(D)$ *and satisfies* $\lim\limits_{x \longrightarrow \partial D} \frac{G_\alpha^D(\varphi(\cdot,w))(x)}{M_\alpha^D 1(x)} = 0$.

(ii) $G^D(\varphi(\cdot, w)) \in C_0(D)$.

(iii) $x \to \delta(x)\varphi(x, w(x)) \in L^1(D)$.

Proof Since the function $x \to \frac{w(x)}{M_\alpha^D 1(x)}$ is positive and continuous in D and satisfies (5.2), it follows that $w \approx M_\alpha^D 1$ in D and so by (2.11), we deduce that $w \approx \delta(\cdot)^{\alpha-2}$.

Then we conclude by the monotonicity of φ that there exists $c > 0$ such that

$$\varphi(x, w(x)) \leq \varphi(x, c\delta(x)^{\alpha-2}), \quad x \in D. \tag{5.3}$$

Put $\psi(x) := \varphi(x, c\delta(x)^{\alpha-2})$, for $x \in D$. Then we have

$$\begin{aligned} G_\alpha^D(\psi)(x) &= \int_D G_\alpha^D(x, y)\psi(y)dy \\ &= \delta(x)^{\alpha-2} \int_D \left(\frac{\delta(y)}{\delta(x)}\right)^{\alpha-2} G_\alpha^D(x, y)\delta(y)^{2-\alpha}\psi(y)dy. \end{aligned}$$

It follows from Theorem 5 that the function

$$x \to \delta(x)^{2-\alpha} G_\alpha^D \psi(x) \in C_0(D).$$

This implies in particular that $G_\alpha^D(\psi)$ is a continuous function in D and consequently by (5.3) and Proposition 5, the function $G_\alpha^D(\varphi(\cdot, w))$ is continuous in D and satisfies

$$\lim_{x \longrightarrow \partial D} \frac{G_\alpha^D(\varphi(\cdot, w))(x)}{M_\alpha^D 1(x)} = 0.$$

To prove (ii), we apply Lemma 5 to the function $G_\alpha^D(\varphi(\cdot, w))$ and we deduce that

$$G^D(\varphi(\cdot, w)) = G_{2-\alpha}^D G_\alpha^D(\varphi(\cdot, w)) \in C_0(D).$$

Finally (iii) holds from (ii). □

Remark 6 Put $\omega = \lambda M_\alpha^D 1$ in Lemma 6, we obtain that the function

$$x \to \frac{1}{M_\alpha^D 1(x)} G_\alpha^D \varphi\left(\cdot, \lambda M_\alpha^D 1\right)(x) \in C_0(D). \tag{5.4}$$

Lemma 7 *Let $\lambda > 0$ and u be a positive continuous function defined on D. Then u is a solution of problem (P_λ) if and only if u satisfies the integral equation*

$$u(x) = \lambda M_\alpha^D 1(x) + \int_D G_\alpha^D(x, y)\varphi(y, u(y))dy, \ x \in D. \tag{5.5}$$

Proof Suppose that the function u satisfies (5.5). Since φ is noninceasing with respect to the second variable, we have obviously $G_\alpha^D(\varphi(\cdot,u)) \leq G_\alpha^D\left(\varphi(\cdot,\lambda M_\alpha^D 1)\right)$. This together with (5.4) implies that $\lim\limits_{x\longrightarrow\partial D}\frac{u(x)}{M_\alpha^D 1(x)}=\lambda$. Now by Lemma 6 (ii), the function $x\rightarrow G^D(\varphi(\cdot,u))(x)$ is in $C_0(D)$. Hence, we apply $(-\Delta_{|D})^{\frac{\alpha}{2}}$ on both sides of (5.5) and we conclude by (2.7) that u is a positive continuous solution of problem (P_λ).

Conversely, suppose that u is a positive continuous solution of problem (P_λ). We claim that u satisfies

$$\begin{cases}\Delta(G_{2-\alpha}^D u - G^D(\varphi(\cdot,u)))=0 \quad (\textit{in the distributional sense})\\ \lim\limits_{x\longrightarrow\partial D}(G_{2-\alpha}^D u(x)-G^D\varphi(\cdot,u)(x))=\lambda.\end{cases}$$

To show the claim, it suffices to remark by Lemma 5 that $G_{2-\alpha}^D u$ is continuous in D and $\lim\limits_{x\longrightarrow\partial D} G_{2-\alpha}^D u(x)=\lambda$ and by Lemma 6 that $G^D(\varphi(\cdot,u))\in C_0(D)$. Thus, the claim holds by (2.6). Furthermore since the function $G_{2-\alpha}^D u - G^D\varphi(\cdot,u)$ is continuous, then by [5, corollary 7, p. 294] it is a classical harmonic function in D satisfying

$$G_{2-\alpha}^D u - G^D\varphi(\cdot,u)=\lambda,\quad on\ \ \partial D.$$

That is $G_{2-\alpha}^D(u-G_\alpha^D\varphi(\cdot,u)-\lambda M_\alpha^D 1)=0$ in D. Hence using the fact that the kernel $G_{2-\alpha}^D$ is injective, we deduce that u satisfies (5.5). This ends the proof. □

Proposition 12 *Let φ be a function satisfying (H_1) and (H_2) and let $0<\mu\leq\lambda$. Then we have*

$$0\leq u_\lambda-u_\mu\leq(\lambda-\mu)M_\alpha^D 1,$$

where u_λ and u_μ are respectively solutions of problems (P_λ) and (P_μ).

Proof Let h be the function defined on D by

$$h(x)=\begin{cases}\frac{\varphi(x,u_\lambda(x))-\varphi(x,u_\mu(x))}{u_\mu-u_\lambda(x)} & \text{if } u_\mu(x)\neq u_\lambda(x)\\ 0 & \text{if } u_\mu(x)=u_\lambda(x).\end{cases}$$

Then $h\in B^+(D)$. Using Lemma 7, we deduce

$$u_\lambda-u_\mu+G_\alpha^D\left(h(u_\lambda-u_\mu)\right)=(\lambda-\mu)M_\alpha^D 1.$$

Furthermore, by (5.4) we conclude that

$$\begin{aligned}G_\alpha^D(h|u_\lambda-u_\mu|)&\leq G_\alpha^D\varphi(\cdot,u_\lambda)+G_\alpha^D\varphi(\cdot,u_\mu)\\&\leq G_\alpha^D\varphi(\cdot,\lambda M_\alpha^D 1)+G_\alpha^D\varphi(\cdot,\mu M_\alpha^D 1)<\infty.\end{aligned}$$

Hence the result holds by Proposition 6. □

Theorem 6 *Let φ be a function satisfying $(H_1)-(H_2)$. Then for each $\lambda > 0$, problem (P_λ) has a unique positive solution $u_\lambda \in C(D)$ satisfying*

$$\lambda M_\alpha^D 1(x) \leq u_\lambda(x) \leq \gamma M_\alpha^D 1(x), \text{ for } x \in D, \tag{5.6}$$

where $\gamma > 0$.

Proof In view of (5.4), the constant

$$\gamma = \lambda + \sup_{x\in D} \frac{1}{M_\alpha^D 1(x)} G_\alpha^D \left(\varphi(\cdot, \lambda M_\alpha^D 1)\right)(x)$$

is finite.

Let Y be the closed convex set given by

$$Y = \left\{ v \in C(D) : \lambda \leq v \leq \gamma, \lim_{x\longrightarrow \partial D} v(x) = \lambda \right\}.$$

We define the integral operator T on Y by

$$Tv(x) := \lambda + \frac{1}{M_\alpha^D 1(x)} \int_D G_\alpha^D(x,y)\varphi\left(y, M_\alpha^D 1(y)v(y)\right) dy.$$

We shall prove that T has a fixed point in Y. First, we have clearly for each $v \in Y, \lambda \leq Tv \leq \gamma$. By same arguments as in the proof of Theorem 5, we obtain that TY is relatively compact in $C(\overline{D})$ with $\lim_{x\longrightarrow\partial D} Tv(x) = \lambda$. In particular $TY \subset Y$. So it remains to prove the continuity of T in Y. Consider a sequence $(v_n)_n$ in Y which converges uniformly to a function v in Y. Then, by (2.11), (H_1) and (H_2), we obtain

$$\begin{aligned}|Tv_n(x) - Tv(x)| \leq c \int_D \left(\frac{\delta(y)}{\delta(x)}\right)^{\alpha-2} G_\alpha^D(x,y)\delta(y)^{2-\alpha} \Big|\varphi(y, M_\alpha^D 1(y)v_n(y)) \\ -\varphi\left(y, M_\alpha^D 1(y)v(y)\right)\Big| dy\end{aligned}$$

and using again the monotonicity of φ, we get

$$\delta(y)^{2-\alpha}|\varphi(y, M_\alpha^D 1(y)v_n(y)) - \varphi(y, M_\alpha^D 1(y)v(y))| \leq 2\psi(y),$$

where $\psi(y) := \delta(y)^{2-\alpha}\varphi(y, \lambda M_\alpha^D 1(y))$. Now, since φ is continuous with respect to the second variable, we deduce by (4.5) and the dominated convergence theorem that

$$\forall x \in \overline{D}, \quad Tv_n(x) \longrightarrow Tv(x), \quad as\ n \longrightarrow \infty.$$

Since TY is a relatively compact family in $C(\overline{D})$, we have the uniform convergence, namely,

$$\|Tv_n - Tv\|_\infty \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Thus, we have proved that T is a compact mapping from Y to itself. Hence by the Schauder fixed-point theorem, T has a fixed point $v_\lambda \in Y$. Put $u_\lambda(x) = M_\alpha^D 1(x) v_\lambda(x)$, for $x \in D$. Then u_λ is a continuous function in D and satisfies

$$u_\lambda(x) = \lambda M_\alpha^D 1(x) + \int_D G_\alpha^D(x, y)\varphi(y, u_\lambda(y))dy, x \in D$$

and

$$\lambda M_\alpha^D 1(x) \le u_\lambda(x) \le \gamma M_\alpha^D 1(x), x \in D.$$

By Lemma 7, we conclude that u_λ is a positive solution of problem (P_λ). The uniqueness follows from Proposition 12. □

Proof of Theorem 3 Let (λ_k) be a sequence of positive real numbers, nonincreasing to zero. For each $k \in \mathbb{N}$, put

$$\gamma_k = \lambda_k + \sup_{x \in D} \frac{1}{M_\alpha^D 1(x)} G_\alpha^D \left(\varphi(\cdot, \lambda_k M_\alpha^D 1)\right)(x)$$

and denote by u_k the solution of problem (P_{λ_k}). Then by Proposition 12, the sequence (u_k) decreases to a function u and so the sequence $\left(u_k - \lambda_k M_\alpha^D 1\right)$ increases to u. Moreover, we have for each $x \in D$

$$\begin{aligned} u(x) &\ge u_k(x) - \lambda_k M_\alpha^D 1(x) \\ &= \int_D G_\alpha^D(x, y)\varphi(y, u_k(y))dy \\ &\ge G_\alpha^D \varphi(\cdot, \gamma_k M_\alpha^D 1)(x) > 0. \end{aligned}$$

Hence applying the monotone convergence theorem, we get by the continuity of φ with respect to the second variable

$$u(x) = \int_D G_\alpha^D(x, y)\varphi(y, u(y))dy, \;\; \forall\, x \in D. \tag{5.7}$$

Let us prove that u is a positive continuous solution of (2.12). It is clear that u is continuous in D. Indeed, we have

$$u = \inf_k u_k = \sup_k (u_k - \lambda_k M_\alpha^D 1)$$

and u_k and $M_\alpha^D 1$ are continuous functions in D.

Furthermore, since $0 < u(x) \le u_k(x)$, for each $x \in D$ and $k \in \mathbb{N}$, we deduce that $\lim_{x \longrightarrow \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = 0$. This implies by Lemma 5 that $G_{2-\alpha}^D u = G^D \varphi(\cdot, u) \in C_0(D)$.

Hence, applying $(-\Delta_{|D})^{\frac{\alpha}{2}}$ on both sides of Eq. 5.7, we conclude by (2.7) that u is a positive continuous solution of problem (2.12). □

Corollary 3 *Let φ be a function satisfying (H_1) and (H_2) and let f be a nonnegative continuous function on ∂D. Then the following problem*

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} u = \varphi(\cdot, u) \;\; in\; D \;\; (in\ the\ distributional\ sense) \\ \lim\limits_{x \longrightarrow z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = f(z), \end{cases} \tag{5.8}$$

has a positive continuous solution in D satisfying

$$u(x) = M_\alpha^D f(x) + G_\alpha^D(\varphi(\cdot, u))(x), x \in D.$$

Proof Let ψ be the function defined on $D \times (0, \infty)$ by

$$\psi(x, t) = \varphi(x, t + M_\alpha^D f(x)).$$

Then ψ satisfies (H_1) and (H_2). Now by Theorem 3, the following problem

$$\begin{cases} (-\Delta_{|D})^{\frac{\alpha}{2}} v = \psi(\cdot, v) \;\; in\; D\; (in\ the\ distributional\ sense) \\ \lim\limits_{x \longrightarrow \partial D} \frac{v(x)}{M_\alpha^D 1(x)} = 0, \end{cases}$$

has a positive continuous solution v satisfying $v = G_\alpha^D(\psi(\cdot, v))$ on D. Then the function

$$\begin{aligned} u &= M_\alpha^D f + v \\ &= M_\alpha^D f + G_\alpha^D(\psi(\cdot, v)) \\ &= M_\alpha^D f + G_\alpha^D(\varphi(\cdot, u)) \end{aligned}$$

is a positive continuous solution of problem (5.8). This completes the proof. □

6 Proof of Theorem 4

Before giving the proof of Theorem 4, some potential theory tools are needed. We are going to recall them in this paragraph and we refer to [4, 10] for more details. For a nonnegative measurable function q in D, we define the potential kernel V_q on $B^+(D)$ by

$$V_q f(x) := \int_0^\infty \widetilde{E}^x \left(e^{-\int_0^t q(Z_\alpha^D(s))ds} f(Z_\alpha^D(t)) \right) dt, x \in D,$$

with $V_0 := V = G_\alpha^D$.

Furthermore if q satisfies $Vq < \infty$, we have the following resolvent equation

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{6.1}$$

In particular, if $u \in B^+(D)$ is such that $V(qu) < \infty$, then we have

$$(I - V_q(q\cdot))(I + V(q\cdot))u = (I + V(q\cdot))(I - V_q(q\cdot))u = u \tag{6.2}$$

The following lemma plays a key role.

Lemma 8 *Let q be a nonnegative function in $K_\alpha(D)$ and h be a positive finite function in $\mathcal{S}_\alpha^D$. Then for all $x \in D$, we have*

$$\exp\left(-a_\alpha(q)\right) h(x) \le h(x) - V_q(qh)(x) \le h(x).$$

Proof Since $h \in \mathcal{S}_\alpha^D$, then by [3, Chap. II, proposition 3.11], there exists a sequence of nonnegative measurable functions $(f_n)_n$ in D such that $h = \sup_n Vf_n$.

Let $x \in D$ and $n \in \mathbb{N}$ be such that $0 < Vf_n(x) < \infty$. Consider $\theta(t) = V_{tq} f_n(x)$, for $t \ge 0$. Then the function θ is completely monotone on $[0, \infty)$ and so $\log \theta$ is convex on $[0, \infty)$. This implies that

$$\theta(0) \le \theta(1) \exp\left(-\frac{\theta'(0)}{\theta(0)}\right)$$

i.e.

$$Vf_n(x) \le V_q f_n(x) \exp\left(\frac{V(qVf_n)(x)}{Vf_n(x)}\right).$$

Since Vf_n is in $\mathcal{S}_\alpha^D$, it follows from (4.3) that

$$Vf_n(x) \le V_q f_n(x) \exp(a_\alpha(q)).$$

Hence by (6.1) we obtain

$$\exp(-a_\alpha(q)) Vf_n(x) \le V_q f_n(x) = Vf_n(x) - V_q(qVf_n)(x) \le Vf_n(x).$$

The result holds by letting $n \longrightarrow \infty$. □

Proof of Theorem 4 We shall convert problem (2.14) into a suitable integral equation. So we aim to show an existence result for the equation

$$u + V(u\varphi(\cdot, u)) = M_\alpha^D f. \tag{6.3}$$

Let $c_0 > 0$ be such that for each $x \in D$

$$M_\alpha^D 1(x) \le c_0 \delta^{\alpha-2}(x).$$

Put $c := c_0 \|f\|_\infty$ and $q := q_c$ be the function in $K_\alpha(D)$ given by (H_4).

Let

$$\Gamma = \{u \in B^+(D) : \exp(-a_\alpha(q)) M_\alpha^D f \leq u \leq M_\alpha^D f\}$$

and let T be the operator defined on Γ by

$$Tu = M_\alpha^D f - V_q(q M_\alpha^D f) + V_q((q - \varphi(\cdot, u))u).$$

We claim that Γ is invariant under T. Indeed, since for all $x \in D$, $M_\alpha^D f(x) \leq c\delta^{\alpha-2}(x)$, then by using hypothesis (H_4), we have for any $u \in \Gamma$

$$0 \leq \varphi(\cdot, u) \leq q. \tag{6.4}$$

Then it follows from Lemma 8 that for $u \in \Gamma$ we have

$$Tu \geq M_\alpha^D f - V_q(q M_\alpha^D f) \geq \exp(-a_\alpha(q)) M_\alpha^D f.$$

Moreover, for $u \in \Gamma$, we have $u \leq M_\alpha^D f$ and consequently

$$Tu \leq M_\alpha^D f - V_q\left(q M_\alpha^D f\right) + V_q(qu) \leq M_\alpha^D f.$$

This shows that $T\Gamma \subset \Gamma$.

Next, we will prove that the operator T has a fixed point in Γ. Let u and v be two functions in Γ such that $u \leq v$. Then from (H_4), we have

$$Tu - Tv = V_q[(q - \varphi(\cdot, u))u - (q - \varphi(\cdot, v))v] \leq 0.$$

Thus, T is nondecreasing on Γ. Now, let (u_n) be the sequence defined by

$$u_0 = \exp(-a_\alpha(q)) M_\alpha^D f \text{ and } u_{n+1} = Tu_n \quad \text{for } n \in \mathbb{N}.$$

We obviously obtain that the function u_n is in Γ and we deduce by the monotonicity of T that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq M_\alpha^D f.$$

Hence by the dominated convergence theorem and (H_4), we conclude that the sequence (u_n) converges to a function $u \in \Gamma$ satisfying

$$u = M_\alpha^D f - V_q(q M_\alpha^D f) + V_q\left[(q - \varphi(\cdot, u))u\right].$$

That is

$$(I - V_q(q\cdot))u + V_q(u\varphi(\cdot, u)) = (I - V_q(q\cdot)) M_\alpha^D f.$$

Applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (6.1) and (6.2), we deduce that u satisfies (6.3).

It remains to prove that u is a positive continuous solution of problem (2.14). Since $q \in K_\alpha(D)$, then by Theorem 5, the function $x \to \delta(x)^{2-\alpha} \int_D G_\alpha^D(x, y)q(y)\delta(y)^{\alpha-2}dy$ is in $C_0(D)$. So using that

$$0 \leq \varphi(\cdot, u)u \leq qu \leq qM_\alpha^D f \leq cq\delta^{\alpha-2},$$

it follows from Proposition 5 that the function $x \to \delta^{2-\alpha}(x)V(u\varphi(\cdot, u))(x)$ is in $C_0(D)$.

Now, going back to (6.3) and applying $(-\Delta_{|D})^{\frac{\alpha}{2}}$ on both sides, we deduce by (2.7) that u is a positive continuous solution of

$$(-\Delta_{|D})^{\frac{\alpha}{2}}u + u\varphi(\cdot, u) = 0 \;\; in \;\; D \;\; (in\ the\ distributional\ sense)$$

and satisfies $\lim\limits_{x \longrightarrow z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = f(z)$. This completes the proof. □

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